Appendix of *A New Approach to Clausification for Intuitionistic Propositional Logic*

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This document contains the omitted proofs of paper [2].

A Omitted proofs

Here we prove Th. 2 of Sec. 3 asserting that the procedure $\texttt{ClauIPL}(\alpha)$ computes a set of IPL-clauses Θ which is is equivalent to the formula α; formally:

Theorem 2. *Let $\Theta = \texttt{ClauIPL}(\alpha)$ and let $\Omega \cup \{\beta\}$ be a set of formulas such that $\mathcal{V}_{\Omega,\beta} \subseteq \mathcal{V}_\alpha$. Then, $\Omega, \alpha \models_{\mathrm{i}} \beta$ iff $\Omega, \Theta \models_{\mathrm{i}} \beta$.*

We exploit Kripke semantics. We recall a well-known property about preservation of forcing (see e.g. [1]) that can be easily proved by induction on formulas:

Lemma A.1 *Let V be a set of propositional variables, let $\mathcal{K}_1 = \langle W, \leq, r, \vartheta_1 \rangle$ and $\mathcal{K}_2 = \langle W, \leq, r, \vartheta_2 \rangle$ be two Kripke models such that, for every $w \in W$, $\vartheta_1(w) \cap V = \vartheta_2(w) \cap V$. For every formula α such that $\mathcal{V}_\alpha \subseteq V$ and every $w \in W$, $\mathcal{K}_1, w \Vdash \alpha$ iff $\mathcal{K}_2, w \Vdash \alpha$.*

The proof of Th. 2 is based on the next three lemmas.

Lemma A.2 *Let $\Theta = \texttt{ClauIPL}(\alpha)$, where α is simplified, then:*

(i) For every $\mathcal{K} = \langle W, \leq, r, \vartheta \rangle$ and for every $w \in W$, if $w \Vdash \Theta$ then $w \Vdash \alpha$.

(ii) Let $\Omega \cup \{\beta\}$ be a set of formulas such that $\mathcal{V}_{\Omega,\beta} \subseteq \mathcal{V}_\alpha$; if $\Omega, \alpha \models_{\mathrm{i}} \beta$, then $\Omega, \Theta \models_{\mathrm{i}} \beta$.

Proof. The proof of Point (i) is by induction on $|\,\alpha\,|$. We only detail some significant cases.

Let $\alpha = (\alpha_1 \to \alpha_2) \to \beta$, where both α_1 and α_2 and β are composite formulas. We have:

$$\Theta = \{\, (\tilde{p}_{\alpha_1} \to \tilde{p}_{\alpha_2}) \to \tilde{p}_\beta \,\} \cup \Theta_1 \cup \Theta_2 \cup \Theta_3 \qquad \Theta_1 \;=\; \texttt{ClauIPL}(\tilde{p}_{\alpha_1} \to \alpha_1)$$
$$\Theta_2 \;=\; \texttt{ClauIPL}(\alpha_2 \to \tilde{p}_{\alpha_2}) \qquad \Theta_3 \;=\; \texttt{ClauIPL}(\tilde{p}_\beta \to \beta)$$

Let $w \in W$ be such that $w \Vdash \Theta$; we show that $w \Vdash \alpha$. Since the formulas $\tilde{p}_{\alpha_1} \to \alpha_1$, $\alpha_2 \to \tilde{p}_{\alpha_2}$ and $\tilde{p}_\beta \to \beta$ are simplified and have size less then the size of α (see Lemma 5), we can apply the induction hypothesis and claim that: $w \Vdash \tilde{p}_{\alpha_1} \to \alpha_1$ and $w \Vdash \alpha_2 \to \tilde{p}_{\alpha_2}$ and $w \Vdash \tilde{p}_\beta \to \beta$. Let $w' \in W$ be such that $w \leq w'$ and $w' \Vdash \alpha_1 \to \alpha_2$. We show that:

(*) $w' \Vdash \tilde{p}_{\alpha_i} \to \tilde{p}_{\alpha_2}$.

Let $w'' \in W$ be such that $w' \leq w''$ and $w'' \Vdash \tilde{p}_{\alpha_1}$. Since $w \Vdash \tilde{p}_{\alpha_1} \to \alpha_1$ and $w \leq w''$, we get $w'' \Vdash \alpha_1$. By the fact that $w' \Vdash \alpha_1 \to \alpha_2$ and $w' \leq w''$, it follows that $w'' \Vdash \alpha_2$. Since $w \Vdash \alpha_2 \to \tilde{p}_{\alpha_2}$ and $w \leq w''$, we get $w'' \Vdash \tilde{p}_{\alpha_2}$, and this concludes the proof of (*). We remark that $w \Vdash (\tilde{p}_{\alpha_1} \to \tilde{p}_{\alpha_2}) \to \tilde{p}_\beta$ (indeed, such a formula belongs to Θ and we are assuming $w \Vdash \Theta$) and $w \Vdash \tilde{p}_\beta \to \beta$. From (*) and the fact that $w \leq w'$, we conclude $w' \Vdash \tilde{p}_\beta$. This proves that $w \Vdash (\alpha_1 \to \alpha_2) \to \beta$.

Let us consider the case

$$
\begin{array}{lcl}
\alpha & \equiv & (\eta_1 \wedge \alpha_1 \wedge \cdots \wedge \alpha_x) \to (\eta_2 \vee \beta_1 \vee \cdots \vee \beta_y) \\
\eta_1 & = & c_1 \wedge \cdots \wedge c_m \wedge (a_1 \to b_1) \wedge \cdots \wedge (a_n \to b_n) \\
\eta_2 & = & d_1 \vee \cdots \vee d_l
\end{array}
$$

where $\alpha_1, \ldots, \alpha_x, \beta_1, \ldots, \beta_y$ are composite formulas. We have

$$
\begin{array}{lcl}
\Theta & = & \{\, (\, \eta_1 \wedge \tilde{p}_{\alpha_1} \wedge \cdots \wedge \tilde{p}_{\alpha_x}\,) \to (\, \eta_2 \vee \tilde{p}_{\beta_1} \vee \cdots \vee \tilde{p}_{\beta_y}\,)\,\} \\
 & & \cup\, \Theta'_1 \,\cup \ldots \cup\, \Theta'_x \,\cup\, \Theta''_1 \,\cup \ldots \cup\, \Theta''_y \\
\Theta'_i & = & \texttt{ClauIPL}(\alpha_i \to \tilde{p}_{\alpha_i}), \quad i \in \{1, \ldots, x\} \\
\Theta''_j & = & \texttt{ClauIPL}(\tilde{p}_{\beta_j} \to \beta_j), \quad j \in \{1, \ldots, y\}
\end{array}
$$

Let $w \in W$ be such that $w \Vdash \Theta$; we show $w \Vdash \alpha$. By the induction hypothesis we get:

$$
w \Vdash \alpha_i \to \tilde{p}_{\alpha_i}, \text{ for every } i \in \{1, \ldots, x\}; \qquad w \Vdash \tilde{p}_{\beta_j} \to \beta_j, \text{ for every } j \in \{1, \ldots, y\}.
$$

Moreover, since $w \Vdash \Theta$, $w \Vdash (\, \eta_1 \wedge \tilde{p}_{\alpha_1} \wedge \cdots \wedge \tilde{p}_{\alpha_x}\,) \to (\, \eta_2 \vee \tilde{p}_{\beta_1} \vee \cdots \vee \tilde{p}_{\beta_y}\,)$. By the above three facts we get $w \Vdash \alpha$.

As for Point (2), let us assume $\Omega, \Theta \not\models_{\mathrm{i}} \beta$. Then, there is a Kripke model $\mathcal{K} = \langle W, \leq, r, \vartheta \rangle$ such that $r \Vdash \Omega \cup \Theta$ and $r \nVdash \beta$. By Point (i), $r \Vdash \alpha$ and hence $\mathcal{K}$ witnesses that $\Omega, \alpha \not\models_{\mathrm{i}} \beta$. ☐

To complete the proof of Th. 2, we have to prove the converse of Lemma A.2.(ii) ($\Omega, \Theta \models_{\mathrm{i}} \beta$ implies $\Omega, \alpha \models_{\mathrm{i}} \beta$). This requires some effort since the converse of Lemma A.2.(ii) (if $w \Vdash \alpha$ then $w \Vdash \Theta$) in general does not hold, due to the presence in Θ of the new variables $\tilde{p}_\delta$. To guarantee this, we need additional conditions on the model $\mathcal{K}$ at hand.

Lemma A.3 *Let* $\Theta = \texttt{ClauIPL}(\alpha)$*, where* α *is simplified, and let* $\mathcal{K} = \langle W, \leq, r, \vartheta \rangle$ *be a Kripke model such that: (KC)* $r \Vdash \tilde{p}_\delta \leftrightarrow \delta$*, for every* $\tilde{p}_\delta \in \mathcal{V}_\Theta$*. For every* $w \in W$*, if* $w \Vdash \alpha$ *then* $w \Vdash \Theta$*.*

Proof. The proof is by induction on $|\alpha|$. We only consider some significant cases.

Let $\alpha = (\alpha_1 \to \alpha_2) \to \beta$, where both α_1 and α_2 and β are composite formulas. We have:

$$\Theta = \{ (\tilde{p}_{\alpha_1} \to \tilde{p}_{\alpha_2}) \to \tilde{p}_\beta \} \cup \Theta_1 \cup \Theta_2 \cup \Theta_3$$

$$\Theta_1 = \texttt{ClauIPL}(\tilde{p}_{\alpha_1} \to \alpha_1) \qquad \Theta_2 = \texttt{ClauIPL}(\alpha_2 \to \tilde{p}_{\alpha_2}) \qquad \Theta_3 = \texttt{ClauIPL}(\tilde{p}_\beta \to \beta)$$

Let us assume that $w \Vdash \alpha$; we prove that $w \Vdash \Theta$. Since the variables $\tilde{p}_{\alpha_1}$, $\tilde{p}_{\alpha_2}$ and $\tilde{p}_\beta$ belong to $\mathcal{V}_\Theta$, by the assumption (KC) of the lemma we get:

- $r \Vdash \tilde{p}_{\alpha_1} \leftrightarrow \alpha_1$ and $r \Vdash \tilde{p}_{\alpha_2} \leftrightarrow \alpha_2$ and $r \Vdash \tilde{p}_\beta \leftrightarrow \beta$.

Since $w \Vdash \alpha$ and $r \leq w$, we get $w \Vdash (\tilde{p}_{\alpha_1} \to \tilde{p}_{\alpha_2}) \to \tilde{p}_\beta$. Note that $\mathcal{V}_{\Theta_1}$ is contained in $\mathcal{V}_\Theta$, hence assumption (KC) holds for the variables $\tilde{p}_\delta$ belonging to $\mathcal{V}_{\Theta_1}$. Accordingly, since $w \Vdash \tilde{p}_{\alpha_1} \to \alpha_1$, we can apply the induction hypothesis on the call $\texttt{ClauIPL}(\tilde{p}_{\alpha_1} \to \alpha_1)$ and we get $w \Vdash \Theta_1$. Similarly, we can prove $w \Vdash \Theta_2$ and $w \Vdash \Theta_3$; we conclude $w \Vdash \Theta$.

Let us consider the case

$$\begin{array}{lcl} \alpha & \equiv & (\eta_1 \wedge \alpha_1 \wedge \cdots \wedge \alpha_x) \to (\eta_2 \vee \beta_1 \vee \cdots \vee \beta_y) \\ \eta_1 & = & c_1 \wedge \cdots \wedge c_m \wedge (a_1 \to b_1) \wedge \cdots \wedge (a_n \to b_n) \\ \eta_2 & = & d_1 \vee \cdots \vee d_l \end{array}$$

where $\alpha_1, \ldots, \alpha_x, \beta_1, \ldots, \beta_y$ are composite formulas. We have

$$\begin{array}{lcl} \Theta & = & \{ (\eta_1 \wedge \tilde{p}_{\alpha_1} \wedge \cdots \wedge \tilde{p}_{\alpha_x}) \to (\eta_2 \vee \tilde{p}_{\beta_1} \vee \cdots \vee \tilde{p}_{\beta_y}) \} \\ & & \cup\, \Theta'_1 \cup \ldots \cup \Theta'_x \cup \Theta''_1 \cup \ldots \cup \Theta''_y \\ \Theta'_i & = & \texttt{ClauIPL}(\alpha_i \to \tilde{p}_{\alpha_i}), \quad i \in \{1, \ldots, x\} \\ \Theta''_j & = & \texttt{ClauIPL}(\tilde{p}_{\beta_j} \to \beta_j), \quad j \in \{1, \ldots, y\} \end{array}$$

We assume $w \Vdash \alpha$ and we prove that $w \Vdash \Theta$. We proceed as in the previous case. By assumption (KC), since $w \Vdash \alpha$ and $r \leq w$, we get:

- $w \Vdash (\eta_1 \wedge \tilde{p}_{\alpha_1} \wedge \cdots \wedge \tilde{p}_{\alpha_x}) \to (\eta_2 \vee \tilde{p}_{\beta_1} \vee \cdots \vee \tilde{p}_{\beta_y})$.

By the induction hypothesis it follows that:

- $w \Vdash \Theta'_i$ and $w \Vdash \Theta''_j$, for every $i \in \{1, \ldots, x\}$ and $j \in \{1, \ldots, y\}$

Accordingly $w \Vdash \Theta$. $\square$

Now we can prove the converse of Lemma A.2.(ii):

Lemma A.4 *Let $\Theta = \texttt{ClauIPL}(\alpha)$, where α is simplified, let $\Omega \cup \{\beta\}$ be a set of formulas such that $\mathcal{V}_{\Omega,\beta} \subseteq \mathcal{V}_\alpha$. If $\Omega, \Theta \models_{\mathrm{i}} \beta$, then $\Omega, \alpha \models_{\mathrm{i}} \beta$.*

Proof. Let us assume $\Omega, \alpha \not\models_{\mathrm{i}} \beta$. There is a Kripke model $\mathcal{K} = \langle W, \leq, r, \vartheta \rangle$ such that $r \Vdash \Omega \cup \{\alpha\}$ and $r \nVdash \beta$. Note that we cannot apply Lemma A.3 to claim that $r \Vdash \Theta$, since we are not guaranteed that $\mathcal{K}$ matches assumption (KC). We show that we can build a valuation ϑ' satisfying the following properties:

(i) $\vartheta'(w) \cap \mathcal{V}_\alpha = \vartheta(w) \cap \mathcal{V}_\alpha$, for every $w \in W$.

(ii) Let $\mathcal{K}' = \langle W, \leq, r, \vartheta' \rangle$. Then $\mathcal{K}', r \Vdash \tilde{p}_\delta \leftrightarrow \delta$, for every $\tilde{p}_\delta \in \mathcal{V}_\Theta$.

By (i) and the fact that $\mathcal{K}, r \Vdash \Omega \cup \{\alpha\}$ and $\mathcal{K}, r \nVdash \beta$ and $\mathcal{V}_{\Omega,\beta} \subseteq \mathcal{V}_\alpha$, we can apply Lemma A.1 to infer that $\mathcal{K}', r \Vdash \Omega \cup \{\alpha\}$ and $\mathcal{K}', r \nVdash \beta$. By (ii), the model $\mathcal{K}'$ matches assumption (KC) with respect to $\mathcal{V}_\Theta$; thus, we can apply Lemma A.3 to claim that $\mathcal{K}', r \Vdash \Theta$. It follows that $\Omega, \Theta \not\models_{\mathrm{i}} \beta$, and this concludes the proof of the assertion.

It remains to show how to define the evaluation ϑ' satisfying (i) and (ii). Let $\tilde{p}_{\delta_1}, \dots, \tilde{p}_{\delta_m}$ be an enumeration of the variables in $\mathcal{V}_\Theta$ respecting the order they have been defined (thus, $\tilde{p}_{\delta_{k+1}}$ has been defined just after $\tilde{p}_{\delta_k}$). We inductively build a sequence of evaluations $\vartheta_0, \dots, \vartheta_m$ having the set of worlds W as domain:

- $\vartheta_0 = \vartheta$.
- Let j be such that $0 \leq j < m$ and let $\mathcal{K}_j = \langle W, \leq, r, \vartheta_j \rangle$. The evaluation ϑ_{j+1} is built from ϑ_j as follows:

$$\vartheta_{j+1}(w) \;=\; \begin{cases} \vartheta_j(w) \,\cup\, \{\, \tilde{p}_{\delta_{j+1}} \,\} & \text{if } \mathcal{K}_j, w \Vdash \delta_{j+1} \\ \vartheta_j(w) \,\setminus\, \{\, \tilde{p}_{\delta_{j+1}} \,\} & \text{otherwise} \end{cases}$$

Let $V_j = \mathcal{V}_\alpha \cup \{\tilde{p}_{\delta_1}, \dots, \tilde{p}_{\delta_j}\}$, where $0 \leq j \leq m$. We prove that every evaluation ϑ_j satisfies the following properties:

(A) $\vartheta_j(w) \cap \mathcal{V}_\alpha = \vartheta(w) \cap \mathcal{V}_\alpha$, for every $w \in W$.

(B) $\mathcal{K}_j, r \Vdash \tilde{p}_\delta \leftrightarrow \delta$, for every $\tilde{p}_\delta \in V_j$.

The proof of point (A) is immediate, since $\mathcal{V}_\alpha$ does not contain any variable $\tilde{p}_\delta$. We prove point (B) by induction on j. The case $j = 0$ is trivial, since $V_0 = \mathcal{V}_\alpha$. Let us assume that point (B) holds for j (with $0 \leq j < m$). We remark that $V_{j+1} = V_j \cup \{\tilde{p}_{\delta_{j+1}}\}$ and $\vartheta_{j+1}(r) \cap V_j = \vartheta_j(r) \cap V_j$. By the induction hypothesis we get:

- $\mathcal{K}_j, r \Vdash \tilde{p}_\delta \leftrightarrow \delta$, for every $\tilde{p}_\delta \in V_j$.

Note that, for every $\tilde{p}_\delta \in V_j$, $\mathcal{V}_\delta \subseteq V_j$. By Lemma A.1 it follows that:

- $\mathcal{K}_{j+1}, r \Vdash \tilde{p}_\delta \leftrightarrow \delta$, for every $\tilde{p}_\delta \in V_j$.

Since $\mathcal{V}_{\delta_{j+1}} \subseteq V_j$, by Lemma A.1 we get:

- $\mathcal{K}_j, w \Vdash \delta_{j+1}$ iff $\mathcal{K}_{j+1}, w \Vdash \delta_{j+1}$, for every $w \in W$.

By definition of $\tilde{p}_{\delta_{j+1}}$, this implies that:

- $\mathcal{K}_{j+1}, w \Vdash \tilde{p}_{\delta_{j+1}}$ iff $\mathcal{K}_{j+1}, w \Vdash \delta_{j+1}$, for every $w \in W$.

It follows that $\mathcal{K}_{j+1}, r \Vdash \tilde{p}_{\delta_{j+1}} \leftrightarrow \delta_{j+1}$, and this concludes the proof of point (B). By (A) and (B), we can set $\vartheta' = \vartheta_m$ and, being $\mathcal{V}_\Theta = V_m$, properties (i) and (ii) are matched. □

As a corollary of lemmas A.2.(ii) and A.4, we get Th. 2.

References

[1] Alexander V. Chagrov and Michael Zakharyaschev. *Modal Logic*, volume 35 of *Oxford logic guides*. Oxford University Press, 1997.

[2] C. Fiorentini and M. Ferrari. A new approach to clausification for intuitionistic propositional logic. In A. Dovier and A. Formisano, editors, *Proceedings of the 38th Italian Conference on Computational Logic - CILC 2023, Udine, Italy, 21-23 June, 2023*, volume 3428 of *CEUR Workshop Proceedings*, pages 1–15. CEUR-WS.org, 2023.