In [1]:
```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import matplotlib
from setup_problem import load_problem
from tqdm import tqdm
import warnings
warnings.filterwarnings('ignore')
```

In [2]:
```
matplotlib.rcParams['figure.figsize'] = [15, 10]
np.random.seed(1337)
```

Load and Featurize Dataset

In [3]:
```
x_train, y_train, x_val, y_val, target_fn, coefs_true, featurize = load_pro
blem('lasso_data.pickle')
x_train_featurized, x_val_featurized = featurize(x_train), featurize(x_val)
```

2. Ridge Regression

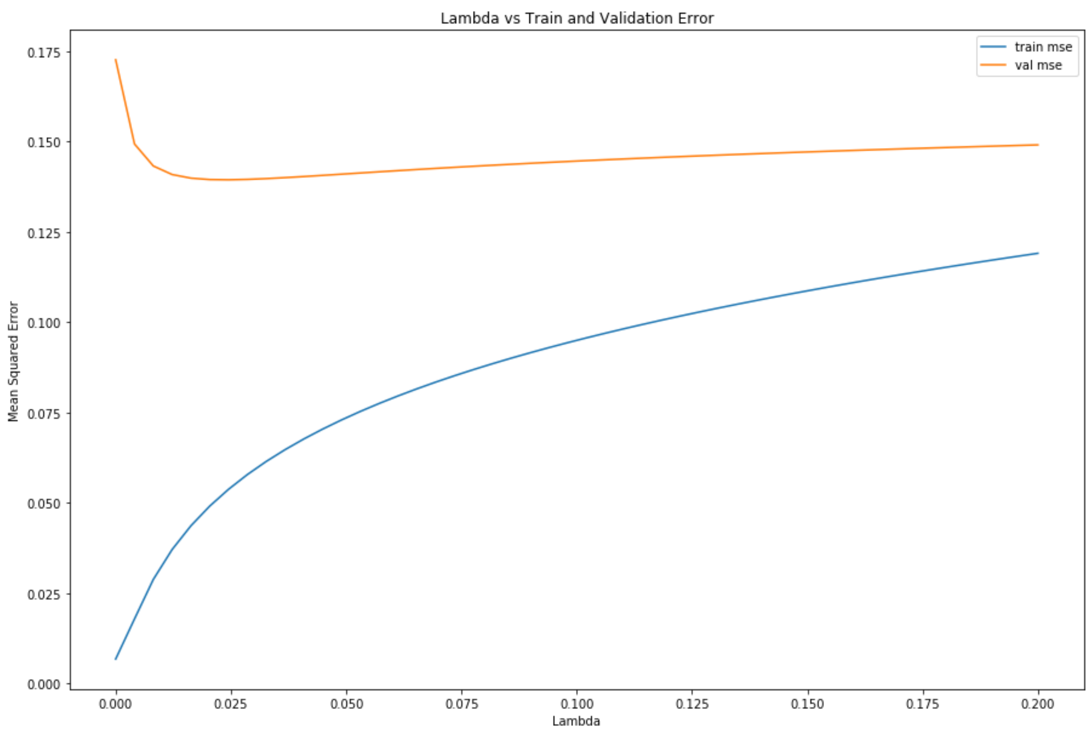
2.1

In [4]:
```
from ridge_regression import RidgeRegression as Ridge
from sklearn.metrics import mean_squared_error
```

In [5]:
```
lambdas = np.linspace(0,0.2, 50)
train_error = []
val_error = []
for l in tqdm(lambdas):
    model = Ridge(l)
    model.fit(x_train_featurized, y_train)
    train_error.append(mean_squared_error(y_train, model.predict(x_train_fe
aturized)))
    val_error.append(mean_squared_error(y_val, model.predict(x_val_featuriz
ed)))

results = pd.DataFrame({
    'lambda': lambdas,
    'train_error': train_error,
    'val_error': val_error
})

plt.plot(lambdas, train_error, label='train mse')
plt.plot(lambdas, val_error, label='val mse')
plt.legend()
plt.title("Lambda vs Train and Validation Error")
plt.xlabel('Lambda')
plt.ylabel('Mean Squared Error')
```

```
100%|██████████| 50/50 [00:36<00:00,  1.13it/s]
```

Out[5]: `Text(0, 0.5, 'Mean Squared Error')`



In [6]:
```
# Table of the results
results
```

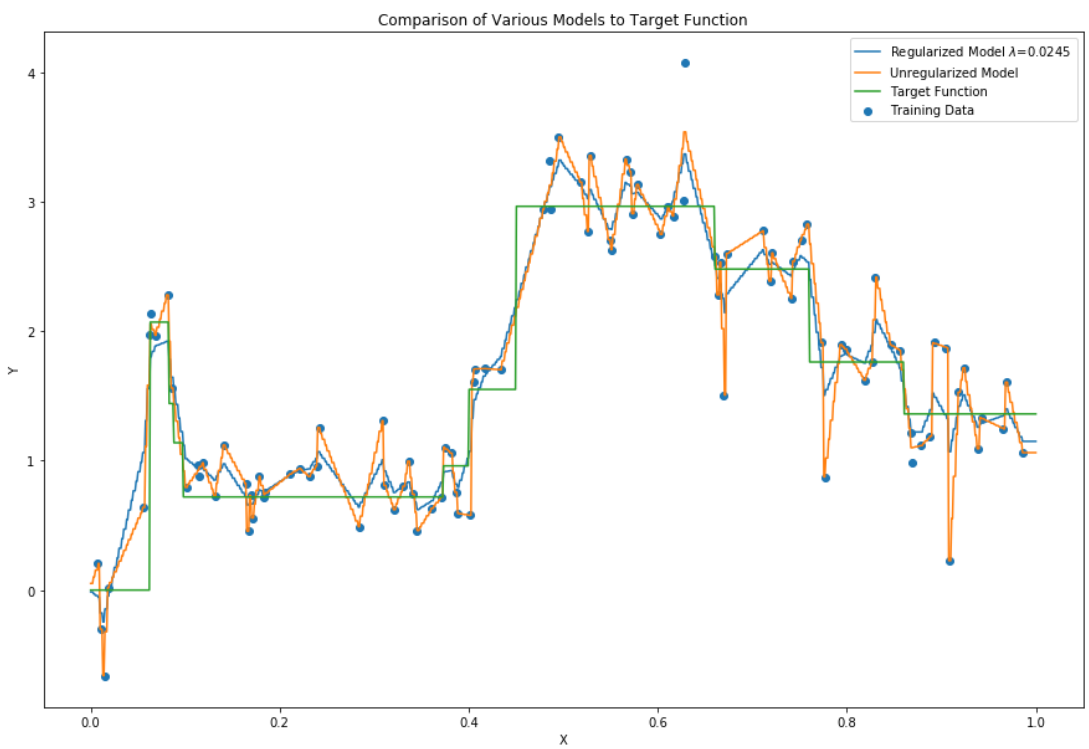
Out[6]:

	lambda	train_error	val_error
0	0.000000	0.006752	0.172592
1	0.004082	0.017843	0.149295
2	0.008163	0.028775	0.143205
3	0.012245	0.037058	0.140837
4	0.016327	0.043637	0.139837
5	0.020408	0.049084	0.139448
6	0.024490	0.053738	0.139384
7	0.028571	0.057810	0.139497
8	0.032653	0.061431	0.139709
9	0.036735	0.064689	0.139979
10	0.040816	0.067661	0.140286
11	0.044898	0.070390	0.140607
12	0.048980	0.072918	0.140936
13	0.053061	0.075274	0.141267
14	0.057143	0.077480	0.141594
15	0.061224	0.079558	0.141916
16	0.065306	0.081518	0.142230
17	0.069388	0.083379	0.142536
18	0.073469	0.085146	0.142834
19	0.077551	0.086832	0.143123
20	0.081633	0.088445	0.143404
21	0.085714	0.089991	0.143675
22	0.089796	0.091476	0.143940
23	0.093878	0.092905	0.144195
24	0.097959	0.094282	0.144444
25	0.102041	0.095613	0.144686
26	0.106122	0.096898	0.144918
27	0.110204	0.098145	0.145146
28	0.114286	0.099354	0.145368
29	0.118367	0.100528	0.145585
30	0.122449	0.101669	0.145796
31	0.126531	0.102779	0.146002
32	0.130612	0.103860	0.146202
33	0.134694	0.104912	0.146396
34	0.138776	0.105941	0.146588

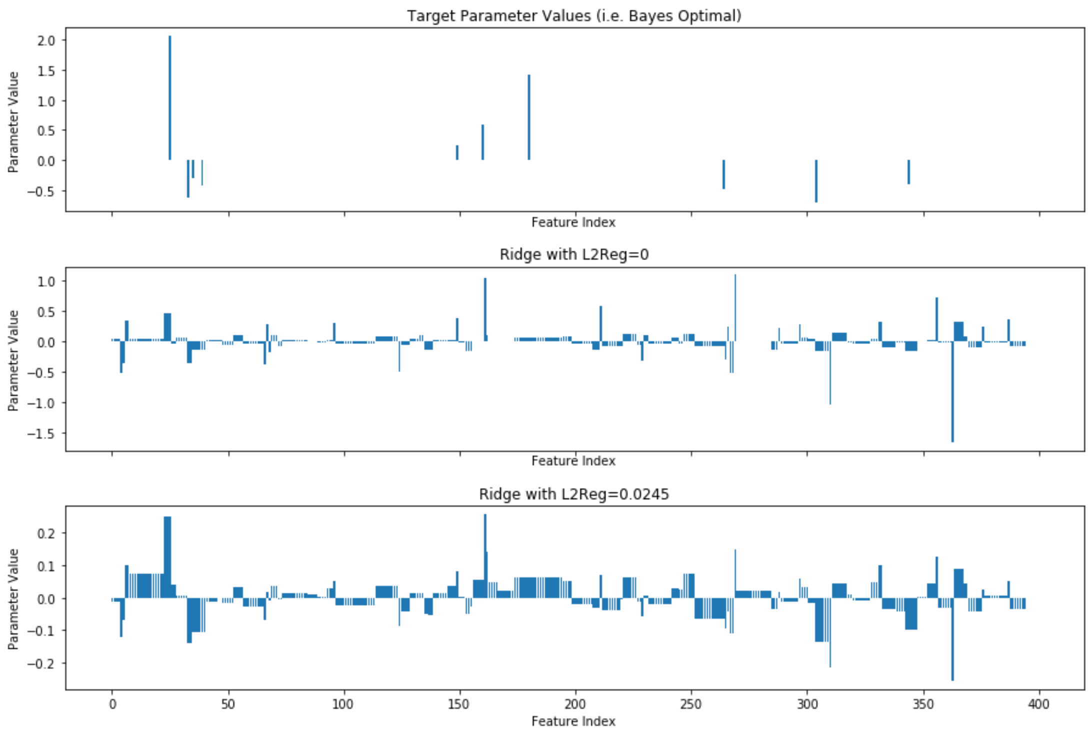
2.2

In [7]:
```
# Get best value for lambda
l = lambdas[np.argmin(val_error)]
best_model = Ridge(l).fit(x_train_featurized, y_train)
unreg_model = Ridge(0).fit(x_train_featurized, y_train)

x = np.linspace(0,1,1000)
x_featurized = featurize(x)

plt.scatter(x_train, y_train, label='Training Data')
plt.plot(x, best_model.predict(x_featurized), label='Regularized Model $\la
mbda$={}'.format(round(l,4)))
plt.plot(x, unreg_model.predict(x_featurized), label='Unregularized Model')
plt.plot(x, target_fn(x), label='Target Function')
plt.legend()
plt.xlabel('X')
plt.ylabel('Y')
plt.title("Comparison of Various Models to Target Function")
```

Out[7]:
```
Text(0.5, 1.0, 'Comparison of Various Models to Target Function')
```



In [8]:
```
from ridge_regression import compare_parameter_vectors
pred_fns = []
name = "Target Parameter Values (i.e. Bayes Optimal)"
pred_fns.append({"name":name, "coefs":coefs_true})
pred_fns.append({"name":'Ridge with L2Reg=0', "coefs":unreg_model.w_})
pred_fns.append({"name":'Ridge with L2Reg={}'.format(round(l,4)), "coefs":b
est_model.w_})

fig = compare_parameter_vectors(pred_fns)
```



With increased regularization we see the scale of the coefficients decrease. It appears that the coefficients with the most weight remain the same, it is only the size of these weights that differ. The graph also shows that the difference in weights between the most important features and other features decreases as regularization is increased.

2.3

In [9]:
```
from sklearn.metrics import confusion_matrix
```

In [10]:
```
coef_true_binary = (np.abs(coefs_true) > 0).astype(int)
```

In [11]:
```
thresholds = [1e-6, 1e-3, 1e-2, 5e-2, 1e-1]
coef_preds = []
for t in thresholds:
    pred = (np.abs(best_model.w_) >= t).astype(int)
    coef_preds.append(pred)
```

In [12]:
```
for thresh, pred in zip(thresholds, coef_preds):
    print('Threshold = {}'.format(thresh))
    print(confusion_matrix(pred, coef_true_binary))
    print()
```

```
Threshold = 1e-06
[[  5   0]
 [385  10]]

Threshold = 0.001
[[ 11   0]
 [379  10]]

Threshold = 0.01
[[ 53   0]
 [337  10]]

Threshold = 0.05
[[277   0]
 [113  10]]

Threshold = 0.1
[[369   5]
 [ 21   5]]
```

3. Coordinate Descent for Lasso

3.1 Experiments with the Shooting Algorithm

3.1.1

Give an expression for computing a_j and c_j using matrix and vector operations, without explicit loops.

$$a_j = 2X_j^T X_j$$
$$c_j = 2(y - Xw + w_j X_j)X_j^T$$

3.1.2

Write a function that computes the Lasso solution for a give λ using the shooting algorithm described above. For convergence criteria, continue coordinate descent until a pass through the coordinates reduces the objective function by less than 10−8 , or you have taken 1000 passes through the coordinates. Compare performance of cyclic coordinate descent to randomized coordinate descent, where in each round we pass through the coordinates in a different random order (for your choices of λ). Compare also the solutions attained (following the convergence criteria above) for starting at 0 versus starting at the ridge regression solution suggested by Murphy (again, for your choices of λ). If you like, you may adjust the convergence criteria to try to attain better results (or the same results faster).

In [13]:
```
def shuffle(X, y):
    tmp = np.hstack((X, y.reshape((len(y),1))))
    np.random.shuffle(tmp)
    return tmp[:,:-1], tmp[:, -1]

def lasso_loss(X, y, w, l):
    # Compute the lasso loss
    return sum(np.square(np.matmul(X, w) - y)) + l*sum(np.abs(w))

def soft(a, b):
    c = np.abs(a) - b
    c_plus = c if c >=0 else 0
    return np.sign(a)*c_plus
```

In [14]:
```
def compute_lasso_solution(X, y, l, thresh=1e-8, max_iter=1000, w_init=Fals
e, randomized=False):
    """
    Args:
        X: self-explanatory
        y: self-explanatory
        thresh (float): stop optimization if loss reduction is less than th
is.
        max_iter (int): max iterations through coordinates before stopping
optimization.
        init_ridge (bool): if True, initialize weights with Ridge regressio
n solution.
        randomized (bool): if True, shuffle data at epoch end.

    Returns:
        w (numpy array): self-explanatory
    """

    n_iter = 0
    n_cols = X.shape[1]
    cols = list(range(n_cols))

    if w_init == 'ridge':
        w = np.matmul(np.matmul(np.linalg.inv(np.matmul(X.T, X) + l*np.eye(
n_cols)), X.T), y)
    elif w_init is False:
        w = np.zeros(n_cols)
    else:
        w = w_init

    old_loss = lasso_loss(X, y, w, l)
    delta_loss = thresh + 1


    while n_iter <= max_iter and delta_loss > thresh:
        if randomized:
            np.random.shuffle(cols)
        for j in cols:
            X_j = X[:,j]
            if not X_j.any():
                # Edge case if column is all 0
                w[j] = 0
            else:
                a = 2 * np.matmul(X_j.T, X_j)
                c = 2 * np.matmul((y - np.matmul(X, w) + w[j]*X_j).T, X_j)
                w[j] = soft(c/a, l/a)
        # end of iteration / epoch
        n_iter += 1
        new_loss = lasso_loss(X, y, w, l)
        delta_loss = old_loss - new_loss
        old_loss = new_loss
    return w
```

Cyclic vs Randomized Coordinate Descent

In [15]:
```
%timeit compute_lasso_solution(x_train_featurized, y_train, 1, thresh=1e-3, randomized=True)
```

```
2.84 s ± 974 ms per loop (mean ± std. dev. of 7 runs, 1 loop each)
```

In [16]:
```
%timeit compute_lasso_solution(x_train_featurized, y_train, 1, thresh=1e-3, randomized=False)
```

```
2.91 s ± 135 ms per loop (mean ± std. dev. of 7 runs, 1 loop each)
```

It appears that randomized coordinate descent converges faster.

Effect of solution initialization.

In [17]:
```
%timeit compute_lasso_solution(x_train_featurized, y_train, 1, thresh=1e-3, randomized=True, w_init='ridge')
```

```
899 ms ± 32.4 ms per loop (mean ± std. dev. of 7 runs, 1 loop each)
```

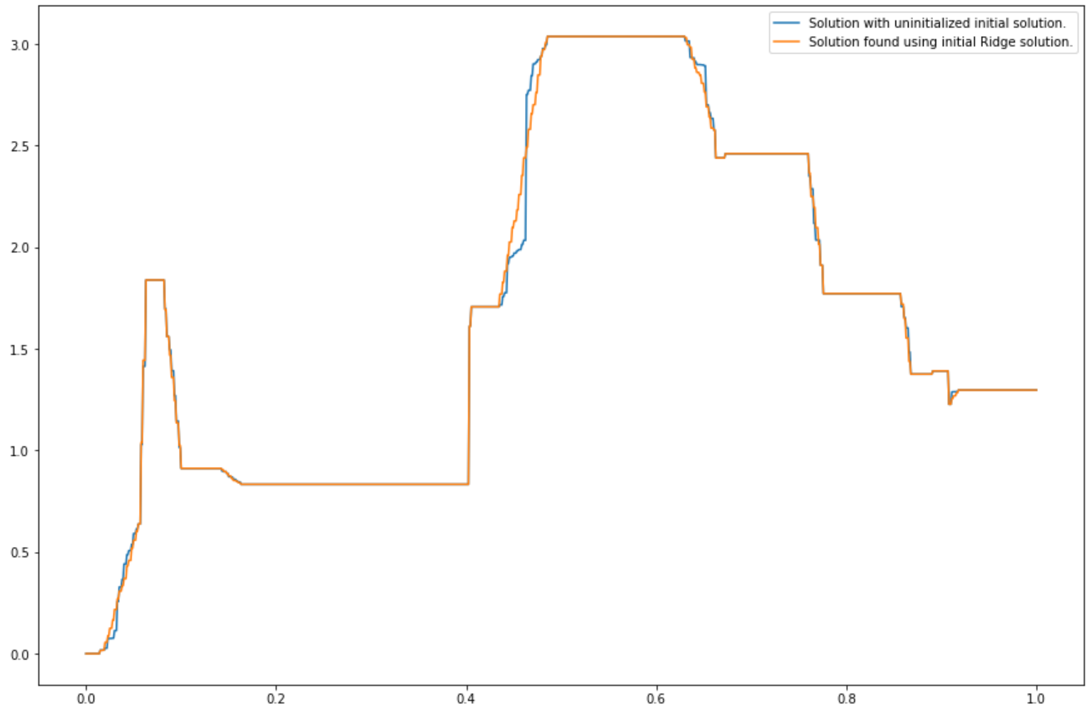
In [18]:
```
%timeit compute_lasso_solution(x_train_featurized, y_train, 1, thresh=1e-3, randomized=True, w_init=False)
```

```
1.78 s ± 329 ms per loop (mean ± std. dev. of 7 runs, 1 loop each)
```

In [29]:
```
w_init = compute_lasso_solution(x_train_featurized, y_train, 1, thresh=1e-8, randomized=True, w_init='ridge')
w = compute_lasso_solution(x_train_featurized, y_train, 1, thresh=1e-8, randomized=True, w_init=False)
```

In [30]:
```
plt.plot(x, np.matmul(x_featurized, w), label='Solution with uninitialized
initial solution.')
plt.plot(x, np.matmul(x_featurized, w_init), label='Solution found using in
itial Ridge solution.')
plt.legend()
```

Out[30]: <matplotlib.legend.Legend at 0x7efc62d52128>



Looking at the timings, we can see that initializing the solution with the Ridge solution results in much faster convergence. Also, by plotting the solutions we can see that both methods result in very similar solutions.

3.1.3

Run your best Lasso configuration on the training dataset provided, and select the λ that minimizes the square error on the validation set. Include a table of the parameter values you tried and the validation performance for each. Also include a plot of these results. Include also a plot of the prediction functions, just as in the ridge regression section, but this time add the best performing Lasso prediction function and remove the unregularized least squares fit. Similarly, add the lasso coefficients to the bar charts of coefficients generated in the ridge regression setting. Comment on the results, with particular attention to parameter sparsity and how the ridge and lasso solutions compare. What's the best model you found, and what's its validation performance?

In [21]:

```
lambdas = [1e-2, 1e-1, 1, 2, 5]
train_losses = []
val_losses = []
best_ridge = best_model
weights = []

for l in lambdas:
    w = compute_lasso_solution(x_train_featurized, y_train, l, thresh=1e-8,
randomized=True, w_init='ridge')
    train_preds = np.matmul(x_train_featurized, w)
    val_preds = np.matmul(x_val_featurized, w)
    train_losses.append(mean_squared_error(train_preds, y_train))
    val_losses.append(mean_squared_error(val_preds, y_val))
    weights.append(w)
```

Table of Parameter Search Results

In [22]:

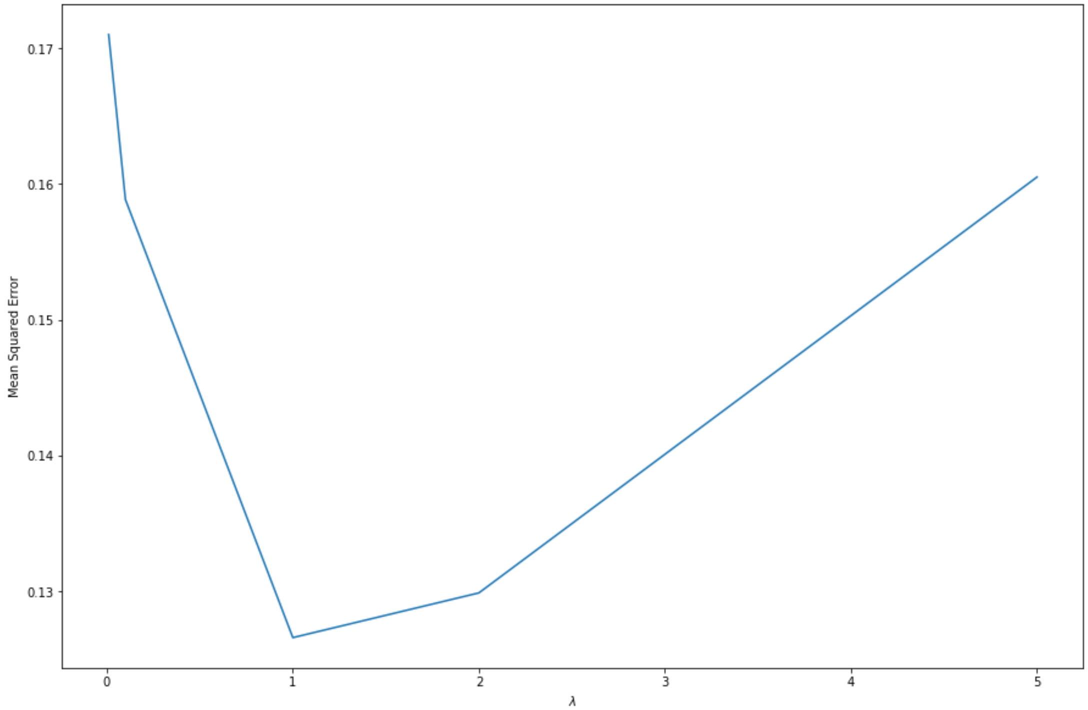
```
l_search_results = pd.DataFrame({
    'lambda': lambdas,
    'train_mse': train_losses,
    'val_mse': val_losses
})
l_search_results
```

Out[22]:

	lambda	train_mse	val_mse
0	0.01	0.006805	0.170998
1	0.10	0.011139	0.158850
2	1.00	0.091950	0.126596
3	2.00	0.105366	0.129889
4	5.00	0.170150	0.160504

In [23]:
```
plt.plot(l_search_results['lambda'], l_search_results.val_mse)
plt.xlabel('$\lambda$')
plt.ylabel('Mean Squared Error')
```

Out[23]: Text(0, 0.5, 'Mean Squared Error')

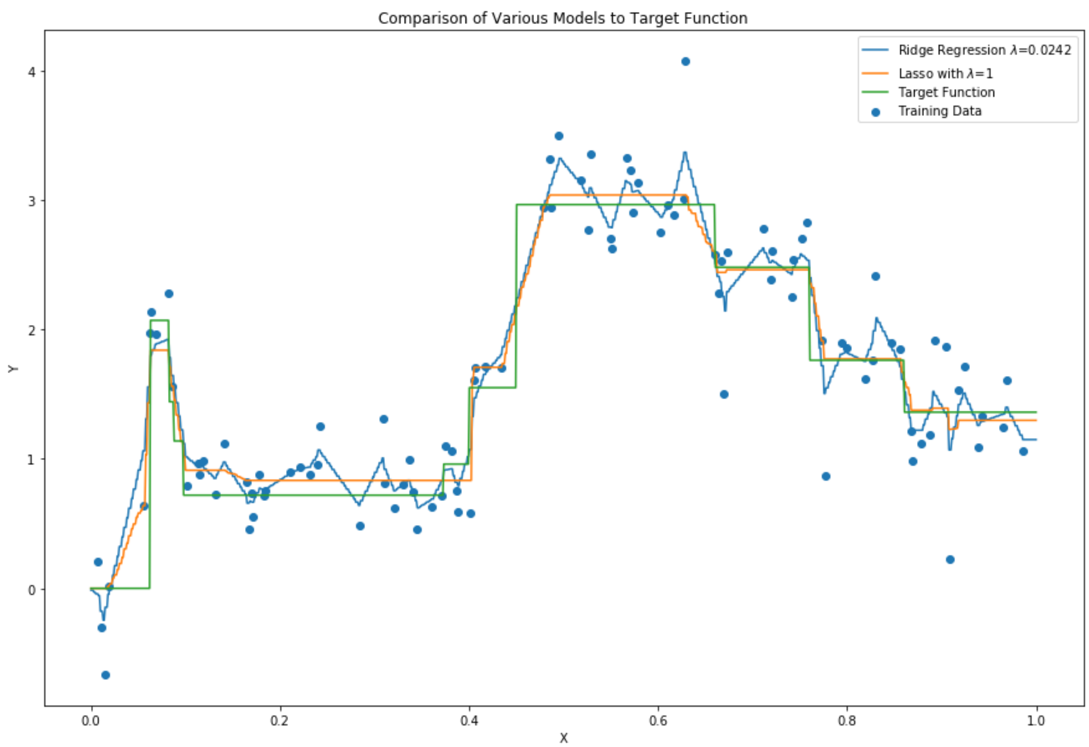


In [24]:
```
best_lasso_w = weights[np.argmin(val_losses)]
l = lambdas[np.argmin(val_losses)]
```

Comparison of Lasso and Ridge to Target Function

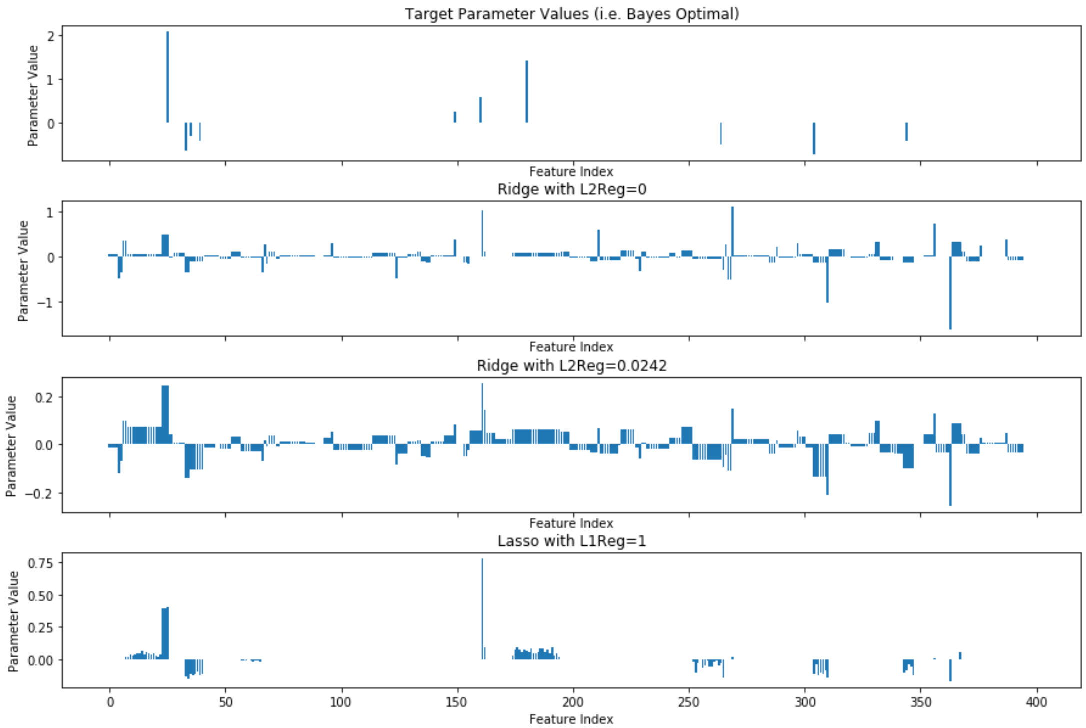
In [25]:
```
plt.scatter(x_train, y_train, label='Training Data')
plt.plot(x, best_ridge.predict(x_featurized), label='Ridge Regression $\lambda$=0.0242')
plt.plot(x, np.matmul(x_featurized, best_lasso_w), label='Lasso with $\lambda$=1')
plt.plot(x, target_fn(x), label='Target Function')
plt.legend()
plt.xlabel('X')
plt.ylabel('Y')
plt.title("Comparison of Various Models to Target Function")
```

Out[25]: Text(0.5, 1.0, 'Comparison of Various Models to Target Function')



Coefficient Chart

In [26]:
```
pred_fns = []
name = "Target Parameter Values (i.e. Bayes Optimal)"
pred_fns.append({"name":name, "coefs":coefs_true})
pred_fns.append({"name":'Ridge with L2Reg=0', "coefs":unreg_model.w_})
pred_fns.append({"name":'Ridge with L2Reg={}'.format(0.0242), "coefs":best_
model.w_})
pred_fns.append({"name":'Lasso with L1Reg=1', "coefs":best_lasso_w})
fig = compare_parameter_vectors(pred_fns)
```



Comparison of Lasso to Ridge Performance

In [27]:
```
lasso_losses = min(val_losses)
ridge_loss = mean_squared_error(best_ridge.predict(x_val_featurized), y_val
)
print("Lasso model loss: {} \nRidge model loss: {}".format(lasso_losses, ri
dge_loss))
```

```
Lasso model loss: 0.1265958720525034
Ridge model loss: 0.13938409974459445
```

Lasso regression resulted in a much sparser solution. In addition, using the lasso loss resulted in improved performance as measured by mean squared error. It outperformed ridge with a mse of 0.126. Also, in this particular case, the lasso solution better exhibits certain characteristics of the Bayes optimal solution such as its sparsity and step function resembelence.

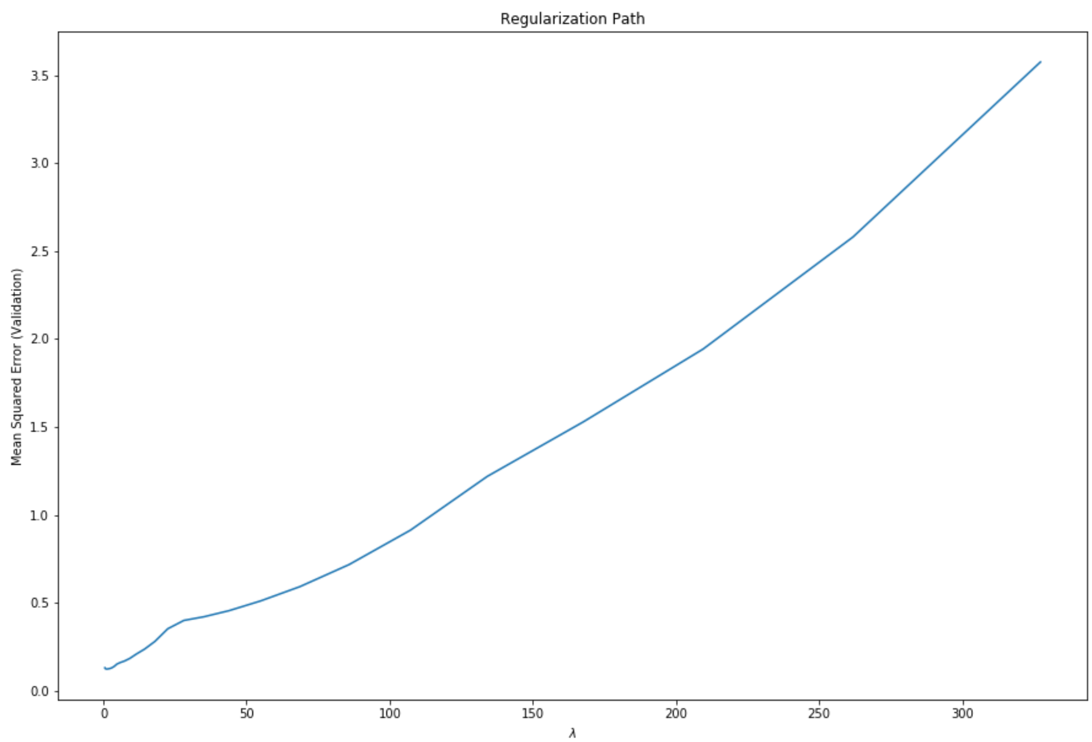
3.1.4

Implement the homotopy method described above. Plot the results (average validation loss vs λ).

In [31]:
```
l_max = 2 * np.linalg.norm(np.matmul(x_train_featurized.T, y_train), ord=np
.inf)
lambdas = [l_max * 0.8**i for i in range(30)]
val_losses = []

w = 'ridge'
for l in lambdas:
    w = compute_lasso_solution(x_train_featurized, y_train, l, thresh=1e-8,
randomized=True, w_init=w)
    val_preds = np.matmul(x_val_featurized, w)
    val_losses.append(mean_squared_error(val_preds, y_val))
```

In [32]:
```
plt.plot(lambdas, val_losses)
plt.xlabel('$\lambda$')
plt.ylabel('Mean Squared Error (Validation)')
plt.title("Regularization Path")
```

Out[32]: `Text(0.5, 1.0, 'Regularization Path')`



3.1.5

Note that the data in Figure 1 is almost entirely nonnegative. Since we don't have an unregularized bias term, we have "pay for" this offset using our penalized parameters. Note also that λmax would decrease significantly if the y values were 0 centered (using the training data, of course), or if we included an unregularized bias term. Experiment with one or both of these approaches, for both and lasso and ridge regression, and report your findings

We will try centering the y values

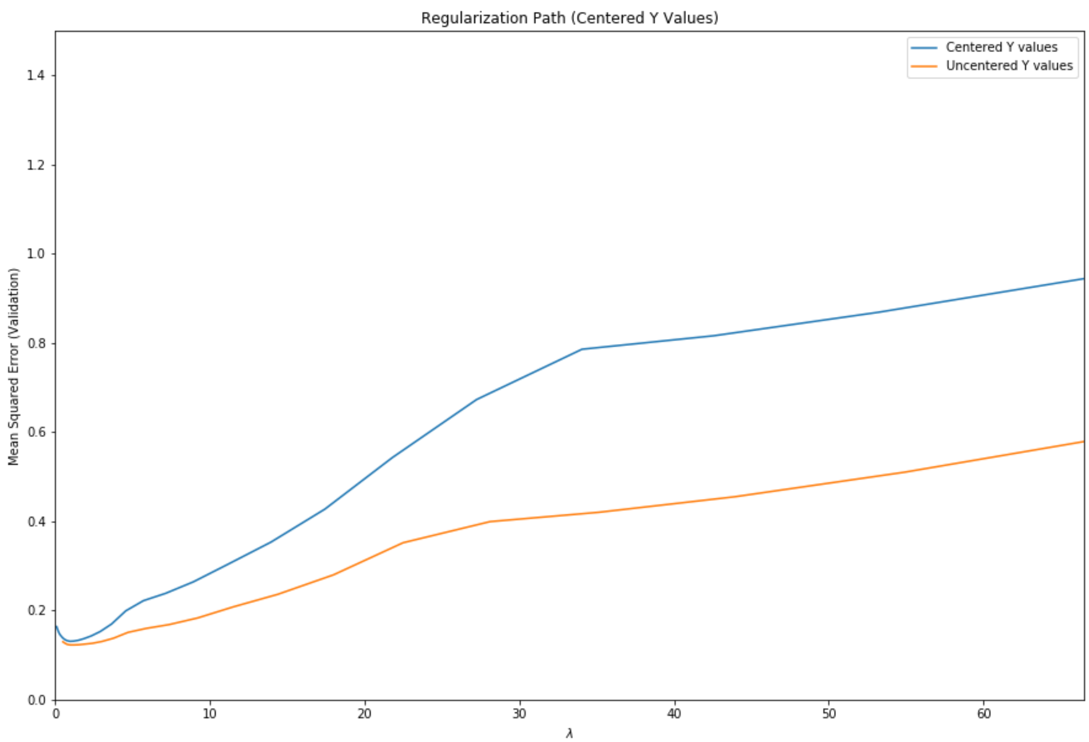
In [33]:
```
y_mean = y_train.mean()
y_train_trans = y_train - y_mean
y_val_trans = y_val - y_mean

l_max_trans = 2 * np.linalg.norm(np.matmul(x_train_featurized.T, y_train_tr
ans), ord=np.inf)
print("With 0 centered y values, l_max = {}".format(l_max_trans))
```

```
With 0 centered y values, l_max = 66.51995371985255
```

In [44]:
```
lambdas_trans = [l_max_trans * 0.8**i for i in range(30)]
val_losses_centered = []

w = 'ridge'
for l in lambdas_trans:
    w = compute_lasso_solution(x_train_featurized, y_train_trans, l, thresh
=1e-8, randomized=True, w_init=w)
    val_preds = np.matmul(x_val_featurized, w)
    val_losses_centered.append(mean_squared_error(val_preds, y_val_trans))
```

In [38]:
```
plt.plot(lambdas_trans, val_losses_centered, label='Centered Y values')
plt.plot(lambdas, val_losses, label='Uncentered Y values')
plt.xlim(0, max(lambdas_trans))
plt.ylim(0,1.5)
plt.xlabel('$\lambda$')
plt.legend()
plt.ylabel('Mean Squared Error (Validation)')
plt.title("Regularization Path (Centered Y Values)")
```

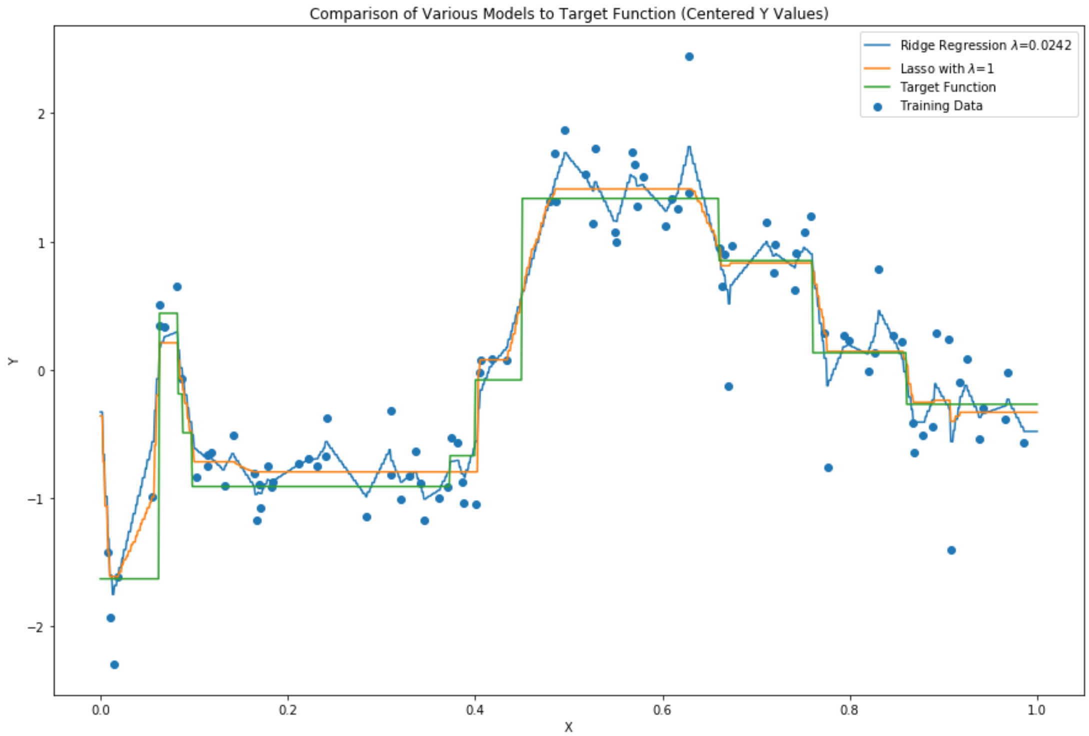
Out[38]:
```
Text(0.5, 1.0, 'Regularization Path (Centered Y Values)')
```



In [39]:
```
best_ridge.fit(x_train_featurized, y_train_trans)
w_lasso = compute_lasso_solution(x_train_featurized, y_train_trans, 1, thre
sh=1e-8, randomized=True, w_init='ridge')
```

In [40]:
```
plt.scatter(x_train, y_train_trans, label='Training Data')
plt.plot(x, best_ridge.predict(x_featurized), label='Ridge Regression $\lam
bda$=0.0242')
plt.plot(x, np.matmul(x_featurized, w_lasso), label='Lasso with $\lambda$=1
')
plt.plot(x, target_fn(x) - y_mean, label='Target Function')
plt.legend()
plt.xlabel('X')
plt.ylabel('Y')
plt.title("Comparison of Various Models to Target Function (Centered Y Valu
es)")
```

Out[40]:
```
Text(0.5, 1.0, 'Comparison of Various Models to Target Function (Centered Y
Values)')
```



Using centered y values did decrease lambda_max from around 330 to 66. The best solutions do not appear to differ much from the best solution using uncentered y values. However, looking at the regularization path for lasso regression, we see that using the centered y values results in higher mean squared error outside of the optimal lambda values.

3.2 Deriving the Coordinate Minimizer for Lasso

3.2.1

First let's get a trivial case out of the way. If $x_{ij} = 0$ for $i = 1, \ldots, n$, what is the coordinate minimizer w_j?

When $X_j = 0$, we have $f(w_j) = \sum_{i=1}^{n} y_i^2 + \lambda|w_j| + \lambda \sum_{k \neq j} |w_k|$. It is trivial to see that, with respect to w_j, this is minimized when $w_j = 0$

3.2.2

Give an expression for the derivative $f(w_j)$ for $w_j \neq 0$.

$$\begin{aligned} f'(w_j) &= 2\sum_{i=1}^{n}\left[(w_j x_{ij} + \sum_{k \neq j} w_k x_{ik} - y_i)x_{ij}\right] + \lambda sign(w_j) \\ &= 2w_j \sum_{i=1}^{n} x_{ij}^2 - 2\sum_{i=1}^{n} w_k x_{ij}(y_i - \sum_{k \neq j} w_k x_{ik}) + \lambda sign(w_j) \\ &= w_j a_j - c_j + \lambda sign(w_j) \end{aligned}$$

3.2.3

If $w_j > 0$ and minimizes f, show that $w_j = \frac{1}{a_j}\left(c_j - \lambda\right)$. Similarly, if $w_j < 0$ and minimizes f, show that $w_j = \frac{1}{a_j}\left(c_j + \lambda\right)$. Give conditions on c_j that imply that a minimizer w_j is positive and conditions for which a minimizer w_j is negative.

Assume that $w_j \neq 0$ and minimizes f. Then

$$\begin{aligned} f'(w_j) &= 0 \\ w_j a_j - c_j + \lambda sign(w_j) &= 0 \\ w_j a_j &= c_j - \lambda sign(w_j) \\ w_j &= \frac{1}{a_j}(c_j - \lambda sign(w_j)) \end{aligned}$$

Now its clear that if $w_j > 0$

$$w_j = \frac{1}{a_j}(c_j - \lambda)$$

And if $w_j < 0$

$$w_j = \frac{1}{a_j}(c_j + \lambda)$$

Now its clear that with respect to c_j , $w_j > 0$ when $c_j > \lambda$

And $w_j < 0$ when $c_j < -\lambda$

3.2.4

Derive expressions for the two one-sided derivatives at $f(0)$, and show that $c_j \in [-\lambda, \lambda]$ implies that $w_j = 0$ is a minimizer.

$\lim_{\epsilon \downarrow 0} \frac{f(\epsilon) - f(0)}{\epsilon} \geq 0 \quad \text{and} \quad \lim_{\epsilon \downarrow 0} \frac{f(-\epsilon) - f(0)}{\epsilon} \geq 0.$

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \frac{f(\epsilon) - f(0)}{\epsilon} &= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \Bigg[\sum_{i=1}^{n} \Big(\epsilon x_{ij} + \sum_{k \neq j} w_k x_{ik} - y_i\Big)^2 + \lambda|\epsilon| + \lambda \sum_{k \neq j} |w_j| - \sum_{i=1}^{n} \Big(\sum_{k \neq j} w_k x_{ik} - y_i\Big)^2 - \lambda \sum_{k \neq j} \\ &= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \Bigg[\sum_{i=1}^{n} \Bigg(\epsilon^2 x_{ij}^2 + 2\epsilon x_{ij} \Big(\sum_{k \neq j} w_k x_{ik} - y_i\Big) + \Big(\sum_{k \neq j} w_k x_{ik} - y_i\Big)^2\Bigg) + \lambda|\epsilon| - \Big(\sum_{k \neq j} w_k x_{ik} - \\ &= \lim_{\epsilon \downarrow 0} \Bigg[\sum_{i=1}^{n} \Bigg(\epsilon x_{ij}^2 + 2x_{ij} \Big(\sum_{k \neq j} w_k x_{ik} - y_i\Big)\Bigg) + \lambda|\epsilon| \Bigg] \\ &= \sum_{i=1}^{n} \Bigg(2x_{ij} \Big(\sum_{k \neq j} w_k x_{ik} - y_i\Big)\Bigg) + \lambda \\ &= \lambda - c_j \end{aligned}$$

$$\begin{aligned}
\lim_{\epsilon\downarrow 0}\frac{f(-\epsilon)-f(0)}{\epsilon} &= \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\Bigg[\sum_{i=1}^{n}\Big(-\epsilon x_{ij}+\sum_{k\neq j}w_kx_{ik}-y_i\Big)^2+\lambda|-\epsilon|+\lambda\sum_{k\neq j}|w_j|-\sum_{i=1}^{n}\Big(\sum_{k\neq j}w_kx_{ik}-y_i\Big)^2\cdot \\
&= \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\Bigg[\sum_{i=1}^{n}\bigg(\epsilon^2x_{ij}^2-2\epsilon x_{ij}\Big(\sum_{k\neq j}w_kx_{ik}-y_i\Big)+\Big(\sum_{k\neq j}w_kx_{ik}-y_i\Big)^2\bigg)+\lambda|\epsilon|-\Big(\sum_{k\neq j}w_kx_{ik} \\
&= \lim_{\epsilon\downarrow 0}\Bigg[\sum_{i=1}^{n}\bigg(\epsilon x_{ij}^2-2x_{ij}\Big(\sum_{k\neq j}w_kx_{ik}-y_i\Big)\bigg)+\lambda|\epsilon|\Bigg] \\
&= \sum_{i=1}^{n}\bigg(-2x_{ij}\Big(\sum_{k\neq j}w_kx_{ik}-y_i\Big)\bigg)+\lambda \\
&= \lambda+c_j
\end{aligned}$$

Note that if $c_j\in[-\lambda,\lambda]$ then $\lambda+c_j\geq 0$ and $\lambda-c_j\geq 0$. Thus $\lim_{\epsilon\downarrow 0}\frac{f(\epsilon)-f(0)}{\epsilon}\geq 0$ and $\lim_{\epsilon\downarrow 0}\frac{f(-\epsilon)-f(0)}{\epsilon}\geq 0.$ Thus $w_j=0$ is a minimizer.

3.2.5

Putting together the preceding results, we conclude the following:

$$w_j=\begin{cases}\frac{1}{a_j}\big(c_j-\lambda\big) & c_j>\lambda\\ 0 & c_j\in[-\lambda,\lambda]\\ \frac{1}{a_j}\big(c_j+\lambda\big) & c_j<-\lambda\end{cases}$$

Show that this is equivalent to the expression given in

Case 1: $c_j\in[-\lambda,\lambda]$

Note that

$$soft\left(\frac{c_j}{a_j},\frac{\lambda}{a_j}\right)=sign\left(\frac{c_j}{a_j}\right)\left(\left|\frac{c_j}{a_j}\right|-\frac{\lambda}{a_j}\right)_+$$

But

$$\begin{aligned}|c_j|-\lambda\leq 0\\ and\\ a_j\geq 0\end{aligned}$$

Thus $\left|\frac{c_j}{a_j}\right|-\frac{\lambda}{a_j}\leq 0$ and so $soft\left(\frac{c_j}{a_j},\frac{\lambda}{a_j}\right)=0$

Case 2 $c_j \geq \lambda$

Since $c_j \geq 0$ and $a_j \geq 0$

$$\begin{aligned} soft\left(\frac{c_j}{a_j}, \frac{\lambda}{a_j}\right) &= sign\left(\frac{c_j}{a_j}\right)\left(\left|\frac{c_j}{a_j}\right| - \frac{\lambda}{a_j}\right)_+ \\ &= \left(\frac{c_j - \lambda}{a_j}\right) \end{aligned}$$

Case 3 $c_j \leq -\lambda$

Since $c_j \leq 0$ and $a_j \geq 0$

$$\begin{aligned} soft\left(\frac{c_j}{a_j}, \frac{\lambda}{a_j}\right) &= sign\left(\frac{c_j}{a_j}\right)\left(\left|\frac{c_j}{a_j}\right| - \frac{\lambda}{a_j}\right)_+ \\ &= -\left(\frac{|c_j| - \lambda}{a_j}\right) \\ &= \frac{-|c_j| + \lambda}{a_j} \\ &= \frac{c_j + \lambda}{a_j} & \text{Since } c_j \leq 0. \end{aligned}$$

4 Lasso Properties

4.1 Deriving λ_{max}

4.1.1

Compute $J'(0; v)$. That is, compute the one-sided directional derivative of $J(w)$ at $w = 0$ in the direction v. [Hint: the result should be in terms of $X, y, \lambda,$ and v.]

$$
\begin{aligned}
J'(0 : v) &= \lim_{h \downarrow 0} \frac{J(hv) - J(0)}{h} \\
&= \lim_{h \downarrow 0} \frac{\|hXv - y\|_2^2 + \lambda h - \| - y\|_2^2}{h} \\
&= \lim_{h \downarrow 0} \frac{h^2 \|Xv\|_2^2 - 2h(Xv)^T y + \lambda h}{h} \\
&= \lim_{h \downarrow 0} h\|Xv\|_2^2 - 2(Xv)^T y + \lambda \\
&= \lambda - 2(Xv)^T y
\end{aligned}
$$

4.1.2

Show that for any $v \neq 0$, we have $J'(0; v) \geq 0$ if and only if $\lambda \geq C$, for some C that depends on $X, y, \text{ and } v$.

$$
\begin{aligned}
J'(0 : v) \geq 0 \\
\lambda - 2(Xv)^T y \geq 0 \\
\lambda &\geq 2(Xv)^T y
\end{aligned}
$$

Thus

$$
C = 2(Xv)^T y
$$

4.1.3

In the previous problem, we get a different lower bound on λ for each choice of v. Show that the maximum of these lower bounds on λ is $\lambda_{\max} = 2\|X^T y\|_\infty$. Conclude that $w = 0$ is a minimizer of $J(w)$ if and only if $\lambda \geq 2\|X^T y\|_\infty$

4.1.3.1 Prove that $\lambda_{\max} = 2\|X^T y\|_\infty$

Observe that

$$
\begin{aligned}
C &= 2v^T X^T y \\
&= 2 \sum_{i=1}^{d} v_i (X^T y)_i \qquad \text{subject to } \|v\|_1 = 1
\end{aligned}
$$

From here it should be clear that $C \leq 2\|X^T y\|_\infty$ and that $C = 2\|X^T y\|_\infty$ when $v = e_j$ where j is the index of $2\|X^T y\|_\infty$ in $X^T y$.\

Thus $\lambda_{\max} = 2\|X^T y\|_\infty$.

4.2.3.1 Conclude that $w = 0$ is a minimizer of $J(w)$ if and only if $\lambda \geq 2\|X^T y\|_\infty$

Assume that $w = 0$ is a minimizer of $J(w)$. Then it follows that $J'(0; v) \geq 0 \ \forall \ v \neq 0$. From 4.1.2 we know that $J'(0; v) \geq 0$ when $\lambda \geq 2(Xv)^T y$. But from 4.1.3.1 we have $2(Xv)^T y \leq 2\|X^T y\|_\infty$. Thus $J'(0; v) \geq 0 \ \forall \ v \neq 0$ when $\lambda \geq 2\|X^T y\|_\infty$

Now assume that $\lambda \geq 2\|X^T y\|_\infty$. Then $\lambda \geq 2(Xv)^T y \ \forall \ v \neq 0$. Then by 4.1.2 we have $J'(0; v) \geq 0 \ \forall \ v \neq 0$ and so $w = 0$ is a minimizer of $J(w)$

4.1.4

Let $J(w, b) = \|Xw + b\mathbf{1} - y\|_2^2 + \lambda\|w\|_1$, where $\mathbf{1} \in R^n$ is a column vector of 1's. Let $\bar{y}$ be the mean of values in the vector y. Show that $(w^*, b^*) = (0, \bar{y})$ is a minimizer of $J(w, b)$ if and only if $\lambda \geq \lambda_{\max} = 2\|X^T(y - \bar{y})\|_\infty$.

Part 1

Observe that

$$\begin{aligned}
J'(0, \bar{y} : v) &= \lim_{h \downarrow 0} \frac{J(hv, \bar{y}) - J(0, \bar{y})}{h} \\
&= \lim_{h \downarrow 0} \frac{\|hXv + (\bar{y} - y)\|_2^2 + \lambda h - \|\bar{y} - y\|_2^2}{h} \\
&= \lim_{h \downarrow 0} \frac{h^2\|Xv\|_2^2 - 2h(Xv)^T(\bar{y} - y) + \lambda h}{h} \\
&= \lim_{h \downarrow 0} h\|Xv\|_2^2 - 2(Xv)^T(\bar{y} - y) + \lambda \\
&= \lambda - 2(Xv)^T(y - \bar{y})
\end{aligned}$$

Part 2

We now claim that for any $v \neq 0$, we have $J'(0, \bar{y}; v) \geq 0$ if and only if $\lambda \geq C$, for some C that depends on $X, y, \bar{y}, \text{ and } v$

Observe that

$$\begin{aligned}
J'(0, \bar{y} : v) \geq 0 \\
\lambda - 2(Xv)^T(y - \bar{y}) \geq 0 \\
\lambda \geq 2(Xv)^T(y - \bar{y}).
\end{aligned}$$

Thus

$$C = 2(Xv)^T(y - \bar{y})$$

Part 3

Now prove that $\lambda_{\max} = 2\|X^T(y-\bar{y})\|_\infty$ Observe that

$$\begin{aligned} C &= 2v^T X^T(y-\bar{y}) \\ &= 2\sum_{i=1}^{d} v_i (X^T(y-\bar{y}))_i \qquad \text{subject to } \|v\|_1 = 1 \end{aligned}$$

From here it should be clear that $C \leq 2\|X^T(y-\bar{y})\|_\infty$ and that $C = 2\|X^T(y-\bar{y})\|_\infty$ when $v = e_j$ where j is the index of $2\|X^T(y-\bar{y})\|_\infty$ in $X^T(y-\bar{y})$.

Thus $\lambda_{\max} = 2\|X^T(y-\bar{y})\|_\infty$.

Part 4

Conclude that $(w^*, b^*) = (0, \bar{y})$ is a minimizer of $J(w, b)$ if and only if $\lambda \geq 2\|X^T(y-\bar{y})\|_\infty$

Assume that $(w^*, b^*) = (0, \bar{y})$ is a minimizer of $J(w, b)$. Then it follows that $J'(0, \bar{y}; v) \geq 0 \;\forall\; v \neq 0$. We know that $J'(0, \bar{y}; v) \geq 0$ when $\lambda \geq 2(Xv)^T(y-\bar{y})$. But $2(Xv)^T(y-\bar{y}) \leq 2\|X^T(y-\bar{y})\|_\infty$. Thus $J'(0, \hat{y}; v) \geq 0 \;\forall\; v \neq 0$ when $\lambda \geq 2\|X^T(y-\bar{y})\|_\infty$

Now assume that $\lambda \geq 2\|X^T(y-\bar{y})\|_\infty$. Then $\lambda \geq 2(Xv)^T(y-\bar{y}) \;\forall\; v \neq 0$. Then we have $J'(0, \bar{y}; v) \geq 0 \;\forall\; v \neq 0$ and so $(w^*, b^*) = (0, \bar{y})$ is a minimizer of $J(w, b)$

4.2 Feature Correlation

4.2.1

Without loss of generality, assume the first two colums of X are our repeated features. Partition X and θ as follows: \

$$X = \begin{pmatrix} x_1 & x_2 & X_r \end{pmatrix} \qquad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_r \end{pmatrix}$$

We can write the Lasso objective function as:

$$\begin{aligned} J(\theta) =& \|X\theta - y\|_2^2 + \lambda\|\theta\|_1 \\ =& \|x_1\theta_1 + x_2\theta_2 + X_r\theta_r - y\|_2^2 + \lambda|\theta_1| + \lambda|\theta_2| + \lambda\|\theta_r\|_1 \end{aligned}$$

With repeated features, there will be multiple minimizers of $J(\theta)$. Suppose that

$$\hat{\theta} = \begin{pmatrix} a \\ b \\ r \end{pmatrix}$$

is a minimizer of $J(\theta)$. Give conditions on c and d such that $\left(c, d, r^T\right)^T$ is also a minimizer of $J(\theta)$. [Hint: First show that a and b must have the same sign, or at least one of them is zero. Then, using this result, rewrite the optimization problem to derive a relation between a and b.]\

4.2.1.1 Claim: a and b must both have the same sign or at least one is zero. In other words, $sign(a)sign(b) \geq 0$

Note that since $x_i = x_j$, any solution that minimizes $J(\hat{\theta})$ can be written as

$$y = (a+b)x_1 + X_r r$$

Thus there exists some k that for all solutions, $a + b = k$

Now, again since $x_i = x_j$,we can write

$$J(\hat{\theta}) = \|(a+b)x_1 + X_r r - y\|_2^2 + \lambda|a| + \lambda|b| + \lambda\|r\|_1$$

Now let

$$F(\hat{\theta}) = \|(a+b)x_1 + X_r r - y\|_2^2 + \lambda\|r\|_1$$

Thus

$$J(\hat{\theta}) = F(\hat{\theta}) + \lambda(|a| + |b|)$$

Note that for some $a + b = k$, $F(\hat{\theta})$ is a constant. Thus $J(\hat{\theta})$ is minimized by minimizing $\lambda(|a| + |b|)$

Now by the triangle inequality

$$|a+b| \leq |a| + |b|$$

Now $|a+b| = |a| + |b|$ only when $sign(a)sign(b) \geq 0$.

Thus $J(\hat{\theta})$ is minimized when $sign(a)sign(b) \geq 0$.

4.2.1.2 Derive the relation between two possible solutions.

We already showed that a and b must have the same sign, thus

$$J(\hat{\theta}) = \|(a+b)x_1 + X_r r - y\|_2^2 + \lambda|a+b| + \lambda\|r\|_1$$

Letting $a + b = k$, its clear to see that $\left(c, d, r^T\right)^T$ is also a minimizer if $c + d = k = a + b$

4.2.2

Using the same notation as the previous problem, suppose

$$\hat{\theta} = \begin{pmatrix} a \\ b \\ r \end{pmatrix}$$

minimizes the ridge regression objective function. What is the relationship between a and b, and why? \

Solution:

Let $\phi = \begin{pmatrix} a \\ b \end{pmatrix}$

Note then that the ridge regression solution can be written as

$$J(\hat{\theta}) = \|(a+b)x_1 + X_r r - y\|_2^2 + \lambda\|\phi\|_2^2 + \lambda\|r\|_2^2$$

Thus $J(\hat{\theta})$ is minimized when $a = b$ since $\lambda\|\phi\|_2^2$ is minimized when $a = b$.

5. The Ellipsoids in the l_1/l_2 Regularization Picture

5.1

Let $\hat{w} = \left(X^T X\right)^{-1} X^T y$. Show that $\hat{w}$ has empirical risk given by $\hat{R}_n(\hat{w}) = \frac{1}{n}\left(-y^T X\hat{w} + y^T y\right)$

First note that

$$\begin{aligned} \hat{w} &= (X^T X^{-1})^{-1} X^T y \\ &= X^{-1}(X^T)^{-1} X^T y \\ &= X^{-1} y \end{aligned}$$

Thus

$$\hat{w}^T = y^T (X^T)^{-1}$$

Now

$$\begin{aligned} \hat{R}_n(\hat{w}) &= \frac{1}{n}\left(X\hat{w} - y\right)^T \left(X\hat{w} - y\right) \\ &= \frac{1}{n}\left(\hat{w}^T X^T X\hat{w} - \hat{w}^T X^T y - y^T X\hat{w} + y^T y\right) \\ &= \frac{1}{n}\left(\hat{w}^T X^T X\hat{w} - 2y^T X\hat{w} + y^T y\right) \\ &= \frac{1}{n}\left(y^T (X^T)^{-1} X^T X\hat{w} - 2y^T X\hat{w} + y^T y\right) \\ &= \frac{1}{n}\left(-y^T X\hat{w} + y^T y\right) \end{aligned}$$

5.2

Show that for any w we have

$$\hat{R}_n(w) = \frac{1}{n}\left(w - \hat{w}\right)^T X^T X\left(w - \hat{w}\right) + \hat{R}_n(\hat{w}).$$

Recall that $\hat{R}_n(\hat{w}) = \frac{1}{n}\left(-y^T X\hat{w} + y^T y\right)$ and $\hat{w} = \left(X^T X\right)^{-1} X^T y$.

Then

$$\begin{aligned}
\hat{R}_n(w) &= \frac{1}{n}(Xw - y)^T\,(Xw - y) \\
&= \frac{1}{n}\left(w^T X^T X w - 2y^T X w + y^T y\right) \\
&= \frac{1}{n}\left(w^T X^T X w - 2(X^T y)^T w + y^T y\right) \\
&= \frac{1}{n}\left[\left(w - (X^T X)^{-1} X^T y\right)^T X^T X \left(w - (X^T X)^{-1} X^T y\right) - y^T X (X^T X)^{-1} X^T y + y^T y\right] && \text{By completing} \\
&= \frac{1}{n}\left[(w - \hat{w}) X^T X (w - \hat{w}) - y^T X\hat{w} + y^T y\right] && \text{Since } \hat{w} = (X \\
&= \frac{1}{n}\left[(w - \hat{w}) X^T X (w - \hat{w})\right] + \hat{R}_n(\hat{w}) && \text{Since } \hat{R}_n(\hat{w}) =
\end{aligned}$$

5.3

Give a very short proof that $\hat{w} = \left(X^T X\right)^{-1} X^T y$ is the empirical risk minimizer.

Since $\hat{R}_n(\hat{w})$ is a constant, $argmin_w \hat{R}_n(w) = argmin_w (w - \hat{w}) X^T X (w - \hat{w})$

Now when $w = \hat{w}$, we have

$$\begin{aligned}
(w - \hat{w}) X^T X (w - \hat{w}) &= (\hat{w} - \hat{w}) X^T X (\hat{w} - \hat{w}) \\
&= 0
\end{aligned}$$

But $X^T X$ is a symmetric positive semidefinite matrix, thus $(w - \hat{w}) X^T X (w - \hat{w}) \geq 0$ for all w.

Thus $argmin_w \hat{R}_n(w) = \hat{w} = \left(X^T X\right)^{-1} X^T y$

5.4

Give an expression for the set of w for which the empirical risk exceeds the minimum empirical risk $\hat{R}_n(\hat{w})$ by an amount $c > 0$. If X is full rank, then $X^T X$ is positive definite, and this set is an ellipse - what is its center?

$\left\{ w \,\middle|\, \frac{1}{n}\left[(w - \hat{w}) X^T X (w - \hat{w})\right] = c \right\}$

The center is at $\hat{w}$ since when c=0, we have $\left\{ w \,\middle|\, \frac{1}{n}\left[(w - \hat{w}) X^T X (w - \hat{w})\right] = 0 \right\} = \hat{w}$

6. Projected SGD via Variable Splitting

6.1

Implement projected SGD to solve the above optimization problem for the same λ's as used with the shooting algorithm. Since the two optimization algorithms should find essentially the same solutions, you can check the algorithms against each other. Report the differences in validation loss for each λ between the two optimization methods. (You can make a table or plot the differences.)

In [ ]:
```
def compute_projected_lasso(X, y, l, alpha=0.1, n_epochs=1000, w_init='ridge'):
    # alpha: step size
    n_cols = X.shape[1]

    if w_init == 'ridge':
        w = np.matmul(np.matmul(np.linalg.inv(np.matmul(X.T, X) + l*np.eye(n_cols)), X.T), y)
    else:
        w = np.zeros(n_cols)

    w_pos = w
    w_neg = -w
    w_pos[w_pos<0] = 0
    w_neg[w_neg<0] = 0
    #w_pos = np.clip(w, a_min=0)
    #w_neg = -np.clip(w, a_max=0)

    #old_loss = lasso_loss(X, y, w, l)
    #delta_loss = thresh + 1


    for epoch in range(n_epochs):
        for i, x in enumerate(X):
            residual = np.dot(x, w) - y[i]
            grad_w_pos = 2 * x * residual + l
            grad_w_neg = -2 * x * residual + l

            w_pos = w_pos - alpha*grad_w_pos
            w_neg = w_neg - alpha*grad_w_neg

            # Project back into valid set
            w_pos[w_pos<0] = 0
            w_neg[w_neg<0] = 0
            #w_pos = np.clip(w_pos, a_min=0)
            #w_neg = np.clip(w_neg, a_max=0)

            w = w_pos - w_neg

    return w
```

In [ ]:
```
w = compute_projected_lasso(x_train_featurized, y_train, 0.015, alpha=0.001
, w_init='ridge', n_epochs=2000)
```

In [ ]:
```
plt.scatter(x_train, y_train, label='Training Data')
#plt.plot(x, best_ridge.predict(x_featurized), label='Ridge Regression $\la
mbda$=0.0242')
plt.plot(x, np.matmul(x_featurized, w), label='Lasso with $\lambda$=1')
plt.plot(x, target_fn(x), label='Target Function')
plt.legend()
plt.xlabel('X')
plt.ylabel('Y')
plt.title("Comparison of Various Models to Target Function (Centered Y Valu
es)")
plt.xlim(0,0.95)
plt.ylim(-1,4)
```

IDK what is wrong here, it seems to perform lasso regression but only with much lower values of lambda