DS-1002: Homework #2

Due on February 12, 2014

Professor Ling

Cody Fizette

Problem 1

Justify the following results.

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
2. $\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$

Solution 1.1
To prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, we must show that:

$$1.\ A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

$$2.\ (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

Let $x \in A \cap (B \cup C)$. Then $x \in A$ and either $x \in B$ or $x \in C$. Without loss of generality assume that $x \in B$. Then $x \in A \cap B$. Since $A \cap B \subseteq (A \cap B) \cup (A \cap C)$, we conclude that $x \in (A \cap B) \cup (A \cap C)$, thus showing that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Next, let $y \in (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Then either $y \in A$ and $y \in B$ or $y \in A$ and $y \in C$. Without loss of generality assume that $y \in A$ and $y \in B$. Then $y \in B \cup C$. Thus $y \in A \cap (B \cup C)$. So therefore $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Since $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$, we conclude that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution 1.2
Observe that

$$\begin{aligned}
x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^c &\iff x \notin \bigcup_{i=1}^{\infty} A_i && \text{By definition of set complement} \\
&\iff x \notin A_i,\ \forall i \in \mathbb{Z}^+ && \text{By definition of set union} \\
&\iff x \in A_i^c,\ \forall i \in \mathbb{Z}^+ && \text{By definition of set complement} \\
&\iff x \in \bigcap_{i=1}^{\infty} A_i^c && \text{By definition of set intersection}
\end{aligned}$$

Thus $\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$

Problem 2

Given a sample space Ω with n elements, compute the total number of distinct subsets of Ω.

Solution 2

When constructing a subset A, $\forall x_i \in \Omega$, either $x_i \in A$ or $x_i \notin A$. Since there are n elements in Ω and two possibilities for each element, by the Fundamental Counting Principle there are 2^n distinct subsets of Ω.

Problem 3

Two players, A and B, alternately and independently flip a coin and the first player to obtain a head wins. Asume player A flips first.

1. If the coin is fair, what is the probability that A wins?
2. Suppose that $\mathbb{P}\left(head\right) = p$. What is the probability that A wins?

Solution 3.1

A wins if the first head occurs on an odd numbered throw. Thus:

$$\begin{aligned}
\mathbb{P}\left(A\,wins\right) &= \sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{2i}\left(\frac{1}{2}\right) \\
&= \frac{1}{2}\sum_{i=0}^{\infty}\left(\frac{1}{4}\right)^{i} \\
&= \frac{1}{2}\left(\frac{1}{1-\frac{1}{4}}\right) && \text{Sum of geometric series} \\
&= \frac{1}{2}\cdot\frac{4}{3} \\
&= \frac{2}{3}
\end{aligned}$$

Solution 3.2

Let $q = 1 - p$

$$\begin{aligned}
\mathbb{P}\left(A\,wins\right) &= \sum_{i=0}^{\infty}\left(1-p\right)^{2i}\left(p\right) \\
&= p\sum_{i=0}^{\infty}\left(q^2\right)^{i} \\
&= \frac{p}{1-q^2} && \text{Sum of geometric series}
\end{aligned}$$