2a) $R = \{x \mid \sum_{i=1}^{n} x_i \geq \frac{n}{2}\} = \{x \mid \bar{x} \geq 0.5\}$

By Central Limit Theorem, $\bar{X} \sim N(\mu_x, \sigma_x^2/n)$

$\mu_x = \theta$
$\sigma_x^2 = \theta(1-\theta)$

So $\bar{X} \sim N\left(\theta, \frac{\theta(1-\theta)}{n}\right)$

Or $\frac{\sqrt{n}(\bar{X}-\theta)}{\sqrt{\theta(1-\theta)}} \sim N(0,1)$

$$\beta(\theta) = P_\theta(\bar{X} \geq 0.5)$$
$$= P\left((\bar{X}-\theta)\sqrt{\frac{n}{\theta(1-\theta)}} \geq (0.5-\theta)\sqrt{\frac{n}{\theta(1-\theta)}}\right)$$
$$= \boxed{1 - \Phi\left((0.5-\theta)\sqrt{\frac{n}{\theta(1-\theta)}}\right)}$$

2b) Type I: $P(\bar{X} \geq 0.5 \mid \theta = 0.49) \leq 0.01$

$$0.99 \leq \Phi\left[(0.5-\theta)\sqrt{\frac{n}{\theta(1-\theta)}}\right]$$

$$\frac{\Phi^{-1}(0.99)}{0.5-\theta} \leq \sqrt{\frac{n}{\theta(1-\theta)}}$$

$$n \geq \left[\frac{\Phi^{-1}(0.99)}{0.5-\theta}\right]^2 \theta(1-\theta)$$

$n \geq 13520.27$

$n \geq 13{,}521$

3a) $f_Y(y) = \frac{ny^{n-1}}{\theta^n} \quad 0 \leq y \leq \theta$

$$\beta(\theta) = P_\theta(Y \geq c)$$

$$= \int_c^\theta \frac{ny^{n-1}}{\theta^n}\,dy$$

$$= \left[\frac{y^n}{\theta^n}\right]_{y=c}^{y=\theta} = \boxed{\begin{matrix} 1 - \frac{c^n}{\theta^n} & 0 \leq c \leq \theta \\ 0 & c > \theta \end{matrix}}$$

3b) $$\sup_{\theta \in \Theta_0} \beta(\theta) = 1 - \frac{c^n}{\theta_0^n}$$

$$\alpha = 1 - \frac{c^n}{\theta_0^n}$$

$$\frac{c^n}{\theta_0^n} = 1 - \alpha$$

$$\boxed{c = \theta_0(1-\alpha)^{\frac{1}{n}}}$$

3c) $$p = P_{\theta_0}(Y \geq 0.48)$$

$$= 1 - \frac{0.48^{20}}{0.5^{20}}$$

$$= \boxed{0.558}$$ We would choose to keep H_0

3d) $$p = P_{\theta_0}(Y \geq 0.52)$$

$$\boxed{= 0}$$ since $0.52 > \theta_0$

We have to reject H_0, it is impossible given the evidence.

4a) $\hat{\mu} = \overline{X}$ $\quad \hat{se} = \frac{\sigma}{\sqrt{n}}$

$$\boxed{\text{Reject if } \theta_0 \notin \left(\overline{X} - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}},\ \overline{X} + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)}$$

4b) $L(\mu|x) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left(\frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma^2}\right)$

$l(\mu|x) = n\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) + \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2$

$R = \left\{x \mid 2\log\left(\frac{L(\hat{\theta})}{L(\theta_0)}\right) \geq c\right\}$

Choose c such that

$P_{\theta_0}\left(2\log\left(\frac{L(\hat{\theta})}{L(\theta_0)}\right) \geq c\right) = \alpha$

So $c = \chi^2_{1,\alpha}$ since $2\log\left(\frac{L(\hat{\theta})}{L(\theta_0)}\right) \sim \chi^2_1$

Now $2\log\left(\frac{L(\hat{\theta})}{L(\theta_0)}\right) = 2n\log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) + \frac{1}{\sigma^2}\sum_{i=1}^n (x_i-\overline{x})^2 - 2n\log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{1}{\sigma^2}\sum_{i=1}^n (x_i-\mu_0)^2$

$$= \frac{1}{\sigma^2}\sum_{i=1}^n\left[(x_i-\overline{x})^2 - (x_i-\mu_0)^2\right]$$

$$\boxed{\text{So } R = \left\{x \mid \frac{1}{\sigma^2}\sum_{i=1}^n\left[(x_i-\overline{x})^2-(x_i-\mu_0)^2\right] \geq \chi^2_{1,\alpha}\right\}}$$

5a) $\pi(\theta) \propto \theta^{a-1}(1-\theta)^{b-1} \quad 0<\theta<1$

$f(\vec{x}|\theta) = (1-\theta)^{\sum x_i - n}\,\theta^n$

$$\boxed{\pi(\theta|\vec{x}) \propto \theta^{a+n-1}(1-\theta)^{b-n-1+\sum x_i}} \sim \text{Beta}\left(a+n,\ b+\sum_i x_i - n\right)$$

5b) $\hat{\theta}_{MMSE} = E\left[\text{Beta}(a+n,\ b+\sum_i x_i - n)\right]$

$$= \frac{a+n}{a+n+b+\sum x_i - n} = \boxed{\frac{a+n}{a+b+\sum x_i}}$$

$$\boxed{\lim_{n\to\infty} \hat{\theta}_{MMSE} = \frac{n}{\sum_i x_i} = \frac{1}{\bar{x}}}$$

5c) $\hat{\theta}_{MAP} = \underset{\theta\in\Theta}{\text{argmax}}\left[\theta^{a+n-1}(1-\theta)^{b-n-1+\sum x_i}\right]$

$$= \underset{\theta\in\Theta}{\text{argmax}}\left[(a+n-1)\ln\theta + (b-n-1+\sum x_i)\ln(1-\theta)\right] = \underset{\theta\in\Theta}{\text{argmax}}\left[\ln(\pi(\theta|\vec{x}))\right]$$

$$\frac{d\ln(\pi(\theta|\vec{x}))}{d\theta} = \frac{a+n-1}{\theta} - \frac{b+n-1+\sum x_i}{1-\theta} = \frac{A}{\theta} - \frac{B}{1-\theta} = 0 \quad A = a+n-1 \quad B = b+n-1+\sum x_i$$

$$A - \theta A - B\theta = 0$$

$$\theta(A+B) = A$$

$$\theta = \frac{A}{A+B} = \boxed{\frac{a+n-1}{2n+a+b+\sum x_i - 1}}$$

6a)

$$\pi(\theta|\vec{X}) = \pi(\theta) f(\vec{X}|\theta)$$

$$\propto \exp\left(-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2\right)\frac{1}{\theta^2+1}$$

$$= \exp\left(-\frac{1}{2}\left(\sum_{i=1}^{n}x_i^2 - 2\theta\sum_{i=1}^{n}x_i + n\theta^2\right)\right)\frac{1}{\theta^2+1}$$

$$= \exp\left(-\frac{n}{2}\left(\frac{\sum_{i=1}^{n}x_i^2}{n} - 2\theta\overline{X} + \theta^2\right)\right)\frac{1}{\theta^2+1}$$

$$= \exp\left(-\frac{n}{2}\left(\frac{\sum x_i^2 - \sum x_i^2}{n} + \overline{X}^2 - \overline{X}^2\right)\right)\exp\left(-\frac{n}{2}\left(\frac{\sum x_i^2}{n} - 2\theta\overline{X} + \theta^2\right)\right)\frac{1}{\theta^2+1}$$

$$= \exp\left(-\frac{n}{2}\left(\frac{\sum x_i^2}{n} - \overline{X}^2\right)\right)\exp\left(-\frac{n}{2}\left(\overline{X}^2 - 2\theta\overline{X} + \theta^2\right)\right)\frac{1}{\theta^2+1}$$

$$\boxed{\propto \exp\left(-\frac{n}{2}(\theta-\overline{X})^2\right)\frac{1}{\theta^2+1}}$$

hw10

December 8, 2018

```
In [60]: import numpy as np
         import matplotlib.pyplot as plt
         import matplotlib
         import seaborn as sns
         matplotlib.rcParams['figure.figsize'] = [15, 8]
```

```
/home/cfizette/anaconda3/lib/python3.6/importlib/_bootstrap.py:219: RuntimeWarning: numpy.dtype
  return f(*args, **kwds)
```

1 6b

1.0.1 Posterior distribution

$$\pi(\theta|x) \propto \exp(-\frac{n}{2}(\theta - \bar{X}_n)^2) \cdot \frac{1}{\theta^2 + 1}$$

```
In [10]: samp = [-1.249, 3.134, 0.825, 0.036, 0.814, -0.105]
         n = len(samp)
```

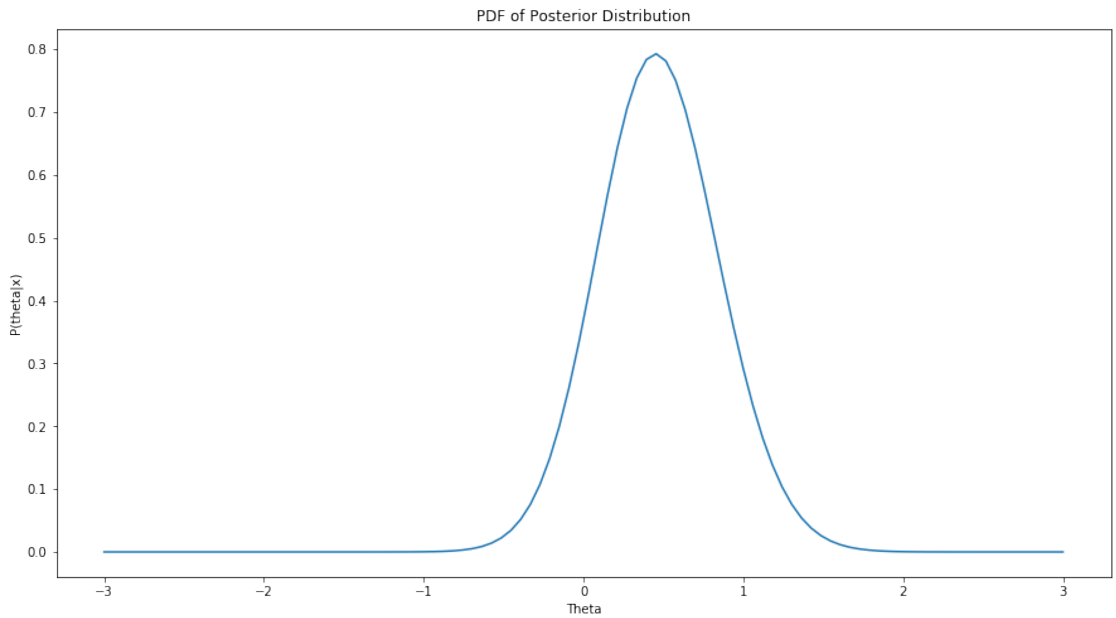
```
In [6]: x_bar = np.mean(samp)
```

```
In [25]: thetas = np.linspace(-3,3,100)
```

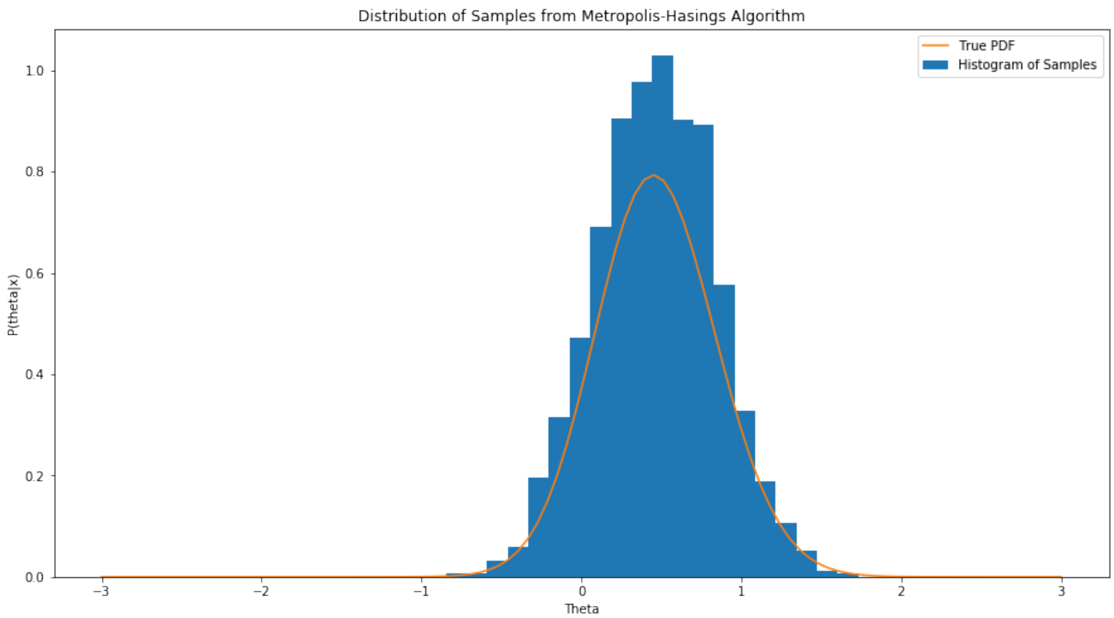
```
In [26]: pdf = np.exp(-(n/2)*(thetas - x_bar)**2)*(1/(np.power(thetas,2) + 1))
```

```
In [27]: plt.plot(thetas, pdf)
         plt.title('PDF of Posterior Distribution')
         plt.xlabel('Theta')
         plt.ylabel('P(theta|x)')
```

```
Out[27]: Text(0,0.5,'P(theta|x)')
```




```
In [32]: def f(x):
             return np.exp(-(n/2)*(x - x_bar)**2)*(1/(np.power(x,2) + 1))

In [33]: def q(x, mu):
             return np.exp(-3*(x-mu)**2)

In [36]: def rho(x,y):
             ratio = (f(y)*q(x,y))/(f(x)*q(y,x))
             return min(ratio, 1)

In [62]: sigma=1/6

In [63]: X = [0]
         for i in range(6000):
             y = np.random.normal(X[i], sigma)
             rho_ = rho(X[i],y)
             X_new = float(np.random.choice([y,X[i]],1, p=[rho_, 1-rho_]))
             X.append(X_new)
         X = np.array(X[1000:])

In [76]: plt.hist(X, normed=True, bins=20, label='Histogram of Samples')
         plt.plot(thetas, pdf, label='True PDF')
         plt.title('Distribution of Samples from Metropolis-Hasings Algorithm')
         plt.xlabel('Theta')
         plt.ylabel('P(theta|x)')
         plt.legend(loc='best')

Out[76]: <matplotlib.legend.Legend at 0x7f86ad890be0>
```



2 6c

```
In [73]: e_x = np.mean(X)

In [85]: p_0_1 = len(X[(0 < X) & (X < 1)])/len(X)

In [87]: print("E(theta|x) = {}".format(e_x))

E(theta|x) = 0.46470057589611125


In [89]: print("P(0 < theta < 1 | x) = {}".format(p_0_1))

P(0 < theta < 1 | x) = 0.8112377524495101


In [91]: np.mean(samp)

Out[91]: 0.5758333333333333
```

We see that the bayesian expected value of theta is less than the frequentist expectation. This is due to us using a non-uniform prior distribution for theta.