COMP 285

Analysis of Algorithms

Welcome to COMP 285

Lecture 11: BSTs + Self-Balancing Trees

Lecturer: Chris Lucas (cflucas@ncat.edu)

HW2

Grades were released!

HW3

Due Today @ 11:59pm

HW4

Released by EoD! Due 10/13!

Midterm Approaching!

10/06 @ 2pm! Lectures 0-11

Midterm Approaching!

10/06 @ 2pm! Lectures 0-11

Written exam, similar to quizzes, homework assignments, coding

Midterm Approaching!

Review topics?

Written exam, similar to quizzes, homework assignments, coding

Extra Credit Opportunities!

- Technical Interview Prep with Meta (Oct.) +0.5%
- Technical Interview Prep with Chris (Wed) +0.5%, up to 1%
- Midpoint survey (link) +1%

Quiz!

www.comp285-fall22.ml

Recall where we ended last lecture…

Binary tree terminology

Each node has two children.

The left child of 3 is 2

The right child of 3 is 4

The parent of 3 is 5

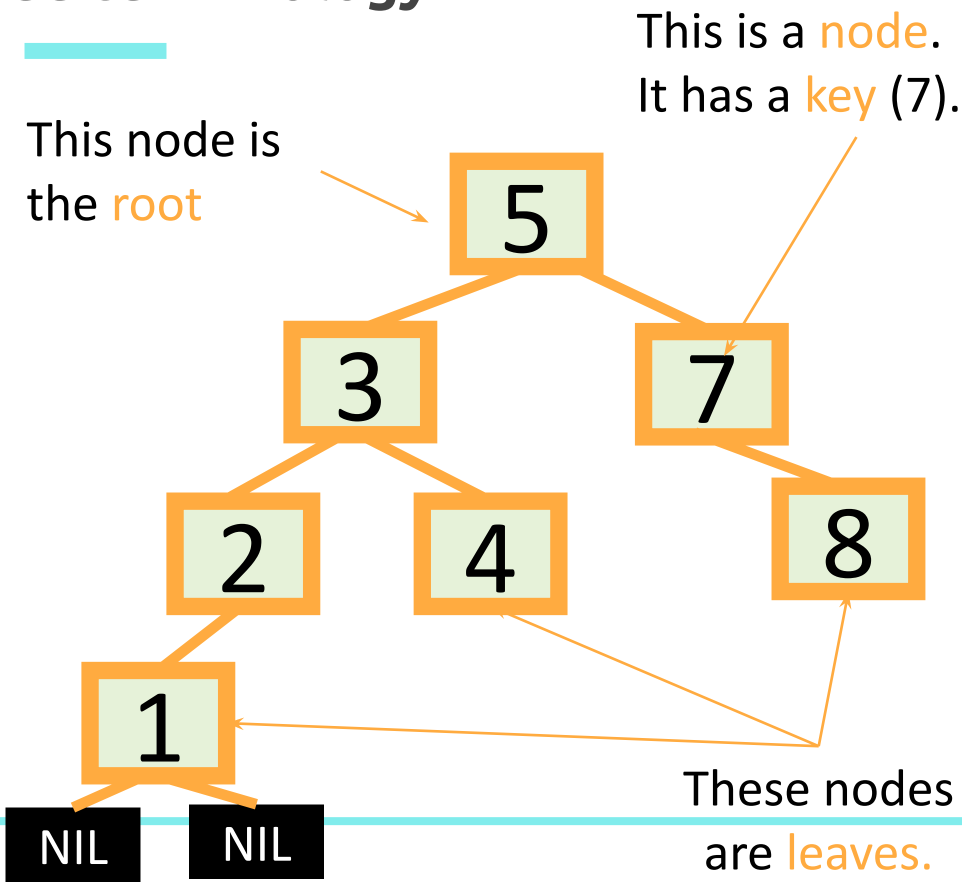
2 is a descendant of 5

Each node has a pointer to its left child, right child, and parent.

Both children of 1 NIL.

(I won't usually draw them).

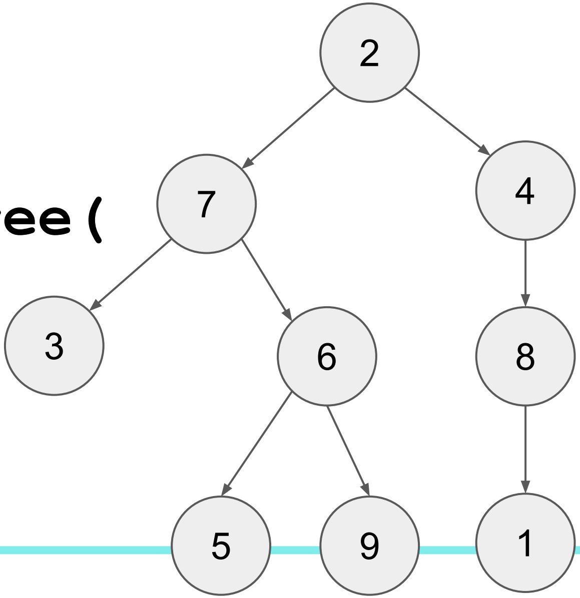
The height of this tree is 3. (Max number of edges from the root to a leaf).



findMaxOfTree

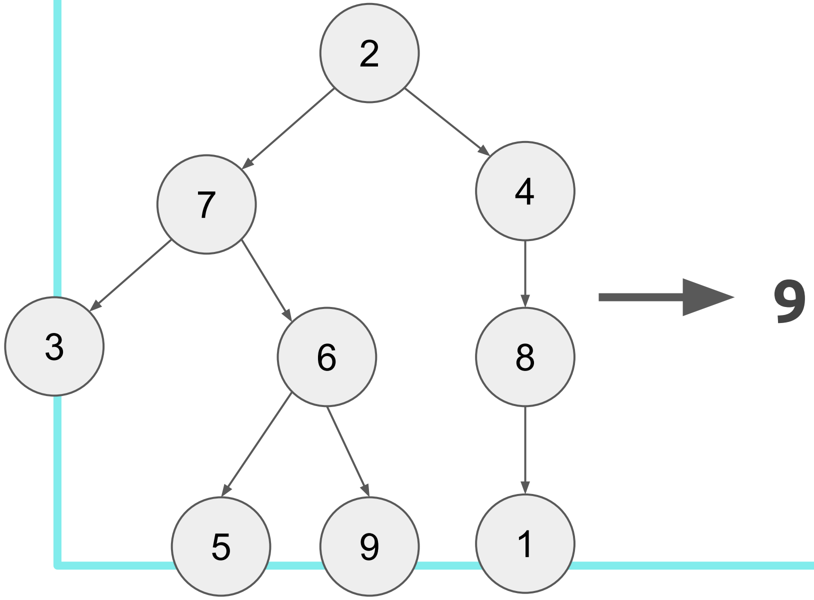
Instructions: *Write an algorithm that takes in a tree of ints, and returns the max of all the values within the tree.*

findMaxOfTree(



) outputs 9

Write an algorithm that takes in a tree of ints, and returns the max of all the values within the tree.

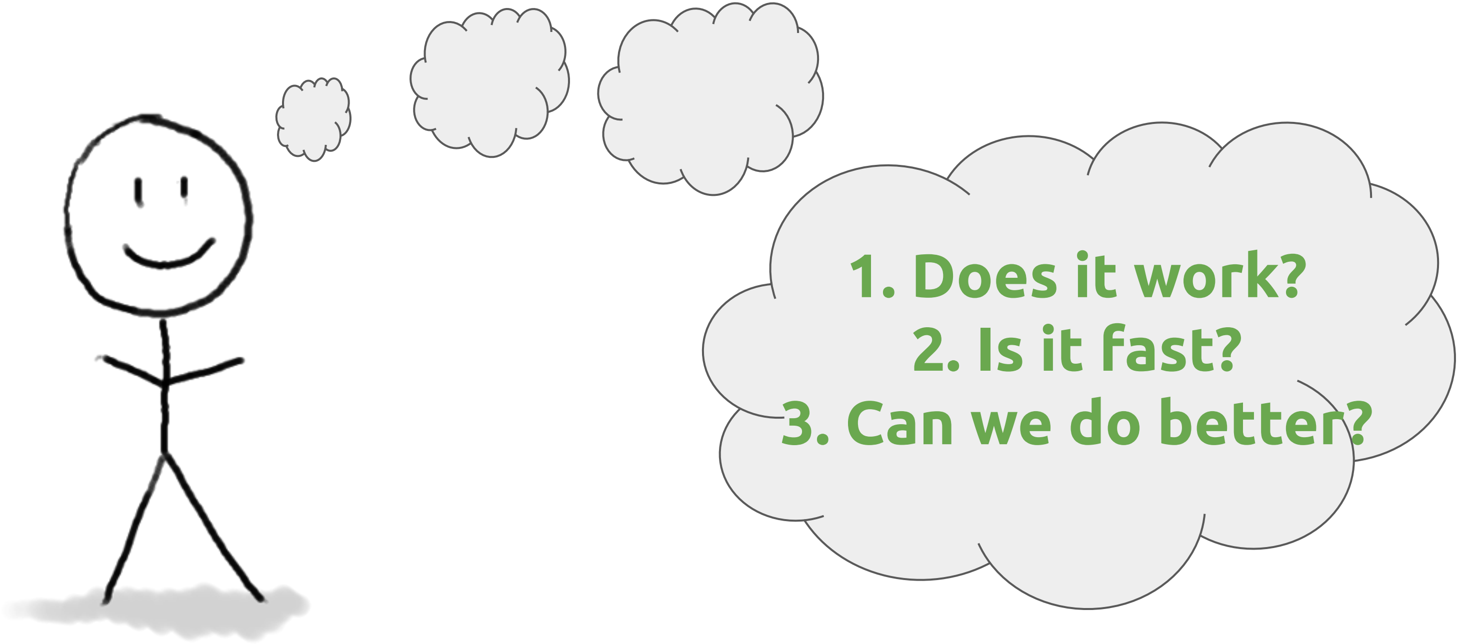



```
int findMaxOfTree(TreeNode<int>* root){
 if (root->isLeaf()) {
   return root->getValue();
 }
 int leftMax = INT_MIN;
 if (root->getLeft()) {
   leftMax = fMOT(root->getLeft());
 }
 int rightMax = INT_MIN;
 if (root->getRight()) {
   rightMax = fMOT(root->getRight());
 }
 return std::max({
             root->getValue(),
             leftMax,
             rightMax
         });
}
```

Big Questions!

- What are/Why binary search trees (BST)?
- Why does balance matter?
- How do we maintain balance?

Our Guiding Questions…



Motivation for Binary Search Trees

	Sorted Arrays	Linked Lists	(balanced) Binary Search Trees
Search			
Delete			
Insert			

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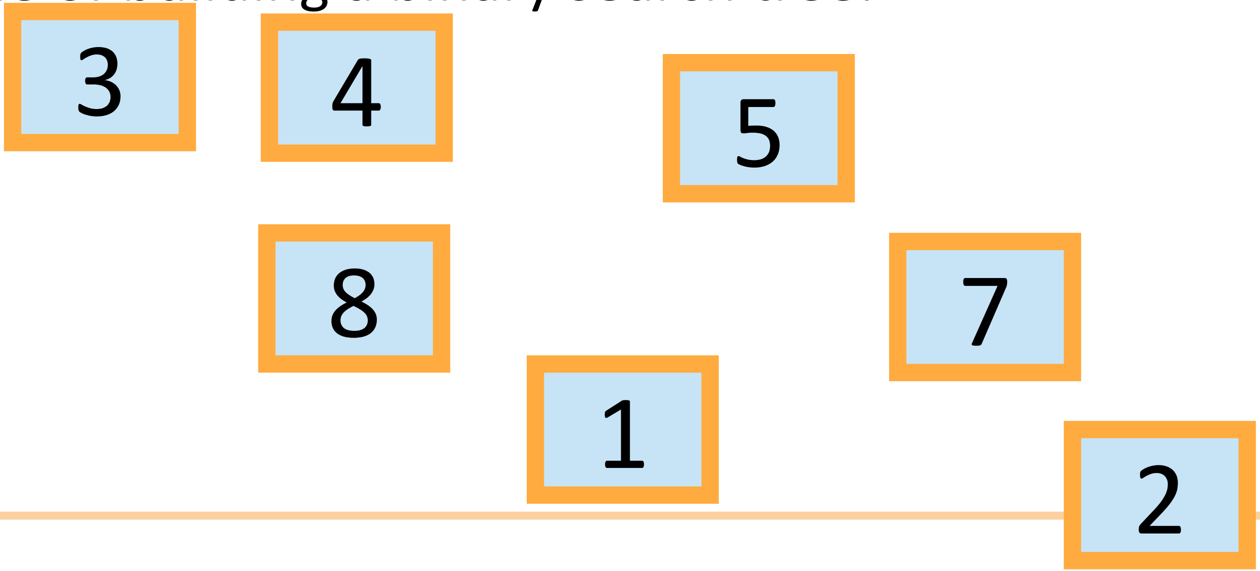
Search	O(log(n))	O(n)	O(log(n))
Delete	O(n)	O(n)	O(log(n))
Insert	O(n)	O(1)	O(log(n))

Binary Search Trees

- A BST is a binary tree so that:
  - Every LEFT descendant of a node has key less than that node.
  - Every RIGHT descendant of a node has key larger than that node.
- Example of building a binary search tree:



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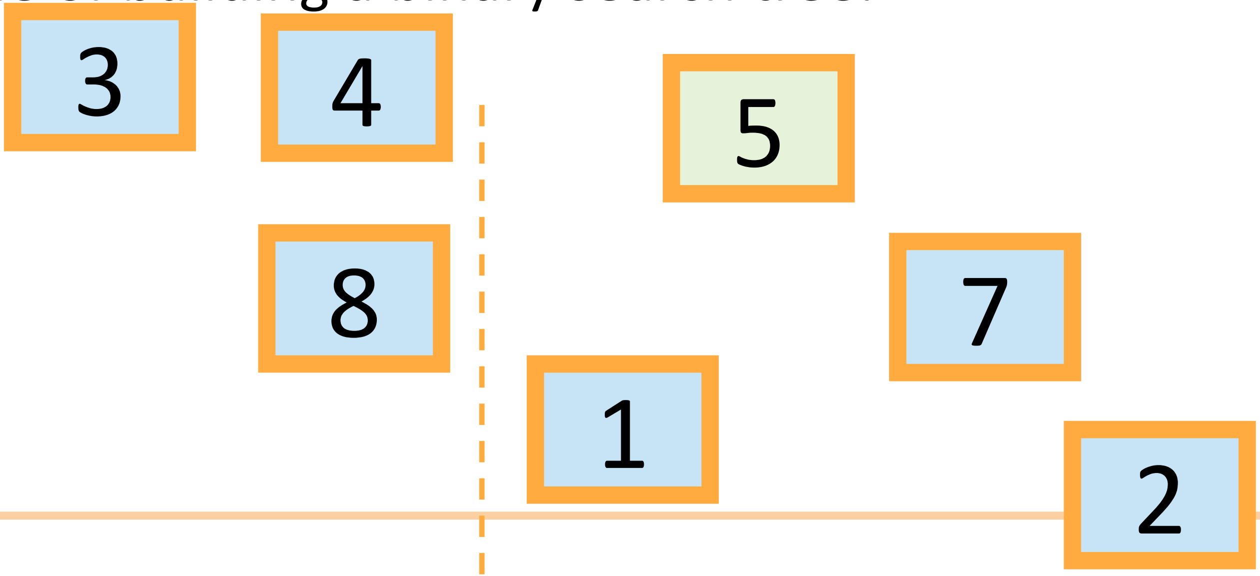

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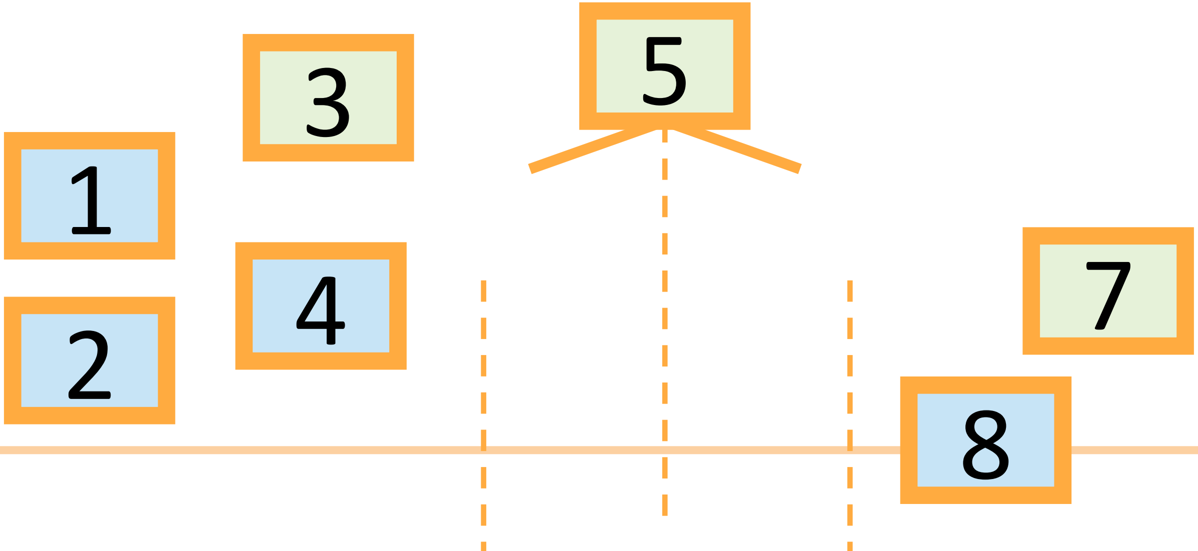

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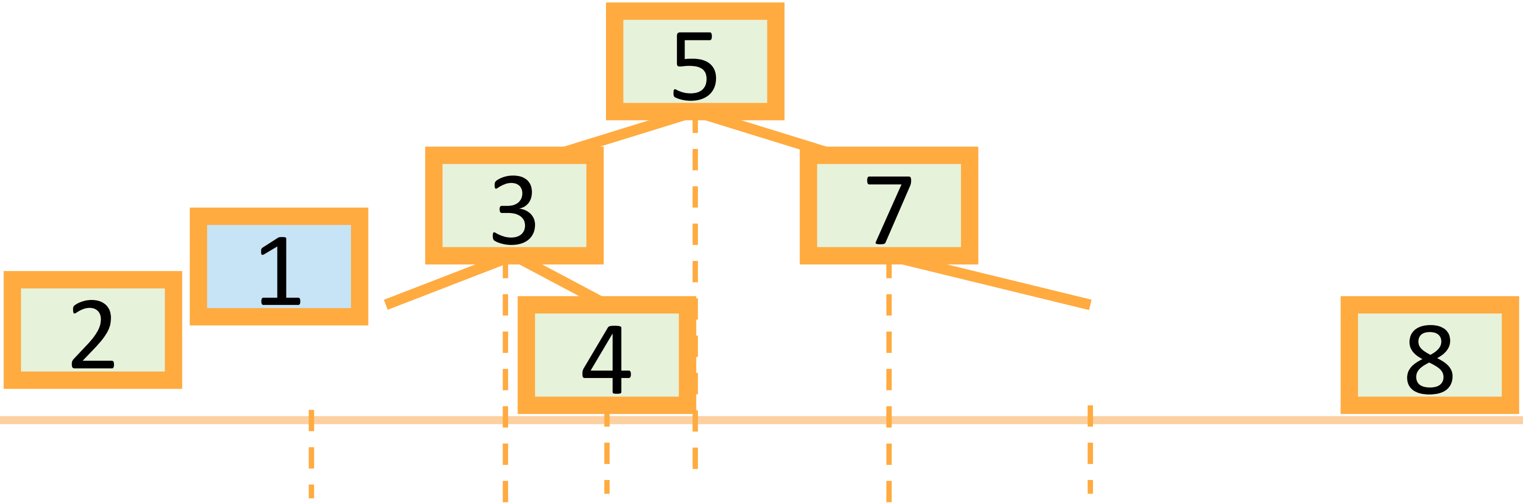

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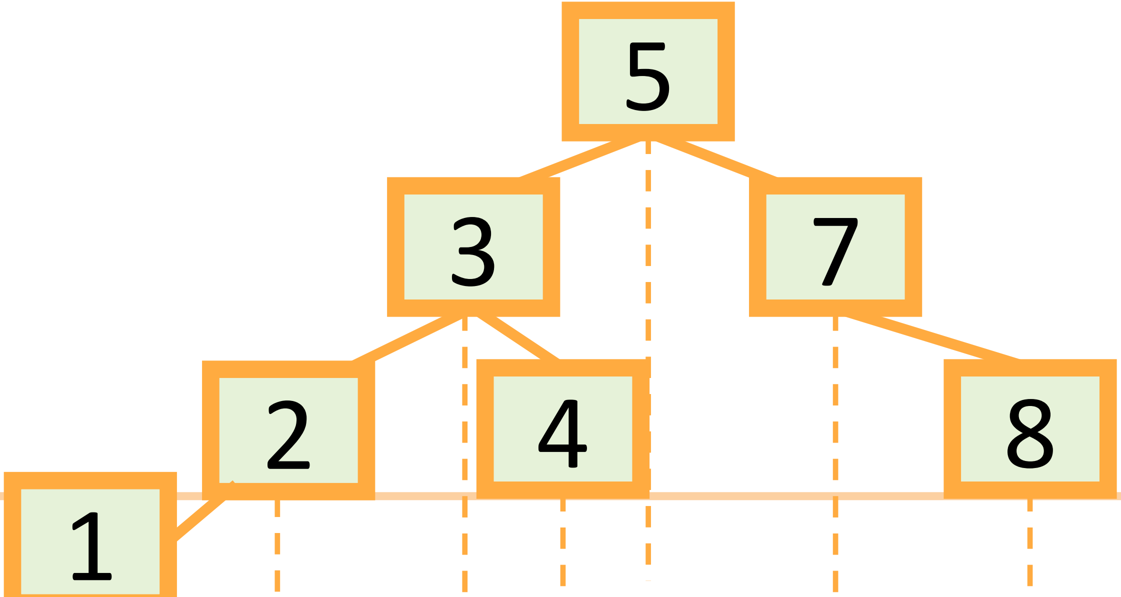
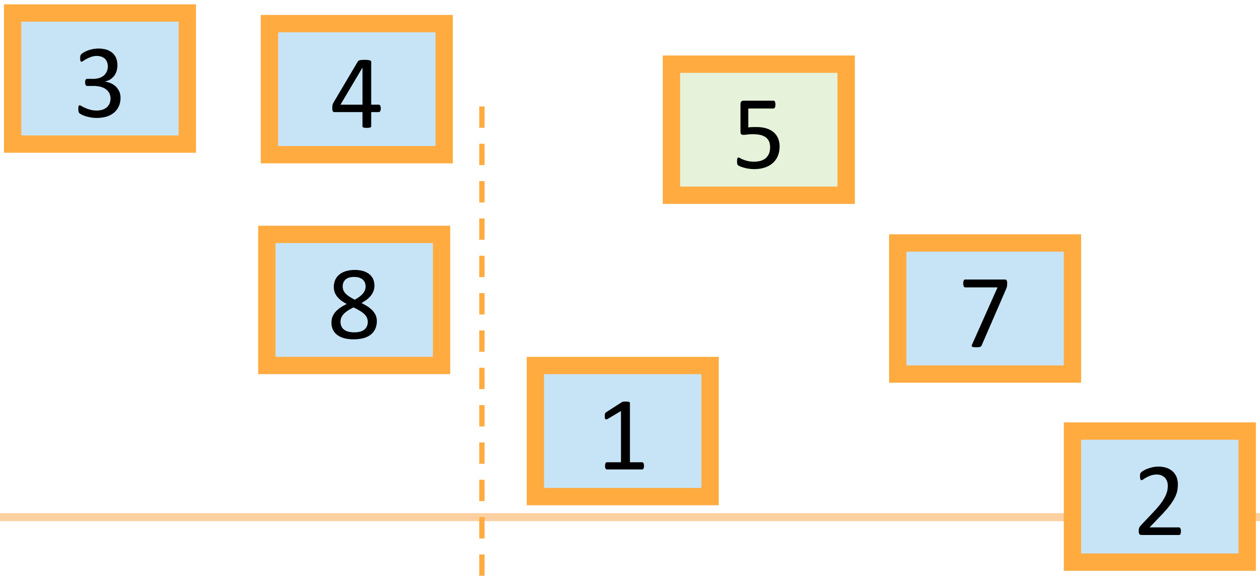


Q: Is this the only binary search tree I could possibly build with these values?

A: **No.** I made choices about which nodes to choose when. Any choices would have been fine.

Aside: this should look familiar

kinda like QuickSort



Are these Binary Search Trees? Yes or No?

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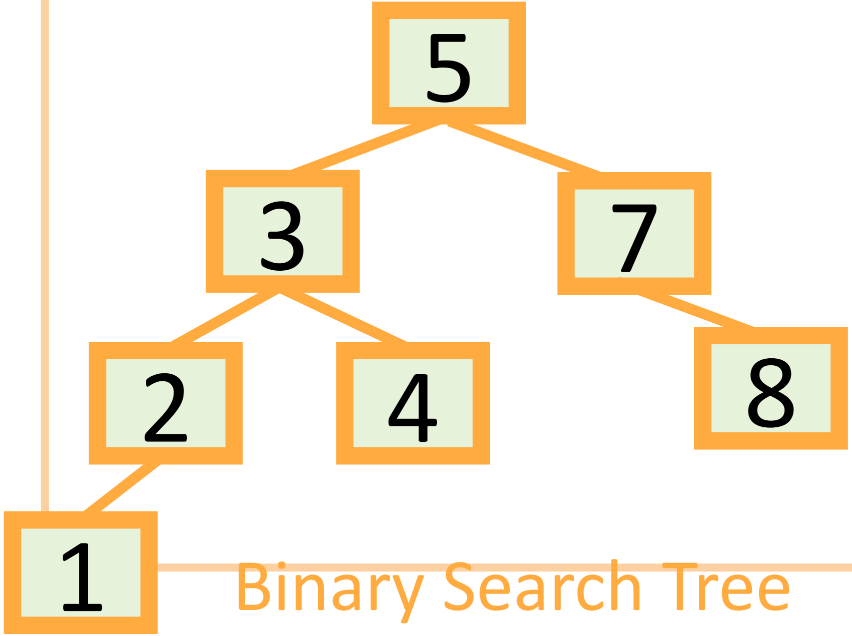
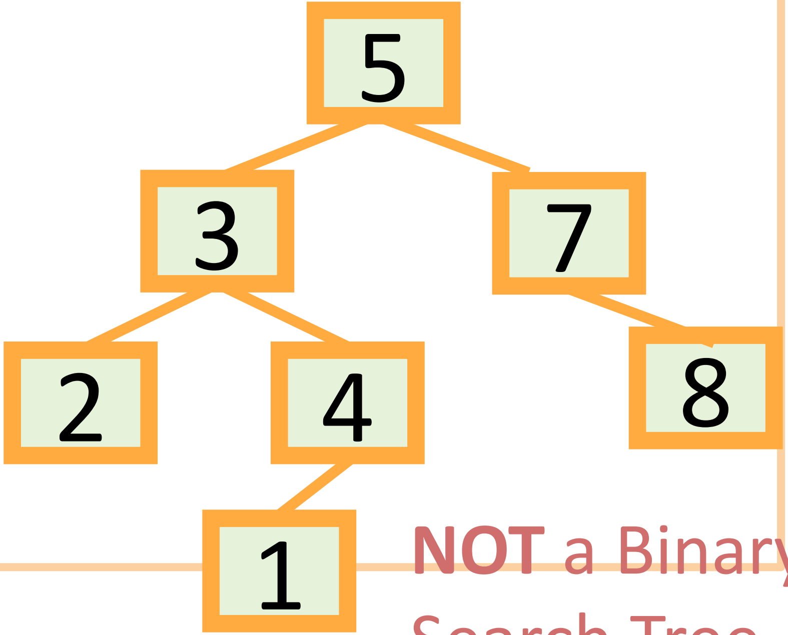


Binary Search Tree



NOT a Binary Search Tree

Are these Binary Search Trees? Yes or

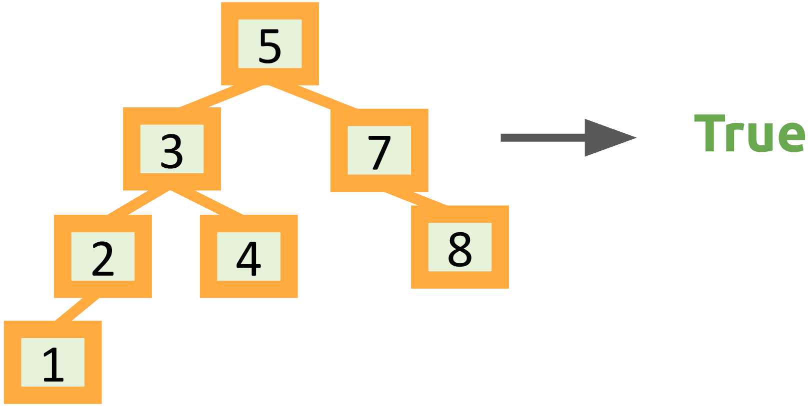
- A BST is a binary tree so that:
  - Every LEFT descendant of
  - Every RIGHT des

Can we do this programmatically?

Binary Search Tree

NOT a Binary Search Tree

Given a root node, determine if it's a BST and return True/False



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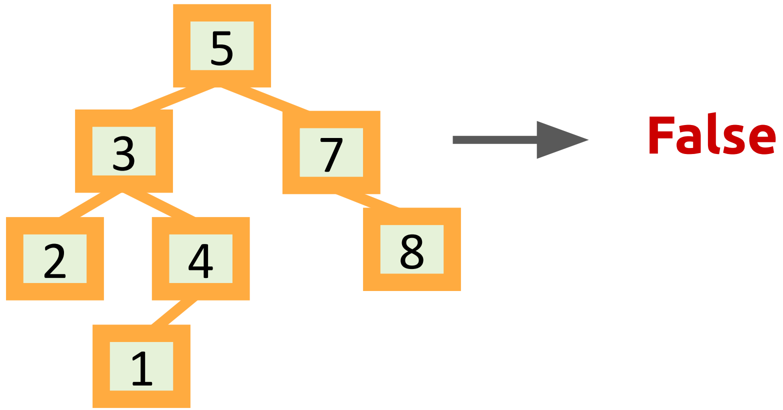

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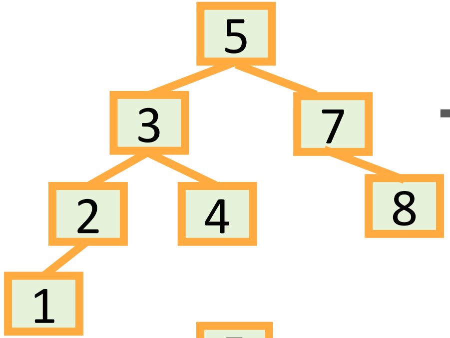
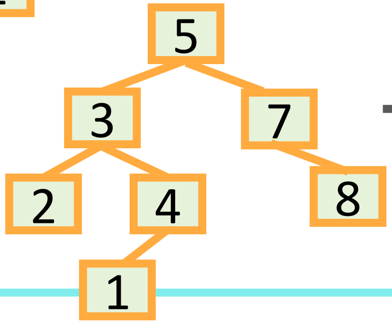


True



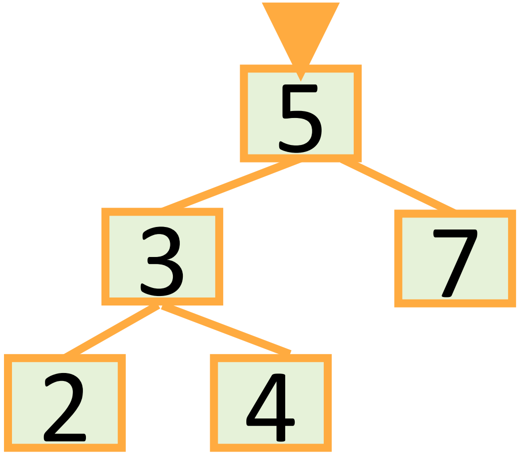
False

Let’s code it!!!

Aside: In-Order Traversal of BSTs

- Output all the elements in sorted order!

- inOrderTraversal(x):
  - if x!= NIL:
    - inOrderTraversal(x.left)
    - print(x.value)
    - inOrderTraversal(x.right)



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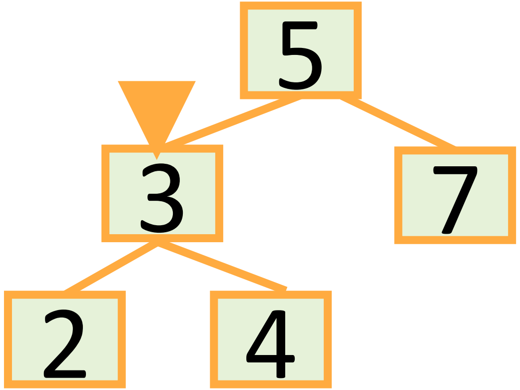

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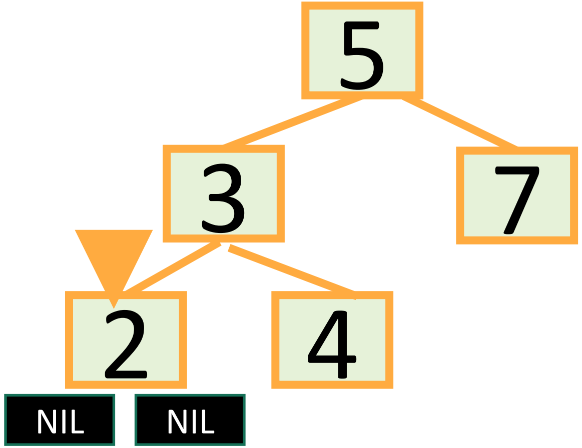

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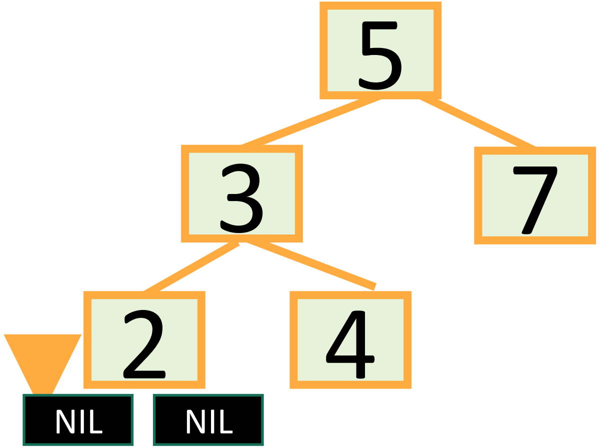

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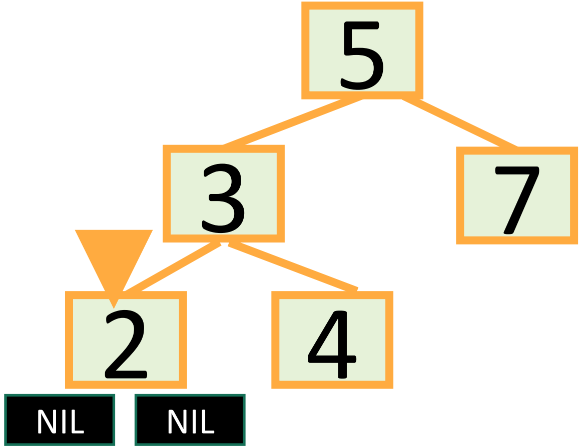

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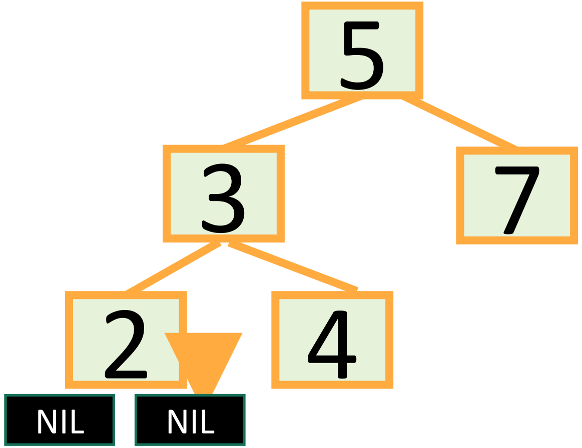

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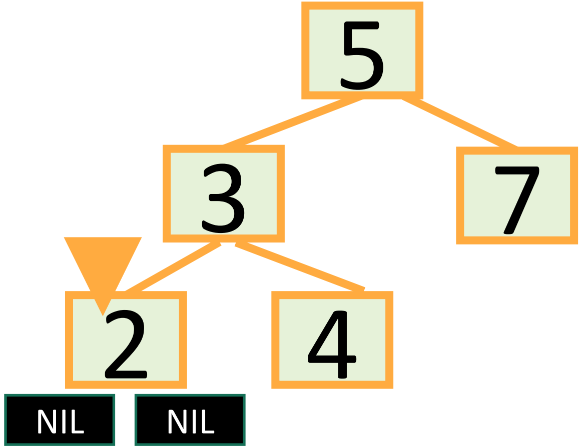

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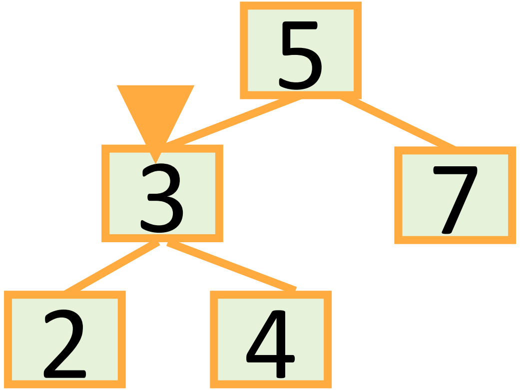

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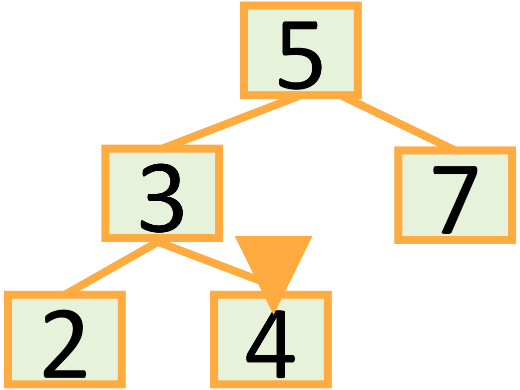

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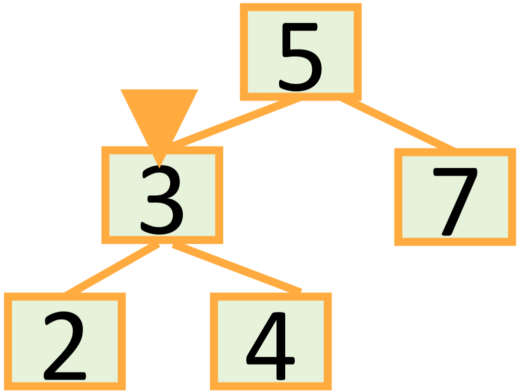

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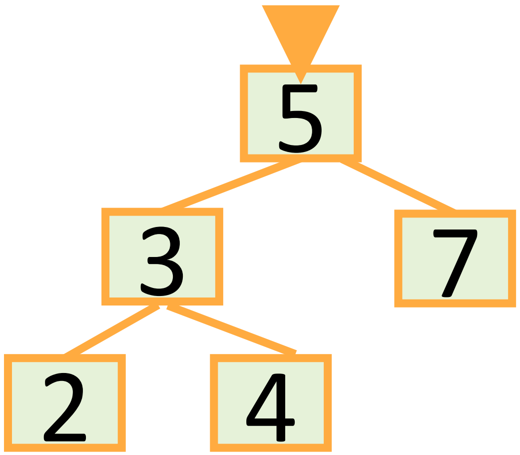

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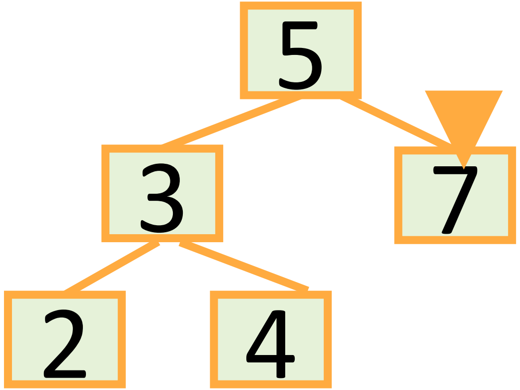

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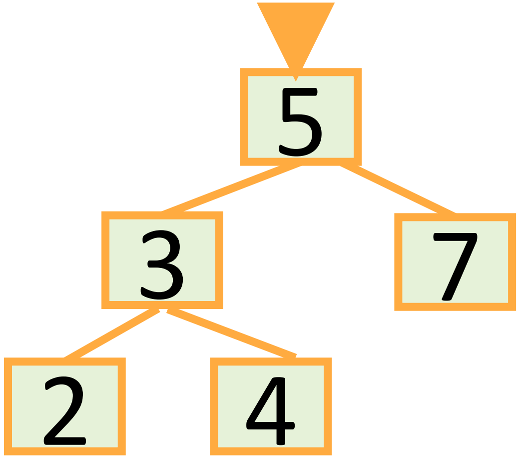


- Runs in time O(n).

2 3 4 5 7 Sorted!

Big Questions!

- What are/Why binary search trees?
- Why does balance matter?
- How do we maintain balance?

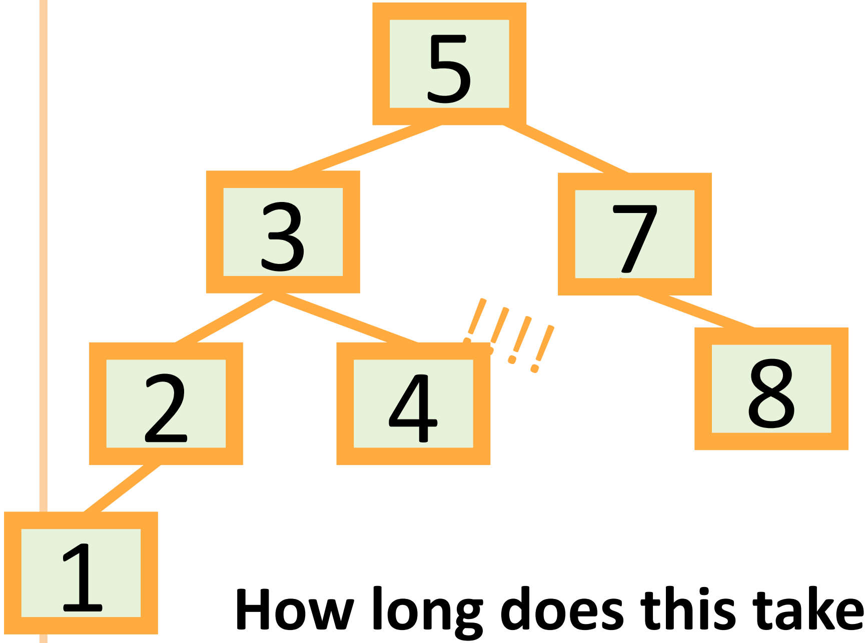
Back to the goal

Fast SEARCH/INSERT/DELETE

Can we do these?

	Sorted Arrays	Linked Lists	Binary Search Trees*
Search	O(log(n)) 😃	O(n) ☹️	O(log(n)) 😃
Delete	O(n) ☹️	O(n) ☹️	O(log(n)) 😃
Insert	O(n) ☹️	O(1) 😃	O(log(n)) 😃

SEARCH in a Binary Search Tree



EXAMPLE: Search for 4.

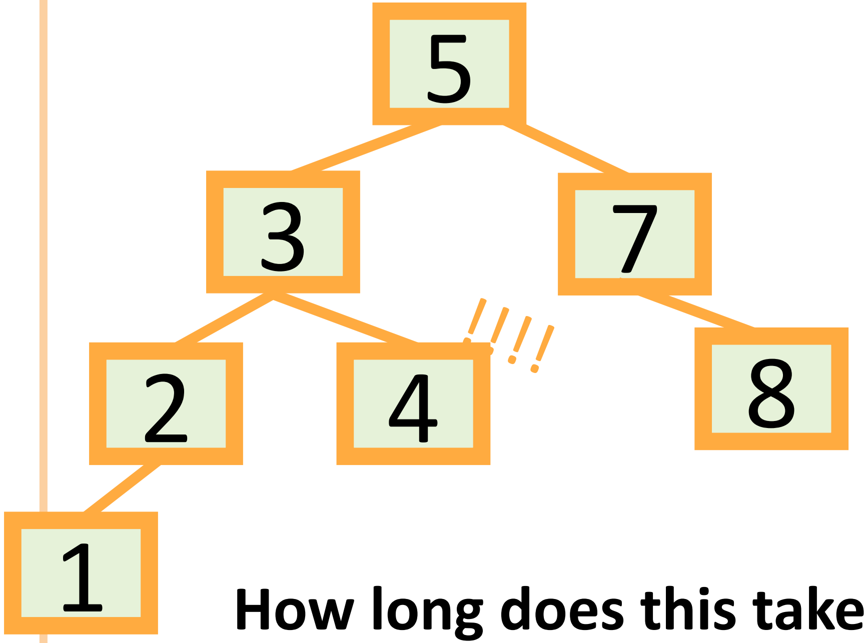
EXAMPLE: Search for 4.5

- It turns out it will be convenient to **return 4** in this case
- (that is, **return** the last node before we went off the tree)

How long does this take?

O(length of longest path) = O(height)

SEARCH in a Binary Search Tree



EXAMPLE: Search for 4.

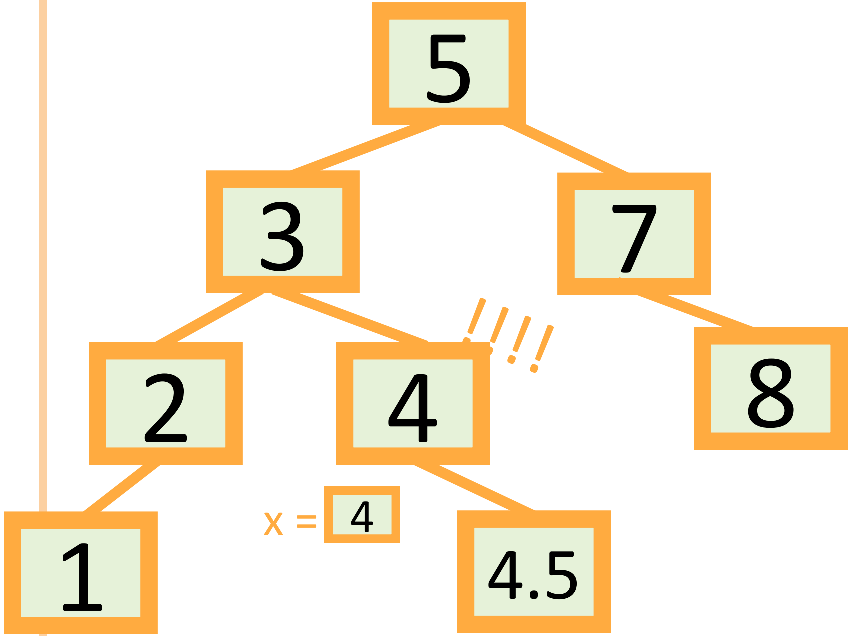
EXAMPLE: Search for 4.5

- if value > x.value:
  - Recurse on x.rightChild
- if value < x.value:
  - Recurse on x.leftChild
- if x.value == value:
  - **Return x**

How long does this take?

O(length of longest path) = O(height)

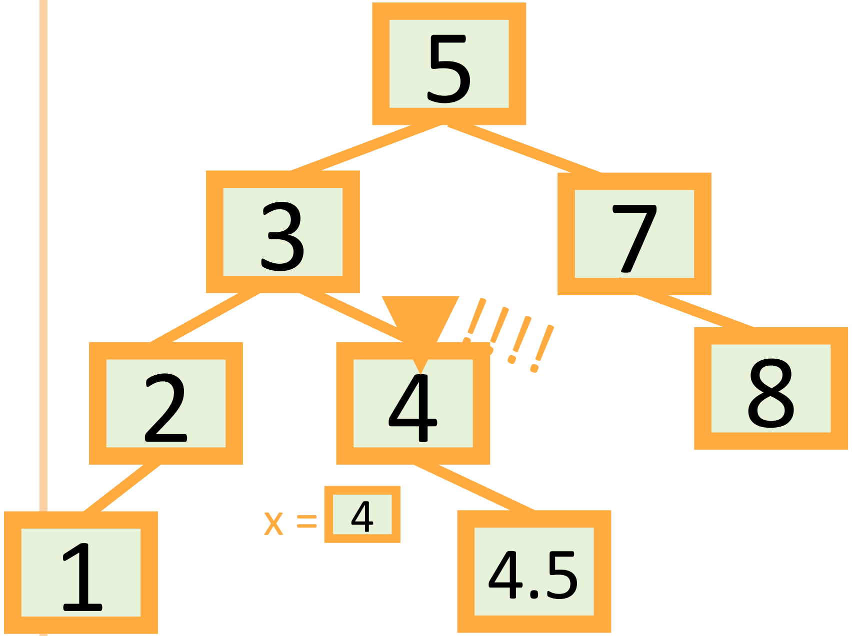
INSERT in a Binary Search Tree



EXAMPLE: Insert 4.5

- INSERT(value):
  - x = SEARCH(value)
  - **Insert** a new node with desired value at x…

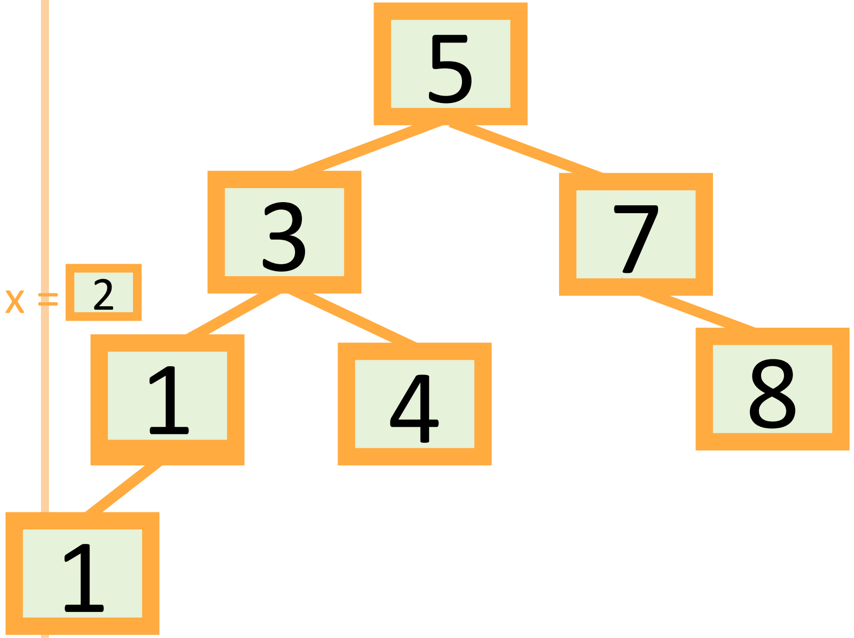
INSERT in a Binary Search Tree



EXAMPLE: Insert 4.5

- INSERT(value):
  - x = SEARCH(value)
  - if value > x.value:
    - Make a new node with the correct value, and put it as the right child of x.
  - if value < x.value:
    - Make a new node with the correct value and put it as the left child of x.
  - if x.value == value:
    - **return**

DELETE in a Binary Search Tree

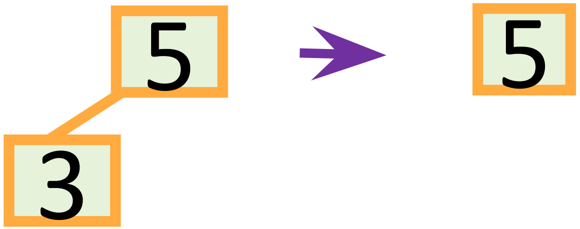


EXAMPLE: Delete 2

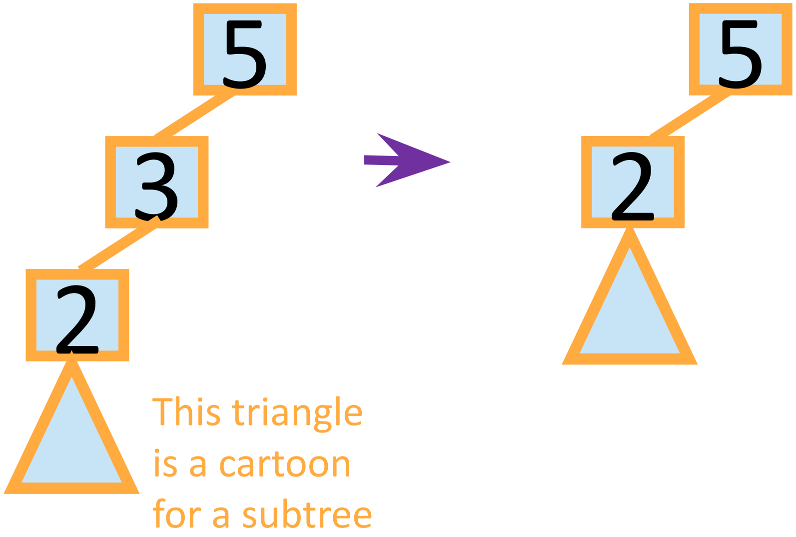
- DELETE(value):
  - x = SEARCH(value)
  - **if** x.value == value:
    - ….delete x….

DELETE in a Binary Search Tree

say we want to delete 3



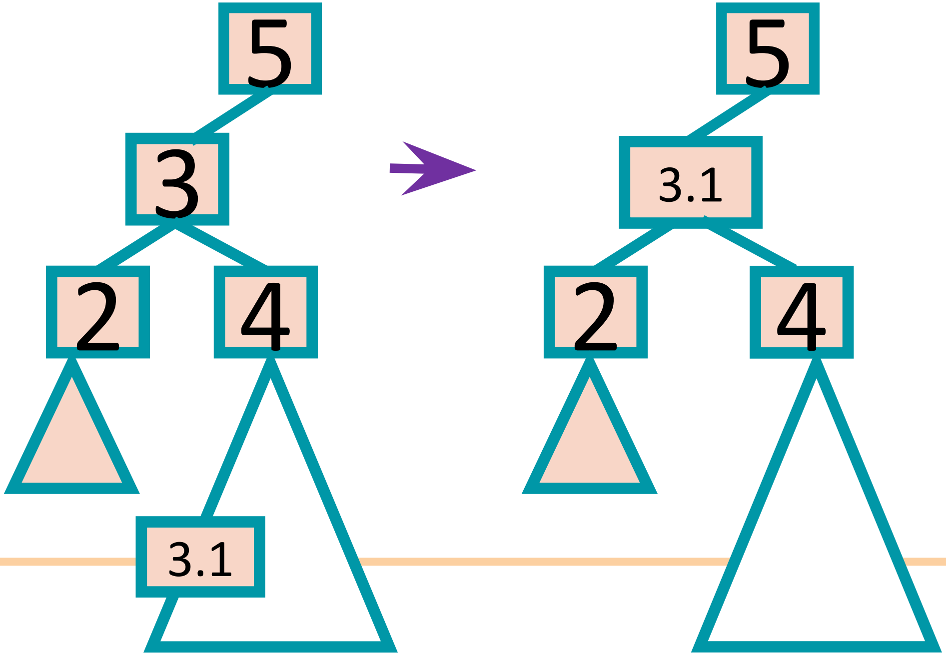
Case 1: if 3 is a leaf,
just delete it.



Case 2: if 3 has just one child,
move that up.

DELETE in a Binary Search Tree (cont.)

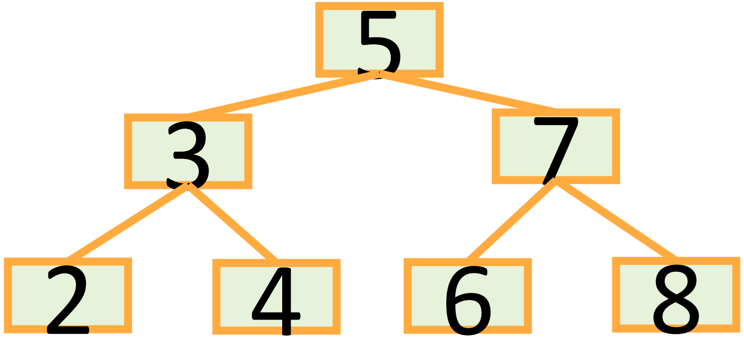
Case 3: if 3 has two children,
replace 3 with it's immediate successor.
(aka, next biggest thing after 3)



- Does this maintain the BST property?
  - Yes.
- How do we find the immediate successor?
  - SEARCH for 3 in the subtree under 3.right
- How do we remove it when we find it?
  - If [3.1] has 0 or 1 children, do one of the previous cases.
- What if [3.1] has two children?
  - It doesn't.

How long do these operations take?

- SEARCH is the big one.
  - Everything else just calls SEARCH and then does some small O(1)-time operation.



Time = O(height of tree)

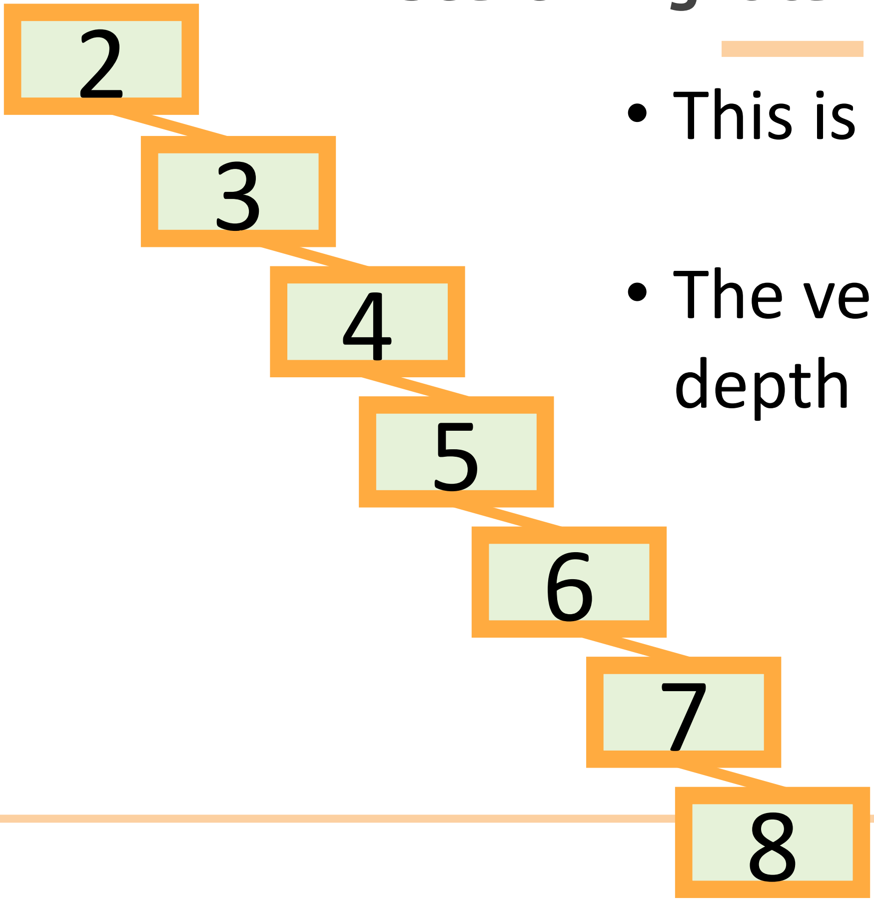
How long does search take?

1 minute think; 1 minute pair+share

Trees have depth O(log(n)). **Done!**

Wait a second…

Search might take time O(n).



- This is a valid binary search tree.
- The version with n nodes has depth n, **not** O(log(n)).

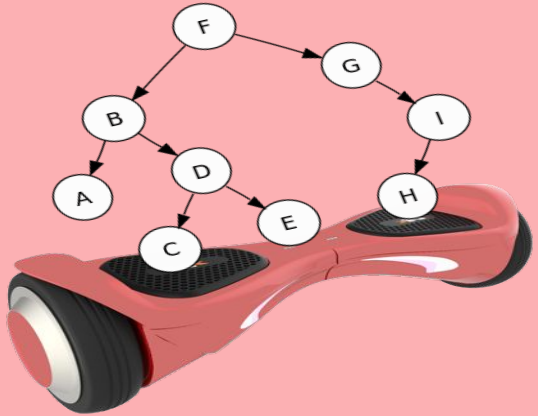
What to do?

- Goal: Fast SEARCH/INSERT/DELETE
- All these things take time O(height)
- And the height might be big!!! ☹️
- Idea!
  - Keep track of how deep the tree is getting.
  - If it gets too tall, re-do everything from scratch.
    - At least $\Omega(n)$ every so often….
- Turns out that's not a great idea, but we're onto something…

Big Questions!

- What are/Why binary search trees?
- Why does balance matter?
- How do we maintain balance?

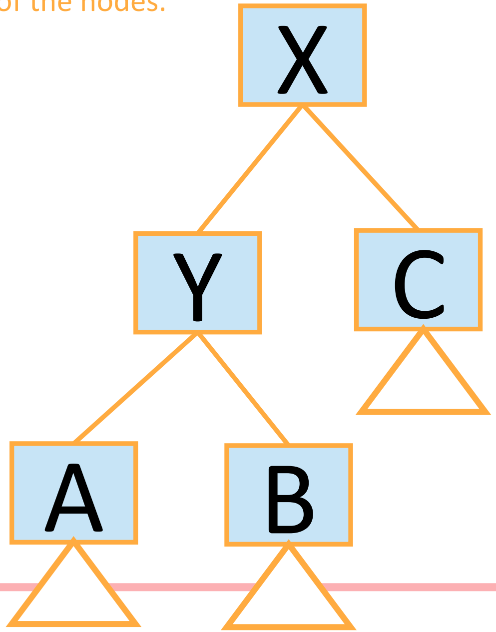
Self-Balancing Binary Search Trees



Idea! Rotations

- Maintain Binary Search Tree (BST) property, while moving stuff around.

Note: A, B, C, X, Y are variable names, not the contents of the nodes.



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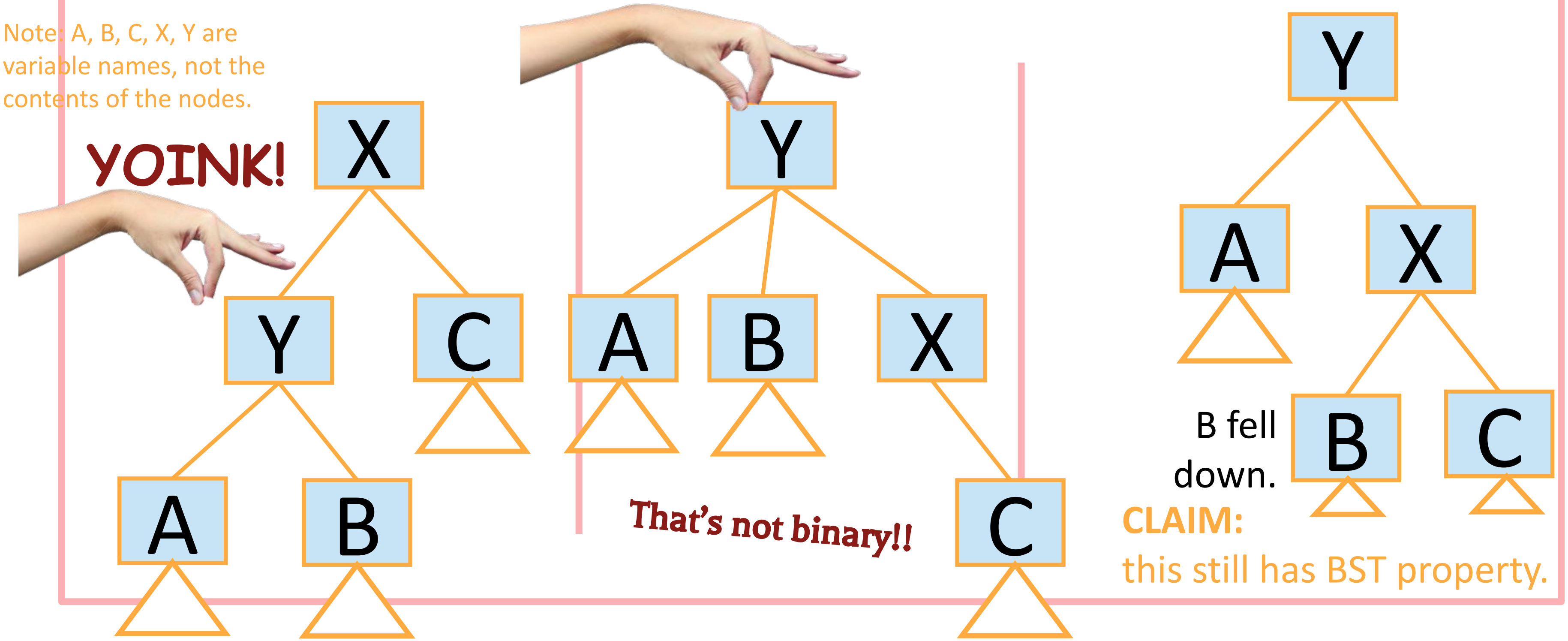
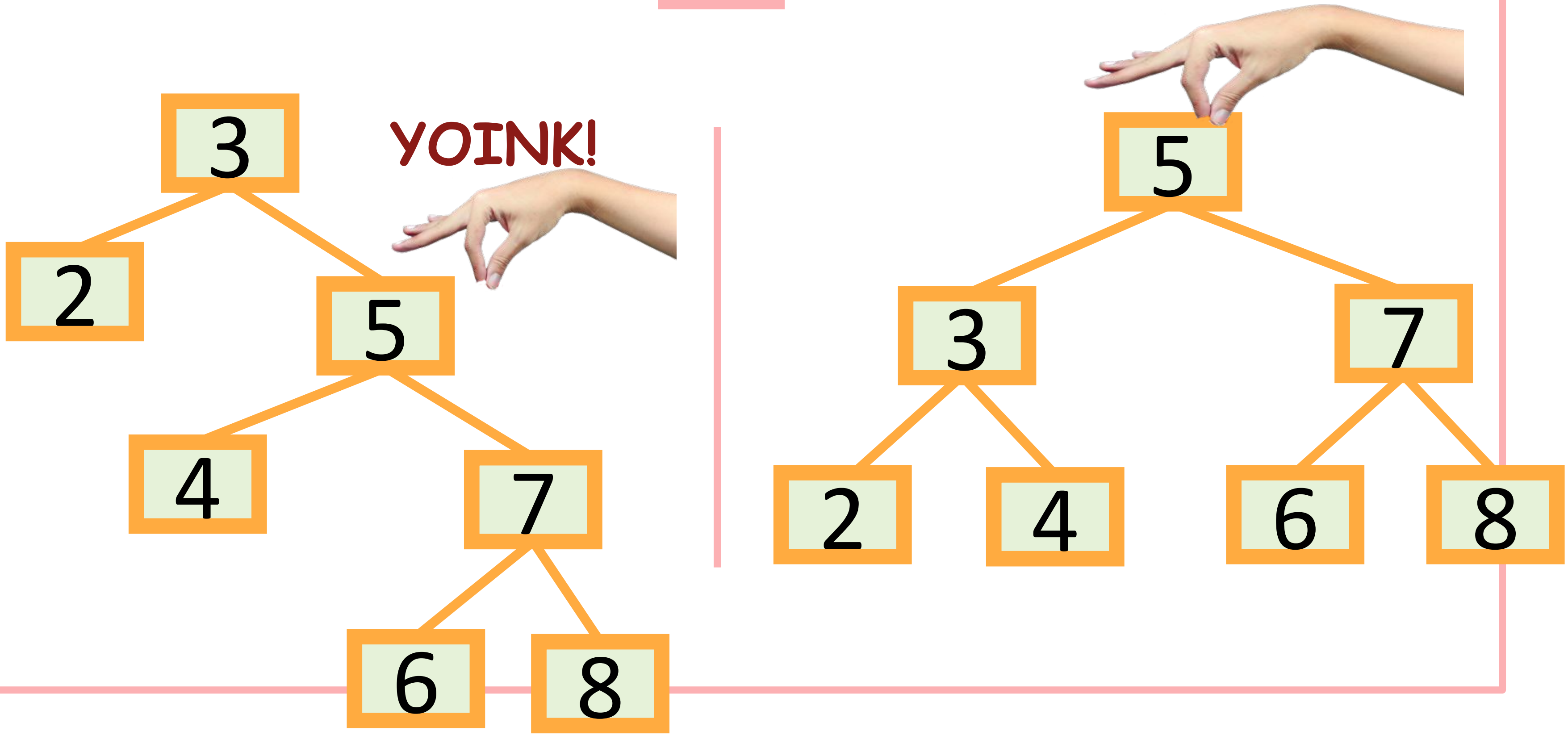


This seems helpful



Next idea! Have some proxy for balance

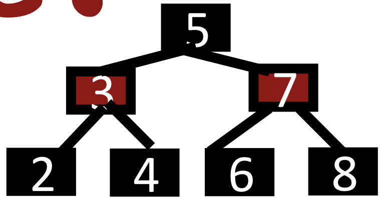
- Maintaining perfect balance is too hard.
- Instead, come up with some proxy for balance:
  - If the tree satisfies [SOME PROPERTY], then it's pretty balanced.
  - We can maintain [SOME PROPERTY] using rotations.

There are actually several ways to do this, but today we'll see…

Red-Black Trees

- A Binary Search Tree that balances itself!
- No more time-consuming by-hand balancing!

Red-Black tree!

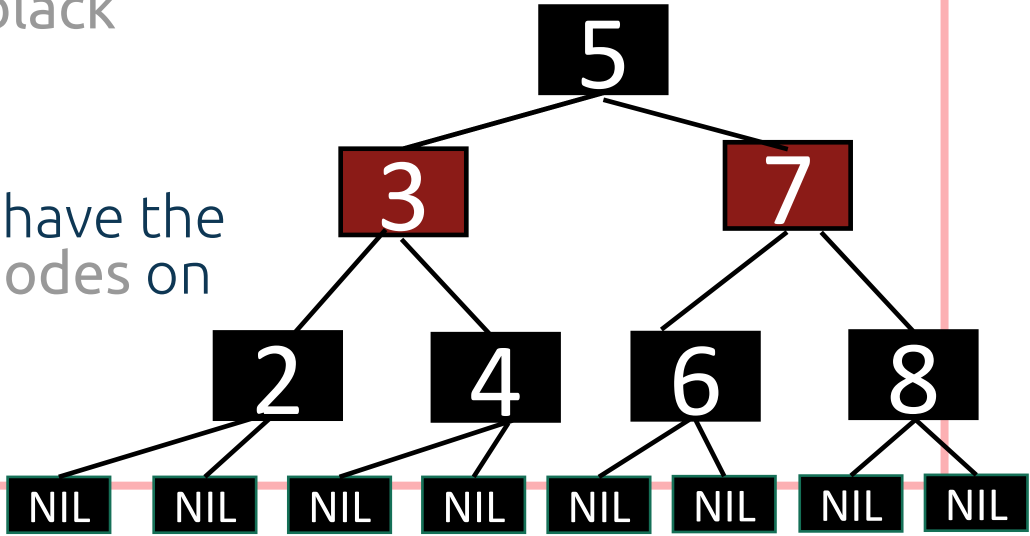


Maintain balance by stipulating that **black nodes** are balanced, and that there *aren't too many* **red nodes.**

Red-Black Trees obey the following rules (which are a proxy for balance)

- Every node is colored red or black.
- The root node is a black node.
- NIL children count as black nodes.
- Children of a red node are black nodes.
- For all nodes x:
  - all paths from x to NIL's have the same number of black nodes on them.

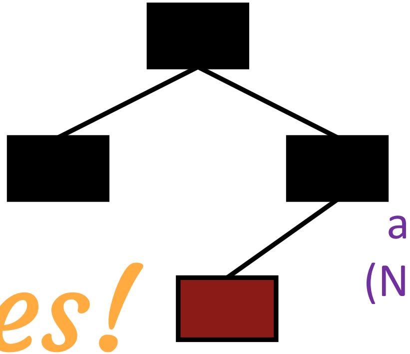
I'm not going to draw the NIL children in the future, but they are treated as black nodes.



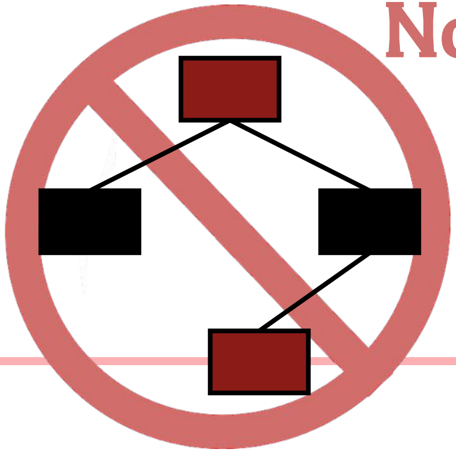
Examples(?)

- Every node is colored **red** or **black.**
- The root node is a **black node**.
- NIL children count as **black nodes**.
- Children of a **red node** are **black nodes**.
- For all nodes x:
  - all paths from x to NIL's have the same number of **black nodes** on them.

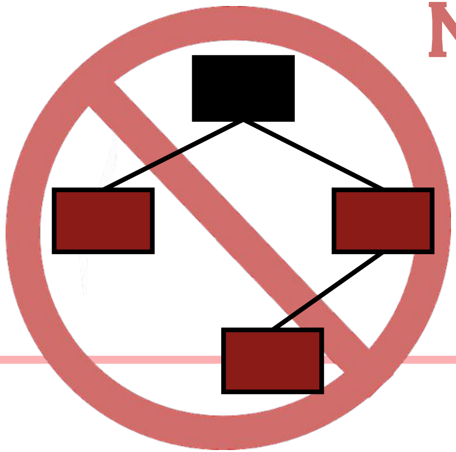
Which of these are red-black trees? (NIL nodes not drawn)



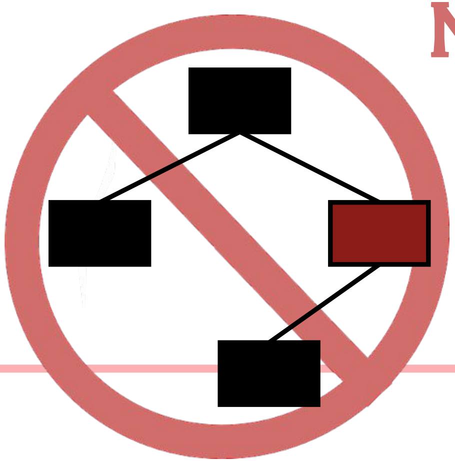
Yes!



No!



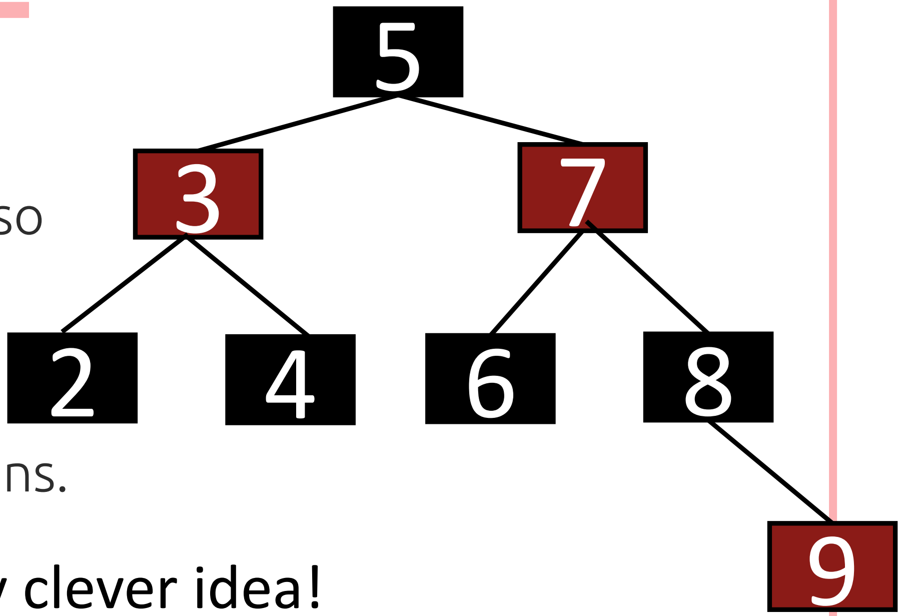
No!



No!

Why these rules???????

- This is pretty balanced.
  - The **black nodes** are balanced
  - The **red nodes** are "spread out" so they don't mess things up too much.
- We can maintain this property as we insert/delete nodes, by using rotations.



This is the really clever idea!

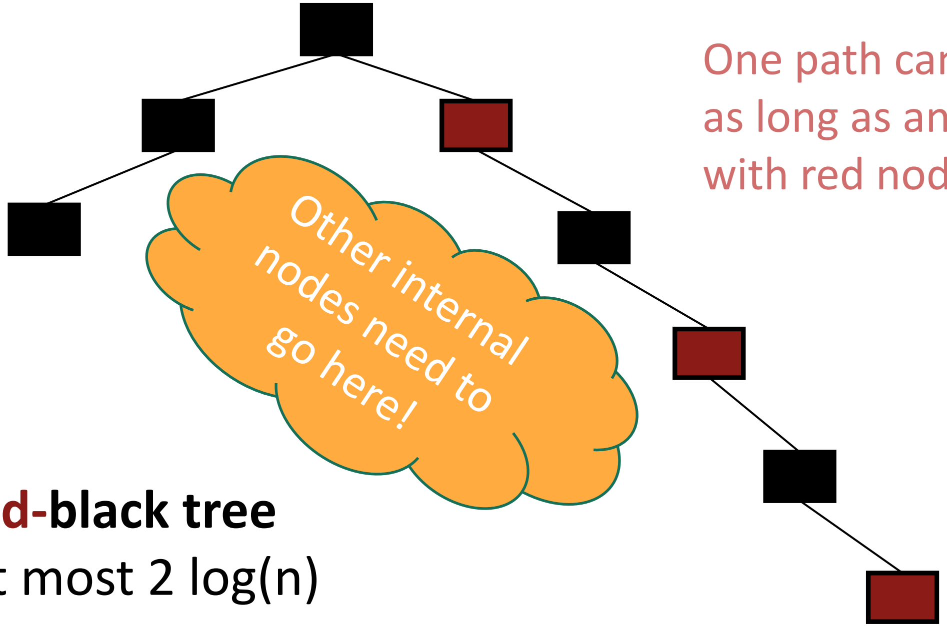
This **Red-Black** structure is a proxy for balance.

It's just a smidge weaker than perfect balance, but we can actually maintain it!

This is “pretty balanced”

- To see why, intuitively, let’s try to build a Red-Black Tree that’s unbalanced.

Note, this is just a conjecture to build intuition! Rigorous proof on the next, skipped slide.

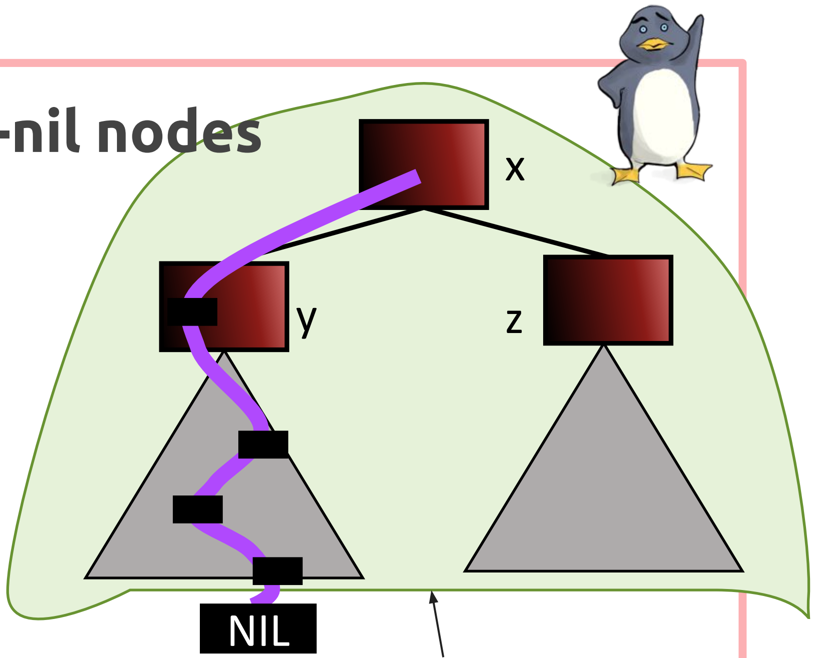


One path can be at most twice as long as another if we pad it with red nodes.

Conjecture:
the height of a **red-black tree** with n nodes is at most 2 log(n)

RBTree with N non-nil nodes

- Define b(x) to be the number of black nodes in any path from x to NIL.
  - (excluding x, including NIL).
- Claim:
  - There are at least $2^{b(x)} - 1$ non-NIL nodes in the subtree underneath x. (Including x).
- [You can proof by induction]



Claim: at least $2^{b(x)} - 1$ nodes in this WHOLE subtree (of any color).

Then:

$$n \geq 2^{b(root)} - 1$$ using the Claim

$$\geq 2^{height/2} - 1$$ b(root) >= height/2 because of RBTree rules.

Rearranging:

$$n + 1 \geq 2^{height/2} \Rightarrow height \leq 2\log(n+1)$$

This is great!

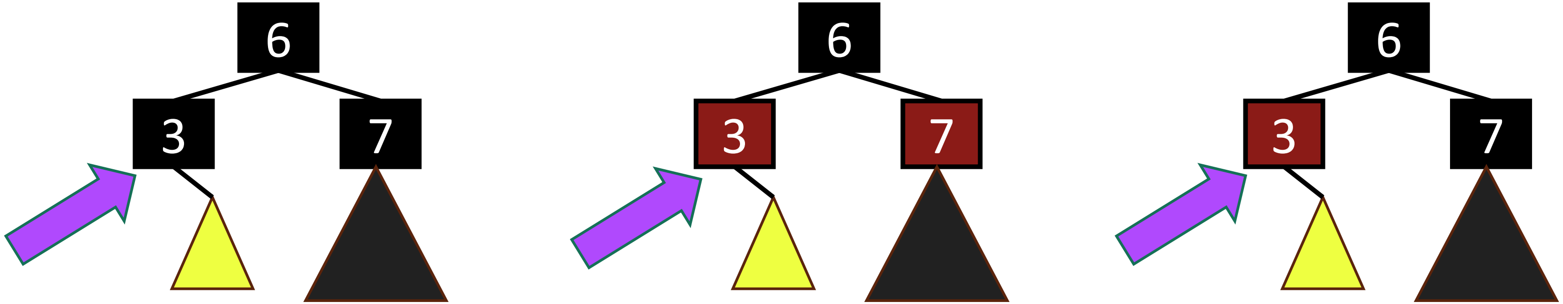
- SEARCH in an RBTree is immediately O(log(n)), since the depth of an RBTree is O(log(n)).

- What about INSERT/DELETE?
  - Turns out, you can INSERT and DELETE items from an RBTree in time O(log(n)), while *maintaining* the RBTree property.
  - That's why this is a good property!

INSERT/DELETE

- I expect we are out of time…
  - There are some slides which you can check out to see how to do INSERT/DELETE in RBTrees if you are curious.
  - See CLRS Ch 13. for even more details.
- You are **not responsible** for the details of INSERT/DELETE for RBTrees for this class.
  - You should know what the “proxy for balance” property is and why it ensures approximate balance.
  - You should know **that** this property can be efficiently maintained, but you do not need to know the details of how.

INSERT: Many cases

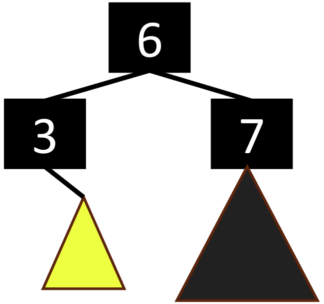


- Suppose we want to insert 0 **here**.

- There are 3 “important” cases for different colorings of the existing tree, and there are 9 more cases for all of the various symmetries of these 3 cases.

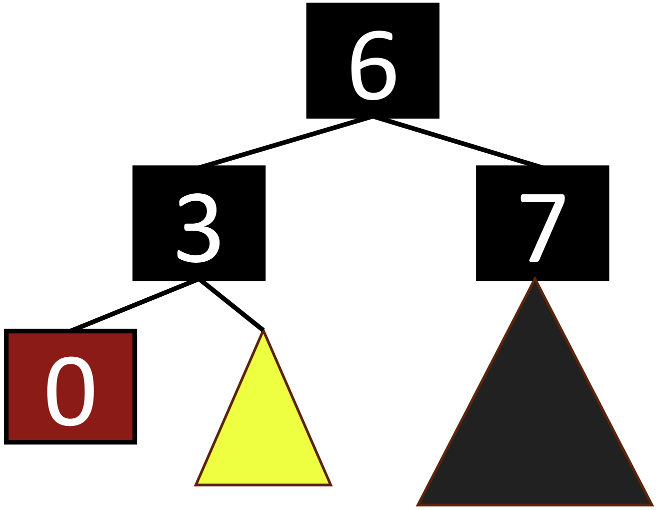
INSERT: Case 1

- Make a new **red node.**
- Insert it as you would normally.

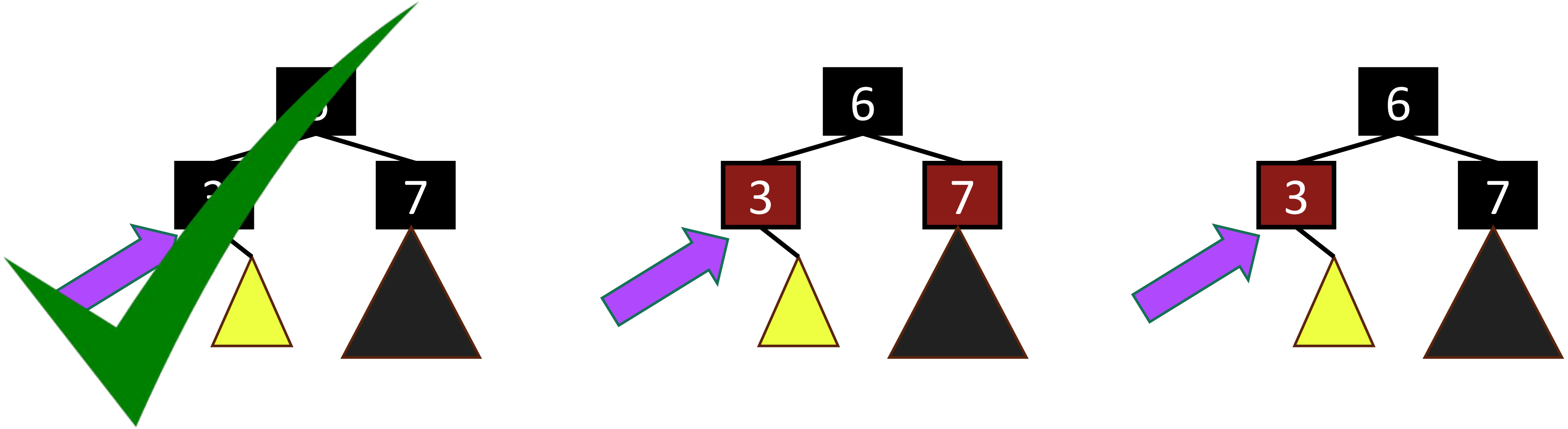


What if it looks like this?

Example: insert 0



INSERT: Many cases

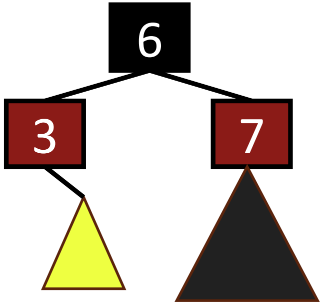


- Suppose we want to insert 0 **here**.

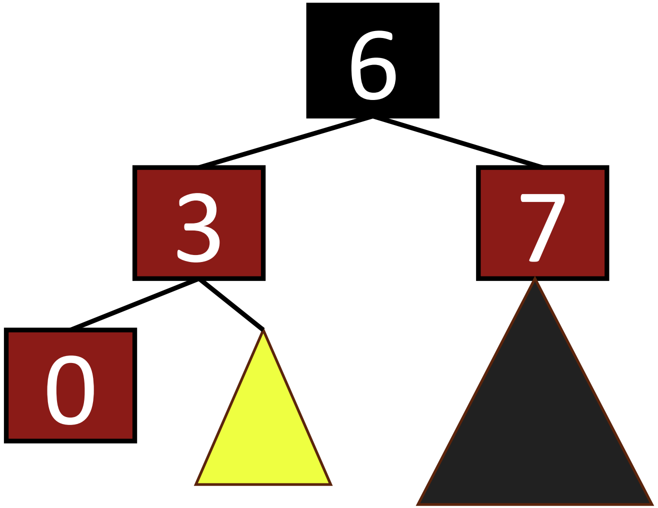
- There are 3 “important” cases for different colorings of the existing tree, and there are 9 more cases for all of the various symmetries of these 3 cases.

INSERT: Case 2

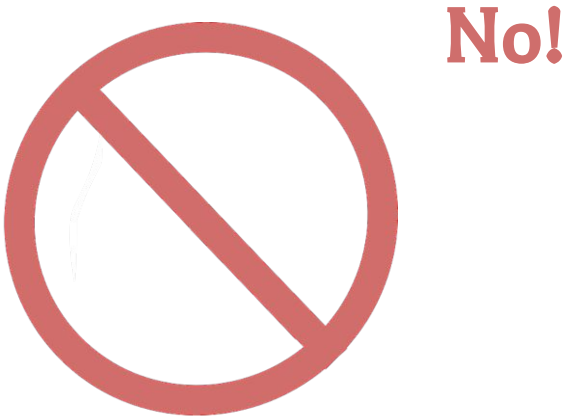
- Make a new **red node.**
- Insert it as you would normally.
- Fix things up if needed.



What if it looks like this?

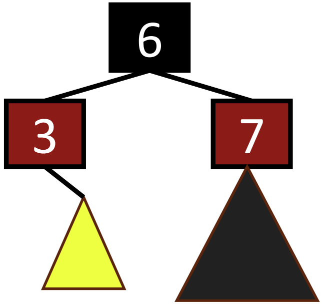


Example: insert 0

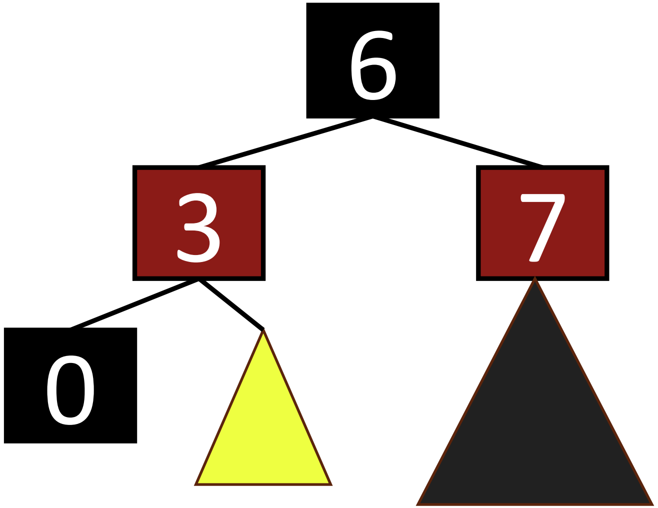


INSERT: Case 2

- Make a new **red node.**
- Insert it as you would normally.
- Fix things up if needed.



What if it looks like this?

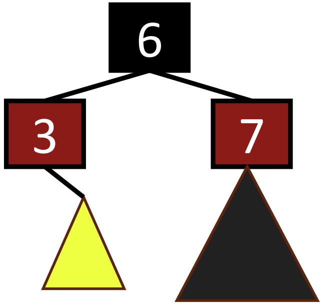


Example: insert 0

Can't we just insert 0 as a **black node?**

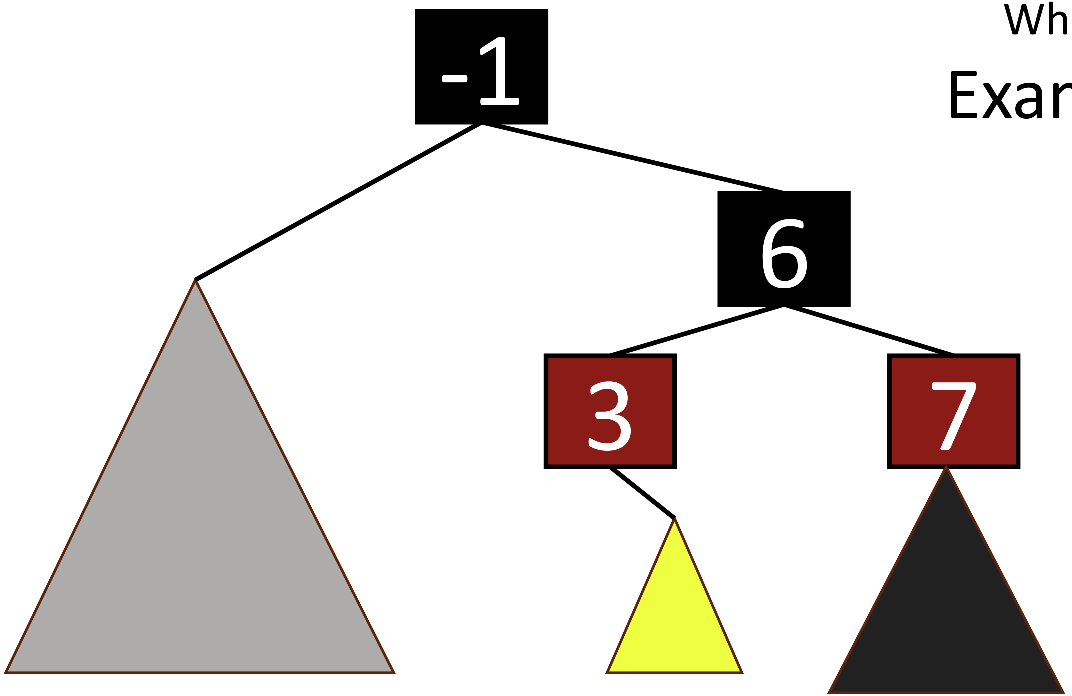
No!

We need a bit more context



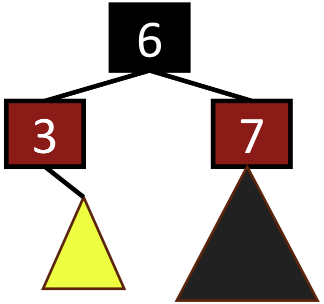
What if it looks like this?

Example: insert 0



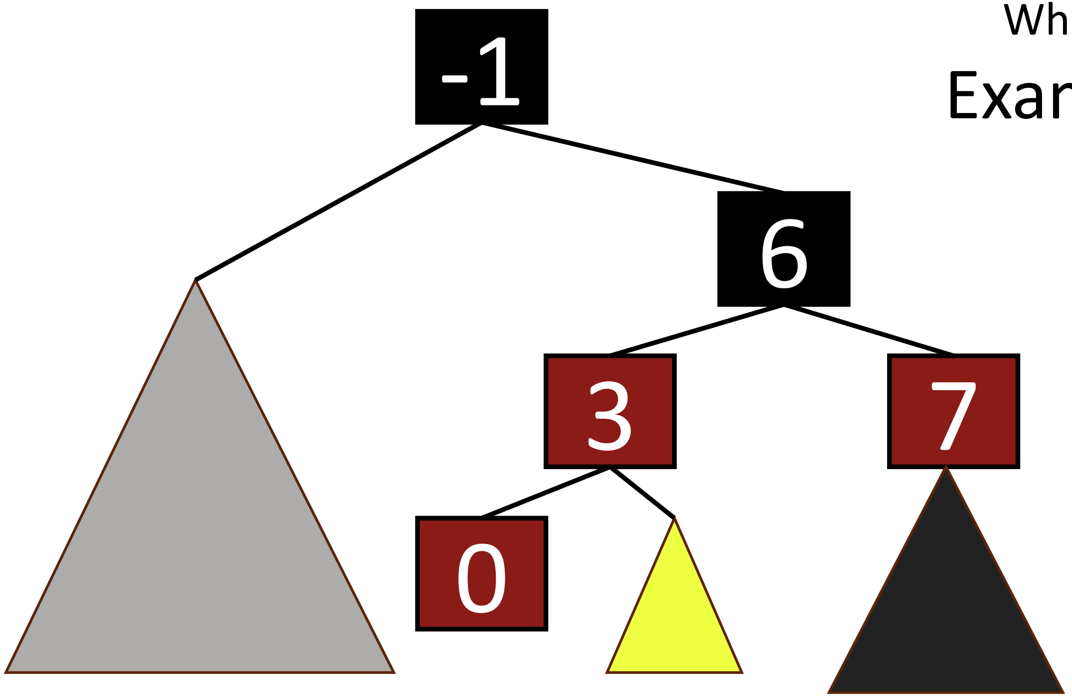
We need a bit more context

- Add 0 as a red node.



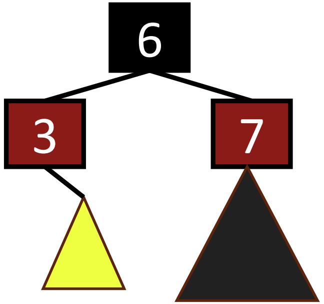
What if it looks like this?

Example: insert 0



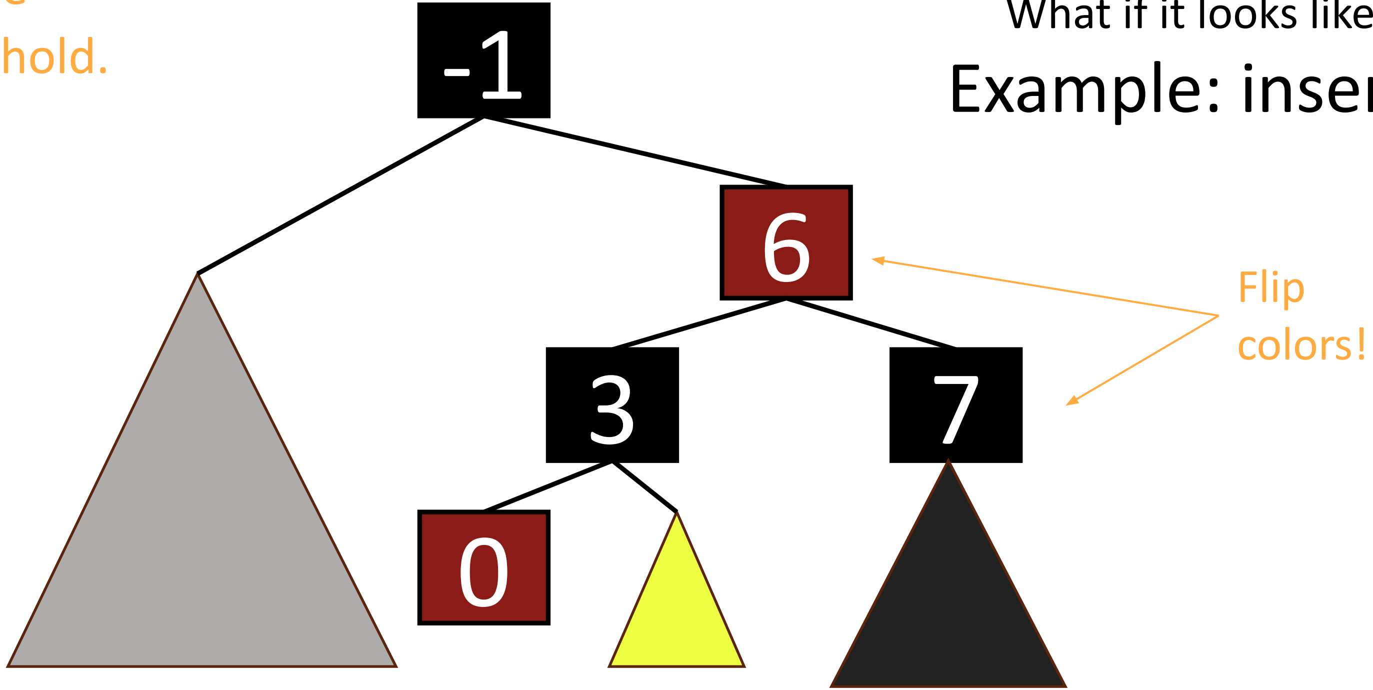
We need a bit more context

- Add 0 as a red node.
- **Claim:** RB-Tree properties still hold.

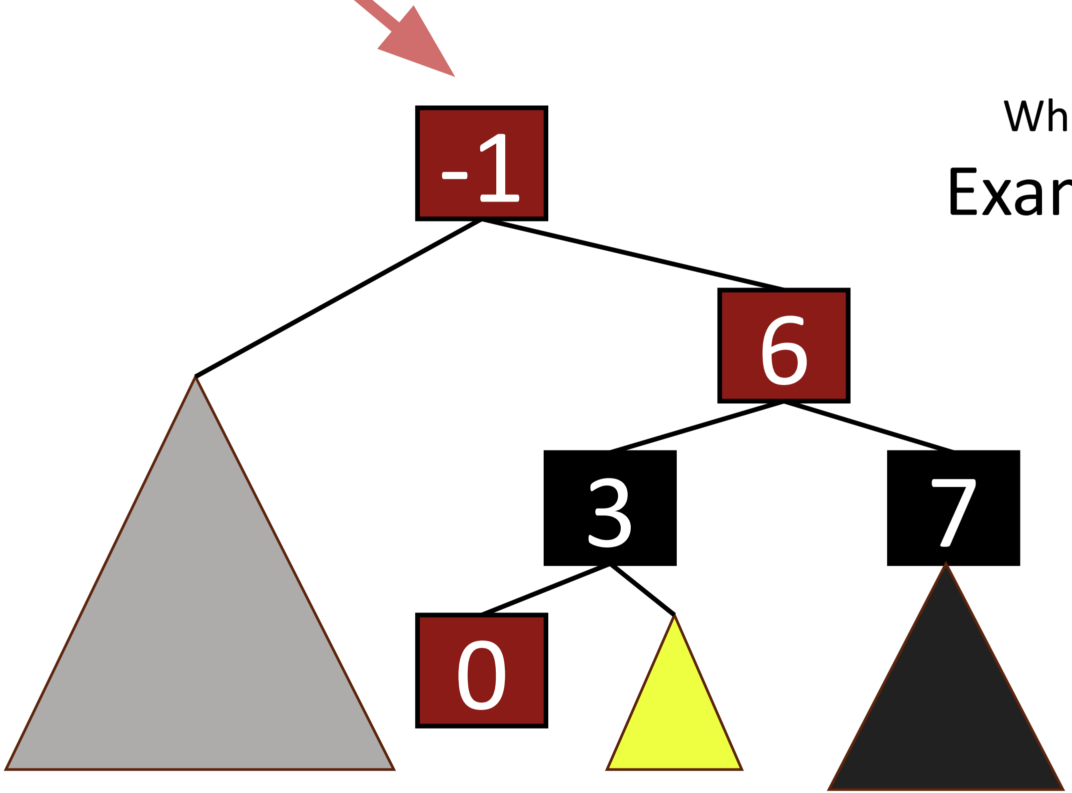


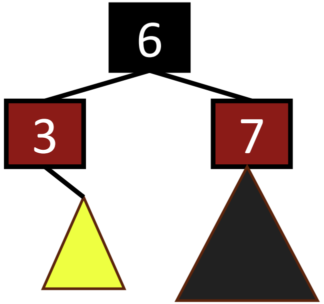
What if it looks like this?

Example: insert 0



But what if that was red?

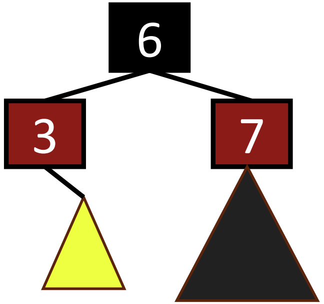




What if it looks like this?

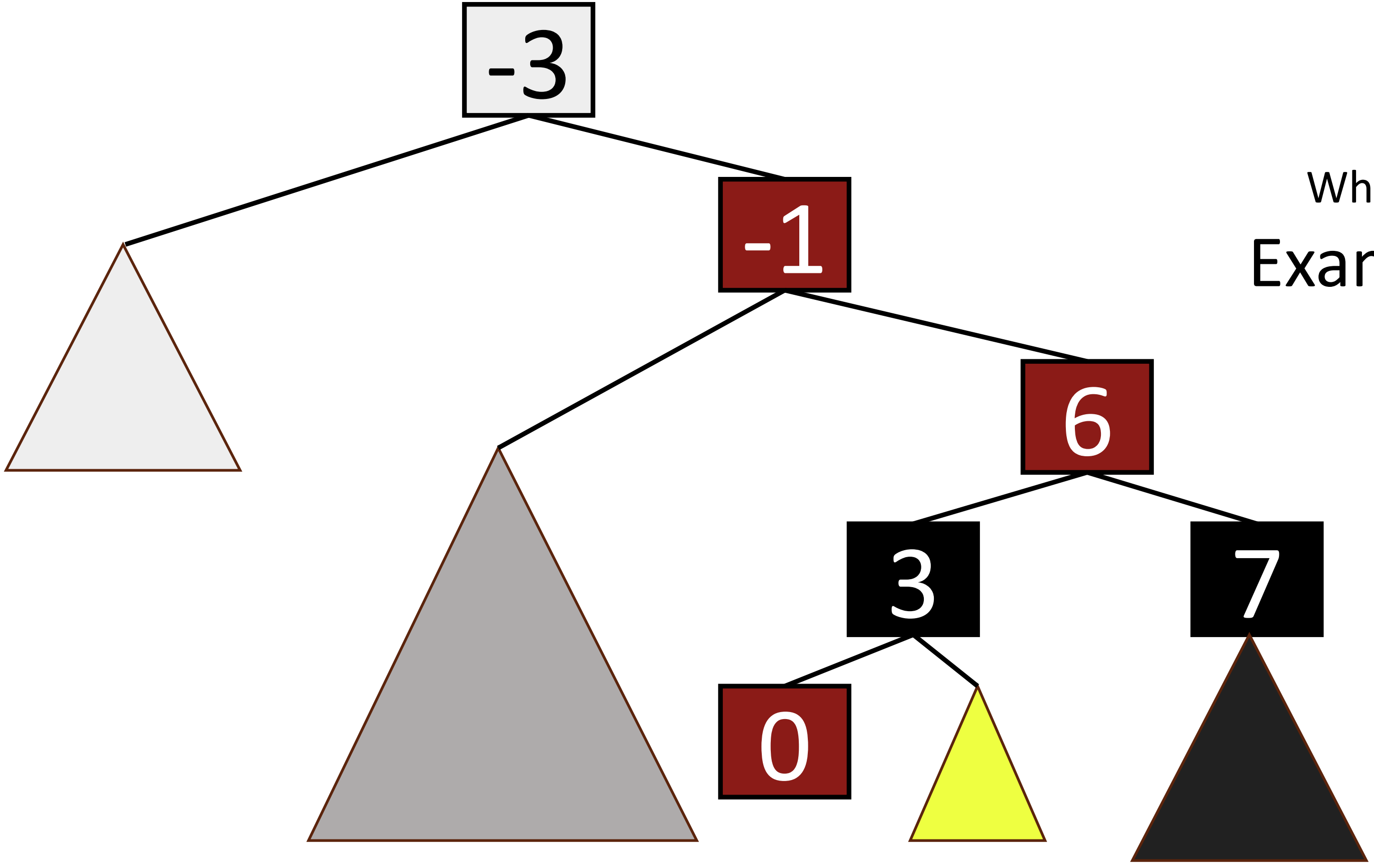
Example: insert 0

More context...

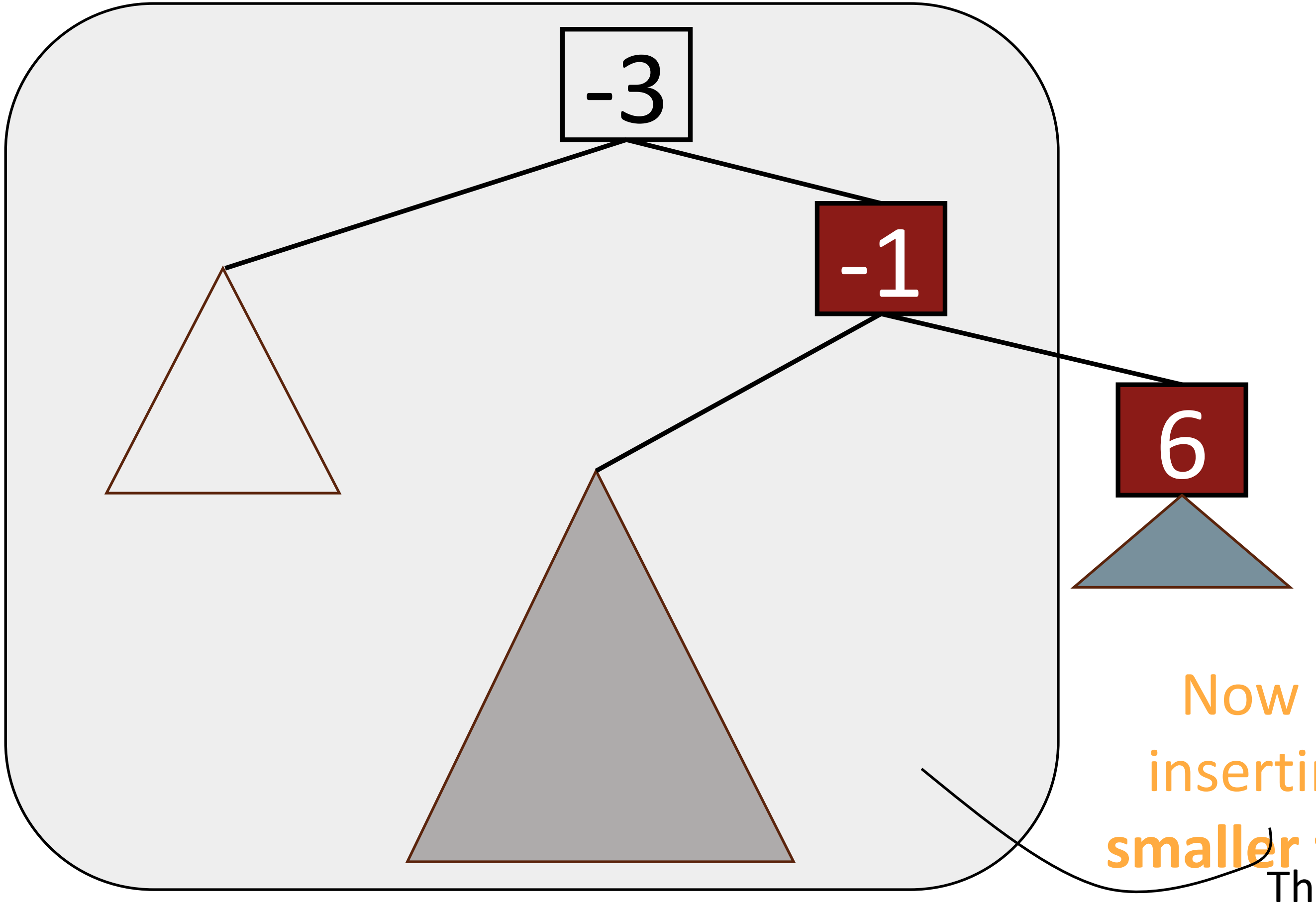


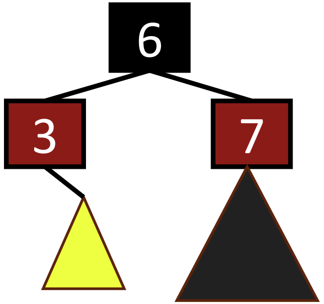
What if it looks like this?

Example: insert 0



More context...





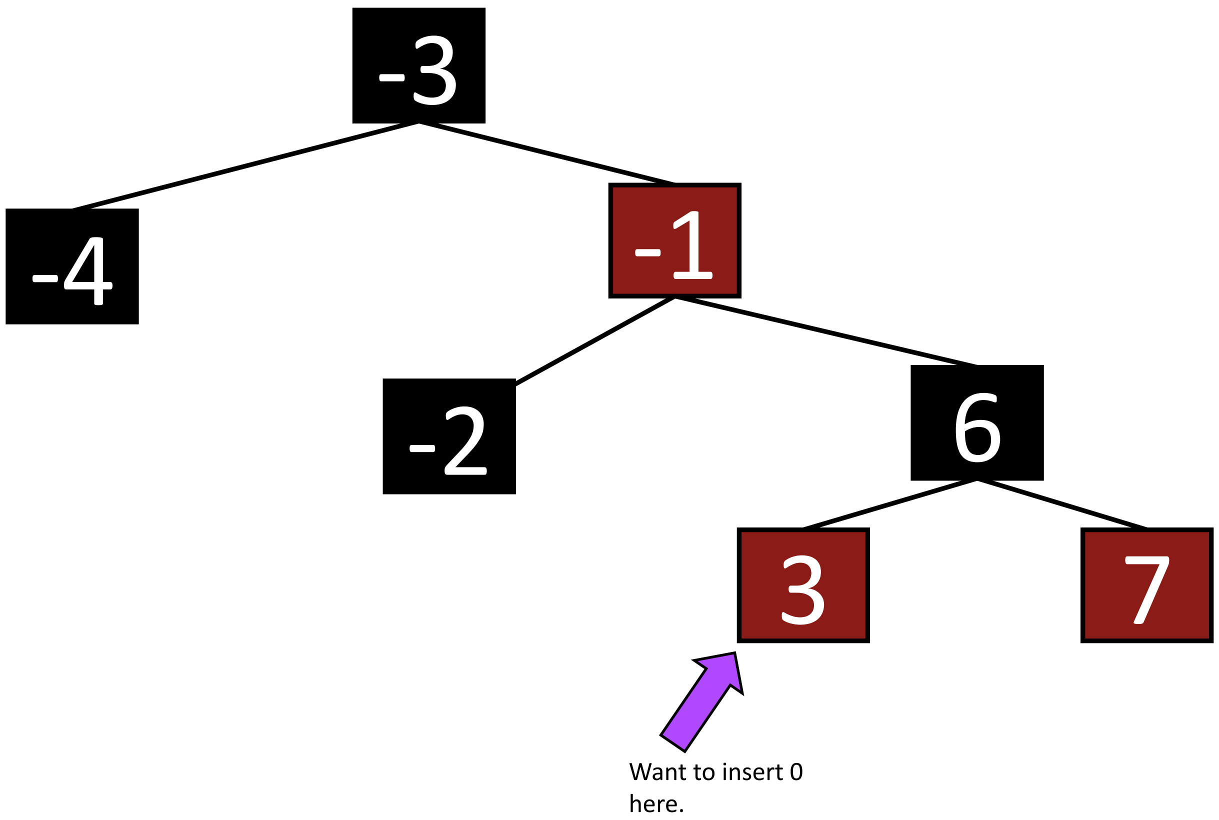
What if it looks like this?

Example: insert 0

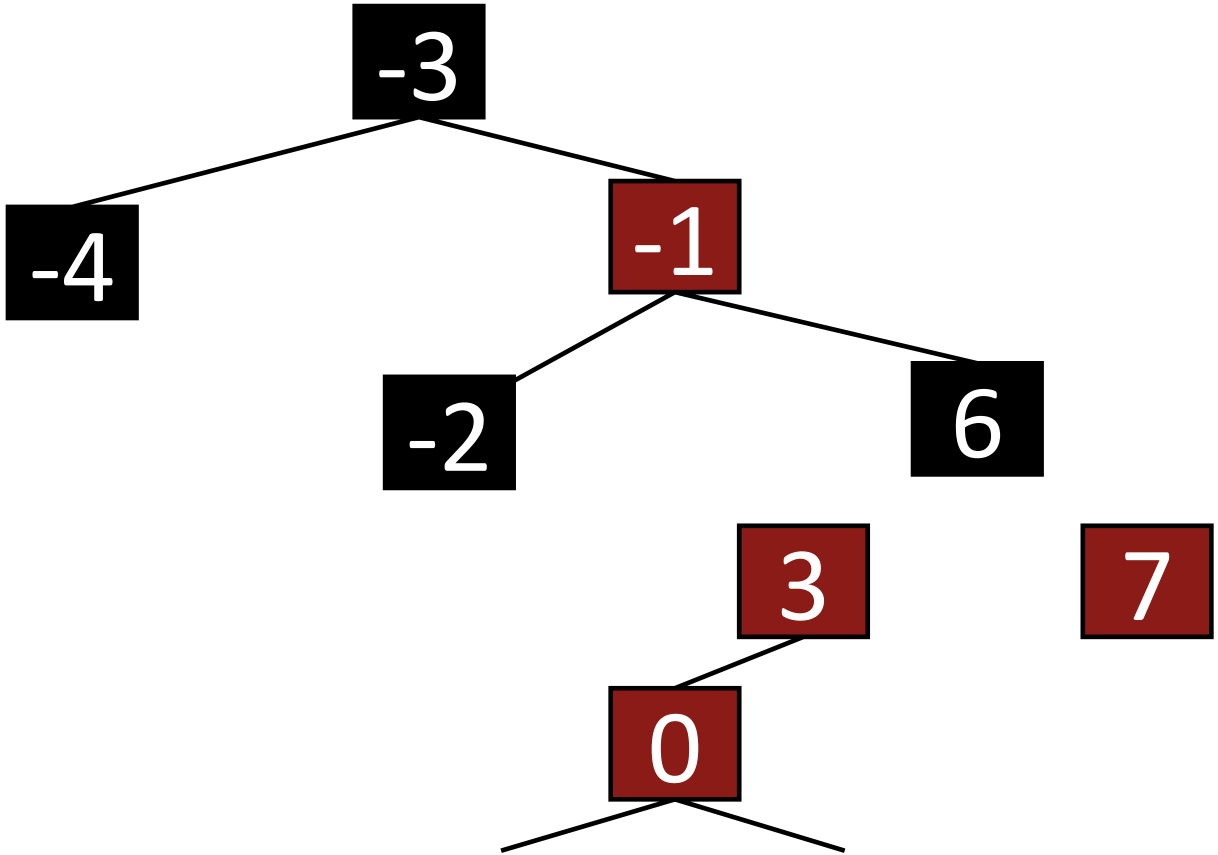
Now we're basically inserting 6 into some **smaller tree**. Recurse!

This one!

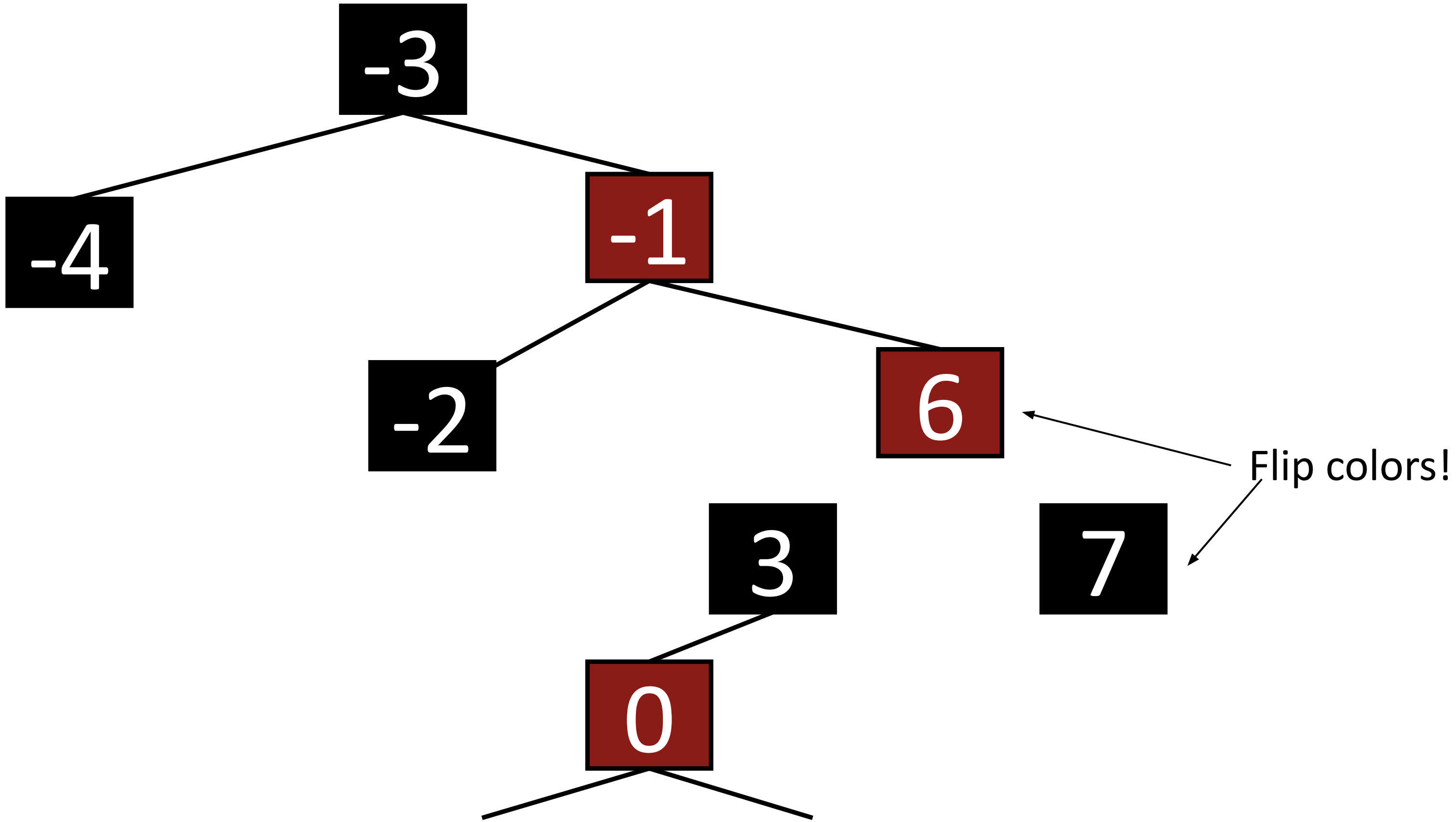
Example, part I



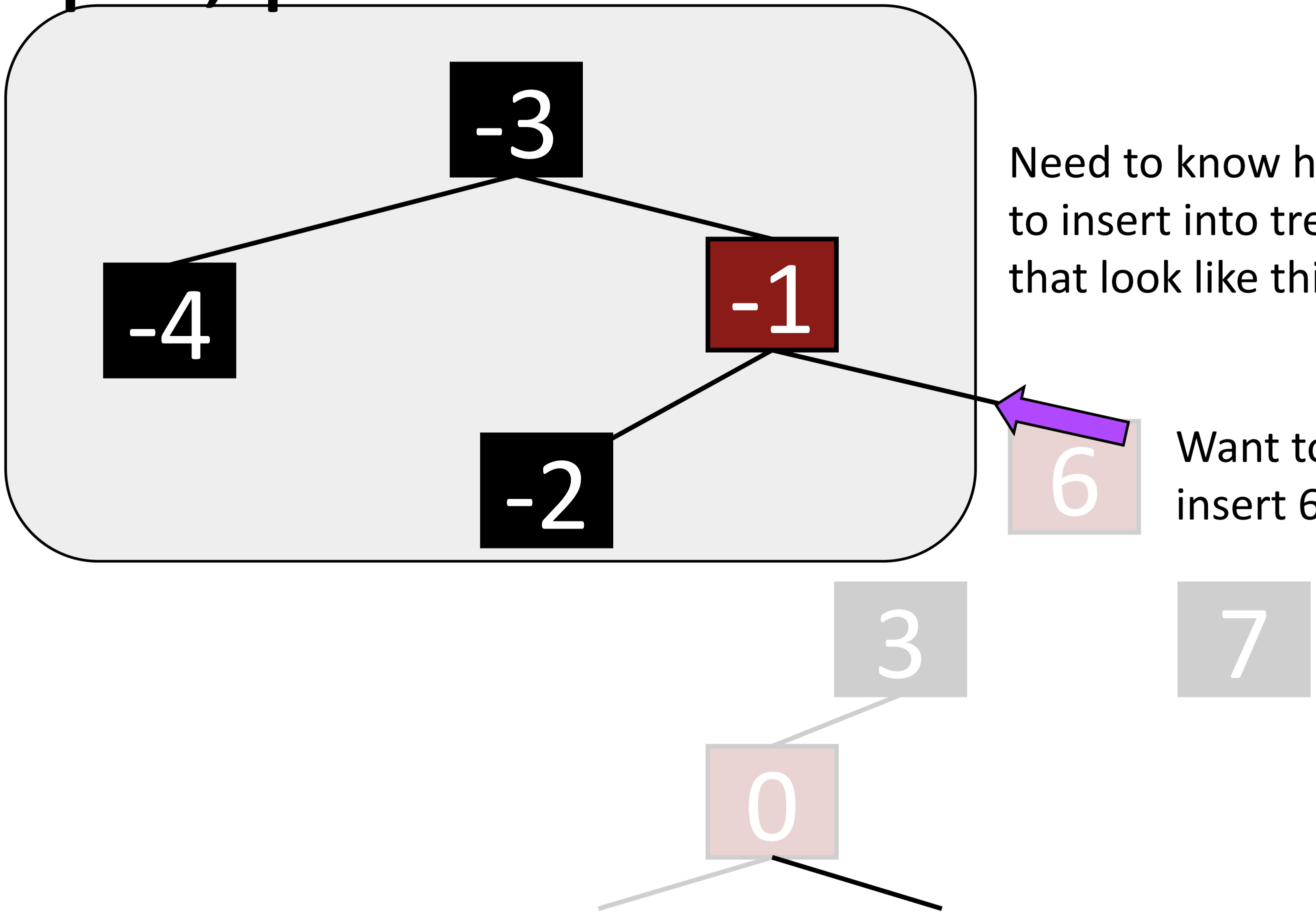
Example, part I



Example, part I



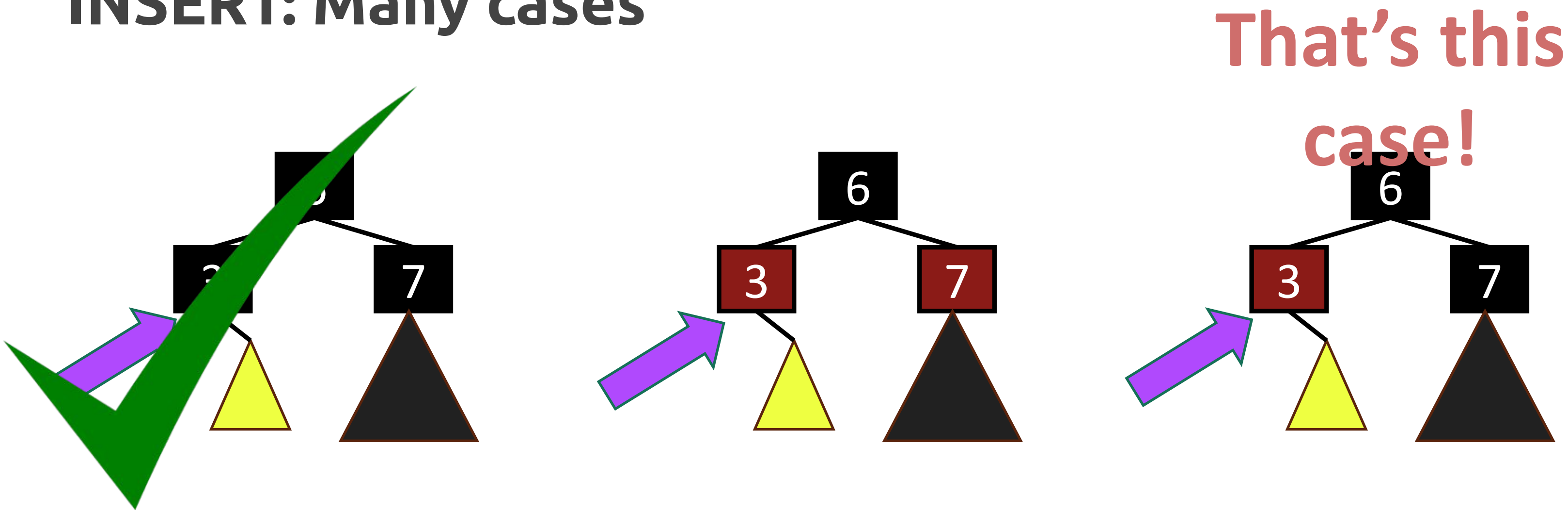
Example, part I



Need to know how to insert into trees that look like this…

Want to insert 6 here.

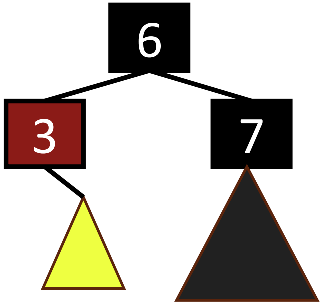
INSERT: Many cases



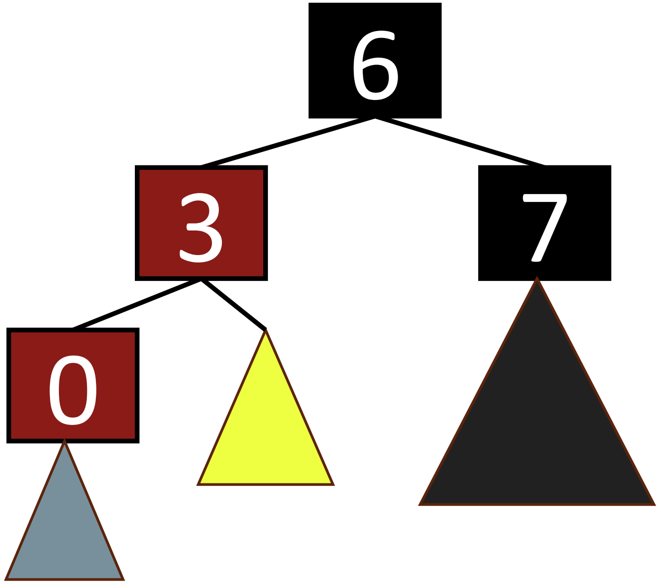
- Suppose we want to insert 0 **here**.
- There are 3 "important" cases for different colorings of the existing tree, and there are 9 more cases for all of the various symmetries of these 3 cases.

INSERT: Case 3

- Make a new **red node.**
- Insert it as you would normally.
- Fix things up if needed.



What if it looks like this?

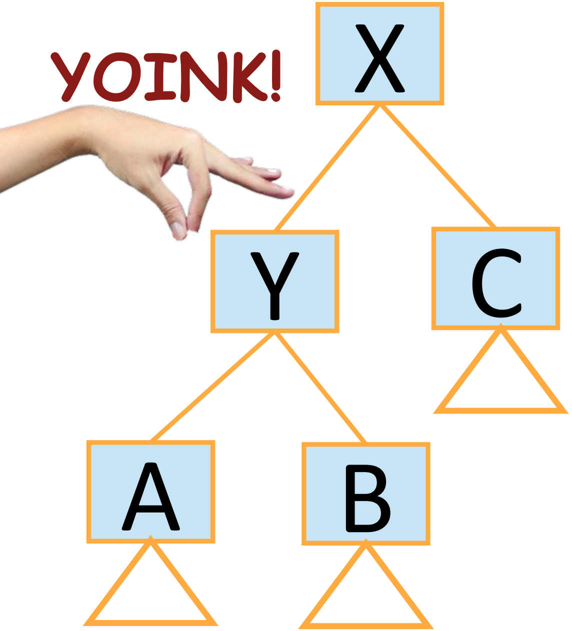


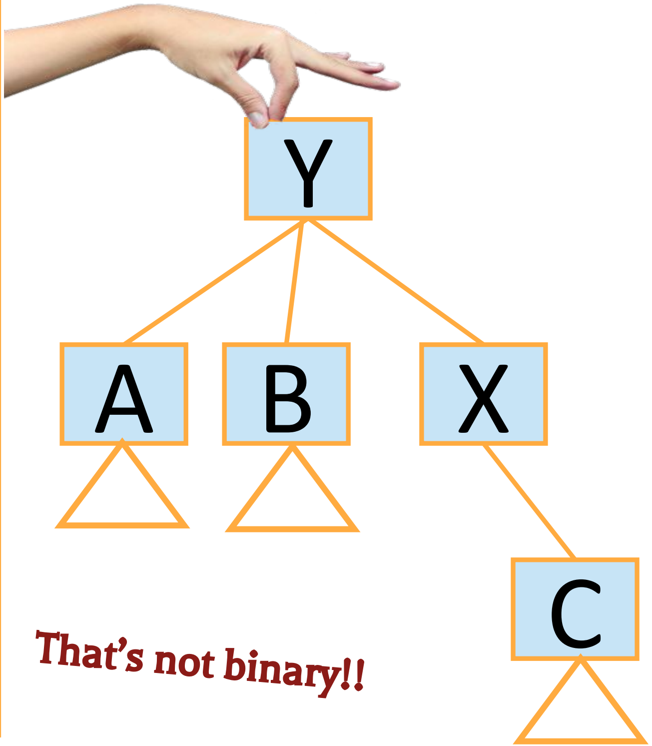
Example: Insert 0.

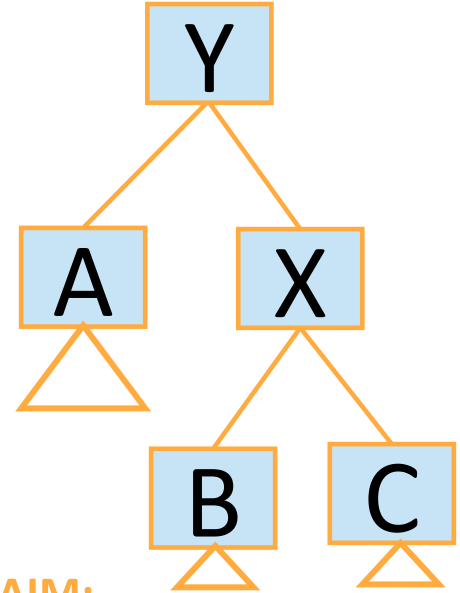
- Maybe with a subtree below it.

Recall Rotations

- Maintain Binary Search Tree (BST) property, while moving stuff around.



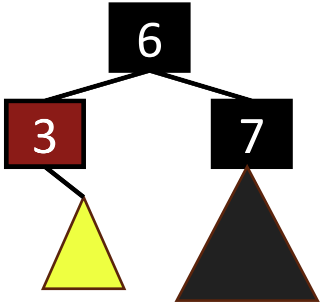




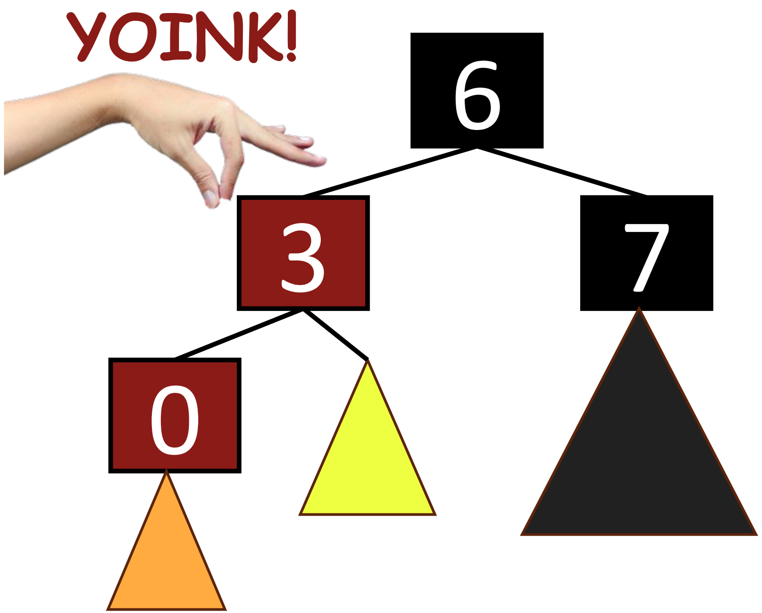
CLAIM:
this still has BST property.

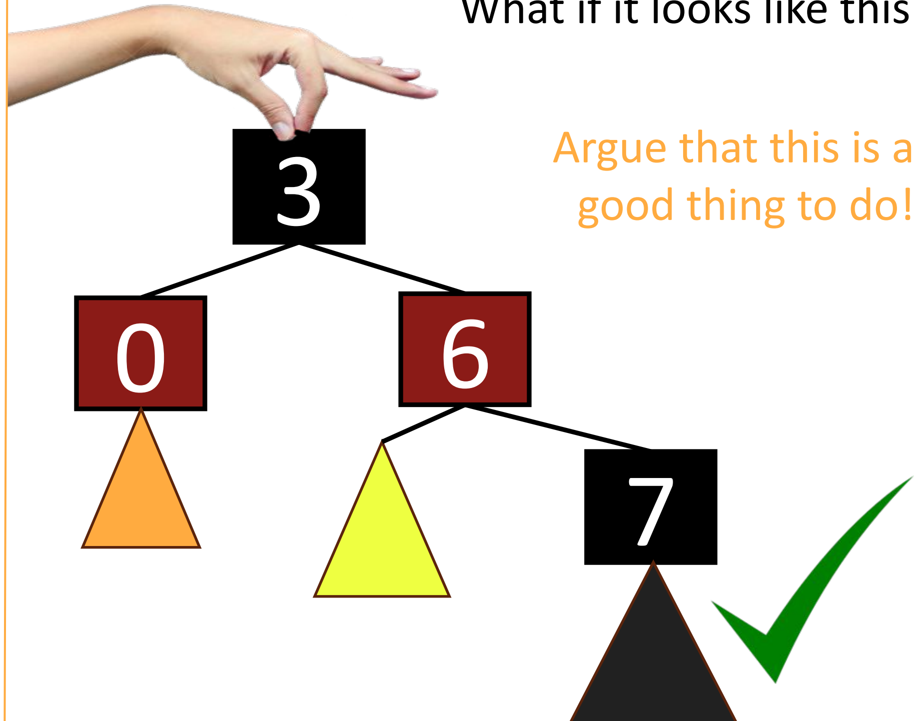
Inserting into a Red-Black Tree

- Make a new **red node.**
- Insert it as you would normally.
- Fix things up if needed.



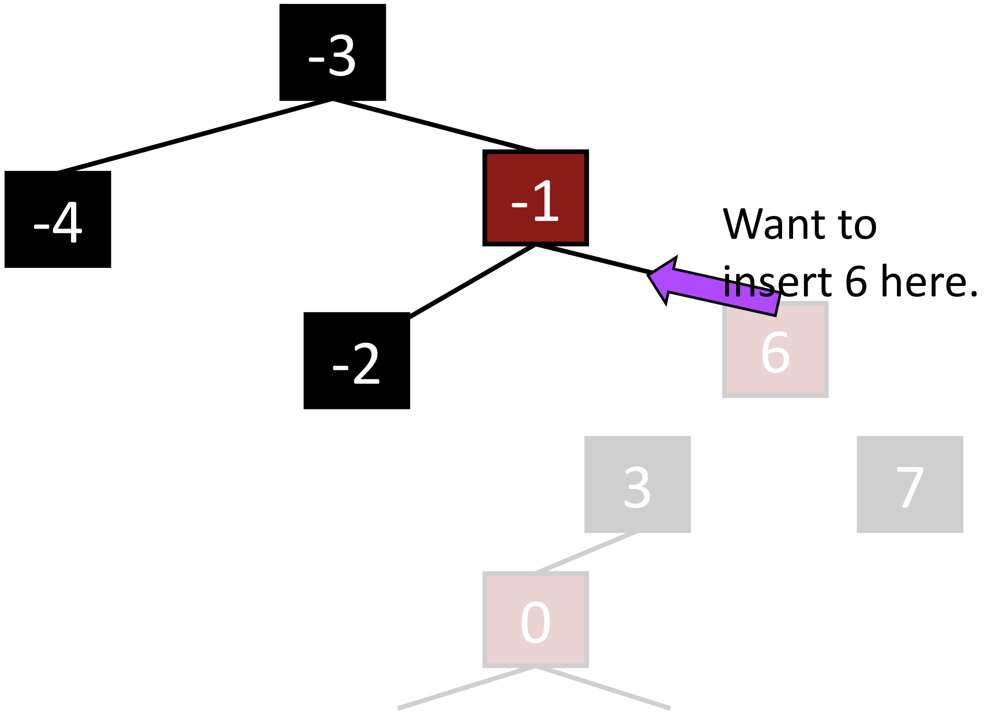
What if it looks like this?



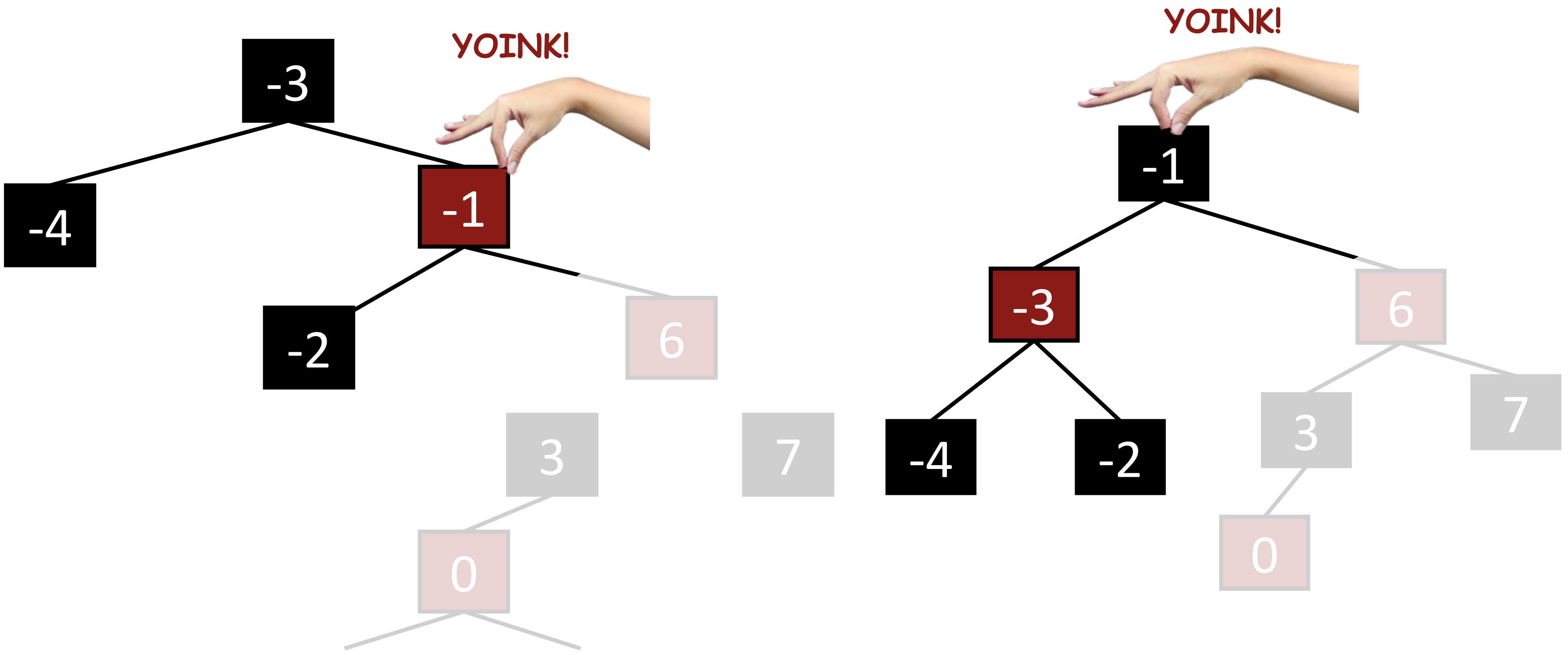


Argue that this is a good thing to do!

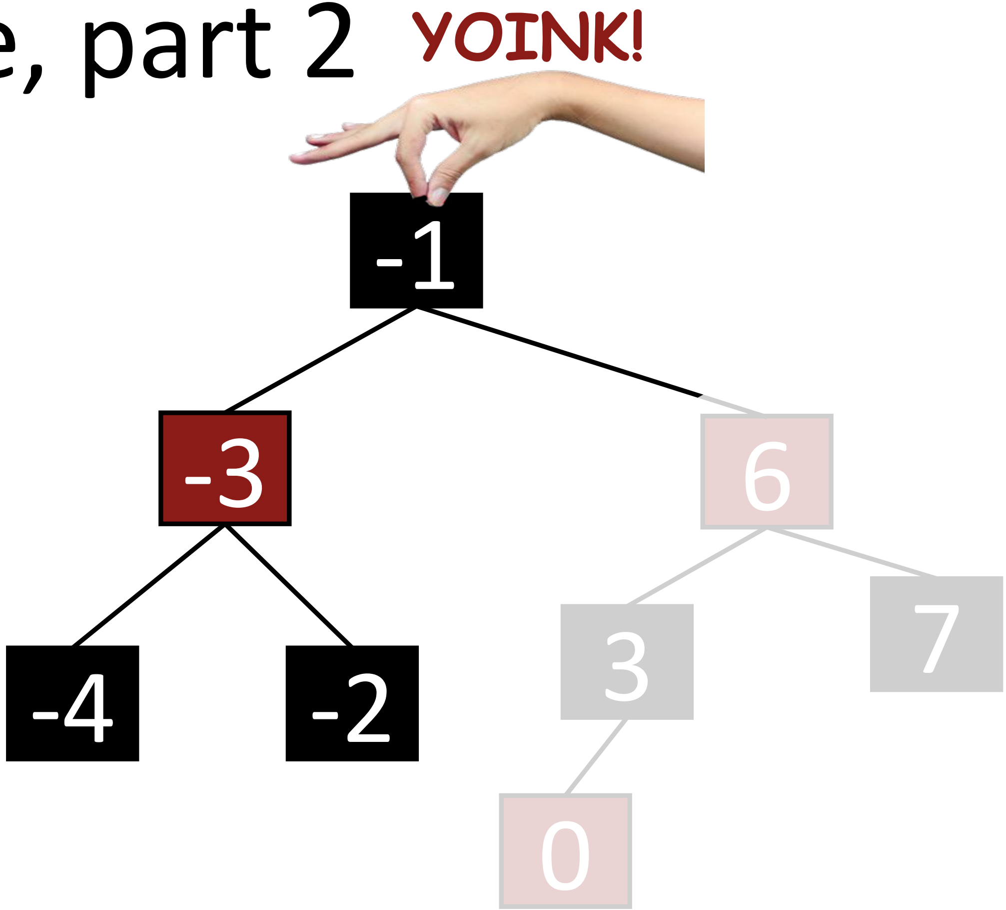
Example, part 2



Example, part 2

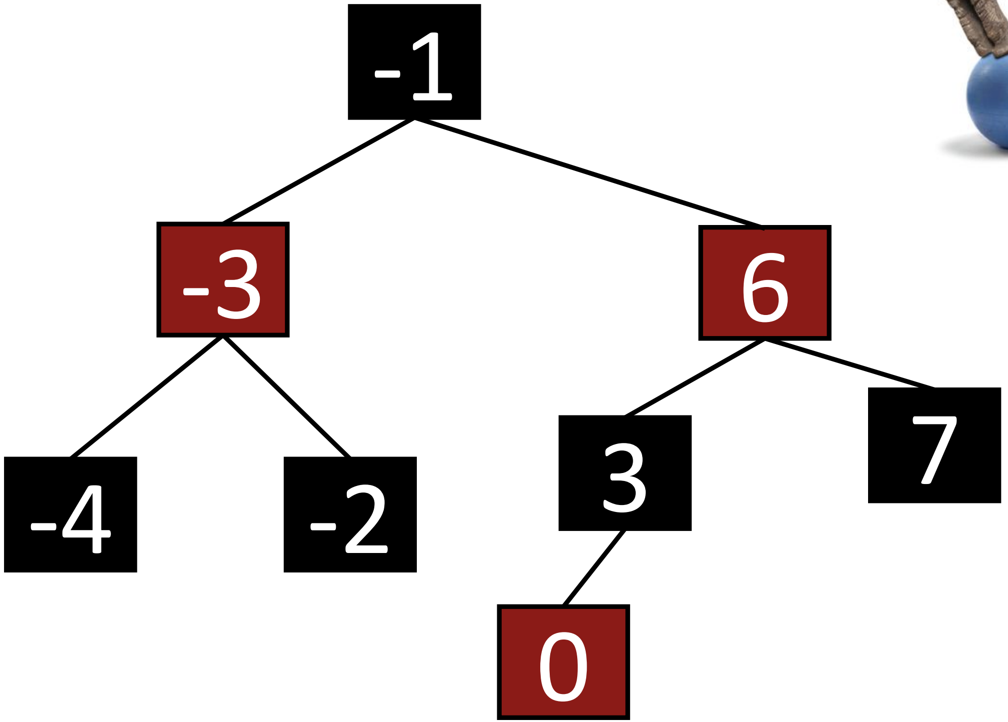


Example, part 2

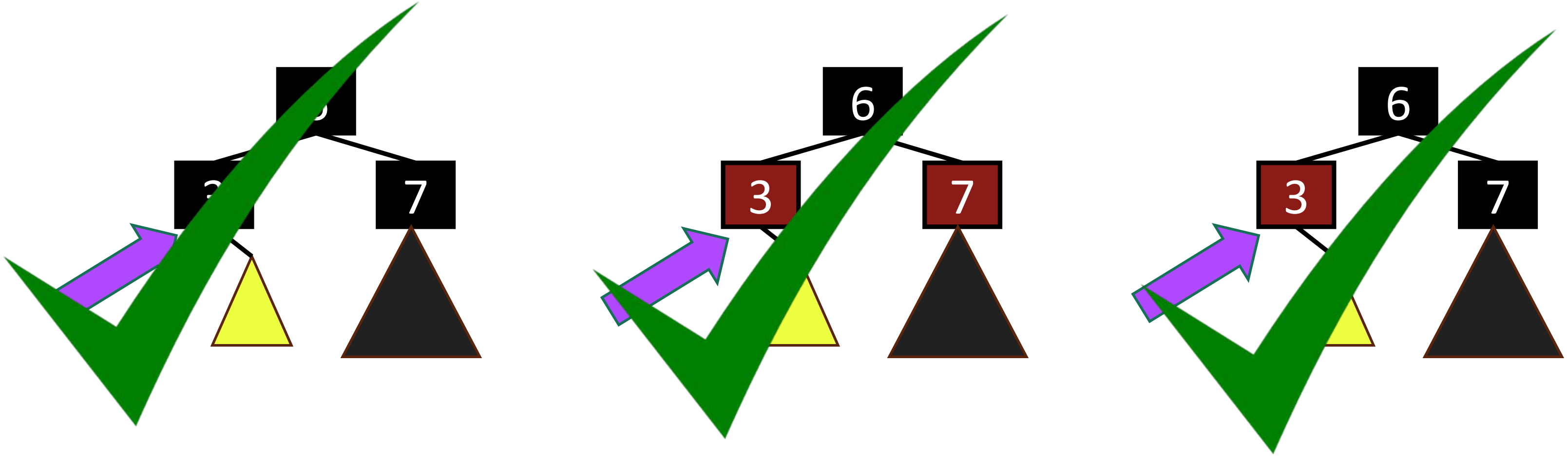


Example, part 2

TA-DA!



Many cases

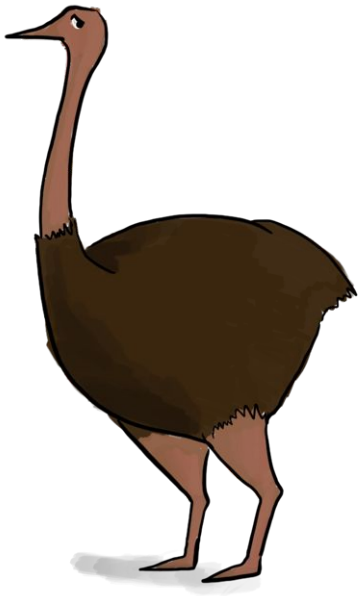


- Suppose we want to insert 0 **here**.

- There are 3 “important” cases for different colorings of the existing tree, and there are 9 more cases for all of the various symmetries of these 3 cases.

Deleting from a Red-Black tree

Fun exercise!



That's a lot of cases!

- You are **not responsible** for the nitty-gritty details of Red-Black Trees. (For this class)
  - Though implementing them is a great exercise!
- You should know:
  - What are the properties of an RB tree?
  - And (more important) why does that guarantee that they are balanced?

What have we learned?

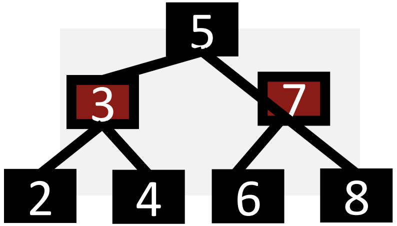
- Red-Black Trees always have height at most 2log(n+1).
- As with general Binary Search Trees, all operations are O(height)
- So all operations with RBTrees are O(log(n)).

Conclusion: The best of both worlds

	Sorted Arrays	Linked Lists	Binary Search Trees*
Search	O(log(n)) 😃	O(n) ☹️	O(log(n)) 😃
Delete	O(n) ☹️	O(n) ☹️	O(log(n)) 😃
Insert	O(n) ☹️	O(1) 😃	O(log(n)) 😃

Recap

- Balanced binary trees are the best of both worlds!
- But we need to keep them balanced.
- **Red-Black Trees** do that for us.
  - We get O(log(n))-time INSERT/DELETE/SEARCH
  - Clever idea: have a proxy for balance



How was the pace today?

COMP 285
Analysis of Algorithms

Welcome to COMP 285

Lecture 11: BSTs + Self-Balancing Trees

Lecturer: Chris Lucas (cflucas@ncat.edu)