MATH 330 – HW #19 Revision

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Proposition 9.15: If $k \in \mathbb{N}$, then $e(k) \in \mathbb{R}_{\geq 0}$.

Proof:
Let the statement $P(k)$ be the statement "If $k \in \mathbb{N}$, then $e(k) \in \mathbb{R}_{\geq 0}$".
<u>Base Case:</u> Let $k = 1_{\mathbb{N}}$. By Axiom 2.1, $1_{\mathbb{Z}}$ as well. By Proposition 9.14, $e(1_{\mathbb{Z}})$ $= 1_{\mathbb{R}}$.
<u>Inductive Case:</u> Assume $P(k)$ holds. We want to prove $P(n+1)$, where "If $n+1 \in \mathbb{N}$, then $e(n+1) \in \mathbb{R}$". By Axiom 2.1, $n + 1_{\mathbb{Z}}$ as well. So, we have $e(n + 1_{\mathbb{Z}}) = e(n) + 1_{\mathbb{Z}}$ by the relationship between $\mathbb{Z}$ and $\mathbb{R}$. $e(n)$ holds by induction hypothesis. This completes our proof by induction. $\square$

Proposition 9.18: The function e preserves multiplication: for all $m, k \in \mathbb{Z}$,

$$e(m \cdot k) = e(m) \cdot e(k),$$

where $\cdot$ on the left-hand side refers to multiplication in $\mathbb{Z}$, whereas $\cdot$ on the right-hand side refers to multiplication in $\mathbb{R}$.
Proof:
Let $m \in \mathbb{Z}$. We first show that for all $n \in (Z_{\geq 0})$ that $e(m \cdot n) = e(m) \cdot e(n)$ by induction on n. First, when $n = 0$,

$$e(m \cdot n) = e(m \cdot 0) = e(0) = 0 = e(m) \cdot 0 = e(m) \cdot e(0) = e(m) \cdot e(n).$$

Now suppose for some $n \geq 0$ that $e(m \cdot n) = e(m) \cdot e(n)$. By induction hypothesis,

$$\begin{aligned} e(m \cdot (n+1)) &= e(m \cdot n + m) \\ &= e(m \cdot n) + e(m) \\ &= e(m) \cdot (e(n) + 1) \\ &= e(m) \cdot e(n+1). \end{aligned}$$

This completes our proof by induction.
It remains to show that for all $n < 0$, $e(m \cdot n) = e(m) \cdot e(m)$. Suppose $n < 0$. Then $-n > 0$. So,

$$e(m \cdot n) = e(-(m(-n))) = -e(m(-n)) = -e(m) \cdot e(-n) = e(m) \cdot e(n).$$

$\square$