MATH 330 – HW #23

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Proposition 10.23:
(iii) $\lim_{k\to\infty}(a_k + b_k) = A + B$.

Proof: Let $\varepsilon > 0$ then there exists $M_1 \in \mathbb{N}$ such that

$$m \geq M_1 \Longrightarrow |a_m - A| < \tfrac{\varepsilon}{2},$$

and there exists $M_2 \in \mathbb{N}$ such that

$$m \geq M_2 \Longrightarrow |b_m - B| < \tfrac{\varepsilon}{2}.$$

We define $M = \max M_1, M_2$. Then, for all $n \leq M$, we have

$$\begin{gathered}|(a_m + b_m) - (A + B)| = |(a_m - A) + (b_m - B)| \\ \leq |a_m - A| + |b_m - B| \text{ (by the triangle inequality)} \\ < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.\end{gathered}$$

$\square$

(v) If $A \neq 0$, then $\lim_{k\to\infty} \frac{1}{a_k} = \frac{1}{A}$.

Proof: Suppose $A \neq 0$ and $M_1 \in \mathbb{N}$ such that, for $m \geq M_1$, we have

$$|a_m - A| \leq \tfrac{|A|}{2} \Longrightarrow |A| - |a_m| \leq ||A| - |a_m|| \leq |A - a_m| < \tfrac{|A|}{2} \Longrightarrow \tfrac{|A|}{2} < |a_m|,$$

where, in the second inequality after the first implication, we used Proposition 10.10 (v). Thus for all $m \geq M_1$, we have

$$0 < \tfrac{|A|}{2} < |a_m| \Longrightarrow 0 < \tfrac{1}{|a_m|} < \tfrac{2}{|A|}.$$

Thus, for $m \geq M_1$,

$$|\tfrac{1}{a_m} - \tfrac{1}{A}| = |\tfrac{A - a_m}{A \cdot a_m}| = \tfrac{|A - a_m|}{|A||a_m} \leq \tfrac{2}{|A|^2} \cdot |A - a_m|.$$

Now let $\varepsilon > 0$. Since $\lim_{k\to\infty} a_k = A$, we can choose $M_2 \in \mathbb{N}$ such that

$$m \leq M_2 \Longrightarrow |A - a_m| < \tfrac{|A|^2}{2} \cdot \varepsilon.$$

We now define $N = \max M_1, M_2$. Then, for $m \geq M$, we have

$$|\tfrac{1}{a_m} - \tfrac{1}{A}| \leq \tfrac{2}{|A|^2} \cdot |A - a_m| < \tfrac{2}{|A|^2} \cdot \tfrac{|A|^2}{2} \cdot \varepsilon = \varepsilon.$$

□