MATH 330 – HW #30

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Question: Let $A = \mathbb{R} \diagdown \{-1\}$, that is, A is the set of all integers excluding -1. We define a binary operation $\circ$ on A as follows:

$$a \circ b = ab + a + b, \text{ for all } a, b \in A$$

Verify the following:
(1) $a \circ b \in \mathbb{A}$ for all $a, b \in \mathbb{A}$.
Proof: A is closed if $a, b \in A$ implies: $a \circ b \in A$. Suppse $a \circ b = ab + a + b = -1$. This means $ab + a + b + 1 = (a+1)(b+1) = 0$. So, either $a = -1$ or $b = -1$. $\square$
(2) for $a, b, c \in A, (a \circ b) \circ c = a \circ (b \circ c)$
Proof: The operation $\circ$ is associative if and only if:

$$(a \circ b) \circ c = a \circ (b \circ c) \text{ for all } a, b, c \in A$$

Let's computer both sides of this equation:

$$(a \circ b) \circ c = (ab + a + b) \circ c = (ab + a + b)c + (ab + a + b) + c = \\ abc + ac + bc + ab + a + b + c$$

$$a \circ (b \circ c) = a \circ (bc + b + c) = a(bc + b + c) + a + (bc + b + c) = abc + ac + bc + ab + a + b + c$$

Since both equations are the same, the operation $\circ$ is associative.$\square$
(3) $a \circ 0 = a = 0 \circ a$ for all $a \in A$
Proof: We want to the operation $\circ$ has an identity element 0. We must show that $a \circ 0 = a = 0 \circ a$.
The left side of the equation $a \circ 0 = a(0) + a + 0 = a$.
The right right of the equation $0 \circ a = (0)a + 0 + a = a$.
Since both equations are the same, $a \circ 0 = a = 0 \circ a$. $\square$

(4) for each $a \in A$, there exits $b \in A$ such that $a \circ b = 0 = b \circ a$
Proof: For a given $a \in A$, we need to find $b \in A$ such that

$$a \circ b = 0 = b \circ a$$

(1) We will first solve for b for the equation on the left.

$$a \circ b = ab + a + b = 0 \tag{1}$$
$$= b(a-1) = -a \tag{2}$$

As $a \in A$, $a \neq -1$, so $a+1 \neq 0$. So $b = -\frac{a}{a+1}$. By a way of contradiction, we will assume $b = -1$.

$$-\frac{a}{a+1} = -1 \tag{3}$$
$$-a = -(a+1) \tag{4}$$
$$-a = -a - 1 \tag{5}$$
$$0 \neq 1 \tag{6}$$

So $b = -\frac{a}{a+1} \neq 1$ or $b \in A$. (2) We will now solve for b for the right side of the equation.

$$b \circ a = -\frac{a}{a+1}a - \frac{a}{a+1} + a \tag{7}$$
$$= -\frac{a^2 - a}{a+1} + a \tag{8}$$
$$= \frac{(-a)(a+1)}{1 \times (a+1)} + a \tag{9}$$
$$= -a + a = 0 \tag{10}$$

Thus, b is the inverse of $a \in A$. $\square$
(5) $a \circ b = b \circ a$ for all $a, b \in A$.
Proof: For all $a, b \in A$,

$$a \circ b = b \circ a$$

So we have,

$$(a \circ b) - (b \circ a) = 0 \tag{11}$$
$$(ab + a + b) - (ba + b + a) = 0 \tag{12}$$
$$ab + a + b - ba - b - a = 0 \tag{13}$$
$$0 = 0 \tag{14}$$

$\square$
Thus, we conclude that $(A, \circ)$ is an abelian group with 0 as an identity element.