MATH 330 – HW #20 Revision

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November 28, 2017

Propostion 10.7: Let $x, y \in \mathbb{Z}$. Then $|x| < |y|$ if and only if $x^2 < y^2$.

Proof: Let $x, y \in \mathbb{Z}$. We want to prove if $|x| < |y|$, then $x^2 < y^2$ (1) and if $x^2 < y^2$, then $|x| < |y|$ (2).

(1) Assume $|x| < |y|$. By Propostion 10.5, $|x| < |y|$ if and only if $|x|^2 < |y|^2$. Proposition 10.6 implies, $|x|^2 < |y|^2$ if and only if $x^2 < y^2$. This proves our statement, if $|x| < |y|$, then $x^2 < y^2$.

(2) Assume $x^2 < y^2$. Then, $y^2 - x^2 \in \mathbb{R}_{>\not{\,}}$. Equivalently, $(y - x) \cdot (y + x)$. We have $x < y$ and $x < -y$. These two inequalities can only hold if $|x| < |y|$, by the definition of absolute value. $\square$

Proposition 10.9: Let $x \in \mathbb{R}$ be such that $0 \leq x \leq 1$, and let $m, n \in \mathbb{N}$ be such that $m \geq n$. Then $x^m \leq x^n$.

Proof: Let $0 \leq x \leq 1$ and a fix $n \in \mathbb{N}$. Let $P(m)$ be the statement $x^m \leq x^n$. We will prove this by induction on m that $P(m)$ is true for all $m \geq n$.

<u>Base Case:</u> When $m = n$ is obviously true since $x^n \leq x^n$.

<u>Induction:</u> Now suppose $k \geq n$ and $P(k)$ is true. Since $0 \leq x \leq 1$, we have $x^n \cdot x \leq x^n \cdot 1 = x^n$. So,

$$x^{k+1} = x^k \cdot x \leq x^n \cdot x \leq x^n,$$

where the first inequality follows from the induction hypothesis and the second inequality also holds since $0 \leq x \leq 1$ and it will make x^n less than or equal to itself. Thus, completing our proof by induction. $\square$