MATH 330 – HW #31

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Questions: Let G be a group and let $a \in G$. Prove that
(a) $\mathrm{C}_G(a) = \{g \in G : ga = ag\}$ is a subgroup of G.
Proof: To show the set $\mathrm{C}_G(a)$ is a group, it needs to be non-empty and satisfy if $g, h \in \mathrm{C}_G(a)$ then $gh^{-1} \in \mathrm{C}_G(a)$.

Let $g, h \in \mathrm{C}_G(a)$. Then

$$\begin{aligned} ahg^{-1} &= hg^{-1}g(h^{-1}ah)g^{-1} && (1)\\ &= hg^{-1}g(h^{-1}ha)g^{-1} && (2)\\ &= hg^{-1}(gag^{-1}) && (3)\\ &= hg^{-1}a && (4)\end{aligned}$$

(1) can be done because $hg^{-1} = (gh^{-1})^{-1}$. (3) is done since $ah = ha$ and $hh^{-1} = 1$. (4) is done since $ag = ga$ and $gg^{-1} = 1$. Since, $hg^{-1}a = ahg^{-1}$ and we have shown if $g, h \in \mathrm{C}_G(a)$, then $hg^{-1} \in \mathrm{C}_G(a)$ and we have a subgroup. So $\mathrm{C}_G(a)$ is a subgroup of G. $\square$

(b) Define $\mathrm{Z}(G) = \{x \in G : xg = gx \text{ for all } g \in G\}$. Show that $\mathrm{Z}(G) = \cap_{a \in G}\mathrm{C}_G(a)$.

Proof: We want to show that $Z(G) \subseteq \cap_{a\in G}\mathrm{C}_G(a)$ (1) and $\cap_{a\in G}\mathrm{C}_G(a) \subseteq Z(G)$ (2), to prove that $\mathrm{Z}(G) = \cap_{a\in G}\mathrm{C}_G(a)$.

(1) Suppose that $a \in Z(G)$. From the definition of center, for all $g \in G : ga = ag$. By the definition of centralizer, this corresponds to for all $g \in G : a \in \mathrm{C}_G(a)$. Therefore, we have, by set intersection, $a \in \cap_{a\in G}\mathrm{C}_G(a)$. Thus, $Z(G) \subseteq \cap_{a\in G}\mathrm{C}_G(a)$.

(2)Suppose now that $a \in \cap_{a \in G} \mathrm{C}_G(a)$. Then by definition of intersection, for all $g \in G : a \in \mathrm{C}_G(g)$. That is, for all $g \in G : ag = ga$, by the definition of centralizer. By the definition of the center, this means $a \in Z(G)$. Thus, $\cap_{a \in G} \mathrm{C}_G(a) \subseteq Z(G)$.

From (1) and (2), we can say that $\mathrm{Z}(G) = \cap_{a \in G} \mathrm{C}_G(a)$. $\square$