MATH 330 – HW #30

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Question: Let $A = \mathbb{R} \diagdown \{-1\}$, that is, A is the set of all integers excluding -1. We define a binary operation $\circ$ on A as follows:

$$a \circ b = ab + a + b, \text{ for all } a, b \in A$$

Verify the following:
(1) $a \circ b \in \mathbb{A}$ for all $a, b \in \mathbb{A}$.
Proof: A is closed if $a, b \in A$ implies: $a \circ b \in A$. Suppse $a \circ b = ab + a + b = -1$. This means $ab + a + b + 1 = (a+1)(b+1) = 0$. So, either $a = -1$ or $b = -1$. $\square$
(2) for $a, b, c \in A$, $(a \circ b) \circ c = a \circ (b \circ c)$
Proof: The operation $\circ$ is associative if and only if:

$$(a \circ b) \circ c = a \circ (b \circ c) \text{ for all } a, b, c \in A$$

Let's computer both sides of this equation:

$$(a \circ b) \circ c = (ab + a + b) \circ c = (ab + a + b)c + (ab + a + b) + c =$$
$$abc + ac + bc + ab + a + b + c$$
$$a \circ (b \circ c) = a \circ (bc + b + c) = a(bc + b + c) + a + (bc + b + c) = abc + ac + bc + ab + a + b + c$$

Since both equations are the same, the operation $\circ$ is associative.$\square$
(3) $a \circ 0 = a = 0 \circ a$ for all $a \in A$
Proof: We want to the operation $\circ$ has an identity element 0. We must show that $a \circ 0 = a = 0 \circ a$.
The left side of the equation $a \circ 0 = a(0) + a + 0 = a$.
The right right of the equation $0 \circ a = (0)a + 0 + a = a$.
Since both equations are the same, $a \circ 0 = a = 0 \circ a$. $\square$

(4) for each $a \in A$, there exits $b \in A$ such that $a \circ b = 0 = b \circ a$
Proof: We must show $a \circ b = 0 = b \circ a$.

$$a \circ b = ab + a + b = 0 = ba + b + a = a \circ b$$

Thus, $a \circ b = 0 = b \circ a$. $\square$
(5) $a \circ b = b \circ a$ for all $a, b \in A$.
Proof: It is implied by (4) that $a \circ b = b \circ a$. $\square$
We conclude that $(A, \circ)$ is an abelian group with 0 as an identity element.