Reference Frame Transformation

Kial Stewart

Research School of Earth Sciences
e: kial.stewart@anu.edu.au

PHYS 3202

Flow in an inertial reference frame

- Continuum fluid motion forced by pressure gradients and gravity:

Flow in an inertial reference frame

- Continuum fluid motion forced by pressure gradients and gravity:
  - Continuity equation (incompressible fluid)

$$\nabla \cdot \mathbf{u} = 0$$

Flow in an inertial reference frame

- Continuum fluid motion forced by pressure gradients and gravity:
  - Continuity equation (incompressible fluid)

$$\nabla \cdot \mathbf{u} = 0$$

  - Navier-Stokes momentum equation

$$\frac{D\mathbf{u}}{Dt} = \mathbf{g} - \frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

Flow in an inertial reference frame

- Continuum fluid motion forced by pressure gradients and gravity:
  - Continuity equation (incompressible fluid)

$$\nabla \cdot \mathbf{u} = 0$$

  - Navier-Stokes momentum equation

$$\frac{D\mathbf{u}}{Dt} = \mathbf{g} - \frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

- For inviscid flow, the horizontal and vertical momentum equations reduce to Euler's equations

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Flow in an inertial reference frame

- Continuum fluid motion forced by pressure gradients and gravity:
  - Continuity equation (incompressible fluid)

$$\nabla \cdot \mathbf{u} = 0$$

  - Navier-Stokes momentum equation

$$\frac{D\mathbf{u}}{Dt} = \mathbf{g} - \frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

- For inviscid flow, the horizontal and vertical momentum equations reduce to Euler's equations

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{Dw}{Dt} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Flow in an inertial reference frame

- Continuum fluid motion forced by pressure gradients and gravity:
  - Continuity equation (incompressible fluid)

$$\nabla \cdot \mathbf{u} = 0$$

  - Navier-Stokes momentum equation

$$\frac{D\mathbf{u}}{Dt} = \mathbf{g} - \frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

- For inviscid flow, the horizontal and vertical momentum equations reduce to Euler's equations

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{Dw}{Dt} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

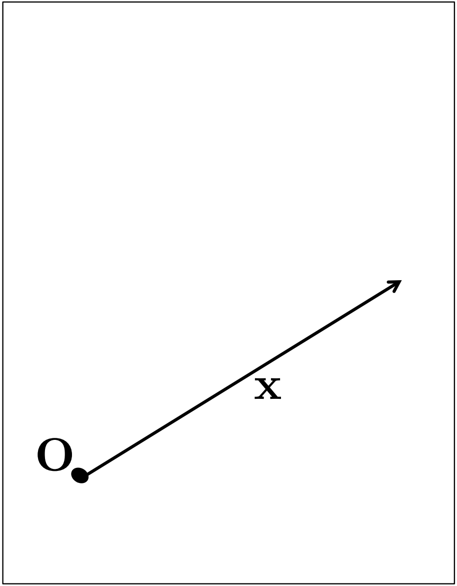
Inertial reference frame

- implies **flow timescale $<<$ planetary rotation timescale**

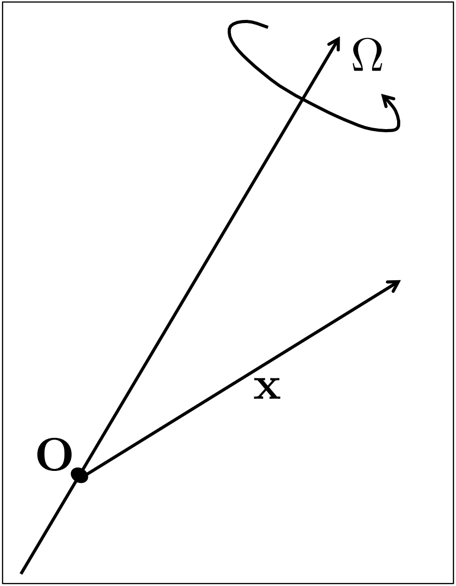
Flow on a Rotating Planet

- Geophysical Flows: atmospheres and oceans, rotation is important

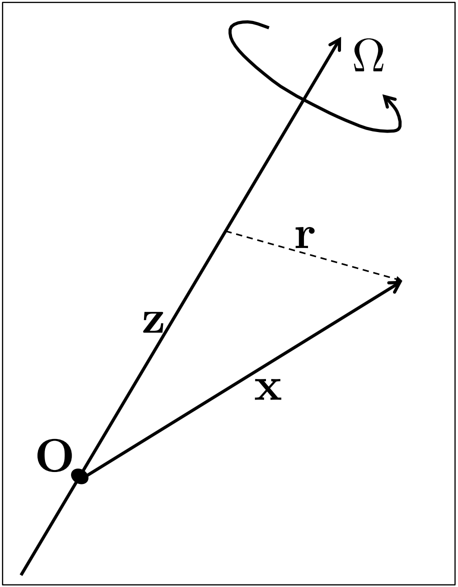
Transformation into Rotating Reference Frame



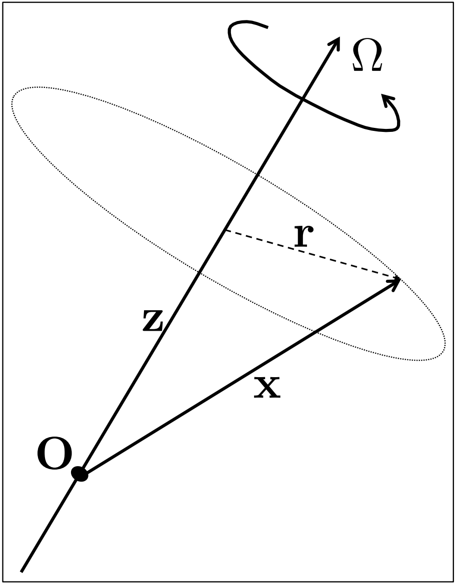
Transformation into Rotating Reference Frame



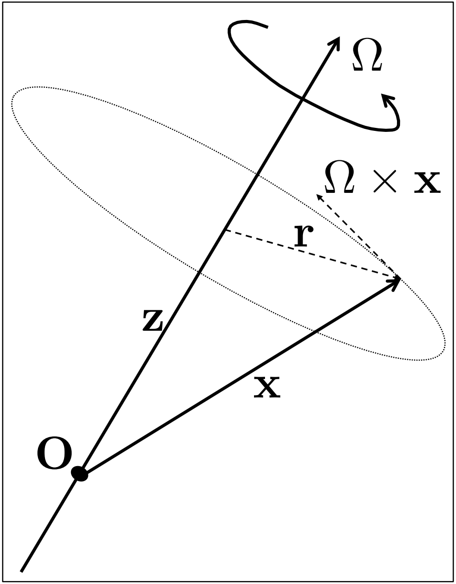
Transformation into Rotating Reference Frame



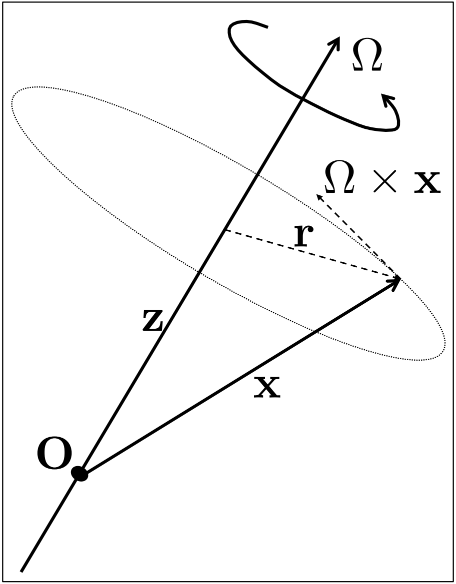
Transformation into Rotating Reference Frame



Transformation into Rotating Reference Frame



Transformation into Rotating Reference Frame



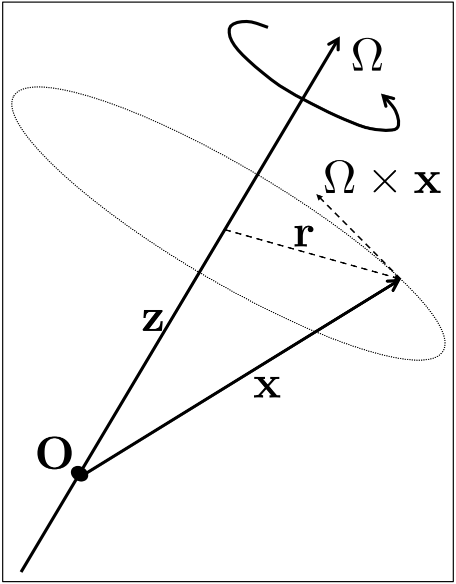
Velocity in R_f: $\mathbf{u}_f$

$$\mathbf{u}_f = \frac{d_f \mathbf{x}}{dt}$$

Velocity in R_r: $\mathbf{u}_r$

$$\mathbf{u}_r = \frac{d_r \mathbf{x}}{dt}$$

Transformation into Rotating Reference Frame



Velocity in R_f: $\mathbf{u}_f$

$$\mathbf{u}_f = \frac{d_f \mathbf{x}}{dt}$$

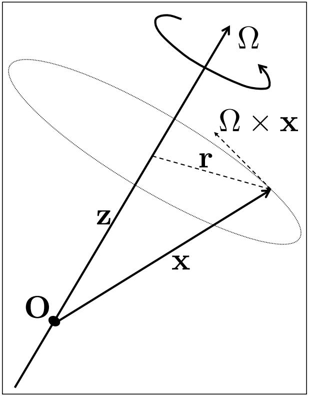
Velocity in R_r: $\mathbf{u}_r$

$$\mathbf{u}_r = \frac{d_r \mathbf{x}}{dt}$$

For some vector $\mathbf{x}$,

$$\frac{d_f \mathbf{x}}{dt} = \frac{d_r \mathbf{x}}{dt} + \Omega \times \mathbf{x}$$

Transformation into Rotating Reference Frame



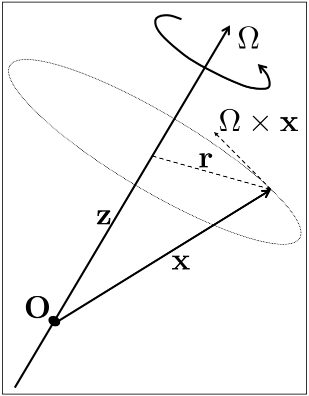
In R_f: $\mathbf{u}_f = \frac{d_f \mathbf{x}}{dt}$

In R_r: $\mathbf{u}_r = \frac{d_r \mathbf{x}}{dt}$

For vector $\mathbf{x}$,

$$\frac{d_f \mathbf{x}}{dt} = \frac{d_r \mathbf{x}}{dt} + \Omega \times \mathbf{x}$$

Transformation into Rotating Reference Frame



In R_f: $\mathbf{u}_f = \frac{d_f \mathbf{x}}{dt}$

In R_r: $\mathbf{u}_r = \frac{d_r \mathbf{x}}{dt}$

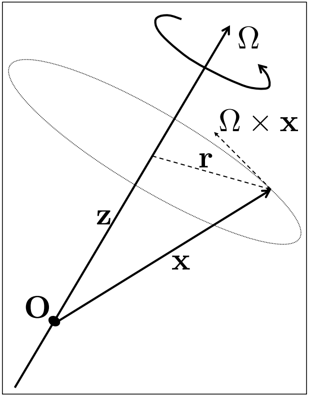
For vector $\mathbf{x}$,

$$\frac{d_f \mathbf{x}}{dt} = \frac{d_r \mathbf{x}}{dt} + \Omega \times \mathbf{x}$$

Take the vector $\mathbf{u}_f$,

$$\frac{d_f \mathbf{u}_f}{dt} = \frac{d_r \mathbf{u}_f}{dt} + \Omega \times \mathbf{u}_f$$

Transformation into Rotating Reference Frame



In R_f: $\quad \mathbf{u}_f = \frac{d_f \mathbf{x}}{dt}$

In R_r: $\quad \mathbf{u}_r = \frac{d_r \mathbf{x}}{dt}$

For vector $\mathbf{x}$,

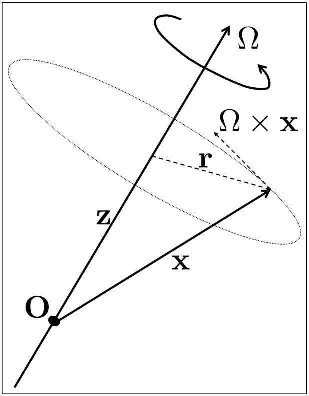
$$\frac{d_f \mathbf{x}}{dt} = \frac{d_r \mathbf{x}}{dt} + \Omega \times \mathbf{x}$$

Take the vector $\mathbf{u}_f$,

$$\frac{d_f \mathbf{u}_f}{dt} = \frac{d_r \mathbf{u}_f}{dt} + \Omega \times \mathbf{u}_f$$

$$= \frac{d_r}{dt}\left[\frac{d_r \mathbf{x}}{dt} + \Omega \times \mathbf{x}\right] + \Omega \times \left[\frac{d_r \mathbf{x}}{dt} + \Omega \times \mathbf{x}\right]$$

Transformation into Rotating Reference Frame



In R_f: $\mathbf{u}_f = \frac{d_f \mathbf{x}}{dt}$

In R_r: $\mathbf{u}_r = \frac{d_r \mathbf{x}}{dt}$

For vector $\mathbf{x}$,

$$\frac{d_f \mathbf{x}}{dt} = \frac{d_r \mathbf{x}}{dt} + \Omega \times \mathbf{x}$$

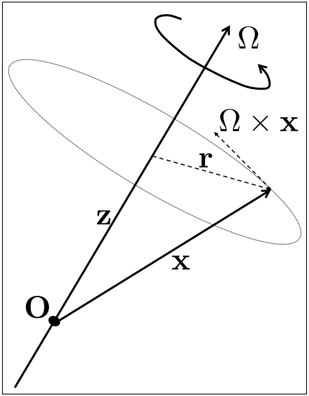
Take the vector $\mathbf{u}_f$,

$$\frac{d_f \mathbf{u}_f}{dt} = \frac{d_r \mathbf{u}_f}{dt} + \Omega \times \mathbf{u}_f$$

$$= \frac{d_r}{dt}\left[\frac{d_r \mathbf{x}}{dt} + \Omega \times \mathbf{x}\right] + \Omega \times \left[\frac{d_r \mathbf{x}}{dt} + \Omega \times \mathbf{x}\right]$$

$$= \frac{d_r}{dt}\left[\mathbf{u}_r + \Omega \times \mathbf{x}\right] + \Omega \times \left[\mathbf{u}_r + \Omega \times \mathbf{x}\right]$$

Transformation into Rotating Reference Frame



In R_f: $\mathbf{u}_f = \frac{d_f \mathbf{x}}{dt}$

In R_r: $\mathbf{u}_r = \frac{d_r \mathbf{x}}{dt}$

For vector $\mathbf{x}$,

$$\frac{d_f \mathbf{x}}{dt} = \frac{d_r \mathbf{x}}{dt} + \Omega \times \mathbf{x}$$

Take the vector $\mathbf{u}_f$,

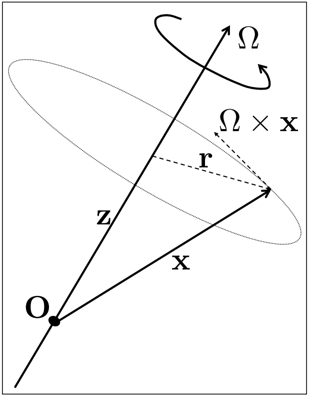
$$\frac{d_f \mathbf{u}_f}{dt} = \frac{d_r \mathbf{u}_f}{dt} + \Omega \times \mathbf{u}_f$$

$$= \frac{d_r}{dt}\left[\frac{d_r \mathbf{x}}{dt} + \Omega \times \mathbf{x}\right] + \Omega \times \left[\frac{d_r \mathbf{x}}{dt} + \Omega \times \mathbf{x}\right]$$

$$= \frac{d_r}{dt}\left[\mathbf{u}_r + \Omega \times \mathbf{x}\right] + \Omega \times \left[\mathbf{u}_r + \Omega \times \mathbf{x}\right]$$

$$= \frac{d_r \mathbf{u}_r}{dt} + \frac{d_r}{dt}\left[\Omega \times \mathbf{x}\right] + \Omega \times \mathbf{u}_r + \Omega \times \Omega \times \mathbf{x}$$

Transformation into Rotating Reference Frame



In R_f: $\mathbf{u}_f = \frac{d_f \mathbf{x}}{dt}$

In R_r: $\mathbf{u}_r = \frac{d_r \mathbf{x}}{dt}$

For vector $\mathbf{x}$,

$$\frac{d_f \mathbf{x}}{dt} = \frac{d_r \mathbf{x}}{dt} + \Omega \times \mathbf{x}$$

Take the vector $\mathbf{u}_f$,

$$\frac{d_f \mathbf{u}_f}{dt} = \frac{d_r \mathbf{u}_f}{dt} + \Omega \times \mathbf{u}_f$$

$$= \frac{d_r}{dt}\left[\frac{d_r \mathbf{x}}{dt} + \Omega \times \mathbf{x}\right] + \Omega \times \left[\frac{d_r \mathbf{x}}{dt} + \Omega \times \mathbf{x}\right]$$

$$= \frac{d_r}{dt}\left[\mathbf{u}_r + \Omega \times \mathbf{x}\right] + \Omega \times \left[\mathbf{u}_r + \Omega \times \mathbf{x}\right]$$

$$= \frac{d_r \mathbf{u}_r}{dt} + \frac{d_r}{dt}\left[\Omega \times \mathbf{x}\right] + \Omega \times \mathbf{u}_r + \Omega \times \Omega \times \mathbf{x}$$

$$\frac{d_f \mathbf{u}_f}{dt} = \frac{d_r \mathbf{u}_r}{dt} + 2\Omega \times \mathbf{u}_r + \Omega \times \Omega \times \mathbf{x}$$

Transformation into Rotating Reference Frame

$$\frac{d_f \mathbf{u}_f}{dt} \quad = \quad \frac{d_r \mathbf{u}_r}{dt} \quad + \quad 2\Omega \times \mathbf{u}_r \quad + \quad \Omega \times \Omega \times \mathbf{x}$$

Transformation into Rotating Reference Frame

$$\frac{d_f \mathbf{u}_f}{dt} \quad = \quad \frac{d_r \mathbf{u}_r}{dt} \quad + \quad 2\Omega \times \mathbf{u}_r \quad + \quad \Omega \times \Omega \times \mathbf{x}$$

accel. in R_f

Transformation into Rotating Reference Frame

$$\frac{d_f \mathbf{u}_f}{dt} \quad = \quad \frac{d_r \mathbf{u}_r}{dt} \quad + \quad 2\Omega \times \mathbf{u}_r \quad + \quad \Omega \times \Omega \times \mathbf{x}$$

accel. in R_f = accel. in R_r

Transformation into Rotating Reference Frame

$$\frac{d_f \mathbf{u}_f}{dt} \quad = \quad \frac{d_r \mathbf{u}_r}{dt} \quad + \quad 2\Omega \times \mathbf{u}_r \quad + \quad \Omega \times \Omega \times \mathbf{x}$$

accel. in R_f = accel. in R_r + Coriolis

Transformation into Rotating Reference Frame

$$
\begin{array}{ccccccc}
\dfrac{d_f \mathbf{u}_f}{dt} & = & \dfrac{d_r \mathbf{u}_r}{dt} & + & 2\Omega \times \mathbf{u}_r & + & \Omega \times \Omega \times \mathbf{x} \\
\text{accel. in } R_f & = & \text{accel. in } R_r & + & \text{Coriolis} & + & \text{centripetal}
\end{array}
$$

Transformation into Rotating Reference Frame

$$\frac{d_f \mathbf{u}_f}{dt} \quad = \quad \frac{d_r \mathbf{u}_r}{dt} \quad + \quad 2\Omega \times \mathbf{u}_r \quad + \quad \Omega \times \Omega \times \mathbf{x}$$

accel. in R_f = accel. in R_r + Coriolis + centripetal

Substitute back into momentum equation for fixed frame of reference:

$$\frac{D_f \mathbf{u}_f}{Dt} = \mathbf{g} - \frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}_f$$

Transformation into Rotating Reference Frame

$$\begin{array}{ccccccc} \dfrac{d_f \mathbf{u}_f}{dt} & = & \dfrac{d_r \mathbf{u}_r}{dt} & + & 2\Omega \times \mathbf{u}_r & + & \Omega \times \Omega \times \mathbf{x} \\ \text{accel. in } R_f & = & \text{accel. in } R_r & + & \text{Coriolis} & + & \text{centripetal} \end{array}$$

Substitute back into momentum equation for fixed frame of reference:

$$\frac{D_f \mathbf{u}_f}{Dt} = \mathbf{g} - \frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}_f$$

$$\frac{D_r \mathbf{u}_r}{Dt} + 2\Omega \times \mathbf{u}_r + \Omega \times \Omega \times \mathbf{x} = \mathbf{g} - \frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}_r$$