Geopotential

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PHYS 3202

Transform from Inertial to Rotating Reference Frame

$$
\begin{array}{ccccccc}
\dfrac{d_f \mathbf{u}_f}{dt} & = & \dfrac{d_r \mathbf{u}_r}{dt} & + & 2\Omega \times \mathbf{u}_r & + & \Omega \times \Omega \times \mathbf{x} \\
\text{accel. in } R_f & = & \text{accel. in } R_r & + & \text{Coriolis} & + & \text{centripetal}
\end{array}
$$

Substitute back into momentum equation for fixed frame of reference:

$$
\frac{D_f \mathbf{u}_f}{Dt} = \mathbf{g} - \frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}_f
$$

$$
\boxed{\frac{D_r \mathbf{u}_r}{Dt} + 2\Omega \times \mathbf{u}_r + \Omega \times \Omega \times \mathbf{x} = \mathbf{g} - \frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}_r}
$$

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Substitute the along-axis ($\mathbf{z}$) and off-axis ($\mathbf{r}$) components for $\mathbf{x}$:

$$\Omega \times \Omega \times \mathbf{x} = \Omega(\Omega \cdot (\mathbf{r} + \mathbf{z})) - (\mathbf{r} + \mathbf{z})(\Omega \cdot \Omega)$$

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Make this quantity a potential in $\mathbf{r}$ by integrating with respect to $\mathbf{r}$:

$$\Omega \times \Omega \times \mathbf{x} = -\nabla \frac{\mathbf{r}^2 \Omega^2}{2}$$

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Incorporate it together with the gravity term:

$$\mathbf{g}^* = \mathbf{g} - \Omega \times \Omega \times \mathbf{x} = \mathbf{g} + \nabla \frac{\mathbf{r}^2 \Omega^2}{2} = -\nabla \Phi$$

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Substitute back into the momentum equation to give,

$$\frac{D_r \mathbf{u}_r}{Dt} + 2\Omega \times \mathbf{u}_r = \mathbf{g}^* - \frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{u}_r$$