# NP-Completeness of the Vehicle-to-Vehicle Charging Model

Cláudio Gomes[1,2*], Second Author[2,3†] and Third Author[1,2†]

[1*]Department, Organization, Street, City, 100190, State, Country.
[2]Department, Organization, Street, City, 10587, State, Country.
[3]Department, Organization, Street, City, 610101, State, Country.

*Corresponding author(s). E-mail(s): claudiogomes@cmu.edu;
Contributing authors: iiauthor@gmail.com; iiiauthor@gmail.com;
†These authors contributed equally to this work.

## 1 Proof of NP-completeness

For the purposes of proving the NP-completeness of our model, we consider *V2V problem* as the *decision problem* in which an answer to a particular instance of the V2V model is always *True* if the constraints $Ax = b$, $l \leq x \leq u$ shown in equation (1) can be satisfied, and *False* otherwise.

$$(IP)_{A,b,l,u,f} : \begin{cases} \min f(x) \\ Ax = b, \quad l \leq x \leq u, \quad l, x, u \in \mathbb{Z}^n \\ A \in \mathbb{Z}^{m \times n}, \quad b \in \mathbb{Z}^m \end{cases}, \tag{1}$$

A proof for the NP-completeness of this model is presented by showing that the V2V problem belongs to the NP class, and by showing that the V2V problem is NP-hard. We prove the NP-hardness of the V2V problem by specifying and illustrating a polynomial reduction from the 3SAT problem to the V2V problem.

### 1.1 NP class

For any given decision problem, a *verifier* is an algorithm capable of verifying whether a solution $\tau$ given to an instance $p$ of the problem is correct. Formally, a verifier is a function $V(p, \tau)$, which returns *True* when $\tau$ is correct and *False* when otherwise.

We define a *witness* $x$ of a decision problem instance $p$ as a solution to $p$ such that $V(p, x) = \mathit{True}$, where $V$ is a verifier of $p$. That is, a witness is a correct solution to a problem instance, which, once verified, let us conclude the problem instance is *True*. In order for the V2V problem to be in NP, we need to show that: there exists a polynomial-time verifier that enables us to verify that a witness attests to a particular instance of the problem to be *True*; and that the witness should be polynomial in the size of the problem instance.

For the V2V problem, we can define a polynomial-time verifier as the algorithm that checks whether the equality and inequality constraints $Ax = b,\ l \leq x \leq u$ shown in equation (1) are satisfied by a solution $x$. Concretely, this verifier of the V2V problem can be defined as

$$V_{\mathrm{V2V}}\left((A, b, l, u), x\right) = \begin{cases} \mathit{True} & \text{if } Ax = b,\ l \leq x \leq u \text{ are satisfied,} \\ \mathit{False} & \mathit{otherwise}, \end{cases} \tag{2}$$

where $(A, b, l, u)$ is a V2V instance and $x$ is a solution to the instance. We remind the reader that this formulation uses the same notation as in equation (1) for simplicity. The matrix $A$ and the vectors $b, l$, and $u$ are represented as a series of constraints and the solution $x$ is represented as the values for the binary decision variables $X$, $Y$, and $Z$, as well as the slack variables. $V_{\mathrm{V2V}}$ performs matrix multiplication and equality and inequality comparisons to check whether a given solution is correct for a given V2V instance. The matrix dimensions have been shown to be polynomial in terms of the number of vehicles, parking stations, meeting points, and timesteps. Moreover, the length of the vectors compared in the equality and inequality comparisons is the same as the number of columns in the matrix. Therefore, since these operations are known to be computable in polynomial time with respect to the dimensions of the matrix and of the vectors, $V_{\mathrm{V2V}}$ is a polynomial-time verifier.

Considering the verifier $V_{\mathrm{V2V}}$, a witness $x$ for it would be a set of values attributed to the binary decision variables $X$, $Y$, and $Z$, as well as the slack variables, that satisfy the constraints of the V2V instance (which is the set of matrices and vectors $A$, $b$, $l$, and $u$). In other words, a witness is a solution for a V2V instance that is able to route all the vehicles to their destinations while also meeting the V2V instance's path constraints, battery constraints, parking station charging constraints, and V2V charging constraints.

Any solution of a V2V instance is a set of values attributed to its binary decision variables $X$, $Y$, and $Z$, as well as the slack variables. By definition, the length of these variables is the same as the number of columns of the matrix $A$ of the instance. Again, by definition, a witness is a solution of a V2V instance. Therefore, a witness of a V2V instance is polynomial in terms of the size of the problem instance ($A$, $b$, $l$, and $u$).

We have shown that there exists a verifier that enables us to verify that a witness attests to a particular instance of the problem to be *True* in time polynomial in the size of the problem instance; and that the size of the witnesses is polynomial in the size of the problem instance. Thus, we have proven that the V2V problem is in NP.

## 1.2 Reduction from the 3SAT problem

A problem is considered NP-hard if we can reduce all the problems in NP to it in polynomial time. To prove that the V2V problem is NP-hard, it suffices to show a polynomial-time reduction to it from a known NP-hard problem. More concretely, we will show that the V2V problem is NP-hard by performing a polynomial-time Karp reduction from the 3SAT problem.

### 1.2.1 3SAT Problem

The 3SAT problem is a decision problem that is known to be NP-hard and defined as follows:

**Definition 1.** *The 3SAT problem accepts as input a Boolean formula in* ***conjuctive normal form*** *such that there are at most three literals in each clause. There may be any number of atoms and clauses. The output is* ***True*** *if the formula is* ***satisfiable*** *and* ***False*** *otherwise.*

We remind the reader that a literal is one of two types in the context of the 3SAT: *positive literal* when it is just the atom; and *negative literal* when it is the negation of the atom. Moreover, a Boolean formula in the conjuctive normal form is a conjuction of one or more clauses, where each clause is a disjunction of literals. Finally, a Boolean formula is considered satisfiable when there is an assignment of values to the atoms that will satisfy it. A Boolean formula is considered to be satisfied by an assignment of values to its atoms when it is *True* under that assignment. We illustrate an instance of the 3SAT next:

$$(x_1) \wedge (x_2 \vee x_3) \wedge (\neg x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \tag{3}$$

Going back to the context of our NP-completeness proof of the V2V problem, we define a verifier for the 3SAT problem as follows:

$$V_{\mathrm{3SAT}}(p, x) = \begin{cases} \textit{True} \text{ if } p \text{ is } \textit{True} \text{ under } x, \\ \textit{False} \text{ otherwise,} \end{cases} \tag{4}$$

where $p$ is the 3SAT instance, defined by its Boolean formula, and $x$ is a solution to the instance, defined as a list of (atom = truth value) pairs, with one pair for each atom. Since the 3SAT problem is in NP, this verifier can be executed in a time polynomial to the number of atoms and clauses (which is easy to see how, given it only needs to compute a polynomial number of bitwise operations). Now that a polynomial-time verifier $V_{\mathrm{3SAT}}$ has been defined, we can define a witness $x$ of a 3SAT instance $p$ as a list of (atom, truth value) pairs such that $V_{\mathrm{3SAT}}(p, x) = \textit{True}$. In other words, the witness $x$ of a 3SAT instance is an assignment of values to the atoms such that it satisfies the Boolean formula of the instance. As an example, $(x_1 = \textit{True}, x_2 = \textit{False}, x_3 = \textit{True})$ is a witness for the 3SAT instance shown in (3).

### 1.2.2 Reduction Steps

We show next the reduction steps from the 3SAT problem to the V2V problem. Let us suppose that we are given an instance of 3SAT with $n$ atoms $x_1, \ldots, x_n$ and $m$

clauses $c_1, \dots, c_m$. Then we construct the space network graph $G = (N, A)$ and the other parameters for the equivalent V2V instance as follows:

- For each clause $c_j$, we create a node $sat_j \in M$, which is a meeting point ($M \subset N$), and a node $f_j \in N$, which are connected by a direct edge from $sat_j$ to $f_j$.
- For each atom $x_i$, we create a starting node $s_i \in N$. Next, we create a node $true_i \in M$ and a node $false_i \in M$, which are meeting points. Afterwards, we create directed edges from $s_i$ to $true_i$ and to $false_i$.
- For each clause $c_j$ and for each atom $x_i$, if $c_j$ has a literal of $x_i$, we create a directed edge from $true_i$ to $f_j$ and from $false_i$ to $f_j$; if $c_j$ has a positive literal of $x_i$, we create a directed edge from $true_i$ to $sat_j$; and if $c_j$ has a negative literal of $x_i$, we create a directed edge from $false_i$ to $sat_j$.
- For each atom $x_i$, we denote as $k_i$ the number of literals it has among all clauses. Next, we create $k_i$ EVs $v_{i,1}$ to $v_{i,k_i}$, all located on $s_i$. For each vehicle $v_{i,o}$, $o \in \{1, \dots, k_i\}$, we set its final destination as $f_j$, where $c_j$ is the clause that contains the $o^{\text{th}}$ literal of the atom $x_i$. $\text{SOC}_{v_{i,o}}$ is 1 if $o \neq 1$ and $3k_i + 1$ if otherwise.
- For each clause $c_j$, we create an EV $v_j^{sat}$ located at $sat_j$ and whose final destination is $f_j$. $\text{SOC}_{v_j^{sat}}$ is 0.
- For each EV $v$, we set $e_i = 1$ (energy that EV $v$ provides to another EV in a timestep) and $\text{MAXSOC}_v$ as a sufficiently large number, such as $3m + 1$.
- All the arcs (or directed edges) created have a time duration of 1 and a traversal cost of 1.
- Finally, we define $T = 3K + 6$, where $K$ is the greatest number of literals any atom $x_i$ has among all clauses ($K = \max(k_1, \dots, k_n)$).

With the construction steps listed above, we have defined all the parameters required for the input of a V2V problem. Therefore, the graph $G$ we just constructed will be transformed into a time-space network graph $G_{TS}$, finalizing the setup of a V2V instance from a 3SAT instance.

### 1.2.3 Proof of Correctness for the Reduction

In order to show that our reduction from the 3SAT problem to the V2V problem is correct, we prove that the constructions steps shown are equivalent to a function $f$ mapping instances of the 3SAT problem to instances of the V2V problem, such that:

$$\text{answer to p is } \textit{True} \iff \text{answer to f(p) is } \textit{True}, \tag{5}$$

where $p$ is an instance of the 3SAT problem. $f$ is also known as a *Karp reduction* from the 3SAT problem to the V2V problem.

Since we have proven that both the 3SAT problem and the V2V problem are in NP, we can leverage the definition of NP to show that a mapping function $f$ is a Karp reduction from the 3SAT problem to the V2V problem if and only if for all instances $p$ of the 3SAT problem the following is true:

$$\begin{array}{c}\text{There exists a polynomial-}\\ \text{size witness } x \text{ such that}\\ V_{\text{3SAT}}(p,x) = \mathit{True}.\end{array} \iff \begin{array}{c}\text{There exists a polynomial-}\\ \text{size witness } z \text{ such that}\\ V_{\text{V2V}}(f(p),z) = \mathit{True}.\end{array} \tag{6}$$

where $V_{\text{3SAT}}$ and $V_{\text{V2V}}$ are the previously defined polynomial-time verifiers for the 3SAT problem and the V2V problem, respectively.

We present a series of lemmas and their proofs next. These lemmas culminate into a theorem that shows that our reduction is a Karp reduction and, consequently, that the V2V problem is NP-hard (and thus NP-complete).

**Lemma 1.** *The reduction from the 3SAT problem to the V2V problem shown in 1.2.2 is polynomial in time and size in terms of the number of atoms and clauses.*

*Proof.* Let us suppose that we are given an instance of 3SAT with $n$ atoms and $m$ clauses. As shown by the construction steps, the reduction creates an instance of the V2V problem that uses a space network graph with $3n + 2m$ nodes and, at most, $2n + 10m$ arcs. As an example to clarify the number of arcs, a clause $c_j$ implies having an arc from $sat_j$ to $f_j$ and may have three atoms $x_i, x_j, x_k$ with positive literals in it and, thus, also have arcs from the nodes $true_i, true_j, true_k, false_i, false_j, false_k$ to $f_j$, as well as arcs from the nodes $true_i, true_j, true_k$ to $sat_j$. □

**Lemma 2.** *If the answer to a 3SAT instance is* ***True****, then the answer to its reduction to a V2V instance is also* ***True****.*

*Proof.* Consider a 3SAT instance $p$ with $n$ atoms and $m$ clauses. Let us assume that the answer to $p$ is *True*. Therefore, there exists a polynomial-size witness $x$ such that $V_{3SAT}(p, x) = \mathit{True}$. We remind the reader that a witness for a 3SAT instance is an assignment of values to the atoms of the Boolean formula of the instance, in the form of a list of (atom = truth value) pairs. Hence, for each variable $x_i$ of the instance, we have the pair $(x_i, \mathit{True})$ or the pair $(x_i, \mathit{False})$ in the list. This witness $x$ can be transformed into a witness $z$ such that we have $V_{\text{V2V}}(f(p), z) = \mathit{True}$:

- For each atom $x_i$, if its pair is *True*, we move all EVs $v_{i,1}, \dots, v_{i,k_i}$ from $s_i$ to $true_i$, otherwise to $false_i$. This step happens from $t = 0$ to $t = 1$ and takes one unit of energy from each vehicle involved.
- For each atom $x_i$ and $o \in \{2, 3, \dots, k_i\}$, vehicle $v_{i,1}$ will charge $v_{i,o}$ with 3 units of energy from $t = 3o - 5$ to $t = 3o - 2$.
- For each atom $x_i$ and $o \in \{1, 2, \dots, k_i\}$, if there is an arc between the current node of $v_{i,o}$ and $sat_j$, where $c_j$ is the clause that contains the $o^{\text{th}}$ literal of the atom $x_i$, then $v_{i,o}$ will move to $sat_j$, charge $v_j^{sat}$ with one unit of energy, and then move to $f_j$; otherwise, if there is no arc between the current node of $v_{i,o}$ and $sat_j$, then $v_{i,o}$ will be in the same node for two timesteps and then move to $f_j$. If $o \geq 2$, $v_{i,o}$ will perform these actions from $t = 3o - 2$ to $t = 3o + 1$. $v_{i,1}$ will perform these actions last, from $t = 3k_i + 1$ to $t = 3k_i + 4$. These actions consume 1 or 3 units of energy from each vehicle involved, depending on whether they go directly to $f_j$.

- After being charged by all possible vehicles, all $v_1^{\text{sat}}, \ldots, v_m^{\text{sat}}$ will move to $f_1, \ldots, f_m$, respectively. This happens during the last timestep, from $t = T - 2$ to $t = T - 1$ (from $t = 3K + 4$ to $t = 3K + 5$, $K = \max(k_1, \ldots, k_n)$).

In order for $V_{\text{V2V}}(f(p), z) = \mathit{True}$, witness $z$ will have to assign the required actions in the V2V instance $f(p)$ such that all the EVs reach their destinations. We prove that this transformation from the witness $x$ is, indeed, a valid witness $z$:

- Since $T = 3K + 6$ and $k_i \leq K, \forall i \in \{1, \ldots, n\}$, all the steps above do not exceed the number of timesteps specified for the V2V instance.
- For each atom $x_i$, if its pair is *True*, $v_{i,1}$ arrives to $\mathit{true}_i$ with $3k_i$ units of energy, all the vehicles $v_{i,2}, \ldots, v_{i,k_i}$ arrive to $\mathit{true}_i$ with 0 units of energy. After arrival, they each get 3 units of energy from $v_{i,1}$, which loses $3(k_i - 1)$ units of energy. All in all, all the vehicles $v_{i,1}, \ldots, v_{i,k_i}$ leave $\mathit{true}_i$ with 3 units of energy. By the same logic, for each atom $x_i$, if its pair is *False*, all the vehicles $v_{i,1}, \ldots, v_{i,k_i}$ leave $\mathit{false}_i$ with 3 units of energy. Hence, for each atom $x_i$, each of the vehicles $v_{i,1}, \ldots, v_{i,k_i}$ has enough of units of energy in case it moves directly to its respective $f_j$, or even if it moves to its respective $\mathit{sat}_j$, charges its respective $v_j^{sat}$, and then moves to its destination $f_j$.
- It remains to prove that, for each clause $c_j$, $v_j^{sat}$ reaches their destination $f_j$, which is equivalent to proving that $v_j^{sat}$ receives at least one unit of energy, as it only needs one unit to move to $f_j$. We have that all the clauses of the 3SAT instance $p$ are satisfied by the list of (atom = truth values) pairs defined by the witness $x$. Therefore, there is at least one literal in each clause $c_j$ that is satisfied by some atom $x_i$ with an assigned truth value. Moreover, considering the construction steps for the instance $f(p)$, there will be an arc between $\mathit{true}_i$ and $\mathit{sat}_j$ if the literal is positive, and an arc between $\mathit{false}_i$ and $\mathit{sat}_j$ if otherwise. We also have that, by mapping, if $o$ is the index of the literal for the atom $x_i$, the vehicle $v_{i,o}$ is moved from $s_i$ to $\mathit{true}_i$ if the value assigned to $x_i$ is *True*, and to $\mathit{false}_i$ if otherwise. Hence, $v_{i,o}$ will move into a node that coincides with the arc that connects that node to $\mathit{sat}_j$. Consequently, $v_{i,o}$ will move to $\mathit{sat}_j$ and charge its respective $v_j^{sat}$ before moving to its respective $f_j$. All in all, it is proven that, for each $v_j^{sat}$, there is at least one vehicle charging one unit of energy to it and, thus, $v_j^{sat}$ is able to reach $f_j$.

□

**Lemma 3.** *If the answer to a V2V instance $f(p)$ reduced from a 3SAT instance is **True**, then all its witnesses, for any atom $x_i$, will have all the vehicles $v_{i,1}$ to $v_{i,k_i}$ **pass through one and only one** of the meeting points $\mathit{true}_i$ or $\mathit{false}_i$.*

*Proof.* We remind the reader that, for any atom $x_i$ of the 3SAT instance, the reduction to a V2V instance will have $k_i$ related vehicles $v_{i,1}$ to $v_{i,k_i}$, where $k_i$ is the number of literals the atom $x_i$ has among all clauses. In order for each of these vehicles to reach their destination $f_j$, they need to traverse at least two arcs, as the shortest path goes from the starting node $s_i$ to one of the meeting points $\mathit{true}_i$ or $\mathit{false}_i$, and then from that meeting point to $f_j$. Therefore, each vehicle needs at least two units of energy to reach their destination. By construction, $v_{i,1}$ has an associated $\text{SOC}_{v_{i,1}} = 3k+1$, while $v_{i,2}$ to $v_{i,k_i}$ have an associated $\text{SOC}_{v_{i,2}} = \cdots = \text{SOC}_{v_{i,k_i}} = 1$. Hence, all vehicles but one need to be charged one unit of energy in order to reach their destination. For this to be possible, they need to traverse to one of the meeting points $true_i$ or $false_i$. When all $k_i$ vehicles move to a meeting point $true_i$ or $false_i$, they consume one unit of energy, all arriving to the meeting point with 0 units available, except for $v_{i,1}$, which has $3 \times k_i$ units available. Hence, in order for all vehicles to reach their destination, at least $k_i$ of the $3k_i$ units stored in $v_{i,1}$ need to be shared among all the vehicles. This implies that vehicles need to be in the same meeting point $true_i$ or $false_i$ as $v_{i,1}$ in order to get a unit of energy, since vehicles in $true_i$ cannot move to $false_i$ and vice-versa (there is no path between $true_i$ and $false_i$, as the only way to enter into these nodes is via the arcs from $s_i$ and there is no arc to enter into $s_i$). □

**Lemma 4.** *If the answer to a V2V instance reduced from a 3SAT instance is **True**, then the answer to that 3SAT instance must be **True**.*

*Proof.* Let us assume that we have a witness $z$ for the V2V instance $f(p)$, such that $V_{\text{V2V}}(f(p), z) = \mathit{True}$:

- Then, by Lemma 3, for each atom $x_i$ of the instance $p$, we have that all EVs $v_{i,1}$ to $v_{i,k_i}$ go from $s_i$ to the same meeting point $true_i$ or $false_i$.
- Moreover, by definition, for each clause $c_j$ of the instance $p$, we have that the vehicle $v_j^{sat}$ arrives to its destination $f_j$.

The second item implies that, for each node $sat_j$, we have at least one arc from some node $true_i$ or $false_i$ to $sat_j$, and that at least one vehicle $v_{i,o}$ moved from that node to $sat_j$ in order to give one unit of energy to $v_j^{sat}$, where $o$ is the index of the literal for the atom $x_i$ of the instance $p$, enabling $v_j^{sat}$ to reach its destination.

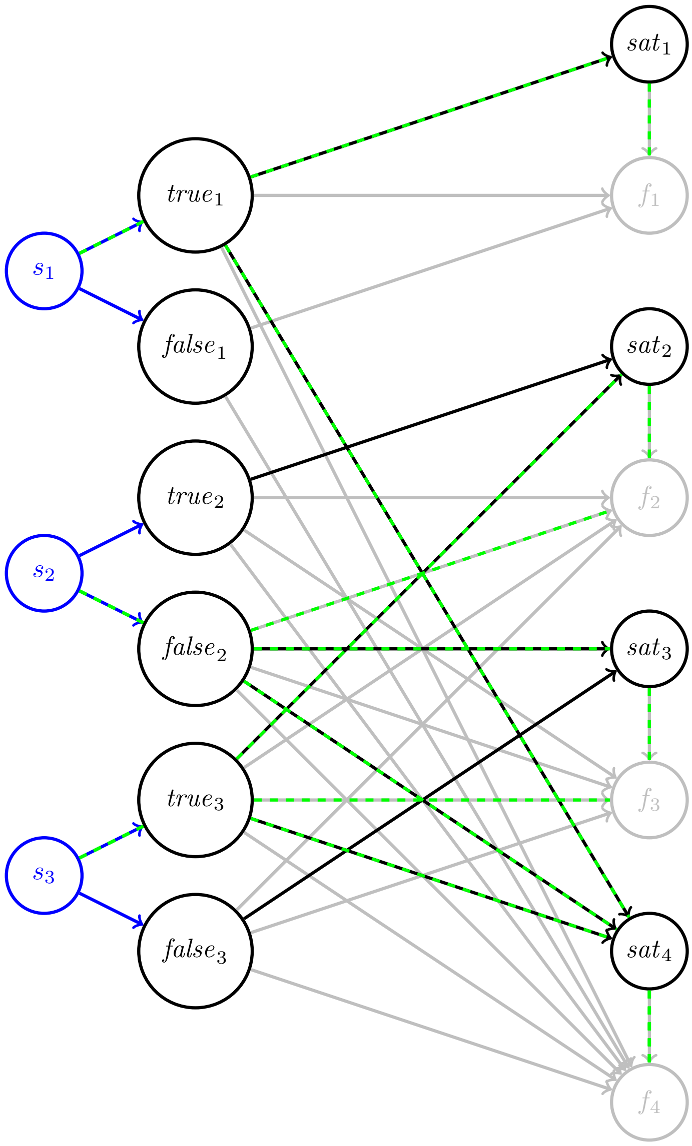
Together with the first item, we have that the meeting points $true_i$ and $false_i$ chosen by a witness $z$ for the vehicles corresponding to each atom $x_i$ can be transformed into a witness $x$ of the 3SAT instance $p$, by assigning a value of *True* to $x_i$ when its vehicles $v_{i,1}, \ldots, v_{i,k_i}$ pass through $true_i$, and a value of *False* when otherwise. This transformation is a witness of $p$, since, by construction, the existence of an arc from some node $true_i$ or $false_i$ to the node $sat_j$ implies that a positive or negative literal of some atom $x_i$ exists in the clause $c_j$, respectively, and that the same clause $c_j$ is satisfied by the atom $x_i$, as it has the same value as that literal. Hence, the existence of a witness for $p$ implies that the answer to $p$ is also *True*, since we have $V_{\text{3SAT}}(p, x) = \mathit{True}$. Thus, the answer to the 3SAT instance $p$ cannot be $False$ when the answer to its reduction to a V2V instance is $True$. □

**Theorem 5.** *The answer to a 3SAT instance p is **True** if and only if the answer to its reduction to a V2V instance $f(p)$ is **True**. The reduction $f$ is a Karp reduction.*

*Proof.* The proof follows from combining Lemmas 1, 2 and 4. Thus, by definition, the reduction $f$ is a Karp reduction. □

## 1.3 Proof of NP-completeness

Now that we have proved that the V2V problem is in NP and is NP-hard, then, by definition, the V2V problem is NP-complete. As an aid for the reader to understand




| $v_i$ | $s_i$ | $f_i$ | $\mathrm{SOC}_i$ |
|---|---|---|---|
| $v_{1,1}$ | $s_1$ | $f_1$ | 7 |
| $v_{1,2}$ | $s_1$ | $f_4$ | 1 |
| $v_{2,1}$ | $s_2$ | $f_2$ | 10 |
| $v_{2,2}$ | $s_2$ | $f_3$ | 1 |
| $v_{2,3}$ | $s_2$ | $f_4$ | 1 |
| $v_{3,1}$ | $s_3$ | $f_2$ | 10 |
| $v_{3,2}$ | $s_3$ | $f_3$ | 1 |
| $v_{3,3}$ | $s_3$ | $f_4$ | 1 |
| $v_1^{sat}$ | $sat_1$ | $f_1$ | 0 |
| $v_2^{sat}$ | $sat_2$ | $f_2$ | 0 |
| $v_3^{sat}$ | $sat_3$ | $f_3$ | 0 |
| $v_4^{sat}$ | $sat_4$ | $f_4$ | 0 |

Meeting points are denoted as **black nodes**.

$\forall i \in \{1,2,3\}$ :
$\mathrm{MAXSOC}_i = 1$,
$e_i = 1$.

$\forall a$ :
$e_a = 1$,
$d_a = 1$.

$T = 15$.

Green dashed lines show the paths of a witness like $(x_1, x_3 = \mathit{True}, x_2 = \mathit{False})$.

**Fig. 1** Illustration of a reduction from a 3SAT instance to a V2V instance.

the proof, Figure 1 shows an illustration of the reduction from the 3SAT instance shown in equation (3).