Interference effects

Cristian Fernando Rodríguez Cruz

Authors:
A. Flórez[1], **C. Rodriguez**[1], J. Reyes-Vega[1],
J. Jones-Pérez[2].

[1]Universidad de los Andes

[2]Pontificia Universidad Católica del Perú

August 15, 2024

Consider the process

$$A(\vec{\boldsymbol{p}}_1) + B(\vec{\boldsymbol{p}}_2) \longrightarrow C(\vec{\boldsymbol{p}}_3) + D(\vec{\boldsymbol{p}}_4), \tag{1}$$

From the Golden Rule, the cross section is given by

$$\sigma = \frac{S(2\pi)^4}{4\sqrt{(\vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}_2)^2 - (m_1 m_2)^2}} \int |\mathcal{M}|^2 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \frac{d^3\vec{\boldsymbol{p}}_3}{(2\pi)^3 2E_3} \frac{d^3\vec{\boldsymbol{p}}_4}{(2\pi)^3 2E_4}. \tag{2}$$

But, in the CM frame, $\vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2 = 0$, where

$$\sqrt{(\vec{\boldsymbol{p}}_1 \cdot \vec{\boldsymbol{p}}_2)^2 - (m_1 m_2)^2} = E_1 E_2 \left|\vec{\boldsymbol{p}}_1\right|, \tag{3}$$

$$\delta^{(4)}(p_1 + p_2 - p_3 - p_4) = \delta\left(E_1 + E_2 - E_3 - E_4\right) \delta^{(3)}(\vec{\boldsymbol{p}}_3 + \vec{\boldsymbol{p}}_4). \tag{4}$$

Thus

$$\sigma = \left(\frac{1}{8\pi}\right)^2 \frac{S}{(E_1 E_2)\left|\vec{\boldsymbol{p}}_1\right|} \int |\mathcal{M}|^2 \frac{\delta\left(E_1 + E_2 - \sqrt{\vec{\mathbf{p}}_3^2 + m_3^2} - \sqrt{\vec{\mathbf{p}}_3^2 + m_4^2}\right)}{\sqrt{\vec{\mathbf{p}}_3^2 + m_3^2}\sqrt{\vec{\mathbf{p}}_3^2 + m_4^2}} \mathrm{d}\vec{\mathbf{p}}_3 \tag{5}$$

Integrating over the radial part $|\vec{\mathbf{p}}_3|$, we get

$$\sigma = \left(\frac{1}{8\pi}\right)^2 \frac{S|\vec{\boldsymbol{p}}_3|}{(E_1+E_2)^2|\vec{\boldsymbol{p}}_1|} \int |\mathcal{M}|^2 \mathrm{d}\Omega, \tag{6}$$

with

$$|\vec{\boldsymbol{p}}_3| = \frac{1}{2} \frac{\sqrt{((E_1+E_2)^2 - m_3^2 - m_4^2)^2 - 4m_3^2 m_4^2}}{E_1+E_2}, \tag{7}$$

the outgoing momentum in the CM frame.

We prefer work with differential cross section as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \frac{S}{(E_1+E_2)^2} \frac{|\vec{\boldsymbol{p}}_3|}{|\vec{\boldsymbol{p}}_1|} |\mathcal{M}|^2. \tag{8}$$

Note that at this point, we don't need to know the explicit form of the matrix element $\mathcal{M}$, so it is a generic result.

Two body scattering

CM-Frame

Defining $\sqrt{s} = E_1 + E_2$, we have

$$|\vec{p}_3| = \frac{1}{2}\frac{\sqrt{(s-m_3^2-m_4^2)^2-4m_3^2m_4^2}}{s}, \quad |\vec{p}_1|\frac{1}{2}\frac{\sqrt{(s-m_1^2-m_2^2)^2-4m_1^2m_2^2}}{s}. \tag{9}$$

so the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\frac{S}{s}\sqrt{\frac{(s-(m_3+m_4)^2)(s-(m_3-m_4)^2)}{(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}}|\mathcal{M}|^2. \tag{10}$$

In general, there are three Lorentz-invariant useful kinematical variables to describe the scattering process, known as Mandelstam variables:

$$\hat{s} = (p_1+p_2)^2 = (p_3+p_4)^2 = m_1^2+m_2^2+2\vec{\mathbf{p}}_1\cdot\vec{\mathbf{p}}_2 = m_3^2+m_4^2+2\vec{\mathbf{p}}_3\cdot\vec{\mathbf{p}}_4, \tag{11}$$

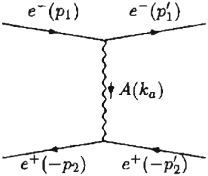
$$\hat{t} = (p_1-p_3)^2 = (p_2-p_4)^2 = m_1^2+m_3^2-2\vec{\mathbf{p}}_1\cdot\vec{\mathbf{p}}_3 = m_2^2+m_4^2-2\vec{\mathbf{p}}_2\cdot\vec{\mathbf{p}}_4, \tag{12}$$

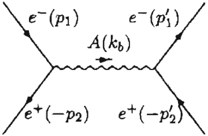
$$\hat{u} = (p_1-p_4)^2 = (p_2-p_3)^2 = m_1^2+m_4^2-2\vec{\mathbf{p}}_1\cdot\vec{\mathbf{p}}_4 = m_2^2+m_3^2-2\vec{\mathbf{p}}_2\cdot\vec{\mathbf{p}}_3. \tag{13}$$

In the CM-frame, $\hat{s} = s = (E_1+E_2)^2$.

QED $e^+e^- \longrightarrow e^+e^-$ scattering

At tree level, there are two Feynman diagrams that contribute to the process





$$\left[\bar{u}_1'(ie\gamma^\mu)u_1\right] iD_{\mu\nu}(k_a) \left[\bar{v}_2(ie\gamma^\nu)v_2'\right]$$
$$= -ie^2 \frac{\left[\bar{u}_1'\gamma^\mu u_1\right] \left[\bar{v}_2\gamma_\mu v_2'\right]}{k_a^2}$$
$$k_a = p_1 - p_1' = p_2' - p_2$$

$$\left[\bar{v}_2(ie\gamma^\mu)u_1\right] iD_{\mu\nu}(k_b) \left[\bar{u}_1'(ie\gamma^\nu)v_2'\right]$$
$$= -ie^2 \frac{\left[\bar{v}_2\gamma^\mu u_1\right] \left[\bar{u}_1'\gamma_\mu v_2'\right]}{k_b^2}$$
$$k_b = p_1 + p_2 = p_2' + p_1'$$

The fermionic exchange between the initial positron and the final electron is the same in both diagrams, so we have a relative minus sign between the two contributions.

$$i\mathcal{M} = ie^2 \left(\frac{\left[\bar{u}_1\gamma^\mu u_1'\right] \left[\bar{v}_2\gamma_\mu v_2'\right]}{(p_1 - p_1')^2} - \frac{\left[\bar{v}_2\gamma^\mu u_1\right] \left[\bar{u}_1'\gamma_\mu v_2'\right]}{(p_1 + p_2)^2} \right), \tag{14}$$

QED $e^+e^- \longrightarrow e^+e^-$ scattering

In terms of the Mandelstam variables, we have

$$i\mathcal{M} = ie^2 \left(\frac{\left[\bar{u}_1 \gamma^\mu u_1'\right] \left[\bar{v}_2 \gamma_\mu v_2'\right]}{\hat{t}} - \frac{\left[\bar{v}_2 \gamma^\mu u_1\right] \left[\bar{u}_1' \gamma_\mu v_2'\right]}{\hat{s}} \right). \tag{15}$$

So, the mean square of the matrix element is

$$\overline{|\mathcal{M}|^2} = \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^2}{4} \left(\frac{T_{11}}{\hat{t}^2} + \frac{T_{22}}{\hat{s}^2} - \frac{T_{12} + T_{21}}{\hat{s}\hat{t}} \right) \tag{16}$$

with

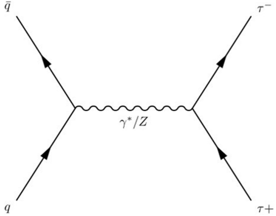
$$T_{11} = 32 \left[(\vec{\mathbf{p}}_1 \cdot \vec{\mathbf{p}}_2)^2 + (\vec{\mathbf{p}}_1 \cdot \vec{\mathbf{p}}_2')^2 + 2m^2(m^2 - \vec{\mathbf{p}}_1 \cdot \vec{\mathbf{p}}_1') \right], \tag{17}$$

$$T_{22} = 32 \left[(\vec{\mathbf{p}}_1 \cdot \vec{\mathbf{p}}_1')^2 + (\vec{\mathbf{p}}_1 \cdot \vec{\mathbf{p}}_2')^2 + 2m^2(m^2 + \vec{\mathbf{p}}_1 \cdot \vec{\mathbf{p}}_2) \right], \tag{18}$$

$$-T_{12} = -T_{21} = 32 \left[(\vec{\mathbf{p}}_1 \cdot \vec{\mathbf{p}}_2') + m^2 \left(\vec{\mathbf{p}}_1 \cdot \vec{\mathbf{p}}_2' + \vec{\mathbf{p}}_1 \cdot \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1 \cdot \vec{\mathbf{p}}_1' \right) + m^4 . \right] \tag{19}$$

Photon and Z-boson interference, $q\bar{q} \longrightarrow \tau^+\tau^-$

At tree level, Consider the process $q\bar{q} \longrightarrow \tau^+\tau^-$ with γ^* and Z-boson contributions in the s-channel.

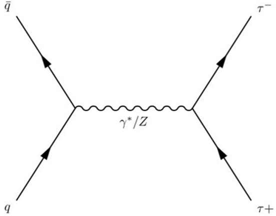


The squared matrix element can be written as

$$
\begin{aligned}
|\mathcal{M}|^2 &= \left|\mathcal{M}_{\gamma^*} + \mathcal{M}_Z\right| \\
&= \left|\mathcal{M}_{\gamma^*}\right|^2 + |\mathcal{M}_Z|^2 + 2\,\mathrm{Re}\left(\mathcal{M}_{\gamma^*}^*\mathcal{M}_Z\right).
\end{aligned}
$$

Photon and Z-boson interference, $q\bar{q} \longrightarrow \tau^+\tau^-$

At tree level, Consider the process $q\bar{q} \longrightarrow \tau^+\tau^-$ with γ^* and Z-boson contributions in the s-channel.



The squared matrix element can be written as

$$
\begin{aligned}
|\mathcal{M}|^2 &= \left|\mathcal{M}_{\gamma^*} + \mathcal{M}_Z\right| \\
&= \left|\mathcal{M}_{\gamma^*}\right|^2 + |\mathcal{M}_Z|^2 + 2\,\mathrm{Re}\left(\mathcal{M}^*_{\gamma^*}\mathcal{M}_Z\right).
\end{aligned}
$$

For the case $q_R\bar{q}_L \longrightarrow \tau_L^+\tau_R^-$, the amplitudes are

$$
\begin{aligned}
\left|\mathcal{M}_{\gamma^*}\right|^2 &= e^4\left[Q^{(f)}Q^{(q)}\right]^2[1+\cos\theta]^2 \\
|\mathcal{M}_Z|^2 &= \frac{s^2 g_Z^4\left[g_R^{(f)}g_R^{(q)}\right]^2}{\left(s-m_Z^2\right)^2+(m_Z\Gamma_Z)^2}[1+\cos\theta]^2 \\
\mathcal{M}^*_{\gamma^*}\mathcal{M}_Z &= \frac{g_Z^2 e^2 Q^{(f)}Q^{(q)}g_R^{(f)}g_R^{(q)}}{\left(s-m_Z^2+i\Gamma_Z\right)}s\,(1+\cos\theta)^2
\end{aligned}
$$