Solutions to Complex Analysis (Stein and Shakarchi)

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Chapter 1

1.1

(a) $|z - z_1| = |z - z_2|$ for $z_1, z_2 \in \mathbb{C}$ describes the line in the complex plane perpendicular the the line segment connecting z_1 and z_2 and passing through its midpoint.
(b) $\frac{1}{z} = \overline{z}$ describes the hyperbola in the complex plane matching $x^2 - y^2 = 1$ in the real plane.
(c) A vertical line intersecting the horizontal, real axis at 3.
(d) All points in the complex plane to the right of the line described in (c), inclusive of the line itself if $\geq$ and exclusive if $>$.
(e) If we let $a = a_0 + ia_1$ and $b = b_0 + ib_1$ be complex numbers, then $Re(ax+b) = a_0x_0 - a_1x_1 + b_0$ for $x = x_0 + ix_1$. Then $Re(ax+b) > 0$ defines half the complex plane divided by the line $a_0x_0 + a_1x_1 + b_0$ which has slope $-a_0/a_1$ and vertical intercept $-b_0/a_1$.
(f) A concave curve matching $y = \sqrt{2x+1}$ in the real plane.
(g) A horizontal line passing through the vertical, imaginary axis at $c \in \mathbb{R}$.

1.2

$$\begin{aligned}
\langle z, w\rangle &= \frac{1}{2}\left((z,w) + (w,z)\right) \\
&= \frac{1}{2}((z,w) + \overline{(z,w)}) \\
&= \frac{1}{2} \cdot 2Re(z,w) \\
&= Re(z,w)
\end{aligned}$$

1.3 Consider solutions of the form $z_k = s^{1/n} e^{i\frac{2\pi k + \phi}{n}}$ for $k = 0, 1, \ldots, n-1$. These are each unique and

$$z_k^n = (s^{1/n})^n \cdot \left(e^{i\frac{2\pi k+\phi}{n}}\right)^n = se^{i(2\pi k + \phi)} = se^{i\phi} = \omega$$

So, there are n total solutions.

1.4 Suppose for purposes of contradiction that $i \succ 0$. Then following the multiplicative rule $-1 = i \cdot i \succ i \cdot 0 = 0$, which in turn implies $-i = -1 \cdot i \succ -1 \cdot 0 = 0$. But this must be a contradiction, since the additive rule implies $0 = -i + i \succ -i + 0 = -i$, and only one version can be true.

The only alternative is that $0 \succ i$, but then $-i = -i + 0 \succ -i + i = 0$. This implies $-1 = -i \cdot -i \succ -i \cdot 0 = 0$, but then $i = -1 \cdot i \succ -1 \cdot 0 = 0$, which contradicts our original assumption. So, no total ordering exists on $\mathbb{C}$.

1.5

(a) Since $\gamma \in \Omega$ and $0 \leq t^* \leq 1$ by its definition, then $z(t^*) \in \Omega$. Suppose that $z(t^*) \in \Omega_1$. Since Ω_1 is open, there exists $\epsilon > 0$ such that $B_\epsilon(z(t^*)) \subset \Omega_1$. The parameterized curve γ is piecewise-smooth and thus continuous, so $\exists \delta > 0$ such that $t \in (t^* - \delta, t^* + \delta)$ implies $z(t) \in B_\epsilon(z(t^*)) \subset \Omega_1$.

But then for any $t \in (t^*, t^* + \delta)$, then $z(s) \in \Omega_1 \ \forall 0 \leq s < t$, which violates t^* being the supremum of the defined set.

If instead $z(t^*) \in \Omega_2$, then the same continuity argument implies $\exists \epsilon, \delta > 0$ such that for $t \in (t^* - \delta, t^* + \delta)$ then $z(t) \in B_\epsilon(z(t^*)) \subset \Omega_2$. But this again contradicts t^* being the supremum of the defined set, since $t \in (t - \delta, t)$ are upper bounds of the set but smaller than t^*. So, Ω cannot be written in this disjoint form and must be connected.

(b) By their definition, $\Omega_1 \cap \Omega_2 = \varnothing$ and $\Omega_1 \cup \Omega_2 = \Omega$. What remains to show is that they are both open sets. Let $z \in \Omega_1$ and denote the curve joining w to z in Ω by γ. Since Ω is open, $\exists \epsilon > 0$ with $B_\epsilon(z) \subset \Omega$. There exists a curve connecting z to any point $u \in B_\epsilon(z)$ that remains entirely in Ω, the simplest being the directly line segment joining them, since it remains entirely in $B_\epsilon(z)$ which itself lies in Ω. For any such u, call the line segment between z and it β, and then there is a curve joining w to u i.e. the composite curve $\gamma \cup \beta$. So, $u \in \Omega_1$, and because u was arbitrary $B_\epsilon(z) \subset \Omega_1$. So, Ω_1 is open.

Now let $z \in \Omega_2$. Again because Ω is open, we can find a ball $B_\epsilon(z) \subset \Omega$. By the same reasoning as above $\forall u \in B_\epsilon(z)$ can be connected to z by a line segment entirely in Ω. Suppose $\exists u \in B_\epsilon(z)$ where $u \in \Omega_1$ i.e. there a curve γ connecting u and w entirely in Ω. This implies that z could also be connected to w with the composite curve of γ and the line segment between it and u, which contradicts the original assumption of $z \in \Omega_2$. Since no such u exists, $B_\epsilon(z) \subset \Omega_2$, and Ω_2 is open. Because Ω is connected, such a decomposition $\Omega = \Omega_1 \cup \Omega_2$ cannot exist, so one of the sets must be empty with the other being Ω itself. We know Ω_1 is non-empty since $w \in \Omega_1$ using the trivial curve connecting w to itself $z(t) = w$ for $t \in [0, 1]$. So $\Omega_1 = \Omega$, which by Ω_1 definition implies Ω is path-connected.

1.6

(a) For any point $w \in \mathcal{C}_z$ then $w \in \Omega$ and there exists ball around it $B_\epsilon(w) \subset \Omega$. Any point in the ball can be connected to z with the composite curve to w plus the line segment connecting it to w. So $B_\epsilon(w) \subset \mathcal{C}_z$ and $\mathcal{C}_z$ is open. By its definition, $\mathcal{C}_z$ is path-connected since any two points can be connected by the composite curves connecting them each to z, which both lie entirely in $\mathcal{C}_z$. Since C_z is open and path-connected, it must be connected by the previous exercise. Clearly, $z \in \mathcal{C}_z$ using the trivial curve connecting z to itself. If $w \in \mathcal{C}_z$, there exists a curve connecting w to z that lies in Ω. That same curve connects z to w in reverse, so $z \in \mathcal{C}_w$. Finally, transitivity follows from forming the composite curve from those connecting w to z and z to η for $w \in \mathcal{C}_z$ and $z \in \mathcal{C}_\eta$.

(b) Suppose there are an infinite number of distinct connected components. Each one is open and nonempty, so $\exists z_\alpha \in \mathcal{C}_{z_\alpha}$ for uncountable collection $\alpha \in \mathcal{A}$ and $\exists B_{\epsilon_\alpha}(z_\alpha) \subset \mathcal{C}_\alpha$. Because rational coordinates are dense in the complex plane, each ball must contain a distinct point with rational coordinates w_α. However, this would imply an uncountable number of rationals, which is a contradiction.

(c) Let $A \subset \mathbb{C}$ be a compact set, so it is closed an bounded. Then there is $K > 0$ such that $A \subset B_K(0)$ and $B_K(0)^c \subset A^c$. Note that any two points $z = re^{i\phi}$ and $w = se^{i\psi}$ in $B_K(0)$ with $r, s \geq K$ can be connected by a curve entirely in $B_K(0)$ i.e. $\gamma(t) = (tr + (1-t)s)e^{i(t\phi + (1-t)\psi)}$ for $t \in [0, 1]$ $(tr + (1-t)s \geq K$ for $r, s \geq K)$. So $B_K(0)^c$ is path-connected and must belong to a single connected component of Ω. Any unbounded component of Ω must intersect $B_K(0)^c$ (otherwise it wouldn't be unbounded), and since connected components are either disjoint or coincide, it must be the same connected component containing $B_K(0)^c$. So, that is the only unbounded component.

1.7

(a)

(b)

1.8 This can shown by expanding both sides of the equation using the definitions for $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ and simplifying to show equality. I don't want to type it all out here.

1.9 Let an arbitrary complex number be described in polar coordinates $z = re^{i\theta}$. We can convert to Euclidean coordinates with the following

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

which implies partial derivatives

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos\theta & \frac{\partial x}{\partial \theta} &= -r\sin\theta \\ \frac{\partial y}{\partial r} &= \sin\theta & \frac{\partial y}{\partial \theta} &= r\cos\theta \end{aligned}$$

Given complex function $f(z) = u(x,y) + iv(x,y)$ for $z = x + iy$, and $x = x(r,\theta)$, $y = y(r,\theta)$, we can write the component functions partials with respect to polar coordinates using the chain-rule

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\ \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}\cdot -r\sin\theta + \frac{\partial u}{\partial y}r\cos\theta \\ \frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta \\ \frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x}\cdot -r\sin\theta + \frac{\partial v}{\partial y}r\cos\theta \end{aligned}$$

By substituting the standard Cauchy-Riemann equations, it is easy to show

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -\frac{1}{r}\frac{\partial v}{\partial r}$$

1.10

$$\begin{aligned} 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} &= \left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) \\ &= \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} + \frac{1}{i}\left(\frac{\partial^2}{\partial y\partial x} - \frac{\partial^2}{\partial x\partial y}\right) \\ &= \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} \\ &= \Delta \end{aligned}$$

since the cross partials are equal $\frac{\partial^2}{\partial x\partial y} = \frac{\partial^2}{\partial y\partial x}$. The calculation for $4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$ leads to a similar result.

1.11 Let $f = u + iv$ where $u(x,y)$ and $v(x,y)$ are the real and imaginary parts for $z = x + iy \in \mathbb{C}$. If f is holomorphic, then u, v satisfy the Cauchy-Riemann equations.

$$\begin{aligned} \Delta u &= \frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} \\ &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \\ &= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial}{\partial y} - \frac{\partial v}{\partial x} \\ &= \frac{\partial^2 v}{\partial x\partial y} - \frac{\partial^2 v}{\partial y\partial x} \\ &= 0 \end{aligned}$$

where the third line follows from substituting in the Cauchy-Riemann equations, and the fifth line follows from the cross derivatives not depending on order. The calculation for imaginary part v is equivalent.

1.12 At the origin

$$\begin{aligned}\frac{\partial u}{\partial x} &= \lim_{h\to 0}\frac{u(h,0)-u(0,0)}{h} = 0\\ \frac{\partial u}{\partial y} &= \lim_{h\to 0}\frac{u(0,h)-u(0,0)}{h} = 0\end{aligned}$$

since $u(\cdot,0) = u(0,\cdot) = 0$. Further, since $f(x+iy) = \sqrt{|x||y|} \in \mathbb{R}$, then $v(x,y) = 0$ and its partials are zero everywhere. So, the Cauchy-Riemann equations are satisfied at these values. Now, consider approaching the origin in the limit along the 45-degree line

$$\lim_{h\to 0}\frac{\sqrt{|h||h|}-\sqrt{|0||0|}}{h+ih} = \frac{1}{1+i} = \frac{1-i}{2}$$

If we consider the limit approaching the origin from the other side along the same 45 degree line

$$\lim_{h\to 0}\frac{\sqrt{|-h||-h|}-\sqrt{|0||0|}}{-h-ih} = -\frac{1}{1+i} = \frac{-1+i}{2}$$

Because the limits differ according to the approach towards the origin of the complex plane, the limit does not exist and so f is not holomorphic at 0.

1.13

(a) $Re(f) = u$ is constant implies $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$. Since f is holomorphic, the Cauchy-Riemann equations hold which implies $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$. Then

$$f'(z) = \frac{\partial f}{\partial z} = \left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)u = 0$$

We now show that $f'(z) = 0$ in an open set implies a constant f. Let $z, w \in \Omega$ and define $u = w - z$. Define for any $\epsilon > 0$

$$E = \{h \in [0,1] : |f(z+hu) - f(z)| \le \epsilon |h||u|\}$$

Since $0 \in E$, the set is nonempty, and we can define $g = \sup E$ with sequence $h_n \uparrow g$. Note that f is holomorphic and thus continuous, so

$$|f(z+gu) - f(z)| \le |f(z+gu) - f(z+h_n u)| + |f(z+h_n u) - f(z)| \le \eta + \epsilon|h_n||u| < \eta + \epsilon|g||u|$$

where η can be made arbitrarily small for large enough n. This implies $g \in E$. Suppose $g < 1$. Then since $f'(z+gu) = 0$, there exists $h \in (g,1)$ such that

$$|f(z+hu) - f(z+gu)| \le \epsilon|h-g||u|$$

Then

$$\begin{aligned}|f(z+hu) - f(z)| &\le |f(z+hu) - f(z+gu)| + |f(z+gu) - f(z)|\\ &\le \epsilon|h-g||u| + \epsilon|g||u|\\ &= \epsilon|h||u|\end{aligned}$$

but this violates $g = \sup E$. So $\sup E = g = 1$ and $|f(w) - f(z)| \le \epsilon|w-z|$. Because ϵ can be made arbitrarily small, this implies $f(w) = f(z)$, and so f is constant on Ω.

(a) $Im(f) = v$ is constant implies $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$. Since f is holomorphic, the Cauchy-Riemann equations hold which implies $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$. Then

$$f'(z) = \frac{\partial f}{\partial z} = \left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)u = 0$$

By the same reasoning as in (a) then f is constant.

(c) Since $|f|$ is constant, so is $|f|^2$, and

$$\begin{aligned}
0 = \frac{\partial}{\partial z}|f|^2 &= 2u\frac{\partial u}{\partial z} + 2v\frac{\partial v}{\partial z} \\
&= u\left(\frac{\partial u}{\partial x} + \frac{1}{i}\frac{\partial u}{\partial y}\right) + v\left(\frac{\partial v}{\partial x} + \frac{1}{i}\frac{\partial v}{\partial y}\right) \\
&= u\left(\frac{\partial u}{\partial x} + \frac{1}{i}\frac{\partial u}{\partial y}\right) + v\left(\frac{-\partial u}{\partial y} + \frac{1}{i}\frac{\partial u}{\partial x}\right) \\
&= u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - i\left(u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x}\right)
\end{aligned}$$

So both $u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0$ and $u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0$ must hold everywhere in Ω. Then

$$\begin{aligned}
0 = u\left(u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x}\right) &= u^2\frac{\partial u}{\partial y} + uv\frac{\partial u}{\partial x} \\
&= u^2\frac{\partial u}{\partial y} + v^2\frac{\partial u}{\partial y} \\
&= (u^2+v^2)\frac{\partial u}{\partial y}
\end{aligned}$$

where the second line comes from substituting in $u\frac{\partial u}{\partial x} = v\frac{\partial u}{\partial y}$. This implies either $u^2+v^2 = 0$ everywhere, so $u = v = 0$ and $f(z) = 0$, which is constant. Or $\frac{\partial u}{\partial y} = 0$. Using the earlier substitution equation, then $u = 0$ or $\frac{\partial u}{\partial x} = 0$. In the former case, if $|f|^2 = u^2+v^2 = v^2$ is constant, then v is constant and so is $f = iv$. In the second case, we have $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ and so $f'(z) = 0$. The reasoning from the previous two parts follows to show that f must be constant.

1.14 We proceed by induction. Note that

$$\begin{aligned}
&a_N B_N - a_{N-1}B_{N-2} - (a_N - a_{N-1})B_{N-1} \\
&= a_N(B_N - B_{N-1}) + a_{N-1}(B_{N-1} - B_{N-2}) \\
&= a_N b_N + a_{N-1}b_{N-1} \\
&= \sum_{n=N-1}^{N} a_n b_n
\end{aligned}$$

This shows the state for the base case $M = N-1$. Suppose it holds for all $k \geq M$ for $M \in [1, N-1]$. We want to show the statement also holds for $M-1$. Denote the sum on the right-hand side by S_M. Then

$$\begin{aligned}
S_{M-1} &= a_N B_N - a_{M-1}B_{M-2} - \sum_{n=M-1}^{N-1}(a_{n+1} - a_n)B_n \\
&= \left(a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1}(a_{n+1} - a_n)B_n\right) + (a_{M-1}B_{M-1} - a_{M-1}B_{M-2}) \\
&= S_M + a_{M-1}b_{M-1} \\
&= \sum_{n=M}^{N} a_n b_n + a_{M-1}b_{M-1} \\
&= \sum_{n=M-1}^{N} a_n b_n
\end{aligned}$$

where the substitution of S_M follows from the inductive step we assumed earlier. This proves the formula for all $M \in [0, N-1]$.

1.15

1.17 Let $\epsilon > 0$ and $N \in \mathbb{N}$ be such that $|\frac{|a_{n+1}|}{|a_n|} - L| < \epsilon$ for all $n \geq N$. Note that for any such $n \leq N$

$$|a_n| = \frac{|a_n|}{|a_{n-1}|}\frac{|a_{n-1}|}{|a_{n-2}|}\cdots\frac{|a_{N+1}|}{|a_N|}|a_N|$$

But each $\frac{|a_i|}{|a_{i-1}|} < |L| + \epsilon = L + \epsilon$ for $i \in [N+1, n]$, so

$$|a_n| < |L+\epsilon|^{n-N}|a_N|$$

or $|a_n|^{1/n} < (L+\epsilon)^{1-N/n}|a_N|^{1/n}$. Because we can make n arbitrarily large, this shows that

$$\lim_{n\to\infty} |a_n|^{1/n} \leq L + \epsilon$$

But ϵ was chosen arbitrarily, so $\lim_{n\to\infty} |a_n|^{1/n} \leq L$. A similar line of reasoning can show that $\lim_{n\to\infty} |a_n|^{1/n} \geq L$ (see that $|L| - |a_i/a_{i-1}| < \epsilon$ and so $|a_i/a_{i-1}| > L - \epsilon$), and so $\lim_{n\to\infty} |a_n|^{1/n} = L$.

1.18 Let z_0 be some point in the disc of convergence. Then:

$$\begin{aligned}
f(z) = f(z_0 + (z - z_0)) &= \sum_{n=0}^{\infty} a_n(z_0 + (z - z_0))^n \\
&= \sum_{n=0}^{\infty} a_n \left(\sum_{k=0}^{n} \binom{n}{k} z_0^k (z - z_0)^{n-k}\right) \\
&= \sum_{n=0}^{\infty} b_n (z - z_0)^n
\end{aligned}$$

where $b_n = \sum_{k=n}^{\infty} a_k \binom{k}{n} z_0^{k-n}$. This defines a power series expansion for f centered around z_0.

1.19

(a) If $S_N = \sum_{n=0}^{N} nz^n$ is the partial sum, then $|S_{N+1} - S_N| = |(n+1)z^{n+1}| = n+1$ since $|z| = 1$ on the unit circle. If the difference between partial sums is unbounded, then the sequence is not Cauchy, and thus it does not converge.

(b) We will prove the sequence of partial sums is Cauchy to show they then converge. Let $\epsilon > 0$, and consider for $N > M$.

$$\begin{aligned}
|S_N - S_M| &= \left|\frac{z^{M+1}}{(M+1)^2} + \frac{z^{M+2}}{(M+2)^2} + \ldots + \frac{z^N}{N^2}\right| \\
&< \left|\frac{z^{M+1}}{(M+1)^2}\right| + \left|\frac{z^{M+2}}{(M+2)^2}\right| + \ldots + \left|\frac{z^N}{N^2}\right| \\
&= \frac{1}{(M+1)^2} + \frac{1}{(M+2)^2} + \ldots + \frac{1}{N^2} \\
&< \frac{1}{(M+1)M} + \frac{1}{(M+2)(M+1)} + \ldots + \frac{1}{N(N-1)}
\end{aligned}$$

Note that for $M \geq 2$ then $\frac{1}{(M+1)M} = \frac{1}{M} - \frac{1}{M+1}$, so the above summation becomes telescoping sequence

$$\begin{aligned}
&\left(\frac{1}{M} - \frac{1}{M+1}\right) + \left(\frac{1}{M+1} - \frac{1}{M+2}\right) + \ldots + \left(\frac{1}{N-1} - \frac{1}{N}\right) \\
&= \frac{1}{M} - \frac{1}{N} \\
&< \frac{1}{M}
\end{aligned}$$

Then for arbitrary $\epsilon > 0$, if $N, M > 1/\epsilon$, then Cauchy sequence condition holds for the partial summations. So, the series converges.

(c) For $z \neq 1$, we can use summation between parts to write the difference in partial sums

$$
\begin{aligned}
S_N - S_M = \sum_M^N \frac{z^n}{n} &= \frac{1}{N}\sum^N z^n - \frac{1}{M}\sum^{M-1} z^n - \sum_M^{N-1}\left(\frac{1}{n+1} - \frac{1}{n}\right)\sum^n z^k \\
&= \frac{1}{N}\cdot\frac{z^{N+1}-1}{z-1} - \frac{1}{M}\cdot\frac{z^M-1}{z-1} - \sum_M^{N-1}\left(\frac{1}{n+1} - \frac{1}{n}\right)\cdot\frac{z^{n+1}-1}{z-1}
\end{aligned}
$$

since $\sum^N z^n = \frac{z^{N+1}-1}{z-1}$ for $z \neq 1$. Note that then

$$
\begin{aligned}
|S_N - S_M| &< \frac{1}{N}\left|\frac{z^{N+1}-1}{z-1}\right| + \frac{1}{M}\left|\frac{z^M-1}{z-1}\right| + \left(\left(\frac{1}{M} - \frac{1}{M+1}\right)\left|\frac{z^{M+1}-1}{z-1}\right| + \ldots + \left(\frac{1}{N} - \frac{1}{N+1}\right)\left|\frac{z^{N+1}-1}{z-1}\right|\right) \\
&< \frac{2}{|z-1|}\left[\frac{1}{N} + \frac{1}{M} + \left(\frac{1}{M} - \frac{1}{M+1}\right) + \ldots + \left(\frac{1}{N} - \frac{1}{N+1}\right)\right] \\
&= \frac{2}{|z-1|}\left(\frac{1}{N} + \frac{1}{M} + \frac{1}{M} - \frac{1}{N+1}\right) \\
&< \frac{2}{|z-1|}\cdot\frac{3}{M}
\end{aligned}
$$

since $|\frac{z^k-1}{z-1}| < \frac{|z^k|+1}{|z-1|} = \frac{2}{|z-1|}$ for any arbitrary $k \geq 1$. For large enough M, this can be made arbitrarily small. So, the sequence of partial sums are Cauchy and convergent for $z \neq 1$. For $z = 1$, the series becomes $\sum_{n=0}^\infty \frac{1}{n}$, which is known to be divergent.

1.20

1.24 We can parametrize the reverse orientation curve γ^- using $w(t) = z(a+b-t)$ for $t \in [a, b]$. Then

$$
\begin{aligned}
\int_{\gamma^-} f(z)dz &= \int_a^b f(w(t))w'(t)dt \\
&= \int_a^b f(z(a+b-t))\,\frac{d}{dt}z(a+b-t)dt
\end{aligned}
$$

Substituting $\tilde{t} = a+b-t$ results in

$$
\begin{aligned}
\int_b^a f(z(\tilde{t}))\,\frac{d}{dt}z(\tilde{t})dt &= -\int_a^b f(z(\tilde{t}))z'(\tilde{t})d\tilde{t} \\
&= -\int_\gamma f(z)dz
\end{aligned}
$$

This proves the result.

1.25

(a) For any $n \neq -1$, the function $f(z) = z^n$ has primitive $F(z) = \frac{1}{n+1}z^{n+1}$ everywhere in the complex plane, excluding the origin. So, in that case

$$
\int_\gamma z^n dz = 0
$$

when γ is a closed circle containing the origin. For $n = -1$, there is no such primitive and

$$
\begin{aligned}
\int_\gamma \frac{1}{z}dz &= \int_0^{2\pi}\frac{ie^{it}}{e^{it}}dt \\
&= \int_0^{2\pi} idt \\
&= 2\pi i
\end{aligned}
$$

(b) Because $f(z) = z^n$ for $n \in \mathbb{Z}$ has primitive $F(z) = \frac{1}{n+1} z^{n+1}$ everywhere excluding the origin, for any closed circle γ not containing the origin

$$\int_\gamma z^n dz = 0$$

(c) We can rewrite

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right)$$

so that the integral becomes

$$\frac{1}{a-b}\left(\int_\gamma \frac{1}{z-a} dz - \int_\gamma \frac{1}{z-b} dz\right)$$

Note that the first integral is equivalent to the integral of $\frac{1}{z}$ around a circle centered at $-a$ of radius r, and the second is the same around a circle centered at $-b$ of radius r. Because $r < |b|$, the second integral follows a circle not containing the origin, so by the previous section, this is equal to zero. However, $r > |a|$, so the first integral does follow a circle containing the origin.

1.26 Fix point $w_0 \in \Omega$, so that for any arbitrary $w \in \Omega$, there exists a path $\gamma \subset \Omega$ connecting w and w_0 (since Ω is open and connected and thus path-connected). For any two primitives F and G for f

$$F(w) - F(w_0) = \int_\gamma f dz = G(w) - G(w_0)$$

which implies $F(w) = G(w) + F(w_0) - G(w_0)$. Since $F(w_0) - G(w_0)$ is constant, this proves the statement.

Chapter 2

Exercises

2.1 Because the function $f(z) = e^{-z^2}$ is entire, the integral

$$\int_\gamma e^{-z^2} dz = 0$$

where γ is the suggested closed contour. We can divide it into component parts

$$\int_\gamma e^{-z^2} dz = \int_{\gamma_1} e^{-z^2} dz + \int_{\gamma_2} e^{-z^2} dz + \int_{\gamma_3} e^{-z^2} dz$$

where γ_1 is the outwards radial path, γ_2 is the arc along the circle, and γ_3 is the inwards radial path returning to the origin. Note that

$$\begin{aligned}
\lim_{R\to\infty} \int_{\gamma_1} e^{-z^2} dz &= \int_0^\infty e^{-x^2} dx \\
&= \frac{1}{2}\int_{-\infty}^\infty e^{-x^2} dx \\
&= \frac{\sqrt{\pi}}{2}
\end{aligned}$$

where the second equality follows from $f(x) = e^{-x^2}$ begin an even function. Next

$$\begin{aligned}
\int_{\gamma_3} e^{-z^2} dz &= \int_R^0 e^{-R^2 e^{i\frac{\pi}{2}}} e^{i\frac{\pi}{4}} dR \\
&= e^{i\frac{\pi}{4}} \int_R^0 \cos(R^2) - i\sin(R^2) dR
\end{aligned}$$

Lastly, we can show the integral along the second leg of the contour disappears in the limit

$$\begin{aligned}\int_{\gamma_2} e^{-z^2} dz &= \int_{\gamma_2} e^{-R^2 e^{i2\theta}} dz \\ &= \int_{\gamma_2} e^{-R^2 \cos(2\theta)} e^{-R^2 i \sin(2\theta)} dz\end{aligned}$$

Then

$$\begin{aligned}\left| \int_{\gamma_2} e^{-z^2} dz \right| &= \left| \int_0^{\pi/4} e^{-R^2 \cos(2\theta)} e^{-R^2 i \sin(2\theta)} iRe^{i\theta} d\theta \right| \\ &\leq R \int_0^{\pi/4} e^{-R^2 \cos(2\theta)} d\theta \\ &\leq R \int_0^{\pi/4} e^{-R^2 (1-\frac{4}{\pi}\theta)} d\theta \\ &= \frac{\pi}{4R}\left(1 - e^{-R^2}\right)\end{aligned}$$

which goes to zero as $R \to \infty$. Note that third line follows from $\cos\theta \geq 1 - \frac{2}{\pi}\theta$ on the interval $\theta \in [0, \frac{\pi}{2}]$. Substituting in our results are rearranging we can show in the limit that

$$\begin{aligned}e^{i\frac{\pi}{4}} \left(\int_\infty^0 \cos(R^2) - i \int_\infty^0 \sin(R^2) dR \right) &= \int_{\gamma_3} e^{-z^2} dz \\ &= -\int_{\gamma_1} e^{-z^2} dz \\ &= -\frac{\sqrt{\pi}}{2}\end{aligned}$$

After rearranging

$$\int_0^\infty \cos(R^2) - i \int_0^\infty \sin(R^2) dR = \frac{\sqrt{\pi}}{2e^i \pi/4} = \frac{\sqrt{2\pi}}{4} - i\frac{\sqrt{2\pi}}{4}$$

Matching real and imaginary parts gives us the desired result.

2.2 Note that the suggested integral can be simplified

$$\begin{aligned}\int_{-\infty}^\infty \frac{e^{ix} - 1}{x} dx &= \int_{-\infty}^\infty \frac{\cos x - 1}{x} dx + i \int_{-\infty}^\infty \frac{\sin x}{x} dx \\ &= i \int_{-\infty}^\infty \frac{\sin x}{x} dx\end{aligned}$$

where the last equality holds because $\frac{\cos x - 1}{x}$ is an odd function. So $\frac{1}{2i}\int_{-\infty}^\infty \frac{e^{ix}-1}{x} dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin x}{x} dx = \int_0^\infty \frac{\sin x}{x} dx$ since $\frac{\sin x}{x}$ is an even function. Consider the integral

$$\int_\gamma \frac{e^{iz} - 1}{z} dz$$

where γ is the suggested indented semicircle contour, which we divide four parts, the two line segments on the real line, the inner semicircle γ_ϵ, and the outer semicircle γ_R. Because the integrand is holomorphic everywhere excluding the origin, the integral and thus the sum of the four integrals along each component contour must equal zero. For the inner semicircle, we can use the power series to rewrite

$$\frac{e^{iz} - 1}{z} = i + \frac{(iz)^2}{2z} + \frac{(iz)^3}{3!z} + \ldots = i + E(z)$$

where $E(z)$ is bounded as $z \to 0$ (which occurs at every point on inner semicircle as $\epsilon \to 0$). This follows since

$$\begin{aligned}|E(z)| &\leq \frac{1}{2}|z| + \frac{1}{3!}|z|^2 + \frac{1}{4!}|z|^3 + \dots \\ &\leq |z| + |z|^2 + |z|^3 + \dots \\ &= \frac{1}{1-|z|}\end{aligned}$$

which equals $\frac{1}{1-\epsilon}$ on γ_ϵ. So

$$\begin{aligned}\left|\int_{\gamma_\epsilon} \frac{e^{iz}-1}{z} dz\right| &= \left|\int_{\gamma_\epsilon} i + E(z) dz\right| \\ &\leq i \cdot \pi\epsilon + \frac{\pi\epsilon}{1-\epsilon}\end{aligned}$$

which goes to zero in the limit. Finally, we can write the outer contour integral

$$\begin{aligned}\int_{\gamma_R} \frac{e^{iz}-1}{z} dz &= \int_0^\pi \frac{e^{iRe^{i\theta}}-1}{Re^{i\theta}} iRe^{i\theta} d\theta \\ &= i\int_0^\pi e^{iRe^{i\theta}} - 1 d\theta \\ &= -i\pi + i\int_0^\pi e^{-R\sin\theta} e^{iR\cos\theta} d\theta\end{aligned}$$

We can bound the remaining integral

$$\begin{aligned}\left|\int_0^\pi e^{-R\sin\theta} e^{iR\cos\theta} d\theta\right| &\leq \int_0^\pi e^{-R\sin\theta} d\theta \\ &= 2\int_0^{\pi/2} e^{-R\sin\theta} d\theta \\ &\leq 2\int_0^{\pi/2} e^{-R\frac{2}{\pi}\theta} d\theta \\ &= -\frac{\pi}{R}\left(e^{-R} - 1\right)\end{aligned}$$

which goes to zero as $R \to \infty$. So

$$\begin{aligned}\int_0^\infty \frac{\sin x}{x} dx &= \frac{1}{2i}\int_{-\infty}^\infty \frac{e^{ix}-1}{x} dx \\ &= \frac{1}{2i} \cdot - \lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}-1}{z} dz \\ &= \frac{1}{2i} \cdot i\pi \\ &= \frac{\pi}{2}\end{aligned}$$

2.3 We consider the integral of $f(z) = e^{-Az}$ around the circle sector of radius R with angle ω, and taking the limit as $R \to \infty$. The integral around the whole closed contour must be equal to zero since e^{-Az} is an entire function. We can look at its component parts, the first of which is the segment along the real axis

$$\int_0^R e^{-Ax} dx = -\frac{1}{A}(e^{-AR} - 1)$$

which goes to $1/A$ as $R \to \infty$. The radial segment at angle ω is

$$\begin{aligned}
\int_R^0 e^{-A\cdot re^{i\omega}} e^{i\omega} dr &= e^{i\omega} \int_R^0 e^{-Ar\cos\omega} e^{-Ari\sin\omega} dr \\
&= e^{i\omega} \int_R^0 e^{-ar}\cos(-br) + ie^{-ar}\sin(-br) dr \\
&= e^{i\omega}\left(- \int_0^R e^{-ar}\cos(br)dr + i\int_0^R e^{-ar}\sin(br)dr \right)
\end{aligned}$$

Taking $R \to \infty$, this has the integrals we are after, which we be able to identify from the real and imaginary parts shortly. The last component is the integral along the arc contour

$$\begin{aligned}
\left| \int_0^\omega e^{-ARe^{i\theta}} iRe^{i\theta} d\theta \right| &\leq R\int_0^\omega e^{-AR\cos\theta} d\theta \\
&\leq R\int_0^\omega e^{-AR(1-\frac{2}{\pi}\theta)} d\theta \\
&= R \cdot \frac{\pi}{2AR}(e^{-AR(1-\frac{2}{\pi}\omega)} - e^{-AR})
\end{aligned}$$

which goes to zero as $R \to \infty$. So, putting all the parts together we have

$$\begin{aligned}
\frac{1}{A} &= e^{i\omega}\left(\int_0^\infty e^{-ar}\cos(br)dr - i\int_0^\infty e^{-ar}\sin(br)dr \right) \\
\Rightarrow \frac{e^{-i\omega}}{A} = \frac{a-ib}{a^2+b^2} &= \int_0^\infty e^{-ar}\cos(br)dr - i\int_0^\infty e^{-ar}\sin(br)dr
\end{aligned}$$

Matching the real and imaginary parts then

$$\begin{aligned}
\int_0^\infty e^{-ar}\cos(br)dr &= \frac{a}{a^2+b^2} \\
\int_0^\infty e^{-ar}\sin(br)dr &= \frac{b}{a^2+b^2}
\end{aligned}$$

2.4 We write $\xi = a + ib$ for $a, b \in \mathbb{R}$. Then

$$\begin{aligned}
\int_{-\infty}^\infty e^{-\pi x^2} e^{2\pi i x\xi} dx &= \int_{-\infty}^\infty e^{-\pi x^2} e^{-2\pi xb} e^{2\pi i xa} dx \\
&= e^{\pi b^2} \int_{-\infty}^\infty e^{-\pi(x+b)^2} e^{2\pi i xa} dx \\
&= e^{\pi b^2 - 2\pi i ab} \int_{-\infty}^\infty e^{-\pi(x+b)^2} e^{2\pi i (x+b)a} dx
\end{aligned}$$

where the second line follows from completing the square with the first two exponential terms, and the third line follows from adding and subtracting to get $(x + b)$ in the last exponential. Because the integral is taken over the entire real line, any translation by b doesn't matter, and it is just equal to $e^{-\pi a^2}$ from the result earlier in the chapter. The integral thus simplifies to

$$\begin{aligned}
e^{\pi b^2 - 2\pi i ab - \pi a^2} &= e^{-\pi(a^2 + 2iab - b^2)} \\
&= e^{\pi(a+ib)^2} \\
&= e^{\pi\xi^2}
\end{aligned}$$

which is the desired result.

2.5 Define complex function $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy \in \mathbb{C}$. Then given that $dz = dx + idy$, we can write the contour integral

$$\begin{aligned}
\int_T f(z)dz &= \int_T (u(x,y) + iv(x,y))(dx + idy) \\
&= \int_T u(x,y)dx - v(x,y)dy + i\int_T v(x,y)dx + u(x,y)dy
\end{aligned}$$

We can apply Green's theorem for each of these two integrals, since f is holomorphic, and thus its real and imaginary parts satisfy the Cauchy-Riemann equations and define an analogous continuously differentiable map in the real plane. Then

$$\begin{aligned}\int_T u(x,y)dx - v(x,y)dy &= \int_{Interior\ T} -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} dxdy = 0 \\ \int_T v(x,y)dx + u(x,y)dy &= \int_{Interior\ T} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} dxdy = 0\end{aligned}$$

where the integrand of both integrals after applying Green's theorem is equal to zero using the Cauchy-Riemann equations. This gives us the final desired result that

$$\int_T f(z)dz = 0$$

2.6

2.7 From the Cauchy integral formulas we have

$$f'(0) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^2} d\zeta$$

for $0 < r < 1$ so that the circle $|\zeta| = r$ is contained in the interior of $\mathbb{D}$. We can substitute in $-\zeta$ into the integral to get a similar equality

$$f'(0) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(-\zeta)}{(-\zeta)^2} d(-\zeta) = \frac{1}{2\pi i}\int_{|\zeta|=r} -\frac{f(-\zeta)}{\zeta^2} d\zeta$$

Adding these two together, we have

$$\begin{aligned}2f'(0) = f'(0) + f'(0) &= \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^2} d\zeta + \frac{1}{2\pi i}\int_{|\zeta|=r} -\frac{f(-\zeta)}{\zeta^2} d\zeta \\ &= \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta) - f(-\zeta)}{\zeta^2} d\zeta\end{aligned}$$

The magnitude of the right-hand side can be bounded as

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta) - f(-\zeta)}{\zeta^2} d\zeta\right| &\leq \frac{1}{2\pi} \cdot \frac{\sup_{\zeta \in C} |f(\zeta) - f(-\zeta)|}{r^2} \cdot 2\pi r \\ &= \frac{\sup_{\zeta \in C} |f(\zeta) - f(-\zeta)|}{r} \\ &\leq \frac{d}{r}\end{aligned}$$

where C denotes the circle $|\zeta| = r$. Since $0 < r < 1$ is arbitrary, taking the limit as $r \to 1$ gives the result $2f'(0) \leq d$.

2.8 Fix some $0 < \epsilon < 1$. Using Cauchy inequalities, we know

$$|f^{(n)}(x)| \leq \frac{n!}{\epsilon^n}\|f\|_C$$

where C is the circle of radius ϵ centered at $x \in \mathbb{R}$. Because $0 < \epsilon < 1$, this circle is contained in the strip indicated in the problem, so that $\|f\|_C = \sup_{z\in C}|f(z)| \leq \sup_{z\in C} A(1+|z|)^\eta$. Suppose $\eta > 0$, so that this supremum is reached by the point on the circle along the real axis furthest from the origin. This is exactly $x + \epsilon$ if $x > 0$ or $x - \epsilon$ if $x < 0$, so that

$$\begin{aligned}\|f\|_C \leq A(1+\epsilon+|x|)^\eta &\leq A(1+\epsilon+|x|+\epsilon|x|)^\eta \\ &= A(1+\epsilon)^\eta(1+|x|)^\eta\end{aligned}$$

So we have $|f^n(x)| \leq \frac{n!}{\epsilon^n} A(1+\epsilon)^\eta (1+|x|)^\eta = A_n(1+|x|)^\eta$ for $A_n = n! A \frac{(1+\epsilon)^\eta}{\epsilon^n}$. If instead $\eta < 0$, the supremum is reached at the point on the circle *closest* to the origin i.e. $x+\epsilon$ for $x<0$ and $x-\epsilon$ for $x>0$. Then

$$\begin{aligned}\|f\|_C \leq A(1+|x|-\epsilon)^\eta &\leq A(1+|x|-\epsilon-\epsilon|x|)^\eta \\ &= A(1-\epsilon)^\eta (1+|x|)^\eta\end{aligned}$$

In this case $|f^n(x)| \leq A_n(1+|x|)^\eta$ for $A_n = n! A \frac{(1-\epsilon)^\eta}{\epsilon^n}$. Finally, the case of $\eta = 0$ is trivial with $|f^n(x)| \leq \frac{n! A}{\epsilon^n}$.

2.9

2.10 No, because we know that polynomials are holomorphic functions, and any function that is converged to uniformly by a sequence of holomorphic functions must also itself be holomorphic. However, we can come up with examples of functions on the closed unit disc which are continuous but not holomorphic such as $f(z) = \overline{z}$. This is a counterexample to the claim.

2.11

(a) Note that $\left|\frac{R^2}{\overline{z}}\right| = \frac{R^2}{|z|} > R$, so w is outside the closed disc of radius R around the origin. Because $\frac{f(\zeta)}{\zeta - w}$ is holomorphic everywhere except at w, then the suggest closed integral around the circle of radius R must be zero. Now expand the real part of the integral of interest

$$\begin{aligned}Re\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right) &= \frac{1}{2}\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z} + \frac{Re^{-i\theta}+\overline{z}}{Re^{-i\theta}-\overline{z}}\right) \\ &= \frac{1}{2}\left(\frac{Re^{i\theta}}{Re^{i\theta}-z} + \frac{z}{Re^{i\theta}-z} + \frac{Re^{-i\theta}}{Re^{-i\theta}-\overline{z}} + \frac{\overline{z}}{Re^{-i\theta}-\overline{z}}\right)\end{aligned}$$

Note that the last expression can be rewritten

$$\frac{\overline{z}}{Re^{-i\theta}-\overline{z}} = \frac{Re^{i\theta}}{\frac{R^2}{\overline{z}} - Re^{i\theta}} = \frac{-Re^{i\theta}}{Re^{i\theta}-w}$$

If we consider the integral involving only this term after separating integrals across the addition we have

$$\int_0^{2\pi} f(Re^{i\theta}) \cdot \frac{-Re^{i\theta}}{Re^{i\theta}-w} d\theta = \int_{|\zeta|=R} \frac{-f(\zeta)}{\zeta - w} d\zeta = 0$$

which is exactly the integral we argued was zero in the beginning. Because that integral equals zero, adding or subtracting that corresponding term in the expansion above won't change the entire integral. We use this below

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta}) \cdot \frac{1}{2}\left(\frac{Re^{i\theta}}{Re^{i\theta}-z} + \frac{z}{Re^{i\theta}-z} + \frac{Re^{-i\theta}}{Re^{-i\theta}-\overline{z}} + \frac{\overline{z}}{Re^{-i\theta}-\overline{z}}\right) d\theta \\ =&\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta}) \cdot \frac{1}{2}\left(\frac{Re^{i\theta}}{Re^{i\theta}-z} + \frac{z}{Re^{i\theta}-z} + \frac{Re^{-i\theta}}{Re^{-i\theta}-\overline{z}} - \frac{\overline{z}}{Re^{-i\theta}-\overline{z}}\right) d\theta \\ =&\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta}) \cdot \frac{1}{2}\left(\frac{2Re^{i\theta}}{Re^{i\theta}-z} + \frac{z-Re^{i\theta}}{Re^{i\theta}-z} + \frac{Re^{-i\theta}-\overline{z}}{Re^{-i\theta}-\overline{z}}\right) d\theta \\ =&\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta}) \frac{Re^{i\theta}}{Re^{i\theta}-z} d\theta \\ =&\frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)}{\zeta - z} d\zeta\end{aligned}$$

where the second line follows from replacing the $\frac{\overline{z}}{Re^{-i\theta}-\overline{z}}$ with its negative since both integrals involving those terms are trivially zero as argued above. The third line follows from adding and subtracting $\frac{Re^{i\theta}}{Re^{i\theta}-z}$, and the remainder simplifies to the original Cauchy integral formula.

(b)

2.12

(a)

(b)

2.13 For any $z \in \mathbb{C}$ and by the assumption let $n \geq 0$ be the smallest integer such that $c_n = 0$ in the power series expansion around z. This implies $f^{(n)}(z) = 0$. Because this assumption works for any $z \in \mathbb{C}$, we can defined a function $N : \mathbb{C} \to \mathbb{N} \cup \{0\}$ that maps z to the smallest zero coefficient in the power series expansion for f centered at z - this is well-defined.

Now fix $z, w \in \mathbb{C}$ and consider the line segment between them in the complex plane, which I denote by $l(z, w) \subset \mathbb{C}$. This is an uncountable set, and the function N restricted to the line segment then maps an uncountable set to the natural numbers (and zero), which is countably infinite. For each $n \in \mathbb{N} \cup \{0\}$, we can consider the corresponding preimage $N^{-1}(n) \subset l(z, w)$. There must exists an $n \in \mathbb{N} \cup \{0\}$ such that $N^{-1}(n)$ has an infinite number of elements. If not, stacking each of the preimages $N^{-1}(0)$, $N^{-1}(1)$, $N^{-1}(2)$, ..., of which there are countably infinite and each of which contains a finite number of elements, we could construct a one-to-one ordering mapping the countable naturals to $l(z, w)$. This is clearly a contradiction, since $l(z, w)$ is uncountable.

Let $n \in \mathbb{N} \cup \{0\}$ be such that $N^{-1}(n)$ has an infinite number of elements on the line segment. Because the line segment is bounded in the complex plane, we can construct a convergent sequence of points in $N^{-1}(n)$ using an argument à la Bolzano-Weierstrass. Because $N^{-1}(n)$ has an infinite number of elements, we can divide the line segment $l(z, w)$ in half and argue that one of the halves must have an infinite number of $N^{-1}(n)$ points on it. If not, then both would have a finite number of points, which would combine to a finite number of $N^{-1}(n)$, contradicting its infinite cardinality. Denote $l^{(1)}(z, w)$ the half with the infinite points (and $l^{(0)}(z, w)$ to be the original entire line segment) and pick a point from that half z_1. The same logic can be applied to divide $l^1(z, w)$ in halves, one of which contains an infinite number of $N^{-1}(n)$ points - $l^{(2)}(z, w)$ - from which we pick another point in the sequence z_2. We can construct a sequence of points $\{z_k\}_{k=0}^{\infty}$ (let z_0 just be the some arbitrary point in $N^{-1}(n)$ on the original line segment) each of which belongs to $N^{-1}(n)$, so that $f^{(n)}(z_k) = 0$. Further, later sections of the sequence fall on increasingly restricted portions of the line segment so that

$$|z_k - z_l| \leq 2^{-m} L$$

for all $k, l \geq m$ where $L = \|z - w\|$ is the length of the original line segment. So, the sequence is Cauchy and thus converges to some $v \in l(z, w)$ since $\mathbb{C}$ is complete and the line segment is closed.

So, we have shown that $f^{(n)}(z_k) = 0$ where $\{z_k\}$ is a sequence of distinct points with limit point v. Thus, $f^{(n)}$ is identically zero, and all higher derivatives $f^{(k)}$ for $k \geq n$ are zero. The power series expansion for f around point $z_0 \in \mathbb{C}$ then must take the form

$$f(z) = a_0 + a_1(z - z_0) + \ldots + a_{n-1}(z - z_0)^{n-1}$$

Since f is entire and this power series has a radius of convergence of the entire complex plane, then f is a polynomial.

2.14

2.15 The approach is to show that f extended to the entire complex plane using the hint is both entire and bounded, in which case we can apply Liouville's theorem to get the desired result. Now suppose $|z|$, $|z_0| > 1$ lie in the exterior of the unit disc, and then $1/|\overline{z}| < 1$ and $1/|\overline{z_0}| < 1$ both lie in the interior of the disc. Because

Problems

2.1

(a)

(b)

2.2