As we will show in the following section, $p = \dot{\phi}$, $q = \dot{\theta}$, and $r = \dot{\psi}$ only at the instant that $\phi = \theta = 0$. Generally, the angular rates p, q, and r are functions of the time derivatives of the attitude angles, $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ and the angles ϕ and θ. The remainder of this chapter is devoted to formulating the equations of motion corresponding to each of the states listed in table 3.1.

3.2 Kinematics

The translational velocity of the MAV is commonly expressed in terms of the velocity components along each of the axes in a body-fixed coordinate frame. The components u, v, and w correspond to the inertial velocity of the vehicle projected onto the $\mathbf{i}^b$, $\mathbf{j}^b$, and $\mathbf{k}^b$ axes, respectively. On the other hand, the translational position of the MAV is usually measured and expressed in an inertial reference frame. Relating the translational velocity and position requires differentiation and a rotational transformation

$$\frac{d}{dt}\begin{pmatrix} p_n \\ p_e \\ p_d \end{pmatrix} = \mathcal{R}_b^v \begin{pmatrix} u \\ v \\ w \end{pmatrix} = (\mathcal{R}_v^b)^\top \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

which using equation (2.5) gives

$$\begin{pmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \tag{3.1}$$

where we have used the shorthand notation $c_x \triangleq \cos x$ and $s_x \triangleq \sin x$. This is a kinematic relation in that it relates the derivative of position to velocity: forces or accelerations are not considered.

The relationship between angular positions ϕ, θ, and ψ and the angular rates p, q, and r is also complicated by the fact that these quantities are defined in different coordinate frames. The angular rates are defined in the body frame $\mathcal{F}^b$. The angular positions (Euler angles) are defined in three different coordinate frames: the roll angle ϕ is a rotation from $\mathcal{F}^{v2}$ to $\mathcal{F}^b$ about the $\mathbf{i}^{v2} = \mathbf{i}^b$ axis; the pitch angle θ is a rotation from $\mathcal{F}^{v1}$ to $\mathcal{F}^{v2}$ about the $\mathbf{j}^{v1} = \mathbf{j}^{v2}$ axis; and the yaw angle ψ is a rotation from $\mathcal{F}^{v}$ to $\mathcal{F}^{v1}$ about the $\mathbf{k}^{v} = \mathbf{k}^{v1}$ axis.

The body-frame angular rates can be expressed in terms of the derivatives of the Euler angles, provided that the proper rotational transformations are carried out as follows:

$$\begin{aligned}
\begin{pmatrix} p \\ q \\ r \end{pmatrix} &= \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + \mathcal{R}^{b}_{v2}(\phi) \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \mathcal{R}^{b}_{v2}(\phi)\mathcal{R}^{v2}_{v1}(\theta) \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \\
&= \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}.
\end{aligned} \tag{3.2}$$

Inverting this expression yields

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \tag{3.3}$$

which expresses the derivatives of the three angular position states in terms of the angular positions ϕ and θ and the body rates p, q, and r.

3.3 Rigid-body Dynamics

To derive the dynamic equations of motion for the MAV, we will apply Newton's second law—first to the translational degrees of freedom and then to the rotational degrees of freedom. Newton's laws hold in inertial reference frames, meaning the motion of the body of interest must be referenced to a fixed (i.e., inertial) frame of reference, which in our case is the ground. We will assume a flat earth model, which is appropriate for small and miniature air vehicles. Even though motion is referenced to a fixed frame, it can be *expressed* using vector components associated with other frames, such as the body frame.

We do this with the MAV velocity vector $\mathbf{V}_g$, which for convenience is most commonly expressed in the body frame as $\mathbf{V}_g^b = (u, v, w)^\top$. $\mathbf{V}_g^b$ is the velocity of the MAV with respect to the ground as expressed in the body frame.

3.3.1 Translational Motion

Newton's second law applied to a body undergoing translational motion can be stated as

$$\mathsf{m}\frac{d\mathbf{V}_g}{dt_i} = \mathbf{f}, \tag{3.4}$$

where m is the mass of the MAV,[1] $\frac{d}{dt_i}$ is the time derivative in the inertial frame, and $\mathbf{f}$ is the sum of all external forces acting on the MAV. The external forces include gravity, aerodynamic forces, and propulsion forces.

The derivative of velocity taken in the inertial frame can be written in terms of the derivative in the body frame and the angular velocity according to equation (2.17) as

$$\frac{d\mathbf{V}_g}{dt_i} = \frac{d\mathbf{V}_g}{dt_b} + \boldsymbol{\omega}_{b/i} \times \mathbf{V}_g, \tag{3.5}$$

where $\boldsymbol{\omega}_{b/i}$ is the angular velocity of the MAV with respect to the inertial frame. Combining (3.4) and (3.5) results in an alternative representation of Newton's second law with differentiation carried out in the body frame:

$$\mathsf{m}\left(\frac{d\mathbf{V}_g}{dt_b} + \boldsymbol{\omega}_{b/i} \times \mathbf{V}_g\right) = \mathbf{f}.$$

In the case of a maneuvering aircraft, we can most easily apply Newton's second law by expressing the forces and velocities in the body frame as

$$\mathsf{m}\left(\frac{d\mathbf{V}_g^b}{dt_b} + \boldsymbol{\omega}_{b/i}^b \times \mathbf{V}_g^b\right) = \mathbf{f}^b, \tag{3.6}$$

[1] Mass is denoted with a sans serif font m to distinguish it from m, which will be introduced as the sum of moments about the body-fixed $\mathbf{j}^b$ axis.

where $\mathbf{V}_g^b = (u, v, w)^\top$ and $\boldsymbol{\omega}_{b/i}^b = (p, q, r)^\top$. The vector $\mathbf{f}^b$ represents the sum of the externally applied forces and is defined in terms of its body-frame components as $\mathbf{f}^b \triangleq (f_x, f_y, f_z)^\top$.

The expression $\frac{d\mathbf{V}_g^b}{dt_b}$ is the rate of change of the velocity expressed in the body frame, as viewed by an observer on the moving body. Since u, v, and w are the instantaneous projections of $\mathbf{V}_g^b$ onto the $\mathbf{i}^b$, $\mathbf{j}^b$, and $\mathbf{k}^b$ axes, it follows that

$$\frac{d\mathbf{V}_g^b}{dt_b} = \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix}.$$

Expanding the cross product in equation (3.6) and rearranging terms, we get

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} rv - qw \\ pw - ru \\ qu - pv \end{pmatrix} + \frac{1}{\mathrm{m}} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}. \tag{3.7}$$

3.3.2 Rotational Motion

For rotational motion, Newton's second law states that

$$\frac{d\mathbf{h}}{dt_i} = \mathbf{m},$$

where $\mathbf{h}$ is the angular momentum in vector form and $\mathbf{m}$ is the sum of all externally applied moments. This expression is true provided that moments are summed about the center of mass of the MAV. The derivative of angular momentum taken in the inertial frame can be expanded using equation (2.17) as

$$\frac{d\mathbf{h}}{dt_i} = \frac{d\mathbf{h}}{dt_b} + \boldsymbol{\omega}_{b/i} \times \mathbf{h} = \mathbf{m}.$$

As with translational motion, it is most convenient to express this equation in the body frame, giving

$$\frac{d\mathbf{h}^b}{dt_b} + \boldsymbol{\omega}_{b/i}^b \times \mathbf{h}^b = \mathbf{m}^b. \tag{3.8}$$

For a rigid body, angular momentum is defined as the product of the *inertia matrix* $\mathbf{J}$ and the angular velocity vector: $\mathbf{h}^b \triangleq \mathbf{J}\boldsymbol{\omega}_{b/i}^b$

where $\mathbf{J}$ is given by

$$\mathbf{J} = \begin{pmatrix} \int (y^2+z^2)\, d\mathrm{m} & -\int xy\, d\mathrm{m} & -\int xz\, d\mathrm{m} \\ -\int xy\, d\mathrm{m} & \int (x^2+z^2)\, d\mathrm{m} & -\int yz\, d\mathrm{m} \\ -\int xz\, d\mathrm{m} & -\int yz\, d\mathrm{m} & \int (x^2+y^2)\, d\mathrm{m} \end{pmatrix}$$

$$\triangleq \begin{pmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{xy} & J_y & -J_{yz} \\ -J_{xz} & -J_{yz} & J_z \end{pmatrix}. \tag{3.9}$$

The diagonal terms of $\mathbf{J}$ are called the *moments of inertia*, while the off-diagonal terms are called the *products of inertia*. The moments of inertia are measures of the aircraft's tendency to oppose acceleration about a specific axis of rotation. For example, J_x can be conceptually thought of as taking the product of the mass of each element composing the aircraft (dm) and the square of the distance of the mass element from the body x axis ($y^2 + z^2$) and adding them up. The larger J_x is in value, the more the aircraft opposes angular acceleration about the x axis. This line of thinking, of course, applies to the moments of inertia J_y and J_z as well. In practice, the inertia matrix is not calculated using equation (3.9). Instead, it is numerically calculated from mass properties using CAD models or it is measured experimentally using equipment such as a bifilar pendulum [17, 18].

Because the integrals in equation (3.9) are calculated with respect to the $\mathbf{i}^b$, $\mathbf{j}^b$, and $\mathbf{k}^b$ axes fixed in the (rigid) body, $\mathbf{J}$ is constant when viewed from the body frame, hence $\frac{d\mathbf{J}}{dt_b} = 0$. Taking derivatives and substituting into equation (3.8), we get

$$\mathbf{J}\frac{d\boldsymbol{\omega}^b_{b/i}}{dt_b} + \boldsymbol{\omega}^b_{b/i} \times \left(\mathbf{J}\boldsymbol{\omega}^b_{b/i}\right) = \mathbf{m}^b. \tag{3.10}$$

The expression $\frac{d\boldsymbol{\omega}^b_{b/i}}{dt_b}$ is the rate of change of the angular velocity expressed in the body frame, as viewed by an observer on the moving body. Since p, q, and r are the instantaneous projections of $\boldsymbol{\omega}^b_{b/i}$ onto the $\mathbf{i}^b$, $\mathbf{j}^b$, and $\mathbf{k}^b$ axes, it follows that

$$\dot{\boldsymbol{\omega}}^b_{b/i} = \frac{d\boldsymbol{\omega}^b_{b/i}}{dt_b} = \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix}.$$

Rearranging equation (3.10), we get

$$\dot{\boldsymbol{\omega}}^b_{b/i} = \mathbf{J}^{-1}\left[-\boldsymbol{\omega}^b_{b/i} \times \left(\mathbf{J}\boldsymbol{\omega}^b_{b/i}\right) + \mathbf{m}^b\right]. \tag{3.11}$$

Aircraft are often symmetric about the plane spanned by $\mathbf{i}^b$ and $\mathbf{k}^b$. In that case $J_{xy} = J_{yz} = 0$, which implies that

$$\mathbf{J} = \begin{pmatrix} J_x & 0 & -J_{xz} \\ 0 & J_y & 0 \\ -J_{xz} & 0 & J_z \end{pmatrix}.$$

Under this symmetry assumption, the inverse of $\mathbf{J}$ is given by

$$\begin{aligned}
\mathbf{J}^{-1} &= \frac{\mathrm{adj}(\mathbf{J})}{\det(\mathbf{J})} \\
&= \frac{\begin{pmatrix} J_y J_z & 0 & J_y J_{xz} \\ 0 & J_x J_z - J_{xz}^2 & 0 \\ J_{xz} J_y & 0 & J_x J_y \end{pmatrix}}{J_x J_y J_z - J_{xz}^2 J_y} \\
&= \begin{pmatrix} \frac{J_z}{\Gamma} & 0 & \frac{J_{xz}}{\Gamma} \\ 0 & \frac{1}{J_y} & 0 \\ \frac{J_{xz}}{\Gamma} & 0 & \frac{J_x}{\Gamma} \end{pmatrix},
\end{aligned}$$

where $\Gamma \triangleq J_x J_z - J_{xz}^2$.

Defining the components of the externally applied moment about the $\mathbf{i}^b$, $\mathbf{j}^b$, and $\mathbf{k}^b$ axes as $\mathbf{m}^b \triangleq (l, m, n)^\top$, we can write equation (3.11) in component form as

$$\begin{aligned}
\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} &= \begin{pmatrix} \frac{J_z}{\Gamma} & 0 & \frac{J_{xz}}{\Gamma} \\ 0 & \frac{1}{J_y} & 0 \\ \frac{J_{xz}}{\Gamma} & 0 & \frac{J_x}{\Gamma} \end{pmatrix} \left[\begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} J_x & 0 & -J_{xz} \\ 0 & J_y & 0 \\ -J_{xz} & 0 & J_z \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} + \begin{pmatrix} l \\ m \\ n \end{pmatrix} \right] \\
&= \begin{pmatrix} \frac{J_z}{\Gamma} & 0 & \frac{J_{xz}}{\Gamma} \\ 0 & \frac{1}{J_y} & 0 \\ \frac{J_{xz}}{\Gamma} & 0 & \frac{J_x}{\Gamma} \end{pmatrix} \left[\begin{pmatrix} J_{xz} pq + (J_y - J_z) qr \\ J_{xz}(r^2 - p^2) + (J_z - J_x) pr \\ (J_x - J_y) pq - J_{xz} qr \end{pmatrix} + \begin{pmatrix} l \\ m \\ n \end{pmatrix} \right] \\
&= \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr + \Gamma_3 l + \Gamma_4 n \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) + \frac{1}{J_y} m \\ \Gamma_7 pq - \Gamma_1 qr + \Gamma_4 l + \Gamma_8 n \end{pmatrix},
\end{aligned} \tag{3.12}$$

where

$$
\begin{aligned}
\Gamma_1 &= \frac{J_{xz}(J_x - J_y + J_z)}{\Gamma} \\
\Gamma_2 &= \frac{J_z(J_z - J_y) + J_{xz}^2}{\Gamma} \\
\Gamma_3 &= \frac{J_z}{\Gamma} \\
\Gamma_4 &= \frac{J_{xz}}{\Gamma} \\
\Gamma_5 &= \frac{J_z - J_x}{J_y} \\
\Gamma_6 &= \frac{J_{xz}}{J_y} \\
\Gamma_7 &= \frac{(J_x - J_y)J_x + J_{xz}^2}{\Gamma} \\
\Gamma_8 &= \frac{J_x}{\Gamma}.
\end{aligned} \tag{3.13}
$$

The six-degree-of-freedom, 12-state model for the MAV kinematics and dynamics are given by equations (3.1), (3.3), (3.7), and (3.12), and are summarized as follows:

$$
\begin{pmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{3.14}
$$

$$
\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} rv - qw \\ pw - ru \\ qu - pv \end{pmatrix} + \frac{1}{\mathrm{m}} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}, \tag{3.15}
$$

$$
\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta} \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} \tag{3.16}
$$

$$
\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) \\ \Gamma_7 pq - \Gamma_1 qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 l + \Gamma_4 n \\ \frac{1}{J_y} m \\ \Gamma_4 l + \Gamma_8 n \end{pmatrix}. \tag{3.17}
$$

Equations (3.14)–(3.17) represent the dynamics of the MAV. They are not complete in that the externally applied forces and moments are not yet defined. Models for forces and moments due to gravity, aerodynamics, and propulsion will be derived in chapter 4. In appendix B, an alternative formulation to these equations that uses quaternions to represent the MAV attitude is given.

3.4 Chapter Summary

In this chapter, we have derived a six-degree-of-freedom, 12-state dynamic model for a MAV from first principles. This model will be the basis for analysis, simulation, and control design that will be discussed in forthcoming chapters.

Notes and References

The material in this chapter is standard, and similar discussions can be found in textbooks on mechanics [14, 15, 19], space dynamics [20, 21], flight dynamics [1, 2, 5, 7, 12, 22] and robotics [10, 23].

Equations (3.14) and (3.15) are expressed in terms of inertially referenced velocities u, v, and w. Alternatively, they can be expressed in terms of velocities referenced to the air-mass surrounding the aircraft u_r, v_r, and w_r as

$$\begin{pmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \end{pmatrix} = \mathcal{R}_b^v(\phi, \theta, \psi) \begin{pmatrix} u_r \\ v_r \\ w_r \end{pmatrix} + \begin{pmatrix} w_n \\ w_e \\ w_d \end{pmatrix} \tag{3.18}$$

$$\begin{pmatrix} \dot{u}_r \\ \dot{v}_r \\ \dot{w}_r \end{pmatrix} = \begin{pmatrix} r v_r - q w_r \\ p w_r - r u_r \\ q u_r - p v_r \end{pmatrix} + \frac{1}{\mathsf{m}} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} - \mathcal{R}_v^b(\phi, \theta, \psi) \begin{pmatrix} \dot{w}_n \\ \dot{w}_e \\ \dot{w}_d \end{pmatrix}, \tag{3.19}$$

where

$$\mathcal{R}_b^v(\phi, \theta, \psi) = (\mathcal{R}_v^b)^\top(\phi, \theta, \psi) = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix}.$$

The choice of which equations to use to express the aircraft kinematics is a matter of personal preference. In equations (3.14) and (3.15), the velocity states u, v, and w represent the aircraft motion with respect to the ground (inertial frame). In equations (3.18) and (3.19), the velocity states u_r, v_r, and w_r represent the aircraft motion with respect to the air mass surrounding the aircraft. To correctly represent the motion of the aircraft in the inertial frame using u_r, v_r, and w_r as states, the effect of wind speed and wind acceleration must be taken into account.