RungeFunktion_lagrange

October 8, 2025

Rungefunktion $f(x) = \frac{1}{1+x^2}$

Lagrange Interpolierende vom Grad ≤ 5 **in 6 Stellen** $(x_1 = -5, x_2 = -3, x_3 = -1, x_4 = 1, x_5 = 3, x_6 = 5)$

[126]:
```
import sympy as sp
sp.init_printing()
from sympy import *

x = symbols("x")
l_1 = Rational(1,␣
 ↪(-5+3)*(-5+1)*(-5-1)*(-5-3)*(-5-5))*((x+3)*(x+1)*(x-1)*(x-3)*(x-5))
l_2 = Rational(1,␣
 ↪(-3+5)*(-3+1)*(-3-1)*(-3-3)*(-3-5))*((x+5)*(x+1)*(x-1)*(x-3)*(x-5))
l_3 = Rational(1,␣
 ↪(-1+5)*(-1+3)*(-1-1)*(-1-3)*(-1-5))*((x+5)*(x+3)*(x-1)*(x-3)*(x-5))
l_4 = Rational(1, (1+5)*(1+3)*(1+1)*(1-3)*(1-5))*((x+5)*(x+3)*(x+1)*(x-3)*(x-5))
l_5 = Rational(1, (3+5)*(3+3)*(3+1)*(3-1)*(3-5))*((x+5)*(x+3)*(x+1)*(x-1)*(x-5))
l_6 = Rational(1, (5+5)*(5+3)*(5+1)*(5-1)*(5-3))*((x+5)*(x+3)*(x+1)*(x-1)*(x-3))
f_interpol =␣
 ↪Rational(1,26)*l_1+Rational(1,10)*l_2+Rational(1,2)*l_3+Rational(1,2)*l_4+Rational(1,10)*l_

f_l = expand(f_interpol) #f = simplify(f_interpol)
f_l
```

[126]: $\frac{x^4}{520} - \frac{9x^2}{130} + \frac{59}{104}$

Da Punkte (x_i, f_i) symmetrisch zu $x = 0$, ergibt sich eine gerade Funktion: Polynom vom Grad 4 mit nur geraden Potenzen statt Polynom vom Grad 5! Spezialfall hier!

[140]:
```
# Test der Interpolation bei x=-5, f(x)=1/26
f_l.subs(x, -5)
```

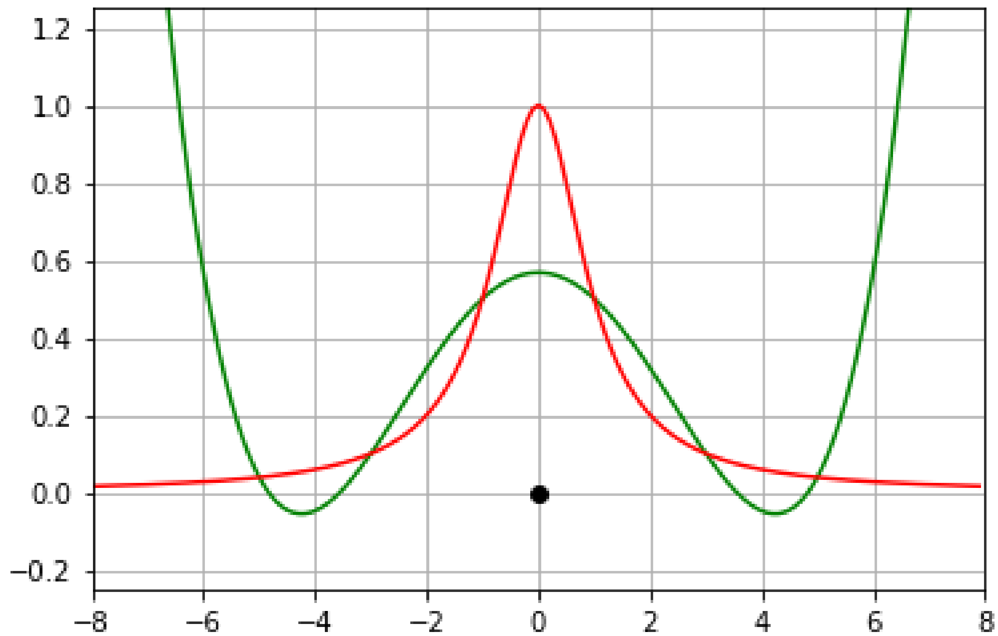
[140]: $\frac{1}{26}$

[141]:
```
import math
import numpy as np              # array Funktionen
import matplotlib.pyplot as plt # plot Funktionen
```

```
xx = np.arange(-8, 8, 0.1) # x-Werte
sf = lambda arg: float(f_l.evalf(subs={x: arg})) # fuer einzelne
vf = np.vectorize(sf)                            # fuer arrays
yy = vf(xx)                # y-Werte der Wertetabelle
plt.plot(xx, yy, 'g-')               # Plot der Lagrange Interpolierenden

plt.plot(xx, 1.0/(1+xx**2), 'r-')      # Plot der Rungefunktion

plt.plot([ 0], [0], 'ko-') # Nullpunkt
plt.axis([-8, 8, -0.25, 1.25])
plt.grid(True)
plt.show()
```



Taylorpolynom vom Grad 3

[167]:
```
f = 1/(1+x**2)
# Stelle x=-2
df_1 = f.diff(x)
df_2 = df_1.diff(x)
df_3 = df_2.diff(x)
f_3=df_3.subs(x, -2)
f_2=df_2.subs(x, -2)
f_1=df_1.subs(x, -2)
f_0=f.subs(x, -2)
```

```
f_m2_taylor= f_0 +f_1*(x+2) +f_2*Rational(1,2)*(x+2)**2␣
↪+f_3*Rational(1,6)*(x+2)**3

# Stelle x=0
df_1 = f.diff(x)
df_2 = df_1.diff(x)
df_3 = df_2.diff(x)
ff_3=df_3.subs(x, 0)
ff_2=df_2.subs(x, 0)
ff_1=df_1.subs(x, 0)
ff_0=f.subs(x, 0)
f_0_taylor= ff_0 +ff_1*x +ff_2*Rational(1,2)*x**2 +ff_3*Rational(1,6)*x**3

[f_0, f_1, f_2, f_3]
```

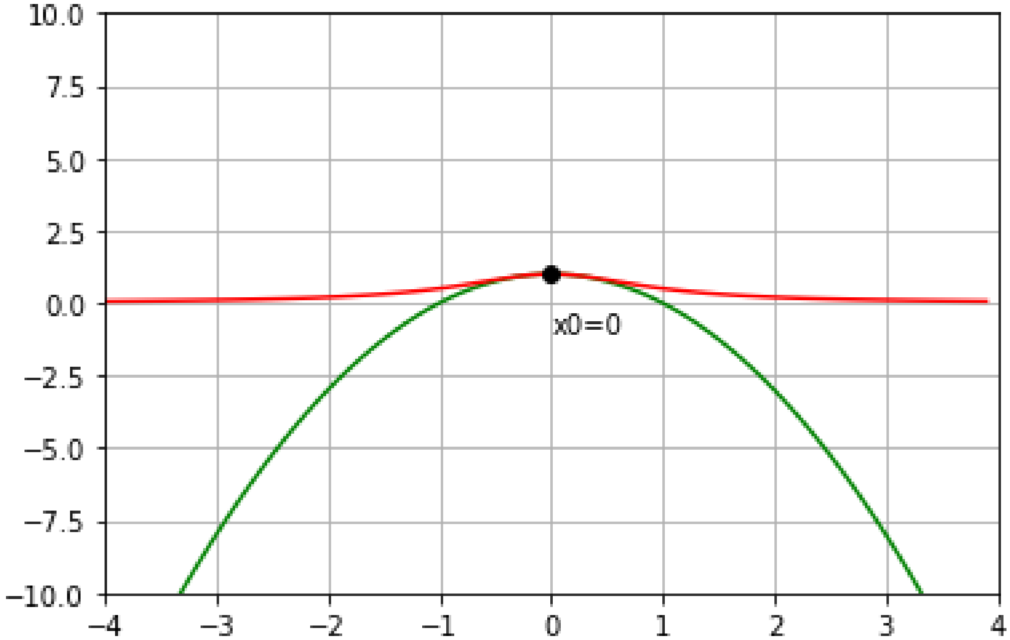
[167]: $\left[\frac{1}{5}, \ \frac{4}{25}, \ \frac{22}{125}, \ \frac{144}{625}\right]$

[168]:
```
import math
import numpy as np              # array Funktionen
import matplotlib.pyplot as plt # plot Funktionen

xx = np.arange(-4, 4, 0.1) # x-Werte
sf = lambda arg: float(f_0_taylor.evalf(subs={x: arg})) # fuer einzelne
vf = np.vectorize(sf)                                  # fuer arrays
yy = vf(xx) # y = f(x)
plt.plot(xx, yy, 'g-')                    # Plot des Taylorpolynom

plt.plot(xx, 1.0/(1+xx**2), 'r-')       # Plot der Rungefunktion

plt.plot([0], [1], 'ko-') # Stelle
plt.text(0, -1, 'x0=0')
plt.axis([-4, 4, -10, 10])
plt.grid(True)
plt.show()
```

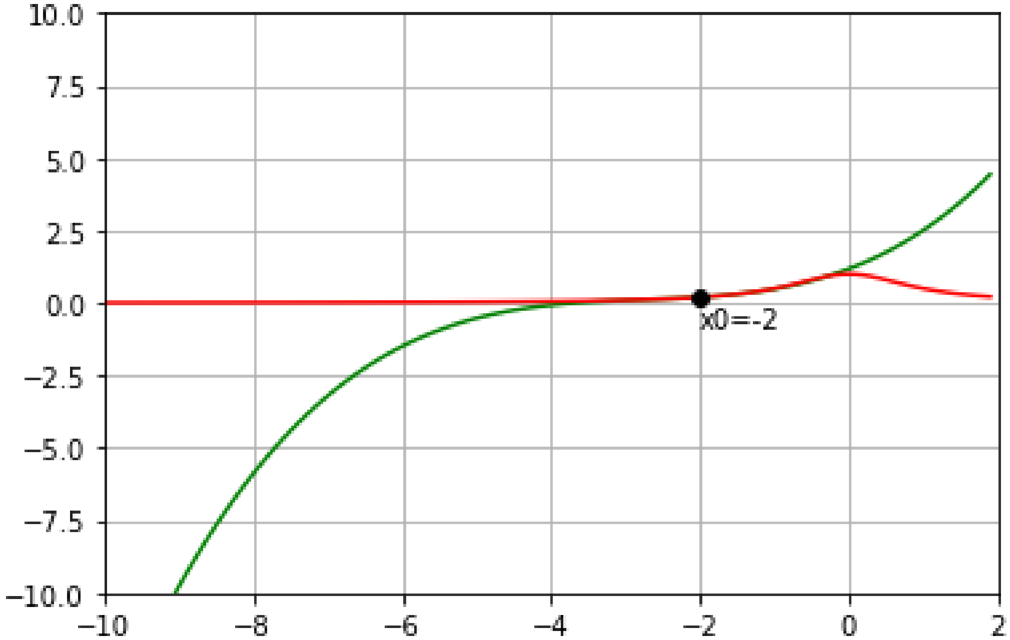


[166]:
```
import math
import numpy as np               # array Funktionen
import matplotlib.pyplot as plt # plot Funktionen

xx = np.arange(-10, 2, 0.1) # x-Werte
sf = lambda arg: float(f_m2_taylor.evalf(subs={x: arg})) # fuer einzelne
vf = np.vectorize(sf)                                      # fuer arrays
yy = vf(xx) # y = f(x)
plt.plot(xx, yy, 'g-')                  # Plot des Taylorpolynom

plt.plot(xx, 1.0/(1+xx**2), 'r-')       # Plot der Rungefunktion

plt.plot([ -2], [1.0/(1+(-2)**2)], 'ko-') # Stelle
plt.text( -2, 1.0/(1+(-2)**2)-1, "x0=-2")
plt.axis([-10, 2, -10, 10])
plt.grid(True)
plt.show()
```


10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
−10
−8
−6
−4
−2
0
2
x0=-2

[ ]: