More on Event-B: Relations

© Michael Butler

University of Southampton

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Ordered Pairs and Cartesian Products

An ordered pair is an element consisting of two parts:
a first part and a second part.

An ordered pair with first part x and second part y is written: $\boxed{x \mapsto y}$

The Cartesian product of two sets is the set of pairs whose first part is in S and second part is in T.

The Cartesian product of S with T is written: $\boxed{S \times T}$

Cartesian Products: Definition and Examples

Defining Cartesian product:

Predicate	Definition
$x \mapsto y \quad \in \quad S \times T$	$x \in S \quad \wedge \quad y \in T$

Examples:

$$\{a, b, c\} \times \{1, 2\} \quad = \quad \{\ a \mapsto 1,\ a \mapsto 2,\ b \mapsto 1,\ b \mapsto 2,\ c \mapsto 1,\ c \mapsto 2,\ \}$$

$$\{a, b, c\} \times \{\} \quad = \quad ?$$

$$\{\ \{a\},\ \{a, b\}\ \} \ \times\ \{1, 2\} \quad = \quad ?$$

Cartesian Product is a Type Constructor

$S \times T$ is a new type constructed from types S and T.

Cartesian product is the type constructor for ordered pairs.

Given $x \in S$, $y \in T$, we have

$$\boxed{x \mapsto y \quad \in \quad S \times T}$$

$$4 \mapsto 7 \quad \in \quad ?$$

$$\{5, 6, 3\} \mapsto 4 \quad \in \quad ?$$

$$\{\ 4 \mapsto 8,\ 3 \mapsto 0,\ 2 \mapsto 9\ \} \quad \in \quad ?$$

Sets of Order Pairs

A database can be modelled as a set of ordered pairs:

$$
\begin{aligned}
directory \quad = \quad \{\ & mary \mapsto 287573, \\
& mary \mapsto 398620, \\
& john \mapsto 829483, \\
& jim \mapsto 398620\ \}
\end{aligned}
$$

directory has type

$$directory \quad \in \quad \mathbb{P}(Person \times PhoneNum)$$

Relations

A relation is a set of ordered pairs.

A relation is a common modelling structure so Event-B has a special notation for it:

$$\boxed{T \leftrightarrow S} \quad = \quad \mathbb{P}(T \times S)$$

So we can write:

$$directory \quad \in \quad Person \leftrightarrow PhoneNum$$

Do not confuse the arrow symbols:

$\leftrightarrow$ combines two sets to form a set.

$\mapsto$ combines two elements to form an ordered pair.

Domain and Range

- The domain of a relation R is the set of first parts of all the pairs in R, written $\boxed{dom(R)}$
- The range of a relation R is the set of second parts of all the pairs in R, written $\boxed{ran(R)}$

Predicate	Definition
$x \in dom(R)$	$\exists y \cdot \; x \mapsto y \; \in \; R$
$y \in ran(R)$	$\exists x \cdot \; x \mapsto y \; \in \; R$

Examples:

$$\mathrm{dom}(directory) = \{mary, john, jim\}$$

$$\mathrm{ran}(directory) = \{287573, 398620, 829483\}$$

Telephone Directory Model

- Phone directory relates people to their phone numbers.
- Each person can have zero or more numbers.
- People can share numbers.

context *PhoneContext*
sets *Person* *PhoneNum*
end

machine *PhoneBook*
variables *dir*
invariants $dir \in Person \leftrightarrow PhoneNum$

initialisation $dir := \{\}$

Extending the Directory

Add an entry to the directory:

$$
\begin{array}{lcl}
\mathit{AddEntry} & \hat{=} & \textbf{any}\ p, n\ \textbf{where} \\
 & & \quad p \in \mathit{Person} \\
 & & \quad n \in \mathit{PhoneNum} \\
 & & \textbf{then} \\
 & & \quad \mathit{dir} \ := \ \mathit{dir} \cup \{p \mapsto n\} \\
 & & \textbf{end}
\end{array}
$$

Relational Image

Assume $R \in S \leftrightarrow T$ and $A \subseteq S$

The relational image of set A under relation R is written $\boxed{R[A]}$

Predicate	Definition
$y \in R[A]$	$\exists x \cdot x \in A \wedge x \mapsto y \in R$

Example:

$$\begin{aligned} directory = \{ & mary \mapsto 287573, \\ & mary \mapsto 398620, \\ & john \mapsto 829483, \\ & jim \mapsto 398620 \} \end{aligned}$$

$$directory[\{mary\}] = \{287573, 398620\}$$

Modelling Queries using Relational Image

Determine all the numbers associated with a person in the directory:

$$
\begin{array}{lcl}
GetNumbers & \hat{=} & \textbf{any } p, ns! \textbf{ where} \\
 & & \quad p \in Person \\
 & & \quad ns! = dir[\ \{p\}\ ] \\
 & & \textbf{end}
\end{array}
$$

Determine all the numbers associated with a set of people:

$$
\begin{array}{lcl}
GetMultiNumbers & \hat{=} & \textbf{any } ps, ns! \textbf{ where} \\
 & & \quad ps \subseteq Person \\
 & & \quad ns! = dir[\ ps\ ] \\
 & & \textbf{end}
\end{array}
$$

Relational Inverse

Given $R \in S \leftrightarrow T$, the relational inverse of R is written $\boxed{R^{-1}}$

Predicate	Definition
$y \mapsto x \in R^{-1}$	$x \mapsto y \in R$

Example:

$$\begin{aligned} directory^{-1} = \{ & 287573 \mapsto mary, \\ & 398620 \mapsto mary, \\ & 829483 \mapsto john, \\ & 398620 \mapsto jim \} \end{aligned}$$

$$directory^{-1}[\{398620\}] = \{ mary, jim \}$$

Inverse Queries

Return all the people associated with a number in the directory:

$$
\begin{array}{lcl}
GetNames & \widehat{=} & \textbf{any } n, ps! \textbf{ where} \\
 & & \quad n \in PhoneNum \\
 & & \quad ps! = dir^{-1}[\ \{n\}\ ] \\
 & & \textbf{end}
\end{array}
$$

Return all the people associated with a set of numbers:

$$
\begin{array}{lcl}
GetMultiNames & \widehat{=} & \textbf{any } ns, ps! \textbf{ where} \\
 & & \quad ns \subseteq PhoneNum \\
 & & \quad ps! = dir^{-1}[\ ns\ ] \\
 & & \textbf{end}
\end{array}
$$

Domain Restriction

Given $R \in S \leftrightarrow T$ and $A \subseteq S$,
the domain restriction of R by A is writen $\boxed{A \lhd R}$

Restrict relation R so that it only contains pairs whose first part is in the set A.

Example:

$$\begin{aligned} \mathit{directory} \quad = \quad \{ \; & \mathit{mary} \mapsto 287573, \quad \mathit{mary} \mapsto 398620, \\ & \mathit{john} \mapsto 829483, \quad \mathit{jim} \mapsto 398620 \; \} \end{aligned}$$

$$\begin{aligned} \{\mathit{john}, \mathit{jim}, \mathit{jane}\} \lhd \mathit{directory} \quad = \quad \{ \; & \mathit{john} \mapsto 829483, \\ & \mathit{jim} \mapsto 398620 \; \} \end{aligned}$$

Domain Subtraction

Given $R \in S \leftrightarrow T$ and $A \subseteq S$,
the domain subtraction of R by A is written $\boxed{A \lhd R}$

Remove those pairs from R whose first part is in A.

Example:

$$\{john, jim, jane\} \lhd directory = \{ mary \mapsto 287573, \\ mary \mapsto 398620 \}$$

Domain and Range, Restriction and Substraction

Assume $R \in S \leftrightarrow T$ and $A \subseteq S$ and $B \subseteq T$

Predicate	Definition	
$x \mapsto y \in A \lhd R$	$x \mapsto y \in R \wedge x \in A$	domain restriction
$x \mapsto y \in A \ntriangleleft R$	$x \mapsto y \in R \wedge x \notin A$	domain subtraction
$x \mapsto y \in R \rhd B$	$x \mapsto y \in R \wedge y \in B$	range restriction
$x \mapsto y \in R \ntriangleright B$	$x \mapsto y \in R \wedge y \notin B$	range subtraction

Removing Entries from the Directory

Remove all the entries associated with a person in the directory:

$$
\begin{array}{lll}
\mathit{RemovePerson} & \widehat{=} & \textbf{any } p \textbf{ where} \\
 & & \quad p \in \mathit{Person} \\
 & & \textbf{then} \\
 & & \quad \mathit{dir} \ := \ \{p\} \lhd\!\!\!- \ \mathit{dir} \\
 & & \textbf{end}
\end{array}
$$

Remove all the entries associated with a number in the directory:

$$
\begin{array}{lll}
\mathit{RemoveNumber} & \widehat{=} & \textbf{any } n \textbf{ where} \\
 & & \quad n \in \mathit{PhoneNum} \\
 & & \textbf{then} \\
 & & \quad \mathit{dir} \ := \ \mathit{dir} \ \rhd\!\!\!- \ \{n\} \\
 & & \textbf{end}
\end{array}
$$

Relational Composition

Given $Q \in S \leftrightarrow T$ and $R \in T \leftrightarrow U$,
the relational composition of Q and R is written $\boxed{Q \; ; R}$

We have that $Q \; ; R \in S \leftrightarrow U$

Predicate	Definition
$x \mapsto z \in (Q \; ; R)$	$\exists y \cdot x \mapsto y \in Q \wedge y \mapsto z \in R$

Example:

$$
\begin{aligned}
M &= \{ a \mapsto l, b \mapsto m, c \mapsto n \} \\
N &= \{ l \mapsto 4, n \mapsto 6, p \mapsto 8 \} \\
M \; ; N &= \; ?
\end{aligned}
$$

Composition and Image

Given $Q \in S \leftrightarrow T$ and $R \in T \leftrightarrow U$ and $A \subseteq S$

$$(Q \,;\, R)[\, A \,] \;=\; R[\, Q[\, A \,] \,]$$

Example:

$$\begin{aligned} M &= \{\, a \mapsto l, b \mapsto m, c \mapsto n \,\} \\ N &= \{\, l \mapsto 4, n \mapsto 6, p \mapsto 8 \,\} \end{aligned}$$

$$\begin{aligned} (M \,;\, N)\,[\, \{a, b\} \,] &= \; ? \\ N[\, M[\, \{a, b\} \,] \,] &= \; ? \end{aligned}$$

Extend directory with friends

variables $dir, friend$
invariants

$$
\begin{aligned}
friend &\in Person \leftrightarrow Person \\
dir &\in Person \leftrightarrow PhoneNum
\end{aligned}
$$

Return the telephone numbers of all friends of p:

$$
\begin{array}{l}
GetFriendNumbers \quad \hat{=} \\
\quad \textbf{any } p, ns! \textbf{ where} \\
\qquad p \in Person \\
\qquad ns! = (friend ; dir)[\ \{p\}\ ] \\
\quad \textbf{end}
\end{array}
$$

Recap

- Cartesian product is the type constructor for pairs of elements.
- A relation is a set of pairs.
- Range of a relation, domain of a relation.
- Relational image, relational inverse.
- Restriction and subtraction.
- Relational composition.