PROOFS OF STRONG NORMALIZATION

R.O. Gandy

Mathematical Institute, 24-29 St. Giles, Oxford OX1 3LB, UK

Dedicated to H.B. Curry on the occasion of his 80th birthday

In this paper I present a rather transparent method for proving that all reduction sequences terminate[1]. The method is explained in detail for the typed lambda calculus and then it is shown how it can be extended to other calculi, including the calculus of primitive recursive functionals of finite type (Godel 1958) and a term calculus, having direct products and sums as types, which corresponds to the calculus of proofs by natural deduction for the intuitionistic propositional calculus as presented in Prawitz (1971). (Details of these systems and of normalization proofs for them will be found in the invaluable and encyclopoedic book edited by Troelstra (1973)). Besides its essential simplicity the method shows one how, given a term of the calculus, one can write down a numerical term whose value is an upper bound for the length of reduction sequences starting with the given term. However, the method does not prove weak normalization; it assumes that numerical terms have numerical values. Nor, as it stands does it cope with permutative reductions or with proofs incorporating a rule of induction.

1 THE MONOTONICITY OF λ-I TERMS

The method is based on the following observation: if one partially orders the ground type(s) and carries this order pointwise up through the types, then, in the hierarchy of hereditarily strictly monotonic functions every term of the typed λ-I calculus denotes a strictly monotonic function.

1.1 For simplicity we consider only a single ground type, with type symbol o. Let T_o be a non-empty set, partially ordered by a relation $<_o$. Let $T = \{T_\alpha : \alpha \text{ a type symbol}\}$ be a type structure: i.e. (a) for each type symbol $(\alpha\to\beta)$, $T_{(\alpha\to\beta)}$ is a non-empty collection of functions from T_α into T_β, and (b) T is closed under application and definition by λ-abstraction. We introduce the collection

$$M = \{M_\alpha : \alpha \text{ a type symbol}\}$$

of hereditarily monotonic members of T, and the relation $<_\alpha$ on M_α as follows.

(1) $(M_o, <_o) = (T_o, <_o)$

(2) $M_{(\alpha\to\beta)} = \{f \in T_{(\alpha\to\beta)} : \forall a, a' \in M_\alpha .\ fa \in M_\beta \wedge (a <_\alpha a' \Rightarrow fa <_\beta fa')\}$.

(3) For $f, g \in M_{(\alpha\to\beta)}$

$$f <_{(\alpha\to\beta)} g \Leftrightarrow \forall a \in M_\alpha \ .\ fa <_\beta ga \ .$$

The M is closed under application and $M_\alpha \subset T_\alpha$; $<_\alpha$ is transitive and, if all the M_α are not empty, a partial ordering.

1.2 Under an assignment of a value x_α^A in M_α to each free variable x_α in a term σ (of the typed λ-calculus) of type β the term σ will denote an element of T_β which may or may not belong to M_β . For example, $\lambda x_\alpha y_\beta$ and $\lambda f_{((\alpha\to\beta)\to\gamma)} f(\lambda x_\alpha y_\beta)$ will not have denotations in M. (Here, and in future, we suppress the type subscript from all occurrences of a bound variable other than the binding occurrence.)

We recall that the class of typed λ-I terms is defined by: (i) variables are λ-I terms; (ii) if σ and τ are λ-I terms of types $(\alpha\to\beta)$ and α, then $(\sigma\tau)$ is a λ-I term of type β; (iii) if τ is a λ-I term of type β which contains a free occurrence of x_α, then $(\lambda x_\alpha \tau)$ is a λ-I term of type $(\alpha\to\beta)$. As is customary, we shall often omit the outermost parentheses. Now we can prove the observation made at the beginning of this section.

1.3 THEOREM (i) Every typed λ-I term ρ has a denotation in M for every assignment of values in M to its free variables. (ii) If the variable x_α actually occurs free in a term ρ of type β and if the assignments A,B agree on all the free variables of ρ except x_α, then

$$x_\alpha^A <_\alpha x_\alpha^B \Rightarrow \rho^A <_\beta \rho^B .$$

Proof. Both (i) and (ii) are trivially true if ρ is a variable. Suppose that ρ is $\sigma\tau$ and that (i) and (ii) hold for σ and τ. Then (i) holds for ρ. Suppose that $x_\alpha^A <_\alpha x_\alpha^B$. Since x_α must occur in σ, in τ, or in both there are three cases. Case A: $\sigma^A <_{(\gamma\to\beta)} \sigma^B$ and $\tau^A = \tau^B$. Then, by the definition of $<_{(\gamma\to\beta)}$, $\rho^A <_\beta \rho^B$. Case B: $\sigma^A = \sigma^B$ and $\tau^A <_\gamma \tau^B$. Then, since by hypothesis $\sigma \in M_{(\gamma\to\beta)}$, $\rho^A <_\beta \rho^B$. Case C: $\sigma^A <_{(\gamma\to\beta)} \sigma^B$ and $\tau^A <_\gamma \tau^B$.

Then

$$\sigma^A \tau^A <_\beta \sigma^B \tau^A <_\beta \sigma^B \tau^B$$

and $\rho^A <_\beta \rho^B$ follows because $<_\beta$ is transitive.

Suppose now that ρ is $\lambda y_\gamma\, \tau$ where τ is of type δ (so that $\beta = (\gamma\to\delta)$). By induction hypothesis, (i) and (ii) (with y_γ as the indicated variable) hold for τ, and so $\rho^A \in M_{(\gamma\to\delta)}$; but (ii) also holds for τ with x_α as the indicated variable, irrespective of the value assigned to y_γ. Hence

$$x_\alpha^A <_\gamma x_\alpha^B \Rightarrow (\lambda y_\gamma \tau)^A <_{(\gamma\to\delta)} (\lambda y_\gamma \tau)^B ,$$

and so (ii) holds also for ρ.

1.4 COROLLARY Let σ be a particular occurrence of a subterm (of type β) of the λ-I term τ (of type α) and let τ_1 be the result of substituting the λ-I term σ_1 (of type β) for σ in τ; and suppose that τ_1 is also a λ-I term. Let $z^1_{\gamma_1}, \ldots, z^r_{\gamma_r}$ be a list of the variables which occur free in σ or σ_1 but which are bound in τ; (we assume they are distinct from the bound variables of σ and σ_1). Let A be an assignment in M for the free variables of σ and σ_1, and suppose that for every assignment B which extends A and which assigns values in M to $z^1_{\gamma_1},\ldots,z^r_{\gamma_r}$:

(1) $\sigma_1^B <_\beta \sigma^B$.

Then $\tau_1^A <_\alpha \tau^A$.

Proof. We proceed by induction on r. For r = 0, the condition (1) is simply $\sigma_1^A <_\beta \sigma^A$; then $\tau_1^A <_\alpha \tau^A$ follows directly from the theorem. Assume now (IH) that the corollary holds when there are fewer than r free variables of σ or σ_1 which occur bound in τ. Assume the premises (and notations) of the corollary and let $z^1_{\gamma_1}$ be the variable whose binding occurrence lies outside the binding occurrences of $z^2_{\gamma_2},\ldots,z^r_{\gamma_r}$. Thus σ occurs in a part π of τ of type $(\gamma_1\to\delta)$ and of the form $\lambda z^1_{\gamma_1}\ \rho$ and τ_1 has a corresponding part $\pi_1 = \lambda z^1_{\gamma_1}\rho_1$. For any assignment C which extends A by assigning a value in M to $z^1_{\gamma_1}$ we have, by (IH) (applied with ρ replacing τ and C replacing A):

$$\rho_1^C <_\delta \rho^C$$

But then, since π and π_1 are both λ-I terms, the theorem tells us that

$$\pi_1^A <_{(\gamma_1\to\delta)} \pi^A \quad ,$$

and further that $\tau_1^A <_\alpha \tau^A$. This completes the inductive step and the proof of the corollary. The slight fussiness arises from the fact that we wish to apply to corollary in cases where some of the 'z's' do not occur in σ_1 .

2 INTRODUCTION OF ADDITION

2.1 We now suppose that $(T_0, <_0)$ is the set of natural numbers with the usual ordering and that addition belongs to $T_{(0\to(0\to0))}$ and hence also to M. We introduce new symbols 0_o, (for zero) S^0 (for successor) and $+_0$ (for addition), and call the resulting system the 'λ^+ calculus'. A λ-I^+ term is a λ-I term which may contain these new symbols. Since S and + are monotonic, 1.3 and 1.4 also hold for λ-I^+ terms.

We carry S and + up through the types:

2.2 DEFINITION

$$S^{(\alpha\to\beta)} = \lambda f_{(\alpha\to\beta)}\, \lambda x_\alpha\, .\; S^\beta(fx),$$

$$+_{(\alpha\to\beta)} = \lambda f_{(\alpha\to\beta)}\, \lambda g_{(\alpha\to\beta)}\, \lambda x_\alpha\, .\; fx +_\beta gx\ ,$$

where for ease of reading we have placed '+' between its arguments. By 1.3, S^α and $+_\alpha$ belong to M; here and elsewhere we do not make any notational distinction between a closed term and its denotation. Our estimate for the height of a reduction tree rests on the following lemma, which is an immediate consequence of the definitions.

2.3 LEMMA

$$x^A_\alpha,\, y^A_\alpha \in M \Rightarrow x^A_\alpha <_\alpha (S^\alpha(x_\alpha +_\alpha y_\alpha))^A\ .$$

The presence of 0_o and $+_0$ ensures that, for every type α, there are closed terms with denotations in M_α .

2.4 DEFINITION

(i) $L_0 = 0_o$. (ii) $L_{(0\to0)} = \lambda x_0 . x$

(iii) $L_{((\alpha\to\beta)\to0)} = \lambda f_{(\alpha\to\beta)} . L_{(\beta\to0)}\,(fL_\alpha)$.

(iv) $L_{(\alpha\to(\beta\to\gamma))} = \lambda x_\alpha\, \lambda y_\beta . L_{(\alpha\to\gamma)}\, x +_\gamma L_{(\beta\to\gamma)} y$.

These are all λ-I^+ terms and so have denotations in M. These definitions may be made more digestible by remarking that if

$$\alpha = (\alpha_1 \to (\alpha_2\to\ldots(\alpha_n \to 0)\ldots))$$

then, omitting brackets with association to the left, we have:

(v) $L_{(\alpha\to0)}\, x_\alpha = x_\alpha\, L_{\alpha_1}\, L_{\alpha_2} \ldots L_{\alpha_n}$

(vi) $L_\alpha x_{\alpha_1} x_{\alpha_2} \ldots x_{\alpha_n} = \sum_{i=1}^{n} p_i (L_{(\alpha_i \to 0)} x_{\alpha_i})$,

where $p_1 = 1$, and for $i > 1$, $p_i = 2^{i-2}$.

2.5 DEFINITION The norm, written $|\tau|^A$, of any term τ of type α with respect to the assignment A of values in M to its free variables, is $L_{(\alpha\to 0)}\tau^A$.

2.6 Assuming weak normalisation for the typed λ-calculus, if τ is a closed term then $L_{(\alpha\to 0)}\tau$ will reduce to a numerical term built up from 0, S^0 and $+_0$, and so $|\tau|$ can be easily computed. Note that for closed terms σ, τ of type α

(1) $\sigma <_\alpha \tau \Rightarrow |\sigma| < |\tau|$.

The converse implication does not hold; for example, if α is $(0\to 0)$ then $|\sigma|$ is the value of the function denoted by σ at 0, so that e.g., $|\lambda x_0 . x + x| < |\lambda x_0 . S^0 x|$.

2.7 *A note on proofs.* For clarity we have used a type structure to present our definitions and proofs. It would however be possible to carry out the arguments in pure numerical terms; variables of type α would then be interpreted as ranging over terms of type α. If one introduced symbols for all the relevant functions (certainly primitive recursive), one could conduct the proofs using just π^0_1-induction, and so, I presume, in primitive recursive arithmetic.

3 STRONG NORMALISATION

3.1 We introduce new abstraction operators λ^* as abbreviations by

(1) $\lambda^* x_\alpha \tau$ stands for $\lambda x_\alpha . S^\beta(\tau +_\beta L_{(\alpha\to\beta)} x)$,

where τ is a term of type β. If τ is a λ-I^+ term (which need not contain any occurrences of x_α) then so in $\lambda^* x_\alpha \tau$. Further, if τ and σ are λ-I^+ terms, and A is any assignment of values in M to the free variables of $(\lambda x_\alpha \tau(x))\sigma$ (which are assumed to be distinct from the bound variables of τ) then, by 2.3,

(2) $\quad (\tau(\sigma))^A <_\beta ((\lambda^* x_\alpha \tau)\sigma)^A$

(where, as usual, $\tau(\sigma)$ is the result of substituting σ at all the free occurrences of x_α in $\tau(x_\alpha)$). This inequality is the nub of our proof of strong normalisation; it only remains to fill in some details.

3.2 Let X be any typed λ-calculus and let K be the set of its primitive type symbols. We define a mapping of terms of X into terms of the λ^+ calculus of §2 as follows:

Type symbols: (i) α^* is o if $\alpha \in K$;

(ii) $(\alpha \to \beta)^*$ is $(\alpha^* \to \beta^*)$.

Terms: (iii) x^*_α is x_{α^*} for any variable x_α of X, and we assume that distinct letters are used for variables in distinct types, so that $\alpha \neq \beta \Rightarrow (x_\alpha)^* \neq (y_\beta)^*$ for any variables x_α, y_β;

(iv) $(\tau\sigma)^*$ is $(\tau^*\sigma^*)$;

(v) $(\lambda x_\alpha \tau)^*$ is $(\lambda^* x_{\alpha^*} \tau^*)$.

Plainly for every term τ of X, τ^* is a λ-I^+ term. Also, by induction on the construction of any term $\tau(= \tau(x_\beta)$, say), one sees that $\tau^*(\sigma^*)$ is $(\tau(\sigma))^*$; we assume all bound variables are chosen so as to avoid collisions.

Now we can prove strong normalisation for X in the following form.

3.3 THEOREM Let ρ be any term of type α of the typed λ-calculus X, and let $\rho_0(=\rho), \rho_1, \ldots, \rho_n$ be any sequence of immediate reductions. Let A be any assignment of values in M to the free variables of ρ; then $n \leq |\rho^*|^A$. In particular, an upper bound for n is obtained by substituting constants L_β of appropriate type for the free variables of ρ^* and computing (as in 2.6) the norm of the resulting term.

<u>Proof</u>. Let ρ_{i+1} be obtained from ρ_i by replacing a part

$((\lambda x_\gamma.\tau(x))\pi)$ (= σ, of type β, say) by $\tau(\pi)$ (= σ_1). Let B be any extension of A to the free variables of σ. Then, by 3.1 (2) $(\sigma_1{*})^B <_\beta (\sigma{*})^B$. But the ρ_i^* are λ-I^+ terms; hence, by 1.4,

$$(\rho_{i+1}^*)^A <_\alpha (\rho_i^*)^A \quad .$$

The theorem now follows by 2.6 (1).

4 EXTENSION TO PRIMITIVE RECURSIVE FUNCTIONALS

4.1 The system T of Gödel (1958) can be obtained by adding recursors R^α, for each type α, to the typed λ^+ calculus of §2; (for details see Troelstra (1973)). For simplicity we keep the now redundant constant $+_0$. R^α is of type $(0\to((0\to(\alpha\to\alpha))\to(\alpha\to\alpha)))$ and satisfies

(1) $$R^\alpha\ 0\ \tau\ \sigma = \sigma \quad ,$$
$$R^\alpha(S^0\rho)\ \tau\ \sigma = \tau\ \rho\ (R^\alpha\rho\tau\sigma) \quad ,$$

for any terms ρ, σ, τ of types 0, α, $(0\to(\alpha\to\alpha))$ respectively. Reductions in this calculus consist of λ-reductions and the substitution of terms on the RHS of (1) for the correpending terms on the LHS.

4.2 We take over the definition of the * translation from §3 , setting of course, $\alpha^* = \alpha$, $0^* = 0$, $(S^0)^* = S^0$, $(+_0)^* = +_0$. If we can find, for each type α, a term $(R^\alpha)^*$ which satisfies:

(1) $$(R^\alpha)^* \in M \ ;$$

(2a) $$G <_\alpha (R^\alpha)^*\ 0\ H\ G$$

(2b) $$Hm((R^\alpha)^*\ m\ HG) <_\alpha (R^\alpha)^*\ (Sm)H\ G$$

for any $m \in T_0$, $G \in M_\alpha$, $H \in M_{(0\to(\alpha\to\alpha))}$; then, *mutatis mutandis*, we can take over the proof of theorem 3.3.

In fact, if we set $(R^\alpha)^*$ =

$$\lambda y_0\ \lambda h_{(0\to(\alpha\to\alpha))}\ \lambda g_\alpha.R^\alpha y(\lambda x_0\ \lambda u_\alpha.S^\alpha(hxu +_\alpha u))(S^\alpha(g +_\alpha hOL_\alpha)),$$

then (1), (2a), (2b) are indeed satisfied. For we have

(3) $(R^\alpha)^*\ 0\,H\,G = S^\alpha(G +_\alpha H\,0\,L_\alpha)$,

(4) $(R^\alpha)^*(Sm)HG = S^\alpha(Hm((R^\alpha)^*m\,H\,G) +_\alpha (R^\alpha)^*m\,H\,G)$.

Now (3) shows that $(R^\alpha)^*\ 0\,H\,G$ in monotonic in H and G; (4) shows by induction that this is also true for $(R^\alpha)^*m\,G\,H$ for any m. Also by (4) and an obvious variant of lemma 2.3. we see that $(R^\alpha)^*m\,H\,G <_\alpha (R^\alpha)^*(Sm)HG$, so that $(R^\alpha)^*$ is monotonic in its first argument. Finally (3) and (4) and lemma 2.3 show that (2a), (2b) are satisfied. This completes the proof of strong normalisation for Gödel's T; we have of course assumed weak normalisation by supposing that a term of type 0 will denote a natural number.

5 EXTENSION TO DIRECT PRODUCT AND DIRECT SUM

5.1 We extend any typed λ-calculus to a typed 'λ-ν-π-calculus' by adding new type symbols, constants and terms as follows.

(A) If α_1 and α_2 are type symbols so are $\alpha_1\times\alpha_2$ and $\alpha_1+\alpha_2$.

(B) $P^{\alpha_1\times\alpha_2}$ (for 'pair') is a constant of type $(\alpha_1\to(\alpha_2\to(\alpha_1\times\alpha_2)))$.

(C) For k = 1,2, $k_\pi^{\alpha_1\times\alpha_2}$ (for 'projection') is a constant of type $((\alpha_1\times\alpha_2)\to\alpha_k)$.

(D) For k = 1,2, $k_\iota^{\alpha_1+\alpha_2}$ (for 'injection') is a constant of type $(\alpha_k\to(\alpha_1+\alpha_2))$.

From now on, k will always range over {1,2}.

(E) If $\sigma_1(=\sigma_1(x_{\alpha_1}))$ and $\sigma_2(=\sigma_2(y_{\alpha_2}))$ are terms of type β, then

$$\nu x_{\alpha_1}\, y_{\alpha_2}[\sigma_1(x_{\alpha_1}),\ \sigma_2(y_{\alpha_2})]$$

is a term of type $((\alpha_1+\alpha_2)\to\beta)$; in it the free occurrences of x_{α_1} in σ_1 and of y_{α_2} in σ_2 are bound by the operator $\nu x_{\alpha_1}\, y_{\alpha_2}$.

5.2 These constants satisfy the following identities:

(1) $\quad {}^{k}\Pi^{\alpha_1\times\alpha_2}(P^{\alpha_1\times\alpha_2}\sigma_1\,\sigma_2) = \sigma_k$.

(2) $\quad (\nu x_{\alpha_1}\, y_{\alpha_2}[\sigma_1(x_{\alpha_1}),\ \sigma_2(y_{\alpha_2})])({}^{k}\iota^{\alpha_1+\alpha_2}\tau_k) = \sigma_k(\tau_k)$,

where τ_k is a term of type α_k, and the free variables of τ_k do not occur bound in σ_k .

An immediate π (resp. ν) reduction consists of substituting for a part of a term having the form of the LHS of (1) (resp. (2)) the corresponding RHS.

5.3 All the above corresponds exact to the usual category-theoretic treatment of $\times$ and $+$. In particular,

$$f_1\times f_2 = \lambda z_\beta\ P^{\alpha_1\times\alpha_2}(f_1 z)(f_2 z) \qquad (f_k \text{ of type } (\beta\to\alpha_k))\ ,$$

$$g_1+g_2 = \nu x_{\alpha_1}\ y_{\alpha_2}\ [g_1 x,\ g_2 y] \qquad (g_k \text{ of type } (\alpha_k\to\beta))\ ,$$

are the maps from β into in $\alpha_1\times\alpha_2$ and from $\alpha_1+\alpha_2$ into β which are given by the universal properties of $\times$ and $+$.

In proof-theoretic terms $\times$ and $+$ correspond to $\wedge$ and $\vee$, and P and ${}^{k}\Pi$ correspond to the introduction and elimination rules for $\wedge$. ${}^{k}\iota$ corresponds to 'strong' $\vee$-introduction; (i.e. in an inference from α to $\alpha\vee\alpha$ the rule indicates whether the proof of α is to be attached to the left or right occurrence of α in $\alpha\vee\alpha$). Finally, $(\nu x_{\alpha_1}\ y_{\alpha_2}[\sigma_1,\ \sigma_2])\tau$ is a (term for a) derivation which ends with the $\vee$-elimination rule, where τ is the derivation of the major premise, σ_1 and σ_2 are the derivations of the minor premises, and x_{α_1}, y_{α_2} are those assumptions of α_1 in σ_1 and α_2 in σ_2 which are cancelled by the application of the rule.[3]

5.4 In order to extend the proof of strong normalisation to the new calculus, we first extend the definition of §1.

For direct product the extension of the notion of monotonicity is unproblematic. We set

(1) $\quad M_{(\alpha_1\times\alpha_2)} = M_{\alpha_1}\times M_{\alpha_2} = \{(a,b)\colon a\in M_{\alpha_1},\ b\in M_{\alpha_2}\}$,

(2) $\quad (a,b) <_{(\alpha_1 \times \alpha_2)} (a',b') \Leftrightarrow a <_{\alpha_1} a' \wedge b <_{\alpha_2} b'$.

For direct sum the most natural definitions are:

(3) $\quad M_{(\alpha_1+\alpha_2)} = \{(k,a) : k = 1,2 \wedge a \in M_{\alpha_k}\}$,

(4) $\quad (k,a) <_{(\alpha_1+\alpha_2)} (k',a') \Leftrightarrow k = k' \wedge a <_{\alpha_k} a'$.

The denotations of the constants P, ${}^{k}\Pi$, ${}^{k}\iota$ are the obvious ones; e.g., if $a \in M_{\alpha_k}$ then ${}^{k}\iota a = (k,a)$; evidently the denotations of ${}^{k}\Pi$ and ${}^{k}\iota$ belong to M. For a ν term:

(5) $$(\nu x_{\alpha_1} y_{\alpha_2} [\sigma_1, \sigma_2])^A a = (\lambda x_{\alpha_1} \sigma_1)^A a' \quad \text{if } a = (1,a'),$$
$$= (\lambda y_{\alpha_2} \sigma_2)^A a' \quad \text{if } a = (2,a').$$

An extension of the notion of λ-I term, namely the property of being ν-I term is obtained by adding to the definition given in 1.2 the following clauses:-

(i') the constants ${}^{k}\Pi$, ${}^{k}\iota$ are ν-I terms;

(iv) if σ_1 and σ_2 are ν-I terms of types α_1 and α_2 which have exactly the same free variables, then ${}^{k}\mathrm{P}^{\alpha_1 \times \alpha_2} \sigma_1\sigma_2$ is a ν-I term of type $(\alpha_1 \times \alpha_2)$;

(v) if σ_1 and σ_2 are ν-I terms of type β then $\nu x_{\alpha_1} y_{\alpha_2} [\sigma_1, \sigma_2]$ (= ρ, say) is a ν-I terms of type $((\alpha_1+\alpha_2) \to \beta)$ provided that:

(a) x_{α_1} occurs free in σ_1 and y_{α_2} occurs free in σ_2 ;

(b) any variable which occurs free in ρ occurs free in both σ_1 and σ_2; note that this precludes, e.g. x_{α_1} from occurring free in σ_2 .

It is now straightforward to extend the proof of 1.3 to obtain:

5.5 THEOREM Under any assignment A of values in M to the free variable of the ν-I term ρ, $\rho^A \in M$ and the value ρ^A is monotonic in the assignment A.

Observe that if ρ is as in (iv) above, then condition (a) ensures that ρ^A belong to $M_{((\alpha_1+\alpha_2)\ \ )}$ and condition (b) ensures that ρ^A is monotonic in the assignment made to any of its free variables.

The corollary 1.4 and its proof are now extended simply by replacing 'λ-I' by 'ν-I'.

5.6 We now extend the work of §2. We will denote by ν^+ the calculus got by extending the λ^+ calculus of §2 (with 0 as the basic type and primitive constants 0, S^0, $+_0$) as in 5.1. The extensions of S and + to the new types are got by adding to the definitions in 2.2 the following clauses:

(1) $$S^{\alpha_1\times\alpha_2} = \lambda z_{(\alpha_1\times\alpha_2)}\ P(S^{\alpha_1}({}^1\Pi z)(S^{\alpha_2}({}^2\Pi z)).$$

(2) $$S^{\alpha_1+\alpha_2} = \nu x_{\alpha_1}\ y_{\alpha_2}\ [{}^1\iota(S^{\alpha_1}x),\ {}^2\iota(S^{\alpha_2}y)]\ .$$

(3) $$+_{\alpha_1\times\alpha_2} = \lambda x_{(\alpha_1\times\alpha_2)}\ y_{(\alpha_1\times\alpha_2)}\ P({}^1\Pi x +_{\alpha_1} {}^1\Pi y)({}^2\Pi x +_{\alpha_2} {}^2\Pi y)$$

Note that we often omit type superscripts and subscripts; note also that the RH sides of the above are ν-I^+ terms (and so have denotations in M). For addition in the direct sum there are several alternatives; the reason for the particular choice made is that it preserves 2.3. We assume (see below) that we have already defined closed ν-I^+ terms $L_{(\alpha_1\to\alpha_2)}$ and $L_{(\alpha_2\to\alpha_1)}$. Then we define

$$L^1_{((\alpha_1+\alpha_2)\to\alpha_1)} = \nu x_{\alpha_1}\ y_{\alpha_2}\ [x,\ L_{(\alpha_2\to\alpha_1)}y]\ ,$$

$$L^2_{((\alpha_1+\alpha_2)\to\alpha_2)} = \nu x_{\alpha_1}\ y_{\alpha_2}\ [L_{(\alpha_1\to\alpha_2)}\ x, y]\ .$$

Now we set

(4) $$+_{\alpha_1+\alpha_2} = \lambda u_{(\alpha_1+\alpha_2)}\ \lambda v_{(\alpha_1+\alpha_2)}.(\nu x_{\alpha_1}\ y_{\alpha_2}\ [\sigma_1,\sigma_2])u\ ,$$

where

$$\sigma_1 \text{ is } {}^1\iota(x +_{\alpha_1} L^1_{((\alpha_1+\alpha_2)\to\alpha_1)}v) ,$$

$$\sigma_2 \text{ is } {}^2\iota(y +_{\alpha_2} L^2_{((\alpha_1+\alpha_2)\to\alpha_2)}v) .$$

By inspection the RHS of (4) is a ν-I^+ term. Two examples may illuminate:-

$$(1,a) +_{\alpha_1+\alpha_2} (2,b) = (1,a +_{\alpha_1} L_{(\alpha_2\to\alpha_1)}b) ,$$

$$(2,b) +_{\alpha_1+\alpha_2} (2,b') = (2,b +_{\alpha_2} b') ,$$

(where $a \in M_{\alpha_1}$, $b,b' \in M_{\alpha_2}$). This addition is not commutative, but as is readily checked, the inequality 2.3 continues to hold.

We now add clauses to the inductive definition 2.4 of the (monotonic) constants.

(v) $$L_{\alpha_1\times\alpha_2} = P\, L_{\alpha_1}\, L_{\alpha_2} .$$

(vi) $$L_{\alpha_1+\alpha_2} = {}^1\iota\, L_{\alpha_1} , \qquad (\text{or} = {}^2\iota\, L_{\alpha_2})$$

(vii) $$L_{((\alpha_1\times\alpha_2)\to 0)} = \lambda z_{(\alpha_1\times\alpha_2)}.L_{(\alpha_1\to 0)}({}^1\Pi z) +_0 L_{(\alpha_2\to 0)}({}^2\Pi z)$$

(viii) $$L_{((\alpha_1+\alpha_2)\to 0)} = \nu x_{\alpha_1}\, y_{\alpha_2}\, [L_{(\alpha_1\to 0)}\, x,\, L_{(\alpha_2\to 0)}\, y] .$$

(ix) $$L_{(\beta\to(\alpha_1\times\alpha_2))} = \lambda y_\beta\, P(L_{(\beta\to\alpha_1)}y)(L_{(\beta\to\alpha_2)}\, y).$$

(x) $$L_{(\beta\to(\alpha_1+\alpha_2))} = \lambda y_\beta.{}^1\iota(L_{(\beta\to\alpha_1)}y) ,$$

$$\text{or} = \lambda y_\beta.{}^2\iota(L_{(\beta\to\alpha_2)}y)).$$

In conjunction with 2.4 (ii)-(iii), (vii) and (viii) give a definition of $L_{(\alpha\to 0)}$ for every type α; then 2.4 (iv), (ix) and (x) give a definition of $L_{(\alpha\to\beta)}$ for all α,β. Finally 2.4 (i), (v), (vi) give a definition of L_γ when γ is not of the form $(\alpha\to\beta)$. By 5.5, the denotations of all these constants belong to M. Of course there is a good deal of arbitrariness in the particular definitions we have made; in particular, $L_{((\alpha_1+\alpha_2)\to\alpha_1)}$ may differ from the L^1 used in the definition of $+_{\alpha_1+\alpha_2}$.

5.7 Now we can extend the results of §3. To avoid a galaxy of stars we consider only the case where 0 is the only basic type; so we take $\alpha^* = \alpha$. To the clauses (iii) to (v) of 3.2 we add:

(vi) $({}^k\Pi)^*$ is ${}^k\Pi$ and $({}^k\iota)^*$ is ${}^k\iota$.

(vii) $(P^{\alpha_1\times\alpha_2})^*$ is $\lambda x_{\alpha_1} y_{\alpha_2}\, P^{\alpha_1\times\alpha_2}(S^{\alpha_1}\sigma_1)(S^{\alpha_2}\sigma_2)$, where

σ_1 is $(x +_{\alpha_1} L_{(\alpha_2\to\alpha_1)}y)$ and σ_2 is $(y +_{\alpha_2} L_{(\alpha_1\to\alpha_2)}x)$.

(viii) If σ_1, σ_2 are of type β then $(\nu x_{\alpha_1} y_{\alpha_2} [\sigma_1(x), \sigma_2(y)])^*$ is $\nu x_{\alpha_1} y_{\alpha_2}\ [S^\beta \pi_1(x), S^\beta \pi_2(y)]$,

where

$\pi_1(x)$ is $\sigma_1^*(x) +_\beta (\sigma_2^*(L_{(\alpha_1\to\alpha_2)}x) +_\beta L_{(\alpha_1\to\beta)}x)$,

$\pi_2(y)$ is $\sigma_2^*(y) +_\beta (\sigma_1^*(L_{(\alpha_2\to\alpha_1)}y) +_\beta L_{(\alpha_2\to\beta)}y)$:

we assume that the bound variable x_{α_1} is chosen so as not to collide with the free variables σ_2^*, similarly for y_{α_2}.

By inspection (and induction) for any term ρ, $(\rho)^*$ is a ν-I^+ term. Further, the operation * commutes with substitution and, as remarked in 5.6, the inequality 2.3 still holds. Hence 3.1(2) holds; and further:

(1) $\quad ((\sigma_k(\tau))^{*A} <_\beta (((\nu x_{\alpha_1} y_{\alpha_2}[\sigma_1(x), \sigma_2(y)])({}^k\iota\tau))^*)^A$,

(2) $\quad (\sigma_k^*)^A <_{\alpha_k} (({}^k\Pi^{\alpha_1\times\alpha_2}(P^{\alpha_1\times\alpha_2}\sigma_1\,\sigma_2))^*)^A$,

for any assignment A of values in M to all the relevant free variables. Thus in any immediate reduction, the value of the * of the reduct is less than the value of the * of the corresponding redex. So we can conclude:

5.8 THEOREM Theorem 3.3 holds for the typed λ-ν-π calculus. (For the computation of the bound for n see the Postscriptum).

5.9 *Discussion.* The reductions we have considered correspond exactly to the $\supset$, & and $\vee$-reductions given in Prawitz (1971); further his 'immediate simplifications')p. 254) correspond to norm-reducing operations on terms. But in order to arrive at proofs without maximal segments, Prawitz also introduces '$\vee$E reductions' (p. 253). In our notation the trickiest of these is represented by the passage from the LHS to the RHS of the equation:

(1) $$\rho((\nu x_{\alpha_1}\, y_{\alpha_2} \ulcorner\sigma_1, \sigma_2\urcorner)\tau) = (\nu x_{\alpha_1}\, y_{\alpha_2} [\rho\sigma_1, \rho\sigma_2])\tau \quad ,$$

where τ is of type $\alpha_1 + \alpha_2$, σ_1 and σ_2 are of type $\beta_1 + \beta_2$ and ρ is a ν-term of type $((\beta_1+\beta_2)\to\gamma)$. Note that both sides of (1) will have the same value. Suppose that ρ is capable of p successive reductions; then the RHS of (1) will be capable of 2p reductions, and so we may expect that the norm of the RHS will be <u>greater</u> than the norm of the LHS. At this point we seem to be paying a price for the conceptual clarity gained by our extensional interpretation: the value of $(\sigma\tau)$ depends only on the values assigned to σ and τ, not on the particular forms of the terms σ and τ. I think I have found a way round this difficulty: one interprets terms in a structure built up by (hereditarily) forming finite sets of elements in the original type structure; in particular, a typical ν-term will have a denotation which is the disjoint union of the sets denoted by σ_1 and σ_2. But the formal complications make the idea rather unattractive.

6. FURTHER EXTENSIONS

6.1 The method extends easily to the intuitionistic predicate calculus. The description of proofs by terms involves the introduction of atomic types indexed by the type I of individuals. But when we pass to the star interpretation all these atomic types are mapped into type 0, and we can take I^* (= κ, say) to consist of a single element: $M_\kappa = \{k\}$. We do not admit κ as a proper type-symbol, but give the rules: if α is a type symbol so are $(\kappa\to\alpha)$ and $(\kappa\times\alpha)$, corresponding respectively to universal and existential quantification. The rules of term formation are extended by: (i) if ρ is a variable x_κ or the constant k_κ and τ is a $[\nu\text{-}I^+]$ term of type $(\kappa\to\alpha)$ then $(\tau\rho)$ is a $[\nu\text{-}I^+]$ term of type α and $(P^{\kappa\times\alpha}\rho)$ is a $[\nu\text{-}I^+]$ term of type $(\alpha\to(\kappa\times\alpha))$; (ii) if τ is a $[\nu\text{-}I^+]$ terms of type α then $(\lambda x_\kappa\ \tau)$ is a $[\nu\text{-}I^+]$ term of type $(\kappa\to\alpha)$. Then $M_{\kappa\to\alpha}$ will consist of <u>all</u> functions from M_κ to M_α, ordered by their unique value. The ordering of $M_{\kappa\times\alpha}$ $(= M_\kappa \times M_\alpha)$ is, naturally, given by:

$$(k,\ a) <_{\kappa\times\alpha} (k,\ a') \Leftrightarrow a <_\alpha a' .$$

Then it is readily checked that Theorem 5.5 holds for the extended system. There is no difficulty in extending the definitions of S, + and < to the new types, nor in proving Theorem 5.8 for the extended system. In applying this result to proofs we first observe that a $\forall$ contraction will correspond to a λ-reduction. Instead of introducing further constants, we suppose that the term corresponding to a proof ending with $\exists$-elimination is formed in such a way that its star has the form:

$$(\lambda x_\kappa\ \lambda y_\alpha\ \tau)({}^1\Pi\sigma)({}^2\Pi\sigma)$$

(But note that if, e.g. σ is a variable of type $\kappa\times\alpha$ corresponding to an assumption $\exists x\ A(x)$, there are no proof-theoretic operations corresponding to ${}^k\Pi\sigma$, so that the λ-reduction is not allowed). Then an $\exists$-contraction corresponds to the replacement of $(\lambda x_\kappa\ y_\alpha\ \tau(x,y))({}^1\Pi(p_{\rho\sigma}))({}^2\Pi(p_{\rho\sigma}))$ by $\tau(\rho,\sigma)$. Hence Thereom 5.8 does yield an upper bound for the number of proof reduction.

6.3 The method can also be applied if $(T_0,<)$ is taken to be an initial segment of the ordinals; 2.3 will hold if $+_0$ is the natural sum. With suitable term forming operations this could, for example, be applied to Howard's (1972) calculus for primitive recursive functionals of finite type over the ordinals.

6.3. In conclusion we comment on the difference between our method of proof and those previously published. They use an inductively defined property of terms (proofs) such as 'strong derivability' or 'computability'; the clauses of the definition involve a universal quantifier over certain collections of terms. In our proof this inductive definition is tucked away in the definitions of 'ϵM_α' and '$<_\alpha$'. This has two advantages: firstly we only require an induction over types, not over terms; and secondly our inductively defined properties correspond to natural and easily graspable concepts, so that one does not have continually to recall the details of the definition when following the proof.

POSTSCRIPTUM

Theorem 5.8 tells us that the number denoted by $|\rho^T|$ is an upper bound to the length of any reduction sequence in a typed λ-ν-π-calculus starting with the term ρ, where ρ^T is a closed term of the ν^+-calculus got by substituting constants L_β of appropriate type for the free variables in ρ^*. We wish to show how $|\rho^T|$ can be computed. In the first place, we have

THEOREM I Any term of a typed λ-ν-π-calculus can be reduced to a normal form (i.e. one for which no immediate reduction can be made).

The proof of this is a straightforward extension of the proof of normalization for the typed λ-calculus (see, e.g., Turing's proof appearing in this volume) and we omit it.

One can also prove a Church-Rosser theorem for the typed λ-ν-π-calculus, but that is not necessary for our purpose. All we need to do is to extend 2.6 and this we now do.

THEOREM II A closed term τ of type 0 of the ν^+-calculus in normal form contains no symbols other than 0, S^0, $+_0$ (and parentheses).

The proof of this is distinctly more tiresome than for the λ-calculus. We apologise in advance for the excessive use of *reduction and absurdum*; more direct proofs seem necessarily to involve much subdivision into cases and several repeated arguments.

Proof of Theorem II. In analysing the forms of terms we ignore parentheses. A term is *peculiar* if it is of type $\alpha_1 \times \alpha_2$ and not of the form $P\sigma_1\sigma_2$, or of the type $\alpha_1 + \alpha_2$ and not of the form ${}^k\iota\sigma$. A term is *standard* if it is a λ term or a ν term or if it is one of the form $P\sigma$, $P\sigma_1\sigma_2$, ${}^k\iota\sigma$. A term is *non-numerical* if it is not 0_0 and if it does not begin with S^0 nor with $+_0$ (which we are now supposing to be written in front of its arguments).

Observe that peculiar and standard terms are non-numerical and cannot be of type 0; also that a peculiar term is not standard.

LEMMA 1 If ρ is a closed non-numerical term in normal form which consists of more than one symbol, then either ρ is standard, or it contains a proper sub-term which is both closed and peculiar.

Proof. Suppose ρ satisfies the premise and is not standard. It cannot be a λ or a ν term, and so must have the form $(\rho_1\ \rho_2)$. Here ρ_1 cannot begin with λ since ρ is in normal form; nor can ρ_1 begin with P or ${}^k\iota$, for then ρ would be standard. Hence either ρ_1 is ${}^k\Pi$ and ρ_2 is a closed peculiar term, or ρ_1 is, or begins with, a ν term σ_1 and then ρ begins with, or is, a term $(\sigma_1\ \sigma_2)$ where σ_2 is closed and peculiar.

COROLLARY 1 A peculiar term in normal form cannot be closed.

For consider, if possible, a minimal closed peculiar term in normal form. It satisfies the premise of Lemma 1 and is not standard, contradicting the conclusion.

COROLLARY 2 If ρ satisfies the premise of the lemma it is standard.

LEMMA 2 If τ is a closed term of type 0 in normal form then it cannot have any standard sub-terms.

Proof. Consider, if possible, a maximal standard sub-term σ of τ. Then σ cannot be a part of a λ term or a ν term and so it is closed and must occur in a closed part ρ of τ having the form $\rho = (\sigma\pi)$ or $\rho = (\pi\sigma)$. By the observation preceeding Lemma 1, σ is non numerical and not of type 0. But then in either case ρ is non numerical, and therefore satisfies the promise of Lemma 1; so, by Corollary 2, ρ is standard, contradicting the maximality of σ.

Proof of Theorem II resumed . Let τ be a closed term of type 0 in normal form. By Lemma 2 τ cannot contain λ or ν , and hence cannot contain any variables. Suppose that τ contains one of the constants P, ${}^{k}\iota$, ${}^{k}\Pi$ and let σ be the leftermost occurrence of such a constant. Since σ is not of type 0 it must occur in a part of τ of the form $(\sigma\rho)$. By Lemma 2, σ cannot be P or ${}^{k}\iota$; and if it were ${}^{k}\Pi$ then ρ would be closed and peculiar, contradicting Corollary 1. Thus τ can only contain the symbols 0_o, S^0, $+_0$, as was to be proved.[2]

FOOTNOTES

1. I was stimulated to discover the idea of the proof presented here by my distaste for the proof given in Prawitz (1971) which he presented when he was visiting Oxford in 1975-76. A manuscript version of the idea (dated March 1976) has been in circulation for sometime; the occasion of Haskell Curry's 80-th birthday provided the necessary spur to produce a printed and fuller version.

2. *Notes added in proof*. (i) It seems that everyone who extends the typed λ-calculus to include cartesian product,

direct sum and so on does it in their own way. The advantage of the ν-terms introduced here is that they follow closely the category-theoretic approach. But other notations may in fact be simpler to handle; in particular one can replace a ν-term by a term built up from two λ-terms. Martin-Löf ((in 1975) and many mimeographed typescripts) is perhaps the ultimate authority on extensions of simple type-theory; but he has been more concerned to bring out the meaning of the various operations than to provide a smooth-running mechanism of notations.

(ii) Since I handed over my manuscript to the typist I have learnt of two other forms of proof of strong normalization.

(a) Nederpelt (1973) replaces β-reduction by 'β_1-reduction' in which each reduction step leaves a trace of the unreduced term behind. The hard work in the proof is then concentrated in proving a Church-Rosser theorem for β_1-reduction. (b) Minc (1979) (of which I have only read the abstract) gives a way of computing the maximum length of a reduction sequence from the length of a standard reduction sequence. In this connection the following remarks may be of interest. Let τ be a closed λ^+-term of type 0, let n be its length (not counting type subscripts) and let p be the length of the type-symbol for the type of the subterm(s) of highest type occurring in τ. By considering the standard reduction sequence it is a straightforward matter to compute an upper bound for the value of τ; it can be given in the form of exponentiating n to the base 2 p' times were p' increases with p. The work of Statman (1979) suggests that a lower bound for the value of τ will have the same form. In other words a function which takes one from a description of a term to its value may not be elementary. On the other hand it is not hard to show that, for τ as above, the length of τ^* is bounded by n.p' where p' is got by exponentiating p a small fixed number of times. The complexity of the function giving the maximum length of a reduction sequence is only slightly greater than the complexity of the function giving the minimum length. It seems to me just possible that this will be false if one allows Prawitz's reductions (see 5.9).

(3) The remark made in 5.3 about 'strong ∨' is ill informed. However in the λ-ν-π^+ calculus, + is strong in the following sense. Set

$$C = \nu x_{\alpha_1} y_{\alpha_2} [0, S0],\ D^2 = \nu x_{\alpha_1} y_{\alpha_2} [x_{\alpha_1}, L_{\alpha_1}],\ D^2 = \nu x_{\alpha_1} y_{\alpha_2} [L_{\alpha_2}, y_{\alpha_2}];$$

then $C({}^1\iota\sigma)$ reduced to 0, $C({}^2\iota\tau)$ reduces to S0, $D^1({}^1\iota\sigma)$ reduces to σ and $D^2({}^2\iota\tau)$ reduces to τ, where σ is of type α_1, τ of type α_2. (This footnote added in proof.)

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