COMBINATORICS AND GEOMETRY

in İstanbul

August 5-6, 2025

Bahçeşehir University

Table of Contents

About CGI 2025

Welcome to the "Combinatorics and Geometry in İstanbul" workshop, taking place on August 5-6 in the enchanting city of İstanbul. This two-day event brings together researchers and enthusiasts working at the intersection of algebra, combinatorics, and geometry, aiming to foster collaboration and the exchange of ideas in a vibrant academic setting.

The program features a rich collection of talks by both invited and contributing participants, highlighting recent developments and open problems across a range of topics. We hope to provide a stimulating environment for sharing ideas, initiating new collaborations, and deepening our understanding of these interconnected fields.

We are delighted to convene this event in such a unique and inspiring location and look forward to two days of stimulating academic exchange and collaboration.

Our Sponsors

We gratefully acknowledge the support of

Navist Mühendislik	TÜBİTAK	Bahçeşehir University	Türk Matematik Derneği MAD programı
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Selçuk Kayacan (Bahçeşehir University)

Local Info

Transportation: İstanbul has two airports. Istanbul Aiport (IST) and Sabiha Gökçen Airport (SAW).

How to come from Istanbul Airport: at the airport exits for a convenient transfer to your destination. Alternatively, you can take the Havaist Airport Buses, which provide reliable and affordable transportation options. For more detailed information, visit the official İstanbul Airport website.

Also, you can take the M11 metro to Gayrettepe, and take a municipal bus to Beşiktaş.

How to come from Sabiha Gökçen Airport: Taxis are available at the airport exits for a direct transfer to your destination. Alternatively, you can take Public Buses or the Havabus Shuttle to reach Kadıköy or Taksim.

- From Kadıköy, you can take a ferry to Beşiktaş
- From Taksim, you can reach Beşiktaş by bus (multiple lines are available) or by Dolmuş (shared taxi).

Another option is to take the M4 Metro to Kadıköy and then transfer to a ferry heading to Beşiktaş.

From İstanbul Bus Station (Esenler Otogar): Please note that Esenler Otogar is the main terminal for intercity buses. However, depending on your bus company, you may have the option to disembark at an earlier station, which could make transferring to Beşiktaş more convenient. Additionally, some bus companies offer complimentary shuttle services directly to Beşiktaş, so be sure to check with your provider.

If you arrive at Esenler Otogar, you can follow this route to reach Beşiktaş:

1. Take the M1A Metro heading to Yenikapı.
2. At Yenikapı, transfer to the M2 Metro and travel to Taksim.
3. From Taksim, take a bus (multiple lines are available) or a Dolmuş (shared taxi) to Beşiktaş.

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You can also manage your İstanbulkart using the official İstanbulkart App, which makes it easy to check your balance, recharge your card, and plan your trips. Download the app here:

- Google Play Store
- Apple App Store

Route Planning Apps:

For planning your routes, you can use the following apps:

İETT "Otobüsüm Nerede" App: This official app by İETT (İstanbul Electric Tramway and Tunnel) helps you find the best bus routes and track buses in real-time. Download the app here:

- Google Play Store
- Apple App Store

Google Maps: Google Maps is also highly reliable for route planning in İstanbul. It provides detailed directions for public transportation, walking, and driving.

Tip: While these apps are helpful, asking a local for directions can sometimes be the best way to navigate İstanbul's public transportation system.

CGI 2025
Tuesday Program

8:30 - 9:15	**Registration**	
9:15 - 9:30	Opening Talk	
9:30 - 10:30	Ayesha Asloob Qureshi (Sabancı University) *Squarefree powers of facet ideals of simplicial trees*	
10:30 - 11:30	Mikhail Ostrovskii (St. John's University) *Low-distortion embeddings of graphs with large girth*	
11:30 - 12:20	Coffee break (20 minutes)	Omar Tout (Sultan Qaboos University) *On the product of two equal conjugacy classes of the symmetric group*
12:20 - 12:50	Ezgi Kantarcı Oğuz (Galatasaray University) *Oriented Posets and Cluster Algebras*	
12:50 - 14:00	**Lunch and collaboration**	
14:00 - 15:00	Russ Woodroofe (University of Primorska FAMNIT) *Extremal set theory as algebraic geometry*	
15:00 - 15:50	Sofiya Ostrovska (Atılım University) *Geometry of the metric on Q+ induced by limit q-Bernstein operators*	Coffee break
15:50 - 16:50	Reymond Akpanya (RWTH Aachen University) *Construction of Toroidal Polyhedra corresponding to perfect Chains of isosceles Tetrahedra*	Coşar Gözükırmızı (İstanbul Beykent University) *Implementation of Enhanced Multivariance Products Representation for Multiway Arrays*
16:50 - 17:40	Marko Pešović (University of Belgrade) *Weighted quasisymmetric functions and stabilization of graph-associated generalized permutohedra*	Coffee break
17:40 - 18:40	Dariush Kiani (Amirkabir Univ. of Technology) *Combinatorial coefficients in data clustering*	Şafak Özden (TED University) *Stability Property of Block Permutations*
18:40 - 19:10	Marie Amalore Nambi (Sabancı University) *Binomial Edge Ideals of Linear Type*	

CGI 2025
Wednesday Program

09:30 - 10:00	Tolga Birdal (Imperial College London) *Topological Deep Learning: Going Beyond Graph Data*	
10:00 - 11:00	Martina Juhnke (University of Osnabrück) *???*	
11:00 - 11:50	Coffee Break (20 minutes)	Abhiram Natarajan (University of Warwick) *Partitioning Theorems for Sets of Semi-Pfaffian Sets, with Applications*
11:50 - 12:50	David Yost (Federation University) *Classification of polytopes with not many edges*	Mehmet Akif Yıldız (Centrum Wiskunde & Informatica (CWI)) *Path Partitions in Regular Digraphs*
12:50 - 14:00	**Lunch & Collaboration**	
14:00 - 15:00	Volkmar Welker (Philipps-Universität Marburg) *Symmetries in Simplicial and Cubical Homology Theories*	
15:00 - 16:00	Qays R. Shakir (Middle Technical University) *Duality of Some Topological Graph Operations on Surface Graphs*	Aslı Tuğcuoğlu Musapaşaoğlu (Sabancı University) *Invariants of Toric Double Determinantal Rings*
16:00 - 16:20	Coffee break	
16:20 - 17:20	Yusuf Civan (Süleyman Demirel University) *Problems surrounding the Hadwiger number of graphs topologically*	
17:20 - 18:20	Damir Ferizović (KU Leuven) *Point distributions on the sphere: a biased introduction*	Ugo Detaille (RWTH Aachen University) *Cycle Double Covers of Cubic Graphs with given Automorphism Groups*
18:20 - 18:50	Mevlüde Alizade (Ankara University) *On Two-Faced Simplicial Surfaces*	
20:00 - ...	**Workshop dinner at Hamdi restaurant**	

Abstracts for Invited Talks

Tolga Birdal
(Imperial College London, UK, `t.birdal@imperial.ac.uk`)

Topological Deep Learning: Going Beyond Graph Data

Topological deep learning is a rapidly growing field that pertains to the development of deep learning models for data supported on topological domains such as simplicial complexes, cell complexes, and hypergraphs, which generalize many domains encountered in scientific computations. In this talk, Tolga will present a unifying deep learning framework built upon an even richer data structure that includes widely adopted topological domains. Specifically, he will begin by introducing combinatorial complexes, a novel type of topological domain. Combinatorial complexes can be seen as generalizations of graphs that maintain certain desirable properties. Similar to hypergraphs, combinatorial complexes impose no constraints on the set of relations. In addition, combinatorial complexes permit the construction of hierarchical higher-order relations, analogous to those found in simplicial and cell complexes. Thus, combinatorial complexes generalize and combine useful traits of both hypergraphs and cell complexes, which have emerged as two promising abstractions that facilitate the generalization of graph neural networks to topological spaces. Second, building upon combinatorial complexes and their rich combinatorial and algebraic structure, Tolga will develop a general class of message-passing combinatorial complex neural networks (CCNNs), focusing primarily on attention-based CCNNs. He will additionally characterize permutation and orientation equivariances of CCNNs, and discuss pooling and unpooling operations within CCNNs. The performance of CCNNs on tasks related to mesh shape analysis and graph learning will be provided. The experiments demonstrate that CCNNs have competitive performance as compared to state-of-the-art deep learning models specifically tailored to the same tasks. These findings demonstrate the advantages of incorporating higher-order relations into deep learning models and shows great promise for AI4Science.

Keywords: topology, deep learning, high-order networks

References:

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[2] Papamarkou, T., Birdal, T., Bronstein, M. M., Carlsson, G. E., Curry, J., Gao, Y., ... & Zamzmi, G., Position: Topological Deep Learning is the New Frontier for Relational Learning, *International Conference on Machine Learning*, 39529-39555 (2024).

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Yusuf Civan
(Süleyman Demirel University, Turkey, `yusufcivan@sdu.edu.tr`)

Problems surrounding the Hadwiger number of graphs topologically

For a (finite and simple) graph G, its Hadwiger number $had(G)$ is defined to be the largest integer h such that G contains the complete graph K_h as a minor. The most long standing and intriguing conjecture of Hugo Hadwiger (1943) claims that the inequality $\chi(G) \leq had(G)$ holds for every graph G, where $\chi(G)$ denotes the chromatic number. As opposed to the Hadwiger number, the chromatic number of graphs is topologically lower bounded. That brings the question of whether these topological bounds are also valid lower bounds to the Hadwiger number. I hope to address these questions in detail as well as a recent conjectural detection of the Hadwiger number due to Holmsen, Kim and Lee (2019) in terms of the homological dimension of hypergraphs of connected covers. Independent of the later conjecture, I will prove that the Helly number of (simple) hypergraphs can be topologically lower bounded.

Martina Juhnke
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title

To solve a linear program, the simplex method follows a path in the graph of a polytope, on which a linear function increases. The length of this path is a key measure of the complexity of the simplex method.

Our starting point is a conjecture by Jesús De Loera stating that the number of paths counted according to their length forms a unimodal sequence.

We give examples (old and new) for which this conjecture is true but we disprove this conjecture by constructing counterexamples for several classes of polytopes. However, we show that De Loera is "statistically correct": We prove that the length of a coherent path on a random polytope (with vertices chosen uniformly on a sphere) admits a central limit theorem.

This is joint work with Germain Poullot.

Mikhail I. Ostrovskii
(Department of Mathematics and Computer Science, St. John's University,Queens, 11439, New York, USA, `ostrovsm@stjohns.edu`)

Low-distortion embeddings of graphs with large girth

There exist families of graphs with indefinitely growing girths which admit uniformly bilipschitz embeddings into L_1, and thus do not weakly contain any families of expanders. This result was proved by the speaker [5], answering the problem raised in [3] and publicized in the 2002 International Congress [2]. During the last decade, this fact has found important applications; see [4]. Recently, interest in the result increased because of its connection to the transportation cost theory and related fields; see [1]. In the talk, I am going to describe the construction and some of the possible directions of further research.

MSC 2010: 46B85, 51F30

Keywords: girth of a graph, low-distortion embedding, transportation cost

References:

[1] C. Gartland, Hyperbolic metric spaces and stochastic embeddings. Forum Math. Sigma **13** (2025), Paper No. e29.

[2] N. Linial, Finite metric spaces–combinatorics, geometry and algorithms, in: *Proceedings of the International Congress of Mathematicians*, Vol. **III** (Beijing, 2002), 573–586, Higher Ed. Press, Beijing, 2002.

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Ayesha Asloob Qureshi
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Squarefree powers of facet ideals of simplicial trees

Let I be a squarefree monomial ideal. The k-th squarefree power $I^{[k]}$ of I is the ideal generated by the squarefree monomials among the generators of I^k. The study of squarefree powers of squarefree monomial ideals is closely connected with the classical theory of matchings in hypergraphs. Moreover, squarefree powers of I provide important information about the ordinary powers of I, since the multigraded minimal free resolution of $I^{[k]}$ appears as a subcomplex of the multigraded minimal free resolution of I^k. We will discuss the squarefree powers of facet ideals associated with simplicial trees (equivalently, totally balanced hypergraphs), focusing on the linearity of their minimal free resolutions, Castelnuovo–Mumford regularity and projective dimension.

Volkmar Welker
(Philipps-Universität Marburg, Germany, welker@mathematik.uni-marburg.de)

Symmetries in Simplicial and Cubical Homology Theories

In this talk we discuss how symmetries of simplcies and cubes can be used to reduce the size of chain complexes computing homology. For simplcial complexes this is a classical fact. We present a cubical homology theory of graphs which exhibits hyperoctahedral symmetries. We then show the in characterisitic 0 the symmetry chain complex is acyclic for all simplicial and cunbical sets with symmetries.

Russ Woodroofe
(University of Primorska FAMNIT, Slovenia, `russ.woodroofe@famnit.upr.si`)

Extremal set theory as algebraic geometry

Extremal set theory is interested in questions such as "what is the largest family of pairwise intersecting k-element subsets of an n-element set?" Many questions of this type can be modeled with exterior algebras, and admit extensions such as "what is the largest self-annihilating subspace of k-forms in an exterior algebra with n variables"? The shifting technique in extremal set theory corresponds to algebraic geometric limits of certain curves in varieties. In this talk, I will overview these connections, and go on to discuss recent joint work with Bulavka and Gandini. In this work, we find a maximum subspace of k-forms in an n-variable exterior algebra that is self-annihilating but that is not annihilated by any 1-form. This extends the Hilton-Milner theorem of extremal set theory to the exterior algebra setting, and answers questions of Scott and Wilmer and of myself.

Abstracts for Contributed Talks

Reymond Akpanya
(RWTH Aachen University, Germany, reymond.akpanya@rwth-aachen.de)

Construction of Toroidal Polyhedra corresponding to perfect Chains of isosceles Tetrahedra

In 1957, Steinhaus conjectured that a chain of regular tetrahedra, meeting face-to-face and forming a closed loop, does not exist [1]; this was later proven by Świerczkowski, see [2]. We show that modifying the statement by requiring the tetrahedra of a chain to be isosceles results in the first examples of closed chains with all tetrahedra being congruent. As a result, we provide a census of toroidal polyhedra arising from closed chains consisting of up to 20 isosceles tetrahedra, see [3]. Moreover, we establish the existence of an infinite family of toroidal polyhedra emerging from chains of isosceles tetrahedra. Finally, we exploit our methods to construct clusters of isosceles tetrahedra that yield polyhedra of higher genera. This is joint work with Vanishree Krishna Kirekod, Alice C. Niemeyer and Daniel Robertz [4].

MSC 2010: 52B05, 05C15

Keywords: Tetrahedral chains, Toroidal polyhedron, polyhedral realization

References:

[1] Hugo Steinhaus. *Problème 175.* In *Colloquium Mathematicum*, volume 4, page 243, 1957.

[2] S. Świerczkowski, *On chains of regular tetrahedra*, Colloq. Math. **7** (1959), 9–10. DOI: 10.4064/cm-7-1-9-10.

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Mevlüde Alizade
(Ankara University, Turkey, 21050024@ogrenci.ankara.edu.tr)

On Two-Faced Simplicial Surfaces

A simplicial surface can be seen as the incidence geometry of the faces, edges and vertices of a triangulated surface [4, 7]. Such a surface is *two-faced*, if it is closed and the automorphism group splits the faces of the surface into exactly two orbits. This work extends the research initiated by Akpanya and Spreer on the classification of face-transitive simplicial surfaces [1].

The primary purpose of this project is to investigate this class of highly symmetric simplicial surfaces. Specifically, we define the *neighbour type* as a pair (n_1, n_2) recording how many faces from one orbit are adjacent to faces in the other orbit, and introduce corresponding *orbit-stabiliser tuples* to distinguish surfaces with similar adjacency patterns. We determine exactly 12 distinct types of two-faced simplicial surfaces, and provide examples by exploiting the existing databases of simplicial surfaces [6] and cubic graphs [2, 8]. These computations are carried out in GAP [3]. Furthermore, we present various examples of two-faced simplicial spheres, and compute embeddings of these spheres into $\mathbb{R}^3$ as polyhedra having all edge lengths equal to 1. These polyhedra are obtained by solving nonlinear equations in Maple [5]. This project is supervised by Alice C. Niemeyer, Reymond Akpanya and Meike Weiß.

MSC 2010: 05E18, 52B70, 05C10

Keywords: Simplicial surfaces, Group actions on surface triangulations, Cubic graphs, Graph colourings

References:

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Ugo Detaille
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Cycle Double Covers of Cubic Graphs with given Automorphism Groups

Many open problems in graph theory, such as the Cycle Double Cover (CDC) Conjecture [1, 2], have been shown to reduce to the case of cubic graphs. Therefore, the study of cubic graphs has attracted a lot of interest in recent years, see [3] for instance. Given a finite group G and a generating set for G, László Babai presents a construction of a cubic graph whose automorphism group is isomorphic to G [4]. We expand his graph construction to obtain a cubic graph also with automorphism group G, which further admits a 1-Cut-CDC, i.e. a CDC in which any two cycles intersect in at most one edge. Moreover, we prove that our constructed CDC is invariant under the action of the group of the graph. We provide a GAP-implementation for the graph construction and its CDC [5, 6]. This talk is based on the Bachelor's thesis in [7] supervised by Alice C. Niemeyer, Meike Weiß and Reymond Akpanya (RWTH Aachen University).

MSC 2010: 05E18, 20B25

Keywords: Cycle Double Cover, Cubic Graphs, Cayley Graphs, Graphs with prescribed Symmetry

References:

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Damir Ferizović
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Point distributions on the sphere: a biased introduction

The start of investigation into uniform distribution on the unit interval originated from the paper of Weyl [1] and since has been generalized to various compact manifolds or groups. In this talk we will give an overview of nicely distributed point sets on the unit 2-sphere of $\mathbb{R}^3$ that are based on my recent papers [2, 3]. Such point sets could for instance be applied to help design spherical detectors as they appear in Fermilab's MiniBooNE neutrino experiment. Expect many pictures.

MSC 2010: 11K38, 52C99

Keywords: Uniform distribution, discrepancy, two sphere

References:

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Implementation of Enhanced Multivariance Products Representation for Multiway Arrays

High Dimensional Model Representation (HDMR) is a representation for functions as sum of terms of increasing number of variables. HDMR has been adapted for the discrete case to decompose multiway arrays. Therefore, there is HDMR for functions and also HDMR for multiway arrays. Enhanced Multivariance Products Representation (EMPR) is a generalization of HDMR where there are univariate supports (univariate functions for representing functions; and vectors for representing multiway arrays). Introducing these parameters into the finite expansion provides the ability for the truncations to better represent the original function or multiway array. Within this work, we limit ourselves to the decomposition of multiway arrays. Therefore, we use HDMR and EMPR to mean HDMR for multiway arrays, and EMPR for multiway arrays respectively. This work looks at Enhanced Multivariance Products Representation (certain generalization of HDMR) from tensor network perspective. The work provides the equalities for computing the components and also the representation itself, using tensor network diagram notation. We implemented a computer program for Enhanced Multivariance Products Representation (EMPR) for multiway arrays in Julia using ITensor library. This is the first publicly available code for EMPR for multiway arrays and can be downloaded from the url `https://gitlab.com/cosargozukirmizi/emprmaj` and can be used freely. The algorithm is quite general and is able to compute all EMPR components, truncations, remainders and quality measurers for any multiway array of floating-point numbers, as long as the computational resources allow it.

Keywords: high dimensional model representation, multidimensional array decomposition, tensor network diagram

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Combinatorial coefficients in data clustering

Data clustering is an unsupervised machine learning technique which puts data into some meaningful groups, or namely clusters, based on certain similarities among data points. By minimizing intra-cluster variation and maximizing inter-cluster differences, clustering helps reveal the underlying structure of complex datasets. Clustering has been widely used in various areas such as image analysis, bioinformatics, market segmentation and anomaly detection.

Besides other approaches, graph theory provides a powerful framework for data clustering. A well known method is based on modeling relationships between data points as a graph, where nodes represent elements and edges encode similarities or distances. One of the useful algorithms based on graph theory is the so-called spectral clustering algorithm which applies the Laplacian matrix of the graph associated to data points. This method often reveal natural groupings that may be difficult to detect using traditional clustering techniques. Indeed, spectral clustering, as a prominent example of clustering methods, leverages the eigenvectors of graph Laplacians to identify non-convex clusters that are not easily captured by traditional clustering algorithms. For more information on some classical applications of graph spectra in computer science, see for example [1, 3].

In this talk, we discuss two other approaches for data clustering based on graphs and certain related algebraic structures. Indeed, we associate certain graphs, based on Delaunay triangulations, to the data. Then we discuss certain approaches for constructing feature vectors associated to each graph. The feature vectors arise from the coefficients of the Ihara polynomial [2] as well as characteristic polynomial of the graph. Then the obtained feature vectors can be used as the input of certain clustering algorithms, like K-means and HAC algorithms.

MSC 2010: 05C90

Keywords: Graphs, clustering, feature vectors

References:
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Invariants of Toric Double Determinantal Rings

This is a joint work with J. Biermann, E. De Negri, O. Gasanova and S. Roy. In this work, we study a class of double determinantal ideals denoted I^r_{mn}, which are generated by minors of size 2, and show that they are equal to the Hibi rings of certain finite distributive lattices. We compute the number of minimal generators of I^r_{mn}, as well as the multiplicity, regularity, a-invariant, Hilbert function, and h-polynomial of the ring R/I^r_{mn}, and we give a new proof of the dimension of R/I^r_{mn}. We also characterize when the ring R/I^r_{mn} is Gorenstein, thereby answering a question of Li in the toric case. Finally, we give combinatorial descriptions of the facets of the Stanley-Reisner complex of the initial ideal of I^r_{mn} with respect to a diagonal term order.

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Binomial Edge Ideals of Linear Type

An ideal I of a commutative ring R is said to be of *linear type* when its Rees algebra and symmetric algebra exhibit isomorphism. In this talk, we discuss the conjecture put forth by Jayanthan, Kumar, and Sarkar in [1] that if G is a tree or a unicyclic graph, then the binomial edge ideal of G is of linear type.

MSC 2010: 05E40, 13F65

Keywords: Rees algebra, linear type, binomial edge ideal

References:
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Partitioning Theorems for Sets of Semi-Pfaffian Sets, with Applications

We generalize the seminal polynomial partitioning theorems of Guth and Katz [2,1] to a set of semi-Pfaffian sets. Specifically, given a set $\Gamma \subseteq \mathbb{R}^n$ of k-dimensional semi-Pfaffian sets, where each $\gamma \in \Gamma$ is defined by a fixed number of Pfaffian functions, and each Pfaffian function is in turn defined with respect to a Pfaffian chain $\vec{q}$ of length r, for any $D \geq 1$, we prove the existence of a polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$ of degree at most D such that each connected component of $\mathbb{R}^n \setminus Z(P)$ intersects at most $\sim \frac{|\Gamma|}{D^{n-k-r}}$ elements of Γ. Also, under some mild conditions on $\vec{q}$, for any $D \geq 1$, we prove the existence of a Pfaffian function P' of degree at most D defined with respect to $\vec{q}$, such that each connected component of $\mathbb{R}^n \setminus Z(P')$ intersects at most $\sim \frac{|\Gamma|}{D^{n-k}}$ elements of Γ. To do so, given a k-dimensional semi-Pfaffian set $\gamma \subseteq \mathbb{R}^n$, and a polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$ of degree at most D, we establish a uniform bound on the number of connected components of $\mathbb{R}^n \setminus Z(P)$ that γ intersects; that is, we prove that the number of connected components of $(\mathbb{R}^n \setminus Z(P)) \cap \gamma$ is at most $\sim D^{k+r}$. Finally, as applications, we derive Pfaffian versions of Szemerédi-Trotter-type theorems, and also prove bounds on the number of joints between Pfaffian curves.

Keywords: o-minimal, incidence geometry, discrete geometry, pfaffian functions, pfaffian sets, semi-pfaffian sets, joints, szemeredi-trotter, polynomial partitioning, pfaffian partitioning, partitioning, guth, katz

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Oriented Posets and Cluster Algebras

Oriented posets are posets with specialized end points, which we can connect to build larger posets. They come with corresponding matrices, where multiplication corresponds to linking posets. We look at the case of labeled fence posets and show that they can be used to calculate cluster expansions effectively. We look at more potential applications and open questions.

Keywords: Posets, Lattices, Diophantine Equations, Cluster Algebras.

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Geometry of the metric on $\mathbb{Q}_+$ induced by limit q-Bernstein operators

In this talk, some geometric properties of the uniformly discrete metric space $(\mathbb{Q}_+, \rho)$, where

$$\rho(a/b, c/d) := 2(m-1)/m, \quad m = \frac{\max\{ad, bc\}}{\gcd(ad, bc)} \tag{1}$$

are discussed. We study this metric space in the spirit of the classical Blumenthal's book [1]. The metric space $(\mathbb{Q}_+, \rho)$ emerges in the study of the limit q-Bernstein operators $B_q, q \in [0,1]$. These operators occur naturally in the investigation of the q-Bernstein polynomials proposed by G. M. Phillips in [3] and they form a one-parametric family of positive linear operators of the unit norm on $C[0,1]$. It is known that the mapping $q \mapsto B_q$ is continuous in the strong operator topology for all $q \in [0,1]$, while, in the uniform operator topology, the mapping is discontinuous for each $q \in [0,1]$.

What is more, the set $\{B_q\}_{q\in[0,1]}$ equipped with the metric $d(B_q, B_r) := \|B_q - B_r\|$, where $\|.\|$ is the operator norm on $C[0,1]$, forms a uniformly discrete metric space, where each operator B_q is an isolated point and $1 \leqslant \|B_q - B_r\| \leqslant 2$ whenever $q \neq r$. See [2].

It is exactly the metric d that induces the metric (1) on $\mathbb{Q}_+$.

MSC 2010: 46B85, 51K05

Keywords: metric basis , isometry, uniformly discrete metric space

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Stability Property of Block Permutations

We investigate the Hecke algebras $H_{n,k} = \mathbb{C}\big[(S_k \wr S_n)\backslash S_{kn}/(S_k \wr S_n)\big]$, $k \geq 2$, arising from the double–coset convolution algebra of the wreath-product pair $(S_{kn}, S_k \wr S_n)$. Using the action of S_{kn} on the set of k-partitions of $\{1, \ldots, kn\}$ we equip $H_{n,k}$ with a natural filtration indexed by orbit data. A graph–theoretic encoding of double cosets—via "red-blue" type graphs and their *modified types*— yield:
i) Stability. The associated filtered algebras $F_{n,k}$ are independent of n; hence the family $\{H_{n,k}\}_{n\geq 1}$ satisfies a Farahat–Higman–style [1] stability property and is governed by a universal filtered algebra $F_{\infty,k}$ [2,3].
ii) Polynomiality. Every structure constant $c^L_{M,N}(n)$ in the canonical basis depends polynomially on n; if the weight equality $\|L\| = \|M\| + \|N\|$ holds, $c^L_{M,N}(n)$ is actually constant
iii) New non-commutative cases. For $k > 2$ the algebras $H_{n,k}$ are non-commutative; our results extend the previously understood commutative case $k = 2$ [4] (the Brauer–symmetric pair (S_{2n}, B_n)) and supply the first full stability theorem for Hecke algebras of higher wreath products.

Keywords: Representation Theory, Gel'fand Pairs , Farahat-Higman Algebra, Block Permutations, Induced Representations, Hecke Algebras

MSC 2010: 20C30, 20C05, 20C07, 19A22

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Weighted quasisymmetric functions and stabilization of graph-associated generalized permutohedra

For a simple graph Γ on n vertices, we define a sequence of generalized permutohedra

$$Q_{\Gamma,1},\, Q_{\Gamma,2},\, \ldots,\, Q_{\Gamma,n-1},$$

where each polytope $Q_{\Gamma,m}$ is the Minkowski sum of simplices indexed by subsets of vertices that induce connected subgraphs, with the constraint that each subset has cardinality at most $m+1$. This sequence interpolates between two well-studied objects: the *graphical zonotope* $Q_{\Gamma,1}$ and the *graph-associahedron* $Q_{\Gamma,n-1}$, both known for their deep combinatorial and geometric properties.

For each polytope $Q_{\Gamma,m}$, we associate a weighted quasisymmetric function F_q, defined as

$$F_q(Q_{\Gamma,m}) := \sum_{\omega=(\omega_1,\omega_2,\ldots,\omega_n)\in\mathbb{Z}_+^n} x_{\omega_1}x_{\omega_2}\cdots x_{\omega_n}\, q^{\dim(C_\omega)},$$

where C_ω denotes the face of the polytope $Q_{\Gamma,m}$ whose normal cone contains the vector ω in its relative interior. This weighted quasisymmetric function encodes the f–polynomial as a special case, and generalizes the Stanley chromatic symmetric function [4] for graphical zonotopes, as well as the chromatic quasisymmetric function [1] for graph-associahedra. Results in [2] include a detailed study of the stabilization behavior of these functions

$$F_q(Q_{\Gamma,1}),\, F_q(Q_{\Gamma,2}),\, \ldots,\, F_q(Q_{\Gamma,n-1})$$

and their associated polytopes in terms of normal equivalences, revealing new structural insights into this family of polytopes and their combinatorial invariants.

In the special case $q=0$, the function $F_0(Q_{\Gamma,m})$ can be expressed as a sum of quasisymmetric enumerators of P-partitions of certain newly introduced $\mathcal{H}$*–posets* associated to the vertices of the polytope $Q_{\Gamma,m}$. For $m=1$, note that these $\mathcal{H}$–posets correspond to the transitive closures of *acyclic orientations* of the graph Γ, whereas for $m=n-1$, they correspond to $\mathcal{B}$*–trees*, defined in [3].

MSC 2010: 05E99, 52B11, 05C99

Keywords: Generalized permutohedron, Quasisymmetric function, Graphs

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Duality of Some Topological Graph Operations on Surface Graphs

A surface graph is an embedding of an abstract graph in a surface without edge crossings, [1] and [3]. The theory that studies such graphs is called topological graph theory. In such a theory, topological graph operations play a significant role in building various classes of surface graphs. Given minimal surface graphs in the class of interest, we can use certain graph operations to build the class, starting from the minimal graphs and then extending them [2]. In a duality setting, it is crucial to know the duality of each topological operation to conduct simultaneous constructing of a surface graph class and its dual class.

Our focus in this work is on some fundamental operations like digon, triangle and quadrilateral operations. We claim that the duals of these three operations are the topological Henneberg operations. We demonstrate our claim via various tools such as rotation system, delta matroid and polar duality.

MSC 2010: 05C10, 05C76

Keywords: Surface graphs , Graph operations, Rotation systems, Delta matroid

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On the product of two equal conjugacy classes of the symmetric group

Let n be a positive integer. A partition of n is a list of non-increasing integers $(\lambda_1, \lambda_2, \cdots, \lambda_r)$ that sum up to n. We will write λ in the multiplicative way $\lambda = (1^{m_1}, 2^{m_2}, \cdots)$ where m_i is the number of times the part i appears in λ. For example, $(1^7, 3^3, 4)$ is a partition of 20. The cycle-type of a permutation σ of n, denoted $\mathrm{ct}(\sigma)$, is the partition of n obtained from the length of the cycles of σ. For example, the cycle type of the permutation $(3, 6, 2)(1, 5)(4, 7)$ is the partition $(2^2, 3)$ of 7. The conjugacy classes of $\mathcal{S}_n$, the symmetric group on n elements, are in one-to-one correspondence with the set of all partitions of n. If λ is a partition of n, its corresponding conjugacy class is C_λ, where $C_\lambda := \{\sigma \in \mathcal{S}_n : \mathrm{ct}(\sigma) = \lambda\}$. The structure coefficients $c^{\rho}_{\lambda\delta}$ of the center of the symmetric group algebra are defined by the equation:

$$\mathbf{C}_\lambda \mathbf{C}_\delta = \sum_{\rho \text{ partition of } n} c^{\rho}_{\lambda\delta} \mathbf{C}_\rho.$$

Computing these coefficients allows us to compute any product in $Z(\mathbb{C}[\mathcal{S}_n])$. However, it is very difficult to give explicit formulas for these coefficients, even in particular cases of partitions, see [1] and [2]. It can be shown, see [3] and [4], that the structure coefficients of $Z(\mathbb{C}[\mathcal{S}_n])$ are polynomial in n. We show explicit formulas for the diagonal structure coefficients $c^{\lambda}_{(2^k)(2^k)}$ and $c^{\lambda}_{(1,2^k)(1,2^k)}$.

MSC 2010: 05E15, 20C08

Keywords: The center of the Symmetric group algebra, diagonal structure coefficients, Product of conjugacy classes, representation theory of the Symmetric group

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Title

The problem of partitioning a graph into as few vertex-disjoint paths as possible was introduced by Ore [6], who showed that every n-vertex graph can be partitioned into at most $n-\sigma_2$ vertex-disjoint paths, where σ_2 denotes the minimum degree sum over all pairs of non-adjacent vertices. Magnant and Martin [4] conjectured that *regularity* significantly improves this bound: every n-vertex d-regular graph can be partitioned into at most $n/(d+1)$ vertex-disjoint paths. The conjecture has been confirmed for $d \leq 5$ in [4], for $d = 6$ in [1], and for $d = \Omega(n)$ in [2]. A recent work [5] has also established an approximate version. A generalization to directed and oriented graphs was proposed in [3], along with a proof for the case $d = \Omega(n)$. In this work, we prove the directed analogue for $d \leq 5$ and the oriented analogue for $d \leq 3$. Our approach also yields a significantly shorter proof of the undirected case for $d = 6$ than the one in [1].

MSC 2010: 05C35, 05C38

Keywords: path, regular, partition

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Classification of polytopes with not many edges

The combinatorial classification of polytopes under reasonable restrictions is a long respected undertaking. The classification of all d-dimensional polytopes with $v = d + 2$ or $d + 3$ vertices is fairly well understood. For $v = d + 4$, the complete classification is known only for $d \leq 5$ [2], and for $v = d + 5$, it was solved only recently for $d = 4$ [1]. Both of these solutions provide a huge database of examples. We propose to go in a different direction, characterising polytopes of all dimensions but with restrictions also on the number e of edges.

Any vertex of a d-dimensional polytope must have degree at least d. It is natural to define the **excess degree** of a vertex as its degree minus the dimension, and the excess degree of a polytope as the sum of the excess degrees of its vertices. The excess degree of a polytope is easily checked to equal $2e - dv$. Polytopes with excess 0 are the simple polytopes; an application of the g-theorem shows that for a simple polytope, v must either equal $d+1, 2d, 3d-3$ or $3d-1$, or exceed $4d-8$. A sample easy consequence of this is that no simple 6-polytope can have 8, 9, 10, 11, 13 or 14 vertices.

It is known [4] that there are no d-polytopes with excess degree in the range $[1, d-3]$.

In any d-polyope with excess $d-2$, there is either a single vertex with excess $d-2$, or there are $d-2$ vertices with excess 1, which form a simplex face. We use this to completely classify all polytopes with excess degree $d-2$ and strictly less than $3d$ vertices. For each $d \geq 5$, there are precisely 6 examples. From these examples, we see that v must satisfy either $v = d+2, 2d-1, 2d+1$, or $3d-2$, or $v \geq 3d$.

We have recently shown that polytopes with excess d, $d+2$ or $2d-6$ also have strong limitations on their structure. In general, the nonsimple vertices all have the same degree, and they form the vertex set of a face. Moreover no polytope at all has excess degree in the range $[d+3, 2d-7]$.

MSC 2010: 52B05, 52B12

Keywords: polytope, excess degree

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