Business Calculus I

Math 1743
University of Oklahoma
Spring 2023

Instructor Workbook

This workbook is adapted from *Calculus Concepts: An Informal Approach to the Mathematics of Change, 5/e* by Latorre et. al.

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Chapter 1

Ingredients of Change: Functions & Limits

Functions: Four Representations

Representations of Change

In mathematics, particularly applied mathematics, we need to be able to interpret real-world phenomena in four ways: numerically, algebraically, verbally, and graphically.

Example 1.1.1. The price of gas at a certain 7-11 in Norman was \$4.39 per gallon on June 26th. Represent this data in four ways.

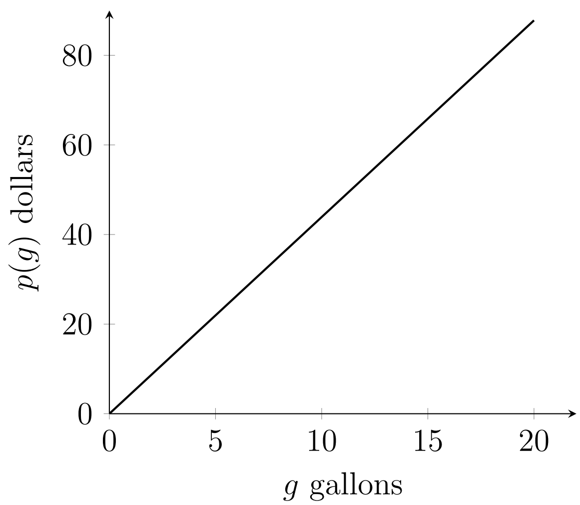
(1) *Numerically*: We can numerically represent the data by placing values in a table.

Gasoline Pumped (in gallons)	0	1	5	10	15	20
Total Cost (in USD)	0	4.39	21.95	43.90	65.85	87.80

(2) *Algebraically*: Since we are paying \$4.39 for every gallon, it is reasonable to express the situation by the function $p(g) = 4.39g$ dollars (total cost), with g gallons pumped.

(3) *Verbally*: The problem is given to us verbally, but using we'll rephrase it to sound more like what we would expect in this class. The price at the pump for gasoline is \$4.39 per gallon of gasoline pumped.

(4) *Graphically*: We may use a graph to display this same information. Since we created the function $p(g) = 4.39g$, we can plot this in order to create a graphical representation of the data.



The process of using information like this to generate something usable is called *mathematical modeling*, and we call the result a **model**. Business Calculus courses place heavy emphasis on developing and deploying models.

Functions & Representations

A **relation** is a rule which links an **input** variable to an **output**; given one piece of information, we can determine the corresponding piece. A special type of relation is one called a function.

Definition 1.1.2 (Function)

A **function** is a rule that assigns a single output per input value. For a given output function f, and given input value x, this is notated $f(x)$.

IT IS VERY IMPORTANT THAT YOU UNDERSTAND THIS NOTATION. One of the most common mistakes in 1743 and 2123 is a misunderstanding of how function notation works. The letters chosen (f, g, h, k, g, A, etc.) indicate the *name* of the function, and the numbers/variables inside the parentheses indicate *what the function is being applied to.* A way to remember this is to read the expression $f(x)$ as "f of x".

Example 1.1.3. Let g be a function. Write the correct notation for the following situations:

(a) g applied to the number 5

(b) g applied to the number 10

(c) g applied to the variable x

(d) g applied to the variable y

(e) g applied to the expression $x + 1$

(f) g applied to the expression $10 - y$

(g) g applied to the expression $x + h$

Example 1.1.4. Evaluate the function $f(x) = 3x - 2$ at the inputs:

(a) $x = 2$

(b) $x = 3$

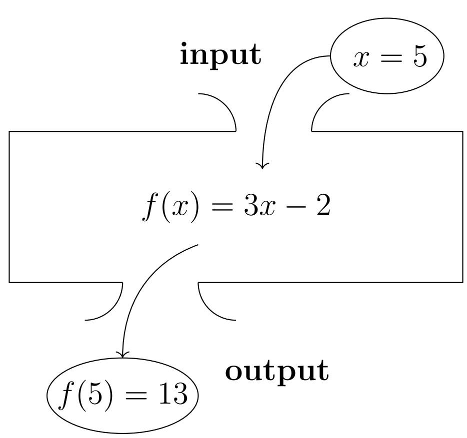
(c) $x = -4$

(d) $x = k$

(e) $x = k + 7$

(f) $x = 3k + 21$

We may also represent functions using an **input/output diagram**. One is given below, for the previous example:



Every function is a relation, but not every relation is a function. If a relation gives more than one output value for even a single input value, then it cannot be a function. This can be determined using a verbal, numerical, or graphical description of the data.

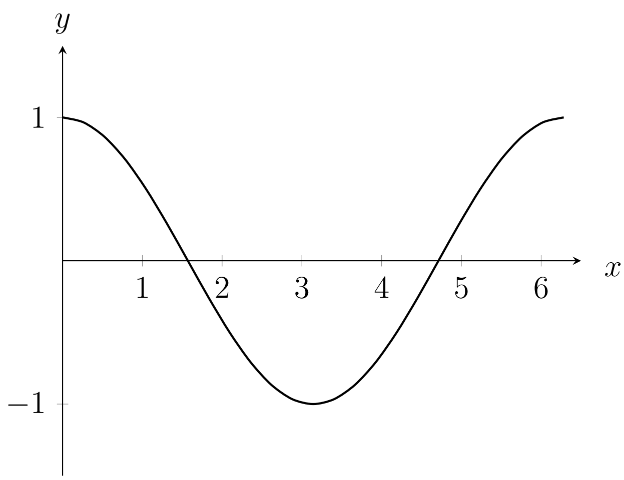
Example 1.1.5. Let $C(t)$ represent the number courses offered campus-wide during the week at time t, and $O(t)$ represent the number of students walking on the South Oval at time t last Monday. Is C a function? What about O?

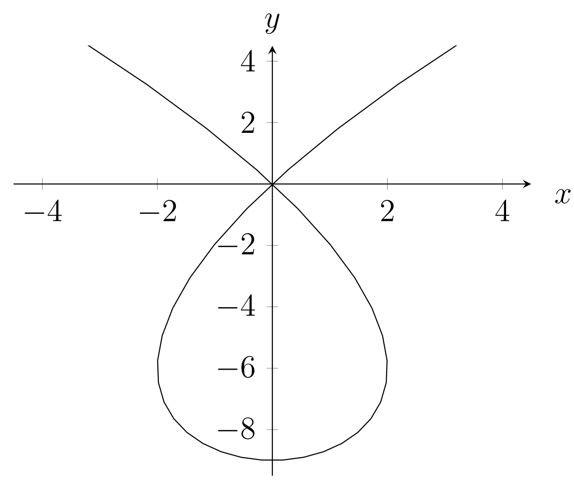
Example 1.1.6. Below are numerical expressions for the functions h and k. Is h a function? What about k?

x	0	1	1	2	5	6
$h(x)$	0	1	2	3	4	5

t	0	1	1	2	5	6
$k(t)$	0	1	1	3	4	5

Example 1.1.7. Are both of these graphs functions? Why or why not?





Model Output and Units of Measure

In real-world applications, the proper units of measure must be attached to a model and every result derived from that model; in this way, we can gain meaningful information from whatever is it we do. The verbal description of a function gives us the units of measure. In our first example, our input unit is *gallons*, and our output unit is *dollars*.

Example 1.1.8. The population of Canada between 1900 and 2010 is given by the model

$$p(t) = 3(1.03^t) \text{ million people}$$

where t is the number of years since the end of 1900.

(a) When did the population reach 155 million people? Write a sentence interpreting the result.

(b) Determine the population in the year 1990. Write a sentence interpreting the result.

(c) Give a description and the unit of measure for both the input and output variables.

(d) Draw an input/output diagram for p, and a graph of p.

Example 1.1.9. Calculate the output value that corresponds to the inputs $t = 4.5$ and $t = -2$ for the function $m(t) = \frac{3}{8}t + 2$.

Example 1.1.10. Calculate the output value that corresponds to the inputs $x = 10$ and $x = -3$ for the function $f(x) = 7x^2 - 2x - 3$.

Example 1.1.11. Let $f(x) = 2.5 \ln x + 3$.

(a) Does the expression $f(x) = 7$ ask to find an input or output?

(b) Solve (a).

Example 1.1.12. Let $f(x) = 6.1x + 3.1^x$.

(a) Does the expression $x = 2.5$ ask to find an input or output?

(b) Solve (a).

Example 1.1.13. Let $u(t) = \dfrac{27.4}{1 + 13e^{2t}}$.

(a) Does the expression $u(t) = 15$ ask to find an input or output?

(b) Solve (a).

Example 1.1.14. The number of donors to the American Red Cross Disaster Relief Fund who donated more than x million dollars during 2005 is represented as $d(x)$.

(a) Write a sentence of interpretation for $d(5) = 2$.

(b) Write the function notation for the statement: "Seventy-five groups donated at least \$500,000 to the Disaster Relief Fund in 2005."

Example 1.1.15. The average number of people standing in the Chick-Fil-A line on Wednesdays can be represented by $p(t)$, where t is the number of hours after 12:00pm.

(a) Write a sentence of interpretation for the expression $p(0) = 32$

(b) Write the function notation for the statement "At 1:15pm, there are an average of 15 people in line at Chick-Fil-A".

Function Behavior and End Behavior Limits

Descriptions of Function Behavior

Especially as we consider longer-term models, we are concerned about what happens to a function as time advances.

Definition 1.2.1 (Increasing, Decreasing, Constant)

Let f be a function defined over some input interval. The function is said to be

- **increasing** if the output values $\boxed{\text{increase on the interval}}$
- **decreasing** if the output values $\boxed{\text{decrease on the interval}}$
- **constant** if the output values $\boxed{\text{remain the same on the interval}}$

Example 1.2.2. On what intervals is $g(x) = -x^3 + 16x - 5$ increasing, decreasing, or constant? Use the calculator to help you find the answer.

Example 1.2.3. Is the function given in the table below increasing, decreasing, or constant? Why?

x	2	4	6	8	10
$h(x)$	5	6	8	12	20

Example 1.2.4. The function $f(x) = c$ is constant. Look at the graph and/or table and explain why.

Direction and Curvature

Definition 1.2.5 (Concavity)

A function f defined over an input interval is said to be

- **concave up** if a graph of the function appears to be $\boxed{\text{bending upward}}$
- **concave down** if a graph of the function appears to be $\boxed{\text{bending downward}}$

The curvature of a function is called $\boxed{\text{concavity}}$.

Example 1.2.6. Describe the concavity of $k(t) = -t^2 + 8t - 13$. Does it appear to ever change? If so, where? Draw a picture.

Example 1.2.7. Describe the concavity of the function $f(z) = \ln z$. Does it ever appear to change? If so, where? Draw a picture.

Example 1.2.8. Describe the concavity of the function $g(x) = -2x$. Draw a picture.

Example 1.2.9. Describe the concavity of $h(d) = d^3 - 11d^2 + 38d - 37$. Does it ever appear to change? Draw a picture.

Definition 1.2.10 (Inflection Point)

A point on a continuous function where the concavity of the function changes is called an **inflection point**.

Example 1.2.11. The function $P(t) = 2t^3 - 10t^2 - 3t + 275$ describes the profit (in hundred dollars) made by a small business after a rash of bad Yelp reviews where t is the number of weeks since the reviews were put online. Let $0 \leq t \leq 5$.

(a) Sketch a picture of the graph.

(b) Estimate the input and output values at the inflection point(s).

(c) Identify the intervals where P is increasing, decreasing, and constant.

(d) Identify the intervals where P is concave up, concave down, or neither.

(e) Use the information from (a)-(c) to describe what was happening to the profit made by the business between the first and sixth week.

Limits and End Behavior

Definition 1.2.12 (End Behavior)

The **end behavior** of a function refers to the behavior of the output values of the function as the input values become larger and larger, or smaller and smaller.

As the input values become larger and larger (more and more positive), we say that the input $\boxed{\text{increases without bound}}$. As they become smaller and smaller (more and more negative), the input $\boxed{\text{decreases without bound}}$.

Example 1.2.13. Consider $h(d) = d^3 - 11d^2 + 38d - 37$.

(a) Sketch the function on the interval $[0, 6]$.

(b) In a sentence, describe the end behavior of h as the input increases without bound.

(c) In a sentence, describe the end behavior of h as the input decreases without bound.

There are **three possibilities** when we consider the end behavior of a function:

- The output values may $\boxed{\text{approach or equal a certain number}}$
- The output values may $\boxed{\text{increase or decrease without bound}}$
- The output values may $\boxed{\text{oscillate and fail to approach any particular number}}$

Example 1.2.14. Draw three functions that have will have each of these three end behaviors.

Example 1.2.15. Determine the end behavior/limit of the function $f(x) = \dfrac{2x}{x-1}$ as the input increases without bound, using numerical estimation. Record your approximations with **full decimal accuracy**, and round the final answer to the hundredths.

x	$f(x) = \dfrac{2x}{x-1}$
10	
100	
1000	
10000	
100000	
End Behavior/ Limit	

Note

When creating a table, you need to stop when the digit **after** the one you're rounding to repeats twice.

Definition 1.2.16 (Limit)

A function $f(x)$ is said to have a **limit** L if the $\boxed{\text{output}}$ of f approaches $\boxed{L}$ as the $\boxed{\text{input}}$ approaches some (possibly infinite) value a. We write this using the following notation:

$$\lim_{x \to a} f(x) = L$$

Example 1.2.17. Rewrite the end behavior/limit from the previous example using limit notation.

Definition 1.2.18 (Horizontal Asymptote)

A horizontal line with the equation $\boxed{y = L}$ is called a **horizontal asymptote**.

Example 1.2.19. Let $f(x) = x^2$ and $g(x) = x^3$.

(a) Write the statement "The limit of $f(x)$ as x approaches ∞ is ∞" in limit notation.

(b) Find $\lim_{x \to -\infty} f(x)$, and write the notation in words (like in (a))

(c) Find the end behaviors of $g(x)$, and write them in limit notation.

Example 1.2.20. Sketch the following functions, and use the sketches to find the limit as the input increases without bound and decreases without bound:

(a) $f(x) = \ln x$

(b) $g(x) = e^x$

(c) $h(x) = \dfrac{1}{1 + e^x}$

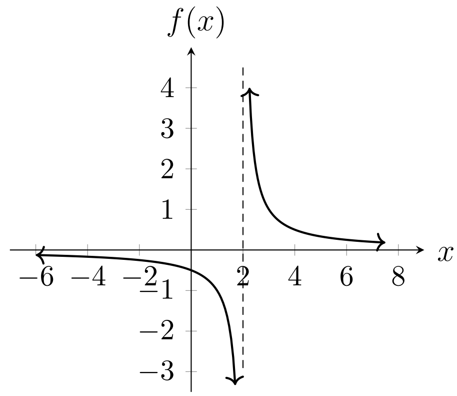
Example 1.2.21. Use numerical estimation to find $\lim_{x\to\infty}(1-0.6^x)$. Make a table showing at least five inputs and the corresponding outputs; write *all* decimals in the table, and round your final answer to two decimal places. Start your input at 2 and double.

Example 1.2.22. Use numerical estimation to find $\lim_{t\to-\infty}\left(1+t^{-2}\right)$. Make a table showing at least five inputs, and the corresponding outputs; write *all* decimals in the table, and round your final answer to the nearest integer. Start your input at -10 and double.

Functions with Unbounded Input

Motivating Example

Consider the function $f(x) = \dfrac{1}{x-2}$, graphed below:



(a) What happens to the output of f as the input increases without bound? Write your answer in limit notation.

(b) What happens to the output of f as the input decreases without bound? Write your answer in limit notation.

(c) What happens to the output of f at $x = 2$?

Left/Right Hand Limits

Definition 1.3.1 (Left/Right Hand Limit)

Let f be a function defined on an interval containing some constant c (except possibly at c itself).

- If $f(x)$ approaches the value of L_1 as x approaches c from the left, then the **left-hand limit** of f is L_1, and is written $\boxed{\lim_{x \to c^-} f(x) = L_1}$
- If $f(x)$ approaches the value of L_2 as x approaches c from the right, then the **right-hand limit** of f is L_2, and is written $\boxed{\lim_{x \to c^+} f(x) = L_2}$

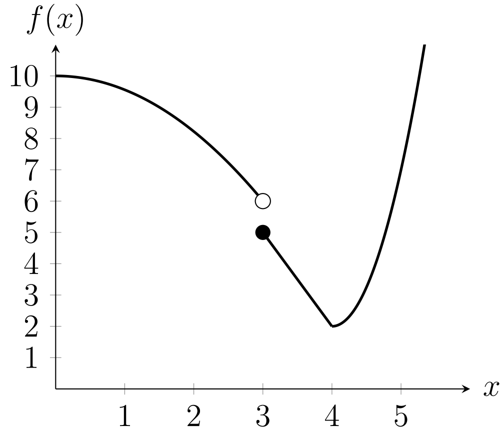
Example 1.3.2. For $f(x) = \dfrac{1}{x-2}$, find $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$.

Example 1.3.3.
Use a calculator to numerically examine the limit behavior of $f(x) = \dfrac{1}{x-2}$ at $x = 2$.

x	$f(x)$	x	$f(x)$
1.9		2.1	
1.99		2.01	
1.999		2.001	
1.9999		2.0001	
1.99999		2.00001	
$\lim_{x \to 2^-} f(x) =$		$\lim_{x \to 2^+} f(x) =$	

$\lim_{x \to 2} f(x) =$ ______________

Example 1.3.4. Use the graph of g to answer the following:



(a) $\lim_{x \to 4^-} g(x) =$

(b) $\lim_{x \to 4^+} g(x) =$

(c) $\lim_{x \to 4} g(x) =$

(d) $\lim_{x \to 3^-} g(x) =$

(e) $\lim_{x \to 3^+} g(x) =$

(f) $\lim_{x \to 3} g(x) =$

Example 1.3.5. Examine the limit behavior of the function $g(t) = \dfrac{3t^2 - 9}{t - 3}$ at $t = 3$. Round to the nearest tenth if necessary.

Example 1.3.6. Use a calculator to examine the limit behavior of the function $r(p) = \dfrac{p^2 - 64}{p + 8}$ at $p = -8$. Round to the nearest thousandth if necessary.

Example 1.3.7. Use a calculator to examine the limit behavior of the function $P(y) = \dfrac{3^y}{2y - 5}$ at $y = 2.5$. Round your answer to 2 decimal places.

Continuity

Definition 1.3.8 (Continuity)

A function $f(x)$, defined on some input interval containing c, is said to be **continuous** at c if and only if the following conditions are satisfied:

(1) $\boxed{\lim_{x \to c} f(x) \text{ exists}}$

(2) $\boxed{f(c) \text{ exists}}$

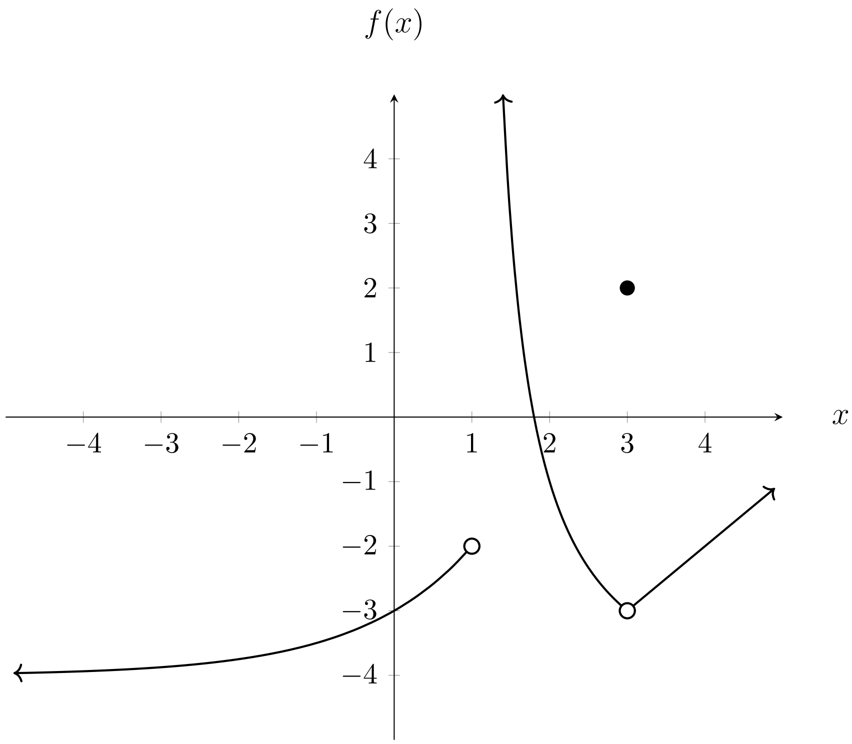
(3) $\boxed{\lim_{x \to c} f(x) = f(c)}$

A function is continuous on any open interval (a, b) if it is continuous at every point inside the interval. If a function is not continuous at the input $x = c$, then we say that f is $\boxed{\text{discontinuous}}$ at c.

Example 1.3.9. Identify any points of discontinuity in the function $f(x) = \dfrac{1}{x-2}$. Explain why the function is discontinuous at those points.

Example 1.3.10. Identify any points of discontinuity in the function $g(x)$ in Example 3.4. Explain why the function is discontinuous at those points.

Example 1.3.11. Use the graph to find the following:



(a) $\lim\limits_{x \to 1^+} f(x)$

(b) $\lim\limits_{x \to 1^-} f(x)$

(c) $\lim\limits_{x \to 1} f(x)$

(d) Is f continuous at $x = 1$?

(e) $\lim\limits_{x \to 3^+} f(x)$

(f) $\lim\limits_{x \to 3^-} f(x)$

(g) $\lim\limits_{x \to 3} f(x)$

(h) Is f continuous at $x = 3$?

(i) $\lim\limits_{x \to 0^+} f(x)$

(j) $\lim\limits_{x \to 0^-} f(x)$

(k) $\lim\limits_{x \to 0} f(x)$

(l) Is f continuous at $x = 0$?

Properties of Limits

Let $f(x)$ and $g(x)$ be continuous functions over some input interval containing c, and k be some arbitrary constant. Then, we have the following properties of limits:

(1) Constant Rule: $\boxed{\lim_{x \to c} k = k}$

(2) Sum Rule: $\boxed{\lim_{x \to c} [f(x) + g(x)]}$

(3) Constant Multiple Rule: $\boxed{\lim_{x \to c} [k \cdot f(x)] = k \lim_{x \to c} f(x)}$

(4) Replacement Rule: If $f(c)$ is defined at c, then $\boxed{\lim_{x \to c} f(x) = f(c)}$

(5) Product Rule: $\boxed{\lim_{x \to c} [f(x) \cdot g(x)] = \left[\lim_{x \to c} f(x)\right] \cdot \left[\lim_{x \to c} g(x)\right]}$

(6) Quotient Rule: $\boxed{\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \text{ (as long as } \lim_{x \to c} g(x) \neq 0)}$

(7) If $f(x)$ can be factored as $f(x) = h(x) \cdot k(x)$, and $g(x)$ can also be factored as $g(x) = j(x) \cdot k(x)$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{h(x) \cdot k(x)}{j(x) \cdot k(x)} = \lim_{x \to c} \frac{h(x)}{j(x)}$$

i.e. common factors may be canceled across fractions under the limit

Example 1.3.12. Algebraically determine the limits of the following:

(a) $\lim_{x \to 5} 9$

(b) $\lim_{z \to 3}(4z - 5)$

(c) $\lim_{t \to -3} \dfrac{t^2 - 4t - 21}{t + 3}$

(d) $\lim_{m \to 13} \dfrac{m}{m^2 + 4m}$

(e) $\lim_{h \to 0} \dfrac{(3 + h)^2 - 9}{h}$

Example 1.3.13. Determine the limit: $\lim_{h \to 0} \dfrac{(5 + h)^2 - 25}{h}$

Example 1.3.14. Let $f(x) = \begin{cases} x^2 & x < -1 \\ 1 & x \geq -1 \end{cases}$. **Algebraically** determine the following limits and answer the questions:

(a) $\lim\limits_{x \to -1^-} f(x)$

(b) $\lim\limits_{x \to -1^+} f(x)$

(c) $f(-1)$

(d) Is f continuous at $x = -1$? Why?

(e) Graph $f(x)$. Do your answers make sense?

Example 1.3.15. Let $h(t) = \begin{cases} 3^t - 9 & t < 2 \\ t^2 - 4 & t \geq 2 \end{cases}$. **Algebraically** determine the following limits and answer the questions:

(a) $\lim\limits_{t \to 2^-} h(t)$

(b) $\lim\limits_{t \to 2^+} h(t)$

(c) $h(2)$

(d) Is h continuous at $t = 2$? Why?

(e) Graph $h(t)$. Do your answers make sense?

Linear Functions & Models

Linear Functions

Remember that a linear function requires two pieces of information- a starting value (b, the y-intercept), and an amount of incremental change in the independent variable (m, the slope of the function). This gives us three ways to describe a linear function:

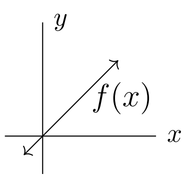
- Verbally: $\boxed{\text{A function with a constant rate of change}}$
- Graphically: $\boxed{\text{There are images below}}$
- Algebraically: $\boxed{f(x) = mx + b}$

$\boxed{\textbf{Question 1.4.1}}$ Given two points (x_1, y_1) and (x_2, y_2), how can we find the slope of the line between them?

Linear Models

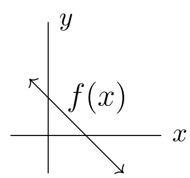
For our general model, $f(x) = ax + b$, we have the following characteristics:

$a > 0$



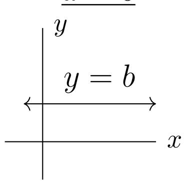
- $\lim\limits_{x\to\infty} f(x) = \boxed{\infty}$
- $\lim\limits_{x\to-\infty} f(x) = \boxed{-\infty}$
- f is always $\boxed{\text{increasing}}$
- f has no concavity

$a < 0$



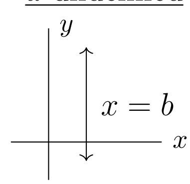
- $\lim\limits_{x\to\infty} f(x) = \boxed{-\infty}$
- $\lim\limits_{x\to-\infty} f(x) = \boxed{\infty}$
- f is always $\boxed{\text{decreasing}}$
- f has no concavity

$a = 0$



- $\lim\limits_{x\to\infty} f(x) = \boxed{b}$
- $\lim\limits_{x\to-\infty} f(x) = \boxed{b}$
- f is always $\boxed{\text{constant}}$
- f has no concavity

a undefined



- $\lim\limits_{x\to\infty} f(x) = \boxed{\text{DNE}}$
- $\lim\limits_{x\to-\infty} f(x) = \boxed{\text{DNE}}$
- Neither inc. nor dec.
- No concavity

For any given graph, the scales ***will*** change; use algebra, don't trust your eyes.

Elements of a Model

From now on, when we refer to a model, we are referring to a specific collection of information. These pieces are listed below; *memorize them*!

(1) Proper and consistent function notation

(2) Model coefficients rounded to **three** decimal places

(3) Output units

(4) Output description

(5) Input units

(6) Input description

Example 1.4.2. The following table gives the percentage of new companies which remained open t years after beginning business.

Years After Opening	5	6	7	8	9	10
Companies Still Open (in %)	50	47	44	41	38	35

(a) Fill in the new inputs if we align the data so that the fifth year corresponds to an input of zero.

Years After Opening						
Companies Still Open (in %)	50	47	44	41	38	35

(b) Use the aligned data to create a **complete** model.

Definition 1.4.3 (Extrapolation)

When using a model, we say that data is **extrapolated** if we find an output value

$\boxed{\text{outside of the interval of the input data.}}$

Definition 1.4.4 (Interpolation)

When using a model, we say that data is **interpolated** if we find an output value

$\boxed{\text{outside of the interval of the input data}}$.

Example 1.4.5. In the example above, predict the number of companies open in the twelfth year of operation. Is this extrapolation or interpolation?

Example 1.4.6. Do the same, but after 8.5 years after opening. Is this extrapolation or interpolation?

Example 1.4.7. The amount of electricity sold by a power company in year x is given below.

Year	2003	2004	2005	2006	2007	2008
Retail Sales (in quadrillion kWh)	1.2	1.23	1.27	1.3	1.33	1.35

(a) Find a **complete** linear model to fit the data.

(b) Write an interpretation the slope of the linear model.

(c) When did retail sales first exceed 1.4 quadrillion kWh? Is this interpolation or extrapolation?

Data Alignment

When using an input value of years, alignment should (usually) happen so that the first year given corresponds to an input of zero.

Example 1.4.8. Find the **complete** linear model to fit the data of the previous example, aligning the input so that the year 2003 corresponds to an input of zero.

Numerical Considerations

Since numerical approximations can vary, we will use the following guidelines:

(1) Use common sense; if a model outputs something like "2.5 people", we would round to 3 people.

(2) The accuracy of the output **must** be the same as the original model's accuracy.

(3) All answers **must** have proper units; answers without labels are useless.

(4) If arriving at your answer requires multiple steps, **do not** round until the *final* answer.

Example 1.4.9. The world's daily demand of oil was recorded in various years, and is listed below.

Year	2004	2005	2006	2007	2008	2009
Oil Demand (in million barrels)	82.327	83.652	84.622	85.385	86.384	87.698

(a) Based on the scatterplot, why is a linear model best?

(b) Align the data so that the year 2000 corresponds to an input of 0, and find the **complete** linear model.

(c) Estimate the demand in the year 2015.

Example 1.4.10. Expenditure on pets in the United States was recorded over the span of several years, and is recorded in the table below.

Year	1994	1996	1998	2001	2002	2003	2004	2005	2006	2007	2008
Expenditure (billion USD)	17	21	23	28.5	29.5	32.4	34.4	36.3	38.5	41.2	43.4

(a) Align the data so that the year 1994 corresponds to an input of zero, and find the **complete** linear model.

(b) Use the model to estimate the expenditure in the year 2013.

Example 1.4.11. The number of successful tax audits performed by a company between 2000 and 2006 can be modeled by $A(t) = -83.9t + 1063$ audits, where t is the number of years since 2000.

(a) Give the rate of change of A. Include units.

(b) Evaluate $A(0)$. Write a sentence interpreting your answer.

(c) Find the number of successful audits in 2005. Is this interpolation or extrapolation?

(d) Find the number of successful audits in 2010. is this interpolation or extrapolation?

Example 1.4.12. The population of a town in selected years is given below.

Year	2005	2006	2007	2008	2009	2010
Population (in thousands)	125.2	128.7	132.4	136.0	139.8	143.6

(a) Find a **complete** model for the population P of the town in year y.

(b) According to your model, what is the constant rate of change of the population of the town?

(c) Use your model to predict the population of the town in 2015.

Example 1.4.13. Honda engineers are designing a new car, and are measuring the distance it takes the car to come to a complete stop on dry pavement. Their measurements are given below.

Speed (mph)	55	60	65	70	75
Distance (ft)	77.6	131.4	186.3	236.7	289.3

(a) Find a **complete** model for the braking distance of the car.

(b) Use your model to find the braking distance needed when the car is traveling at 77 miles per hour; write your answer using function notation.

(c) Find another **complete** model, aligning the data so that a speed of 50 mph corresponds to an input of 0.

(d) Repeat part (b).

(e) How fast is the car traveling if it requires 156 ft to come to a complete stop?

Exponential Functions & Models

Exponential Functions

As with the linear model, we have three descriptions of an exponential model:

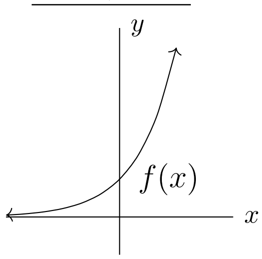
- Algebraic: An exponential model has an equation of the form $\boxed{f(x) = ab^x}$ The percentage change over one unit input is $\boxed{(b-1)\cdot 100\%}$, and $\boxed{a}$ is the *initial value*, the output corresponding to an input of zero.

- Verbally: An exponential model has $\boxed{\text{a constant percent change}}$.
- Graphically: An exponential model will look like the pictures below.

Exponential Models

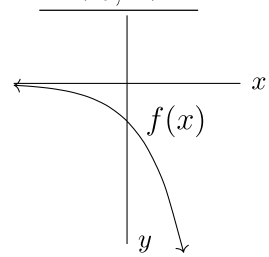
For exponential models, we have the following information:

$a > 0, b > 1$



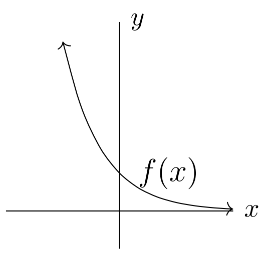
- $\lim_{x\to\infty} f(x) = \boxed{\infty}$
- $\lim_{x\to-\infty} f(x) = \boxed{0}$
- f is always $\boxed{\text{increasing}}$
- f is concave $\boxed{\text{up}}$

$a < 0, b > 1$



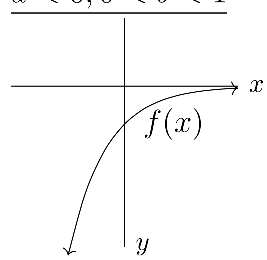
- $\lim_{x\to\infty} f(x) = \boxed{-\infty}$
- $\lim_{x\to-\infty} f(x) = \boxed{0}$
- f is always $\boxed{\text{decreasing}}$
- f is concave $\boxed{\text{down}}$

$a > 0, 0 < b < 1$



- $\lim_{x\to\infty} f(x) = \boxed{0}$
- $\lim_{x\to-\infty} f(x) = \boxed{\infty}$
- f is always $\boxed{\text{decreasing}}$
- f is concave $\boxed{\text{up}}$

$a < 0, 0 < b < 1$



- $\lim_{x\to\infty} f(x) = \boxed{0}$
- $\lim_{x\to-\infty} f(x) = \boxed{-\infty}$
- f is always $\boxed{\text{increasing}}$
- f is concave $\boxed{\text{down}}$

For us, an exponential model will always have an asymptote at $\boxed{y = 0}$.

Formulas and Examples

There are two formulas which will be useful to memorize. For exponential models, we have a constant percent change; this is given above as

Percent Change (Exponential)

$$\%\text{ change} = (b-1)\cdot 100\%$$

For *every other model*, we calculate the percent change between two input values x_1, x_2 as

Percent Change (Other Models)

$$\%\text{ change} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

Example 1.5.1. iPod sales were 7.68 million units in 2006, and increased by 9.1% each year between 2006 and 2008.

(a) Write an exponential model for this situation.

(b) Explain why the exponential model is best.

(c) Use the model to predict the number of iPods sold in 2010.

Example 1.5.2. The population of Northern cod in a certain body of water is given in the table below.

Decade (since 1963)	0	1	2	3	4
Population (in billions)	1.72	0.63	0.24	0.085	0.032

(a) Identify which model (linear/exponential) is best for this data.

(b) Find the **complete** model.

(c) Find the percent change of the model.

Example 1.5.3. Early in the millennium, it was predicted that United States imports of petroleum products would be 4.81 quadrillion Btu, and increase by 5.47% each year through 2020.

(a) Find the associated exponential model.

(b) When will imports exceed 10 quadrillion Btu?

(c) Describe the end behavior of your model.

Example 1.5.4. According to the Social Security Advisory Board, the number of workers per beneficiary of the Social Security program was 3.3 in 1995 and is projected to decline by 1.46% each year until 2030.

(a) Write a model for the number of workers per beneficiary from 1995 through 2030.

(b) What does the model predict the number of workers per beneficiary will be in 2030?

Example 1.5.5. A social media website collected data on its users. Below are the users of a certain age and gender, as a percentage of total users.

Age (years)	27	29	31	33	35	37	39	41	43	45
Females (as %)	9.6	7.8	6.1	5.1	4.3	3.8	2.4	2.1	1.2	1.1
Males (as %)	8.8	7.6	6.0	4.6	4.0	4.4	2.7	1.9	1.5	1.3

(a) Align the input data to the number of years after 27. Write an exponential model for the female user data.

(b) According to the model in part (a), what is the percentage change in your model? Write a sentence interpreting your answer.

(c) What percentage of female users are 30 years old? What about 48 years old? Are these interpolation or extrapolation?

(d) Write the exponential model for the male user data.

(e) According to your model in part (d), what is the percentage change in your model? Write a sentence interpreting your answer.

(f) What percentage of male users are 30 years old? What about 48 years old?

Doubling Time and Half Life

Definition 1.5.6 (Doubling Time)

For an exponential function f, the **doubling time** is defined to be the amount of time it takes an initial quantity to double (or grow by 100%).

Definition 1.5.7 (Half Life)

For an exponential function f, the **half life** is defined to be the amount of time it takes an initial quantity to decay to half of its original size (or decrease by 50%).

Example 1.5.8. Albuterol is used to calm bronchospasm. It has a biological half-life of 7 hours and is normally inhaled as a 1.25 mg dose.

(a) Find a model for the amount of albuterol left in the body after an initial dose 1.25 mg.

(b) How much albuterol is left in the body after 24 hours?

Example 1.5.9. The amount of money Frank has in a particular investment is given by $f(t) = Pe^{.06t}$, where P is the principal invested and t is the amount of time (in years) the investment has been active.

(a) If Frank began the investment 15 years ago, and currently has \$25,500 in the account, what was the principal that he invested?

(b) If Frank currently has \$14,250 in the account and invested \$2,500 to start, how long as the investment been active?

(c) Compute the doubling time for an investment of \$1000.

(d) How long will it take an investment to *triple* instead of double?

Models in Finance

Definition 1.6.1 (Future Value/Present Value)

The **future value** of an investment/loan at time t is the sum of the prevent value and all accumulated interest; this is denoted F or FV. The **present value**, denoted $F(0) = P$ (principal) is the value "today", or at $t = 0$.

Simple Interest

Definition 1.6.2 (Simple Interest)

Simple interest is interest earned on $\boxed{\text{the present value only}}$ the rate (as a decimal) is called the $\boxed{\text{annual percentage rate (APR)}}$, or nominal rate.

We have two formulas for simple interest:

$$I(t) = Prt \text{ dollars}$$

$$F_s(t) = P(1 + rt) \text{ dollars}$$

where P is the $\boxed{\text{principal}}$, r is the $\boxed{\text{rate (as a decimal)}}$, and t is the $\boxed{\text{time (in years)}}$.

Example 1.6.3. A family friend offers to loan you \$10,000 to cover your outlandishly high tuition this year. She wants to earn 5.75% interest on the loan.

(a) If you pay the loan back in 1 year, how much interest does the friend make?

(b) What about if you pay the loan back in 3 years?

(c) What about 4 months?

Example 1.6.4. I invest \$500 at 8.5%. How much is the investment worth in 5 years?

Discretely Compounding Interest

Definition 1.6.5 (Discretely Compounding Interest)

Discretely compounding interest is interest earned on the balance at discrete time intervals.

We have two formulas for discretely compounding interest:

$$
\begin{aligned}
I &= \frac{r}{n} \\
F_d(t) &= P\left(1 + \frac{r}{n}\right)^{nt} \text{ dollars}
\end{aligned}
$$

where P is the $\boxed{\text{principal}}$, r is the $\boxed{\text{rate (as a decimal)}}$, t is the $\boxed{\text{time (in years)}}$, and n is the $\boxed{\text{number of compounds (in a year)}}$.

Example 1.6.6. You take out a \$16,750 loan for a new car. Find the value of the loan (assuming no payments were made) with:

(a) $r = 12.5\%$, monthly

(b) $r = 6.2\%$, $n = 12$

(c) $r = 12.5\%$, yearly

(d) $r = 3.79\%$, quarterly

(e) $r = 3.79\%$, $n = 6$

(f) $r = 7.2\%$, daily

Definition 1.6.7 (Annual Percentage Yield)

The **annual percentage yield** of an investment (also called the $\boxed{\text{effective rate}}$) gives the return on investment in one year. APY for discretely compounding interest is calculated with the formula

$$APY_{\mathrm{D}} = \left[\left(1 + \frac{r}{n}\right)^n - 1\right] \cdot 100\%$$

Example 1.6.8. Calculate the APY for each of the situations from the last example. Round each to the nearest tenth:

(a) $r = 12.5\%$, monthly

(b) $r = 6.2\%$, $n = 12$

(c) $r = 12.5\%$, yearly

(d) $r = 3.79\%$, quarterly

(e) $r = 3.79\%$, $n = 6$

(f) $r = 7.2\%$, daily

Example 1.6.9. OU Federal Credit Union offers an APR of 6.35% (compounded monthly) for an investment opportunity, while First Fidelity offers you an APY of 5.95%. Which option will give the highest return after one year?

Continuously Compounding Interest

Definition 1.6.10 (Continuously Compounding Interest)

Interest earned on the balance at any given time t is called **continuously compounding interest**, and has the future value formula given by

$$F_c(t) = Pe^{rt} \text{ dollars}$$

where P is the principal, r is the rate, and t is the time.

We also have a formula for the APY of continuously compounding interest:

$$APY_{\mathrm{C}} = (e^r - 1) \cdot 100\%$$

Example 1.6.11. Determine the amount that must be invested in the following situations to get \$7000 payable in 4 years:

(a) 3% APR, compounded continuously

(b) 3.9% APR, compounded monthly

(c) 15.1% APR, simple interest

(d) 10% APR, compounded weekly.

Example 1.6.12. Find the APY for the examples above, rounding to the nearest hundredth.

(a) 3% APR, compounded continuously

(b) 3.9% APR, compounded monthly

(c) 15.1% APR, simple interest

(d) 10% APR, compounded weekly.

Constructed Functions

Definitions

Definition 1.7.1 (Fixed Cost)

A **fixed cost** is a cost which remains the same, no matter how much of a product is produced.

Definition 1.7.2 (Variable Cost)

A **variable cost** is a cost which changes depending on the number of units produced.

Definition 1.7.3 (Total Cost)

The **total cost** is the sum of the fixed cost and variable cost.

Definition 1.7.4 (Revenue)

Revenue is the product of the selling price (per unit) and the number of units sold, $R = \text{price} \cdot \text{quantity}$

Definition 1.7.5 (Profit)

Profit is the difference between revenue and cost, $P = R - C$.

Definition 1.7.6 (Break-Even Point)

The **break-even point** is the point when total cost equals total revenue, i.e. when profit is zero

Function Operations

There are five operations which we will need to be familiar with in order to move on.

- **Addition**: $\boxed{h(x) = (f + g)(x) = f(x) + g(x)}$, *if the output units of f and g are* $\boxed{\text{exactly the same}}$.

- **Subtraction**: $\boxed{j(x) = (f - g)(x) = f(x) - g(x)}$, *if the output units of f and g are* $\boxed{\text{exactly the same}}$.

- **Multiplication**: $\boxed{k(x) = (f \cdot g)(x) = f(x) \cdot g(x)}$, *if the output units of f and g are* $\boxed{\text{compatible}}$.

- **Division**: $\boxed{\ell(x) = \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\ (g(x) \neq 0)}$, *if the output units of f and g are* $\boxed{\text{compatible}}$.

- **Composition**: $\boxed{m(x) = (f \circ g)(x) = f(g(x))}$, *if* $\boxed{\text{the output of } g \text{ is the input of } f}$.

Addition creates total cost from fixed and variable costs by adding the two; profit is created using subtraction. Variable cost (and revenue) are created by multiplication, and division gives us average cost $\overline{C}$.

Examples

Example 1.7.7. The number of student tickets sold for a home basketball game at OU is represented by $S(w)$ hundred tickets when w is the winning percentage of the team. The number of non student tickets sold for the same game is represented by $N(w)$ hundred tickets where w is the winning percentage of the team. Combine the functions to construct a new function giving the total number of tickets sold for a home basketball game at OU.

Example 1.7.8. Sales of 12-ounce bottles of sparkling water are modeled as $D(x) = 287.411(0.266^x)$ million bottles, when the price is x dollars per bottle. Write a model for the revenue from the sale of 12-ounce bottles of sparkling water.

Example 1.7.9. The profit from the supply of a certain commodity is modeled as $P(q) = 30 + 60 \ln q$ thousand dollars, where q is the number of units produced in millions. Write a model for the average profit when q units are produced.

Example 1.7.10. A travel agency offers spring break cruise packages. The agency advertises a cruise to Cancun for \$1200 per person. To promote the cruise among student organizations on campus, the agency offers a discount for student groups selling the cruise to over 50 of their members. The price per student will be discounted by \$10 for each student in excess of 50 (for example, if an organization had 55 members go on the cruise, each of those students would pay \$1150). Write a model for the travel agency's revenue that depends on the number of students from a student organization.

Example 1.7.11. The sales of a certain brand of backpack is modeled by $f(s) = 1.56s + 4.3$ million dollars, when s is the number of stores that sell the brand of backpack. The number of stores that sell the brand of backpack is modeled by $s(t) = 3t + 5.4$ stores, t months since the beginning of 2000. Write a model for the sales of a certain brand of backpack with respect to time.

Example 1.7.12. The level of contamination in groundsoil is $f(p) = \sqrt{p}$ parts per million when the population of the surrounding community if p people. The population of the surrounding community in year t is modeled as $p(t) = 400t^2 + 2500$ people, t years since 2000.

(a) Why can we use function composition?

(b) Find a model for the contamination of the groundsoil.

Example 1.7.13. It costs a company \$19.50 to produce 150 glass bottles. Write a model for $\overline{C}(q)$, the average cost of producing a bottle when q units are produced.

Example 1.7.14. Write the following functions as composite functions, and then evaluate the composite at an input of 2.

(a) $f(t) = 3e^t$, $t(p) = 4p^2$

(b) $h(p) = \dfrac{4}{p}$, $p(t) = 1 + 3e^{-0.5t}$

(c) $g(x) = \sqrt{7x^2}$, $x(w) = 4e^w$

(d) $c(x) = 3x^2 - 2x + 5$, $x(t) = 2e^t$

Logarithmic Functions & Models

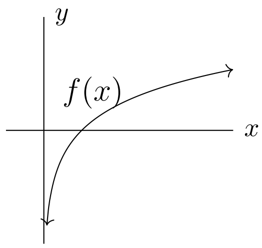
Logarithmic Functions

We have the following descriptions for a logarithmic function:

- Algebraically: A logarithmic model has an equation of the form $\boxed{f(x) = a\ln x + b}$, where $a, b \neq 0$ are constants, and $x > 0$.
- Verbally: A log function has a $\boxed{\text{vertical asymptote}}$ at $x = 0$, and continues to grow (or decay) as x increases without bound.
- Graphically: The graph of a log model takes a form as below.

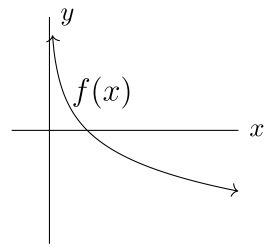
Logarithmic Models

$b > 0$



- $\lim\limits_{x\to\infty} f(x) = \boxed{\infty}$
- $\lim\limits_{x\to 0^+} f(x) = \boxed{-\infty}$
- f is always $\boxed{\text{increasing}}$
- f is concave $\boxed{\text{down}}$

$b < 0$



- $\lim\limits_{x\to\infty} f(x) = \boxed{-\infty}$
- $\lim\limits_{x\to 0^+} f(x) = \boxed{\infty}$
- f is always $\boxed{\text{decreasing}}$
- f is concave $\boxed{\text{up}}$

Logarithmic Behavior

Example 1.8.1. The percentage of viewers that have watched a DVR'd show before a certain number of days have passed is give in the table below.

Time (in days)	1	2	3	4	5	6	7	8
Viewers (in %)	46	62	76	84	91	95	98	100

(a) Why is a logarithmic model best here? Use a scatterplot to help you develop your reasons.

(b) Find the model corresponding to your answer in part (a). Write the complete model.

(c) Explain why the exponential model does not work.

Example 1.8.2. The average length of the ears of men after a certain age is given in the table below.

Age (in years)	0	20	70
Ear Length (in inches)	2.04	2.55	3.07

(a) Find the complete logarithmic model for the data. Do you encounter any problems?

(b) Align the data so that age 0 corresponds to an input of 10, and find the complete logarithmic model for the data.

(c) Use the model to find the ear length of a 45 year old man.

Example 1.8.3. The table below shows the life expectancy for women in Ireland between 1945 and 2011

Year	1945	1955	1965	1975	1985	1995	2005	2011
Expectancy (years)	65.2	71.1	73.1	74.7	77.4	78.8	79.3	80.2

(a) Align the data so that 1940 corresponds to an input of zero.

(b) What type of concavity does the scatter plot suggest?

(c) Describe the end behavior suggested by the scatterplot as the input increases without bound.

(d) Find the complete logarithmic model for the data.

(e) Using your model, find the year in which the life expectancy for Irish women was exactly 76.3 years.

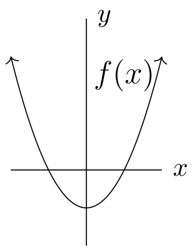
(f) What was the life expectancy of an Irish woman in the year 1979?

Quadratic Functions & Models

Quadratic Models

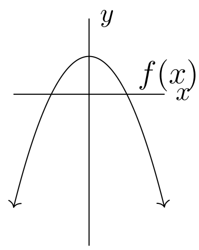
The model is given by $f(x) = ax^2 + bx + c$, where a, b, c are constants ($a \neq 0$). The function has an absolute maximum if $a < 0$, and absolute minimum if $a > 0$.

$\underline{a > 0}$



- $\lim_{x\to\infty} f(x) = \boxed{\infty}$
- $\lim_{x\to-\infty} f(x) = \boxed{\infty}$
- f is decreasing then increasing
- f is concave up

$\underline{a < 0}$



- $\lim_{x\to\infty} f(x) = \boxed{-\infty}$
- $\lim_{x\to-\infty} f(x) = \boxed{-\infty}$
- f is increasing then decreasing
- f is concave up then down

Choosing Models

At this point, we have five models to choose from when analyzing a data set. The process of choosing a model should go as follows:

- Does the scatterplot show any sort of concavity? If yes, then go to the next step. If not, try a **linear** model.
- If the scatterplot shows concavity, does it appear to *change concavity*? If yes, then the model could be **logistic** or **cubic**. If not, then the model could be **exponential**, **logarithmic**, or **quadratic**.
    1. If the scatterplot changes concavity, then does it have an asymptote? If yes, then the model is **logistic**. If no, then the model is **cubic**.
    2. If the scatterplot does not change concavity, then look at the end behavior and for asymptotes. If there is an asymptote at $x = 0$, then the model is **logarithmic**; if it is at $y = 0$, then the model is **exponential**; if there is no asymptote, then it is **quadratic**.
- If it is still difficult to determine between exponential and quadratic, then use the method of second differences (described below). If second differences gives roughly constant values, then the model is **quadratic**; if it does not, then, it is **exponential**.
- If in doubt, one can develop multiple models and compare the fit of each model against the data.
- It is never a bad idea to apply common sense to models.

Example 1.9.1. Draw a decision tree/diagram for choosing a model.

Example 1.9.2. The table below shows the profit (in millions of dollars) that American Airlines makes on tickets between Dallas and Chicago when tickets are set at a certain price:

Ticket Price (dollars)	200	250	300	350	400	450
Profit (million dollars)	3.08	3.52	3.76	3.82	3.7	3.38

(a) Give two reasons why a quadratic model is more appropriate than a log or exponential model.

(b) Find a quadratic model for the data.

(c) Why doesn't the airline profit increase as the ticket price increases?

(d) At what price does the airline begin posting a loss?

Example 1.9.3. The table below gives the braking distance required for a vehicle to come to a complete stop, given the initial velocity of the vehicle.

Speed (mph)	10	20	30	40	50	60	70	80	90
Distance (feet)	27	63	109	164	229	304	388	481	584

(a) Find the second differences of the data above.

(b) Find a quadratic model for stopping distance.

(c) What other factors besides the initial speed would impact the stopping distance?

(d) What speed is the vehicle moving if its braking distance is exactly 412 feet? Round your answer to two decimal places, if needed.

Example 1.9.4. The ratios of public school students to instructional computers with Internet access for years between 1998 and 2004 are given below:

Year	1998	1999	2000	2001	2002	2003	2004
Ratio	9.1	6.1	3.6	2.4	1.8	1.4	1.8

(a) Align the input so that 1998 corresponds to an input of 0.

(b) Write the complete quadratic model for the data.

(c) Write the complete exponential model for the data.

(d) Which model best fits the data: (b) or (c)?

(e) Give two reasons why an exponential model might be best for this data.

Logistic Functions & Models

Logistic Functions

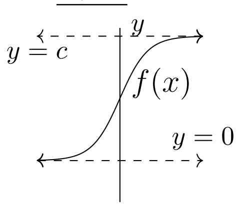
A logistic has the following descriptions:

- Algebraically: A logistic model has an equation of the form $\boxed{f(x) = \dfrac{c}{1 + ae^{bx}}}$ where $a, b \neq 0$ are constants, and $c > 0$ is the $\boxed{\text{carrying capacity/limiting value}}$.
- Graphically: See below; logistics have two horizontal asymptotes at $\boxed{y = 0, y = c}$.

Logistic Models

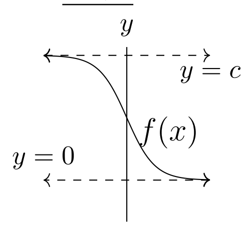
For logistic models, we have the following information:

$b > 0$



- $\lim\limits_{x\to\infty} f(x) = \boxed{c}$
- $\lim\limits_{x\to-\infty} f(x) = \boxed{0}$
- f is always $\boxed{\text{increasing}}$
- f is $\boxed{\text{concave up then down}}$

$b < 0$



- $\lim\limits_{x\to\infty} f(x) = \boxed{0}$
- $\lim\limits_{x\to-\infty} f(x) = \boxed{c}$
- f is always $\boxed{\text{decreasing}}$
- f is $\boxed{\text{concave down then up}}$

Examples

Example 1.9.5. The number of NBA players taller than a given height are listed in the table below.

Height (in inches)	**Number of Players**	**Height** (in inches)	**Number of Players**
68”	490	80”	203
70”	487	82”	86
72”	467	84”	13
74”	423	86”	2
76”	367	88”	1
78”	293		

(a) Using the scatterplot, explain why a logistic model is best for this data.

(b) Align the data so that 68” corresponds to an input of 0, and find the complete logistic model.

(c) Describe (using limit notation) the end behavior of the model as height increases.

Example 1.9.6. The narrow band residential internet access, as a percentage of total residential internet access, is given below.

Year	**Narrow Band Users** (in %)	**Year**	**Narrow Band Users** (in %)
2000	89.4	2008	9.6
2001	80.7	2009	7.3
2002	70.9	2010	4.3
2003	58.3	2011	3.0
2004	45.9	2012	2.5
2005	35.3	2013	1.5
2006	21.5	2014	1.0
2007	12.2		

(a) Based on the scatterplot, explain why a logistic model is best.

(b) Align the model so that 2000 corresponds to an input of 0. Find the complete logistic model for the data.

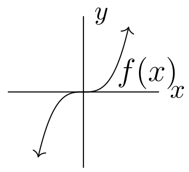
(c) Write the equations for the two asymptotes.

(d) Estimate the location of the inflection point

Cubic Functions & Models

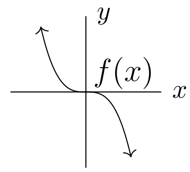
Cubic Models

$\underline{a > 0}$



- $\lim_{x \to \infty} f(x) = \boxed{\infty}$
- $\lim_{x \to -\infty} f(x) = \boxed{-\infty}$
- f is increasing, decreasing, then increasing
- f is concave down then up

$\underline{a < 0}$



- $\lim_{x \to \infty} f(x) = \boxed{-\infty}$
- $\lim_{x \to -\infty} f(x) = \boxed{\infty}$
- f is decreasing, increasing, then decreasing
- f is concave up then down

Examples

Example 1.10.1. A car company's profit on SUV's is given below.

SUV's sold (in millions)	10	20	30	40	50	60	70
Profit (in trillion dollars)	0.9	3.1	4.3	5.2	5.8	6.4	6.9

(a) Use the scatterplot to determine the best model for the data. Give two reasons for your choice.

(b) Write the complete model.

(c) Find the profit when 37 million SUV's are sold. Write a sentence of interpretation for your answer.

Example 1.10.2. A manufacturing company recorded the production of toys when a certain amount of capital is invested in the production run.

Capital Invested (in million dollars)	6	18	24	30	42	48
Units Produced (in billions)	19	38	42	45	60	77

(a) Use the scatterplot to determine the best model for the data. Give two reasons for your choice.

(b) Write the complete model.

(c) Find the capital needed to produce 50 billion units. Write a sentence interpreting your answer.

Chapter 2

Describing Change: Rates

Measures of Change over an Interval

Formulas

Let f be a function with input values x_1, x_2 such that $x_1 < x_2$.

Change

$$\boxed{f(x_2) - f(x_1)}$$

Percent Change

$$\boxed{\frac{f(x_2) - f(x_1)}{f(x_1)}}$$

Average Rate of Change (AROC)

$$\boxed{\frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

When giving interpretations, we have four considerations:

- **When** is this event happening? Be sure to specify the interval.
- **What** is happening? Specify the quantity which is changing.
- **How** is it changing? Specify whether or not the quantity is increasing or decreasing.
- By **how much** is it changing? Include proper units.

Example 2.1.1. If f denotes the number of students enrolled in Math 1743 and x is the number of academic years after the 2000-2001 academic year, interpret the expression $f(10) = 1552$.

Examples

Example 2.1.2. The average temperature in Norman during the last week of September is given in the table below:

Time	**Temperature** (° F)	**Time**	**Temperature** (° F)
7am	49	1pm	80
8am	58	2pm	80
9am	66	3pm	78
10am	72	4pm	74
11am	76	5pm	69
noon	79	6pm	62

(a) Give the average rate of change in temperature between 11am and 4pm. Write a sentence interpreting your result.

(b) Find the percent change in temperature between 9am and noon, and round your answer to the nearest hundredth. Write a sentence interpreting your result.

Example 2.1.3. Airtran posted a revenue of $603.7 million dollars in the second quarter of 2009 compared with revenue of $693.4 million during the second quarter of 2008. Write a sentence interpreting each of the following:

(a) Find the change in revenue between the second quarter of 2008 and the second quarter of 2009.

(b) Find the percent change between the second quarter of 2008 and second quarter end of 2009.

(c) Find the average rate of change between the second quarter of 2008 and the second quarter of 2009.

Example 2.1.4. The American Indian, Eskimo, and Aleut populations in the United States was 362 thousand in 1930, and 4.5 million in 2005. Write a sentence interpreting each of the following, and round to two decimals if necessary:

(a) Find the change in population between 1930 and 2005.

(b) Find the percent change between between 1930 and 2005.

(c) Find the average rate of change between 1930 and 2005.

Example 2.1.5. OU Parking Services commissioned a projection of its profit (in thousands of dollars) when commuter parking passes are set certain prices.

Price (dollars)	200	250	300	350	400	450
Profit (thousand dollars)	2080	2520	2760	2820	2700	2380

(a) Find a model for the data.

(b) Calculate the average rate of change of profit when the parking pass price rises from \$200 to \$350.

(c) Calculate the average rate of change of profit when the parking pass price rises from \$350 to \$450.

(d) Calculate the percent change for parts (b) and (c).

Example 2.1.6. The CDC modeled the number of Zika cases diagnosed in Brazil between January and July of 2016 with the formula

$$z(t) = 2.75(1.04^t) \text{ thousand cases}$$

where t is the number of months since January 2016.

(a) Calculate and write a sentence of interpretation for the average rate of change in the number of Brazilians diagnosed with Zika between January 2016 and July 2016.

(b) Calculate the percentage change in part (a).

Example 2.1.7. The function $c(t)$ represents the number of students in line at Chick-Fil-A, t hours after 11:00am, and $q(t)$ represents the number of students in line at Quizno's, t hours after 11:00am. Write a sentence interpreting the following expressions.

(a) $c(3) = 15$

(b) $q(1) = 8$

(c) $(c + q)(0) = 12$

Measures of Change at a Point - Graphical

Tangent Lines and Secant Lines

Definition 2.2.1 (Secant Line)

Let f be continuous and smooth function on the some input interval $[a, b]$. Let $x_1 < x_2$ be two points in $[a, b]$. The **secant line** through x_1 and x_2 is the line whose slope is $\boxed{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}}$

The characteristic of a secant line is that it "intentionally" goes through f in two points.

Question 2.2.2 The formula for the slope of a secant line should look familiar; what's the other name we used for this?

Example 2.2.3. Let $f(x) = x^2$. Find the secant line through $x_1 = -\dfrac{1}{2}$ and $x_2 = 2$, then sketch $f(x)$ and the secant line.

Average Change vs. Instantaneous Change

Secant lines are drawn between two points on a graph. As the distance between the points decreases, we get values of the independent axis which are closer and closer together; we can use a limiting process to find slope of the line through *a single point.* This is the called the *tangent* line to the point.

Definition 2.2.4 (Rate of Change)

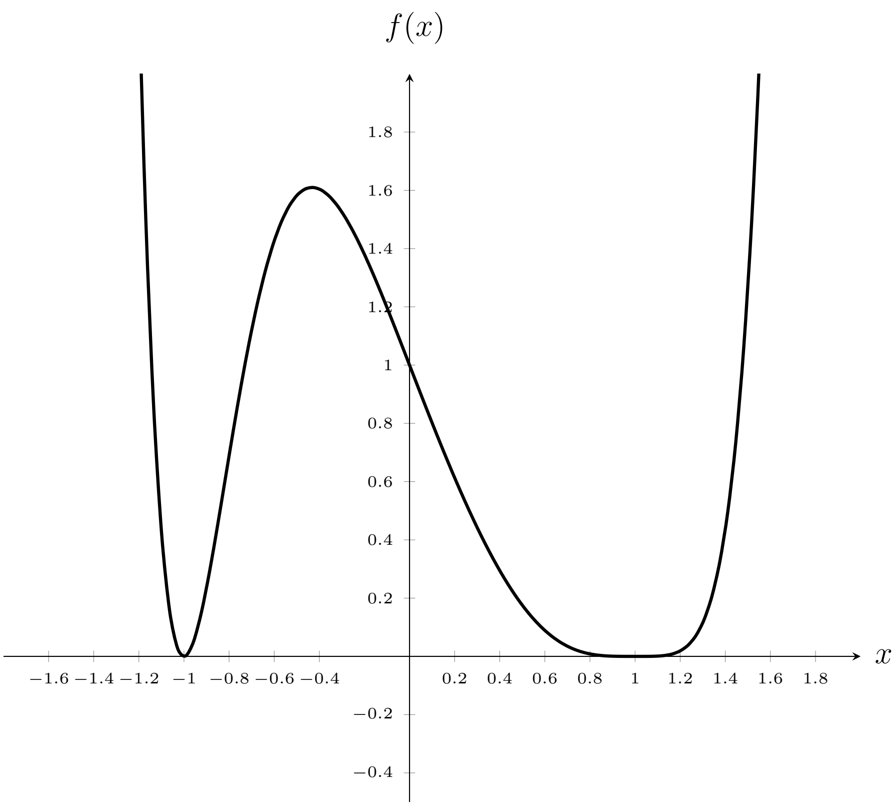
For smooth, continuous function f, the **rate of change** of the function at a particular point x_0 is given by $\boxed{\text{the slope of the tangent line to the graph of } f}$; the rate of change of f at point x_0 is denoted $\boxed{f'(x_0)}$.

Definition 2.2.5 (Instantaneous ROC)

The **instantaneous rate of change** of a function f measures $\boxed{\text{slope of the graph}}$ $\boxed{\text{of a function at a single point}}$.

Example 2.2.6. Estimate the slope of the tangent line to the curve $f(x) = x^2$ at $x = -\dfrac{1}{2}$ by drawing successive secant lines and computing the slope.

Example 2.2.7. For the curve below, use successive secant lines to estimate $f'(0.4)$.



Definition 2.2.8 (Percentage Rate of Change)

For smooth, continuous function f, if the rate of change $f'(x_0)$ exists for input value x_0 and $f(x_0) \neq 0$, then we define the **percentage rate of change** as $\dfrac{f'(x_0)}{f(x_0)}$

The output for percentage rate of change is $\boxed{\text{\% per unit input}}$.

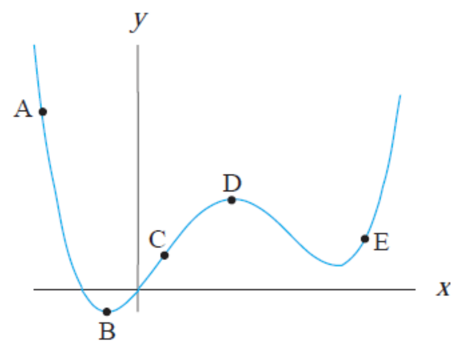
Example 2.2.9. The rate of change of the student population at OU is 2000 students per year, and the current body is 40,000 students. Find the percent rate of change of the student population.

Example 2.2.10. For the following, use the picture:

(a) At each labeled point, identify whether the instantaneous rate of change is positive, negative, or zero.

(b) Is the graph steeper at point C or point E?

(c) Is the graph steeper at point A or at point C?

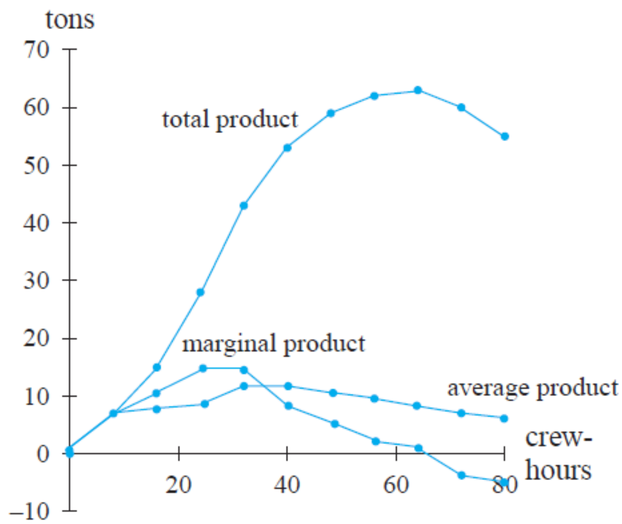


Example 2.2.11. The number of monthly Spotify listeners of an obscure shoegaze band is given by the function $f(x)$, where x is the number of months since the release of their first album.

(a) If the band had 3000 listeners four months after the release of their first album, and the percent rate of change was 36% per month, find the rate of change of listeners in the fourth month to the nearest whole number. Give a sentence of interpretation for your answer.

(b) A year after the release, the band had 12000 listeners. Find the average rate of change of the band's Spotify listeners between four months and a year after the release of the first album. Round to the nearest whole number, if necessary. Give a sentence of interpretation for your answer.

Example 2.2.12. For the following, use the picture below:



(a) For a crew working 32 crew-hours, the rate of change of which two quantities are positive?

(b) For a crew working 30 crew-hours, which two quantities can be improved by adding crew hours?

(c) At 56 crew-hours, give a relationship between the slopes of total product, average product, and marginal product.

(d) Starting with 24 crew-hours, which quantity has negative slope?

(e) The slope of which quantity is positive for less than 64 hours?

(f) For a crew working 75 crew-hours, the slope of which two quantities are nearly equal?

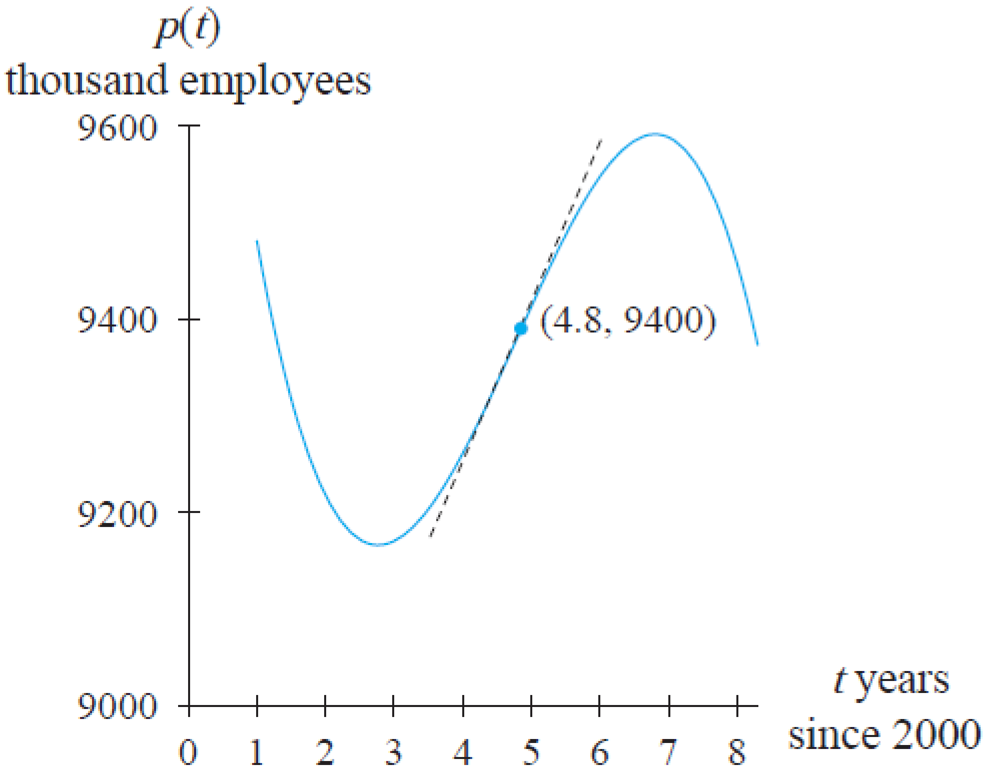
(g) For a crew working 24 crew-hours, which quantity is near its greatest slope?

Example 2.2.13. For the picture below, do the following:

(a) Estimate a second point on the tangent line

(b) Calculate the rate of change of the function at the labeled point; include units and round to 2 decimal places if necessary.

(c) Calculate the percentage rate of change of the function at the labeled point; include units and round to 2 decimal places if necessary.



Rates of Change: Notation & Interpretation

Average ROC vs. Instantaneous ROC

Average Rate of Change

- Measures how rapidly a quantity changes *over an interval.*
- Slope of the secant line between two points.

(Instantaneous) Rate of Change

- Measures how rapidly a quantity is changing at *a specific point.*
- Slope of the tangent line at a single point.
- Function must be smooth and continuous.

Notation and Terminology

Rate of change at a specific point a is often referred to as any of the following (given a function f):

- the derivative of f at a
- the slope of a graph of f at point $(a, f(a))$
- the slope of the line tangent to a graph of f at a point $(a, f(a))$
- the rate of change of f at a

We also have two different notations for the derivative of f at point a:

- $f'(a)$. This is read "f prime of a".
- $\left.\dfrac{df}{dx}\right|_{x=a}$. This is read "d-f d-x, evaluated at a", or as "the derivative of f with respect to x, evaluated at a".

Note: We will freely interchange between *any* of the above terminology or notations.

Examples

Example 2.3.1. The function f gives weekly profit, in thousands of dollars, that an airline makes on flights from Boston to Washington, D.C. when the ticket price is p dollars. Write a sentence interpreting the following:

(a) $f(65) = 15$

(b) $f'(65) = 1.5$

(c) $f'(90) = -2$

Example 2.3.2. The function C gives the number of bushels of corn produced on a tract of farmland that is treated with f pounds of nitrogen per acre.

(a) Is it possible for $C(90)$ to be negative? Why?

(b) What are the units of $\left.\dfrac{dC}{df}\right|_{f=90}$?

(c) Is it possible for $\left.\dfrac{dC}{df}\right|_{f=90}$ to be negative? Why?

(d) Give an alternate notation for the statement $\left.\dfrac{dC}{df}\right|_{f=90}$.

Example 2.3.3. Sketch a possible graph of $t(x)$, given that:

- $t(3) = 7$
- $t(4.4) = t(8) = 0$
- $t'(6.2) = 0$
- t has no change in concavity

Example 2.3.4. The function w gives a certain Business Calculus instructor's weight (in pounds) t weeks after he begins a diet. Write a sentence of interpretation for each of the following statements:

(a) $w(0) = 180$ and $w(12) = 165$

(b) $w'(1) = -2$ and $w'(9) = -1$

(c) $\left.\dfrac{dw}{dt}\right|_{t=12} = 0$ and $\left.\dfrac{dw}{dt}\right|_{t=15} = 0.25$

Example 2.3.5. Sketch a possible graph of the function m with input t, given that

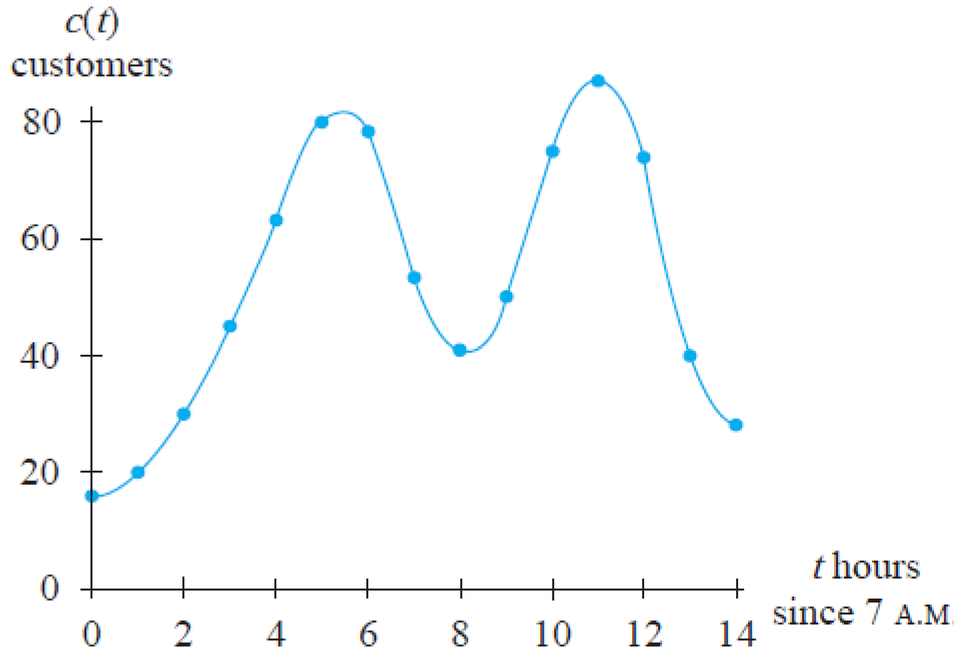
- $m(4) = 8$
- $m'(4)$ is greater than any other slope.
- $m'(0) = m'(6) = 0$
- The graph of m has no direction changes.

Example 2.3.6. The function g gives the fuel efficiency in miles per gallon of a car traveling v miles per hour. Write a sentence of interpretation for each of the following:

(a) $g(55) = 32$ and $g'(55) = -0.25$

(b) $g'(45) = 0.15$ and $g'(51) = 0$

Example 2.3.7. The figure below depicts the number of customers that a fast-food restaurant serves each hour on a typical weekday:



(a) Estimate the average rate of change of the number of customers between 7am and 11am. Interpret your answer.

(b) Estimate the instantaneous rate of change and percentage rate of change of the number of customers at 4pm. Interpret your answers.

Rates of Change: Numerical Limits & Nonexistence

Derivative: Numerical Definition

Let a be fixed, and let x be some point on f other than a. Then, the slope of the secant line is given by

$$\frac{f(x) - f(a)}{x - a}$$

Taking limits, we have the definition for the derivative of f at a point a:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example 2.4.1. Find the derivative of the function $f(x) = \sqrt{2x}$ at $x = 3$ using the numerical method. Round your final answer to the thousandths place, if necessary.

x	$\frac{f(x) - f(3)}{x - 3}$	x	$\frac{f(x) - f(3)}{x - 3}$
2.9		3.1	
2.99		3.01	
2.999		3.001	
2.9999		3.0001	
2.99999		3.00001	
$f'(3) \approx$		$f'(3) \approx$	

$f'(3) \approx$ ______________________

Example 2.4.2. A multinational corporation invests \$32 billion in assets, resulting in the future value $F(t) = 32(1.12^t)$ billion dollars after t years.

(a) By how much is the investment growing in the fourth year? Write a sentence interpreting your answer, and round to the nearest hundredth.

t	$\frac{F(t)-F(4)}{t-4}$
$\lim\limits_{x\to 4^-} \frac{F(t)-F(4)}{t-4}$	

t	$\frac{F(t)-F(4)}{t-4}$
$\lim\limits_{x\to 4^+} \frac{F(t)-F(4)}{t-4}$	

$$F'(4) \approx \underline{\hspace{6cm}}$$

(b) Find the percent rate of change in the fourth year. Round to 2 decimal places.

Derivative: Existence

The derivative of a function does not always exist; the definition requires that the function be smooth and continuous. Formally, we say that a function is *differentiable* when the derivative exists for all x in some interval (a, b). We have three cases for nonexistence:

- Corner/cusp
- Vertical asypmtote
- Discontinuity

Exercises

Example 2.4.3. Numerically estimate the derivative of the function $f(x) = -x^2 + 4x$ at $x = -1$. Round your final answer to the nearest tenth.

x	

x	

Example 2.4.4. Numerically estimate the derivative of the function $g(y) = 5\ln y$ at $x = 5$. Round your final answer to the nearest hundredth.

x	

x	

Example 2.4.5. The annual number of passengers going through the Atlanta airport between 2000 and 2008 can be modeled as $p(t) = -0.102t^3 + 1.39t^2 - 3.29t + 79.25$ million passengers, t years since 2000.

(a) Estimate $p'(6)$ numerically to the nearest thousandth.

t	

t	

(b) Write an interpretation of $p'(6)$.

(c) Find the percent rate of change in 2006, to the nearest hundredth.

Example 2.4.6. The average weekly sales (in million dollars) for Abercrombie & Fitch between 2004 and 2008 is given in the table below.

Year	2004	2005	2006	2007	2008
Sales (in million dollars)	38.87	53.56	63.81	72.12	68.08

(a) Align the data so that the year 2000 corresponds to an input of 0. Determine and write the most appropriate model for the data using this alignment.

(b) Estimate the rate of change of average weekly sales in the year 2007 and interpret your answer.

x	

x	

The Derivative, Algebraically

Definition 2.5.1 (Derivative (Algebraic Definition))

Let $f(x)$ be a function defined on the open interval (a,b), and $x \in (a,b)$. Then, the derivative of f at point x is given by the formula

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Question 2.5.2 Why is this definition the same as the one in §2.4?

It is useful to remember a few things from algebra when doing these calculations:

- $\boxed{\sqrt{x+y} \neq \sqrt{x}+\sqrt{y}}$
- $\boxed{(a+b)^2 \neq a^2+b^2}$
- $\boxed{\frac{a}{b+c} \neq \frac{a}{b}+\frac{a}{c}}$

When we algebraically find the derivative of a function, there is a four-step process which makes the algebra much simpler, and the derivative easier to find. The steps are:

1. $\boxed{\text{Find and simplify } f(x+h)}$
2. $\boxed{\text{Find and simplify } f(x+h)-f(x)}$
3. $\boxed{\text{Find and simplify } \frac{f(x+h)-f(x)}{h}}$
4. $\boxed{\text{Compute } \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$

This is demonstrated below.

Example 2.5.3. Algebraically find the derivative of the function $f(x) = x^2$ using the four-step process.

The Derivative, Algebraically

Example 2.5.4. Algebraically find the derivative of the function $f(x) = 5x - 2$ using the four-step process.

The Derivative, Algebraically

Example 2.5.5. The time it takes an average athlete to swim 100 meters freestyle at age x years can be modeled as

$$t(x) = 0.181x^2 - 8.463x + 147.376 \text{ seconds}$$

(a) Calculate the swim time at age 13 to the nearest second.

(b) Use the algebraic method to develop a formula for the derivative of t (ie, find $t'(x)$).

(c) How quickly is the time to swim 100 meters freestyle changing for an average 13-year-old athlete? Round to the nearest hundredth and interpret the result.

(d) Compute the percent rate of change of swimmers' time at age 13, to the nearest tenth.

The Derivative, Algebraically

Example 2.5.6. Algebraically determine the derivative of $f(t) = \frac{1}{2}t^2 - \frac{1}{3}$, and evaluate $\left.\frac{df}{dt}\right|_{t=1}$

Example 2.5.7. Algebraically determine the derivative of $k(r) = r^2 - 2r^3$, and evaluate $k'(0)$

Rate of Change Graphs

We will use the following terminology interchangeably:

- slope graph
- rate of change graph
- derivative graph

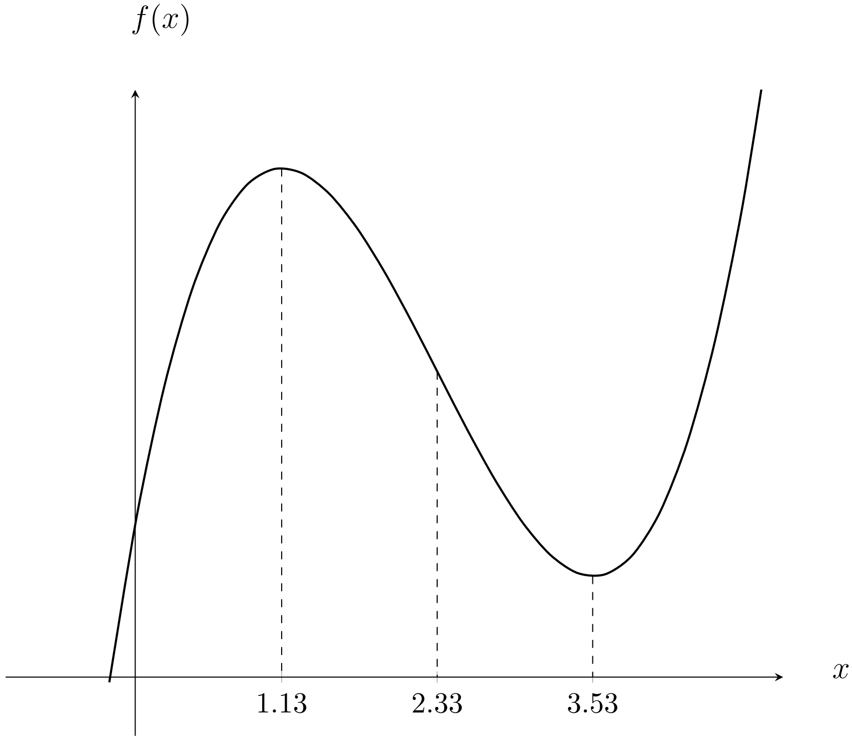
The following information will be useful when constructing slope graphs:

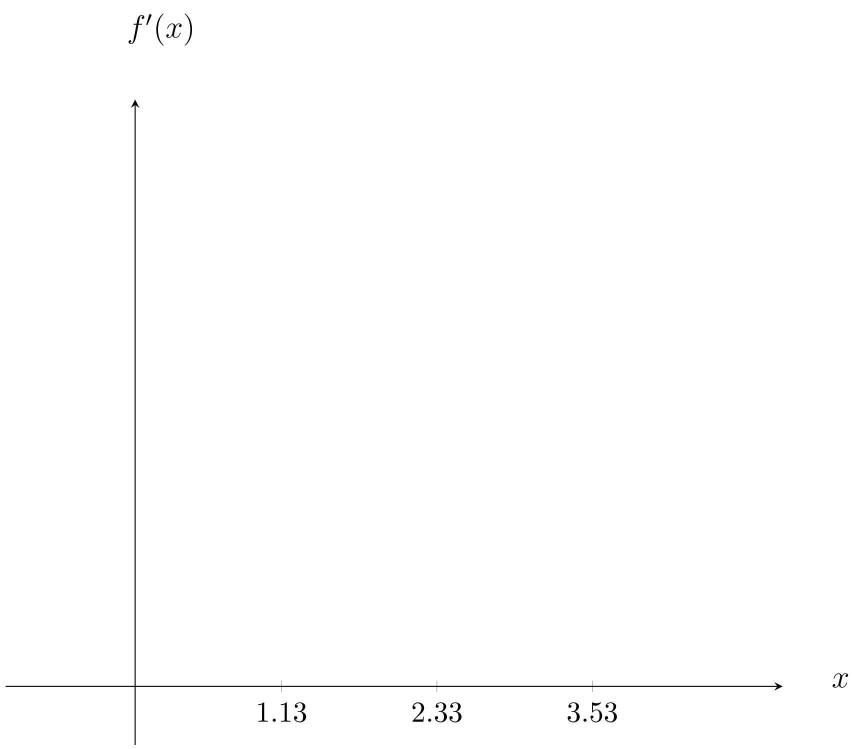
Derivative is:	**Function graph is:**	**Slope graph is:**
Positive		
Zero		
Negative		

Things to keep in mind when plotting the slope graph:

- Horizontal tangents result in a max or min *on the derivative graph.*
- Derivatives may fail to exist at some points; there may be a vertical asymptote, discontinuity, or corner
- The slope graph reports slopes of the original graph; a negative slope results in a negative value, and a positive slope results in a positive value.

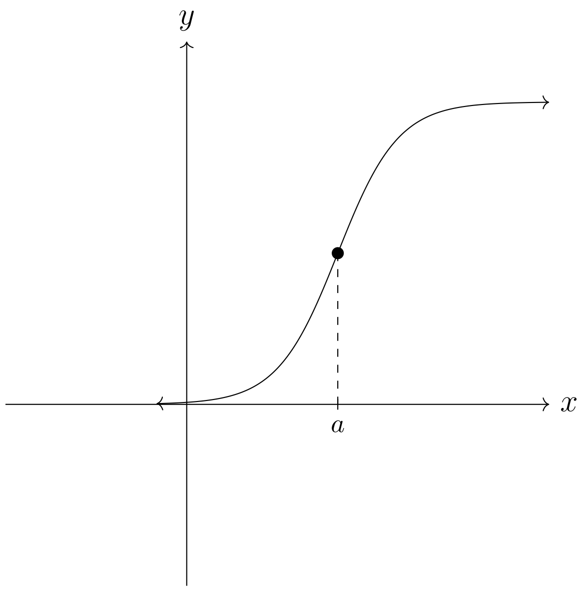
Example 2.6.1. Sketch the rate of change graph for the function

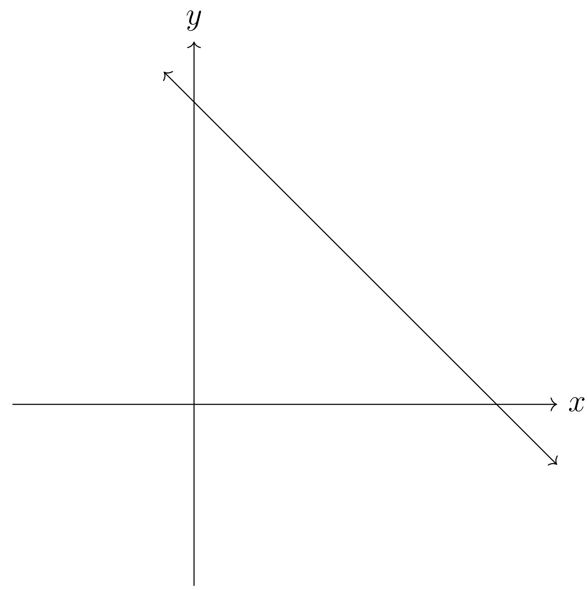




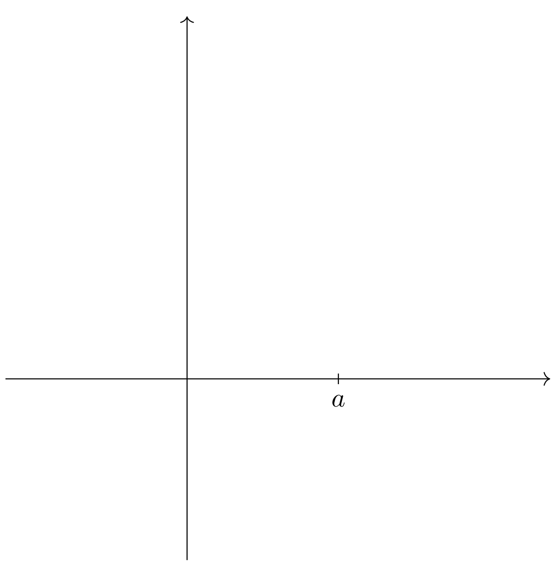
 Rate of Change Graphs

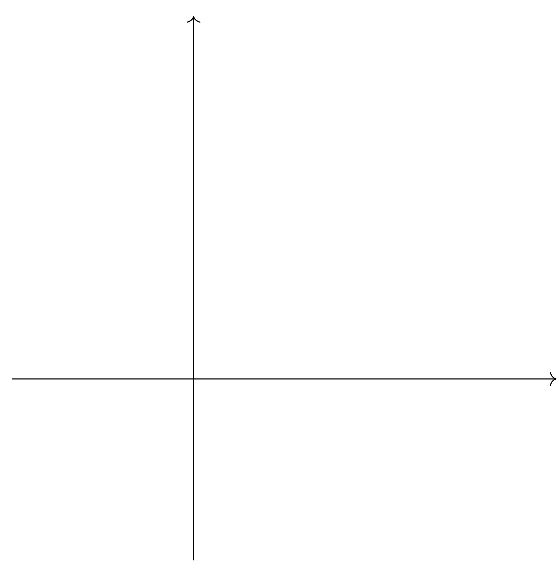
Example 2.6.2. Sketch (and label) the slope graphs of the following functions:

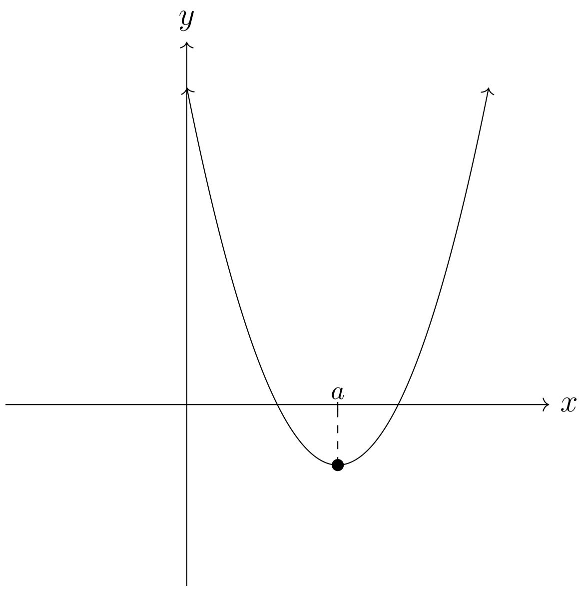


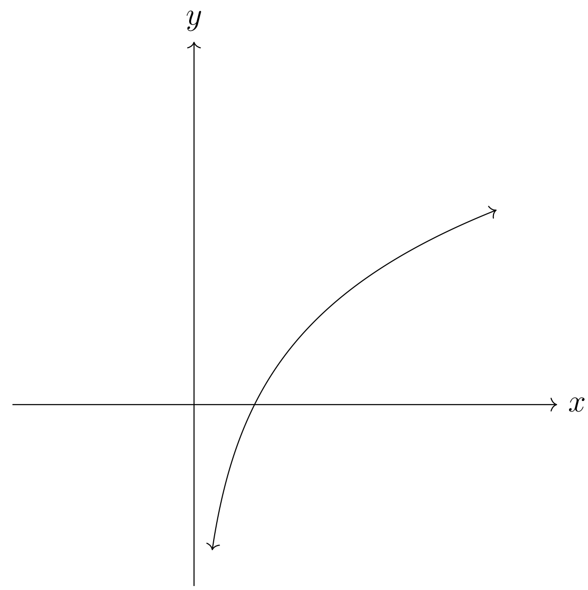


(a)









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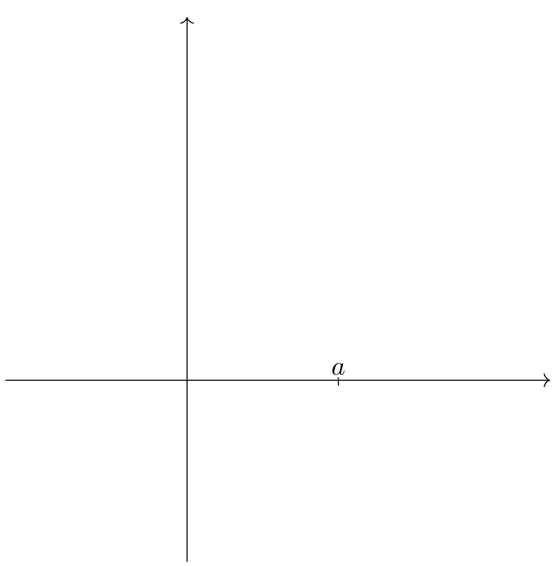

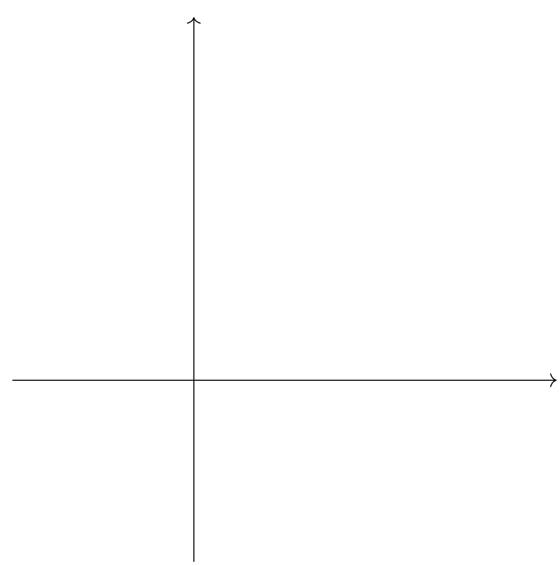

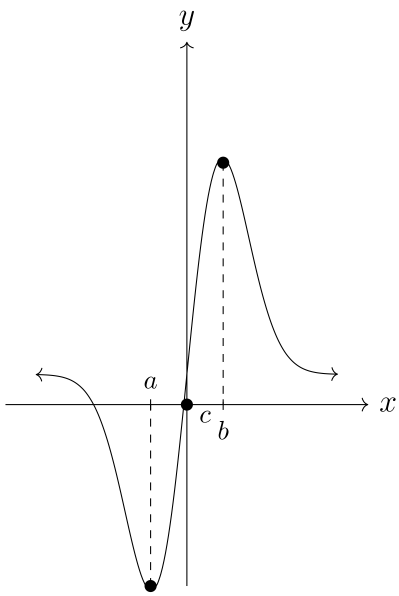

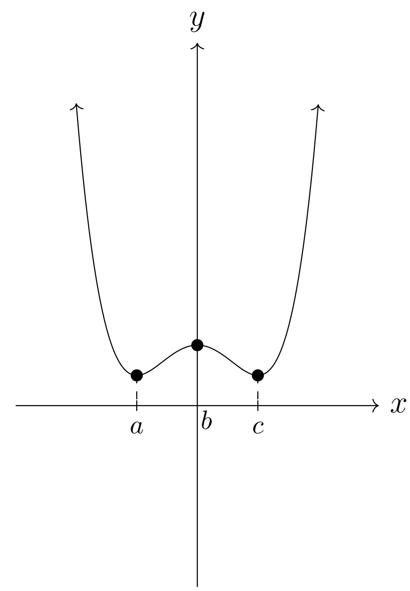

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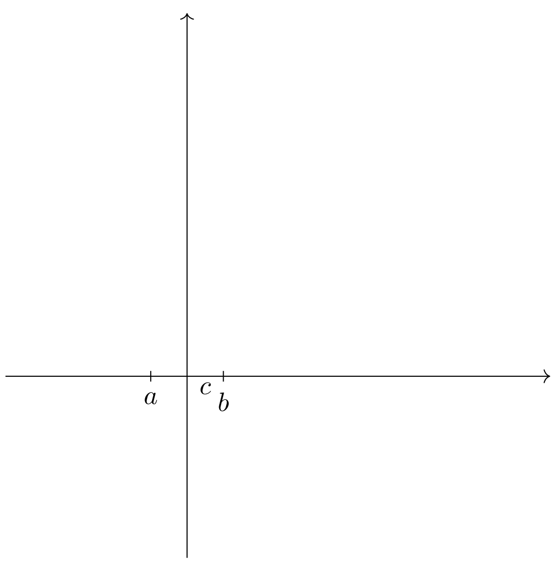
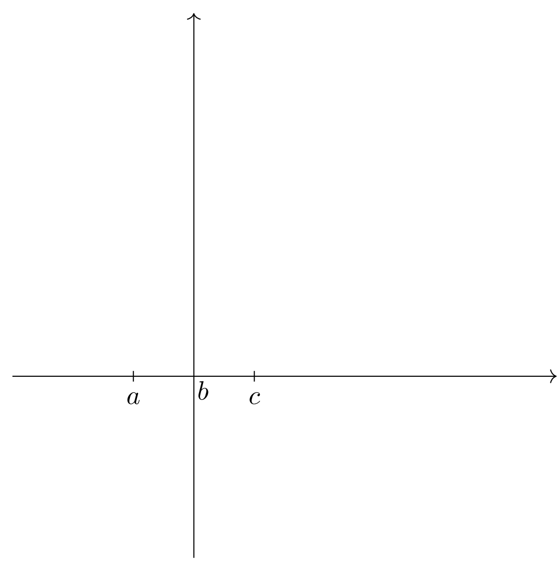
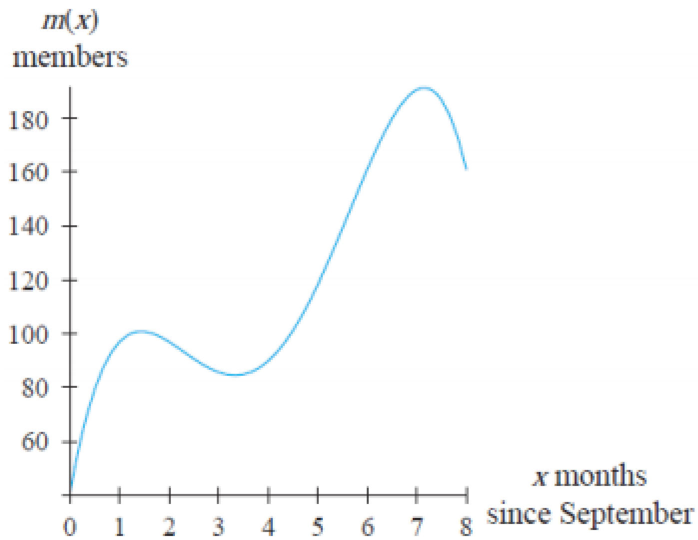




Example 2.6.3. The figure below shows the membership in a campus organization during its first year. Round all answers to the nearest member.

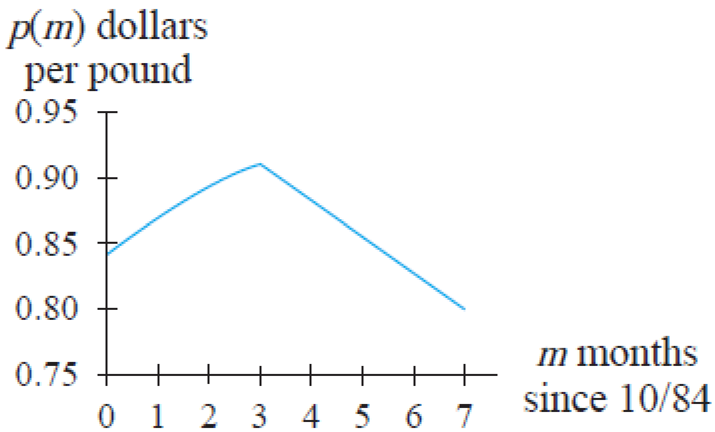


(a) Estimate the average rate of change of membership from September through May.

(b) Estimate the instantaneous rate of change in October, December, and April.

(c) Sketch a rate-of-change graph for membership. Label both axes.

Example 2.6.4. The figure below shows cattle prices (for choice of 450-pound steer calves) from October 1994 through May 1995.



(a) For which input value does the derivative fail to exist? Give a clear, mathematical reason why.

(b) Sketch a slope graph of p. Label both axes.

Example 2.6.5. Sketch the slope graph of a function f with input t that meets the following criteria:

- $f(-2) = 5$
- the slope is positive for $t < 2$
- the slope is negative for $t > 2$
- $f'(2)$ does not exist

Example 2.6.6. Sketch the slope graph of a function g with input x that meets these criteria:

- $g(3)$ does not exist
- $g'(0) = -4$
- $g'(x) < 0$ for $x < 3$
- g is concave down for $x < 3$
- $g'(x) > 0$ for $x > 3$
- g is concave up for $x > 3$
- $\lim_{x \to 3^+} g(x) \to \infty$
- $\lim_{x \to 3^-} g(x) \to -\infty$

Chapter 3

Determining Change: Derivatives

Simple Rate of Change Formulas

The Formulas

Instead of calculating derivatives by hand every time, we can develop sets of rules which will help us more easily calculate them. These are given below; let c be a constant, and $f(x), g(x)$ be functions:

Name	Function	Derivative
Constant Rule	$f(x) = b$	$f'(x) = \boxed{0}$
Power Rule	$f(x) = x^n$	$f'(x) = \boxed{nx^n}$
Constant Multiplier Rule	$c \cdot f(x)$	$f'(x) = \boxed{c \cdot f'(x)}$
Sum Rule	$f(x) + g(x)$	$\boxed{f'(x) + g'(x)}$
Difference Rule	$f(x) - g(x)$	$\boxed{f'(x) - g'(x)}$

Examples

Example 3.1.1. Write the formula for the derivative of the function.

(a) $f(x) = x^2$

(b) $g(x) = 3x^4$

(c) $h(t) = 0.2t^{50} - 10t + 1$

(d) $x(t) = t^{2\pi}$

(e) $f(x) = 3x^3$

(f) $q(x) = zx^{n+2}$

(g) $f(x) = 12x^{0.4} + 2x^{56} + 5$

(h) $g(x) = -3.2x^{-3.5} + 6.1x^{5/2} - 5.3$

(i) $f(x) = 7x^{-3}$

(j) $g(x) = -\dfrac{9}{x^2}$

(k) $f(x) = 4\sqrt{x} + 3.3x^5$

(l) $k(x) = \dfrac{4x^2 + 19x + 6}{x}$

(m) $g(t) = 5.8t^3 + 2t^{-1.2} - 5$

Example 3.1.2. Find the derivative of $h(x) = x^2(x^3 + 1)$

Example 3.1.3. The temperature (in $^\circ F$) of Norman on Wednesday can be modeled by $t(x) = -0.8x^2 + 11.6x + 38.2$ degrees Fahrenheit, x hours after 6 A.M.

(a) Write the **complete** rate of change model for the temperature.

(b) By how much is the temperature changing at 10 A.M.? Round your answer to the nearest hundredth.

(c) Compute and interpret $\left.\dfrac{dt}{dx}\right|_{x=10}$. Round your answer to the nearest tenth.

(d) Compute the percent rate of change of temperature at 4:00pm. Round your answer to the nearest hundredth.

Example 3.1.4. The table shows the metabolic rate of a typical 18- to 30-year-old male according to his weight:

Weight (lbs)	88	110	125	140	155	170	185	200
Metabolic Rate (kCal/day)	1291	1444	1551	1658	1750	1857	1964	2071

(a) Find a **complete** linear model for the metabolic rate of a typical 18- to 30-year-old male.

(b) Write the derivative model for the formula in part (a).

(c) Write a sentence which interprets the derivative of the metabolic rate model of a 26-year-old male. Round your answer to the nearest whole number.

Exponential & Logarithmic Rate of Change Formulas

The Formulas

For exponential and logarithmic functions, we have the following formulas:

Name	Function	Derivative
General Exponential Rule	$f(x) = b^x$	$f'(x) = \boxed{\ln b \cdot b^x}$
Exponential Rule	$f(x) = e^x$	$f'(x) = \boxed{e^x}$
Logarithm Rule	$f(x) = \ln x$	$f'(x) = \boxed{\dfrac{1}{x}}$

Examples

Example 3.2.1. Write the formula for the derivative of the function.

(a) $h(x) = 3 - 7e^x$

(b) $f(x) = 6(0.8)^x$

(c) $f(a) = 10\left(1 + \dfrac{0.05}{4}\right)^{4a}$

(d) $g(x) = 4\ln x - e^{\pi}$

(e) $f(x) = 3.7e^x - 2\ln x$

(f) $y(x) = -\ln x + 2e^x$

(g) $f(g) = 4\sqrt{g} + 5(1.2)^g$

(h) $k(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

Example 3.2.2. For the first two hours after yeast dough has been kneaded, it doubles in volume approximately every 42 minutes. If 1 quart of yeast dough is left to rise in a warm room, its growth can be modeled as $v(h) = e^h$ quarts, where h is the number of hours the dough has been allowed to rise.

(a) How many minutes will it take the dough to attain a volume of 2.5 quarts?

(b) Write a model for the rate of growth of the yeast dough.

Example 3.2.3. The weight of a laboratory mouse between 3 and 11 weeks of age can be modeled as $w(t) = 11.3 + 7.37 \ln t$ grams, where the age of the mouse is $t + 2$ weeks.

(a) What is the weight of a 9-week-old mouse? Round to the nearest hundredth.

(b) Write a rate of change model for the weight of the mouse, and determine how rapidly its weight is changing at 9 weeks.

(c) What is the average rate of change in the weight of the mouse between ages 7 and 11 weeks? Round to the nearest hundredth.

(d) Does the rate at which the mouse is growing increase or decrease as the mouse gets older? Why?

Example 3.2.4. Suppose the managers of a dairy company have modeled weekly production costs as $c(u) = 3250 + 75 \ln u$ dollars for u units of dairy products. Weekly shipping cost for u units is given by $s(u) = 50u + 1500$ dollars.

(a) Write the formula for the total weekly cost of production and shipping of u units.

(b) Write the rate of change model of the total weekly cost of producing and shipping u units.

(c) Calculate the total cost to produce and ship 5000 units in 1 week.

(d) Calculate and interpret the rate of change in the total cost to produce and ship 5000 units in 1 week.

Example 3.2.5. An individual has \$45,000 to invest. \$32,000 will be put into a low-risk mutual fund averaging 6.2% interest compounded monthly, and the remainder will be invested in a high-yield bond fund averaging 9.7% interest, compounded continuously.

(a) Write an equation for the total amount in the two investments, using $I(t)$ as your function.

(b) Write the rate of change model for the low-risk fund, using $L(t)$ as your function.

(c) Write the rate of change model for the high-yield fund, using $H(t)$ as your function.

(d) Write the rate of change model for the combined investment.

(e) Calculate and interpret $\frac{dI}{dt}$ after 8 months, and after 18 months.

Rate of Change of Composite Functions

Review: Composite Functions

Two functions $f(x)$ and $x(t)$ can be composed *if and only if* the output of $x(t)$ is the input of $f(x)$. Notice how the notation is suggestive; f inputs x, which is exactly what $x(t)$ outputs. We write the composition either as $(f \circ x)(t)$ or $f(x(t))$. The new input is now the input of x (ie, t), and the new output is the output of f (namely, f).

Example 3.3.1. Identify the functions which make up the composite functions given below.

(a) $f(x) = \dfrac{1}{x+2}$

(b) $g(x) = \ln(x^2)$

(c) $h(t) = e^{5t}$

(d) $q(x) = (2x+1)^5$

(e) $n(f) = \left(3 + \dfrac{1}{f}\right)^3$

(f) $s(h) = \ln\left(5h^2 + \dfrac{1}{h}\right)$

(g) $y(r) = \dfrac{5.317}{(2r^5 + 1.7)^2}$

(h) $w(c) = \sqrt[3]{\dfrac{c}{1+c}}$

(i) $f(x) = 1 - \sqrt{e^x + 5x}$

The Chain Rule

The *chain rule* is a rule for finding the derivative of composite functions. Let $h(x) = f(g(x))$, where the output of g is the input of f. Then,

$$h'(x) = f'(g(x)) \cdot g'(x)$$

The best way to learn the chain rule is with practice **inside and outside of class**.

Examples

Example 3.3.2. For $f(t) = 3t^2$ and $t(x) = 4 + 7\ln x$, find the rate of change function $(f \circ t)'(x)$ with respect to x.

Example 3.3.3. Let $c(x) = 3x^2 - 2$ and $x(t) = 4 - 6t$. Find $c'(t)$

Example 3.3.4. Consider the following functions:

$$f(g) = \ln g \qquad g(h) = 5h + 2 \qquad h(j) = e^j \qquad j(x) = 4x^{-1}$$

Find $f(x)$ and $f'(x)$.

Example 3.3.5. Find the derivative of $f(x) = \dfrac{1}{x+2}$

Example 3.3.6. Find the derivative of $f(x) = \ln(x^2)$

Example 3.3.7. Find the derivative of $f(x) = (\ln x)^3$

Example 3.3.8. Find the derivative of $f(x) = e^{5x}$

Example 3.3.9. Find the derivative of $f(x) = (e^x)^4$

Example 3.3.10. Find the derivative of $f(x) = 7 + 5\ln(4x^2 + 3)$

Example 3.3.11. If $s(t) = 3e^{5t}$, find $s'(t)$

Example 3.3.12. Find the derivative of $k(x) = 3e^{4x^2}$

Example 3.3.13. Find the derivative of $p(t) = (5 + 6e^{2t})^3$

Example 3.3.14. $f(x) = 6(4x^2 + 3)^5$

Example 3.3.15. $f(x) = -12\ln(6x^2 + 3^x)$

Example 3.3.16. $f(x) = 2e^{0.5x} - 2x$

Example 3.3.17. $f(x) = \dfrac{7.2}{(4x^3+1)^4}$

Example 3.3.18. $f(x) = 3\sqrt{x^3 + 2\ln x}$

Example 3.3.19. Find the derivative of $f(x) = e^{kx}$

Example 3.3.20. Compute the derivative of $e^{f(x)}$

Example 3.3.21. Find the derivative of the function $\dfrac{1.356}{1 + 20.5e^{-4.6t}}$

Example 3.3.22. Compute the derivative of $j(x) = \ln(\ln(\ln(x^2 - e^{3x})))$

Example 3.3.23. The number of children under 18 living in households headed by a grandparent can be modeled as

$$p(t) = 2.111e^{0.04t} \quad \text{million children}$$

where t is the number of years since 1980.

(a) Write the rate-of-change formula for p.

(b) How rapidly was the number of children living with their grandparents growing in 2010?

Example 3.3.24. The tuition x years from now at OU is projected to be $t(x) = 24072e^{0.056x}$ dollars.

(a) Write the rate-of-change formula for tuition.

(b) What is the rate of change in tuition four years from now?

Rate of Change of Product Functions

Product Rule

The *product rule* allows us to take derivatives of product functions, i.e. functions that look like $h(x) = f(x) \cdot g(x)$. The product rule is

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

Example 3.5.1. Find the derivative of $h(x)$, where $h(x) = [4(3^x)] \cdot [5x^2]$.

Example 3.5.2. Find the derivative of $g(y) = \dfrac{\ln(2y)}{1+y^2}$.

Example 3.5.3. Compute the derivative of the function $g(x) = 3x^{-0.7} \cdot 5^x$

Example 3.5.4. Let $f(x) = 3\ln(2 + 5x)$, and $h(x) = \dfrac{1}{\ln x}$. Find $(f \cdot h)'(x)$.

Example 3.5.5. The profit from the supply of a certain commodity is modeled by $P(q) = 72qe^{-0.2q}$ dollars, when q units are produced.

(a) Write the complete rate of change model for profit.

(b) At what production level(s) is the rate of change of profit zero? If necessary, round your answer to two decimal places.

(c) What is profit at the production level for the value in part (b)?

Quotient Rule

Alternately, for a quotient function, we may employ the *quotient rule*. If $k(x) = \dfrac{f(x)}{g(x)}$, then

$$k'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

This can be remembered as

$$\frac{\text{low} \cdot d(\text{high}) - \text{high} \ \cdot d(\text{low})}{(\text{low})^2}$$

Example 3.5.6. Use the quotient rule to find the derivative of $g(y) = \dfrac{\ln(2y)}{1+y^2}$

Examples

Example 3.5.7. The production level at a plant manufacturing radios can be modeled as $f(x) = 10.54x^{0.5}(2 - 0.13x)^{0.3}$ thousand radios, where x thousand dollars has been spent on modernizing plant technology.

(a) Identify two functions g and h that, when multiplied, form the production model- one of the functions will need the chain rule.

(b) Using function notation, write the notation for the production model and for the rate of change of the production model.

(c) Write a model for the rate of change of production

Example 3.5.8. Write the product function and rate-of-change function for the given functions.

(a) $g(x) = 5x^2 - 3$ and $h(x) = 1.2^x$

(b) $g(x) = 6e^{-x} + \ln x$ and $h(x) = 4x^{2.1}$

Example 3.5.9. Find the derivatives of the following functions.

(a) $f(x) = (\ln x)e^x$

(b) $g(x) = (x+5)e^x$

(c) $t(x) = (5.7x^2 + 3.5x + 2.9)^3(3.8x^2 + 5.2x + 7)^{-2}$

(d) $f(x) = \dfrac{2x^3 + 3}{2.7x + 15}$

(e) $f(x) = (8x^2 + 13)\left(\dfrac{39}{1 + 15e^{-0.09x}}\right)$

(f) $f(x) = [\ln(15.7x^3)] \cdot [e^{15.7x^3}]$

(g) $f(x) = \dfrac{4(3^x)}{\sqrt{x}}$

(h) $f(x) = \dfrac{14x}{1 + 12.6e^{-0.73x}}$

Example 3.5.10. A store has determined that the number of cookies sold monthly is approximately $c(x) = 6250(0.929^x)$ cookies, where x is the average price of a cookie (in dollars).

(a) Write a model for revenue as a function of price.

(b) If each cookie costs the store \$1, write a model for profit as a function of price.

(c) Write the complete rate of change model for revenue as a function of price.

Example 3.5.11. Find the derivative of $t(x) = (6x^3)(\ln x)(e^{2x})$.

Chapter 4

Analyzing Change: Applications of Derivatives

Relative Extreme Points

Definitions

Definition 4.2.1 (Relative Extrema)

Let $f(x)$ be a function defined on an input interval $[a, b]$. Let $a < c < b$.

We say that f has a $\boxed{\text{relative maximum}}$ at c if the output $f(c)$ is $\boxed{\text{bigger than}}$ any other output in some interval around c. Likewise, f has a $\boxed{\text{relative minimum}}$ at c if the output $f(c)$ is $\boxed{\text{smaller than}}$ any other output in some interval around c.

Relative maxima/minima are also referred to as $\boxed{\text{local maxima/minima or local extrema}}$

Definition 4.2.2 (Critical Point)

A **critical point** of a continuous function f is a pair $\boxed{(c, f(c))}$ at which f is $\boxed{\text{either not differentiable or has } f'(c) = 0.}$ The input value c of a critical point is called the $\boxed{\text{critical input}}$.

Example 4.2.3. Find the critical points of $f(x) = 4x^3 + 8x^2 - 20x - 21$ on the interval $[-5, 5]$.

Theorem 4.2.4 (First Derivative Test)

Suppose c is a critical input of a continuous function f.

• If f' changes from positive to negative at c, then $\boxed{f \text{ has a relative maximum at } c}$ • If f' changes from negative to positive at c, then $\boxed{f \text{ has a relative minimum at } c}$ • If f' does not change its sign at c, then $\boxed{f \text{ has a horizontal tangent line at } c}$

Theorem 4.2.5 (Conditions Where Extreme Points Exist)

For a function f with input x, a relative extremum can occur at $x = c$ only if $f(c)$ exists/is defined. Further,

• A relative extremum exists where $\boxed{f'(c) = 0}$ and the graph of $f'(x)$ $\boxed{\text{crosses}}$ the input axis at $x = c$.

• A relative extremum *can* exist where $f(x)$ exists, but $f'(x)$ does not exist; further investigation will be needed.

Theorem 4.2.6 (Old Derivative Facts)

Let $f(x)$ be a function defined on an input interval $[a, b]$, and let $a < c < b$.

- If $\boxed{f'(x) > 0}$, then $f(x)$ is increasing at $x = c$.
- If $\boxed{f'(x) < 0}$, then $f(x)$ is decreasing at $x = c$.
- If $\boxed{f'(x) = 0}$, then the line tangent to $f(x)$ at $x = c$ is horizontal.

Methods of Finding Extrema

Let $f(x)$ be differentiable on an open interval (a, b).

Algebraic Method:

1. Calculate the derivative $f'(x)$
2. Set $f'(x) = 0$ and solve for x. All solutions will be horizontal asymptotes; individual analysis (corresponding to 1st Derivative Test) will determine if $f'(c)$ is a maximum, minimum, or neither.

Calculator, Method 1:

1. Input $f(x)$ into Y_1
2. Plot $f(x)$, and do `Zoom`→`0:ZoomFit`
3. If you are finding a **local maximum**, press `2nd`→`Trace`→`4:maximum`. If you are finding a **local minimum**, press `2nd`→`Trace`→`3:minimum`.
   (a) The calculator will prompt for a left bound. Input a number *slightly less than* where you expect the maximum/minimum to be.
   (b) The calculator will then prompt for a right bound. Input a number *slightly greater than* where you expect the maximum/minimum to be.
   (c) The calculator will then prompt for a guess. Input a guess, or press enter.
   (d) The max/min will be displayed as a coordinate pair. If the pair is needed, use appropriate rounding guidelines.
   (e) If you forget the value of the max/min, the calculator will store the $x-$coordinate in the variable X. In order to recall it, on the home screen, press X and the calculator will return the appropriate value.

Calculator, Method 2:

1. Input $f(x)$ into Y_1
2. In Y_2, use the `nDeriv` command by pressing `Math`→`nDeriv(`. The complete entry *must* look like

$$Y_2 = \texttt{nDeriv(Y1(X),X,X)}$$

   This will have the calculator graph the derivative as well as the original function
3. The local extrema are given by wherever the derivative graph touches the $x-$axis. According to 1st Derivative Test, a local max occurs when the derivative crosses from positive to negative, and a local min occurs when the derivative crosses from negative to positive.

Examples

Example 4.2.7. The percentage of people in the United States (aged 15 and above) who are sleeping at a given time of night can be modeled as

$$s(t) = -2.63t^2 + 29.52t + 13.52 \text{ percent, } t \text{ number of hours after 9pm}$$

(a) Find the critical input values of s on the interval $0 \leq t \leq 11$, and calculate the output value for any critical point.

(b) Graph $s(t)$ and $s'(t)$. Label and interpret the critical inputs.

Example 4.2.8. Sketch a graph such that

- $f'(x) > 0$ for $x < -1$
- $f'(x) < 0$ for $x > -1$
- $f'(-1) = 0$

Example 4.2.9. Sketch a graph such that

- f has a relative maximum at $x = 3$
- f has a relative minimum at $x = -1$
- $f'(x) < 0$ for $x < -1$ and $x > 3$
- $f'(x) > 0$ for $-1 < x < 3$
- $f'(-1) = f'(3) = 0$

Example 4.2.10. Consider the function $f(x) = x^2 + 2.5x - 6$.

(a) Write the derivative formula

(b) Locate and classify each critical point.

Example 4.2.11. Consider the function $f(x) = 0.3x^3 + 1.2x^2 - 6x + 4$.

(a) Write the derivative formula

(b) Locate and classify each critical point.

Example 4.2.12. Consider the function $f(x) = 5e^{-x} + \ln x$ (for $x > 0$).

(a) Write the derivative formula

(b) Locate and classify each critical point.

Example 4.2.13. Consider the function $f(x) = \dfrac{10}{1+2e^{-0.5x}}$.

(a) Write the derivative formula

(b) Locate and classify each critical point.

Example 4.2.14. For the following graphs, determine which of the following statements are true:

(1) $f'(x) > 0$ for $x < 2$

(2) $f'(x) > 0$ for $x > 2$

(3) $f'(x) = 0$ for $x = 2$

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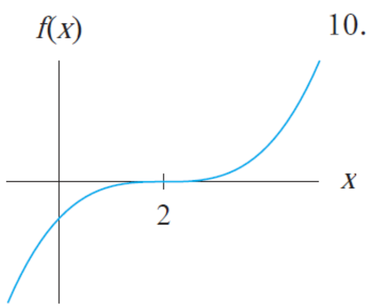

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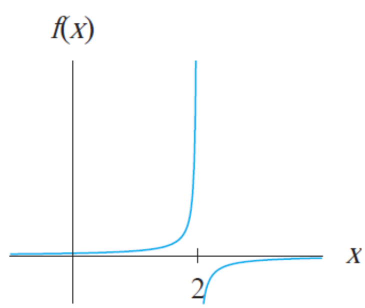

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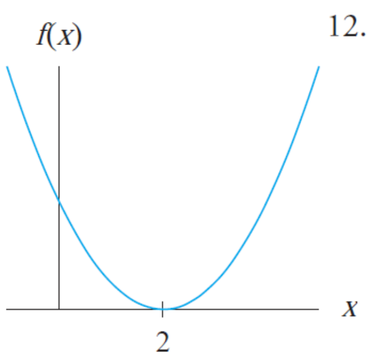
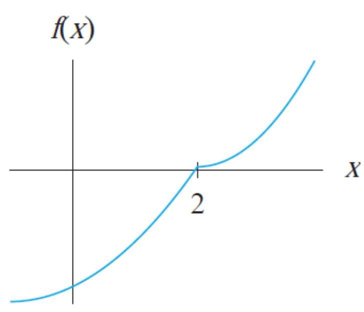


12.



Absolute Extreme Points

Definition 4.3.1 (Absolute Extrema)

A function f has an **absolute maximum** at input c if $\boxed{f(c) \text{ is greater than or equal to all other outputs}}$. Similarly, f has an **absolute minimum** at c if $\boxed{f(c) \text{ is less than or equal to all other outputs}}$.

In practice, there is very little distinction between what we did in 4.2 and what we do here in 4.3; the key difference is determining whether or not the particular max/min is *the greatest* or *the least* output value.

Examples

Example 4.3.2. Consider the function $f(x) = 6x^4 - 6x^3 - 5x^2 + 5x - 1$

(a) Locate any extreme values of the function on the interval $-2 \leq x \leq 2$.

(b) Classify the extreme values you found in part (a)

Example 4.3.3. Consider the function $g(t) = -0.37t^3 + 5.34t^2 - 9.66t + 96.93$

(a) Locate any extreme values of the function on the interval $0 \leq x \leq 11$.

(b) Classify the extreme values you found in part (a)

Example 4.3.4. Consider the function $h(p) = (e^{2-p})(3^p - p^2)$

(a) Locate any extreme values of the function on the interval $-1 \leq x \leq 4$.

(b) Classify the extreme values you found in part (a)

Example 4.3.5. Consider the function $y(x) = 0.75x^4 - 3.86x^2 + 10.18x + 22.186$

(a) Locate any extreme values of the function on the interval $(-\infty, \infty)$

(b) Classify the extreme values you found in part (a)

Example 4.3.6. Find and classify the absolute and relative maxima/minima for the function $f(x) = 3x^4 - 16x^3 + 18x^2$ on $[-1, 4]$. If necessary, round to the nearest hundredth.

Example 4.3.7. Find the absolute and relative extrema for the function $f(x) = x^3 - 3x^2 + 1$ on $-\frac{1}{2} \leq x \leq 4$. If necessary, round to the nearest hundredth.

Example 4.3.8. Find and classify all extrema of the function $f(x) = 12 + 4x - x^2$ on $[0, 5]$. If necessary, round to the nearest hundredth.

Example 4.3.9. Find and classify all extrema of the function $f(t) = (t^2 - 4)^3$ on $[-2, 3]$. If necessary, round to the nearest hundredth.

Example 4.3.10. Find the relative and absolute maxima and minima of the function $g(x) = \dfrac{x}{x^2 - x + 1}$ on $[0, 3]$. If necessary, round to the nearest hundredth.

Example 4.3.11. The sales of a new Starbucks drink are approximated by the function $S(x) = -.002x^4 + .093x^3 - 1.38x^2 + 6.573x + 5.393$ thousand dollars, x months after its introduction. Round your answers to the nearest hundredth.

(a) The absolute **maximum** of drink sales between month 1 and month 15 was ____________

and occurred ____________ months after release.

(b) The absolute **minimum** of drink sales between month 1 and month 15 was ____________

and occurred ____________ months after release.

(c) Calculate the percent rate of change 4 months after introduction, to the nearest hundredth as a percent.

Example 4.3.12. The quantity of a drug in the bloodstream t hours after a tablet is swallowed is given by $q(t) = 20(e^{-t} - e^{-3t})\ \mu g$.

(a) How much of the drug is in the bloodstream at time $t = 0$?

(b) Over the first twelve hours, at what time is the amount of drug in the bloodstream at its highest? What is the maximum amount?

Inflection Points & Second Derivatives

Definitions

Definition 4.4.1 (Inflection Point)

An **inflection point** is the point on the graph of a function where $\boxed{\text{the function changes concavity}}$.

An inflection point gives the point of $\boxed{\text{most/least rapid}}$ change of the function.

Theorem 4.4.2 (Properties of the Second Derivative)

Let $f(x)$ be a twice-differentiable function defined on input interval $[a, b]$, with $a < c < b$.

- f has an inflection point at $x = c$ if and only if $\boxed{f''(c) = 0 \text{ and changes sign}}$
- If $f''(c)$ is positive, then $\boxed{f \text{ is concave up}}$
- If $f''(c)$ is positive, then $\boxed{f' \text{ is increasing}}$
- If $f''(c)$ is negative, then $\boxed{f \text{ is concave down}}$
- If $f''(c)$ is negative, then $\boxed{f' \text{ is decreasing}}$

Theorem 4.4.3 (Second Derivative Test)

Let $f(x)$ be a twice-differentiable function defined on input interval $[a, b]$, with $a < c < b$.

- $\boxed{\text{If } f'(c) = 0 \text{ and } f''(c) > 0\text{, then } f \text{ has a relative minimum at } c}$
- $\boxed{\text{If } f'(c) = 0 \text{ and } f''(c) < 0\text{, then } f \text{ has a relative maximum at } c}$

Examples

Example 4.4.4. The percentage of people living in California in 2007 who were born in the state can be modeled as

$$P(x) = -0.0016x^3 + 0.224x^2 - 10.577x + 204.8 \text{ percent}$$

where x is the age of the resident.

(a) Find the inflection point of the function P.

(b) Give a sentence of interpretation for the age between 20 and 70 at which the percentage of California residents who were born in the state was decreasing least rapidly.

Example 4.4.5. For the function

$$f(t) = -2.1t^2 + 7t$$

(a) Write the first and second derivative.

(b) Identify any inflection points, and label them as the point of *least rapid* or *most rapid* change.

Example 4.4.6. The percentage of new material that an average college student will retain after studying for t hours without a break can be modeled as

$$p(t) = \frac{83}{1 + 5.94e^{-0.969t}} \text{ percent}$$

(a) Find when the retention rate is increasing most rapidly.

(b) Determine the rate of change of retention as well as the percentage of retention at the input found in item (a).

(c) Describe the difference between the direction of p and p' to the right of the input found in item (a).

(d) Explain what happens to the student's retention rate after the input found in item (a).

Example 4.4.7. For the function

$$g(s) = 32s^3 + 2.1s^2 + 7s$$

(a) Write the first and second derivative.

(b) Identify any inflection points, and label them as the point of *least rapid* or *most rapid* change.

Example 4.4.8. For the function

$$h(x) = e^{3x} - \ln 3x$$

(a) Write the first and second derivative.

(b) Identify any inflection points, and label them as the point of *least rapid* or *most rapid* change.

Example 4.4.9. For the function

$$k(t) = \frac{16}{1 + 2.1e^{3.9t}}$$

(a) Write the first and second derivative.

(b) Identify any inflection points, and label them as the point of *least rapid* or *most rapid* change.

Example 4.4.10. For the function

$$f(x) = -x^3 + 12x^2 + 36x + 45$$

(a) Write the first and second derivative.

(b) Identify any inflection points, and label them as the point of *least rapid* or *most rapid* change.

Example 4.4.11. The table below shows the monthly revenue levels associated with various monthly levels of advertising by a furniture store.

Advertising (in hundreds of dollars)	1	4	7	10	13	16	19
Revenue (in thousands of dollars)	114	210	265	299	338	449	632

(a) Find the **complete** cubic model $R(x)$ for the data (do not align the input).

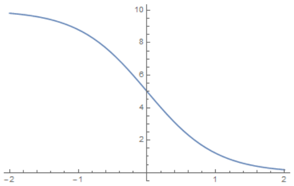
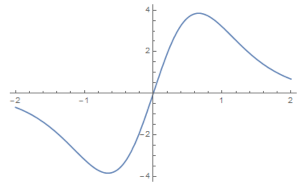
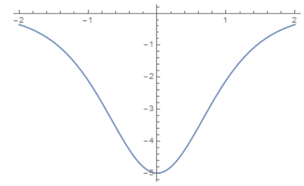
(b) Write the **complete** rate of change model for $R(x)$.

(c) Find $R''(x)$.

(d) Find the inflection point of $R(x)$ on the interval $[0, 19]$. Round both coordinates to the hundredths place and be sure to label them with units.

(e) Find the rate of change at the inflection point. Round to the hundredths place and include units in your answer.

Example 4.4.12. Consider the following graphs:

A  **B**  **C** 

(a) Write "True" or "False" to the left of the following statements:

__________ The graph of the derivative of **A** will have no $x-$intercepts.

__________ The graph of the derivative of **B** will have exactly one $x-$intercept.

__________ The graph of the second derivative of **C** will have exactly two $x-$intercepts.

__________ The graph of the second derivative of **A** will always be negative.

(b) Which of the following describes the relationship between these three graphs? *Mark an X to the left of your choice.*

__________ **B** is $f(x)$, **C** is $f'(x)$, and **A** is $f''(x)$.

__________ **A** is $f(x)$, **C** is $f'(x)$, and **B** is $f''(x)$.

__________ **C** is $f(x)$, **A** is $f'(x)$, and **B** is $f''(x)$.

__________ **B** is $f(x)$, **A** is $f'(x)$, and **C** is $f''(x)$.

Optimization of Constructed Functions

Strategy for Optimization

Step 1: Identify the quantity to be optimized (the output) and the quantities on which the output quantity depends (the input).

Step 2: Sketch and label a diagram of the situation.

Step 3: Construct a model with a single input variable.

Step 4: Locate the optimal point (minimum/maximum) for the model.

Examples

Example 4.6.1. In 2009, airlines had a 45-inch restriction on the maximum linear measurement (length + width + height) of carry-on luggage with the height restricted to 10 inches. Passengers concerned with keeping their travel costs down seek to maximize the capacity of their carry-on bag; what are the optimal measurements to maximize capacity?

Step 1: We want to maximize capacity of the carry-on, so we want to maximize volume. Because volume is given by $V = lwh$, our input variables are l, w, and h. In particular, we're told that $h = 10$; thus, $V = 10lw$.
Step 2: Fill in the sketch below:

Step 3: In order to write the model with a single input variable, we need a second equation to eliminate one of the variables. We know that $V = 10lw$. But, we are also told that the maximum linear measurement is given by $l + w + h = 45$. Since $h = 10$, this becomes $l + w + 10 = 45$, so $l + w = 35$. Now we can solve for either l or w. Choose to solve for length. Thus, $l = 35 - w$. We can substitute this into the equation for volume:

$$\begin{aligned} V &= 10lw \\ &= 10(35 - w)w \\ &= -10w^2 + 350w \end{aligned}$$

This gives us an equation in terms of a single variable, one which we can optimize.

Step 4: In order to optimize the equation, we want to find the max or min. Since we want the *most* volume, we're going to find the maximum. We also have an interval, since $0 < l < 35$ (from the maximum linear measurement). Then,

$$V' = -20w + 350$$

Solving for w gives $w = 17.5$. Plugging this in to the linear measurement, we have $l + 17.5 = 35$, so $l = 17.5$. This means that our maximum dimensions are 17.5" $\times$ 17.5" $\times$ 10", and the maximized volume is $V = (17.5)(17.5)(10) = 3062.5$ in^3.

Example 4.6.2. A rectangular-shaped garden has one side along the side of a house. The other three sides are to be enclosed with 60 feet of fencing. What is the largest possible area of such a garden?

Example 4.6.3. A mason has enough brick to build a 46 foot wall. The homeowners want to use the wall to enclose an outdoor patio. The patio will be along the side of the house and will include a 4-foot opening for a door. What dimensions will maximize the area of the patio?

Example 4.6.4. A cylindrical can is made to hold one liter of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can, accurate to the nearest hundredth.

Example 4.6.5. A certain orchard in Florida has found that when 14 orange trees are planted, their yield is 80 oranges per tree. For each tree added to the orchard, each tree's yield decreases by 2 oranges per tree. For example, if there are 15 trees planted in the orchard, the yield per tree drops to 78 oranges per tree. Find the number of trees needed to maximize the total number of oranges produced.

Example 4.6.6. During one calendar year, a year-round elementary school cafeteria uses 42,000 styrofoam plates/packets, each containing a fork, spoon, and napkin. The smallest amount the cafeteria can order from the supplier is one case containing 1000 plates and packets. Each order costs \$12, and the cost to store a case for the whole year is \$4. Use x to represent the number of cases ordered at one time in the following:

(a) Write equations for: (1) the number of times the manager will need to order during one calendar year, and (2) the annual cost to the cafeteria.

(b) Assume that the average number of cases stored throughout the year is half the number of cases in each order. Write an equation for the total storage cost for 1 year.

(c) Write a model for the combined ordering and storage costs for 1 year.

(d) What order size minimizes the total yearly cost? How many times a year should the manager order? What will the minimum total ordering and storage costs be for the year?

Example 4.6.7. A student organization is planning a bus trip to the Cotton Bowl to cheer on OU football in the playoffs. The bus they charter seats 44 and charges a flat rate of \$350 plus \$35 per person. However, for ever empty seat on the bus, the charge per person is increased by \$2. There is a minimum of 10 passengers. The organization decides that each person going on the trip will pay \$35; the organization will pay the flat rate and any additional amount about \$35 per person.

(a) Construct a model for the revenue made by the bus company as a function of the number of passengers.

(b) Construct a model for the amount the organization pays as a function of the number of passengers.

(c) For what number of passengers will the bus company's revenue be greatest? For what number of passengers will the bus company's revenue be least?

Example 4.6.8. A software developer is planning the launch of a new program. The current version of the program could be sold for \$100. Delaying the release will allow the developers to package add-ons with the program that will increase the selling price by \$2 for each day of delay. However, for each day of delay, the company will lose customers. The company could sell 400,000 copies now, but for each day that release is delayed, they will sell 2,300 fewer copies of the software.

(a) If t is the number of days the company delays the release, write a model for P, the price charged for the product.

(b) If t is the number of days the company will delay the release, write a model for Q, the number of copies they will sell.

(c) If t is the number of days the company will delay the release, write a model for R, the revenue generated from the sale of the product.

(d) How many days should the company delay the release in order to maximize revenue? What is the maximum possible revenue?