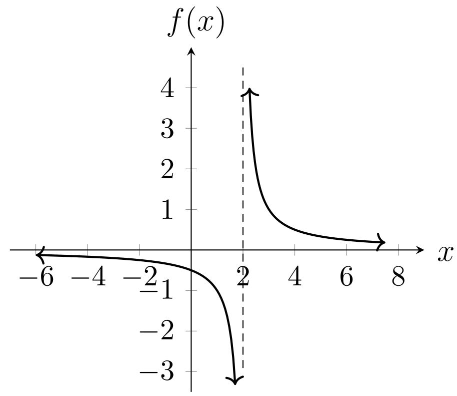
Functions with Unbounded Input

Motivating Example

Consider the function $f(x) = \dfrac{1}{x-2}$, graphed below:



(a) What happens to the output of f as the input increases without bound? Write your answer in limit notation.

(b) What happens to the output of f as the input decreases without bound? Write your answer in limit notation.

(c) What happens to the output of f at $x = 2$?

Left/Right Hand Limits

Definition 1.3.1 (Left/Right Hand Limit)

Let f be a function defined on an interval containing some constant c (except possibly at c itself).

- If $f(x)$ approaches the value of L_1 as x approaches c from the left, then the **left-hand limit** of f is L_1, and is written

- If $f(x)$ approaches the value of L_2 as x approaches c from the right, then the **right-hand limit** of f is L_2, and is written

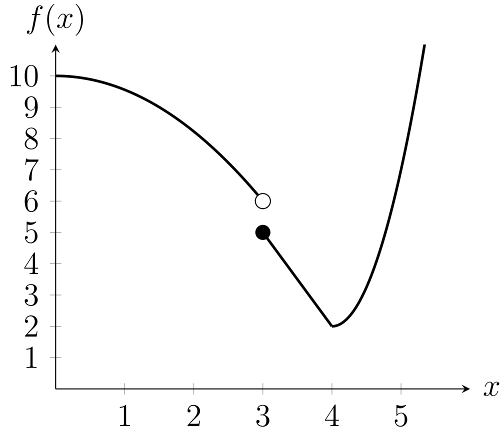
Example 1.3.2. For $f(x) = \dfrac{1}{x-2}$, find $\lim\limits_{x\to 2^-} f(x)$ and $\lim\limits_{x\to 2^+} f(x)$.

Example 1.3.3.
Use a calculator to numerically examine the limit behavior of $f(x) = \dfrac{1}{x-2}$ at $x = 2$.

x	$f(x)$	x	$f(x)$
1.9		2.1	
1.99		2.01	
1.999		2.001	
1.9999		2.0001	
1.99999		2.00001	
$\lim\limits_{x\to 2^-} f(x) =$		$\lim\limits_{x\to 2^+} f(x) =$	

$\lim\limits_{x\to 2} f(x) =$ ______________

Example 1.3.4. Use the graph of g to answer the following:



(a) $\lim_{x\to 4^-} g(x) =$

(b) $\lim_{x\to 4^+} g(x) =$

(c) $\lim_{x\to 4} g(x) =$

(d) $\lim_{x\to 3^-} g(x) =$

(e) $\lim_{x\to 3^+} g(x) =$

(f) $\lim_{x\to 3} g(x) =$

Example 1.3.5. Examine the limit behavior of the function $g(t) = \dfrac{3t^2 - 9}{t - 3}$ at $t = 3$. Round to the nearest tenth if necessary.

Example 1.3.6. Use a calculator to examine the limit behavior of the function $r(p) = \dfrac{p^2 - 64}{p + 8}$ at $p = -8$. Round to the nearest thousandth if necessary.

Example 1.3.7. Use a calculator to examine the limit behavior of the function $P(y) = \dfrac{3^y}{2y - 5}$ at $y = 2.5$. Round your answer to 2 decimal places.

Continuity

Definition 1.3.8 (Continuity)

A function $f(x)$, defined on some input interval containing c, is said to be **continuous** at c if and only if the following conditions are satisfied:

(1)

(2)

(3)

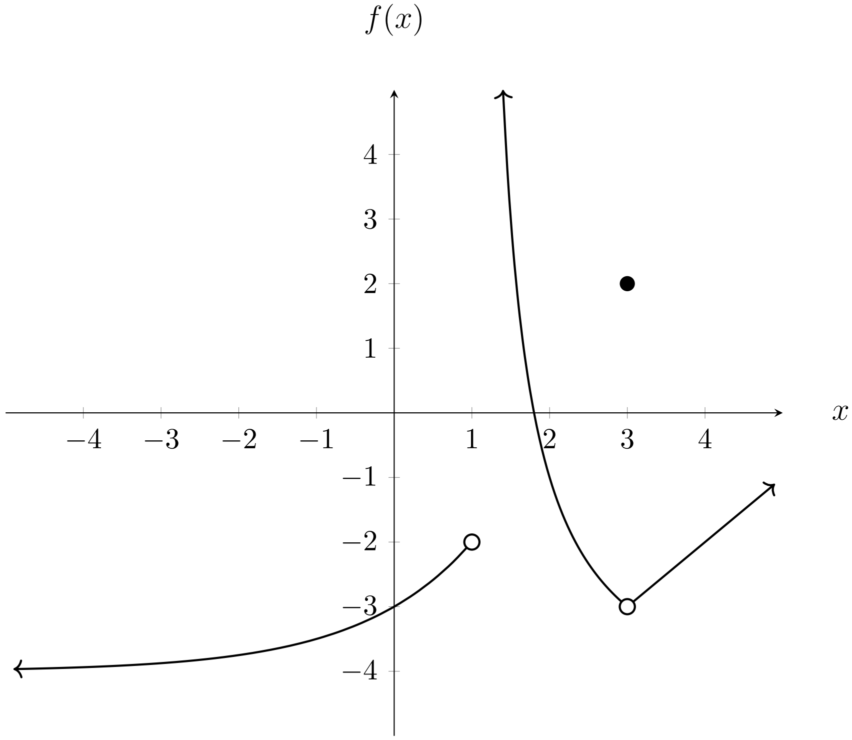
A function is continuous on any open interval (a, b) if it is continuous at every point inside the interval. If a function is not continuous at the input $x = c$, then we say that f is

__________________________ at c.

Example 1.3.9. Identify any points of discontinuity in the function $f(x) = \dfrac{1}{x-2}$. Explain why the function is discontinuous at those points.

Example 1.3.10. Identify any points of discontinuity in the function $g(x)$ in Example 3.4. Explain why the function is discontinuous at those points.

Example 1.3.11. Use the graph to find the following:



(a) $\lim\limits_{x\to 1^+} f(x)$

(b) $\lim\limits_{x\to 1^-} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) Is f continuous at $x = 1$?

(e) $\lim\limits_{x\to 3^+} f(x)$

(f) $\lim\limits_{x\to 3^-} f(x)$

(g) $\lim\limits_{x\to 3} f(x)$

(h) Is f continuous at $x = 3$?

(i) $\lim\limits_{x\to 0^+} f(x)$

(j) $\lim\limits_{x\to 0^-} f(x)$

(k) $\lim\limits_{x\to 0} f(x)$

(l) Is f continuous at $x = 0$?

Properties of Limits

Let $f(x)$ and $g(x)$ be continuous functions over some input interval containing c, and k be some arbitrary constant. Then, we have the following properties of limits:

(1) Constant Rule:

(2) Sum Rule:

(3) Constant Multiple Rule:

(4) Replacement Rule: If $f(c)$ is defined at c, then

(5) Product Rule:

(6) Quotient Rule:

(7) If $f(x)$ can be factored as $f(x) = h(x) \cdot k(x)$, and $g(x)$ can also be factored as $g(x) = j(x) \cdot k(x)$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{h(x) \cdot k(x)}{j(x) \cdot k(x)} = \lim_{x \to c} \frac{h(x)}{j(x)}$$

i.e. common factors may be canceled across fractions under the limit

Example 1.3.12. Algebraically determine the limits of the following:

(a) $\lim\limits_{x \to 5} 9$

(b) $\lim\limits_{z \to 3} (4z - 5)$

(c) $\lim\limits_{t \to -3} \dfrac{t^2 - 4t - 21}{t + 3}$

(d) $\lim\limits_{m \to 13} \dfrac{m}{m^2 + 4m}$

(e) $\lim\limits_{h \to 0} \dfrac{(3 + h)^2 - 9}{h}$

Example 1.3.13. Determine the limit: $\lim\limits_{h \to 0} \dfrac{(5 + h)^2 - 25}{h}$

Example 1.3.14. Let $f(x) = \begin{cases} x^2 & x < -1 \\ 1 & x \geq -1 \end{cases}$. **Algebraically** determine the following limits and answer the questions:

(a) $\lim\limits_{x \to -1^-} f(x)$

(b) $\lim\limits_{x \to -1^+} f(x)$

(c) $f(-1)$

(d) Is f continuous at $x = -1$? Why?

(e) Graph $f(x)$. Do your answers make sense?

Example 1.3.15. Let $h(t) = \begin{cases} 3^t - 9 & t < 2 \\ t^2 - 4 & t \geq 2 \end{cases}$. **Algebraically** determine the following limits and answer the questions:

(a) $\lim_{t \to 2^-} h(t)$

(b) $\lim_{t \to 2^+} h(t)$

(c) $h(2)$

(d) Is h continuous at $t = 2$? Why?

(e) Graph $h(t)$. Do your answers make sense?