Calculus with Parametric Curves

Before Class

Tangents

Derivative of a Parametric Curve

If $x = f(t)$ and $y = g(t)$ are the parametric equations for a curve C, then the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} =$$

Provided that ____.

Proof

Example 0.1.1. For the circle $x = \cos t$, $y = \sin t$, what is the rate of change when $\theta = \frac{\pi}{3}$?

Example 0.1.2. What is the general formula for the rate of change of an ellipse, whose parametrization is given by $x = a\cos t$, $y = b\sin t$ $(0 \le t \le 2\pi)$?

Example 0.1.3. Find an equation for the tangent line to the curve $x = t^3 + 1$, $y = t^4 + t$ at the point corresponding to the parameter value $t = -1$.

Second Derivative of a Parametric Curve

If $x = f(t)$ and $y = g(t)$ are the parametric equations for a curve C with derivative $\frac{dy}{dx}$, then the second derivative $\frac{d^2y}{dx^2}$ is given by

$$\frac{d^2y}{dx^2} =$$

Provided that ____.

Proof

Example 0.1.4. Find the value of the second derivative for the circle $x = \cos t$, $y = \sin t$ when $\theta = \frac{\pi}{3}$.

Example 0.1.5. Let C be a curve defined by the parametric equations $x = 2t^2$, $y = t^3 - t$.

(a) Show that C has two tangents at the points $(2, 0)$, and find their equations.

(b) Find the points on C where the tangent is either horizontal or vertical.

(c) Determine when the curve is concave up or concave down.

(d) Sketch the curve using the information above.

Pre-Class Activities

Example 0.1.6. For the curve defined parametrically by $x = 1 + \sqrt{t}$, $y = e^{t^2}$, find an equation of the tangent line to the curve at the point $(2, e)$. Then, eliminate the parameter to find a Cartesian expression for the curve.

Example 0.1.7. For the following functions, find the first and second derivative.

(a) $x = t^3 + 1$, $y = t^2 - t$

(b) $x = t^2 - 1$, $y = e^t - 1$

(c) $x = \cos 2t$, $y = \sin t$, $0 < t < \pi$

In Class

Example 0.1.8. When a circle rolls on a flat surface, a fixed point on the circle will trace out a curve called a *cycloid*. The parametrization for a cycloid is given by $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$, where r is the radius of the circle.

(a) Does the value of the tangent depend on the radius of the circle?

(b) Compute the slope of the tangent line when $\theta = \dfrac{\pi}{6}$.

(c) At what points is the tangent horizontal? What about when it's vertical?

Example 0.1.9. At what point(s) on the curve $x = 3t^2 + 1$, $y = t^3 - 1$ does the tangent line have slope exactly $\dfrac{1}{2}$?

Areas

Area Under a Parametric Curve

Let C be a curve traced out *exactly once* by the parametric equation $x = f(t)$ and $y = g(t)$. Then, the area under C between $x = a$ and $x = b$ is given by

$$A = \qquad\qquad\qquad\qquad \text{or}$$

where ________________________ or ________________________, depending on direction of travel.

Proof

Example 0.1.10. Use the parametrization $x = r\cos\theta$, $y = r\sin\theta$ $(0 \le t \le 2\pi)$ to show that the (unsigned) area of a circle is exactly πr^2.

Example 0.1.11. Show that the area under one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ is exactly three times the area of the generating circle.

Example 0.1.12. Find the area enclosed by the curve $x = t^2 - 2t$, $y = \sqrt{t}$ and the $y-$axis.

Example 0.1.13. Use the parametric equations $x = a\sin\theta$, $y = b\cos\theta$, $0 \le \theta \le 2\pi$, to show that the area contained in an ellipse is πab.

Arc Length

Arc Length of a Parametric Curve

If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, for $\alpha \le t \le \beta$, where g' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t ranges from α to β, then the length of C is given by

$$L =$$

Proof

Example 0.1.14. Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$

Example 0.1.15. Prove that the circumference of a circle of radius r is $2\pi r$.

Example 0.1.16. Find the exact lenth of the curve $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \leq t \leq 1$.

Example 0.1.17. Find the exact length of the curve $x = e^t \cos t$, $y = e^t \sin t$, $0 \leq t \leq \pi$

After Class Activities

Example 0.1.18. Thomas is practicing this section, and decides to parametrize a circle of radius 6 by the equations $x = 6\cos 2\pi t$, $y = 6\sin 2\pi t$. What time interval should he use in order to get the precise area or circumference of the circle? Why?

Example 0.1.19. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the curve $x = t - \ln t$, $y = t + \ln t$. For which values of t is the curve concave up?

Example 0.1.20. Find the equation of the tangent to the curve $x = \sin \pi t$, $y = t^2 + t$ at the point $(0, 2)$.

Example 0.1.21. Find the area enclosed by the $x-$axis and the curve $x = t^3 + 1$, $y = 2t - t^2$.

Example 0.1.22. Find the exact length of the curve $t \sin t$, $y = t \cos t$ on the interval $0 \leq t \leq 1$.