Exponential Functions & Derivatives

Before Class

Exponential Functions

Definition 0.1.1 (Exponential Function)

An **exponential function** is a function of the form ______________________________, where b is a positive constant.

Properties of Exponential Functions

Let $f(x) = b^x$. Then, $f(x)$ has the following properties:

- Domain: ____________________
- Range: ____________________
- If ________________________, then $f(x)$ is increasing
- If ________________________, then $f(x)$ is decreasing
- $\lim\limits_{x\to\infty} f(x) = \begin{cases} \_\_\_\_\_\_\_\_ & \text{if } \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_ & \text{if } \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \end{cases}$
- $\lim\limits_{x\to-\infty} f(x) = \begin{cases} \_\_\_\_\_\_\_\_ & \text{if } \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_ & \text{if } \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \end{cases}$

Example 0.1.2. For the following functions, find the limits and sketch the graph:

(a) $f(x) = 2(1.2^x) + 3$

(b) $g(x) = 3^{-x} - 1$

Definition 0.1.3 (Euler's Constant (e))

e is defined to be the number for which $\lim_{h \to 0} \dfrac{e^h - 1}{h} = 1$

Calculus of Exponentials

Derivative of an Exponential (First Attempt)

If $f(x) = b^x$, then $f'(x) = f'(0)b^x$

Proof

This means we have the following interpretation of $f(x) = e^x$:

Special Meaning of e

$f(x) = e^x$ is the unique exponential function whose tangent line at the point $(0, 1)$ is exactly 1, i.e. $f'(0) = 1$.

Derivative of e^x

$$\frac{d}{dx}\left[e^x\right] =$$

Antiderivative of e^x

$$\int e^x \, dx =$$

Pre-Class Activities

Example 0.1.4. Write the domain of the function:

(a) $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{4-x^2}}$

(b) $g(x) = \dfrac{1 + x}{3^{\sin x}}$

(c) $h(t) = \sqrt{4^t - 16}$

Example 0.1.5. Find the indicated limit:

(a) $\lim\limits_{x \to \infty} (1.0001)^x$

(b) $\lim\limits_{x \to \infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

(c) $\lim\limits_{x \to \infty} (e^{-2x} \sin x)$

Example 0.1.6. Find the derivative of the function:

(a) $f(x) = e^4$

(b) $g(r) = e^r + r^e$

(c) $f(x) = \dfrac{e^x}{1 + e^x}$

Example 0.1.7. Find the equation of the tangent line to the curve $y = xe^x$ at the point $(1, e)$.

Question 0.1.8 Use this space to write any questions or concerns you have from the pre-class portion of this section.

In Class

Examples

Example 0.1.9. Compute $f'(x)$, if $f(x) = e^{\tan x}$

Example 0.1.10. Compute $f'(x)$, if $f(x) = \tan(e^x)$

Example 0.1.11. Find y' if $y = e^{-6x}\cos(2x)$

Example 0.1.12. Find the absolute maximum and absolute minimum of $y = xe^{-x}$

Example 0.1.13. Find $\dfrac{dy}{dx}$, if $e^{x/y} = y - x$

Example 0.1.14. Compute the derivatives:

(a) $y = x^2 e^{-1/x}$

(b) $g(x) = e^{x^2 - x}$

(c) $f(t) = \sqrt{1 + te^{-2t}}$

Example 0.1.15. Find the absolute maximum and absolute minimum of $f(x) = xe^{-x^2/8}$ on $[-1, 4]$

Example 0.1.16. Evaluate the integral:

(a) $\displaystyle\int_0^1 (x^e + e^x)\,dx$

(b) $\displaystyle\int x^3 e^{x^4}\,dx$

(c) $\displaystyle\int e^x\sqrt{1 + e^x}\,dx$

Example 0.1.17. Compute $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

Example 0.1.18. Find $f(x)$ if $f''(x) = 3e^x + 5\sin x$, $f(0) = 1$, and $f'(0) = 2$

Example 0.1.19. The *error function*, $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}\,dt$ is a useful function in probability, statistics, and engineering. Show that $\displaystyle\int_a^b e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}\,[\operatorname{erf}(b) - \operatorname{erf}(a)]$.

After Class Activities

Example 0.1.20. Show that the function $y = e^x + e^{-x/2}$ satisfies the differential equation $2y'' - y' - y = 0$.

Example 0.1.21. Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

Example 0.1.22. Compute $\dfrac{d^{1000}}{dx^{1000}}\left[xe^{-x}\right]$

Example 0.1.23. If $f(x) = 3 + x + e^x$, find $(f^{-1})'(4)$

Example 0.1.24. Evaluate $\lim_{x \to \pi} \dfrac{e^{\sin x} - 1}{x - \pi}$

Example 0.1.25. Find the volume of the solid obtained by rotating the region bounded by the curves $y = e^x$, $y = 0$, $x = 0$, and $x = 1$ about the $x-$axis.