Chapter 1

Inverse Functions

Inverse Functions

Before Class

Inverse Functions & Properties

Example 1.1.1. The table below gives the population $P(t)$ of a bacterial culture, t hours after it is introduced to an agar-filled petri dish.

t **hours**	0	1	2	3	4	5	6	7	8
$N = P(t)$ **bacteria**	150	165	182	200	220	243	267	294	324

The *inverse function* $P^{-1}(N)$, gives the time elapsed since a bacterial culture was introduced to an agar-filled petri dish, when the population is N bacteria. Use this information to fill out the table below.

$N = P(t)$ **bacteria**	150	165	182	200	220	243	267	294	324
$t = P^{-1}(N)$ **hours**									

Definition 1.1.2 (One-to-one Function)

A function f is said to be **one-to-one** if ____________________________________,

or in notation, ____________________________________.

Theorem 1.1.3 (Horizontal Line Test)

Example 1.1.4. Is $f(x) = x^5$ one-to-one? Why or why not?

Example 1.1.5. Is $f(x) = x^2$ one-to-one? Why or why not?

Question 1.1.6 Let $f(x) = x^k$, where k is an even number. Using the previous exercise, do you think this function is one-to-one? Why or why not?

Definition 1.1.7 (Inverse Function)

Let f be a one-to-one function with domain A and range B. The **inverse function** is notated f^{-1}, with domain ______ and range ______. The inverse function is defined by the equation

Domain and Range of Inverse Functions

Notation Alert!

f^{-1} is a special notation to indicate the *function inverse*; you should not confuse this with the notation for the *multiplicative inverse/reciprocal*, such as x^{-1}. That is,

- $f^{-1}(x)$ denotes the inverse of a function
- x^{-1} denotes the multiplicative inverse of a variable, i.e. $x^{-1} = \dfrac{1}{x}$

The reciprocal of $f(x)$ is written as $[f(x)]^{-1}$. **Notice the placement of the** -1.

Example 1.1.8. Use the table below to answer the questions. If an answer does not exist, write DNE.

x	$f(x)$	$g(x)$
0	5	10
1	8	7
2	-1	3
3	13	1
4	5	9
5	3	-2

(a) $g^{-1}(3)$

(b) $f^{-1}(5)$

(c) $f(f^{-1}(13))$

(d) $(g^{-1} \circ f^{-1})(8)$

Cancellation Property

Let f be a function with domain A and range B, and let f^{-1} be its inverse function. Then, we have the following properties:

Example 1.1.9. If $f(x) = x^5$, what is $f^{-1}(x)$? Use the cancellation properties to check your answer.

Example 1.1.10. Find the inverse function of $g(y) = y^3 - 3$.

There is a graphical interpretation of algebraically finding an inverse:

Pre-Class Activities

Example 1.1.11. If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f(f^{-1}(2))$.

Example 1.1.12. Find the inverse formula for the function $f(x) = \dfrac{4x-1}{2x+3}$

Example 1.1.13. Find the inverse formula for the function $f(x) = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$

In Class

Calculus of Inverse Functions

Continuity of Inverses

If f is a one-to-one continuous function defined on the interval I, then ____________________

____.

Question 1.1.14 If a one-to-one function f is differentiable on the interval I, is it necessarily true that f^{-1} is also differentiable?

Derivative of Inverses (at a Point)

If f is a one-to-one, differentiable function at $x = a$ with inverse function f^{-1} and

__________________________, then the inverse function is differentiable at a and

Proof

If we replace a in the above formula with x, we get another formula for the derivative of the inverse:

Derivative of Inverses (as a Function)

Example 1.1.15. Let $f(x) = 3x - \sin x$. Find $(f^{-1})'(0)$.

Example 1.1.16. Let $g(x) = \sqrt{x-2}$ and $a = 2$.

(a) Show that g is one-to-one.

(b) Find $(g^{-1})'(a)$ using the formula above.

(c) Find $(g^{-1})'(x)$, and give its domain and range.

Example 1.1.17. Let $h(x) = 2x^2 - 8x$.

(a) $h(x)$ is not one-to-one. Sketch it and determine an interval on which it can be made one-to-one. This is called the *restricted domain*.

(b) Complete the square on $h(x)$ and use it to find the inverse function on your restricted domain.

(c) Find $(h^{-1})'(x)$ using your answer in (b).

(d) Find $(h^{-1})'(x)$ using formulas from this section. Compare the two answers.

Example 1.1.18. Find $(f^{-1})'(a)$ for the given functions:

(a) $f(x) = 3x^3 + 4x^2 + 6x + 5$, $a = 5$

(b) $f(x) = \sqrt{x^3 + 4x + 4}$, $a = 3$

Example 1.1.19. Suppose f^{-1} is the inverse function of a differentiable function f with $f(4) = 5$ and $f'(4) = \dfrac{2}{3}$. Find $(f^{-1})'(5)$.

After Class Activities

Example 1.1.20. Find $(f^{-1})'(2)$ for $f(x) = x^3 + 3\sin x + 2\cos x$

Example 1.1.21. Suppose f^{-1} is the inverse function of a differentiable function f, and let $G(x) = \dfrac{1}{f^{-1}(x)}$. If $f(3) = 2$ and $f'(3) = \dfrac{1}{9}$, find $G'(2)$.

Example 1.1.22. If $f(x) = \displaystyle\int_3^x \sqrt{1+t^3}\,dt$, find $(f^{-1})'(0)$.