Assignment - 1

1) Total number of arrangements = $N!$

~~for all the letters to go different env~~

A_1 = atleast one letter to go in same envelope

A_2 = No letter in same envelope

$$P(A_1) = 1 - P(A_2) \qquad P(A_2) = \frac{\text{Dearrangement } (N)}{N!}$$

$$= 1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} \cdots \frac{(-1)^n}{N!}\right) \qquad = \frac{N!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdots \frac{(-1)^n}{N!}\right)}{N!}$$

$$= \frac{1}{1!} - \frac{1}{2!} \cdots - \frac{(-1)^n}{N!}$$

For $\rightarrow N = 50$

$$P(A) = \frac{1}{1!} - \frac{1}{2!} \cdots - \frac{(-1)^n}{50!}$$

$$= 1 - \frac{1}{2} + \frac{1}{6} - \cdots$$

$$\approx \frac{4}{6} \approx \frac{2}{3} \text{ Ans}$$

2) Case A_1: Present 1 has gift.

$P(A_1) = \frac{1}{3}$ → Present 2 & 3 are empty

If you switch → winning prob = 0

$\Rightarrow P(A_1)_f = 0$

Case A_2: Present 1 does not have the gift.

$P(A_2) = \frac{2}{3}$

If you switch → winning prob = 1

$P(A_2)_f = \frac{2}{3}$

Amount = $\left(\frac{1}{3}(0) + \frac{2}{3}\right) \times 1000 = \cancel{666.7}$

$= 666.67$

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3) a) LHS = $P(A \cap B | C) = \dfrac{P(A \cap B \cap C)}{P(C)}$

RHS = $P(A | B \cap C) \cdot P(B | C)$

$= \dfrac{P(A \cap B \cap C)}{P(B \cap C)} \cdot \dfrac{P(B \cap C)}{P(C)} = \dfrac{P(A \cap B \cap C)}{P(C)}$

$\boxed{\text{True}}$

b) $P(A \cap B | C) = \dfrac{P(A \cap B \cap C)}{P(C)}$

RHS = $P(A/C) \cdot P(B | C)$

$= \dfrac{P(A \cap C)}{P(C)} \cdot \dfrac{P(B \cap C)}{P(C)} = \dfrac{P(A \cap B \cap C)}{(P(C))^2}$

LHS $\neq$ RHS $\boxed{\text{False}}$

c) $P(A | D \cap B^c) = \dfrac{P(A \cap D \cap B^c)}{P(D \cap B^c)}$ $\quad$ $P(A | D^c \cap B^c) = \dfrac{P(A \cap D^c \cap B^c)}{P(D^c \cap B^c)}$

$P(A | D \cap B) = \dfrac{P(A \cap D \cap B)}{P(D \cap B)}$ $\quad$ $P(A | D^c \cap B) = \dfrac{P(A \cap D^c \cap B)}{P(D^c \cap B)}$

$P(A | B) = \dfrac{P(A \cap B)}{P(B)}$ $\quad$ $P(A | B^c) = \dfrac{P(A \cap B^c)}{P(B^c)}$

$\dfrac{P(A \cap D \cap B^c)}{P(D \cap B^c)} > \dfrac{P(A \cap D \cap B)}{P(D \cap B)}$ $\quad$ $\dfrac{P(A \cap D^c \cap B^c)}{P(D^c \cap B^c)} > \dfrac{P(A \cap D^c \cap B)}{P(D^c \cap B)}$

$\dfrac{P(A \cap D \cap B^c)}{P(A \cap D \cap B)} > \dfrac{P(D \cap B^c)}{P(D \cap B)}$ $\quad$ $\dfrac{P(A \cap D^c \cap B^c)}{P(A \cap D^c \cap B)} > \dfrac{P(D^c \cap B^c)}{P(D^c \cap B)}$

in D $\quad$ outside D

$\boxed{\text{False}}$

$\dfrac{P(A \cap B^c)}{P(A \cap B)} > \dfrac{P(B^c)}{P(B)} \Rightarrow \dfrac{P(A \cap B^c)}{P(B^c)} > \dfrac{P(A \cap B)}{P(B)}$

Ans $\Rightarrow \boxed{\text{False}}$

4a) we want distributions, where values towards infinity are small for $E[X]$ but not for $E[X^2]$

trying $P(X=n) = 1/n \rightarrow$ diverges

trying $P(X=n) = 1/n\log n \rightarrow$ diverges

$P(X=n) = 1/n^2\log n \rightarrow$ converges

$$\therefore \quad E[X] = \sum_{n=2}^{\infty} n \times \frac{1}{n^2 \log n}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n\log n}$$

convergence for $\int_2^{\infty} \frac{1}{n\log n} dn = \int_{e^2}^{\infty} \frac{du}{u}$

$$= \log - \log(\infty)$$

$\times$

trying $P(X=n) = \frac{1}{n^2\log^2 n} \rightarrow$ converges

$$E[X] = \sum_{n=2}^{\infty} \frac{1}{n\log^2 n}$$

$$\int_2^{\infty} \frac{1}{n\log^2 n} dn = \int_{e^2}^{\infty} \frac{1}{u^2} du$$

$$= -\frac{2}{u}\Big]_{e^2}^{\infty}$$

$$= -\left[\frac{2}{e^2} - 0\right]$$

$$= -\frac{2}{e^2}$$

but $E[X^2]$ is not finite

$$\therefore \boxed{\frac{1}{n^2\log^2 n}}$$

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b) $f_x(x) = \frac{1}{x^2 \log^2 x}$

c) $E[X] = 1$

$E[e^{-x}] < 1/3$

as e^{-x} is convex, use Jensen's inequality

$$E[e^{-x}] \geq e^{-E[X]}$$

$$E[e^{-x}] \geq e^{-1} \geq 0.36$$

$\therefore$ this kind of random variable distribution doesn't exist

A9. $(F * G)(a) = P(X+Y \leq a)$

let $a < b$

$X+Y \leq a \subseteq X+Y \leq b$

$\therefore$ $P(X+Y \leq a) \leq P(X+Y \leq b)$

$\therefore$ $F*G$ is non-decreasing — ①

following this

$\lim_{a \to -\infty} (F*G)(a) = 0$, $\lim_{a \to b} (F*G)(a) = 1$ $(\because ($

9) $P \bigcap_{i=1}^{n} (A_i)^c = \prod_{i=1}^{n} (1 - P(A_i))$

We know
$$1 - x \le e^{-x} \qquad x \in [0,1]$$

$\therefore \quad 1 - P(A_i) \le e^{-P(A_i)}$

$$\Rightarrow \quad P\left(\bigcap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} (1 - P(A_i)) \le \prod_{i=1}^{n} e^{-P(A_i)}$$

$$\boxed{P\left(\bigcap_{i=1}^{n} A_i^c\right) \le e^{-\sum_{i=1}^{n} P(A_i)}}$$ Proved

10) $I = \int_{\Omega} \int_{0}^{\infty} \mathbb{I}_{[0, X(\omega)]}(x)\, dx\, dP(\omega) = \cancel{E[}$

$$\int_{0}^{\infty} \mathbb{I}_{[0, X(\omega)]}(x)\, dx\, \cancel{dP(\omega)} = \int_{0}^{X(\omega)} 1\, dx = X(\omega)$$

$$I = \int_{\Omega} X(\omega)\, dP(\omega) = \int_{\Omega} \cancel{X(\omega)\, dP(\omega)}\ E[X] \quad \text{— A}$$

By reversing the order of Integral

$I = \int_{0}^{\infty} \int_{\Omega} \mathbb{I}_{[0, X(\omega)]}(x)\, dP(\omega)\, dx$ $\qquad F(x) = P(X \le x)$

$$\int_{\Omega} \mathbb{I}_{[0, X(\omega)]}\, x\, dP(\omega) = P(\omega: X > x) = 1 - F(x)$$

$$I = \int_0^\infty (1 - F(x))\,dx \qquad \text{— } B$$

from A & B

$$\boxed{\int_0^\infty (1 - F(x))\,dx = E[X]}$$

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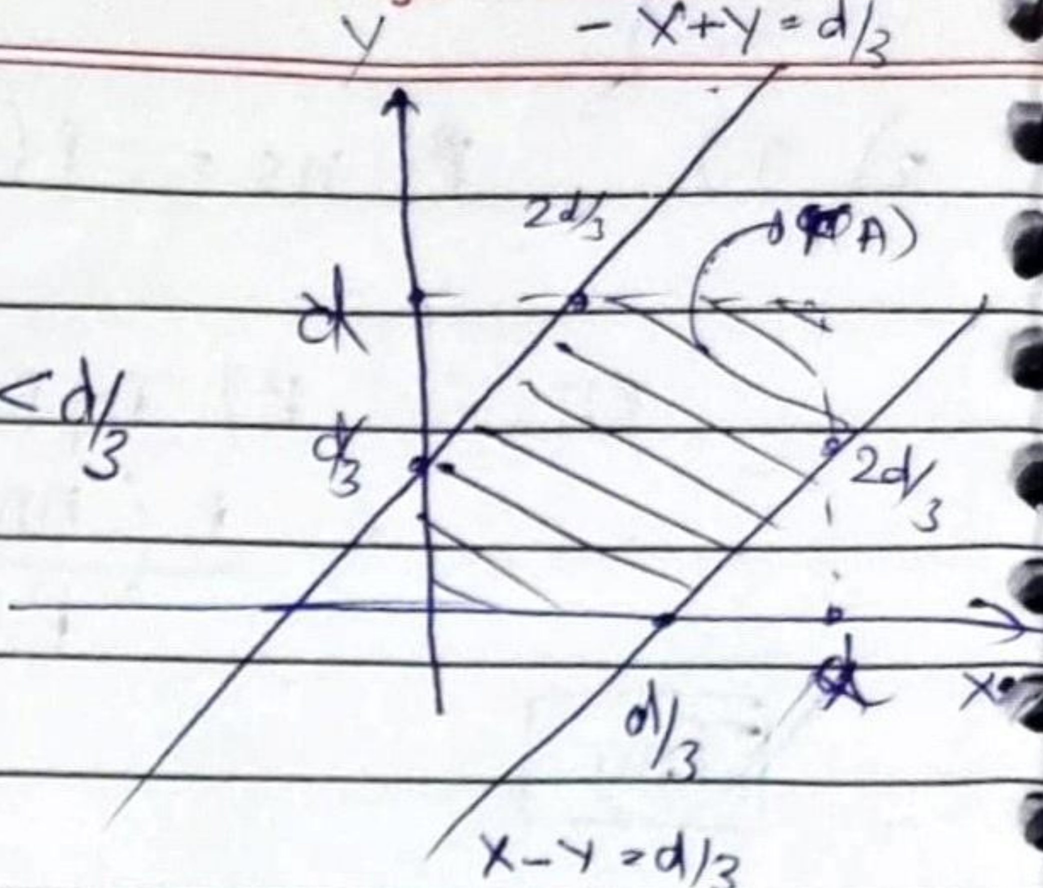
~~(1,1)~~

6) $|X - Y| \leq d/3$



$X - Y < d/3$, $Y - X < d/3$

$0 < X < d$

$0 < Y < d$

$$P(A) = \frac{d^2 - \left(\frac{1}{2} \times \left(\frac{2d}{3}\right)^2\right)}{d^2} = 1 - \frac{4}{9} = \frac{5}{9} \text{ Ans}$$

11 (a) $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$E[e^{ux}] = \int_{-\infty}^{\infty} \frac{e^{ux}}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ux - \frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$\cancel{= \frac{1}{\sigma\sqrt{2\pi}} \left[\frac{e^{ux - \frac{(x-\mu)^2}{2\sigma^2}}}{u - \frac{x-\mu}{\sigma^2}} \right]_{-\infty}^{\infty}}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{2\sigma^2 ux - x^2 - \mu^2 + 2x\mu}{2\sigma^2}} du$$

$$= e^{u\mu + \frac{1}{2}u^2\sigma^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu-\mu\sigma^2)^2}{2\sigma^2}}}_{1}$$

(ii) $E[\phi(x)] = e^{u\mu + \frac{1}{2}u^2\sigma^2}$

$\phi(E[x]) = e^{\mu u}$

$\therefore E[\phi(x)] > \phi(E[x])$