Astro 204 Problem Set 4

Charles Gannon

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1 Collapse and outflow

The force on a patch of mass (dm) in the cloud from gravity is

$$F_G = \frac{GMdm}{r^2}, \tag{1}$$

where r is the radial distance of the patch from the center of cloud, and M is the mass enclosed in radius r. If the cloud was not pressure supported the radial acceleration would be

$$\ddot{r} = -\frac{GM}{r^2}. \tag{2}$$

For a quick estimate of the time for the cloud to collapse, we can approximate the acceleration $\ddot{r}$ as constant. Under this assumption, we can derive a relation for the t_{dyn} in terms of the initial radius of the patch r_0

$$r_0 = \frac{1}{2}\frac{GM}{r_0^2}t_{dyn}^2, \tag{3}$$

which solving for t_{dyn} gives

$$t_{dyn} = \sqrt{\frac{2r_0^3}{GM}}. \tag{4}$$

If no shells cross during spherical collapse, the mass enclosed within a collapsing spherical shell (M) will remain constant. Therefore, we can do better and solve 2 exactly without assuming a density profile. Assuming eq. 2 has a solution of the form $r = a(t - t_0)^n$, then the equation becomes

$$an(n-1)(t-t_0)^{n-2} = \frac{GM}{a^2(t-t_0)^{2n}}, \tag{5}$$

which to be valid forces $n = 2/3$. Solving for a yields

$$a = -\left(\frac{9}{2}GM\right)^{1/3}. \tag{6}$$

By plugging into our initial guess for the solution, we can again we can derive a relation for the dynamical time in terms of the initial radius

$$t_{dyn} = \frac{1}{3}\sqrt{\frac{2r_0^3}{GM}}. \tag{7}$$

This solution doesn't have v(t=0)=0 ! Note that the exact solution and estimate only differ by a factor of 1/3 The time for sound to reach the center of the cloud if it starts at $r = r_0$ (sound crossing time, t_c) and the speed of sound in the gas c_s is constant is

$$t_c = \frac{r_0}{c_s} \tag{8}$$

A gas under hydrostatic equilibrium obeys the equation

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{GM}{r^2} \tag{9}$$

where P is the pressure of the gas and ρ is it's density. The right hand of eq. 9 can be re-written in terms of t_{dyn}

$$\frac{GM}{r^2} \sim -\frac{r}{t_{dyn}^2}, \tag{10}$$

where I have dropped all constant coefficients. Likewise, the left hand of the equation can be written in terms of the crossing time. Using the ideal gas law

$$P = c_s^2\rho, \tag{11}$$

we can rewrite the pressure of the gas in terms of the density. In order to proceed, I assume the density can be written as a power law in terms of the radius

$$\rho = br^{-\gamma}. \tag{12}$$

Under these assumptions, the left hand of 9 becomes

$$\frac{1}{\rho}\frac{dP}{dr} \sim -\frac{c_s^2}{r} = -\frac{r}{t_c^2}, \tag{13}$$

where I again have dropped all constant coefficients. Putting everything back together gives

$$t_c \sim t_{dyn}. \tag{14}$$

For the cloud to collpase, the absolute value of the force of gravity acting on a patch must be greater than the absolute value of the force from the pressure

$$\frac{dP}{dr} < \frac{GM\rho}{r^2}. \tag{15}$$

Using our previously previous results this implies

$$\frac{1}{t_c^2} < \frac{1}{t_{dyn}^2} \tag{16}$$

which can be simplified to

$$t_{dyn} < t_c \tag{17}$$

meaning if the dynamical time is less than the sound crossing time the cloud will collapse. Likewise, for the cloud to fly appart

$$\frac{1}{t_c^2} > \frac{1}{t_{dyn}^2} \tag{18}$$

which rewriting in terms of the escape velocity

$$v_{esc} = \sqrt{\frac{2GM}{r}} \sim \frac{r}{t_{dyn}} \tag{19}$$

and speed of sound in the cloud gives the condition

$$c_s > v_{esc}. \tag{20}$$

Therefore, for the cloud to fly apart the speed of sound in the medium must be larger than the escape velocity.

2 Parker Winds

a) Bernoulli's constant along a stream line is

$$B = \frac{1}{2}v^2 + h + \phi \tag{21}$$

. For a parker wind the potential is gravitational

$$\phi = -\frac{GM}{r} \tag{22}$$

The enthalpy of a system is

$$dH = dU + pdV, \tag{23}$$

which can be rewritten in term of enthalpy per mass

$$dh = \frac{1}{\rho}dP + Tds, \tag{24}$$

or for constant entropy

$$h = \int_S \frac{dP}{\rho} \tag{25}$$

. For our equation of state we have

$$P = c_s^2\rho, \tag{26}$$

which when gives mean our system has entahlpy given by

$$h = c_s^2 ln\rho + C. \tag{27}$$

Plugging everything in,

$$\frac{1}{2}v^2 + ln\rho - \frac{GM}{r} = \text{const} \tag{28}$$

to further simplify we can use the mass loss rate of our system

$$\dot{M} = 4\pi r^2 \rho v \tag{29}$$

where our mass loss rate, $\dot{M}$ is constant to maintain a steady state solution. Solving for ρ and plugging into our expression for the Bernoulli constant gives

$$\frac{1}{2}v^2 - c_s^2(\ln r - 2\ln v) - \frac{GM}{r} = \text{const} \tag{30}$$

Rearanging and substituting $r_s = \frac{GM}{2c_s^2}$ gives

$$\ln v - \frac{1}{2}\frac{v^2}{c_s^2} = -2\ln r - 2\frac{rs}{r} + \text{const}, \tag{31}$$

which taking the exponent gives

$$ve^{-v^2/(2c_s^2)} = Cr^{-2}e^{-2r_s/r}. \tag{32}$$

By imposing $v(r_s) = c$ we get

$$c_s e^{-1/2} = Cr_s^{-2}e^{-2}, \tag{33}$$

which when solved for C gives

$$C = c_s r_s^2 e^{3/2}. \tag{34}$$

Putting everything together we have

$$ve^{-v^2/(2c_s^2)} = c_s\left(\frac{r_s}{r}\right)^2 e^{(3/2)-2r_s/r}. \tag{35}$$

b) The mass equation is

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \tag{36}$$

Since a parker wind is in steady state, $\frac{\partial\rho}{\partial t} = 0$. Furthermore, the velocity of a park wind only radial compont so $\vec{\nabla}\cdot(\rho\vec{v}) = \frac{1}{r^2}(r^2\rho v)$. Therefore the mass equation becomes

$$0 = \frac{1}{r^2}(r^2\rho v) = r^2 v_r\frac{\partial\rho}{\partial r} + 2r\rho + r^2\rho\frac{\partial v}{\partial r} \tag{37}$$

The momentum equation is

$$\rho\frac{d\vec{v}}{dt} = -\vec{\nabla}P + \rho\frac{GM}{r^2}. \tag{38}$$

We know $P = c_s^2\rho$, aditionlly from the chain rule $\frac{d\vec{v}}{dt} = \vec{v}\cdot\frac{\partial v}{\partial r}$. Therefore the momentum equation becomes

$$\rho v\frac{\partial v}{\partial r} = -c_s^2\frac{\partial\rho}{\partial r} - \rho\frac{GM}{r^2}. \tag{39}$$

Plugging in the mass equation we get

$$\rho v\frac{\partial v}{\partial r} = -c_s^2\rho(-\frac{2}{r} - \frac{1}{v}\frac{\partial v}{\partial r}) - \rho\frac{GM}{r^2}. \tag{40}$$

After some simplification and noting that the partial derivatives are equivalent to the total derivative the equation becomes

$$\frac{1}{v}\frac{dv}{dr}(v^2 - c_s^2) = \frac{2c_s^2}{r} - \frac{GM}{r^2} \tag{41}$$

c) See the fig 1 for plots, and problem2c.py for code.

d) The mass change rate for parker wind is

$$\dot{M} = 4\pi r^2 \rho v, \tag{42}$$

so

$$\rho = \frac{\dot{M}}{4\pi r^2 v} = \rho_s \left(\frac{r_s}{r}\right)^2 \frac{c_0}{v}. \tag{43}$$

We may rewrite and get an expression for velocity

$$v = c_s \left(\frac{r_s}{r}\right)^2 \frac{\rho_s}{\rho} \tag{44}$$

which we can plug into our trancendental velocity equation. After some simplifying I arrive at

$$\frac{\rho}{\rho_0} e^{\frac{1}{2}\left(\frac{r_0^2 \rho_0 v_0}{r^2 \rho c_S}\right)^2} = e^{\frac{-2r_s}{r_0}(1 - r_0/r) + \frac{1}{2}(v_0/c_s)^2} \tag{45}$$

which is the same as the hydrostatic solution if $v_0 << c_S$ and $r_0^2 \rho_0 v_0 < r^2 \rho c_s$.

e) The outflow timescale is $t_s \sim r_s/c_s$, a quick way of seeing this is that our equation have one length scale: r_s and one velocity scale v_s. Using dimensional analysis, we can combine these scales to get a time scale by dividing them. Plugging everything in with $M_e = 10^{24}$ kg, and the speed of sound for hydrogen at 10000 K, $c_S \approx 2.9 \cdot 10^3$ m/s I get 2.16 hours (appologies for the SI units). The actual value was 4 hours so this is very close!

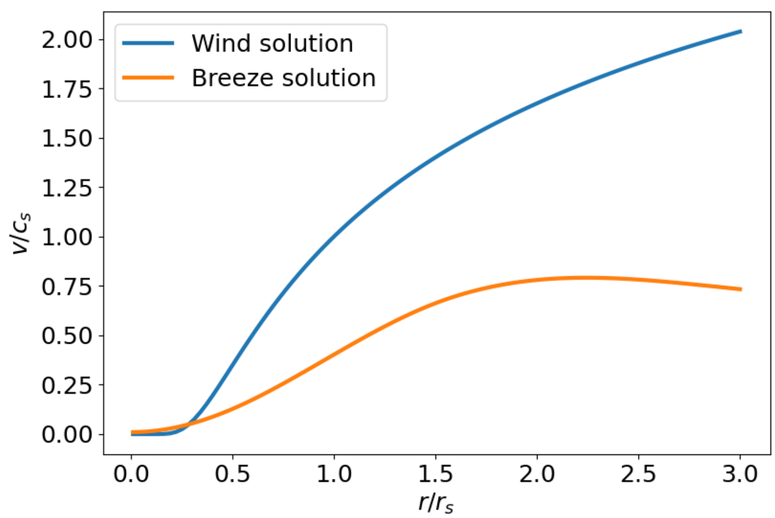

Figure 1: The numerical solutions for part (c).