Astro 204 Problemset 9

Charles Gannon

December 9, 2024

Contents

1 The Dynamical Time

(a) Using the form of Kepler's 3'rd law

$$GM = \Omega^2 a^3, \tag{1}$$

the period is equal to

$$P = \frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{a^3}{GM}}. \tag{2}$$

The free fall time is just half the orbital time

$$t_{ff} = \frac{P}{2} = \pi\sqrt{\frac{a^3}{GM}}. \tag{3}$$

(b) For a circular orbit

$$\frac{GM}{r^2} = \frac{v_{\text{circ}}^2}{r} \tag{4}$$

therefore,

$$v_{\text{circ}} = \sqrt{\frac{GM}{r}} \tag{5}$$

Using Kepler's third law,

$$v_{\text{circ}} = \Omega r \tag{6}$$

or equivalently

$$\Omega_{\rm circ} = \frac{v_{\rm circ}}{r}, \tag{7}$$

which gives an equivalent orbital period of

$$P_{\rm circ} = 2\pi\sqrt{\frac{r^3}{GM}}. \tag{8}$$

(c) See python file / plots directory. The orbital period is 1 year.

(d) See python file / plots directory. The orbital period is about 1/2 a year (looking at the plots).

(e) The Escape velocity can be quickly found by setting the kinetic energy equal to the gravitational potential energy

$$\frac{1}{2}mv_{\rm esc}^2 = \frac{GMm}{r}, \tag{9}$$

which gives an escape velocity

$$v_{\rm esc} = \sqrt{\frac{2GM}{r}}. \tag{10}$$

At constant velocity, the time it takes to travel distance r is

$$t_{\rm esc} = \frac{r}{v_{\rm esc}} = \sqrt{\frac{r^3}{2GM}}. \tag{11}$$

(f) See python file / plots directory. The orbit gets larger with increasing velocity, up to about 4 times longer at $v_0 = 1.4v_{circ}$.

(g) This is the time scale gravitation works on. The free-fall time, collapse time of a cloud, orbital time are all within a factor of unity.

(h) Substituting $M = \frac{4}{3}\pi r^3$ into the free fall time I get

$$t_{\rm collapse} = \pi\sqrt{\frac{3}{4G\rho}} \sim (G\rho)^{-1/2}. \tag{12}$$

2 Tidal Gravity

The Swarzschild radius of black hole is

$$r_s = \frac{2GM_{bh}}{c^2}. \tag{13}$$

Neglecting relativistic effects, and dropping order the Roche limit of the black hole is

$$r_t \sim R_* \left(\frac{M_{bh}}{M_*} \right)^{1/3}, \tag{14}$$

where M_{bh} is the mass of the black hole, M_* is the mass of the star and R_* is the radius of the star.L If the Roche limit is greater than the Swarzschild radius the star will be visibly tidally disrupted. On the other hand if $r_t < r_s$ the star will be swallowed whole. Dropping order of unity coefficients, for the star to be swallowed whole,

$$M_{bh} > \left(\frac{R_* c^2}{G M^{1/3}} \right)^{3/2}. \tag{15}$$

For a stellar mass star ($M_* = M_\odot$ and $R_* = 1R_\odot$) I get that the black hole must have mass $M_{bh} > 3 \cdot 10^8$ to swallow the star whole.