Problem Set 2

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1. Hydrostatic atmospheres

   (a) To calculate the scale height of an atmosphere with pressure

   $$\frac{dP}{dz} = P_0 e^{-z/H}, \tag{1}$$

   we first calculate

   $$\frac{dP}{dz} = -\frac{P_0}{H} \cdot e^{-z/H} \tag{2}$$

   then take the ratio

   $$P(z)/\left|dP(z)/dz\right| = H \tag{3}$$

   (b) The hydrostatic balance equation is

   $$\frac{1}{\rho}\frac{dP}{dr} = -G\frac{M}{r^2}. \tag{4}$$

   The density, ρ can be expressed in terms of pressure

   $$P = \rho c_s^2, \tag{5}$$

   assuming an ideal gas. Furthermore, for the isothermal case, c_s is constant with respect to radius. Recasting eq. 4 in a more easily integrable form gives

   $$\frac{1}{\rho}\frac{dP}{dr} = -\frac{GM}{c_s^2}\frac{1}{r^2} = \frac{1}{H_0}\left(\frac{R_0}{r}\right)^2 \tag{6}$$

   which can be integrated to give

   $$P = Ae^{\frac{R_0^2}{H_0}\frac{1}{r}}. \tag{7}$$

   Imposing the condition $P(R) = P_0$ gives

   $$P = P_0 \exp\left[\left(\frac{R}{H_0}\right)\left(\frac{R}{r} - 1\right)\right] \tag{8}$$

(c) Rewriting the exponent in eq. 8 in terms of z = r - R gives

$$-\frac{R}{H_0}\left[z/R(1+z/R)^{-1}\right], \tag{9}$$

which we can Taylor expand for $z << R$

$$z/R(1+z/R)^{-1} \sim z/R(1-z/R) \sim z/R, \tag{10}$$

which can be substituted into eq. 8 to give

$$P = P_0 \exp\left[-z/H_0\right] \approx P_0 exp\left[-z/R\right], \tag{11}$$

which is identical to the plane parallel case.

(d)

$$\lim_{t\to\infty} P = P_0 \exp\left[-R/H_0\right] \neq 0 \tag{12}$$

2. Optical depth and scale lengths (see jupyter notebook)

3. The Lyman-α forest. For a cloud to a part of the forest, the optical depth, τ must be $\tau \approx 1$. The optical depth of a Lyman α cloud is

$$\tau = N_{HI}\sigma_0. \tag{13}$$

Therefore, for the cloud to be in the forest

$$N_{HI} = 1/\sigma_0, \tag{14}$$

From the notes, the cross-section of Lyman-α absorption is

$$\sigma_0 = \frac{1}{8\pi}\frac{g_1}{g_2}\frac{A_{21}}{\Delta v}\Lambda^2 \tag{15}$$

for an order of magnitude estimate assume states have equal weighting so $g_1 \approx g_2$, therefore

$$\sigma_0 \approx \frac{A_{21}}{\Delta v}\Lambda^2 \tag{16}$$

The speed of sound is $c_s \approx \sqrt{\frac{kT}{\nu}} \approx 10^6 cm/s$ Plugging in $\Lambda \approx 1.2 \cdot 10^{-5} cm$, $\delta_v \approx c_s/\Lambda_0 \approx 10^{11} 1/s$ the crosssection is $\sigma_0 \approx 10^{-12} cm^2$. This gives a required column density of $N_{HI} \approx 10^{12}$ cm^{-2}.

4. Rate coefficients for chemical reactions. To estimate the boor radius, take the angular momentum as the quantized quantity

$$\hbar = m_e a_0 v_c \tag{17}$$

where the orbital velocity, v_c can be estimated from uniform circular motion,

$$k\frac{e^2}{a_0^2} = m_e v_c^2/a_0. \tag{18}$$

Solving for a_0 gives

$$a = \frac{\hbar^2}{ke^2 m_e} \approx\sim 1 \cdot 10^{-8}[cm] \tag{19}$$

The number of collisions occurring per time is

$$n = \sigma v n_1 n_2, \tag{20}$$

therefore

$$k = v\sigma, \tag{21}$$

where v is the thermal velocity

$$1/2mv^2 = 3/2k_bT, \tag{22}$$

and σ is the cross-section of hydrogen

$$\sigma \sim \pi a_0^2, \tag{23}$$

which can approximated using our estimate for the Bohr radius, which gives an approximate cross-section of $\sigma \approx 3 \cdot 10^{-16}$ cm^2. The thermal velocity for the 10^4 K gas is $v \approx 1 \cdot 10^-6$ cm/s which gives a final estimate for k as $k \approx 10^{-10} cm^3/s$ which is close to the q_{21} value provided.