Astro 204 Problemset 7

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1 Effective Viscosity

(a) The viscosity for a fluid is

$$\nu = v_{cs}\lambda_{\mathrm{mfp}}. \tag{1}$$

We can extend this to our effective fluid, assigning an effective speed of sound v_{eddy} and mean free path L_{eddy}. The effective speed of sound cannot exceed the actual speed of sound in the fluid, therefore

$$v_{eddy} < c_s. \tag{2}$$

Additionally, the eddy must be physically contained within the disk, so the length scale associated with each eddy, L_{eddy} must be less than the scale height of the disk

$$L_{\mathrm{eddy}} < H. \tag{3}$$

Combining these two observations, we can write the effective viscosity as

$$\nu = \alpha c_s H, \tag{4}$$

with $\alpha < 1$ parameterizing our ignorance of the underlying physics.
(b) The energy per wavenumber is $kE_k(k) \sim k^{-2/3}$. This energy should be

proportional to the kinetic energy so $k^{-2/3} \sim \frac{1}{2}mv^2$, so $v \sim k^{-1/3}$. The effective viscosity is proportional to the velocity and the length of the scale so

$$\nu = v_{\text{eddy}} L_{\text{eddy}} \sim k^{-1/3}/k = k^{-4/3} \sim l^{4/3} \tag{5}$$

so larger length scales contribute more to the velocity.
(c) Plugging in gives a viscosity

$$\nu = \alpha c_s H = (10^{-2})(2 \cdot 10^6 cm/s)(8 \cdot 10^{11} m) \sim 10^{15} cm^2/s. \tag{6}$$

At $r = 1AU$ This corresponds to a timescale of

$$t = r^2/\nu = \left(1.5 \cdot 10^{13} cm\right)^2 / \left(10^{15} cm^2/s\right) \sim 10^{11} s \sim 10^3 y \tag{7}$$

which is much more reasonable.

2 Temperature of an accretion disk

(a) A generic form of the energy equation for fluids is

$$\rho T \frac{ds}{dt} = \psi - \lambda, \tag{8}$$

integrating over z gives

$$\Sigma T \frac{ds}{ds} = 2H\psi - 2H\lambda = Q_+ - Q_-. \tag{9}$$

We can convert the total derivative to a partial derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}. \tag{10}$$

Noting that in steady state $\frac{\partial}{\partial t} = 0$ and that under the assumption of disk symmetry we have $\vec{v} \cdot \vec{\nabla} = v_r \frac{d}{dr}$ we have

$$\Sigma v_r T \frac{ds}{dr} = Q_+ - Q_-. \tag{11}$$

(b) Since we can expect orbits to be circular because of damping, the total gravitational energy of the system is proportional to the mass squared

$$E \sim -\frac{GM^2}{r}. \tag{12}$$

Taking the derivative gives

$$\dot{E} \sim -\frac{GM\dot{M}}{r}. \tag{13}$$

The area of the orbit is proportional to r^2, therefore

$$Q_+ \propto r^2, \tag{14}$$

therefore

$$Q_+ \sim \frac{GM\dot{M}}{r^3}. \tag{15}$$

Plugging in the expression from Kepler's 3rd law $GM = \Omega^2 r^3$ I get

$$Q_+ \sim \Omega^2 M \tag{16}$$

(d) From thermodynamics $Tds = d\epsilon - \frac{P}{\rho^2} d\rho$, where $\epsilon = \frac{1}{\gamma-1} c_s^2$. Converting to 2d, and plugging in a polytropic equation of state gives

$$Tds = \frac{1}{\gamma - 1} c_s^2 - c_s^2 \frac{d\Sigma}{\Sigma} \tag{17}$$

Therefore we have

$$\Sigma v_r T \frac{ds}{dr} = \frac{1}{\gamma - 1} \Sigma \frac{dc_s^2}{dr} - c_s^2 v_r \frac{d\Sigma}{dr}. \tag{18}$$

Both terms are order of magnitude $\frac{\Sigma v_r c_s^2}{r}$, and multiplying by r on the top and bottom we get and noting $r\Sigma v_r \sim \dot{M}$

$$\frac{r\Sigma v_r c_s^2}{r^2} \sim \frac{\dot{M} c_s^2}{r^2}. \tag{19}$$

From part b, we have $Q_+ \sim \frac{\dot{M} v_k^2}{r^2}$. For a thin disk, $c_s << v_k$, therefore

$$\Sigma v_r T \frac{ds}{dr} << Q_+. \tag{20}$$

(e) The total amount of radiative cooling is equal to the total lumnosity output. Therefore, the total luminosity is

$$L = \int_{R_*}^{\infty} 2\pi r Q_- dr \sim \frac{GM\dot{M}}{2R_*}. \tag{21}$$

(e) We can relate the total temperature of the disk to the radiative cooling

$$Q_- = 2\sigma_{sb}T^4. \tag{22}$$

Since we have $Q_- \sim Q_+$, the temperature as a function of radius is

$$T^4 = \frac{3GM\dot{M}}{8\pi\sigma_{sb}r^3} = \frac{3\Omega^2\dot{M}}{8\pi\sigma_{sb}}. \tag{23}$$

Plugging in $\dot{M} = 10^{-8}M_\odot/yr$ at 1 AU (the earth's orbit, so $\Omega = 1yr^{-1}$), I get $T = 52.6K$ so heating wins. (f) If the photons went straight, it would take them $t = H/c$ to diffuse out of the disk. However, they diffuse so the time to diffuse is $(H/\Lambda_{\rm mfp})^2/c$ to diffuse out of the disk. The optical depth of the disk is

$$\tau = n\sigma H = H/\lambda_{\rm mfp}, \tag{24}$$

therefore, the Luminosity is a factor of τ larger than the Luminosity at the surface. Since $L \sim \sigma_{\rm sb}T^4$, the relationship between the temperature at the surface and the midplane is

$$T_m \sim T_s\tau^{1/4} \tag{25}$$

3 Time evolution of an accretion disk

(a) The mass equations is

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0, \tag{26}$$

integrating over z and imposing symmetry gives

$$\frac{\partial\Sigma}{\partial t} + \frac{1}{r}(r\Sigma v_r) = 0 \tag{27}$$

The momentum equation is

$$\rho\frac{d\vec{v}}{dt} = -\vec{\nabla}P + \rho\vec{f}, \tag{28}$$

we can rewrite the left hand side and drop terms, and dropping terms with time dependence

$$\rho\frac{d\vec{v}}{dt} = \rho(\vec{v}\cdot\vec{\nabla})\vec{v} \tag{29}$$

Plugging in the viscous force per mass, Keppler's law, dropping all other terms on the right and integrating over mass gives

$$\Sigma v_r \frac{\partial}{\partial r}(r^2\Omega) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\Sigma \nu r^3 \frac{\partial \Omega}{\partial r}\right). \tag{30}$$

Keppler's 3rd law gives into the momentum equation and solving for v_r gives

$$v_r = -3\Sigma^{-1} r^{-1/2} \frac{\partial}{\partial r}(\Sigma \nu r^{1/2}) \tag{31}$$

which we can plug into the mass equation to get

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(\Sigma \nu r^{1/2}\right)\right] \tag{32}$$

(b) Plugging in $\gamma = 1$, $R = \frac{r}{r_0}$, i$T = \frac{t}{t_{visc}} + 1$ and $C = \Sigma_0 e$ we get

$$\Sigma = CR^{-1}T^{-3/2}e^{-R/T}. \tag{33}$$

Plugging in our substitution variables T and R into the main equation we get

$$\frac{\partial \Sigma}{\partial T} = \frac{1}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left(\Sigma \nu R^{1/2}\right)\right]. \tag{34}$$

Now, we can plug everything in. The left hand side of the equation becomes

$$CR^{-1}\left(-\frac{3}{2}T^{-5/2}e^{-R/T} + T^{-3/2}e^{-R/T}RT^{-2}\right) = \tag{35}$$

$$CR^{-1}T^{-3/2}e^{-R/T}\left(-\frac{3}{2}T^{-1} + RT^{-2}\right) = \tag{36}$$

$$\Sigma\left(-\frac{3}{2}T^{-1} + RT^{-2}\right) \tag{37}$$

The right hand side becomes

$$\frac{1}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left(CT^{-3/2}R^{1/2}e^{-R/T}\nu R^{1/2}\right)\right] = \tag{38}$$

$$CT^{-3/2}\frac{1}{R}\frac{\partial}{\partial R}\left[\frac{1}{2}e^{-R/T} - RT^{-1}e^{-R/T}\right] = \tag{39}$$

$$CT^{-3/2}\frac{1}{R}\frac{\partial}{\partial R}\left[e^{-R/T}\left(\frac{1}{2} - RT^{-1}\right)\right] = \tag{40}$$

$$CT^{-3/2}R^{-1}\left[-T^{-1}e^{-R/T}\left(\frac{1}{2}-RT^{-1}\right)-T^{-1}e^{-R/T}\right]= \tag{41}$$

$$CT^{-3/2}R^{-1}e^{-R/T}\left(-\frac{3}{2}T^{-1}+RT^{-2}\right)= \tag{42}$$

$$\Sigma\left(-\frac{3}{2}T^{-1}+RT^{-2}\right). \tag{43}$$

Both sides match which means our solution is valid.
(c) See jupyter notebook