Astro 204

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Contents

1 Streamlines

a) The streamline in spherical coordinates are determined by the equation

$$\frac{dr}{v_r} = r\frac{d\theta}{v_\theta} = \frac{dz}{v_z}. \tag{1}$$

For the case $v_r = a$, $v_\theta = b$, $v_z = 0$, the equation of the stream lines becomes

$$\frac{dr}{d\theta} = r\frac{a}{b}, \tag{2}$$

a first order differential equation. The equation has the solution

$$r(\theta) = Ce^{(a/b)\theta}, \tag{3}$$

where C is a constant of integration. This is the form of a logarithmic spiral, which is plotted in fig. 1.

b) For the case $v_r = ar^2$, $v_\theta = br^2$, $v_z = 0$, the equation of the stream lines remains the same

$$\frac{dr}{d\theta} = r\frac{a}{b}, \tag{4}$$

which is identical to case (a).

c) The change in mass, $\dot{M}$ along a circular shell is

$$\dot{M} = 2\pi r \Sigma v_r \tag{5}$$

Rearranging gives

$$\Sigma = \dot{M}/(2\pi r v_r), \tag{6}$$

which becomes $\Sigma(r) = \dot{M}/(2\pi b r)$ for part (a) and $\Sigma = \dot{(M)}/(2\pi b r^3)$ for part (b).

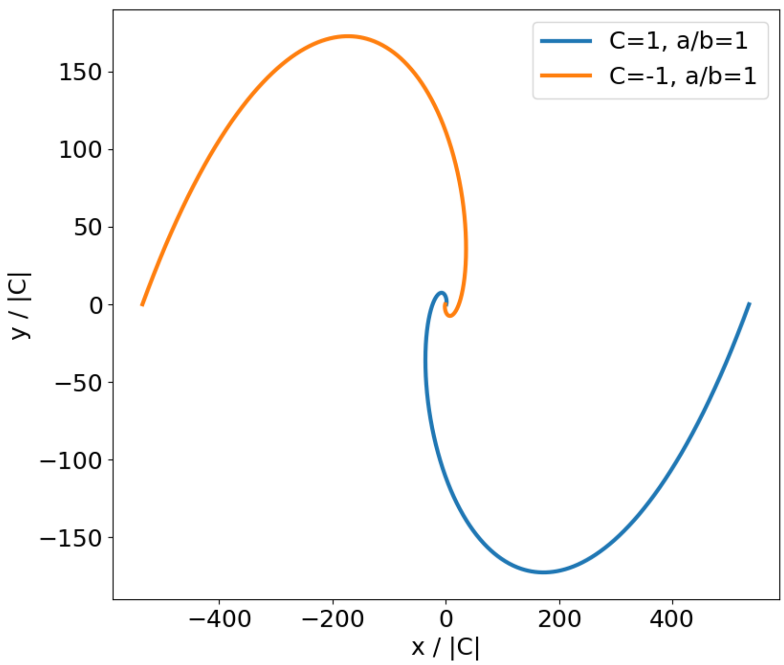

Figure 1: The streamlines for part (a) and part (b)

2 The Energy Equation

Since the no work is being done on the system by external forces $F \cdot \vec{v} = 0$, which can be stated as

$$\frac{\partial E}{\partial t} + \vec{\nabla} \cdot (F_E \vec{v}) = 0 \tag{7}$$

where $F_E = E + P$ and E $= \frac{1}{2}\rho v^2 + \rho\epsilon$. Plugging in yields

$$\begin{aligned} 0 = \frac{1}{2}v^2\frac{\partial\rho}{\partial t} + \rho\vec{v}\cdot\frac{\partial\vec{v}}{\partial t} + \rho\frac{\partial\epsilon}{\partial t} + \epsilon\frac{\partial\rho}{\partial t} + \frac{1}{2}\rho v^2\vec{\nabla}\cdot\vec{v} + \vec{v} \\ \cdot\vec{\nabla}(\frac{1}{2}\rho v^2) + \rho\epsilon\vec{\nabla}\cdot\vec{v} + \vec{v}\cdot\vec{\nabla}(\rho\epsilon) + P\vec{\nabla}\cdot\vec{v} + \vec{v}\cdot\vec{\nabla}P. \end{aligned} \tag{8}$$

Grouping like terms and doing algebra gives

$$\begin{aligned} &\left(\rho\frac{\partial\epsilon}{\partial t} + \rho\vec{v}\cdot\vec{\nabla}\epsilon + P\vec{\nabla}\cdot\vec{v}\right) + \epsilon\left(\vec{v}\cdot\vec{\nabla}\rho + \frac{\partial\rho}{\partial t} + \rho\vec{\nabla}\cdot\vec{v}\right) \\ &+ \frac{1}{2}v^2\left(\frac{\partial\rho}{\partial t} + \rho\vec{\nabla}\cdot\vec{v} + \vec{v}\cdot\vec{\nabla}\rho\right) + \vec{v}\cdot\left(\vec{\nabla}P + \frac{\partial\vec{v}}{\partial t} + \rho\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}\right), \end{aligned} \tag{9}$$

which can be further simplified by realizing the ϵ and $\frac{1}{2}v^2$ term is 0 by continuity ($\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$)and the $\vec{v}$ term is 0 by momentum conservation. Therefore,

$$\rho\frac{\partial\epsilon}{\partial t} + \rho\vec{v}\cdot\vec{\nabla}\epsilon + P\vec{\nabla}\cdot\vec{v} = 0 \tag{10}$$

The fundamental equation of thermodynamics is $d\epsilon = Tds + \frac{P}{\rho^2}d\rho$, which can be rewritten as

$$\rho T\frac{ds}{dt} = \rho\frac{d\epsilon}{dt} - \frac{P}{\rho}\frac{d\rho}{dt} \tag{11}$$

Using the chain rule on $\frac{d\epsilon}{dt}$ gives

$$\frac{d\epsilon}{dt} = \frac{\partial\epsilon}{\partial t} + \vec{v}\cdot\vec{\nabla}\epsilon, \tag{12}$$

likewise using the chain rule for $\frac{d\rho}{dt}$ and plugging in the continuity equation gives

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \vec{v}\cdot\vec{\nabla}\rho = -\rho\vec{\nabla}\cdot\vec{v}. \tag{13}$$

Plugging into eq. 11 gives

$$\rho T\frac{ds}{dt} = \rho\frac{\partial\epsilon}{\partial t} + \rho\vec{v}\cdot\vec{\nabla}\epsilon + P\vec{\nabla}\cdot\vec{v} \tag{14}$$

Which when plugged into eq. 10 gives

$$\frac{ds}{dt} = 0 \tag{15}$$

3 Entropy

a) Starting from the statistical mechanics definition of entropy $\sigma = \log g$, where g is the number of available of states and given the distribution of particles per unit volume in 6D phase space $f(\vec{x}, \vec{v}, t)$ we can understand the provided equation (4). Assuming each constant volume element in phase space can hold an equal number of states, the amount of volume occupied by a particle is equal to the number of equivalent states for that particle. Therefore, the number of states per particle can be thought of as f^{-1} and the entropy per particle becomes $\sigma = -\log f$. The total entropy of the system per unit volume at a point in space and time can be calculated by averaging over all velocity states, and weighting by the number of particles per state. Calculating the thermodyamic entropy $S = k\sigma$, and writing the weighted statistical average explicitly gives

$$s = -k \int_{\text{all}\vec{v}} f \ln f d^3v, \tag{16}$$

where s is the entropy per unit volume of the system.

b) For a maxwellian, $f(\vec{x}, \vec{v}, t) = nf_{\text{Max}}(\vec{v})$ plugging into equation eq. 16 gives

$$S = -4ka^3\pi^{-3/2}n \int_{\mathbb{R}} v^2 e^{-(av)^2} \ln \left[n \left(\frac{a^2}{\pi} \right)^{3/2} e^{-(av)^2} \right] dv \tag{17}$$

where $a = \sqrt{m/(2kT)}$, and I have made the substitution $d^3v = r^2 dr d\theta d\phi$ and integrated over θ and ϕ. Further simplication can be made by making the substitution $u = av$ giving and rewriting interms of entropy per unit mass

$$s = -4k\pi^{-1/2}m^{-1} \int_{\mathbb{R}} u^2 e^{-u^2} \left[\ln \left(na^3\pi^{-3/2} \right) - u^2 \right] dv. \tag{18}$$

Notice that all pressure / density dependance is in the variable a, since s needs to be calculated to an additive constant, we can drop all terms with no a dependence resulting in

$$s = -4k\pi^{-1/2}m^{-1} \ln \left(na^3 \right) \int_{\mathbb{R}} u^2 e^{-u^2} dv. \tag{19}$$

where evaluating the integral $\int_{\mathbb{R}} u^2 e^{-u^2} dv = \sqrt{\pi}/2$ gives

$$s = -km^{-1} \ln \left(na^3 \right) \tag{20}$$

where we can plug the definition of a and $n = \rho/m$ in to get

$$s = -\frac{k}{m}\ln\left(\frac{\rho}{m}\frac{m}{2kT}\right) \tag{21}$$

using the Ideal gas law and dropping all constants gives

$$s = -\frac{k}{m}\ln\left(\rho P^{-3/2}\rho^{3/2}\right) \tag{22}$$

which can be simplified to

$$s = \frac{3k}{2m}\ln\left(P\rho^{-5/3}\right) \tag{23}$$

4 The Lane-Emden Equation for Stellar Structure

a) The condition for hydrostatic equilibrium is

$$\frac{1}{\rho}\frac{dP}{dr} = -G\frac{M}{r^2} \tag{24}$$

where M is the enclosed mass, which satisfies the differential equation

$$\frac{dM}{dr} = 4\pi r^2\rho. \tag{25}$$

To use plug eq. 25 into eq. 24 we must take the derivative of both sides of eq. 24

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dM}{dr} = -4\pi G r^2\rho. \tag{26}$$

Since $P = K\rho^\gamma$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = \frac{d}{dr}\left(r^2 K\gamma\rho^{\gamma-2}\frac{d\rho}{dr}\right) = K\frac{\gamma}{\gamma-1}\frac{d}{dr}\left(r^2\frac{d\rho^{\gamma-1}}{dr}\right), \tag{27}$$

Upon substituting $\gamma = 1 + \frac{1}{n}$, $\xi = r/a$ and $\theta = (\rho/\rho_c)$ eq. 24 becomes

$$\frac{1}{a^2}K(n+1)\rho^{1/n}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = 4\pi G\xi^2\theta^n\rho_c \tag{28}$$

which is equivalent to

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = \theta^n\left(a^2\frac{4\pi G}{K}\xi^2\rho_c^{1-1/n}\right) \tag{29}$$

which can be simplified to

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = \theta^n \tag{30}$$

upon substitution of $a^2 = (n+1)K\rho_c^{1/(n-1)}(4\pi G)^{-1}$

b) See fig. 2 and code provided in problem4.py. As the polytropic index increases the density profile becomes less concentrated.

c) The ideal gas law is $PV = nkT/\mu$ so $kT/\mu = K\rho^{1/n}$. The OOM estimate then is good if a $GM/r \approx KT/\mu = K\rho^{1/n}$. A good estimate from our numerical models is that the density reaches zero at $\xi = 3a$. We can plug this in $GM/R = GM/3a$ and numerically integrate M. We can also estimate M if assume $M \approx 4/3\pi a^3 p$ and get $GM/3a \approx Ga^2\rho$. How well our estimate does, will depend on the value of K.

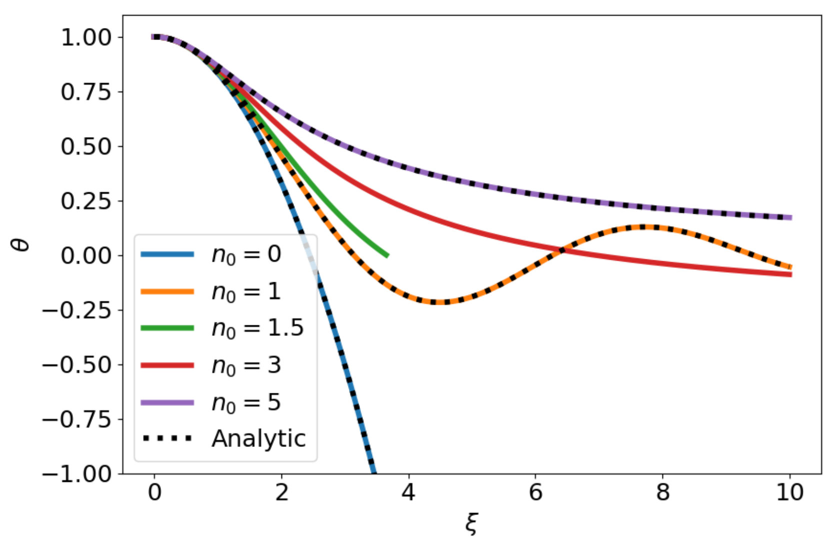

Figure 2: The numerical solutions for part (b). Negative solutions are unphysical.