Astro 204 Problem Set 4

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Contents

1 Numerical Time-Evolution to a Parker Wind

(a) The mass conservation equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \tag{1}$$

simplifies to

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\left(r^2 \rho v\right) = 0, \tag{2}$$

under the assumption of spherical symmetry, which can be expanded to

$$\frac{\partial \rho}{\partial t} + \frac{2}{r}\rho v + v\frac{\partial \rho}{\partial r} + \partial\frac{\partial v}{\partial r} = 0, \tag{3}$$

using the chain rule. The momentum conservation equation is

$$\rho\frac{d\vec{v}}{dt} = -\vec{\nabla}P + \rho\vec{f} \tag{4}$$

which can be expanded by applying the chain rule

$$\frac{d\vec{v}}{dt} = \left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}\right)\hat{r}, \tag{5}$$

Additionally upon substituting the isothermal equation of state $P = c_s^2 \rho$ the gradient of pressure becomes

$$\vec{\nabla}P = c_s^2\frac{\partial \rho}{\partial r}\hat{r}. \tag{6}$$

Combining everything, we get

$$\rho\frac{\partial v}{\partial t} + \rho v\frac{\partial v}{\partial r} = -c_s^2\frac{\partial \rho}{\partial r} - \rho\frac{GM}{r^2} \tag{7}$$

(b) Upon substitution of $x = lnr$ the equation (1) becomes

$$\frac{\partial \rho}{\partial t} + e^{-2x}\frac{d}{dx}(e^{2x}\rho v) = 0 \tag{8}$$

and equation 2 becomes

$$e^x\left(e^x\rho\frac{\partial v}{\partial t} + \rho v\frac{\partial v}{\partial x} + c_s^2\frac{\partial \rho}{\partial x}\right) = -GM\rho. \tag{9}$$

(c) See jupyter notebook

2 Wave Breaking in the Upper Atmosphere

(a) For our, the mass conservation equation is

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \tag{10}$$

and momentum conservation is

$$\rho\frac{d\vec{v}}{dt} = -\vec{\nabla}P + g\rho\hat{z}, \tag{11}$$

if we take the near surface limit for the gravitational force. Now, we add perturbations P_1, ρ_1, v_1 to a set of know solutions, P_0, ρ_0 and v_0, like the derivation in class, I choose $v_0 = 0$. First, I calculate the scale height of the atmosphere. Under the assumption $v_0 = 0$

$$\vec{\nabla}P_1 = \frac{dP_1}{d\rho_1}\vec{\nabla}\rho_1 = g\rho_1\hat{z}, \tag{12}$$

which upon substituting $a_0^2 = \frac{dP}{d\rho}$ and expanding $\vec{\nabla}\rho$ the equation becomes

$$a_0^2\frac{\partial \rho_1}{\partial z}\hat{z} = g\rho_1\hat{z}, \tag{13}$$

which gives a scale height

$$H = \rho/\left|\frac{\partial \rho_1}{\partial z}\right| = \frac{a_0^2}{g}. \tag{14}$$

Applying perturbation theory to the first equation, gives the same result as for the case without considering gravity (the solution we derived in class)

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{v}_1 = 0. \tag{15}$$

Applying perturbation theory to the second equation we and applying similar arguments as were applied in class get

$$\rho_0 \frac{d\vec{v}}{dt} = -a_0^2 \vec{\nabla} \rho_1 - g\rho_1 \hat{z} \tag{16}$$

Combining the energy and momentum equation into the wave equation, we get

$$\frac{\partial^2 \rho_1}{\partial t^2} - a_0^2 \nabla^2 \rho_1 - \frac{\partial \rho_1}{\partial z} = 0. \tag{17}$$

Assuming the solution

$$\rho = \delta \rho e^{i(\vec{k}\cdot\vec{x} - \omega t)} \tag{18}$$

and plugging we get

$$-\omega^2 + a_0^2 k^2 - igk_z = 0 \tag{19}$$

which for a wave traveling in the z direction becomes

$$\omega^2 = a_0^2 \left(k^2 - i\frac{k}{H} \right) \tag{20}$$

after substituting $H = a_0^2/g$ (b) Rearranging for k, we have

$$k^2 - \frac{ik}{H} - \frac{\omega^2}{a_0^2} = 0, \tag{21}$$

solving using the quadratic equation we get

$$k = \frac{i}{2H} \pm \sqrt{-\frac{1}{(2H)^2} + \frac{\omega^2}{a_0^2}}. \tag{22}$$

Plugging into our assumed solution, we get

$$\rho_1 = \delta \rho e^{-z/(2H)} e^{\text{Imaginary Part}} \tag{23}$$

the fractional amplitude is

$$\frac{\Delta \rho}{\rho_0} = \frac{\delta}{\rho_0} e^{-z/(2H)} \tag{24}$$

however the background density decreases with height as $\rho \propto e^{-z/H}$ so the fractional amplitude is proportional to

$$\frac{\Delta \rho}{\rho_0} \propto \frac{\Delta \rho}{\rho_0} e^{-z/(2H)} e^{z/H} = e^{z/(2H)} \tag{25}$$

so the fractional density amplitude increases with height.