Solving the Incompressible Navier-Stokes Equations using "Stable Fluids"

1 Introduction

This document describes the method to solve the incompressible Navier-Stokes equations in a closed box scenario with a forcing function that creates a bloom. The solution strategy follows the "Stable Fluids" approach.

2 Governing Equations

The equations governing the flow are:

2.1 Momentum Equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \tag{1}$$

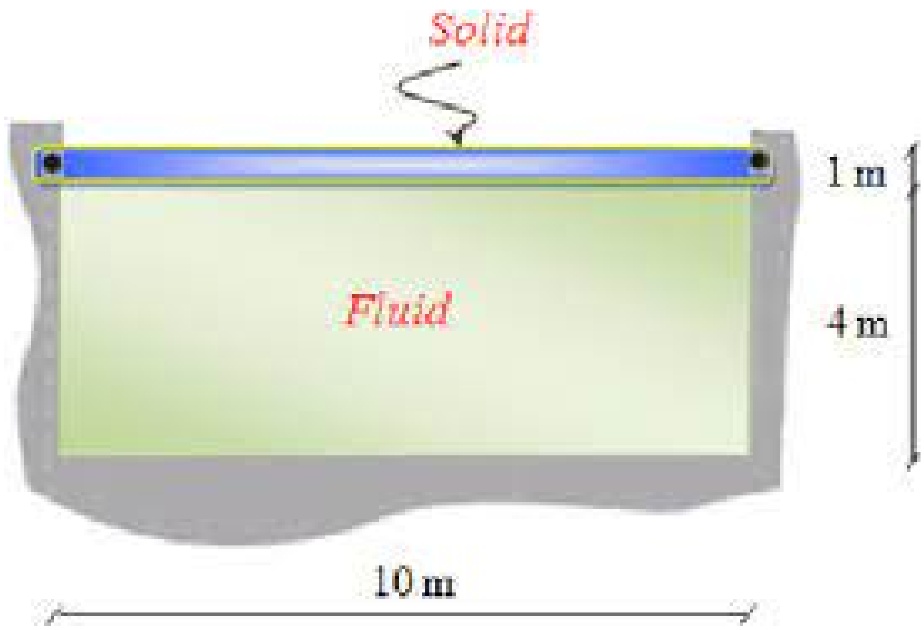
2.2 Incompressibility Condition

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

where:

- $\mathbf{u}$: Velocity (2D vector)
- p: Pressure
- $\mathbf{f}$: Forcing
- ν: Kinematic Viscosity
- ρ: Density
- t: Time
- ∇: Nabla operator (defining nonlinear convection, gradient, and divergence)
- ∇^2: Laplace Operator

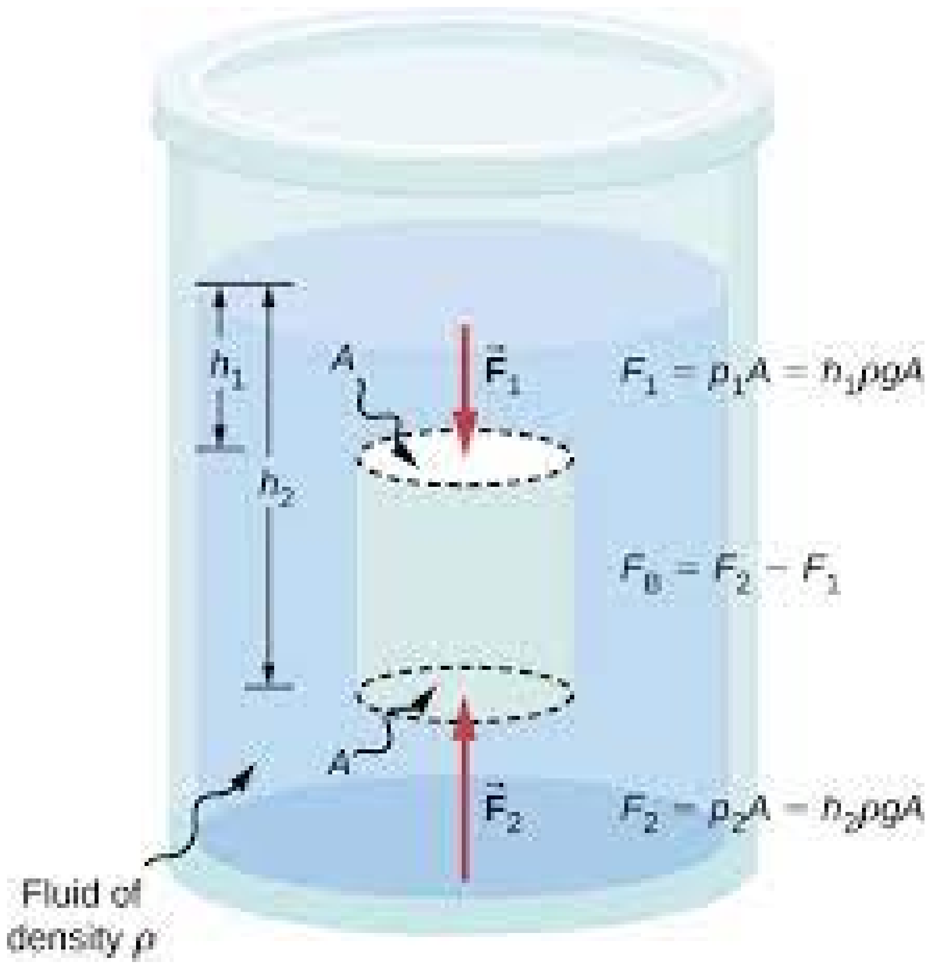
3 Closed Box Scenario



- Homogeneous Dirichlet Boundary Conditions are applied everywhere.
- The velocity $\mathbf{u}$ and pressure p have zero initial conditions.

4 Forcing Function

The forcing function creates an upward force in the lower center of the domain.



5 Solution Strategy

The solver follows these steps:

5.1 Initialization

Start with zero velocity everywhere: $\mathbf{u} = [0, 0]$.

5.2 Add Forces

$$\mathbf{w}_1 = \mathbf{u} + \Delta t\mathbf{f} \tag{3}$$

5.3 Self-Advection (Convection)

Convect by self-advection (set the value at the current location to be the value at the position backtracked on the streamline). This step is unconditionally stable.

$$\mathbf{w}_2 = \mathbf{w}_1(\mathbf{p}(\mathbf{x}, -\Delta t)) \tag{4}$$

5.4 Implicit Diffusion

Diffuse implicitly by solving a linear system matrix-free using Conjugate Gradient. This step is unconditionally stable.

$$(I - \nu \Delta t \nabla^2)\mathbf{w}_3 = \mathbf{w}_2 \tag{5}$$

5.5 Pressure Correction

1. Compute a pressure correction by solving a linear system matrix-free using Conjugate Gradient.
$$\nabla^2 p = \nabla \cdot \mathbf{w}_3 \tag{6}$$

2. Correct velocities to be incompressible.
$$\mathbf{w}_4 = \mathbf{w}_3 - \nabla p \tag{7}$$

5.6 Advance to Next Time Step

$$\mathbf{u} = \mathbf{w}_4 \tag{8}$$

6 Boundary Conditions

The boundary conditions are prescribed indirectly using the discrete differential operators.

7 Stability

The solver is unconditionally stable; hence, all parameters can be chosen arbitrarily. However, high timesteps can make the advection step highly incorrect.

8 Output