$$|\mathbf{A} - \lambda\mathbf{I}| = \left|\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right|$$
$$= \begin{vmatrix} 1-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) = 0$$

implies that there are two roots, $\lambda_1 = 1$ and $\lambda_2 = 3$. The eigenvalues of $\mathbf{A}$ are 3 and 1. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

are $\lambda_1 = 1$ and $\lambda_2 = 3$. The eigenvectors associated with these eigenvalues can be determined by solving the following equations:

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\mathbf{Ax} = \lambda_1\mathbf{x}$$
$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\mathbf{Ax} = \lambda_2\mathbf{x}$$

From the first expression,

$$x_1 = x_1$$
$$x_1 + 3x_2 = x_2$$

or

$$x_1 = -2x_2$$

There are many solutions for x_1 and x_2.

Setting $x_2 = 1$ (arbitrarily) gives $x_1 = -2$, and hence,

$$\mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to the eigenvalue 1. From the second expression,

$$x_1 = 3x_1$$
$$x_1 + 3x_2 = 3x_2$$

implies that $x_1 = 0$ and $x_2 = 1$ (arbitrarily), and hence,

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to the eigenvalue 3. It is usual practice to determine an eigenvector so that it has length unity. That is, if $\mathbf{Ax} = \lambda\mathbf{x}$, we take $\mathbf{e} = \mathbf{x}/\sqrt{\mathbf{x}'\mathbf{x}}$ as the eigenvector corresponding to λ. For example, the eigenvector for $\lambda_1 = 1$ is $\mathbf{e}_1' = \left[-2/\sqrt{5},\ \ 1/\sqrt{5}\right]$.