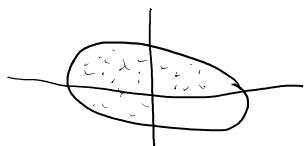


## Regiones de confianza

Sea  $\theta$  un vector de parámetros poblacionales

Sea  $\Theta$  un conjunto de posibles valores de  $\theta$



Decimos que una región de confianza es una región en  $\Theta$  y la denotamos como  $R(\mathbf{X})$  ( $\mathbf{X}$  es la matriz de datos)

$$\text{obs} \rightarrow \begin{pmatrix} & & & \\ X_{11} & \dots & X_{1p} \\ X_{21} & & \vdots \\ \vdots & & \\ X_{n1} & & X_{np} \end{pmatrix}$$

Decimos que  $R(\mathbf{X})$  es una región de confianza con conf  $(1-\alpha) \cdot 100\%$  si

$$P(R(\mathbf{X}) \text{ contiene al verdadero } \theta) = 1-\alpha$$

Si la población es normal  $p$ -variada,  $R(\mathbf{X})$  para  $\mu$  se deduce de:

$$P\left(n(\bar{\mathbf{X}} - \mu)' S^{-1} (\bar{\mathbf{X}} - \mu)\right) \leq \underbrace{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)}_{(*)} = 1-\alpha$$

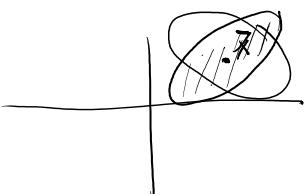
Es decir,  $\bar{\mathbf{X}}$  estará a una distancia menor o igual a (\*) de  $\mu$ . Con prob  $1-\alpha$ .

Para una muestra se calculan  $\bar{x}$  y  $s$ , y con esto podemos obtener  $R(\bar{x})$

$$n(\bar{x} - \mu_0)^T S^{-1} (\bar{x} - \mu_0) \leq \frac{(n-1)p}{(n-p)} F_{p, n-p}(2) \quad (1)$$

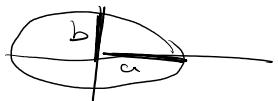
Observaciones:

- $R(\bar{x})$  es una hiperelipsode centrada en  $\bar{x}$   
p.e. 2 dimensiones



- $R(\bar{x})$  son todos los valores de  $\mu_0$  que cumplen (1)
- Vamos a ver si  $\mu_0$  es un valor posible, es decir, está dentro de la hiperelipsode.
- $R(\bar{x})$  es el conjunto de valores de  $\mu_0$  para los cuales el test  $T^2$  hace que NO se rechace la  $H_0$ .
- Los ejes de la hiperelipsode  $R(\bar{x})$  son en dirección a los vectores propios de  $nS^{-1}$  (mismos,  $s$ ) y las longitudes se determinan usando: valores  $d_1, d_2, \dots$  de  $S$   
propios  $n_1, n_2, \dots$  de  $nS^{-1}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$n(\bar{x} - \mu_0)^T S^{-1} (\bar{x} - \mu_0) \leq c^2 \approx \frac{p(n-1)}{(n-p)} F_{p, n-p}(2)$$

las Longitudes son

las longitudes son

$$\sqrt{\frac{\lambda_i}{n}} \cdot c_i = \sqrt{\frac{\lambda_i}{n}} \cdot \sqrt{\frac{p(n-1)}{(n-p)} F_{p,n-p}(\alpha)}$$

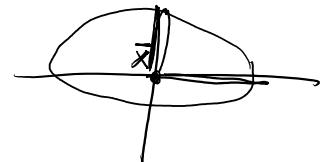
Ej. 2.11

Fig. 2.6.

Entonces, los ejes de confianza son (centrados en  $\bar{x}$ )

$$\bar{x} \pm \sqrt{\lambda_i} \sqrt{\frac{p(n-1)}{n(n-p)} F_{p,n-p}(\alpha)} \cdot e_i$$

longitud.



### WAIS Example

Equation for the  $(1 - \alpha)100\%$  confidence region:

$$n(\bar{x} - \mu)' \mathbf{S}^{-1} (\bar{x} - \mu) \leq \frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)$$

$$\text{or } T^2 \leq \frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)$$

The confidence region is an ellipse (ellipsoid for  $p > 2$ ) centered at  $\bar{x}$  with axes

$$\bar{x} \pm \sqrt{\lambda_i} \sqrt{\frac{p(n-1)}{n(n-p)} F_{p,n-p}(\alpha)} e_i$$

where  $\lambda_i$  and  $e_i$  are the eigenvalues and eigenvectors, respectively, of  $\mathbf{S}$  ( $\lambda_i$  is not Wilk's lambda).

For the WAIS data,

$$\begin{cases} \lambda_1 = 299.982, & e'_1 = (.818, .576) \\ \lambda_2 = 30.238, & e'_2 = (-.576, .818) \end{cases}$$

$$n = 101$$

$$p = 2$$

$$\bar{x} = \begin{pmatrix} 55, 24 \\ 34, 97 \end{pmatrix}$$

### WAIS Example: Finding Major and Minor

$$\bar{x} \pm \sqrt{\lambda_i} \sqrt{\frac{p(n-1)}{n(n-p)} F_{p,n-p}(\alpha)} e_i$$

The major axis:

$$\begin{pmatrix} 55.24 \\ 34.97 \end{pmatrix} \pm \sqrt{299.982} \sqrt{\frac{2(101-1)}{101(101-2)} 3.11} \begin{pmatrix} .818 \\ .576 \end{pmatrix}$$

which gives us (51.71, 32.48) and (58.77, 37.46).

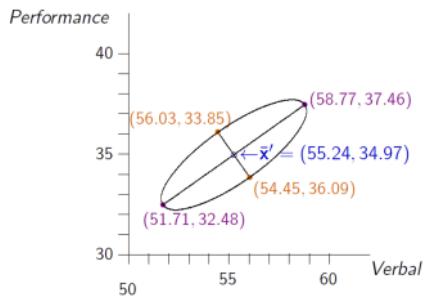
The minor axis:

$$\begin{pmatrix} 55.24 \\ 34.97 \end{pmatrix} \pm \sqrt{30.238} \sqrt{\frac{2(101-1)}{101(101-2)} 3.11} \begin{pmatrix} -.576 \\ .818 \end{pmatrix}$$

which gives us (56.03, 33.85) and (54.45, 36.09)

### Graph of 95% Confidence Region

Length of major = 8.64 (half-length= 4.32)  
Length of minor = 2.74 (half-length= 1.37)



Alternativas para calcular las Regiones de confianza:

I) Intervalos de confianza uno a uno

II) Intervalos  $T^2$ .

III) Bonferroni.

I) Intervalos de confianza uno a uno.

Centrizar I.C para cada variable individualmente, para así crear una  $R(\mathbf{x})$  conjunta.

Objetivo: ver la relación entre  $R(\mathbf{x})$  y los I.C. individuales.

$$\text{Sea } \mathbf{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad Z = \underbrace{a_1 X_1 + \dots + a_p X_p}_{\substack{\downarrow \\ \text{variable aleatoria}}} = \underbrace{a' \mathbf{X}}_{\substack{\downarrow \\ \text{v.a.}}}$$

$$Z \sim N(a' \boldsymbol{\mu}, a' \boldsymbol{\Sigma} a)$$

$$\mu_Z = E(Z) = a' \boldsymbol{\mu}$$

$$\sigma^2 = \text{var}(Z) = a' \boldsymbol{\Sigma} a$$

Ahora, si se tiene una muestra aleatoria  $X_1, \dots, X_n$ , creamos la muestra  $Z_1, \dots, Z_n$

$$\underline{Z_j} = \underbrace{\alpha_1 X_{j1} + \dots + \alpha_p X_{jp}}_{\text{valor}}.$$

$$\bar{Z}_j = \frac{\underline{Z_j}}{n} \quad S_{\bar{Z}}^2 = \frac{\underline{S_{ij}^2}}{n}$$

El IC con confianza  $1-\alpha$  de  $\mu_2 = a' \mu$  se construye con:

$$z = \frac{\bar{Z} - \mu_2}{S_{\bar{Z}}/\sqrt{n}} = \frac{\sqrt{n} (\bar{Z} - \mu_2)}{\sqrt{a' S_a}} \sim t_{n-1}$$

Los IC's quedan definidos por los límites:

$$\bar{Z} \pm t_{n-1}(\alpha/2) \frac{S_{\bar{Z}}}{\sqrt{n}}$$

número de variables  $a' S_a = \frac{S_{11}}{n}$

$a = (1, 0, \dots, 0)$

$$\rightarrow \bar{Z} \pm t_{n-1}(\alpha/2) \frac{\sqrt{a' S_a}}{\sqrt{n}}$$

$\bar{X} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_p \end{pmatrix}$

Si la muestra tiene  $p$  variables

$$a = (1, 0, 0, \dots, 0)$$

$$a = (0, 1, \dots, 0)$$

### WAIS Example: One-at-a-time Intervals

#### Univariate Confidence Intervals

$$\bar{x}_i \pm t_{n-1,(\alpha/2)} \sqrt{s_{ii}/n}$$

We'll let  $\alpha = .05$  (for a 95% confidence interval), so  $t_{100,.025} = 1.99$ .

For verbal score:

$$55.24 \pm 1.99 \sqrt{210.54/101}$$

$$55.24 \pm 2.87 \rightarrow (52.37, 58.11)$$

$$S_{11} = 210.54$$

$$S_{22} = 119.68$$

$$n = 101$$

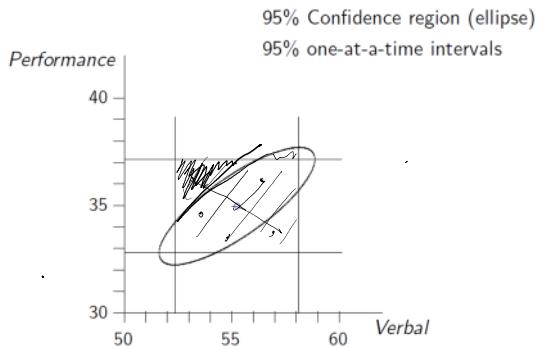
$$P = 2$$

$$\bar{X} = \begin{pmatrix} 55, 24 \\ 34, 97 \end{pmatrix} \rightarrow \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$

### Graph of one-at-a time intervals

### Graph of one-at-a time intervals

Multivariate versus Univariate:



Intervalos  $T^2$ :

Intuitivamente, con  $R(\bar{X})$  buscamos que IC "colectivo" de  $1-\alpha$  pueda ser generado ya contenga a  $\mu$  de forma simultánea

Problema: Crear intervalos más grandes  $\Rightarrow$  más imprecisión

Sean  $X_1, \dots, X_n$  una muestra aleatoria  $N_p(\mu, \Sigma)$ ,  $\Sigma$  def positiva, entonces para toda  $\alpha$ , el IC es

$$\hat{\alpha}' \bar{X} \pm \sqrt{\frac{p(n-1)}{n(p-p)}} F_{p, n-p}(\alpha) \hat{\alpha}' S \hat{\alpha}$$

contiene  $\mu$  con prob.  $1-\alpha$

Demonstración:

$$T^2 \leq c^2 \Rightarrow \frac{n(\hat{\alpha}' \bar{X} - \hat{\alpha}' \mu)^2}{\hat{\alpha}' S \hat{\alpha}} \leq c^2$$

$$\rightarrow \hat{\alpha}' \bar{X} - c \sqrt{\frac{\hat{\alpha}' S \hat{\alpha}}{n}} \leq \hat{\alpha}' \mu \leq \hat{\alpha}' \bar{X} + c \sqrt{\frac{\hat{\alpha}' S \hat{\alpha}}{n}}$$

$$\text{siendo } C^2 = \frac{p(n-1)}{(n-p)} F_{p, n-p}(\alpha)$$

### WAIS Example

$$S_{11} = 210,54$$

$$S_{22} = 119,68$$

For the verbal score:

$$55.24 \pm \sqrt{\frac{100(2)}{99}}(3.11) \sqrt{210.54/101} = 55.24 \pm 3.62 \rightarrow (51.62, 58.86)$$

For the performance score:

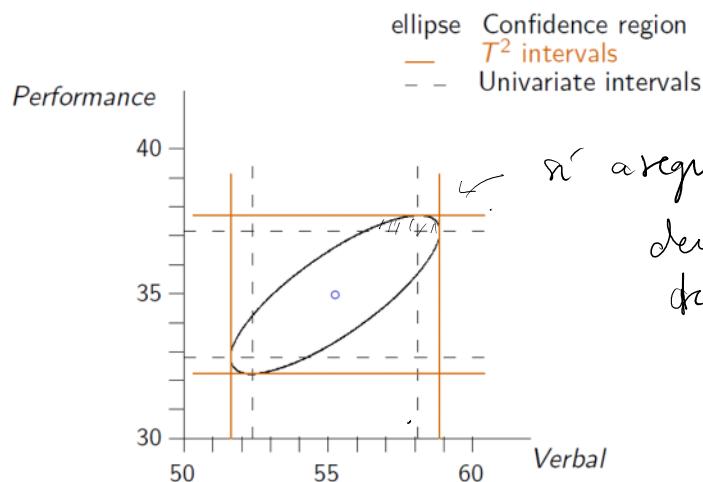
$$34.97 \pm \sqrt{\frac{100(2)}{99}}(3.11) \sqrt{119.68/101} = 34.97 \pm 2.73 \rightarrow (32.24, 37.70)$$

$$n = 101$$

$$P = 2$$

$$\bar{X} = \begin{pmatrix} 55,24 \\ 34,97 \end{pmatrix} \rightarrow \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$

### WAIS: Comparison



### Intervalos de Bonferroni:

Van a ser más estrechos que los  $T^2$  pero menos que los uno a uno.

Salen de la Desigualdad de Bonferroni:

$$\begin{aligned} P(\text{todos los int válidos}) &= 1 - P(\text{por lo menos un intervalo sea inválido}) \geq \\ &\geq 1 - \sum_{i=1}^m P(i\text{-esimo intervalo es falso}) = \end{aligned}$$

$$= 1 - \sum_{\alpha_i} \underbrace{\left(1 - P(\text{í-ésimo sea verdadero})\right)}_{\alpha_i}$$

$$= 1 - \sum_{i=1}^m \alpha_i$$

$$P(\text{todos los int. válidos}) \geq 1 - \sum_{i=1}^m \alpha_i$$

$\boxed{\alpha_i = \frac{\alpha}{m}}$  Le damos el mismo peso a todos los  $\alpha$ .  
 Para ello, dividimos entre el n.º de variables

$$P(\text{todos los int. válidos}) \geq 1 - (\alpha/m + \alpha/m + \dots + \alpha/m) = \\ \geq 1 - \alpha$$

### WAIS & Bonferroni Intervals

**WAIS example:** We'll only consider  $\mathbf{a}_1' = (1, 0)$  and  $\mathbf{a}_2 = (0, 1)$  (i.e., the component means).

$$S_{11} = 210,54$$

$$df = n - 1 = 101 - 1 = 100$$

$$\alpha = .05 \rightarrow \alpha/2 = .025$$

$$t_{100, (.025/2)} = 2.2757$$

$$S_{22} = 119,68$$

Verbal Scores:

$$\begin{aligned} 55.25 &\pm 2.2757 \sqrt{210.54/101} \\ &\pm 2.2757(1.4438) \\ &\pm 3.2856 \rightarrow (51.95, 58.53) \end{aligned}$$

$$n = 101$$

$$\mathbf{a}' \bar{x} \pm t_{n-1, (\alpha/2m)} \sqrt{\frac{\mathbf{a}' \mathbf{S} \mathbf{a}}{n}}$$

Performance Scores:

$$\begin{aligned} 34.97 &\pm 2.2757 \sqrt{119.68/101} \\ &\pm 2.2757(1.08855) \\ &\pm 2.477 \rightarrow (32.49, 37.45) \end{aligned}$$

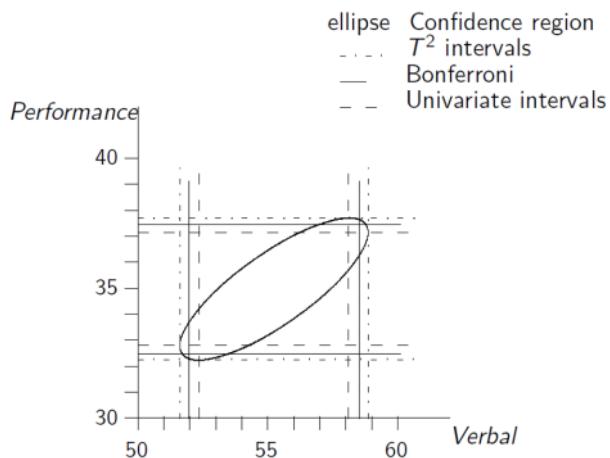
$$P = 2$$

$$\bar{x} = \begin{pmatrix} 55,24 \\ 34,97 \end{pmatrix} \rightarrow \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$

Univariate Confidence Intervals

$$\bar{x}_i \pm t_{n-1, (\alpha/2)} \sqrt{s_{ii}/n}$$

## WAIS: All four Confidence Methods



### Observación

Hemos asumido  $X_i \sim N_p(\mu, \Sigma)$

¿Qué pasa si No es normal?

Si  $n$  es grande (+30 obs.)

Sea  $X_1, \dots, X_n$  una muestra aleatoria (cualq. distribución)

con media  $\mu$  y cov.  $\Sigma$

$$T^2 = n(\bar{X} - \mu_0)' S^{-1} (\bar{X} - \mu_0) \approx \chi_p^2$$

$$P(n(\bar{X} - \mu_0)' S^{-1} (\bar{X} - \mu_0) \leq \chi_p^2(\alpha)) \approx 1 - \alpha$$

$$\frac{(n-1)_p}{(n-p)} F_{p, n-p} \xrightarrow{\text{approx}} \chi_p^2$$

Test de hipótesis:

$$H_0: \mu = \mu_0$$

Rechazamos  $H_0$  si  $T^2 > \chi_p^2(\alpha)$

$$\text{Intervalo} \quad T^2 : \bar{x} \pm \sqrt{\chi_p^2(\alpha)} \cdot \sqrt{\frac{\bar{x}'S\bar{x}}{n}}$$

### WAIS: Large-Sample

- WAIS example with  $n = 101$ ,

$$F_{p,n-p}(\alpha) = F_{2,99}(.05) = 3.11$$

$$\frac{(n-1)p}{n-p} F_{p,n-p} = \frac{100(2)}{99} (3.11) = 6.28$$

$$\chi_2^2(.05) = 5.99$$

The value 6.28 is fairly close to 5.99.

- It's generally true that the more you assume, the more powerful your test (more precise estimates).
- The larger  $n \rightarrow$ , the more power... This is generally true.