Example 2.11 (A positive definite matrix and quadratic form) Show that the matrix for the following quadratic form is positive definite:

$$3x_1^2 + 2x_2^2 - 2\sqrt{2}\, x_1 x_2$$

To illustrate the general approach, we first write the quadratic form in matrix notation as

$$[x_1 \quad x_2]\begin{bmatrix} 3 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}'\mathbf{A}\mathbf{x}$$

By Definition 2A.30, the eigenvalues of $\mathbf{A}$ are the solutions of the equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$, or $(3 - \lambda)(2 - \lambda) - 2 = 0$. The solutions are $\lambda_1 = 4$ and $\lambda_2 = 1$. Using the spectral decomposition in (2-16), we can write

$$\begin{aligned} \underset{(2\times2)}{\mathbf{A}} &= \lambda_1 \underset{(2\times1)}{\mathbf{e}_1}\underset{(1\times2)}{\mathbf{e}_1'} + \lambda_2 \underset{(2\times1)}{\mathbf{e}_2}\underset{(1\times2)}{\mathbf{e}_2'} \\ &= 4\underset{(2\times1)}{\mathbf{e}_1}\underset{(1\times2)}{\mathbf{e}_1'} + \underset{(2\times1)}{\mathbf{e}_2}\underset{(1\times2)}{\mathbf{e}_2'} \end{aligned}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the normalized and orthogonal eigenvectors associated with the eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 1$, respectively. Because 4 and 1 are scalars, premultiplication and postmultiplication of $\mathbf{A}$ by $\mathbf{x}'$ and $\mathbf{x}$, respectively, where $\mathbf{x}' = [x_1, x_2]$ is any *nonzero* vector, give

$$\begin{aligned} \underset{(1\times2)}{\mathbf{x}'}\underset{(2\times2)}{\mathbf{A}}\underset{(2\times1)}{\mathbf{x}} &= 4\underset{(1\times2)}{\mathbf{x}'}\underset{(2\times1)}{\mathbf{e}_1}\underset{(1\times2)}{\mathbf{e}_1'}\underset{(2\times1)}{\mathbf{x}} + \underset{(1\times2)}{\mathbf{x}'}\underset{(2\times1)}{\mathbf{e}_2}\underset{(1\times2)}{\mathbf{e}_2'}\underset{(2\times1)}{\mathbf{x}} \\ &= 4y_1^2 + y_2^2 \geq 0 \end{aligned}$$

with

$$y_1 = \mathbf{x}'\mathbf{e}_1 = \mathbf{e}_1'\mathbf{x} \quad \text{and} \quad y_2 = \mathbf{x}'\mathbf{e}_2 = \mathbf{e}_2'\mathbf{x}$$

We now show that y_1 and y_2 are not both zero and, consequently, that $\mathbf{x}'\mathbf{A}\mathbf{x} = 4y_1^2 + y_2^2 > 0$, or $\mathbf{A}$ is *positive definite*.

From the definitions of y_1 and y_2, we have

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\underset{(2\times1)}{\mathbf{y}} = \underset{(2\times2)}{\mathbf{E}}\underset{(2\times1)}{\mathbf{x}}$$

Now $\mathbf{E}$ is an orthogonal matrix and hence has inverse $\mathbf{E}'$. Thus, $\mathbf{x} = \mathbf{E}'\mathbf{y}$. But $\mathbf{x}$ is a nonzero vector, and $\mathbf{0} \neq \mathbf{x} = \mathbf{E}'\mathbf{y}$ implies that $\mathbf{y} \neq \mathbf{0}$. ■