Matched Filtering

Matched Filtering in Astronomy

- Point source detection in surveys
- Asteroid detection
- Gravitational wave detection
- Galaxy detection
- Galaxy cluster finding
- Matched filtering approach for signal acquisition in radio-pulsar navigation
- Large scale structure in the universe
- Cluster detection in databases
- Source finding in images
- Precision localization (time, frequency, space)
    - Pulsar timing
    - Radial velocities and exoplanets
    - Astrometry

Matched Filtering and Detection

- MF gives the optimal S/N and, if required, the best point estimates for localization.
- *Localization* tasks (timing, Doppler shifts) are less concerned with the actual amplitude. Different cases include:
  - Template known: find translation parameter via least-squares or by cross correlation, paying attention to any needed interpolation. The highest precision is far easier in the Fourier domain
  - Template not known: can use a family of templates (a template bank ordered by one or more parameters) and find maximum S/N over the set.
- *Detection* is more dependent on S/N than on localization precision.
  - Detection criteria:
    - S/N > $m\sigma$ where m corresponds to some false-alarm probability if the underlying PDF is known or can be assumed to be Gaussian.
    - Want $N_{trials}P_{fa} < 1$ in an analysis with N_{trials} statistical trials.
  - Issue: If the template is not known, a large template bank increases N_{trials} significantly.
  - E.g. detection of chirped waveforms in LIGO data (NS-NS, NS-BH binaries): templates ~known from GR, but depend on unknown masses; the occurrence time is also not known so the search parameters include: t_0 = event time and the chirp mass; also the amplitude
  - One can calculate the probability of detection P_{det} for different detection criteria used on the same data set. Similarly for the null case (no signal) one can calculate the false-alarm probability for each detection scheme. A plot of P_{det} vs P_{fa} is an ROC curve.

Matched Filtering

Template too wide

Template too narrow

Template matched

Matched Filtering Methods

- What is it generically?
- Formal derivation (two domains)
- Examples
- Issues related to sampling and interpolation

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Localization in Space, Time, or Frequency

Matched Filtering

Matched filtering is an optimal method for *detecting* a signal of known shape in the presence of additive noise. It also plays a role in *estimation* of signal parameters.

We will use a time-domain signal as a prototype. Consider the model where A is deterministic and known and n is **additive, zero-mean WSS noise with arbitrary spectrum**:

$$x(t) = a_0 A(t) + n(t).$$

We want a filter that whose output maximizes the signal-to-noise ratio of the output.

Matched filtering is different from Wiener filtering, which yields an estimate for a signal that has a minimum least-squares error from the true signal.

Signals in other domains can be treated identically:

Frequency domain:	$x(\nu) = a_0 A(\nu) + n(\nu).$
Image domain:	$I(\theta) = a_0 A(\theta) + n(\theta).$
Spatial domain:	$I(\mathbf{x}) = a_0 A(\mathbf{x}) + n(\mathbf{x}).$
Arbitrary domain:	$U(\mathbf{y}) = a_0 A(\mathbf{y}) + n(\mathbf{y})$ where (e.g.) $\mathbf{y}) = (\mathbf{x}, \nu, t).$

Template

Data = scalefactor * template + white noise

Test statistic = cross correlation function of template with data

Figure 2: Matched filtering of a narrower Gaussian pulse. One realization of the template and pulse are shown while ten realizations of the CCF are shown. The SNR of the pulse is 5 (peak to rms noise). The pulse width (HW @ 1/e) is 10.3 samples.

Detection: find CCF_{max} and rms(CCF) and apply S/N threshold
→ Detection probability and false-alarm probability

Localization: find *location* of CCF maximum
General result: rms localization = width of CCF / (S/N of CCF)

ROC Curves:

A plot of P_d vs. P_{fa} is called a *receiver operating characteristics* curve, named after radar detection schemes.

Figure 5: ROC plot for a pulse with SNR 5 (peak to rms noise) and width of 10.3 samples. Matched filtering is used.

Comments and variations on the theme:

1. In simple cases, all we need to know is the time-domain shape (e.g. a pulse shape) to find the optimal filter for estimating arrival times; the spectral shape for Doppler velocities; the image shape to do astrometry; the object shape to localize spatially.
2. $A(t)$ could be stochastic but if we know its shape (somehow), the matched filter solution is the same as for the deterministic case.

   A more complicated situation might be where we use one kind of tracer to detect sources and use the results as a matched filter to detect the sources in another tracer.

   Example: Use locations of galaxies from a continuum survey as a matched filter for detecting a spectral line in the same galaxies by *stacking*.
3. Any departure of actual data from the assumed model will have consequences:

   (a) the filter is no longer optimal

   (b) detection probabilities are likely to be smaller

   (c) estimated parameters will have larger errors (random and systematic)
4. Types of departures include:

   (a) $A(t)$ not known.

   (b) a_0 not constant, e.g. $a_0(t)$.

   (c) a_0 could itself be stochastic due to multiplicative effects such as atmospheric seeing, plasmas, gain variations in instrumentation.

Matched filter:

We want the linear filter $h(t)$ that maximizes the SNR of the output and thus maximizes the detection probability. Usually we would convolve the filter with the data but it is convenient to define h so that we *cross correlate* it with the data instead. [The equivalent filter that would be convolved with the data is $h(-t)$.]

Assume initially that $h(t)$ is aligned with the signal $A(t)$ so that we only need consider the SNR of the output when h and A are aligned to give the maximum correlation. More generally we would have a signal

$$x(t) = a_0 A(t - t0) + n(t)$$

where the event time t_0 is not known and we want to estimate it.

Thus while we would generally consider the full correlation function,

$$y(\tau) = h(t) * x(t) = \int dt\, x(t) h(t + \tau) \tag{1}$$

(where '*' means correlation), here we initially consider the special time $\tau = 0$ where, by construction, we say that y is maximized if $t_0 = 0$.

In the following we use the noise statistics

$$\langle n(t) \rangle = 0 \qquad \langle n(t) n(t' \rangle = R_n(t - t').$$

The mean and variance of the output correlation y are

$$\begin{aligned}
\langle y \rangle &= a_0 \int dt\, A(t) h(t) \\
\sigma_y^2 &= \iint dt dt'\, \langle n(t) n(t') \rangle h(t) h(t') = \iint dt dt'\, R_n(t-t') h(t) h(t').
\end{aligned}$$

Define the SNR as

$$S = \frac{\langle y \rangle}{\sigma_y} = \frac{a_0 \int dt\, A(t) h(t)}{\sigma_y}.$$

We want the smoothing function $h(t)$ that maximizes S (or S^2). We can write the variational problem as the change in S^2 due to a change in the filter function, δh. We could substitute $h(t) = h_{\rm opt}(t) + \delta h(t)$ where $h_{\rm opt}$ is the optimal filter that, by definition, maximizes S. By doing so, we can collect terms and show that any departure $\delta h(t)$ satisfies:

$$\delta(S^2) = 0 = \delta\left(\frac{\langle y \rangle^2}{{\sigma_y}^2}\right) = \frac{{\sigma_y}^2 \delta(\langle y \rangle^2) - \langle y \rangle^2 \delta({\sigma_y}^2)}{{\sigma_y}^4}. \tag{2}$$

This implies

$$\begin{aligned}
2{\sigma_y}^2 \langle y \rangle \delta(\langle y \rangle) - \langle y \rangle^2 \delta({\sigma_y}^2) &= 2{\sigma_y}^2 \langle y \rangle a_0 \int dt\, A(t) \delta h(t) - \langle y \rangle^2 \iint dt dt'\, R_n(t-t') \delta\left[h(t) h(t')\right] \\
&= 2{\sigma_y}^2 \langle y \rangle a_0 \int dt\, A(t) \delta h(t) - 2\langle y \rangle^2 \iint dt dt'\, R_n(t-t') \delta h(t) h(t') \\
&= 0
\end{aligned}$$

Collecting terms,

$$\int dt\, \delta h(t) \left[\sigma_y^2 a_0 A(t) - \langle y \rangle \int dt'\, R_n(t-t') h(t') \right] = 0.$$

If the solution $h(t)$ yields a maximum SNR then the integrand factor that multiplies $\delta h(t)$ is zero for any $\delta h(t)$, so

$$\int dt'\, R_n(t-t') h(t') = \frac{a_0 \sigma_y^2}{\langle y \rangle} A(t). \tag{3}$$

To use this equation we have to know (or assume) the shape of the autocorrelation function for the noise as well as the signal shape $A(t)$.

White Noise:

This case, the simplest, has $R_n(t-t') = \sigma_n^2 W_n \delta(t-t')$ where W_n has time units to keep the overall units correct. This gives

$$\langle y \rangle = a_0 \int dt\, A(t) h(t)$$

$$\sigma_y{}^2 = \sigma_n^2 W_n \int dt\, h^2(t)$$

and the maximum S/N is given by the expression,

$$h(t) = A(t) \frac{\int dt\, h^2(t)}{\int dt\, A(t) h(t)}.$$

While appearing to be an implicit solution for $h(t)$, the integral quantities are just a proportionality constant.

A particular solution is $h(t)$ = $A(t)$, as is anything proportional to $A(t)$. It is now obvious why this approach is called matched filtering.

Interpretation:

- The optimal smoothing function is the shape of the signal when the noise is white.
- When the signal has an unknown time shift t_0, the entire cross-correlation function is calculated and its maximum found and assessed as statistically significant or not.

Demonstration that the MF solution gives the best signal to noise ratio: The S/N for an arbitrary filter is

$$S = \left(\frac{a_0}{\sigma_n}\right) \frac{\int dt\, A(t)h(t)}{\left[\iint dtdt'\, \rho_n(t-t')h(t)h(t')\right]^{1/2}}.$$

Let S_0 be the S/N for the MF solution.

Let the filter be $h(t) = A(t) + b(t)$ where $b(t)$ is an arbitrary function but we will consider b and A to be orthogonal functions, i.e. $\int dt\, A(t)b(t) = 0$. (If they weren't to begin with we could use the Gram-Schmidt orthogonalization procedure to define them as such.)[1]

Evaluate S and calculate the ratio S/S_0, which is generally

$$\begin{aligned}
\frac{S}{S_0} &= \frac{\int dt\, A(A+b)}{\left[\int dt\, A^2 \int dt\, (A+b)^2\right]^{1/2}} \\
&\underset{\text{orthogonal}}{\rightarrow} \frac{\int dt\, A^2}{\left[\int dt\, A^2 \int dt\, (A^2+b^2)\right]^{1/2}} \\
&= \left[\frac{\int dt\, A^2}{\int dt\, (A^2+b^2)}\right]^{1/2} \\
&\leq 1
\end{aligned}$$

[1]For two functions u, v, we can define two orthogonal functions as $p = u$ and $q = v - u(u,v)/(u,u) = v - u \int dt\, uv / \int dt\, u^2$.

Low-pass Noise: Non-white noise with a short correlation time (relative to the width of the signal) can be treated in the same way.

Let $R_n(t-t') = \sigma_n^2 \rho_n(t-t')$ where $\rho_n(0) = 1$ and its width $\ll$ width of A.

Returning to our original expression for S, we have

$$\begin{aligned} S &= \left(\frac{a_0}{\sigma_n}\right) \frac{\int dt\, A(t)h(t)}{\left[\iint dtdt'\, \rho_n(t-t')h(t)h(t')\right]^{1/2}} \\ &\approx \left(\frac{a_0}{\sigma_n}\right) \frac{\int dt\, A(t)h(t)}{\left[\int d\tau\, \rho_n(\tau) \int dt\, h^2(t)\right]^{1/2}} \\ &\approx \left(\frac{a_0}{\sigma_n}\right) \frac{\int dt\, A(t)h(t)}{\left[W_n \int dt\, h^2(t)\right]^{1/2}}, \end{aligned}$$

where the characteristic time scale of the noise is

$$W_n = \int d\tau\, \rho_n(\tau).$$

We can now apply the Cauchy-Schwarz inequality for the numerator

$$\left[\int dt\, A(t)h(t)\right]^2 \leq \int dt\, A^2(t) \int dt\, h^2(t)$$

which means that

$$
\begin{aligned}
S &= \left(\frac{a_0}{\sigma_n}\right) \frac{\int dt\, A(t)h(t)}{\left[W_n \int dt\, h^2(t)\right]^{1/2}} \\
&\leq \left(\frac{a_0}{\sigma_n}\right) \frac{\left[\int dt\, A^2(t) \int dt\, h^2(t)\right]^{1/2}}{\left[W_n \int dt\, h^2(t)\right]^{1/2}} \\
&\leq \left(\frac{a_0}{\sigma_n}\right) \frac{\left[\int dt\, A^2(t)\right]^{1/2}}{\sqrt{W_n}}
\end{aligned}
$$

We have equality when $h(t) = A(t)$, i.e. when the filter matches the signal.

The Cauchy-Schwartz inequality generalizes the statement that the sum of two vectors has a length that is less than the sum of the magnitudes of the two vectors:

$$
|\mathbf{x} + \mathbf{y}|^2 \leq (x + y)^2
$$

$$
x^2 + y^2 + 2\mathbf{x} \cdot \mathbf{y} \leq x^2 + y^2 + 2x\, y
$$

$$
\mathbf{x} \cdot \mathbf{y} \leq x\, y
$$

This can be written as

$$
\sum_j x_j y_j \leq \left[\sum_j x_j^2 \sum_j y_j^2\right]^{1/2} \quad \text{or in integral form} \quad \int dt\, x(t)\, y(t) \leq \left[\int dt\, x^2(t) \int dt\, y^2(t)\right]^{1/2}
$$

Simplification and Interpretation

For the matched filter case (equality) we can consider $A(t)$ to be dimensionless with $A(0) = 1$. Then the integral

$$\int dt\, A^2(t) \equiv W_A$$

defines the characteristic time scale W_A. We can also define the signal to noise of the time series as the signal maximum a_0 divided by the rms noise σ_n,

$$\mathrm{S}_t \equiv \left(\frac{a_0}{\sigma_n}\right)$$

and then rewrite the SNR of the correlation function as

$$S = \mathrm{S}_t \left(\frac{W_A}{W_n}\right)^{1/2} = \mathrm{S}_t \sqrt{N_{\mathrm{eff}}}.$$

- As we have seen in other contexts, the ratio W_A/W_n represents the number of independent noise fluctuations that have been averaged to get the correlation amplitude. We then see that the correlation function enhances the SNR in the time series by a factor $\sqrt{N_{\mathrm{eff}}}$ where N_{eff} is the effective number of independent fluctuations.
- Note that this simple result holds for WSS noise with a correlation function that is narrow compared to the length of a time series.
- For red noise with a steep power-law spectrum, longer data sets have larger variance because more of the power spectrum is sampled. For these processes, $N_{\mathrm{eff}} \sim 1$.

Arbitrary WSS Noise: Generally, the solution is gotten by Fourier transforming Equation 3. Let the noise spectrum be $R_n(\tau) \Longleftrightarrow S_n(f)$ and define the FTs of A and h as $\tilde{A}$ and $\tilde{h}$. Using 3 (repeated here),

$$\int dt'\, R_n(t-t')h(t') = \frac{a_0\sigma_y^2}{\langle y\rangle}A(t),$$

we have

$$\mathrm{FT}\left\{\int dt'\, R_n(t-t')h(t')\right\} \;=\; S_n(f)\tilde{h}(f),$$

where

$$\sigma_y^2 = \iint dtdt'\, R_n(t-t')h(t)h(t') \;=\; \int df\, S_n(f)|\tilde{h}(f)|^2$$

and

$$\langle y\rangle = a_0\int dt\, A(t)h(t) = a_0\int df\, \tilde{A}(f)\tilde{h}^*(f).$$

so the expression for h becomes

$$\tilde{h}(f) = \frac{\tilde{A}(f)}{S_n(f)}\frac{a_0\sigma_y^2}{\langle y\rangle} = \frac{\tilde{A}(f)}{S_n(f)}\frac{\int df\, S_n(f)|\tilde{h}(f)|^2}{\int df\, \tilde{A}(f)\tilde{h}^*(f)}.$$

and the solution is

$$\tilde{h}(f) = \text{constant} \times \frac{\tilde{A}(f)}{S_n(f)}.$$

This filter “whitens” the data, in effect, to yield an optimal test statistic

Comments on the general matched filtering solution

- For white noise, $S_n(f) =$ constant and we get our previous result. More generally, the matched filter favors frequencies where the ratio of signal to noise is larger.
- Equivalently the filter reduces the noise in the output y at frequencies where the noise spectrum is large and increases it where it is small.
- The amplitude of the signal a_0 does not appear in the solution. It can be considered to be part of the proportionality constant (normalization).

  Also the actual noise variance cancels in the solution for $\tilde{h}(f)$.

  Thus the amplitudes of both the signal and the noise factor out of the solution.
- The matched filter $\tilde{h}(f)$ has an inverse Fourier transform $h(t)$ that is the **optimal smoothing function**. It maximizes the S/N of the output.
  - The OSF depends on both the signal and the noise.
  - For red noise with a steep power spectrum, the low frequencies are de-weighted in the filtering compared to the high frequencies.
  - Usage of Fourier methods is still susceptible to leakage effects so **prewhitening** methods need to be used.

“Original” = original waveform, “Whitened” = waveform / (noise spectrum)

https://inspirehep.net/record/812025/files/plot40.png

Practical Applications in Signal Detection:

Example: we know the shape of the function $A(t)$ and the autocorrelation function of the noise (or its spectrum). The optimal detection scheme is to construct the filter using this information and investigate the cross correlation function Eq. 1.

Output amplitudes can be tested against a threshold y_T that is some multiple of σ_y. Over an ensemble one can then define the *detection* probability P_d and the *false-alarm* probability $P_{\rm fa}$.

By changing the threshold both P_d and $P_{\rm fa}$ will change. Ideally one would like $P_d = 1$ with $P_{\rm fa}$ but in reality there is a tradeoff between the two.

Periodic Signals: The matched filter for a periodic signal is simply a periodic train of the pulse shape. When correlated with the measurements, the output is the same thing as "folding" the data with the underlying period.

Spectra lines: Change $t \to \lambda$ or ν and the results are the same.

Source detection in images: Change $A(t) \to I(\boldsymbol{\theta})$. This might be the PSF if point sources are of interest or it might be a specific kind of source (e.g. spiral galaxies).

Multidimensional: $A(t) \to A(\mathbf{x})$ where $\mathbf{x}$ is N-dimensional.

Figure 1: Matched filtering of Gaussian pulse with itself as a template. One realization of the template and pulse are shown while ten realizations of the CCF are shown. The SNR of the pulse is 5 (peak to rms noise). The pulse width (HW @ 1/e) is 25.3 samples.

Figure 2: Matched filtering of a narrower Gaussian pulse. One realization of the template and pulse are shown while ten realizations of the CCF are shown. The SNR of the pulse is 5 (peak to rms noise). The pulse width (HW @ 1/e) is 10.3 samples.

Figure 3: Matched filtering of a narrow Gaussian pulse with a broader template. One realization of the template and pulse are shown while ten realizations of the CCF are shown. The SNR of the pulse is 5 (peak to rms noise). The pulse width (HW @ 1/e) is 10.3 samples while the template is 45.3 samples wide.

Figure 4: Matched filtering of Gaussian pulse with a narrower template. The SNR of the pulse is 5 (peak to rms noise). The pulse width (HW @ 1/e) is 10.3 samples while the template is 4.3 samples wide.

ROC Curves:

A plot of P_d vs. P_{fa} is called a *receiver operating characteristics* curve, named after radar detection schemes.

Figure 5: ROC plot for a pulse with SNR 5 (peak to rms noise) and width of 10.3 samples. Matched filtering is used.

When the Template is Unknown

Matched filtering is based on knowing the shape of the signal in the appropriate domain.

What if you don't know the shape?

1. Guess!
   (a) While the MF is optimal, a detection experiment may not depend critically on the detailed shape.
   (b) Maybe you know the shape but not the size (width, etc.). In that case, use a *family* with different shape parameters.
   (c) You may not even know the actual shape. In that case, use a *family* of default shapes with different shape parameters.
2. Perhaps you are searching for a wide variety of shapes (e.g. stars, galaxies, nebulae, filaments, etc.) in images. One goal might be that you are trying to detect and classify everything you see in an image or data cube.
   (a) You will have to use multiple families of shapes with a range of parameters for each.
   (b) What are the consequences of seeking a wider variety of shapes? (Hint: number of statistical trials.)
   (c) Think about how you would go about setting up a detection pipeline for this kind of survey.

A template bank to search for gravitational waves from inspiralling compact binaries I: physical models

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Gravitational waves from coalescing compact binaries are searched using the matched filtering technique. As the model waveform depends on a number of parameters, it is necessary to filter the data through a template bank covering the astrophysically interesting region of the parameter space. The choice of templates is defined by the maximum allowed drop in signal-to-noise ratio due to the discreteness of the template bank. In this paper we describe the template-bank algorithm that was used in the analysis of data from the Laser Interferometer Gravitational Wave Observatory (LIGO) and GEO 600 detectors to search for signals from binaries consisting of non-spinning compact objects. Using Monte-Carlo simulations, we study the efficiency of the bank and show that its performance is satisfactory for the design sensitivity curves of ground-based interferometric gravitational wave detectors GEO 600, initial LIGO, advanced LIGO and Virgo. The bank is efficient to search for various compact binaries such as binary primordial black holes, binary neutron stars, binary black holes, as well as a mixed binary consisting of a non-spinning black hole and a neutron star.

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Additive Noise in MF Applications

- Generally the matched filter depends on both the signal A and the noise n
- Specifically, the signal A is assumed to be deterministic while the noise is considered WSS with a known spectrum S_n.
  - In some applications, A is stochastic but has a well-defined ensemble average <A>
  - S_n is the *ensemble average* spectrum of the noise
- What if we do not know S_n a priori?
- How do we proceed?

Noise Spectrum in Matched Filtering Contexts

- Guess the shape of the noise spectrum
- Argue that the CLT applies:
  - Time series: measurement errors often Gaussian
  - Images:
    - Photon counts in detector >> 1: Poisson → Gaussian
    - Images from visibility functions via van Cittert-Zernike theorem: FT → sum of large N random variables
- Other cases:
  - May need to bootstrap from measured data (use the same data to estimate S_n (tricky … chicken and egg issues)
  - Ancillary data → estimate for S_n
    - E.g. Confusion noise from astrophysical sources
      - Power-law amplitude distribution
      - Spectrum = shape of resolution function (dirty beam)
  - Coupling of temporal phenomena with image formation
    - Calibration errors that confound self-calibration algorithms

MF Examples

Timing Error from Radiometer Noise

rms TOA error from template fitting with additive noise:

$$\Delta t_{\mathrm{S/N}} = \frac{\left[\int\int dt\, dt'\, \rho(t-t')U'(t)U'(t')\right]^{1/2}}{\mathrm{SNR}\int \mathrm{dt}\,[\mathrm{U}'(\mathrm{t})]^2} = \frac{W_{\mathrm{eff}}}{\mathrm{SNR}}\left(\frac{\Delta}{W_{\mathrm{eff}}}\right)^{1/2}$$

Gaussian shaped pulse:

$$\Delta t_{\mathrm{S/N}} = \frac{W}{(2\pi \ln 2)^{1/4}\mathrm{SNR}_1\sqrt{\mathrm{N}}}\left(\frac{\Delta}{W}\right)^{1/2}$$

Low-DM pulsars: DISS (and RISS) will modulate SNR

$$\boxed{\Delta t_{\mathrm{S/N}} = 0.69\mu s W_{\mathrm{ms}} N_6^{-1/2}\mathrm{SNR}_1^{-1}(\Delta/\mathrm{W})^{1/2}}$$

$N_6 = N / 10^6$

Interstellar pulse broadening, when large, increases $\Delta t_{S/N}$ in two ways:

- SNR decreases by a factor $W / [W^2+\tau_d^2]^{1/2}$
- W increases to $[W^2+\tau_d^2]^{1/2}$

→ Large errors for high DM pulsars and low-frequency observations

Timing Error from Pulse-Phase Jitter

$$U(\phi) \propto \int d\phi' \, f_\phi(\phi') a(\phi - \phi')$$

$$\Delta t_{\rm J} = N_i^{-1/2} (1 + m_I^2)^{1/2} \, P \langle \phi^2 \rangle^{1/2}$$

$$= N_i^{-1/2} (1 + m_I^2)^{1/2} \, P \left[\int d\phi \, \phi^2 f_\phi(\phi) \right]^{1/2}$$

- f_ϕ = PDF of phase variation
- $a(\phi)$ = individual pulse shape
- N_i = number of independent pulses summed
- m_I = intensity modulation index ≈ 1
- f_J = fraction jitter parameter = ϕ_{rms} / W ≈ 1

Gaussian shaped pulse:

$$\Delta t_{\rm J} = \frac{f_J W_i (1 + m_I^2)^{1/2}}{2 (2 N_i \ln 2)^{1/2}}$$

$N_6 = N_i / 10^6$

$$\boxed{\Delta t_{\rm J} = 0.28 \mu s \, W_{i,\rm ms} N_6^{-1/2} \left(\frac{f_J}{1/3} \right) \left(\frac{1 + m_I^2}{2} \right)^{1/2}}$$

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Transdet: a matched-filter based algorithm for transit detection – application to simulated COROT light curves

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Abstract. We present a matched-filter based algorithm for transit detection and its application to simulated COROT light curves. This algorithm stems from the work by Bordé, Rouan & Léger (2003, A&A 405, 1137). We describe the different steps we intend to take to discriminate between planets and stellar companions using the three photometric bands provided by COROT. These steps include the search for secondary transits, the search for ellipsoidal variability, and the study of transit chromaticity. We also discuss the performance of this approach in the context of blind tests organized inside the COROT exoplanet consortium.

1. Introduction

COROT (Baglin 2003), scheduled for launch by the end of 2006, will be the first space mission with the capability to detect extrasolar planets with sizes down to a couple of Earth radii (Rouan et al. 2000). It will observe a total of 60,000 stars in 5 runs of 150 days each. In a previous paper, we worked out a theoretical estimate of the planet detection capability of COROT (Bordé et al. 2003). In this paper, we first describe our transit detection algorithm, dubbed *Transdet*, then we report its performance on simulated light curves (LCs), and finally, we discuss the tools we are developing to discriminate between genuine exoplanets and astrophysical false positives (binary stars).

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2. Key ideas about Transdet

At low signal-to-noise (SNR), a transit signal can be approximated by a rectangular signal oscillating between a high level H (outside the transit) and a low level L (during the transit). The amplitude of this signal depends on the ratio of the planet surface to that of the star: $H - L = \varepsilon H$, where $\varepsilon \equiv (R_p/R_\star)^2$. The time spent at level L, or transit duration τ, is much shorter than the transit period T. We note t_0 the date of the beginning of the first transit falling inside the observation window (150 days for COROT).

In order to detect the transit signal in a noisy LC, we adopt the matched-filter approach that consists in correlating the LC with rectangular signals of the type described above. We call *test-signals* these rectangular signals. Each of them is characterized by a parameter triplet $(\hat{T}, \hat{\tau}, \hat{t_0})$. The correlation reaches a peak when $(\hat{T}, \hat{\tau}, \hat{t_0}) = (T, \tau, t_0)$. Practically, LCs must be first high-pass filtered to remove irrelevant low-frequency variations (stellar fluctuations) that would bias the correlation, and we take the negative of the LCs so that transits would produce positive correlation peaks.

For, this method to work, the three-dimensional space $(\hat{T}, \hat{\tau}, \hat{t_0})$ must be explored with a sufficient resolution in the given range accessible to COROT ($1 \leq \hat{T} \leq 75$ days, $1 \leq \hat{\tau} \leq 10$ hrs, $0 \leq \hat{t_0} \leq T$). The choice of this resolution results from a trade-off between computing time and detection efficiency. Typically, we use steps $(\delta\hat{T}, \delta\hat{\tau}, \delta\hat{t_0})$ small enough so that the maximum correlation signal would be no less than 75 % of its theoretical maximum value. This requirement translates into the computation of 14×10^6 correlations per LC, a task completed in 90 s with a fully IDL-coded algorithm on a Pentium M at 1.6 Ghz. However, preliminary tests indicate that an optimized IDL–C hybrid code could cut down the computing time by a factor 7–8, which would make possible a higher detection efficiency.

3. Detection threshold

3.1. Theoretical detection threshold

Practically, we use as our primary detection metric not the correlation itself, but the SNR on the correlation defined as

$$SNR_1 \equiv (k\tau)^{1/2}\, \varepsilon H/\sigma, \tag{1}$$

where σ is the standard deviation of the noise affecting the high-pass filtered LC, and k is the number of transits in the observation window.

A theoretical detection threshold can be simply set assuming: (1) a total number of LCs, (2) a false detection rate, and (3) Gaussian white noise (see Bordé 2003, p. 45). Requiring less than 1 false detection, we obtain $SNR_1 > 6.5$ for 1,000 LCs, and $SNR_1 > 7.0$ for 60,000 LCs.

3.2. Empirical detection threshold

In order to train the co-investigators of the COROT exoplanet consortium and compare the performances of existing transit detection algorithms, a first transit detection blind test (BT1) was organized inside the consortium in 2004

Plasma Propagation Effects

- Radio propagation phenomena (many)
- Dedispersion methods
- Kolmogorov-like microstructure in the ionized ISM
- Astronomical impacts:
  - Pulsar surveys
  - Precision pulsar timing to detect gravitational waves
  - Pulsars orbiting Sgr A* (GR tests, dark matter)
  - Scattering in the intergalactic medium

Matched Filtering and Deconvolution of the Interstellar Medium

- Dispersion in the ISM
  - Frequency dependent index of refraction → change in the phase of EM waves
  - δ(phase) → time delay
  - The phase itself is frequency dependent so the time delay is too
- Dispersion acts as a linear filter
- It can be removed exactly once a single parameter is known (the 'dispersion measure' DM)

Nanoshots from the Crab Pulsar

Image courtesy of NRAO/AUI and Joeri van Leeuwen (UC Berkeley) / ESO / AURA

Dispersion

$$t_{\mathrm{DM}}(\nu) = \frac{4.15 \times \mathrm{DM\ \ ms}}{\nu^2_{\mathrm{GHz}}}$$

High frequencies arrive earlier than low frequencies

DM is unique to each pulsar

DM changes with time due to motion of the pulsar and turbulence in the interstellar medium

Starting Point: Index of Refraction

- Dispersion relation in free space:
  - ω = kc
  - phase velocity = group velocity = ω/k = c
- Cold plasma (collisions not important):
  - No magnetic field:

$$\omega = kc/n_r(\omega) \qquad n_r(\omega) = \left(1 - \omega_p^2/\omega^2\right)^{1/2}$$

$$\omega_p = \text{plasma frequency} = \frac{4\pi n_e e^2}{m}$$

  - phase velocity = $c/n_r(\omega) > c$ (speed of phase fronts)
  - group velocity = $d\omega/dk = cn_r(\omega) < c$ (speed of a wave packet)
  - With magnetic field: birefringent (left, right)

$$n_{l,r} \approx 1 - \omega_p^2/2\omega^2 \mp \omega_p^2 \omega_{B_\parallel}/2\omega^3$$

  - cyclotron frequency

$$\omega_{B_\parallel} = \frac{eB\cos\theta}{m_e c}$$

Propagation through the interstellar plasma

Refractive indices for cold, magnetized plasma

$$n_{\ell,r} \sim 1 - \nu_p^2/2\nu^2 \mp \nu_p^2 \nu_{B_\parallel}/2\nu^3$$

$$\nu \gg \nu_p \sim 2\,\text{kHz} \qquad \nu \gg \nu_{B_\parallel} \sim 3\,\text{Hz}$$

Propagation velocities are frequency dependent:

Phase velocity: $v_p = \frac{\omega}{k} = \frac{c}{n_{\ell,r}}$

Group velocity: $v_g = \frac{\partial\omega}{\partial k} = \frac{\partial}{\partial k}\left(\frac{kc}{n_{\ell,r}}\right)$

Group delay = Δ(Time of Arrival)

$$\boxed{\begin{aligned} t &= t_{\text{DM}} \pm t_{\text{RM}} \\ t_{\text{DM}} &= 4.15\,\text{ms}\,\text{DM}\,\nu^{-2} \\ t_{\text{RM}} &= 0.18\,\text{ns}\,\text{RM}\,\nu^{-3} \end{aligned}}$$

birefringence

Dispersion Measure DM $= \int ds\, n_e$ units: pc cm^{-3}

Rotation Measure RM $= 0.81 \int ds\, n_e B_\parallel$ units: rad m^{-2}

Dedispersion

Two methods:

Coherent:

- operates on the voltage proportional to the electric field accepted by the antenna, feed and receiver
- computationally intensive because it requires sampling at the rate of the total bandwidth
- "exact"

Post-detection:

- operates on intensity = $|\text{voltage}|^2$
- computationally less demanding
- an approximation

Basic data unit = a dynamic spectrum

$10^6 – 10^8$ samples x 64 μs

64 to 2024 channels

Frequency

time

Fast-dump spectrometers:

- **Analog filter banks**
- **Correlators**
- **FFT (hardware)**
- **FFT (software)**
- **Polyphase filter banks**

Post-detection Dedispersion:

Sum intensity over frequency after correcting for dispersion delay

Residual time smearing:

$$\Delta t = \left[\Delta t_{DM}^2 + (1/\Delta\nu)^2\right]^{1/2}$$

$$= \left[(a\Delta\nu)^2 + (\Delta\nu)^{-2}\right]^{-1/2}$$

$\Longrightarrow$ minimum smearing time across a channel when

$$\Delta\nu = \left[8.3\ \mu s\, DM \nu^{-3}\right]^{-1/2}$$

A consequence of the uncertainty principle for Fourier transforms: $\Delta v \Delta t \sim 1$

Dispersed Pulse

Coherently dedispersed pulse

Temporal Coherence

- Temporal coherence: describes the relationship of different Fourier components that make up a signal vs. time:

Coherent: a delta function!

Incoherent: white noise

Coherent Dedispersion

pioneered by Tim Hankins (1971)

Dispersion delays in the time domain represent a phase perturbation of the electric field in the Fourier domain:

$$\vec{E}_{\mathrm{measured}}(\omega) = \vec{E}_{\mathrm{emitted}}(\omega) e^{ik(\omega)z}$$

Coherent dedispersion involves multiplication of Fourier amplitudes by the inverse function,

$$e^{-ik(\omega)z}$$

An example of matched filtering

For the non-uniform ISM, we have

$$k(\omega)z \rightarrow \int dz\, k(\omega) \propto \omega^2\, \mathrm{DM} + \mathrm{constant}$$

which is known to high precision for known pulsars.

One parameter: DM

The algorithm consists of

$$\mathrm{IFFT}\{\mathrm{FFT}\,[E_{\mathrm{measured}}(t)] \times e^{-ik(\omega)z}\} \approx E_{\mathrm{emitted}}(t)$$

Application requires very fast sampling to achieve usable bandwidths.

Micropulses coherently dedispersed
(Hankins 1971)

Backend stage of mixing: quadrature baseband mixing

Real part

Imaginary part

$$\epsilon(t) = I(t) + iQ(t) = \text{complex baseband signal}$$

voltage at intermediate frequency in heterodyned system:

$$v(t) = \mathcal{R}e\left\{\epsilon(t)e^{2\pi i f_0 t}\right\} = \text{real}$$

Coherent Dedispersion

pioneered by Tim Hankins (1971)

Coherent dedispersion works by explicit deconvolution:

$$E_{measured}(t) = E_{emitted}(t) * FFT\{e^{ik(\omega)z}\}$$
$$\Rightarrow E_{emitted}(t) \approx E_{measured}(t) * FFT\{e^{-ik(\omega)z}\}$$

<u>Comments and Caveats</u>:

- Software implementation with FFTs to accomplish deconvolution (Hankins 1971)
- Hardware implementations: real-time FIR filters (e.g. Backer et al. 1990s-present)
- Resulting time resolution: 1 / (total bandwidth)
- Requires sampling at Nyquist rate of 2 samples $\times$ bandwidth

  $\Rightarrow$Computationally demanding
- Actual time resolution often determined by <u>interstellar scattering</u> (multipath)
- **Most useful for low-DM pulsars and/or high-frequency observations**

Localization Using Matched Filtering

This handout describes localization of an object in a parameter space. For simplicity we consider localization of a pulse in time. The same formalism applies to localization of a spectral feature in frequency or to an image feature in a 2D image. The results can be extrapolated to a space of arbitrary dimensionality.

I. First consider finding the *amplitude* of a pulse when the *shape* and *location* are known.

Let the data be

$$I(t) = aA(t) + n(t),$$

where a = the unknown amplitude and $n(t)$ is zero mean noise. The known pulse shape is $A(t)$.

Let the model be $\hat{I}(t) = \hat{a}A(t)$.

Define the *cost function* to be the integrated squared error,

$$Q = \int dt \left[I(t) - \hat{I}(t) \right]^2 .$$

Taking a derivative, we can solve for the estimate of the amplitude, $\hat{a}$:

$$\begin{aligned}
\partial_{\hat{a}} Q &= -2 \int dt \left[I(t) - \hat{I}(t) \right] \partial_{\hat{a}} \hat{I}(t) = 0 \\
&\Rightarrow \int dt \left[I(t) - \hat{I}(t) \right] A(t) = 0 \\
&\Rightarrow \int dt \hat{I}(t) A(t) = \int dt I(t) A(t) \\
&\Rightarrow \hat{a} \int dt A^2(t) = \int dt I(t) A(t) \\
&\Rightarrow \hat{a} = \frac{\int dt I(t) A(t)}{\int dt A^2(t)}.
\end{aligned}$$

Note that:

a. The model is linear in the sole parameter, $\hat{a}$

b. The numerator is the zero lag of the crosscorrelation function (CCF) of $I(t)$ and $A(t)$.

c. The denominator is the zero lag of the autocorrelation function (ACF) of $A(t)$.

II. Now consider the case where we don't know the location of the pulse in time (the time of arrival, TOA) and that it is the TOA we wish to estimate. We still know the pulse shape, *a priori*.

Let the data, model and cost function be

$$
\begin{aligned}
I(t) &= aA(t - t_0) + n(t) \\
\hat{I}(t) &= \hat{a}A(t - \hat{t}_0). \\
Q &= \int dt \left[I(t) - \hat{I}(t) \right]^2 .
\end{aligned}
$$

Note that the model is **linear** in $\hat{a}$ but is **nonlinear** in $\hat{t}_0$.

Minimizing Q with respect to $\hat{a}$, we have

$$\begin{aligned}
\partial_{\hat{a}} Q &= -2 \int dt \left[I(t) - \hat{I}(t) \right] \partial_{\hat{a}} \hat{I}(t) = 0 \\
&\Rightarrow \int dt \left[I(t) - \hat{I}(t) \right] A(t - \hat{t}_0) = 0 \\
&\Rightarrow \int dt \hat{I}(t) A(t - \hat{t}_0) = \int dt I(t) A(t - \hat{t}_0) \\
&\Rightarrow \hat{a} \int dt A^2(t - \hat{t}_0) = \int dt I(t) A(t - \hat{t}_0) \qquad (4) \\
&\Rightarrow \hat{a} = \frac{\int dt I(t) A(t - \hat{t}_0)}{\int dt A^2(t - \hat{t}_0)}.
\end{aligned}$$

This last equation has the same form as in I. except that the estimate for the arrival time $\hat{t}_0$ is involved.

Now, minimizing Q with respect to $\hat{t}_0$, we have

$$\begin{aligned}
\partial_{\hat{t}_0} Q &= -2 \int dt \left[I(t) - \hat{I}(t) \right] \partial_{\hat{t}_0} \hat{I}(t) = 0 \\
&\Rightarrow -\hat{a} \int dt \underbrace{\hat{I}(t)}_{\equiv \hat{a} A(t-\hat{t}_0)} A'(t-\hat{t}_0) = -\hat{a} \int dt\, I(t) A'(t-\hat{t}_0) \\
&\Rightarrow \hat{a} \int dt\, A(t-\hat{t}_0) A'(t-\hat{t}_0) = \int dt\, I(t) A'(t-\hat{t}_0). \qquad (5)
\end{aligned}$$

Grid Search: One approach to finding the arrival time is to search over a 2D grid of $\hat{a}, \hat{t}_0$ to find the values that satisfy equations 4 and 5. This approach is inefficient. Instead, one can search over a 1D space for the single nonlinear parameter, $\hat{t}_0$, and then solve for $\hat{a}$ using either equation 4 or 5.

Linearization + Iteration: Another method is to find solutions for $\hat{a}$ and $\hat{t}_0$, we can *linearize* the equations in $\hat{t}_0 - t_0$ by using Taylor-series expansions for $A(t-\hat{t}_0)$ and $A'(t-\hat{t}_0)$.

Let $\hat{t}_0 = t_0 + \delta\hat{t}_0$. Then, to first order in $\delta\hat{t}_0$:

$$\begin{aligned}
A(t-\hat{t}_0) &\approx A(t-t_0) - A'(t-t_0)\delta\hat{t}_0 \\
A'(t-\hat{t}_0) &\approx A'(t-t_0) - A''(t-t_0)\delta\hat{t}_0 \\
A^2(t-\hat{t}_0) &\approx A^2(t-t_0) - 2A'(t-t_0)A(t-t_0)\delta\hat{t}_0.
\end{aligned}$$

Now equations (4) and (5) become

$$\hat{a}\int dt\,[A^2(t-t_0) \;-\; 2\delta\hat{t}_0 A(t-t_0)A'(t-t_0)] = \int dt\, I(t)[A(t-t_0)-\delta\hat{t}_0 A'(t-t_0)]$$

$$\hat{a}\int dt\,[A(t-t_0)A'(t-t_0) \;-\; \delta\hat{t}_0 A(t-t_0)A''(t-t_0) - \delta\hat{t}_0 A'^2(t-t_0)] = \int dt\, I(t)[A'(t-t_0)-\delta\hat{t}_0 A''(t-t_0)].$$

Consider the integral

$$\int dt\, A(t-t_0)A'(t-t_0).$$

The integrand may be written as

$$A(t-t_0)A'(t-t_0) = \frac{1}{2}\frac{d}{dt}A^2(t-t_0)$$

and so the integral equals

$$\frac{1}{2}\,A^2(t-t_0)\Big|_{t_1}^{t_2} \to 0$$

in the limit of (e.g.) $t_{1,2} = \mp T/2$ with $T \gg$ pulse width.

We then have

$$\hat{a}\int dt\, A^2(t-t_0) \;=\; \int dt\, I(t)[A(t-t_0)-\delta\hat{t}_0 A'(t-t_0)]$$

$$-\delta\hat{t}_0\,\hat{a}\int dt\,[A(t-t_0)A''(t-t_0) \;+\; A'^2(t-t_0)] =$$

$$\int dt\, I(t)[A'(t-t_0)-\delta\hat{t}_0 A''(t-t_0)].$$

Solving for $\hat{a}$ in both cases we have

$$\hat{a} \;=\; \frac{\int dt\,[I(t)A(t-t_0)-\delta\hat{t}_0 I(t)A'(t-t_0)]}{\int dt\, A^2(t-t_0)} \tag{6}$$

$$\hat{a} \;=\; \frac{\int dt\, I(t)\left[-A'(t-t_0)+\delta\hat{t}_0 A''(t-t_0)\right]}{\delta\hat{t}_0\int dt\,\left[A(t-t_0)A''(t-t_0)+A'^2(t-t_0)\right]}. \tag{7}$$

Using the notation

$$
\begin{aligned}
i_0 &\equiv \int dt\, I(t) A(t-t_0) \\
i_1 &\equiv \int dt\, I(t) A'(t-t_0) \\
i_2 &\equiv \int dt\, I(t) A^2(t-t_0) \\
i_3 &\equiv \int dt\, I(t) A''(t-t_0) \\
i_4 &\equiv \int dt\, I(t) \left[A(t-t_0) A''(t-t_0) + A'^2(t-t_0) \right].
\end{aligned}
\tag{8}
$$

we have

$$
\begin{aligned}
\hat{a} &= \frac{i_0 - \delta\hat{t}_0 i_1}{i_2} \\
\hat{a} &= \frac{-i_1 + \delta\hat{t}_0 i_3}{\delta\hat{t}_0\, i_4}.
\end{aligned}
$$

Solving for $\delta\hat{t}_0$ (to first order) we have

$$
\delta\hat{t}_0 = \frac{-i_1 i_2}{i_0 i_4 + i_2 i_3}.
$$

Iterative Solution for $\hat{t}_0$

This equation can be solved iteratively for $\delta\hat{t}_0$:

0. choose a starting value for $\hat{t}_0$.
1. calculate $\delta\hat{t}_0$ using the linearized equations.
2. is $\delta\hat{t}_0 = 0$?

3a. if yes, stop.

3b. if no, update $\hat{t}_0 \rightarrow \hat{t}_0 + \delta\hat{t}_0$ and go back to step 1.

For the best fit value for $\hat{t}_0$, the change is zero, $\delta\hat{t}_0 = 0$ (top of the hill) and $\hat{a}$ can be calculated using one of the equations 6 or 7.

Correlation Function Approach

The iterative solution for $\hat{t}_0$ is similar to the following procedure that uses a crosscorrelation approach more directly:

1. cross correlate the template $A(t)$ with $I(t)$ to get a CCF.
2. find the lag of peak correlation as an estimate for the arrival time, $\hat{t}_0 = \tau_{max}$.
3. calculate $\hat{a}$ if needed.

Subtleties of the Cross Correlation Method

The CCF is calculated using sampled data and therefore is itself a discrete quantity. Often one wants greater precision on the arrival time than is given by the sample interval. I.e. we want a

floating point number for $\hat{t}_0$, not an integer index. Therefore we want to calculate the peak of the CCF by interpolating near its peak. The interpolation should be done properly by using the appropriate interpolation formula for sampled data (using the sinc function). Using parabolic interpolation yields excessive errors for the arrival time.

In practice, the proper interpolation is effectively done in the frequency domain by calculating the *phase shift* of the Fourier transform of the CCF, which is the product of the Fourier transform of the template and the Fourier transform of the data.

Arrival Time Errors

Here we wish to localize the occurrence of a function $A(t)$. We will consider this to be a pulse whose arrival time t_0 we want to estimate, along with its expected error.

Let the signal be

$$I(t) = a_0 A(t - t_0) + n(t),$$

where a_0 is the amplitude and $n(t)$ is zero mean noise.

We will find the time of arrival (TOA) by cross correlating the presumed known pulse shape $A(t)$ with the signal:

$$C_{AI}(\tau) = \int dt\, I(t) A(t - \tau).$$

First, assume that the signal has been coarsely shifted so that the template is already aligned with the signal and that template is centered on $t = 0$. This way we can assume that the arrival time estimate is a small correction to the coarse estimate.

Slides 62 – 64 should be removed and replaced (here or elsewhere with Notes_A6523_ArrivalTimeEstimation_2017

AN OPTIMIZED ALGORITHM FOR DETERMINING PULSAR ARRIVAL TIMES

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Berkeley Workshop on Pulsar Timing, June 7–9, 1990

Binary and millisecond pulsars are thought to be relatively old objects, recycled into active status by the accretion of mass and angular momentum from an evolving companion star. They have relatively short periods and small period derivatives, and they make particularly attractive targets for precision timing experiments — partly because they are even better time-keepers than ordinary pulsars, and partly because of the special characteristics associated with their complicated evolutionary histories. My collaborators and I have recently been using accurate measurements of pulse times of arrival (TOA's) for a wide range of applications including milli-arcsecond astrometry, tests of fundamental physical laws, cosmology, and timekeeping metrology. Accumulated experience has only strengthened our motivation for achieving the highest possible accuracies in pulsar timing measurements. In this paper, I describe an algorithm used at Princeton for the optimum extraction of information from pulsar timing data.

To a reasonably good approximation, the integrated profiles for a given pulsar at a particular observing frequency are all the same except for a DC bias, a multiplicative scale factor, a shift of time origin, and additive random noise. In other words, an observed profile $p(t)$ for a given pulsar is related to the corresponding standard profile, $s(t)$, by an equation of the form

$$\boxed{p(t) = a + b\,s(t-\tau) + g(t)\,,} \tag{1}$$

where a, b, and τ are constants and $g(t)$ is a random variable representing radiometer and background noise. To measure TOA's we want to determine τ as accurately as possible in the presence of a given amount of noise. This might be done by using a "least squares" or "maximum likelihood" method to fit a scaled, shifted version of the standard profile to the observed profile, obtaining optimized estimates of the bias, scale factor, and time offset. The parameter uncertainties would also be calculated; their magnitudes would depend on the pulse shape and the signal-to-noise ratio.

In practice, observed profiles are digitized representations of the detected signals, with a well defined sampling interval and known amount of instrumental smoothing. Let's assume that $p(t)$ and $s(t)$ have been recorded at N equally spaced intervals covering the full period, say $t_j = j\Delta t$, $j = 0, 1, \ldots, N-1$. Good experiment design requires that the detected signals be low-pass filtered at a cutoff frequency $f_c \leq$

Notice that the sample interval, $\Delta t = P/N$, appears nowhere in Eqs. (3–12). Our formalism places no limit on timing accuracy expressed as a fraction of the sampling interval. In contrast, experience has shown that the time-domain methods widely in use do not readily produce arrival-time accuracies much smaller than about $0.1\Delta t$ (see, for example, Rawley 1986).

Figure 1 illustrates the results of fitting a set of 1000 artificially generated pulse profiles, with known time offsets, using both time-domain and frequency-domain algorithms. The frequency-domain uncertainties, represented by filled circles in Figure 1, scale faithfully with the signal-to-noise ratio, reaching values around $0.005\Delta t$ at signal-to-noise-ratios of several hundred. On the other hand, the time-domain solutions, in which τ was estimated by parabolic interpolation of the χ^2 minimum, achieve accuracies no better than about $0.05\Delta t$ even at very high signal-to-noise ratios.

Figure 1: Average uncertainties obtained for arrival times from simulated observations, plotted as a function of signal-to-noise ratio. Open circles represent TOA's determined by parabolic interpolation in the time domain; filled circles are based on data using the algorithm described in this paper.

m4

1. Least squares in the time domain:

- Sampled profiles, $s_j = s(t_j) = s(j\Delta t)$, $j = 0, \ldots, N-1$ $\quad (N\Delta t = \text{period})$.

- Find minimum with respect to τ of

$$\chi^2(a, b, \tau) = \sum_{j=0}^{N-1} \left(\frac{p_j - a - b s_{j-\tau}}{\sigma} \right)^2$$

- Equivalent: find maximum of CCF,

$$c(\tau) = p_j \star s_j$$

- Problem: profiles are quantized in time; cannot get analytic derivative $\partial\chi^2/\partial\tau$.

We can expand the signal as

$$I(t) \approx a_0 A(t) - a_0 t_0 A'(t) + n(t).$$

We also expand the template, but to second order in τ because we will be taking a derivative to find the lag of maximum correlation:

$$\begin{aligned} A(t-\tau) &\approx A(t) - \tau A'(t) + \frac{\tau^2}{2} A''(t) \\ A'(t-\tau) &\approx A'(t) - \tau A''(t). \end{aligned}$$

Then we can write

$$\begin{aligned} C'_{IA}(\tau) &= \frac{d}{d\tau} C_{AI}(\tau) = 0 \\ &= \int dt\, I(t) A'(t-\tau) \\ &\approx C_{IA'}(0) - \tau C_{IA''}(0) \end{aligned}$$

An estimator for τ is then

$$\tau = \frac{C_{IA'}(0)}{C_{IA''}(0)} = \frac{a_0 C_{AA'}(0) - a_0 t_0 C_{A'A'}(0) + C_{nA'}(0)}{a_0 C_{AA''} - a_0 t_0 C_{A'A''}(0) + C_{nA''}(0)}.$$

Using previous approximations we encounter the terms $C_{AA'}$ and $C_{A'A''}(0)$ that vanish for pulses that are zero at $\pm\infty$. Also the $C_{nA''}$ term in the denominator yields a second-order term that can be ignored. Then the TOA estimator becomes

$$\tau = \frac{-a_0 t_0 C_{A'A'}(0) + C_{nA'}(0)}{a_0 C_{AA''}(0)}.$$

Noiseless case:

When there is no noise we have

$$\tau = -t_0 \frac{C_{A'A'}(0)}{C_{AA''}(0)}.$$

Using a trial function, such as a Gaussian shape, it can be shown that $\tau = t_0$, as we would expect!

Even better, the denominator can be integrated by parts to show that $C_{AA''}(0) = -C_{A'A'}(0)$ for pulses that vanish at $\pm\infty$, so the equality is general.

With noise:

We can now write

$$\tau = t_0 + \frac{C_{nA'}(0)}{a_0 C_{AA''}(0)}.$$

Then the mean-square TOA error is, using $\delta\tau = \tau - t_0$,

$$\begin{aligned}
\left\langle (\delta\tau)^2 \right\rangle &= \frac{C^2_{nA'}(0)}{a_0^2 C^2_{AA''}(0)} \\
&= \frac{C^2_{nA'}(0)}{a_0^2 C^2_{A'A'}(0)} \\
&= \frac{\int\int dt dt' \, \langle n(t) n(t') \rangle A(t) A(t')}{a_0^2 C^2_{A'A'}(0)}.
\end{aligned}$$

Now assume *white noise* (for specificity) so that

$$\langle n(t)n(t')\rangle = \sigma_n^2 w_n \delta(t-t')$$

where w_n is a short characteristic time scale (such as an inverse bandwidth) to keep the units correct. Then

$$\begin{aligned}\left\langle (\delta\tau)^2 \right\rangle &= \left(\frac{\sigma_n}{a_0}\right)^2 \frac{w_n \int dt A'^2(t)}{C^2_{A'A'}(0)} \\ &= \left(\frac{\sigma_n}{a_0}\right)^2 \frac{w_n}{C_{A'A'}(0)}\end{aligned}$$

We can then write this out as a TOA error

$$\begin{aligned}\sigma_\tau &= \left(\frac{\sigma_n}{a_0}\right)\left[\frac{w_n}{\int dt\ [A'(t)]^2}\right]^{1/2} \\ &= \frac{1}{\mathrm{SNR}}\left[\frac{w_n}{\int dt\ [A'(t)]^2}\right]^{1/2}.\end{aligned}$$

We see that the error scales as the inverse of the *signal to noise ratio* (SNR). The denominator also involves the integral of the squared derivative of the pulse shape, suggesting that *sharper* pulses with larger derivatives will produce smaller arrival time errors.

Localization: Time vs. Frequency Domains

For a Nyquist sampled, bandlimited process with bandwidth B the sampling theorem implies that the continuous-time signal can be reconstructed from the sampled data as

$$x(t) = \sum_{-\infty}^{\infty} x_n \operatorname{sinc}(t - n\Delta t)/\Delta t$$

where $\Delta t = 1/B$ and, as usual, $\operatorname{sinc} x \equiv (\sin \pi x)/x$.

Consider a model appropriate for matched filtering,

$$x(t) = aA(t - t_0) + n(t),$$

where a is the amplitude, $A(t)$ is the known template, and n is noise.

Suppose we have sampled versions of the data and template, x_n, A_n.

One approach is to calculate the discrete CCF

$$C_{xA}(\ell) = \frac{1}{N} \sum_n x_n A_{n-\ell}$$

and find the maximum to determine an esimate $\hat{t}_0 = \ell_{max}$.

However, we really want the CCF of the *continuous time* quantities, $x(t)$ and $A(t)$,

$$C_{xT}(\tau) = \int dt\, x(t) A(t - \tau).$$

We cannot get the true lag of maximum correlation, $\tau_{\max}$, by interpolating the sampled correlation function unless we interpolate according to the sampling theorem. If we interpolate differently, we will get biased results.

Another approach: the frequency domain

Take the FT of the model equation to get

$$\tilde{X}(f) = a\tilde{A}(f)e^{-2\pi i f t_0} + \tilde{n}(f).$$

No noise: Write the FT in terms of its real and imaginary parts and find the phase:

$$\begin{aligned}
\tilde{X}(f) &= \tilde{X}_r(f) + i\tilde{X}_i(f) = \left|\tilde{X}(f)\right| e^{i\phi(f)} \\
\phi(f) &= \tan^{-1}\left[\frac{\tilde{X}_i(f)}{\tilde{X}_r(f)}\right] = \tan^{-1}\left[\frac{-\sin 2\pi f t_0}{\cos 2\pi f t_0}\right] = -2\pi f t_0.
\end{aligned}$$

See example.

With noise: The phase will have a noise-like error that is nonlinearly related to $\tilde{n}(f)$. In the limit of large SNR, the rms phase error will scale as 1/SNR.

Working directly with the phase to determine t_0, however, is numerically problematic because the phase will wrap for large offsets and for low SNR.

W = 2 t0 = 0 samples SNR = 50
Amplitude
1
0.5
0
−400
−200
0
200
400
Time
Amplitude
1
0.5
0
0
200
400
600
800
1000
Time (samples)
|FFT|
4
2
0
0
0.1
0.2
0.3
0.4
0.5
Frequency
Phase φ
2
0
−2
0
0.1
0.2
0.3
0.4
0.5
Frequency

W = 2 t0 = 1.990000248 samples SNR = 100000
Amplitude
1
0.5
0
−400
−200
0
200
400
Time
Amplitude
1
0.5
0
0
200
400
600
800
1000
Time (samples)
|FFT|
3
2
1
0
0
0.1
0.2
0.3
0.4
0.5
Frequency
Phase φ
2
0
−2
0
0.1
0.2
0.3
0.4
0.5
Frequency

W = 2 t_0 = 1.990000248 samples SNR = 50
Amplitude
1
0.5
0
−400
−200
0
200
400
Time
Amplitude
1
0.5
0
0
200
400
600
800
1000
Time (samples)
|FFT|
4
2
0
0
0.1
0.2
0.3
0.4
0.5
Frequency
Phase ϕ
2
0
−2
0
0.1
0.2
0.3
0.4
0.5
Frequency

W = 2 t0 = 8 samples SNR = 10
Amplitude
1
0.5
0
−400
−200
0
200
400
Time
Amplitude
1
0.5
0
0
200
400
600
800
1000
Time (samples)
|FFT|
5
0
0
0.1
0.2
0.3
0.4
0.5
Frequency
Phase φ
2
0
−2
0
0.1
0.2
0.3
0.4
0.5
Frequency

W = 2 t0 = 8 samples SNR = 50
Amplitude
1
0.5
0
−400
−200
0
200
400
Time
Amplitude
1
0.5
0
0
200
400
600
800
1000
Time (samples)
|FFT|
4
2
0
0
0.1
0.2
0.3
0.4
0.5
Frequency
Phase φ
2
0
−2
0
0.1
0.2
0.3
0.4
0.5
Frequency

W = 5 t0 = 8 samples SNR = 50
Amplitude
1
0.5
0
−400
−200
0
200
400
Time
Amplitude
1
0.5
0
0
200
400
600
800
1000
Time (samples)
|FFT|
5
0
0
0.1
0.2
0.3
0.4
0.5
Frequency
Phase φ
2
0
−2
0
0.1
0.2
0.3
0.4
0.5
Frequency

Best approach: Fit to the complex FFT rather than to the phase.

Use as a model

$$\tilde{M}(f) \;=\; \tilde{A}(f)e^{-2\pi i f \hat{t}_0}$$

Then define the product

$$\tilde{J}(f) = \tilde{X}(f)\tilde{M}^*(f) = a|\tilde{A}(f)|^2 e^{-2\pi i f(t-\hat{t}_0)} + \tilde{n}(f)\tilde{A}^*(f)e^{2\pi i f \hat{t}_0}$$

and integrate over frequency:

$$\begin{aligned} S \;&\equiv\; \int_{\Delta f} df\; \tilde{J}(f) \\ &=\; a\int_{\Delta f} df\; |\tilde{A}(f)|^2 e^{-2\pi i f(t_0-\hat{t}_0)} + \int_{\Delta f} df\; \tilde{n}(f)\tilde{A}^*(f)e^{2\pi i f \hat{t}_0}. \end{aligned}$$

This quantity can be maximized vs. $\hat{t}_0$. For no noise, we expect $\hat{t}_0 = t_0$. The maximum can be found using standard search methods over the nonlinear parameter $\hat{t}_0$ (grid search, linearization, etc.).

Note that the integrand is naturally weighted by the actual signal.

An equivalent approach is to use a different test statistic,

$$S' = \int_{\Delta f} df\; \left|\tilde{X}(f) - \tilde{M}(f)\right|^2,$$

that we would minimize to find $\hat{t}_0$.

Note that we have used continuous notation here. With sampled data we can reconstruct the continuous FT as

$$\tilde{X}(f) = \Pi(f/B) \sum_n x_n e^{-2\pi i f n \Delta t},$$

which can be implemented using the DFT.