Amenability

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Discrete Groups

Amenability is a property defined on topological groups.
We will only consider countable groups with the discrete topology, but in general locally compact groups are studied.

For the duration of the talk, let G be such a topological group.

Extension Theorems

Theorem (Hahn-Banach)

If V is a normed vector space, S a subspace of V, and $x_0 \in V$. Then there exists an $f \in V^*$ such that $f(S) = \{0\}$, $f(x_0) = 1$, and $\|f\| = \frac{1}{d(S, x_0)}$.

This is only one version of the Hahn-Banach theorem; There are several other.
We use this is to find a linear functional that annihilates a certain subspace.

Function Spaces

Definition

$$\|f\|_1 = \sum_{x \in G} f(x)$$

then $\ell^1(G) = \{f : G \to \mathbb{R} \mid \|f\|_1 < \infty\}$.

Definition

$$\|f\|_\infty = \sup_{x \in G}\{f(x)\}$$

then $\ell^\infty(G) = \{f : G \to \mathbb{R} \mid \|f\|_\infty < \infty\}$.

Definition

$\ell^\infty(G)^* = \{f : \ell^\infty(G) \to \mathbb{R}\}$ is the space of continuous linear functionals on $\ell^\infty(G)$.

The Weak* Topology

Definition (Hat map)

The map $\widehat{\cdot}: V^* \to \mathbb{R}$ defined by:

$$\widehat{x}(\varphi) = \varphi(x)$$

For $x \in V$, $\varphi \in V^*$.

Definition (Weak* topology)

Consider V^* for some normed vector space V. The weak* topology is the weakest topology on V^* such that $\widehat{x} \in V^{**}$ is continuous for all $x \in V$.

The Weak* Topology

Theorem (Alaoglu-Banach)

If V is a normed vector space, then the closed unit ball in V^* is weak-compact.*

In this case, the weak* topology is not *first-countable*. This means we must use nets rather than sequences.

Invariant Probability Measures

- It is easy to find a probability measure on any given space. However these are usually not translation-invariant.
- The Haar measure is a measure invariant under translation by the group defined on the Borel sets of a locally compact group. However in general it is not a probability measure.
- Question: Can we find a translation-invariant probability measure defined on all subsets in general?

Invariant Probability Measures

- Answer: Yes, but we must abandon countable additivity, and the group must be *Amenable*.
- Much of this work was done by Banach in the '20s and '30s.
- These have limited applications relative to their countably-additive counterparts. However the fact that they are defined on all subsets is powerful.

Invariant Means (Definitions)

Definition (Invariant Mean)

A linear functional $\lambda : \ell^\infty(G) \to \mathbb{R}$ is called a *translation-invariant mean* if it is:

1. $f \geq 0 \Rightarrow \lambda(f) \geq 0, \quad$ for all $f \in \ell^\infty(G)$
2. $\lambda(\mathbf{1}) = 1 \quad$ where $\mathbf{1} = \mathbf{1}_G \in \ell^\infty(G)$
3. $\lambda(\tau_x f) = \lambda(f) \quad$ for all $x \in G, \;\; f \in \ell^\infty(G)$

Invariant Means (Definitions)

Definition (Invariant Mean)

A linear functional $\lambda : \ell^\infty(G) \to \mathbb{R}$ is called a *translation-invariant mean* if it is:

1. $\inf\limits_{x \in G}\{f(x)\} \leq \lambda(f) \leq \sup\limits_{x \in G}\{f(x)\})$
2. $\lambda(\tau_x f) = \lambda(f) \qquad$ for all $x \in G, \;\; f \in \ell^\infty(G)$

Invariant Means (Construction)

1. Define
$S = \{f \in \ell^\infty(G) \;:\; \texttt{s.t. } f = g - \tau_x g,\; \exists x \in G, g \in \ell^\infty(G)\}$.
2. Show S is a subspace of $\ell^\infty(G)$.
3. Show that distance between S and $\mathbf{1}$ is 1.
4. Apply Hahn-Banach theorem to find a λ that is 0 on S and 1 on the constant 1 function.

Approximating with Finite Means

Definition (Finite Mean)

A function $f : G \to \mathbb{R}$ that is:

1. $f \geq 0$
2. finitely supported
3. $\|f\|_1 = 1$, so $f \in \ell^1(G)$

Theorem

A group G is amenable if for every finite set F in G and $\varepsilon > 0$, there exists a finite mean ν such that $\|\nu - \tau_x \nu\|_1 \leq \varepsilon$ for all $x \in F$.

Approximating with Finite Means

1. Recall the hat map.
2. If amenable, there exists a net of finite means $\nu_\lambda \in \ell^1(G)$, indexed by $\lambda \in \Lambda$, such that are each *almost* translation-invariant.
3. The hat map is an isometry. Thus the net of νs maps into the closed unit ball in $\ell^\infty(G)^*$.
4. By Alaoglu-Banach theorem, there is a convergent subnet, converging to some limiting function $\mu \in L^\infty(G)^*$.
5. This limiting function is translation-invariant because the finite means get arbitrarily translation-invariant. It also maps $\mathbf{1}$ to 1, since all finite means do. Finally it is also positive.

Følner Sequences

Definition (Følner Sequence)

A Følner sequence is a sequence $\{A_n\}$ of non-empty finite subsets of G such that

$$\frac{|(xA_n)\triangle A_n|}{|A_n|} \to 0 \tag{1}$$

as $n \to \infty$, for all $x \in S$.

Theorems

Theorem

All finite groups are amenable.

Theorem

All abelian groups are amenable.

Theorems II

Theorem

If G is amenable and $H \leq G$, then H is amenable.

Corollary

If H is not amenable and $H \leq G$ for some G, then G is not amenable as well.

Theorem

If $H \trianglelefteq G$ and G/H are both amenable, then G is amenable.

Theorems III

Theorem

All virtually solvable groups are amenable.

Theorem

If G_λ is amenable for all $\lambda \in \Lambda$, where Λ is a directed set, then

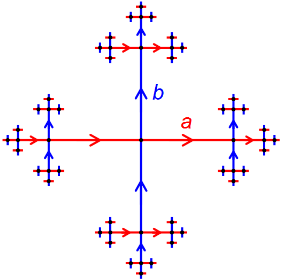
$$\bigcup_{\lambda \in \Lambda} G_\lambda$$

Is amenable.

Elementary Amenable Groups

Elementary amenable groups are all the groups which are amenable by the above theorems.

The Free Group On Two Generators



https://commons.wikimedia.org/w/index.php?curid=49361383

Definition (Free Group on Two Generators)

The group F_2 of all reduced words on the two characters $\{a, b\}$, along with the empty word ε. The group operation is concatenation (with reduction if necessary).

Proof F_2 is Not Amenable

The proof is by contradiction. Suppose λ is an invariant mean on F_2.

1. Define E_a, E_b, $E_{a^{-1}}$, $E_{b^{-1}}$.
2. Show that each of these is contained in a "larger" set.
3. Then $\lambda(\mathbf{1}_{E_a}) = 0$, ..., $\lambda(\mathbf{1}_{\{\varepsilon\}}) = 0$.
4. Thus $\lambda(\mathbf{1}) = 0$, which contradicts $\lambda(\mathbf{1}) = 1$.

Banach-Tarski Paradox

SO(3) has F_2 as a subgroup, by considering rotations by irrational angles. Thus SO(3) is not amenable. Thus we cannot find a finitely-additive measure defined on all subsets that is invariant under the action of SO(3).
Introduced by Banach and Tarski in 1938 while they worked together on invariant means.

https://commons.wikimedia.org/wiki/File:Banach-Tarski_Paradox.svg

Von Neumann's Conjecture

Proposition

All non-amenable groups have F_2 as a subgroup.

Really introduced by Mahlon Day in 1957, so it is sometimes called the von Neumann-Day problem.

Counter-examples to the Von Neumann Conjecture

The first counter-example was found
by Aleksandr Olshansky in 1980 using Tarski monster groups (*scary*).

In 2013, Lodha and Moore found a very simple non-amenable subgroup of this group. It is finitely generated by 3 homeomorphisms of the real line. It is also torsion-free.

Tits Alternative

However, in the case of linear groups, the von Neumann conjecture does hold.

Named for Jacques Tits.

Applications

- Ergodicity and dynamical systems
- Harmonic analysis
- Ramsey theorems

Conclusion

In summary, amenable groups represent a broad category of topological groups. On them we can find invariant means, which are useful linear functionals.

Interesting things also happen when groups fail to be amenable. Many pathological properties can occur, the most famous of which is the Banach-Tarski paradox.

Acknowledgements & References

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