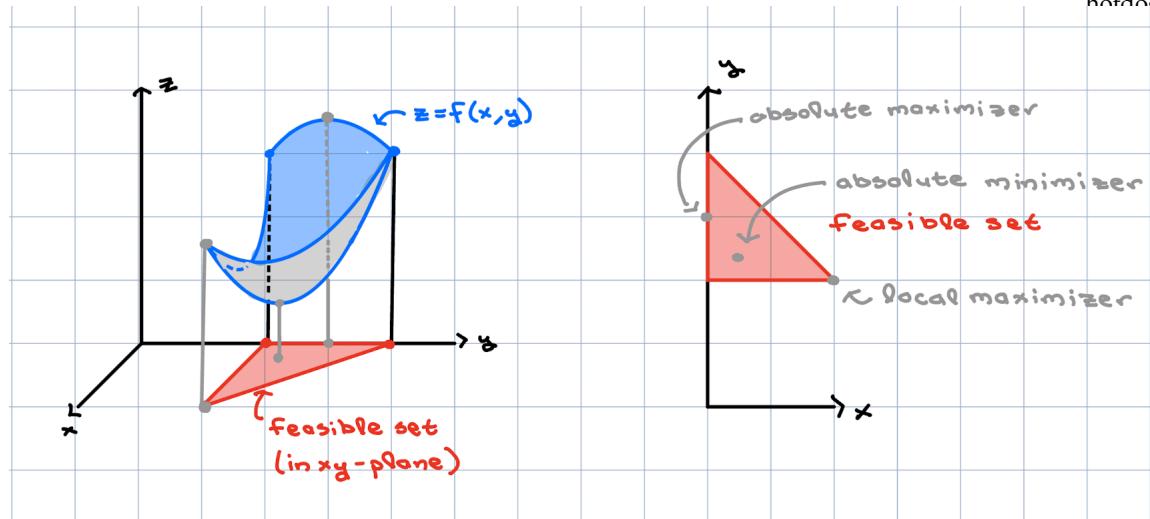


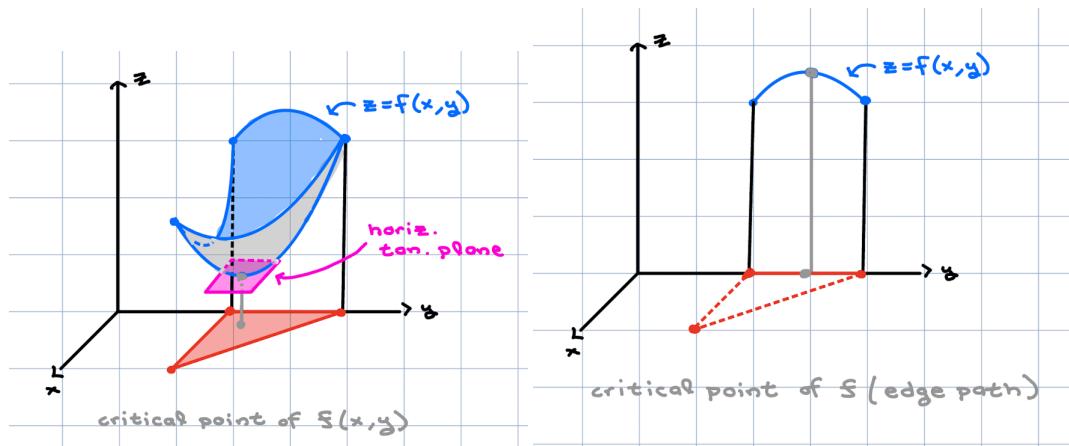
## Lecture 13. B1 – Multivariable Optimization.

**A. Global Extremums.** We would like to maximize and minimize a multivariable function  $f(x, y)$  subject to the restriction that  $(x, y)$  belong to a given **feasible set**. We graph  $z = f(x, y)$  for  $(x, y)$  in the feasible set.



On the graph we have identified **absolute extremums** and **local extremums** and in the feasible set the corresponding **absolute extremerizers** and **local extremerizers**.

Let us identify the different types of locations where we have extremerizers.



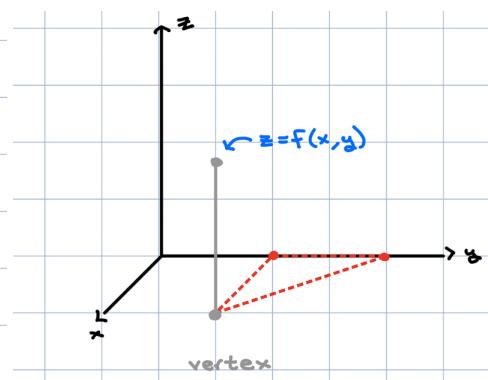
A multivariable **critical point** of a differentiable function  $f$  is an input  $P$  so:

The word feasible comes from applications, where  $x$  and  $y$  might represent something like number of hamburgers and hotdogs sold, and it may, for example, only be feasible to sell between 0 and 12 hamburgers and 10 hotdogs.

A differentiable function  $f(x, y)$  on a feasible set only has extremerizers at:

- multivariable critical points of  $f(x, y)$
- 1-variable critical points of  $f$ (edge paths)
- vertices of the feasible set

An extremum is either a maximum or minimum. An extremum is **absolute** if it is most extreme among all points in the feasible set. An extremum is **local** if it is most extreme among all **nearby** points in the feasible set. The **extremum** is the point on the graph, while the **extremerizer** is the corresponding input from the feasible set.



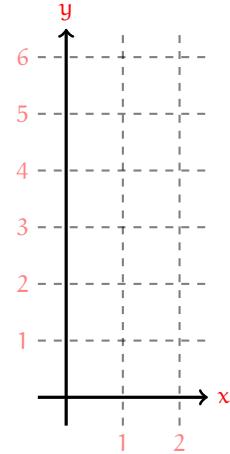
A critical point is a place where the tangent plane is horizontal, i.e. its normal vector is vertical. Earlier we found a normal vector to the tangent plane to  $z = f(x, y)$  to be  $\langle -f_x, -f_y, 1 \rangle$ . This vector will be vertical when  $f_x = f_y = 0$ , i.e. when  $\vec{\nabla}f = 0$ .

**Example 1.** Consider  $f(x, y) = x^2 - xy + y^2 - 3y$  on feasible set:

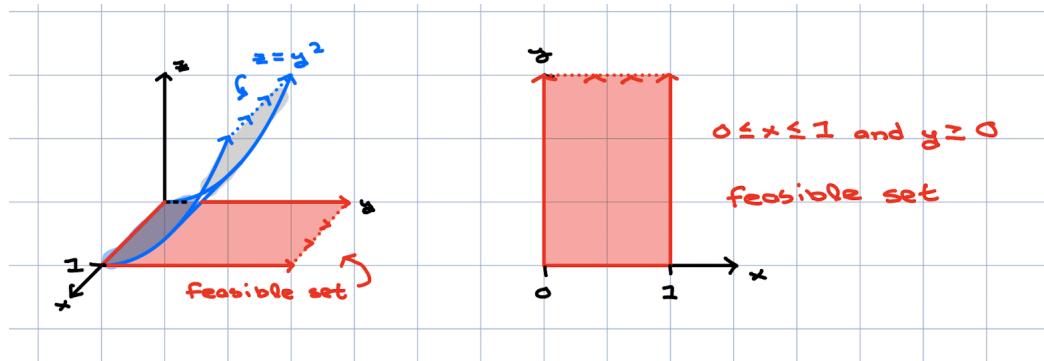
$$S = \{(x, y) \mid 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 3x\}$$

Locate all candidate extremizers of  $f(x, y)$  on the feasible set.

The notation on the left is called **set notation**. It says that  $(x, y)$  is in the feasible set if and only if  $0 \leq x \leq 2$  and  $0 \leq y \leq 3x$ .

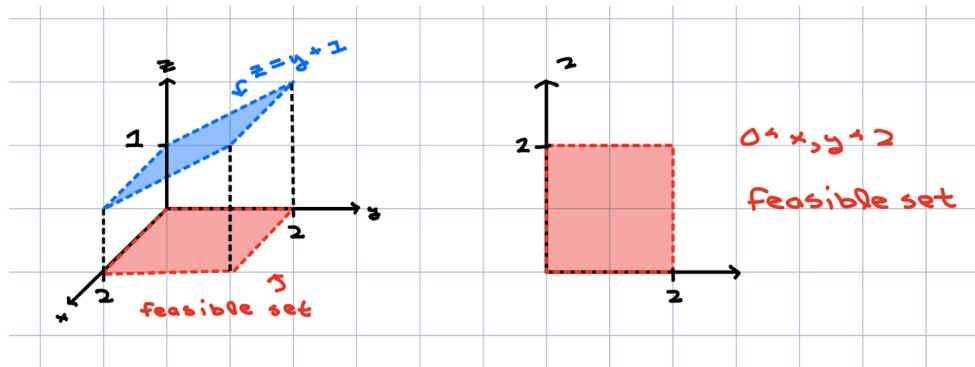


**B. Extreme Value Theorem.** Generally it is not guaranteed that a given function will achieve absolute extreme values on a given feasible set.



One reason we do not have this guarantee is because the function could fail to be continuous. In this course, however, virtually every function we encounter will be continuous where defined. We have not even discussed what it means for a multivariable function to be continuous. Suffice it to say, that functions built from continuous one-variable functions, but with extra variables and using simple algebraic operations, will be continuous where defined.

A set is **bounded** if there is a finite upper limit to how far apart points in the set can be. Otherwise, it is **unbounded**.



A straight dashed line indicates that the edge is **not** part of the set being considered

Intuitively, the **boundary** of a 2D set consists of the “fence” that contains the set. If the boundary of the set is also part of the set, then the set is said to be **closed**.

**Extreme Value Theorem.** A function that is **continuous** on a feasible set that is **closed** and **bounded** is **guaranteed** to achieve both absolute extrema on that feasible set.

The terminology for a set that is both **closed** and **bounded** is **compact**. Hence the extreme value theorem says that a continuous function on a compact feasible set is guaranteed to achieve absolute extrema.

**Example 2.** Earlier we had considered  $f(x, y) = x^2 - xy + y^2 - 3y$  on feasible set:

$$S = \{(x, y) \mid 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 3x\}$$

Is  $f$  guaranteed to have absolute extrema on  $S$ ?

