A. **Repeated Eigenvalues.**

Let A be a 2×2 matrix with repeated eigenvalue λ having:

eigenvector: $\mathbf{v}$

associated generalized eigenvector: $\mathbf{v}_g$

Show that the following solves $\mathbf{x}' = A\mathbf{x}$.

$\mathbf{x} = e^{\lambda t}(\mathbf{v}_g + t\mathbf{v})$

Remember: to be the associated generalized eigenvector meant that:

$(A - \lambda I)\mathbf{v}_g = \mathbf{v}$

In the scenario above, a fundamental set of solutions to $\mathbf{x}' = A\mathbf{x}$ is:

$\mathbf{x}_1 =$

$\mathbf{x}_2 =$

and the general solution is a linear combination of these: $\mathbf{x} = C_1\mathbf{x}_1 + C_2\mathbf{x}_2$.

Example 1. Solve the initial value problem $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$ where:

$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{x}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

B. **Variation of Parameters.**

We now extend to the **nonhomogeneous** linear scenario:

$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$

The matrix $A(t)$ does not need constant coefficients for variation of parameters.

Suppose $A(t)$ is 2×2 and that you have addressed the associated **homogeneous**:

$\mathbf{x}' = A(t)\mathbf{x}$

by locating fundamental **homogenous** solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$.

In the above scenario: the **fundamental matrix** for the system is:

$M(t) =$

In other words, the columns of the fundamental matrix are simply the fundamental set of **homogeneous** solutions. There are really many possible fundamental matrices, because there are many possible sets of fundamental solutions.

Let us compute:

$M'(t) =$

If M is a fundamental matrix of A, then: $M' =$

We next find a **particular** solution to the **inhomogenous** system of the form:

$\mathbf{x}_p = M(t)\mathbf{w}(t)$

This selection for the form of a particular solution may seem shockingly random. But when the matrix multiplication is carried out, this form really is:

$\mathbf{x}_p = w_1(t)\mathbf{x}_1(t) + w_2(t)\mathbf{x}_2(t)$

or in other words, the particular solution is a linear combination of fundamental solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ but with **variable** coefficients. This is like every other application of variation of parameters we have seen.

Variation of Parameters. A **particular** solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ has form:

$\mathbf{x}_p = M(t)\mathbf{w}(t)$

where $M(t)$ is a fundamental matrix and $\mathbf{w}(t) =$

The **general** solution is then:

$\mathbf{x} =$

Remember that $\mathbf{x}_1$, $\mathbf{x}_2$ are fundamental solutions to the associated **homogeneous** system $\mathbf{x}' = A(t)\mathbf{x}$, and the fundamental matrix $M(t)$ has these fundamental solutions as its columns.

Example 2. Use variation of parameters to find the general solution to $\mathbf{x}' = A\mathbf{x}+\mathbf{f}$ where:

$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{f} = \begin{pmatrix} -8e^{-t} \\ 0 \end{pmatrix}$

Remember the formula for the inverse. If:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where $\det A = ad - bc$.

Example 3. Use variation of parameters to find a solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ where:

$A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$ and $\mathbf{f} = \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$

When we calculate an inverse it can help to use the rule:

$(kA)^{-1} = k^{-1}A^{-1}$

if k is a scalar.

Integration by parts can be used to show:

$$\int e^{at}\cos bt\ dt = \frac{e^{at}\left(a\cos bt + b\sin bt\right)}{a^2+b^2}$$

$$\int e^{at}\sin bt\ dt = \frac{e^{at}\left(-b\cos bt + a\sin bt\right)}{a^2+b^2}$$