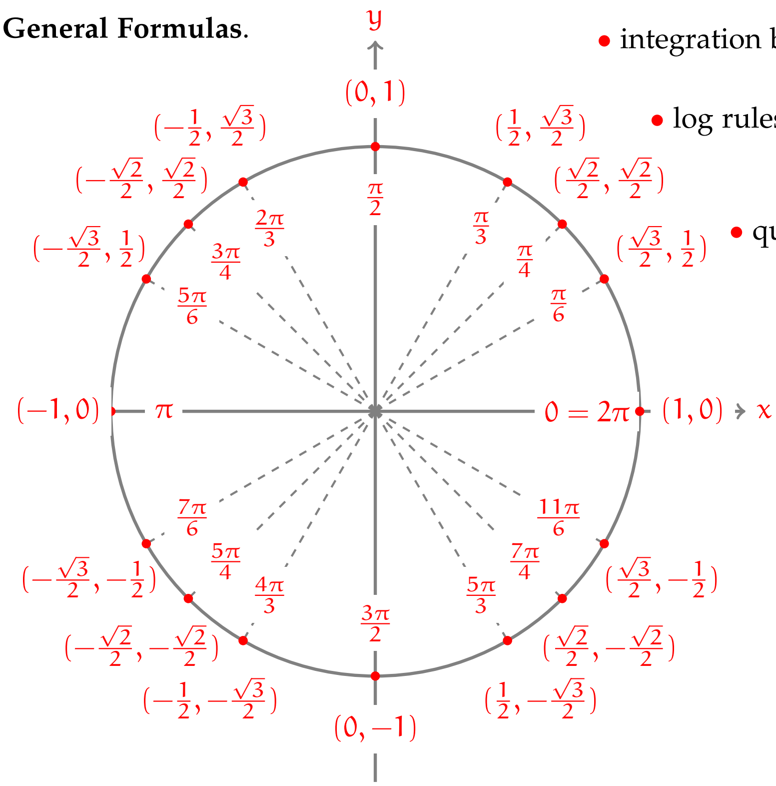
Math 245 – Formula Sheet. **Do not write on this sheet.**

General Formulas.



- integration by parts: $\int u\, dv = uv - \int v\, du$
- log rules: $\ln(A) + \ln(B) = \ln(AB)$, $\ln(A) - \ln(B) = \ln\left(\frac{A}{B}\right)$, $c\ln(A) = \ln\left(A^c\right)$
- quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- $\displaystyle\int e^{at} \cos bt\, dt = \frac{e^{at}\left(a \cos bt + b \sin bt\right)}{a^2 + b^2}$
- $\displaystyle\int e^{at} \sin bt\, dt = \frac{e^{at}\left(-b \cos bt + a \sin bt\right)}{a^2 + b^2}$
- $\cos A \cos B = \frac{1}{2}\left(\cos(A - B) + \cos(A + B)\right)$
- $\sin A \sin B = \frac{1}{2}\left(\cos(A - B) - \cos(A + B)\right)$
- $\sin A \cos B = \frac{1}{2}\left(\sin(A + B) + \sin(A + B)\right)$

A2 Formulas.

- for $y' + a(x)y = f(x)$ we have:
  - integrating factor $I(x) = e^{\int a(x)\,dx}$
  - variation of parameters: $y = uy_h$
    - where $y_h = e^{-\int a(x)\,dx}$ and $u = \int \frac{f(x)}{y_h}\, dx$
- circuits
  - Kirkhoff's Voltage Law: $E = RI + LI' + Q/C$
  - derivative of charge is current: $I = Q'$
- Bernoulli equation: $y' + a(t)y = f(t)y^n$
  - substitute $u = y^{1-n}$
  - converts to $u' + (1 - n)a(t)u = (1 - n)f(t)$

A3 Formulas.

- fundamental solutions for $y'' + py' + qy = 0$ are:
  - distinct real roots $\lambda = a, b$: $y_1 = e^{at}$, $y_2 = e^{bt}$
  - repeated root $\lambda = a$: $y_1 = e^{at}$, $y_2 = te^{at}$
  - complex roots $\lambda = a \pm bi$: $y_1 = e^{at} \cos bt$, $y_2 = e^{at} \sin bt$
- spring with no external force: $my'' + \mu y' + ky = 0$
- Wronskian: $W(y_1, y_2) = y_1 y_2' - y_1' y_2$
- Abel's formula: for $y'' + p(t)y' + q(t)y = 0$
  - $W(y_1, y_2) = Ae^{-\int p(t)dt}$ if y_1, y_2 are solutions
  - if y_1 is a solution, then Abel's formula with $A = 1$ yields:
    - so is $y_2 = uy_1$ where $u = \displaystyle\int \frac{e^{-\int p(t)dt}}{y_1^2}\, dt$

A4 Formulas.

forcing term	trial solution y_p
$[\text{deg } m \text{ poly}]e^{at}$	$t^n[\text{deg } m \text{ poly}]e^{at}$
linear combo of: $[\text{deg } m \text{ poly}]e^{at}\cos(bt)$ and $[\text{deg } m \text{ poly}]e^{at}\sin(bt)$	$t^n[\text{deg } m \text{ poly}]e^{at}\cos(bt)$ + $t^n[\text{deg } m \text{ poly}]e^{at}\sin(bt)$

note: n is the # of times $\lambda = a$ (or $\lambda = a + bi$) is a root of the characteristic equation.

A5 Formulas.

- variation of parameters for $y'' + p(t)y' + q(t)y = f(t)$:
  - $y = u_1 y_1 + u_2 y_2$ where:
    - y_1, y_2 are fundamental homogeneous solutions
    - $u_1 = -\displaystyle\int \frac{y_2 f}{W}\, dt$ and $u_2 = \displaystyle\int \frac{y_1 f}{W}\, dt$

function	Laplace transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}f(t)$	$F(s-a)$ ← shift theorem
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}[F(s)]$ ← time multiplication
$y^{(n)}$	$s^n Y - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0)$
$H(t-c)f(t-c)$	$e^{-cs}F(s)$ ← translation
$H(t-c)f(t)$	$e^{-cs}\mathcal{L}\left\{f(t+c)\right\}(s)$ ← truncation
$H(t-c)$	$\frac{e^{-cs}}{s}$
periodic $f(t)$	$\frac{\mathcal{L}(f_T)(s)}{1-e^{-Ts}}$ period T, window f_T
$f(t)\delta(t-c)$	$f(c)e^{-cs}$
$f * g$	$F \cdot G$

function	Inverse Laplace transform
$F(s)$	$f(t)$
$\frac{1}{(s-a)^n}$	$\frac{t^{n-1}e^{at}}{(n-1)!}$
$\frac{C(s-a)+D}{(s-a)^2+b^2}$	$Ce^{at}\cos bt + \frac{De^{at}\sin bt}{b}$
$e^{-cs}F(s)$	$H(t-c)f(t-c)$

B4 Formulas.

- unit impulse response $e(t)$:
  - set initial values $= 0$
  - set forcing function $= \delta(t)$

B5 Formulas.

- $(f * g)(t) = \int_0^t f(u)g(t-u)\, du$:
- $ay'' + by' + cy = f(t)$ has:
  - state-free solution: $y_s = e(t) * f(t)$
  - input-free solution:

    $y_i = ay(0)e'(t) + (ay'(0) + by(0))\, e(t)$