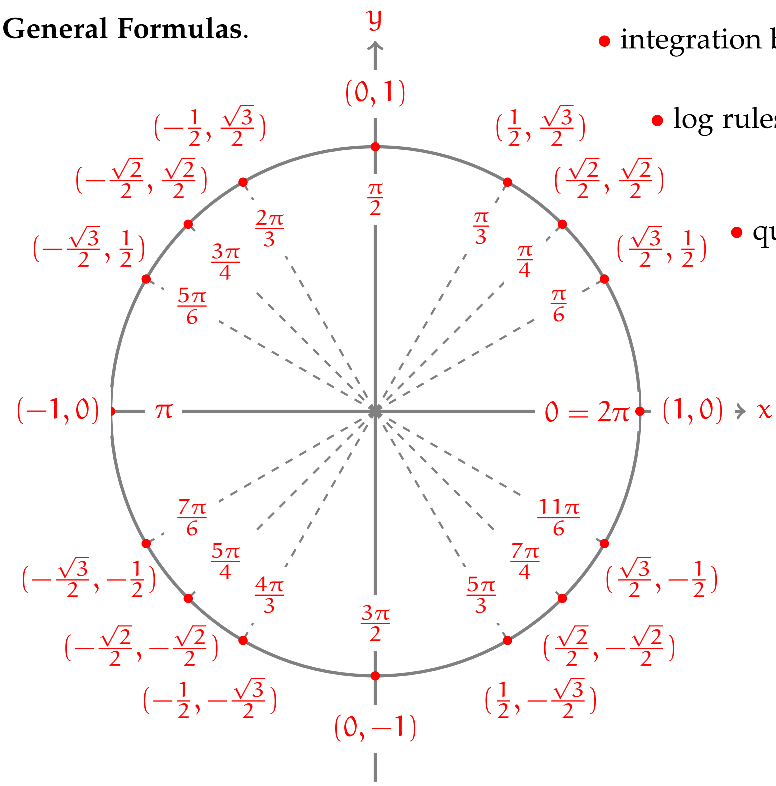
Math 245 – Formula Sheet. **Do not write on this sheet.**

General Formulas.



- integration by parts: $\int u\,dv = uv - \int v\,du$
- log rules: $\ln(A)+\ln(B)=\ln(AB)$, $\ln(A)-\ln(B)=\ln\left(\frac{A}{B}\right)$, $c\ln(A)=\ln\left(A^c\right)$
- quadratic formula: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$
- $\displaystyle\int e^{at}\cos bt\,dt=\frac{e^{at}\left(a\cos bt+b\sin bt\right)}{a^2+b^2}$
- $\displaystyle\int e^{at}\sin bt\,dt=\frac{e^{at}\left(-b\cos bt+a\sin bt\right)}{a^2+b^2}$
- $\cos A\cos B=\frac{1}{2}\left(\cos(A-B)+\cos(A+B)\right)$
- $\sin A\sin B=\frac{1}{2}\left(\cos(A-B)-\cos(A+B)\right)$
- $\sin A\cos B=\frac{1}{2}\left(\sin(A+B)+\sin(A-B)\right)$

A2 Formulas.

- for $y'+a(x)y=f(x)$ we have:
  - integrating factor $I(x)=e^{\int a(x)\,dx}$
  - variation of parameters: $y=uy_h$
    - where $y_h=e^{-\int a(x)\,dx}$ and $u=\int\frac{f(x)}{y_h}\,dx$
- circuits
  - Kirkhoff's Voltage Law: $E=RI+LI'+Q/C$
  - derivative of charge is current: $I=Q'$
- Bernoulli equation: $y'+a(t)y=f(t)y^n$
  - substitute $u=y^{1-n}$
  - converts to $u'+(1-n)a(t)u=(1-n)f(t)$

A3 Formulas.

- fundamental solutions for $y''+py'+qy=0$ are:
  - distinct real roots $\lambda=a,b$: $y_1=e^{at}$, $y_2=e^{bt}$
  - repeated root $\lambda=a$: $y_1=e^{at}$, $y_2=te^{at}$
  - complex roots $\lambda=a\pm bi$: $y_1=e^{at}\cos bt$, $y_2=e^{at}\sin bt$
- spring with no external force: $my''+\mu y'+ky=0$
- Wronskian: $W(y_1,y_2)=y_1y_2'-y_1'y_2$
- Abel's formula: for $y''+p(t)y'+q(t)y=0$
  - $W(y_1,y_2)=Ae^{-\int p(t)dt}$ if y_1,y_2 are solutions
  - if y_1 is a solution, then Abel's formula with $A=1$ yields:
    - so is $y_2=uy_1$ where $u=\displaystyle\int\frac{e^{-\int p(t)dt}}{y_1^2}\,dt$

A4 Formulas.

forcing term	trial solution y_p
[deg m poly]e^{at}	t^n[deg m poly]e^{at}
linear combo of: [deg m poly]$e^{at}\cos(bt)$ and [deg m poly]$e^{at}\sin(bt)$	t^n[deg m poly]$e^{at}\cos(bt)$ $+$ t^n[deg m poly]$e^{at}\sin(bt)$

note: n is the # of times $\lambda=a$ (or $\lambda=a+bi$) is a root of the characteristic equation.

A5 Formulas.

- variation of parameters for $y''+p(t)y'+q(t)y=f(t)$:
  - $y=u_1y_1+u_2y_2$ where:
    - y_1,y_2 are fundamental homogeneous solutions
    - $u_1=-\displaystyle\int\frac{y_2f}{W}\,dt$ and $u_2=\displaystyle\int\frac{y_1f}{W}\,dt$

function	Laplace transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\, dt$
1 t^n e^{at} $t^n e^{at}$ $\cos bt$ $\sin bt$ $e^{at}\cos bt$ $e^{at}\sin bt$	$\frac{1}{s}$ $\frac{n!}{s^{n+1}}$ $\frac{1}{s-a}$ $\frac{n!}{(s-a)^{n+1}}$ $\frac{s}{s^2+b^2}$ $\frac{b}{s^2+b^2}$ $\frac{s-a}{(s-a)^2+b^2}$ $\frac{b}{(s-a)^2+b^2}$
$e^{at}f(t)$ $t^n f(t)$ $y^{(n)}$	$F(s-a) \leftarrow$ shift theorem $(-1)^n \frac{d^n}{ds^n}[F(s)] \leftarrow$ time multiplication $s^n Y - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0)$
$H_c(t)f(t)$ $H_c(t)$	$e^{-cs}\mathcal{L}\left\{f(t+c)\right\}(s) \leftarrow$ truncation $\frac{e^{-cs}}{s}$
periodic $f(t)$	$\frac{\mathcal{L}(f_T)(s)}{1-e^{-Ts}}$ period T, window f_T
$f(t)\delta(t-c)$	$f(c)e^{-cs}$
$f * g$	$F \cdot G$

function	Inverse Laplace transform
$F(s)$	$f(t)$
$\frac{1}{(s-a)^n}$ $\frac{C(s-a)+D}{(s-a)^2+b^2}$	$\frac{t^{n-1}e^{at}}{(n-1)!}$ $Ce^{at}\cos bt + \frac{De^{at}\sin bt}{b}$
$e^{-cs}F(s)$	$H_c(t)f(t-c)$

B4 Formulas.

- unit impulse response $e(t)$:
  - set initial values $= 0$
  - set forcing function $= \delta(t)$

B5 Formulas.

- $(f * g)(t) = \int_0^t f(u)g(t-u)\, du$:
- $ay'' + by' + cy = f(t)$ has:
  - state-free solution: $y_s = e(t) * f(t)$
  - input-free solution:

    $y_i = ay(0)e'(t) + (ay'(0) + by(0))\, e(t)$

C1 Formulas.

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- characteristic polnomial is $\det(A - \lambda I)$
- eigenvector $\mathbf{v}$ with eignvalue $\lambda = c$
  - satisfies $(A - cI)\mathbf{v} = \mathbf{0}$
  - generalized eigenvector $\mathbf{v}_g$ for $\mathbf{v}$:
    - satisfies $(A - cI)\mathbf{v}_g = \mathbf{v}$

C3 Formulas.

- homogeneous system $\mathbf{x}' = A\mathbf{x}$:
  - evalue λ and evec $\mathbf{v} \implies \mathbf{x} = e^{\lambda t}\mathbf{v}$
    - generalized evec $\mathbf{v}_g \implies \mathbf{x} = e^{\lambda t}(\mathbf{v}_g + t\mathbf{v})$
  - $\lambda_1, \lambda_2 > 0 \implies$ nodal source
  - $\lambda_1, \lambda_2 < 0 \implies$ nodal sink
  - $\lambda_1 > 0, \lambda_2 < 0 \implies$ saddle
  - $\lambda = 0 \pm bi \implies$ center
  - $\lambda = a \pm bi$ and $a > 0 \implies$ spiral source
  - $\lambda = a \pm bi$ and $a < 0 \implies$ spiral sink

C4 Formulas.

- $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$
  - variation of parameters:
    - $\mathbf{x}_p = M(t)\int M(t)^{-1} f(t)\ dt$
  - undetermined coefficients:
    - if $\mathbf{f}(t) = e^{ct}\mathbf{v}$, then trial $\mathbf{x}_p = e^{ct}\mathbf{a}$, assuming $\lambda = c$ not an evalue of A

C5 Formulas.

- $e^{tA} = I + tA + \dfrac{t^2A^2}{2!} + \dfrac{t^3A^3}{3!} + \cdots$
- if A and B commute: $e^{tA+tB} = e^{tA}e^{tB}$
- if $M(t)$ is the fundamental matrix of A, then:
  - $e^{tA} = M(t)M(0)^{-1}$
- IVP $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ with $\mathbf{x}(0) = \mathbf{x}_0$ has solution:
  - $\mathbf{x} = \left[e^{tA}\mathbf{x}_0\right] + \left[e^{At}\right] * \left[f(t)\right]$