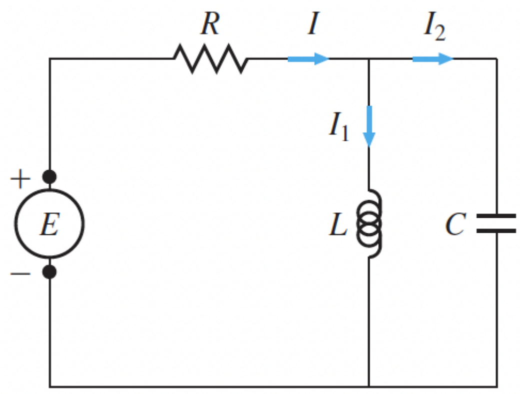
Example 1. Consider the circuit below. Assume R, L, and C are constant.



Find a differential equation in variables I_1 and I_2 that models the system.

In addition to the reminded laws on the right, you will need:

> **Kirkhoff's Current Law.**
>
> At any juncture, the current in equals the current out.

Some reminders about electrical circuits.

Kirkhoff's Voltage Law:

- directed sum of voltages around any closed loop equals 0

Impeding voltages across components:

- resistor: Ohm's Law $E_R = RI$
- capacitor: Capacitance Law $E_C = \frac{Q}{C}$
- inductor: Faraday's Law $E_L = L\frac{dI}{dt}$

Current is derivative of charge:

- $I = \frac{dQ}{dt}$

We apply Kirkhoff's voltage law to the left loop and the outer loop.

Here: Q_2 stands for the charge built up on the capacitor by curent I_2.

A. **Autonomous 2D Systems.** Consider an autonomous 2D system like:

By 2D, we mean involving two variables x and y. Autonomous meant that t does not appear.

$$\begin{cases} x' = \sin\left(\frac{\pi y}{4}\right) \\ y' = \cos\left(\frac{\pi x}{4}\right) \end{cases}$$

We can think of the solution:

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \text{ as a:}$$

in the xy–plane, which here we call the **phase plane**.

We can think of its derivative:

$$\mathbf{x}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \text{ as the:}$$

This relationship between a parametrized curve $\mathbf{x}(t)$ and its tangent vector $\mathbf{x}'(t)$ is something that you learned in Calculus III. How does Caclulus III keep sneaking in here.

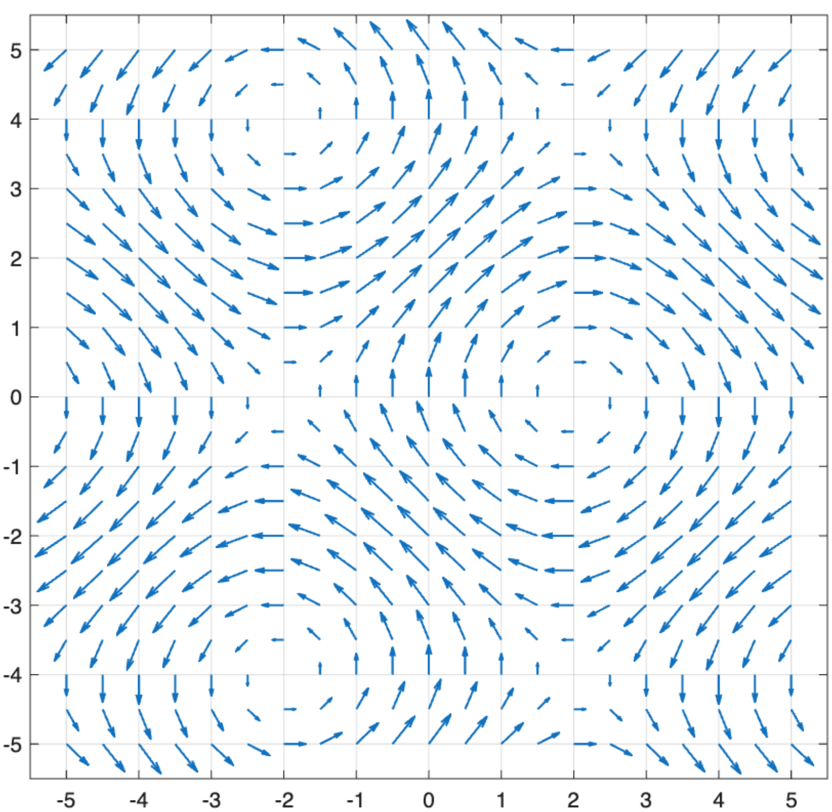
In our case: $\mathbf{x}'(t) = \mathbf{f}(x, y)$ depends only on x and y.

So: rooted at each point (x, y) we sketch tangent vector $\mathbf{x}'(t) = \mathbf{f}(x, y)$ to the solution curve constructing the **direction field** for this planar autonomous system, scaling if necessary to prevent overlap.

It is called a direction field, because each sketched vector is tangent to a solution curve, hence indicates the **direction** of the solution.

In this picture, vectors have been scaled to prevent overlap.

Below we confirm correctness of the sketch by considering a particular (x, y).



Above: sketch the solution with initial values $x(0) = -1$ and $y(0) = 0$.

B. **Homogeneous Linear Systems with Constant Coefficients.**

Consider a 1st–order homogeneous linear system with constant matrix A:

$\mathbf{x}' = A\mathbf{x}$

Suppose we have located eigenvector $\mathbf{v}$ of A with eigenvalue λ.

Let us verify that a solution to to the system is:

$\mathbf{x} = e^{\lambda t}\mathbf{v}$

An example of such a system is:

$$\begin{cases} x' = -3x + 2y \\ y' = x - 2y \end{cases}$$

Notice that the coefficients of x and y are constants, i.e. do not involve t.

The reason we might suspect that an $\mathbf{x}$ of this form is a solution is that: on the one hand, multiplication by A scales $\mathbf{v}$ by λ:

$A\mathbf{v} = \lambda\mathbf{v}$

On the other hand, differentiating $e^{\lambda t}$ also leads to scaling by λ:

$\left(e^{\lambda t}\right)' = \lambda e^{\lambda t}$

Homogoneous Linear Solution if Eigenbasis.

If A is 2×2 and has eigenbasis $\mathbf{v}_1$, $\mathbf{v}_2$, with eigenvalues λ_1, λ_2 then:

$\mathbf{x}' = A\mathbf{x}$

has **fundamental set** of solutions:

$\mathbf{x}_1 =$

$\mathbf{x}_2 =$

which means the general solution to the system has form:

$\mathbf{x} =$

A property of **homogeneous** linear systems is: any linear combination of solutions is also a solution, which is why we express general solution $\mathbf{x}$ as a linear combination of the fundamental set of solutions.

The fact that we only need 2 fundamental solutions is tied to the theory of homogeneous linear systems, and is specifically required because the size of A is 2×2.

Example 2. Find the general solution to the system:

$$\begin{cases} x' = -3x - y \\ y' = -x - 3y \end{cases}$$