A. **Chain Rule.** Consider the function:

$$z = x^2 y$$

but suppose x and y further depend on variables s and t.

$$\begin{cases} x = st \\ y = e^{st} \end{cases}$$

This gives us the following tree of dependencies:

This looks just like the tree in my backyard. I need to have it dealt with.

Goal: find $\partial z/\partial s$ and $\partial z/\partial t$ **without** first writing z in terms of s and t.

Why tie our hands in this way? Restrictions force us to be creative and think in new ways. Try imposing some restrictions in your personal life.

> **Chain Rule.** To find $\partial z/\partial \star$ you should identify each path in the tree starting at z and ending at $\star$, convert that path into a product of partial derivatives, and then add the results.

The description to the left is perhaps painfully succinct, such that it has lost all meaning. First: we can convert each edge in the tree to a partial derivative. Like the edge $z \longrightarrow x$ can be converted to $\partial z/\partial x$.

Then: the paths from z to s in our tree are:

$z \longrightarrow x \longrightarrow s$ converts to $(\partial z/\partial x) \cdot (\partial x/\partial s)$

$z \longrightarrow y \longrightarrow s$ converts to $(\partial z/\partial y) \cdot (\partial y/\partial s)$

$\partial z/\partial s$ is the sum of the terms on the right.

So:

$$\frac{\partial z}{\partial s} =$$

Why does the chain rule work? Well it helps to think of partial derivatives as rates of change. If s increases by 1 unit, then x is increased $(\partial x/\partial s)$ times (1) units. And if x is increased by $\partial x/\partial s$ units, then z is increased by $(\partial z/\partial x)$ times $(\partial x/\partial s)$ units. This is where the $z \longrightarrow x \longrightarrow s$ part of the chain rule comes from.

$$\frac{\partial z}{\partial t} =$$

Example 1. Let $z = f(x, y)$ be a differentiable function and let:

$$\begin{cases} x = s + t \\ y = s - t \end{cases}$$

Show that:

$$\frac{\partial z}{\partial s}\frac{\partial z}{\partial t} = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

Note here that we are not told what $f(x, y)$ is, beyond the fact that it is differentiable. We need to verify that the given equality is true, independent of what $f(x, y)$ is.

B. **Chain Rule and Gradient.** We will view the chain rule a different way. Consider:

$$f(x, y, z) = xe^{y/z}$$

but the variables x, y, and z further depend on variable s and t.

$$\begin{cases} x = 2s + t^2 \\ y = s - t \\ z = s + 2t \end{cases}$$

which we instead think about as a vector–valued function:

$$\mathbf{r}(s, t) =$$

Vector–valued means the outputs are vectors instead of scalars. Paths $\mathbf{r}(t)$ are examples of vector–valued functions.

We now write the following using dot products:

$$\frac{\partial f}{\partial t} =$$

The **gradient** of $f(x, y, z)$ is:

$$\nabla f(x, y, z) =$$

The chain rule says that:

$$\frac{\partial}{\partial t_i}\Big[f(\mathbf{r}(\mathbf{t})\Big] =$$

Here $\mathbf{t} = \langle t_1, \ldots, t_n \rangle$ represents a vector of variables.

Example 2. Let $f(x, y) = \cos x \ln y$ and let $\mathbf{r}(t)$ be a differentiable path so:

$$\mathbf{r}(0) = \langle 0, 1 \rangle$$

$$\mathbf{r}'(0) = \langle -1, 1 \rangle$$

Let $g(t) = f(\mathbf{r}(t))$ and find $g'(0)$.

Remember: a path is an example of a vector–valued function.