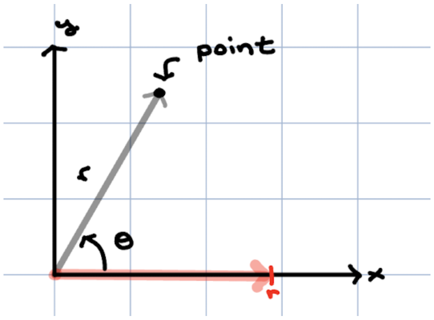
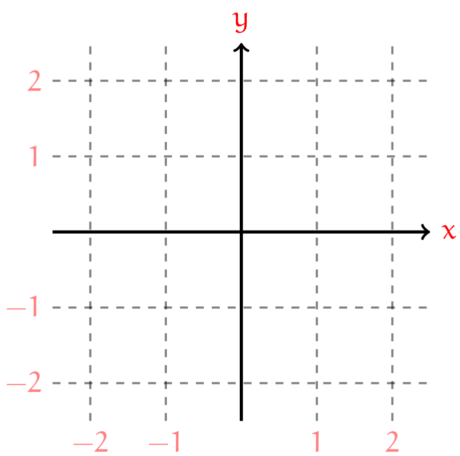
A. **Polar Coordinates.** The **standard** coordinates x and y are not always the most convenient coordinates to describe a particular region in the xy–plane. A useful alternative is captured by **polar coordinates**.



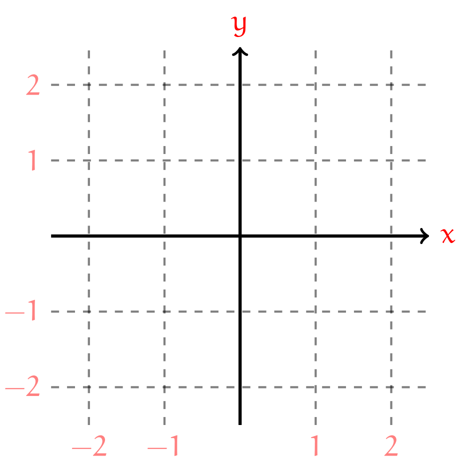
When r is positive, r equals the distance of the point from the origin, and θ measures the counterclockwise angle of the position vector of the point from the positive x–axis.

The point with polar coordinates r and θ has position vector obtained by rotating the vector $\langle r, 0\rangle$ counterclockwise by angle θ.

Sketch the point with polar coordinates $r = -2$ and $\theta = \frac{\pi}{2}$.

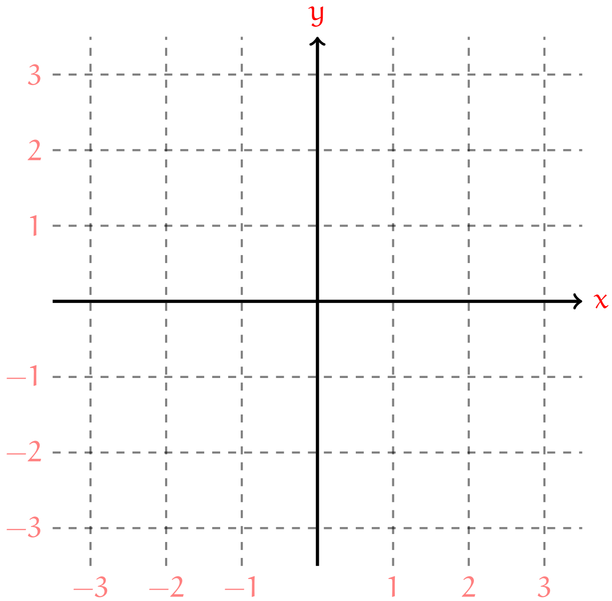


Assuming r is nonnegative, the level sets $\theta = A$ and $r = C$ look as follows.

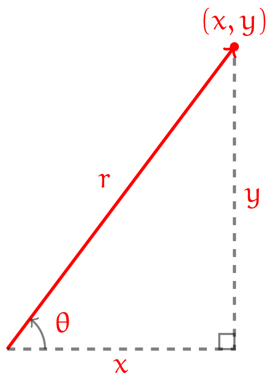


The level set $\theta = A$ is the ray emanating from the origin obtained by rotating the positive x–axis counterlockwise by θ. The level set $r = C$ is the circle of radius C centered at the origin.

Example 1. Sketch the region determined by $2 \leqslant r \leqslant 3$ and $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$.



B. **Polar Conversion.** We convert between standard and polar coordinates.



By standard coordinates are the x- and y-coordinates.

Key Relations Between Polar and Standard Coordinates.

x in terms of polar:

y in terms of polar:

relation bewtween r and standard:

relation between θ and standard:

When r is nonnegative we may write:

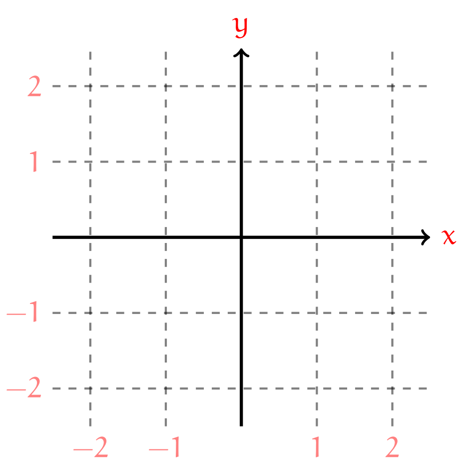
$r = \sqrt{x^2 + y^2}$

The relation between θ and standard is more often written:

$\tan\theta = \frac{y}{x}$

but the relation presented here is more symmetric, and also applies even when tangent is undefined.

Example 2. Graph $r = 2\cos\theta$ by first converting to standard coordinates.

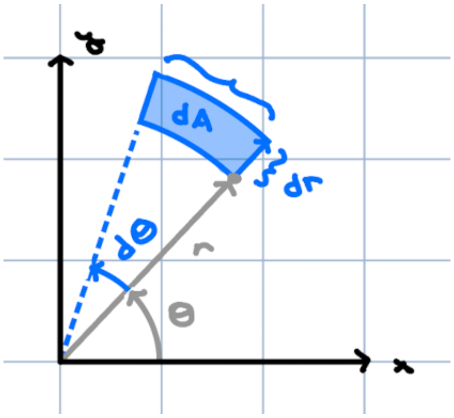


The right side is quite nearly the expression for $2x$ in terms of polar, however the r is missing in front of $\cos\theta$. Our first step is introducing the missing r.

The next step will involve completing the square:

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

C. **Polar Integration.** Now we will discuss polar integration. Imagine the infinitesimal bit of area dA obtained from the point with polar coordinates r and θ by increasing θ by $d\theta$ and r by dr.



The goal in this process is to try to understand how small changes dr and $d\theta$ are related to small changes dA in area. This dA is exactly what appears in double–integration.

For example, we know that small changes dx and dy effect the change $dA = dxdy$ in area, essentially because the picture we would sketch in this case would be a rectangle with sides dx and dy.

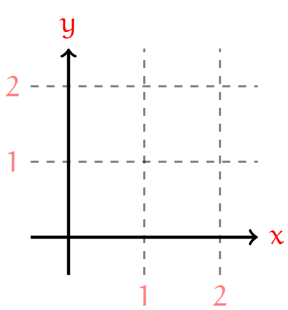
It is a standard fact from geometry that the arc of a circle of radius r traced out by a positive angle θ, measured in radians, has length equal to $r \cdot \theta$.

In polar coordinates:

$dA =$

Example 3. Let D be the region in the first quadrant bounded by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

The first quadrant is defined by $x \geqslant 0$ and $y \geqslant 0$.



and find: $\iint_D e^{x^2+y^2}\, dA$