Math 226 Final Exam Spring 2025

Name: ________________________ **Student ID:** ____________________

Directions. Fill out your name and student ID number on the lines above **right now** before starting the exam! Also, check the box next to the class for which you are registered.

- ☐ 9am Haskell
- ☐ 10am Haskell
- ☐ 11am Tokorcheck
- ☐ 12pm Tokorcheck
- ☐ 2pm Reardon

- **You must show all your work and justify your methods to obtain full credit.** Do not use scratch paper; if more space is needed, use the extra page provided on the back of the test. If you write on this page, let the grader know that there is work to be found there by writing the page number where it says "MY SOLUTION CONTINUES ON PAGE _______"
- Do not write outside the margins.
- Simplify your answers to a reasonable degree. Any fraction should be written in lowest terms. Known trig identities should be simplified. You need not evaluate expressions such as $\ln 5$, $e^{0.7}$ or $\sqrt{226}$.
- No calculators are allowed. **Turn off your cell phone**.
- You may use the sheet of notes that you brought with you, this may be no more than one sheet of $8\frac{1}{2} \times 11$" paper. You may have anything written on it (on both sides), but it must be written in your own handwriting.
- Remember, USC considers cheating to be a serious offense; the minimum penalty is failure for the course. Cheating includes "straying eyes" and failing to stop writing when told to do so at the end of the exam.

1 (14 pts)	2 (12 pts)
3 (10 pts)	4 (10 pts)
5 (10 pts)	6 (10 pts)
7 (10 pts)	8 (12 pts)
9 (20 pts)	10 (12 pts)

120 points total

Question 1. Consider the four surfaces

$$\mathcal{S}_1: \quad 3x+4y-5z=1 \qquad \mathcal{S}_2: \quad x-y+5z=2$$
$$\mathcal{S}_3: \quad z=\sqrt{x^2+y^2} \qquad \mathcal{S}_4: \quad (x-1)^2+y^2=1$$

(a) Indicate if each statement is True or False. Briefly justify your answer.

True ☒ False ☐ (A) The surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ intersect at a line.
Justification:

they are non-parallel planes

True ☐ False ☒ (B) The surface $\mathcal{S}_3$ is an upward paraboloid centered at the origin.
Justification:

$\mathcal{S}_3: z=r \rightarrow$ a cone!

True ☒ False ☐ (C) The surface $\mathcal{S}_4$ is a cylinder of radius 1.
Justification:

$x^2+y^2=1 \xrightarrow[\text{+1 in } x]{\text{translate}} (x-1)^2+y^2=1$

cyl. of radius 1 — still a cyl. of radius 1

True ☐ False ☒ (D) The surface $\mathcal{S}_2$ is tangent to $\mathcal{S}_4$.
Justification:

$\mathcal{S}_4 \rightarrow$ vertical cylinder

$\rightarrow$ horizontal normals : $\vec{n}=\langle *,*,0\rangle$

$\mathcal{S}_2 \rightarrow$ normal : $\langle 1,1,-5\rangle$

True ☒ False ☐ (E) The surface $\mathcal{S}_1$ contains the line $x=2t-1$, $y=t+1$, $z=2t$
Justification: plug in:

$3(2t-1)+4(t+1)-5(2t)=1$

$1=1$

true!

(b) Find a parameterization $\mathbf{r}(t)$ for the curve of intersection of $\mathcal{S}_3$ and $\mathcal{S}_4$. **Be sure to include a range for t.**

$\mathcal{S}_4: (x-1)^2+y^2=1 \xrightarrow{\text{translated polar}} x=1+\cos t,\ y=\sin t,\ 0\le t\le 2\pi$

$\mathcal{S}_2: x-y+5z=2 \longrightarrow z=\frac{2}{5}-\frac{1}{5}x+\frac{1}{5}y$

$\longrightarrow z=\frac{2}{5}-\frac{1}{5}(1+\cos t)+\frac{1}{5}(\sin t)$

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Question 2. Let $f(x,y) = 2y + \cos \pi x - \sqrt{x}$. Consider the point $P = (4,-1)$.

(a) In which direction does f increase the fastest starting from P?

$$\vec{\nabla} f = \left\langle -\pi \sin \pi x - \frac{1}{2\sqrt{x}} \, , 2 \right\rangle$$

$$\vec{\nabla} f(4,-1) = \left\langle -\frac{1}{4} \, , 2 \right\rangle$$

make vec have whole # entries

$$\text{unit dirn } \vec{u} = \frac{\vec{\nabla} f(4,-1)}{\|\vec{\nabla} f(4,-1)\|} = \frac{4\,\vec{\nabla} f(4,-1)}{\|4\vec{\nabla} f(4,-1)\|} = \frac{\langle -1, 8 \rangle}{\sqrt{68}}$$

(b) Find the rate of change of f at P in the direction of $\underbrace{\frac{4}{5}\hat{\mathbf{i}} + \frac{3}{5}\hat{\mathbf{j}}}_{\vec{u}}$.

$$D_{\vec{u}} f(4,-1) = \vec{\nabla} f(4,-1) \cdot \vec{u}$$
$$= \left\langle -\frac{1}{4}, 2 \right\rangle \cdot \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$$
$$= -\frac{1}{5} + \frac{6}{5}$$
$$= 1$$

(c) Find an equation of the tangent plane of f at P.

$$z = f(4,-1) + \vec{\nabla} f(4,-1) \cdot \langle x-4, y+1 \rangle$$

$$z = -2 + 1 - 2 - \frac{1}{4}(x-4) + 2(y+1)$$

$$z = -3 - \frac{1}{4}(x-4) + 2(y+1)$$

(d) Use linear approximation to estimate the value of f at $Q = (4.4, -1.1)$.

$$f(4.4,-1.1) \approx -3 - \frac{1}{4}(4.4-4) + 2(-1.1+1)$$
$$\approx -3 - 0.1 - 0.2$$
$$\approx -3.3$$

(e) Suppose also that $\underbrace{x = -4s\cos t \text{ and } y = se^{2t}}_{\vec{x}}$. Find $\dfrac{\partial f}{\partial s}$ at $(s,t) = (-1,0)$.

$$\frac{\partial f}{\partial s} = \vec{\nabla} f(x,y) \cdot \frac{\partial \vec{x}}{\partial s}$$ (tree method also fine)

$$= \vec{\nabla} f(-4s\cos t, se^{2t}) \cdot \langle -4\cos t, e^{2t} \rangle$$

next, plug in $s = -1, t = 0$:

$$= \vec{\nabla} f(4,-1) \cdot \langle -4, 1 \rangle$$
$$= \left\langle -\frac{1}{4}, 2 \right\rangle \cdot \langle -4, 1 \rangle$$
$$= 3$$

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Question 3. Find the critical points of $f(x, y)$ and classify each one as a local maximum, local minimum, or saddle point where

$$f(x,y) = xy + \frac{16}{x} + \frac{4}{y}, \qquad x > 0,\ y > 0$$

I. $f_x = y - \frac{16}{x^2} \overset{\text{set}}{=} 0 \longrightarrow y = \frac{16}{x^2}$

II. $f_y = x - \frac{4}{y^2} \overset{\text{set}}{=} 0$

plug I into II: $x - \frac{4}{\left(\frac{16}{x^2}\right)^2} = 0$

$\left[x - \frac{x^4}{64} = 0 \right] \div x$ note, told $x > 0$

$1 - \frac{x^3}{64} = 0$

$64 = x^3$

III. $4 = x$

plug into I. $y = \frac{16}{4^2} = 1$

crit pt: $(4, 1)$

Hessian: $Hf(4,1) = \begin{pmatrix} \frac{32}{4^3} & 1 \\ 1 & \frac{8}{1^3} \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & 8 \end{pmatrix}$

det: $4 - 1 = 3 \longrightarrow +$

trace: $8 + \frac{1}{2} = \frac{17}{2} \longrightarrow +$

conclusion: $(4,1)$ is a local minimizer

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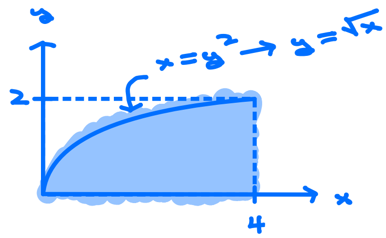
Question 4. Find $\int_0^2 \int_{y^2}^4 \frac{ye^x}{x} dx dy$



change order:

$$\begin{cases} 0 \le y \le \sqrt{x} \\ 0 \le x \le 4 \end{cases}$$

$$\text{integral} = \int_0^4 \int_0^{\sqrt{x}} \frac{ye^x}{x} \, dy \, dx$$

$$= \int_0^4 \frac{1}{2} \frac{(\sqrt{x})^2 e^x}{x} \, dx$$

$$= \frac{1}{2} \int_0^4 e^x \, dx$$

$$= \frac{1}{2} \left[e^4 - 1 \right]$$

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Question 5. Evaluate the following integral over the region D bounded by the lines $x+y=2$, $x+y=-1$, $x-y=1$, and $x-y=-1$:

$$\iint_D 3(x+y)e^{(x-y)}\,dx\,dy.$$

not relevant to our exam

optional topic: change of variables

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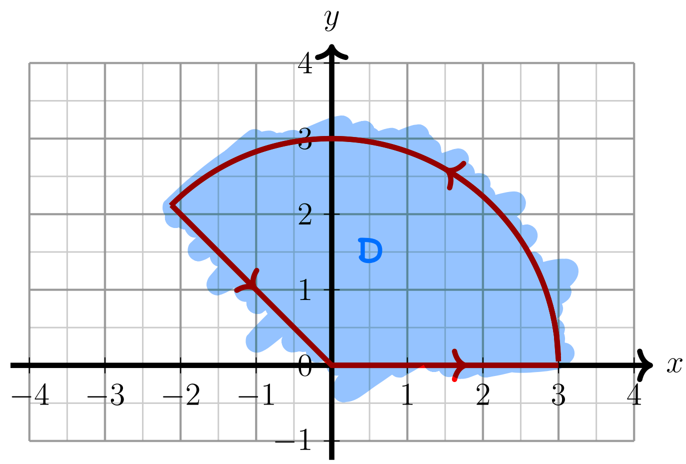
Question 6. Compute

$$\int_C \underbrace{\boxed{(y^2 + xe^x)}}_{P}\, dx + \underbrace{\boxed{(x^2y + 2xy + \sin(y^2))}}_{Q}\, dy$$

where C is the counterclockwise oriented piecewise curve comprised of $y = \sqrt{9 - x^2}$, $y = -x$, and $y = 0$ as shown in the graph below.



Green's: integral $= \iint_D Q_x - P_y \, dA$

$= \iint_D 2xy + 2y - 2y \, dA$

$= \iint_D 2xy \, dA$

polar

$D: 0 \le r \le 3,\ 0 \le \theta \le \frac{3\pi}{4}$

$= \int_0^{\frac{3\pi}{4}} \int_0^3 2r\cos\theta \, r \sin\theta \cdot r \, dr \, d\theta$

$= \left[\int_0^{\frac{3\pi}{4}} 2\cos\theta \sin\theta \, d\theta\right]\left[\int_0^3 r^3 \, dr\right]$

$= \sin^2\left(\frac{3\pi}{4}\right) \cdot \frac{81}{4}$

$= \frac{1}{2} \cdot \frac{81}{4} = \frac{81}{8}$

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Question 7. Consider the vector field $\mathbf{F}$ and curve C given by

$$\mathbf{F}(x,y,z) = \langle e^x,\, -\cos z,\, y\sin z\rangle$$
$$\mathbf{r}(t) = \langle \sin t,\, t\cos t,\, t\rangle, \quad 0 \le t \le 4\pi.$$

(a) Is the vector field $\mathbf{F}$ conservative? If so, find a potential function f such that $\mathbf{F} = \nabla f$. If not, explain why it is not.

- domain of $\vec{F}$ is $\mathbb{R}^3$, which is simply connected
- curl $\vec{F} = \langle \sin z - \sin z, 0, 0\rangle = \vec{0}$, so $\vec{F}$ is irrotational

so: $\vec{F}$ is conservative

find a potential:

I. $f_x = e^x$
II. $f_y = -\cos z$
III. $f_z = y\sin z$

guess and check: $f = e^x - y\cos z$ → I ✓ II ✓ III ✓

(b) Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

fundamental thm of line ints:

integral $= f(\vec{r}(4\pi)) - f(\vec{r}(0))$

$= f(0, 4\pi, 4\pi) - f(0,0,0)$

$= [1 - 4\pi] - [1]$

$= -4\pi$

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Question 8. Consider the vector field

$$\mathbf{F}(x, y, z) = \langle e^z,\ x + z,\ e^{xy+z} \rangle$$

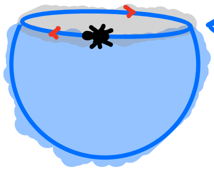
and the goldfish bowl surface S given by

sphere shifted down 1 unit

$$x^2 + y^2 + (z+1)^2 = 2 \quad \text{and} \quad z \leq 0$$

oriented so that the normal points from inside the bowl to outside the bowl. Calculate

$$\iint_S \text{Curl } \mathbf{F} \cdot d\mathbf{S}.$$



$\leftarrow \partial S: z = 0 \rightarrow x^2 + y^2 + (0-1)^2 = 1$

$\rightarrow x^2 + y^2 = 1$

$\leftarrow S$: outwards normals

(on visible blue face $\rightarrow$ normals point at u!)

based on pic, for Stokes, ∂S oriented clockwise, viewed from above

Stokes $\rightarrow$ integral $= \oint_{\partial S} \vec{F} \cdot d\vec{s}$

$\partial S: \vec{r}(\theta) = \langle \cos\theta, \sin\theta, 0 \rangle, \ 0 \leq \theta \leq 2\pi$

$d\vec{r} = \underset{\text{clockwise}}{-} \langle -\sin\theta, \cos\theta, 0 \rangle d\theta$

$\vec{F}(\theta) = \langle e^0, \cos\theta + 0, * \rangle = \langle 1, \cos\theta, * \rangle$

$$= \int_0^{2\pi} \langle 1, \cos\theta, * \rangle \cdot \langle \sin\theta, -\cos\theta, 0 \rangle \, d\theta$$

$$= \int_0^{2\pi} \sin\theta - \cos^2\theta$$

$$= \int_0^{2\pi} \cancelto{0}{\sin\theta} - \frac{1}{2} - \frac{1}{2}\cancelto{0}{\cos 2\theta} \, d\theta$$

$$= -\pi$$

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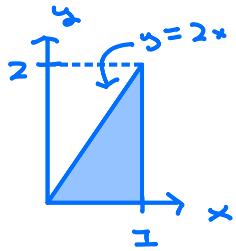
Question 9. Consider the surface S defined by the equation

$$z = 1 + x^2 + 2y$$

and where (x, y) lies in the triangle in the plane with vertices $(0,0)$, $(1,0)$ and $(1,2)$.

(a) Find the surface area of S.

xy-shadow:



$$\begin{cases} 0 \le x \le 1 \\ 0 \le y \le 2x \end{cases}$$

call this $g(x,y)$

$S: \; z = \boxed{1 + x^2 + 2y}$, (x,y) in shadow

shortcut: $dS = \sqrt{g_x^2 + g_y^2 + 1}\; dy\,dx$

$= \sqrt{4x^2 + 4 + 1}\; dy\,dx$

$= \sqrt{4x^2 + 5}\; dy\,dx$

area $= \iint_S dS$

$= \int_0^1 \int_0^{2x} \sqrt{4x^2+5}\; dy\,dx$

$= \int_0^1 2x\sqrt{4x^2+5}\; dx$

$$\begin{cases} u = 4x^2 + 5 \\ du = 8x\,dx \to \frac{1}{4}du = 2x\,dx \\ x = 0 \to u = 5 \\ x = 1 \to u = 9 \end{cases}$$

$= \frac{1}{4}\int_5^9 \sqrt{u}\; du$

$= \frac{1}{4}\cdot\frac{2}{3}\left[9^{3/2} - 5^{3/2}\right] = \frac{1}{6}\left[27 - 5\sqrt{5}\right]$

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Question 9. (CONTINUED) Recall the surface S from (a) defined by the equation

$$z = \boxed{1 + x^2 + 2y} \leftarrow \text{recall: } f(x,y)$$

and where (x, y) lies in the triangle in the plane with vertices $(0, 0)$, $(1, 0)$ and $(1, 2)$.

(b) Consider the vector field $\mathbf{F}(x, y, z) = \langle x,\ 1,\ x + 2z\rangle$. Find the flux of $\mathbf{F}$ across S, oriented upwards in the direction of increasing z.

shortcut : $d\vec{S} = \langle -f_x, -f_y, 1\rangle \, dy\,dx$

$= \langle -2x, -2, 1\rangle \, dy\,dx$

$\vec{F}(x,y) = \langle x, 1, x + 2(1 + x^2 + 2y)\rangle$

$= \langle x, 1, x + 2 + 2x^2 + 4y\rangle$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^{2x} \cancel{-2x^2} \cancel{-2} + \cancel{x} + \cancel{x} + \cancel{2} + \cancel{2x^2} + 4y \, dy\,dx$$

$$= \int_0^1 2x^2 + 2(2x)^2 \, dx$$

$$= \int_0^1 10x^2 \, dx$$

$$= \frac{10}{3}$$

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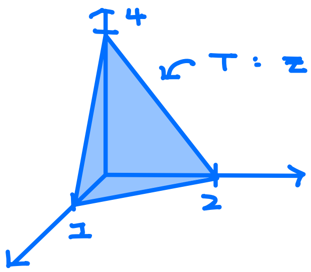
Question 10. Suppose that T is the solid tetrahedron cut from the first octant by the plane $4x + 2y + z = 4$, and let S be its two-dimensional boundary. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = (x + z)\,\hat{\mathbf{i}} - xy\,\hat{\mathbf{j}} + y\,\hat{\mathbf{k}}.$$

Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ with S oriented out, away from the interior of T.



$T: z = 4 - 4x - 2y$

$\hookrightarrow$ xy-shadow : $0 \le y \le 2 - 2x$, $0 \le x \le 1$

divergence theorem

$$\oiint_S \vec{F} \cdot d\vec{S} \overset{\downarrow}{=} \iiint_T \operatorname{div}\vec{F}\, dV$$

$$\left\{\begin{array}{l} \operatorname{div}\vec{F} = 1 - x \\ T:\ 0 \le z \le 4 - 4x - 2y \\ \quad\ \ 0 \le y \le 2 - 2x \\ \quad\ \ 0 \le x \le 1 \end{array}\right.$$

$$= \int_0^1 \int_0^{2-2x} \int_0^{4-4x-2y} 1 - x \; dz\, dy\, dx$$

$$= \int_0^1 \int_0^{2-2x} 4(1-x)^2 - 2y(1-x)\; dy\, dx$$

$$= \int_0^1 8(1-x)^3 - 4(1-x)^3\, dx$$

$$= \int_0^1 4(1-x)^3\, dx$$

$$\left\{\begin{array}{l} u = 1 - x \\ du = -dx \\ x = 0 \to u = 1 \\ x = 1 \to u = 0 \end{array}\right.$$

$$= \int_0^1 4u^3\, du$$

$$= 1$$

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