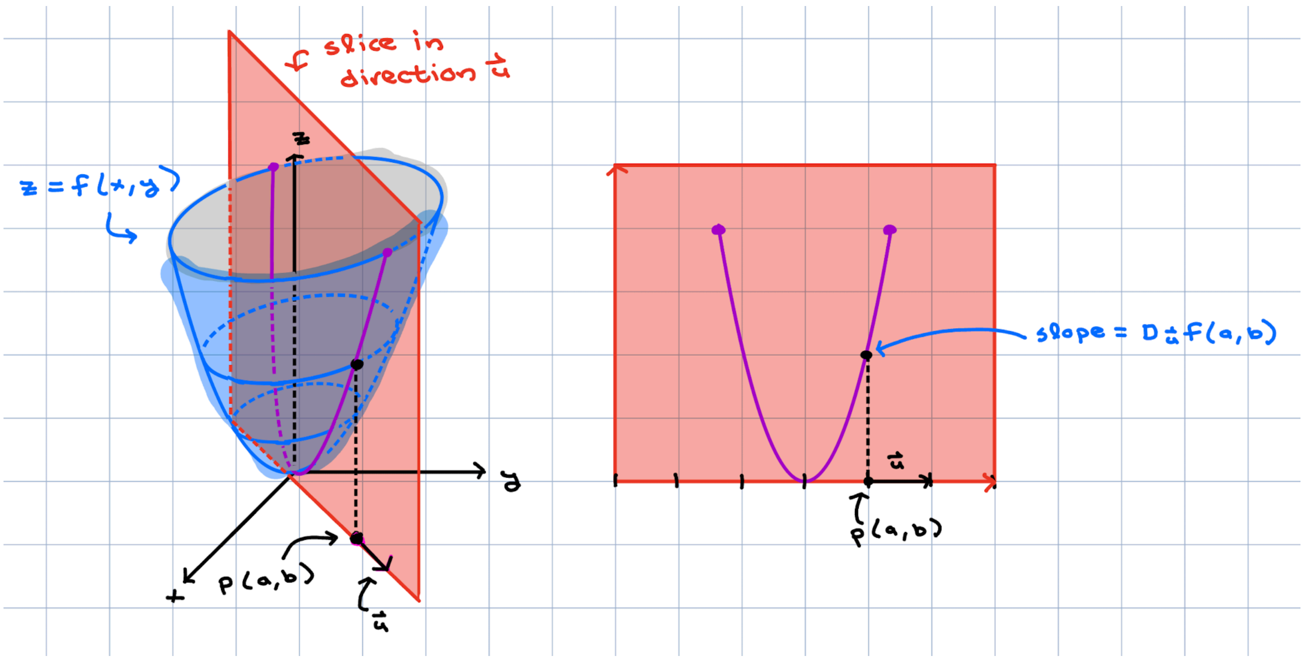
A. **Directional Derivatives.** We can measure rates of change in directions besides the coordinate directions. Consider function $f(x, y)$ and a point $P(a, b)$ along with a unit direction $\mathbf{u}$.

The rates of change in the coordinate directions are the partial derivatives.

By a unit direction $\mathbf{u}$ we mean a unit vector $\mathbf{u}$.



The rate of change of f, beginning at input P and moving in unit direction $\mathbf{u}$, is called the **u–directional derivative** of f at P and is denoted $D_{\mathbf{u}}f(P)$.

To find a formula for the directional derivative, we parametrize the $\mathbf{u}$–axis:

$\mathbf{r}(t) =$

and then calculate the rate of change of f along this parametrization:

$$\frac{d}{dt}\left[f(\mathbf{r}(t))\right] =$$

A formula for the directional derivative is $D_{\mathbf{u}}f(P) = \nabla f(P) \cdot \mathbf{u}$.

Here $\mathbf{u}$ is required to be a unit vector.

Example 1. Calculate the directional derivative of $f(x,y) = xye^{y}$ at $P(-2,-1)$ in the direction determined by $\mathbf{v} = \langle -1, -1 \rangle$.

Because $\mathbf{v}$ is not a unit vector, we first need to convert it to one. Directional derivatives are only defined for unit vectors. If you use the formula for a non–unit vector, then what you are computing is not a slope.

Example 2. T measures temperature (in degrees fahrenheit) and (x, y) measures coordinates (in cm) on a hot pan.

An ant is at point $P(2, 1)$ on the pan. If the ant:

- moves right, then the temperature increases at a rate of 12 degrees/cm
- moves up, then the temperature decreases at a rate of 5 degrees/cm

Rates of change in particular directions are measured by directional derivatives. When that direction is to the right, this is an x-partial derivative. When that direction is up, this is a y-partial derivative.

(a) Find $\nabla T(2, 1) =$

(b) At what rate does the temperature change if the ant moves from $P(2, 1)$ in the unit direction $\mathbf{u} = \langle 3/5, 4/5 \rangle$.

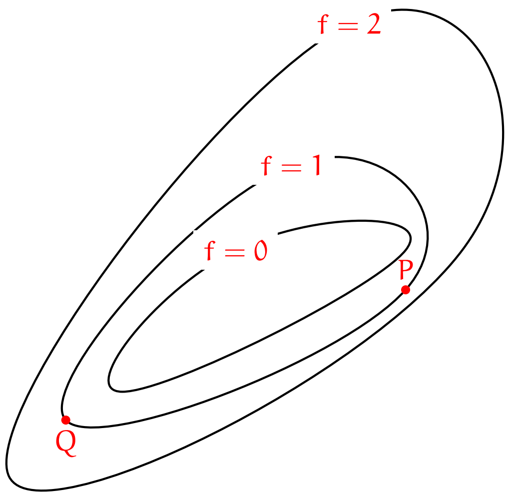
(c) In which unit direction $\mathbf{u}$ from $P(2, 1)$ should the ant move to cool off most rapidly? What is the rate of change of temperature in that direction?

(d) In which unit direction $\mathbf{u}$ should from $P(2, 1)$ should the ant move to remain roughly at the same temperature.

The directional derivative of f from input P is maximized at value $\|\nabla f(P)\|$ in direction $\nabla f(P)$, is minimized at value $-\|\nabla f(P)\|$ in direction $-\nabla f(P)$, and is zero in the directions $\perp$ to $\nabla f(P)$.

B. **Gradients and Level Sets.** Consider the contour diagram for the function $f(x, y)$ depicted below. We will estimate of $\nabla f(Q)$ and $\nabla f(P)$.

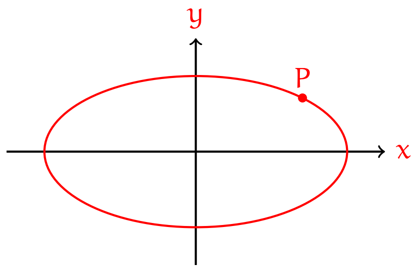
Remember that each contour is called a **level curve** or **level set**.



Remember: the gradient points orthogonal to the direction of "sameness" of f. Further it points in the direction of maximum increase of f, and its length is determined by how quickly f changes.

The gradient $\nabla f(P)$ is, when rooted at P, orthogonal to the level set of f that contains P.

Example 3. Consider the ellipse: $x^2 + 4y^2 = 9$.



(a) Interpret this as a level curve.

(b) Parametrize the **normal line** to the ellipse at $P = (2, 1)$.

The normal line is the line through the point that is orthogonal (aka normal) to the curve.

(c) Use the gradient to find an equation for the tangent line to the ellipse at P.

In 3D, if you have a normal vector $\mathbf{n}$ and a point P, you can find the equation of a plane using $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$. If the vector and point are in 2D, then what you will obtain is the equation of a line instead of a plane.