A. **Stars and Bars.** Consider the equation:

$x_1 + x_2 + x_3 = 5$

The **positive** (> 0) integer solutions are:

We symbolize each solution using **stars and bars** above.

So a solution is found by picking a subset of the gaps between the 5 stars to place 2 bars. Therefore the number of solution is:

Stars and Bars.

The number of **positive** (> 0) integer solutions to $x_1 + x_2 + \cdots + x_r = n$ is:

Another way of thinking about stars and bars is that it counts the number of ways of putting n indistinguishable objects into r bins. x_1 represents the number of objects assigned to the first bin, x_2 the number of objects assigned to the second, and so on. If you use the formula on the left, each bin must contain at least one object. If you use the formula on the next page, it allows for the possibility that bins contains no objects.

Example 1. Count the number of **nonnegative** $(\geqslant 0)$ integer solutions to:

$x_1 + x_2 + \cdots + x_r = n$

We cannot directly use the formula from the last page, because it tells us the number of **positive** integer solutions. Nonnegative includes 0, but positive does not.

We can either figure out how to indirectly use the formula from the previous page, or we can create a new stars and bars argument: we have the same number of stars (n) and bars ($r-1$), except now we allow for the possibility that bars can occur consecutively. In other words any list of length $n+r-1$ made of n stars and $r-1$ bars is possible. Each list is uniquely determined by the locations of the bars. The number of choices of such locations is:

$$\binom{n+r-1}{r-1}$$

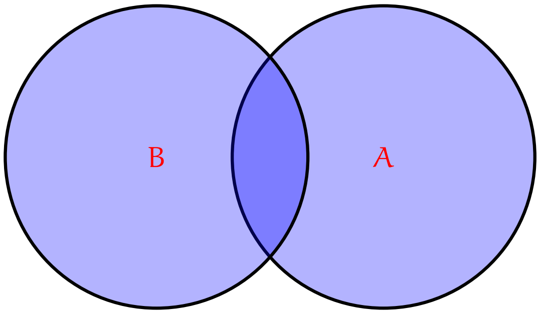
Example 2. 20 blackboards are to be divided among 4 schools.

If the blackboards are identical, and each school must get at least two blackboards, then how many divisions are possible?

B. **Unions and Intersections.** For sets A and B, we can construct their:

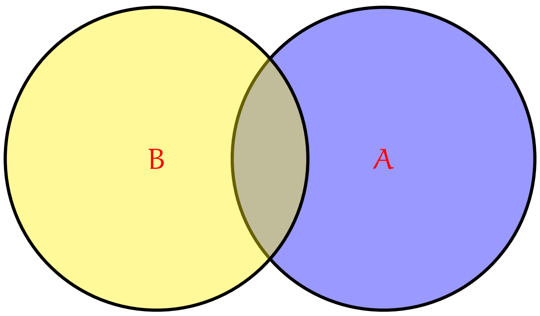
union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Remember that $\in$ is read "in". In math, "or" means "and/or".



$\{1,2,3,4\} \cup \{3,4,5,6\} =$

intersection: $A \cap B = AB = \{x \mid x \in A \text{ and } x \in B\}$



$\{1,2,3,4\} \cap \{3,4,5,6\} =$

Unions and intersections satisfy basic laws.

Laws of Unions and Intersections.

If you intuitively associate "$\cup$" with "$+$", and "$\cap$" with "$\times$", then the following laws can be easily memorized.

commutativity $\Longrightarrow$ $A \cup B = B \cup A$ and $A \cap B = B \cap A$

associativity $\Longrightarrow$ $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$

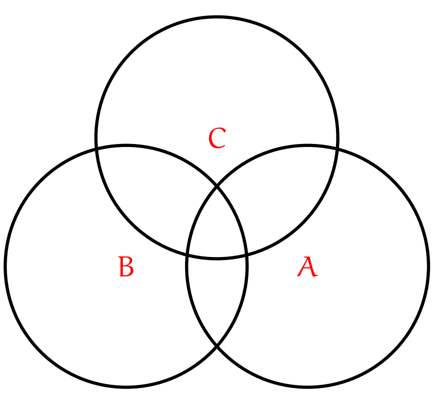
distributivity of $\cap$ over $\cup$ $\Longrightarrow$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

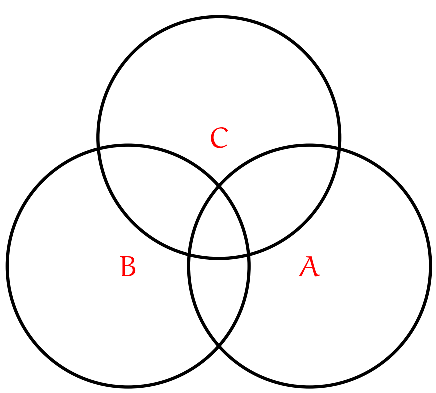
There are some laws that do not follow this intuitive association with ordinary algebra.

distributivity of $\cup$ over $\cap$ $\Longrightarrow$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

This intuitive association is helpful, and can be formalized with the notion of a **Boolean algebra**.

Let's "verify" the last law using venn diagrams.





C. **Singletons and Countability.**

A **singleton** is a set containing a single element. For example: $\{0\}$.

Every set can be written as a, possibly infinite, union of singletons.

$\mathbb{N} = \{\text{set of natural numbers}\} = \{0, 1, 2, 3, \ldots\} =$

$\mathbb{Z} = \{\text{set of integers}\} = \{0, 1, -1, 2, -2, 3, -3, \ldots\} =$

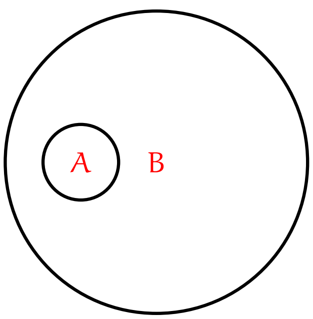
The sets $\mathbb{N}$ and $\mathbb{Z}$ are **countable** because their elements can be arranged in a list that can be counted using natural numbers. In fact, they are **countably infinite**, because infinitely many natural are required. **Finite** set are also countable, but they are not countably infinite.

The set $\mathbb{R}$ of all real numbers is not countable.

D. **Containment.** For sets A and B, we can discuss:

containment: $A \subseteq B$ means: every x that is in A is also in B.

This is read, "A is a subset of B".

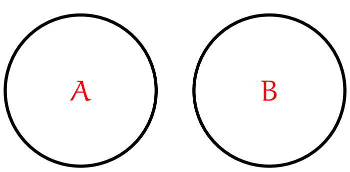


$\{\text{even natural numbers}\} = \{0, 2, 4, 6, \ldots,\} \subseteq \mathbb{N} \subseteq \mathbb{Z}$

E. **Disjointness and the Empty Set.** For sets A and B, we can discuss:

disjointness: A and B are **disjoint** if $A \cap B = \emptyset = \{\text{set with no elements}\}$

We call $\emptyset$ the empty set, because it contains no elements. It may seem odd to discuss it, but it is important in the same way 0 is an important number



$\{\text{even integers}\} \cap \{\text{odd integers}\} = \emptyset$

When taking unions of **disjoint** sets we often use the "square–cup" $\sqcup$ in place of the "cup" $\cup$, just as a reminder that the sets involved are disjoint.

That is: $A \sqcup B = A \cup B$, but also tells us that A and B are disjoint.

So: $\mathbb{Z} =$

F. **Axioms of Probability.** Recall, a **sample space** Ω is a set of outcomes.

For example, consider a weighted 6-sided die, with even numbers twice as likely to appear as odd numbers:

sample space $\implies \Omega = \{1, 2, 3, 4, 5, 6\}$

To particular subsets of Ω, which we call **events**, we assign a probability:

$\mathbb{P}(\text{any number}) = \mathbb{P}(\{1, 2, 3, 4, 5, 6\}) =$

$\mathbb{P}(\text{even number}) = \mathbb{P}(\{2, 4, 6\}) =$

$\mathbb{P}(\text{odd number}) = \mathbb{P}(\{1, 2, 3\}) =$

$\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \mathbb{P}(\{3\}) =$

Kolmogorov Axioms of Probability.

A **probability space** consists of a sample space Ω, an **event space** made up of particular subsets of Ω called events, and a probability function $\mathbb{P}$ that assigns probabilities (values from 0 to 1) to events, such that the following axioms are satisfied.

When the sample space is countable, the event space consists of **all** subsets of the sample space. If the sample space is not countable, the situation is more subtle.

Nonnegativity: For any event E, $\mathbb{P}(E) \geqslant 0$.

Total Probability: $\mathbb{P}(\Omega) = 1$.

This says that at least one outcome must occur.

Countable Additivity: For countably many events:

This includes a countably infinite, or finitely many.

$E_1, E_2, \ldots$

that are **mutually exclusive**, meaning any pair of them is **disjoint**, then:

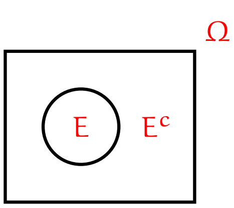
In other words, mutually exclusive means the events are **pairwise disjoint**, i.e. $E_i \cap E_j = \emptyset$ whenever $i \neq j$

$$\mathbb{P}\left(\bigsqcup_i E_i\right) = \sum_i \mathbb{P}(E_i).$$

Remember the $\bigsqcup$ is an indicator that the union is of disjoint sets. More precisely, when used this manner, the sets should be pariwise disjoint.

The **complement** of an event E in the sample space Ω is defined to be:

$E^c = \Omega \setminus E = \{x \in \Omega \mid x \notin E\}$



In the weighted die example:

Complementary Probability. For any event E:

$\mathbb{P}(E^c) =$

Example 3. At a certain school, students wear accessories from among: necklaces, earrings, and bracelets. 10% of students wear all three, 20% wear a necklace and a bracelet, 20% wear a necklace and earrings, 15% wear an earrings and a bracelet, and 87% wear at least one of these accessories. The percent of students that wears bracelets is equal to the percent that wears earrings, but is $1/2$ the percent that wears necklaces.

If a student is selected uniformly at random, find the probability of each of the events N that the student wears a necklace, E that the student wears earrings, and B that the student wears a bracelet.