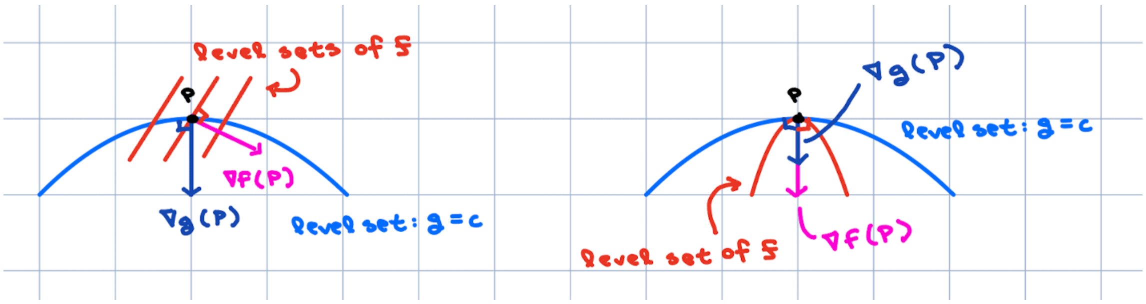
A. **Lagrange Multipliers.** We consider the problem of optimizing a multivariable function $f(\mathbf{x})$ subject to a level set constraint $g(\mathbf{x}) = c$.

Here $\mathbf{x}$ could denote (x, y) or (x, y, z).

We consider different possible orientations of levels sets of f and g.



Lagrange Multipliers. Suppose $f(\mathbf{x})$ and $g(\mathbf{x})$ are differentiable and have continuous first partial derivatives.

An optimizer P of $f(\mathbf{x})$ subject to level set constraint $g(\mathbf{x}) = c$ must:

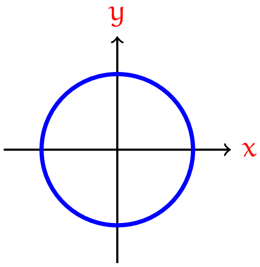
- either satisfy the Lagrange equation: $\nabla f(P) =$
- or be **degenerate**, which means: $\nabla g(P) =$

Here an optimizer refers to an extremizer: a maximizer or minimizer.

The idea behind Lagrange multipliers is that, if the gradient $\nabla f(P)$ is not a multiple of $\nabla g(P)$, then it will be possible to move along the level set $g = c$ in directions that will either increase or decrease f, meaning P is not an optimizer.

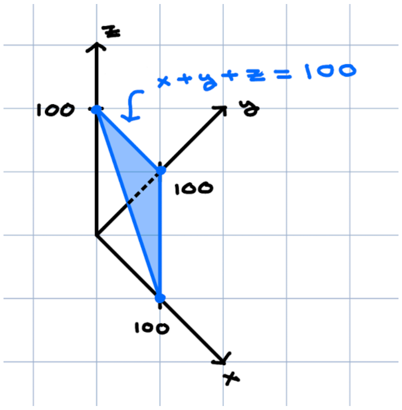
The **degenerate** case is considered rare, in the sense that it will almost never be realized in the examples we encounter. In this case, the gradient $\nabla g(P) = \mathbf{0}$ does not reveal information about the shape of the level set $g = c$.

Example 1. If they exist, use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = xy$ subject to $x^2 + y^2 = 1$.



Of course we should always address whether extremizers are guaranteed to exist. In this case, the feasible set is compact because it is closed (actually equal to its own boundary in this case) and bounded, and f is continuous as a polynomial.

Example 2. Find the closest point on the plane $x + y + z = 100$ to the origin.



We cannot directly apply the extreme value theorem to conclude that extremizers are guaranteed to exist, because the feasible set is unbounded. Intuitively, there **should** be a closest point. Evidently, if there were one, it would, in this case, be contained in the little triangle formed by the intercepts. This triangle is compact, and so we could apply the extreme value theorem to this smaller feasible set to be confident that extremizers exist. And, by inspection, the closest point would not be on an edge of this triangle

A general rule: if you would like to optimize a distance function, then instead minimize the distance–squared function. The reason? The distance–squared will involve no square root, and so will be easier to do calculus with. And of course, a optimizer for the distance–squared function will also be a minimizer for the distance function.

Example 3. Find all solutions to the Lagrange equation for $f(x, y) = 2x^2 + \frac{y^3}{3} - y$ subject to $x^2 + (y-1)^2 = 1$.