A. **Fundamental Matrix Method.**

Let $X(t)$ represent a square matrix, and consider initial value problem:

$X' = AX$ with $X(0) = I$

We can find one solution to this initial value problem using the exponential.

We can find another solution to this problem using the fundamental matrix $M(t)$.

By uniqueness of solutions to initial value problems:

Fundamental Matrix Formula for Exponential.

If A has a fundamental matrix $M(t)$ then:

$e^{tA} =$

Remark. Because it solves $X' = AX$, and is invertible, the matrix e^{tA} is itself a:

If you forgot, a fundamental matrix has columns equal to fundamental solutions:

$\mathbf{x}' = A\mathbf{x}$

so:

$M(t) = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix}$

An important property we derived was:

$M'(t) = AM(t)$

which simply came from:

$M'(t) = \begin{pmatrix} \mathbf{x}_1' & \mathbf{x}_2' \end{pmatrix} = \begin{pmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 \end{pmatrix} = \cdots$

where we have used that $\mathbf{x}_1$, $\mathbf{x}_2$ solve $\mathbf{x}' = A\mathbf{x}$. Then continuing:

$\cdots = A \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} = AM(t)$

where we have used that matrix multiplication can be thought of as multiplying the left matrix A by each column of the right matrix $M(t)$.

Remember that there are many different fundamental matrices, depending on what fundamental set of solutions we select. The matrix e^{tA} is specifically the unique fundamental matrix $M(t)$ with the property that $M(0) = I$.

Example 1. Consider matrix:

$$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$$

and find e^{tA} using the fundamental matrix method.

B. **Homogeneous Exponential Matrix Method.**

Consider **homogeneous** initial value problem:

$\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$

Show that the solution is given by:

$\mathbf{x} = e^{tA}\mathbf{x}_0$

Exponential Solution Formula: Homogeneous.

The solution to **homogeneous** initial value problem:

$\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$

is: $\mathbf{x} =$

Example 2. Solve initial value problem $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = x_0$ where:

$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$ and $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

We already computed the exponential of this matrix in an earlier example, and found:

$e^{tA} = \begin{pmatrix} 4e^{2t} - 3e^{t} & 6e^{2t} - 6e^{t} \\ -2e^{2t} + 2e^{t} & -3e^{2t} + 4e^{t} \end{pmatrix}$

C. **General Exponential Method.**

Now we suppose we have a **nonhomogeneous** initial value problem:

$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$

The **input-free** solution (forcing term set equal to $\mathbf{0}$) is:

$\mathbf{x}_i =$

The **state-free** solution (initial values set equal to $\mathbf{0}$) can be found using variation of parameters:

$\mathbf{x} =$

Variation of parameters said:

$\mathbf{x} = M(t)\int M(t)^{-1}\mathbf{f}(t)\ dt$

In this "convolution," the order of multiplication is very important inside the integral. It is essential we interpret:

$e^{At} * f(t) = \int_0^t e^{Au}\ \mathbf{f}(t-u)\ du$

with the matrix e^{Au} coming before the vector $\mathbf{f}(t-u)$, because only multiplication of matrix times vector in this order makes sense.

Exponential Solution Formula: Non–Homogeneous.

If A is constant, the solution to **non-homogoneous** initial value problem:

$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ is

is the sum of the input–free and state–free solutions:

$\mathbf{x} = \mathbf{x}_i + \mathbf{x}_s =$

Example 3. Solve initial value problem $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ with $\mathbf{x}(0) = x_0$ where:

$$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix} \text{ and } \mathbf{f} = \begin{pmatrix} e^t \\ e^t \end{pmatrix} \text{ and } \mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We already computed the exponential of this matrix in an earlier example, and found:

$$e^{tA} = \begin{pmatrix} 4e^{2t} - 3e^t & 6e^{2t} - 6e^t \\ -2e^{2t} + 2e^t & -3e^{2t} + 4e^t \end{pmatrix}$$

We also already computed the input–free solution:

$$\mathbf{x}_i = e^{tA}\mathbf{x}_0 = \begin{pmatrix} 10e^{2t} - 9e^t \\ -5e^{2t} + 6e^t \end{pmatrix}$$

All that is left is to find the state–free soluton:

$$\mathbf{x}_s = e^{At} * \mathbf{f}(t)$$