

**Sum Formulas.**

- $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
- $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
- $1^3 + 2^3 + \cdots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2$
- if  $|x| < 1$ , then:
 
$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

**A1 Formulas.**

- $k$ -permutations:  ${}_nP_k = \frac{n!}{(n-k)!}$
- $k$ -combinations:  ${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$
- multiset permutations:  $\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$
- binomial theorem:  $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$
- multinomial theorem:
 
$$(x_1 + \cdots + x_r)^n = \sum_{\substack{n_1 + \cdots + n_r = n \\ n_1, \dots, n_r \geq 0}} \binom{n}{n_1, \dots, n_r} x_1^{n_1} \cdots x_r^{n_r}$$
- the # of solutions  $(x_1, \dots, x_r)$  to  $x_1 + \cdots + x_r = n$  is:
  - $\binom{n-1}{r-1}$  if only positive integers are allowed
  - $\binom{n+r-1}{r-1}$  if only nonnegative integers are allowed

**A2 Formulas.**

- inclusion–exclusion:

$$\mathbb{P}(E_1 \cup \cdots \cup E_n) = \sum_{r=1}^n (-1)^{r+1} \sum_{1 \leq i_1 < \cdots < i_r \leq n} \mathbb{P}(E_{i_1} \cdots E_{i_r})$$

**A3 Formulas.**

- conditional probability:  $\mathbb{P}(E | F) = \frac{\mathbb{P}(EF)}{\mathbb{P}(F)}$
- independence:  $E \perp F$  if and only if  $\mathbb{P}(EF) = \mathbb{P}(E)\mathbb{P}(F)$
- law of total probability: if  $F_1 \sqcup F_2 \sqcup \cdots \sqcup F_n = \Omega$ , then:
 
$$\mathbb{P}(E) = \sum_{i=1}^n \mathbb{P}(E | F_i) \cdot \mathbb{P}(F_i)$$
- Bayes's formula:  $\mathbb{P}(E | F) = \frac{\mathbb{P}(F|E)\mathbb{P}(E)}{\mathbb{P}(F)}$
- multiplication rule:
 
$$\mathbb{P}(E_1 E_2 \cdots E_n) = \mathbb{P}(E_1) \mathbb{P}(E_2 | E_1) \cdots \mathbb{P}(E_n | E_1 E_2 \cdots E_{n-1})$$

**Discrete Random Variables.**

- uniform random variable:  $X \sim \text{Uniform}(\{1, \dots, n\})$ 
  - $\mathbb{P}(X = k) = \frac{1}{n}$  if  $k \in \{1, \dots, n\}$
  - $\mathbb{E}[X] = \frac{n+1}{2}$  and  $\text{Var}(X) = \frac{n^2-1}{12}$
- Bernoulli random variable:  $X \sim \text{Bernoulli}(p)$ 
  - $\mathbb{P}(X = 1) = p$  and  $\mathbb{P}(X = 0) = 1 - p$
  - $\mathbb{E}[X] = p$  and  $\text{Var}(X) = p(1 - p)$

**Discrete Random Variables Continued.**

- binomial random variable:  $X \sim \text{Binom}(n, p)$ 
  - $\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$
  - $\mathbb{E}[X] = np$  and  $\text{Var}(X) = np(1-p)$
- geometric random variable:  $X \sim \text{Geom}(p)$ 
  - $\mathbb{P}(X = k) = \binom{n}{k} (1-p)^{k-1} p$
  - $\mathbb{E}[X] = \frac{1}{p}$  and  $\text{Var}(X) = \frac{1}{p^2} - \frac{1}{p}$
- Poisson random variable:  $X \sim \text{Poisson}(\lambda)$ 
  - $\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$
  - $\mathbb{E}[X] = \lambda$  and  $\text{Var}(X) = \lambda$
- indicator function of an event  $E$ :  $\mathbb{1}_E$ 
  - $\mathbb{E}[\mathbb{1}_E] = \mathbb{P}(E)$  and  $\text{Var}(\mathbb{1}_E) = \mathbb{P}(E)\mathbb{P}(E^c)$

**A5 Formulas.**

- $\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X = x)$
- $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$
- $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
- if  $X \perp Y$  then:
  - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
  - $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$
  - $\text{Cov}(X, Y) = 0$

