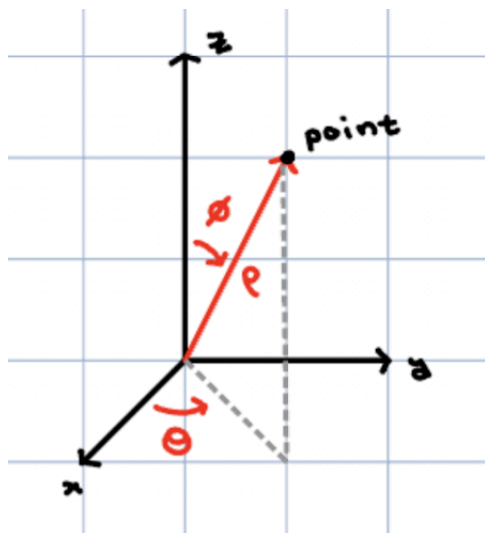


A. **Spherical Coordinates.** There is another coordinate system that is excellent for objects that are symmetric about the origin, like spheres centered at the origin. These are the spherical coordinates ρ , ϕ , and θ . The spherical coordinate θ is the same as the cylindrical coordinate.

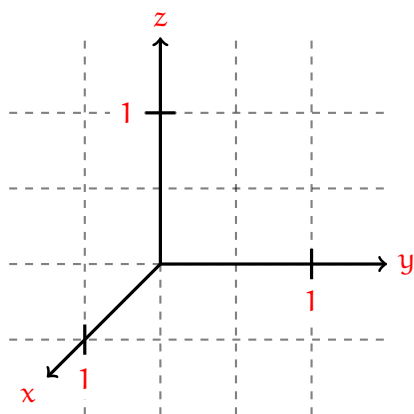
ρ is read “roe” and ϕ is read “fee”.



The point with spherical coordinates ρ , ϕ , and θ has position vector obtained by rotating the vector $\langle 0, 0, \rho \rangle$ down by angle ϕ in the vertical half-plane at angle θ . Every point can be described using coordinates in the range:

$$\begin{aligned} \rho &\geq 0 \\ 0 &\leq \theta < 2\pi \\ 0 &\leq \phi < \pi \end{aligned}$$

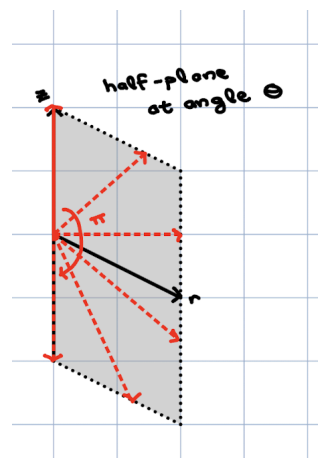
For example we sketch the surface S defined by $\rho = 1, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq \frac{\pi}{2}$.



Because of the way spherical coordinates are defined, if ρ is nonnegative, then it equals the distance of the point from the origin.

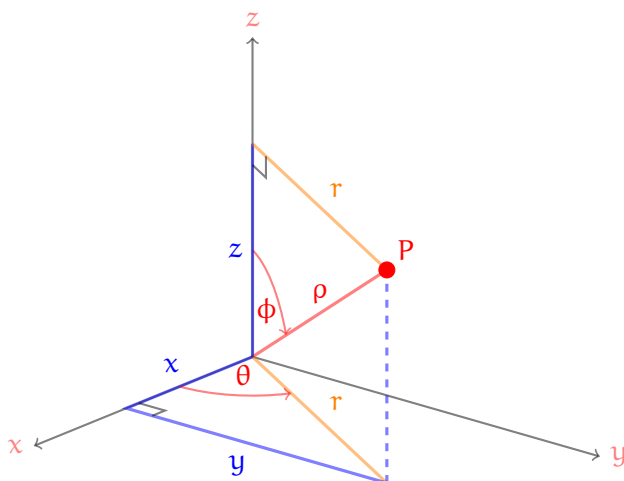
Specifically: the vertical half-plane is the level set of θ assuming that $r \geq 0$. We sketched these level sets when we first introduced cylindrical coordinate.

Every vertical half-plane can be described by θ in the range 0 to 2π . Every ray in that half-plane can be described by an angle ϕ down from the z -axis in range 0 to π .



On a sphere, the coordinates ϕ and θ are analogous to latitude and longitude.

B. Spherical Conversion. We develop algebraic relationships between spherical coordinates and the other coordinate systems.



The two triangles formed by ϕ and by θ in this picture contain all the relations we describe below.

[Spherical \leftrightarrow Cylindrical] and [Spherical \leftrightarrow Standard]

r in terms of spherical coordinates:

x in terms of spherical coordinates:

y in terms of spherical coordinates:

z in terms of spherical coordinates:

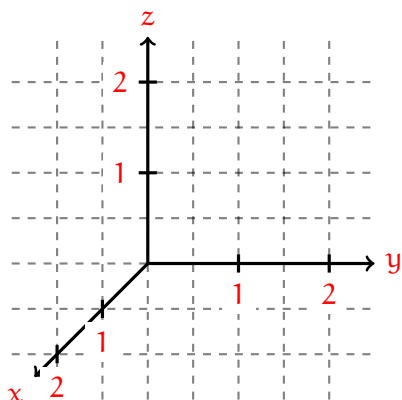
relation between ρ and standard/cylindrical coordinates:

relation between ϕ and standard/cylindrical coordinates:

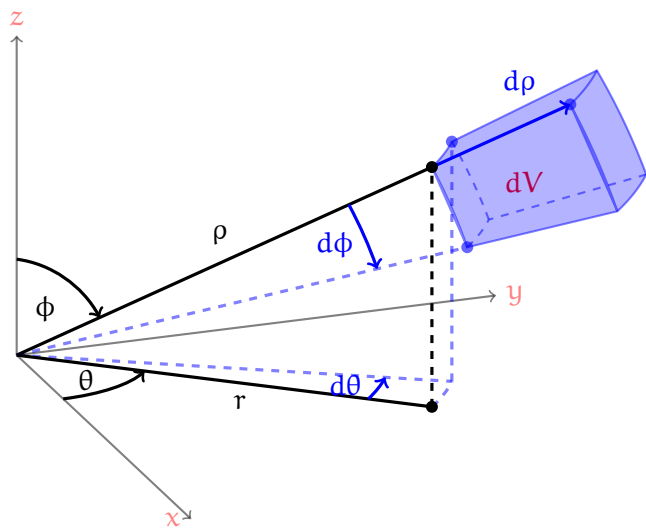
Example 1. Convert the cylindrical equation:

$$z = \sqrt{3}r$$

into a spherical equation, and sketch this surface assuming $r \geq 0$.



C. Spherical Integration. Now we will discuss integration in spherical coordinates. Imagine the infinitesimal bit of area dV obtained from the point with spherical coordinates ρ, ϕ, θ by increasing ρ by $d\rho$, ϕ by $d\phi$, and θ by $d\theta$.



The goal in this process is to try to understand how small changes $d\rho$, $d\phi$, and $d\theta$ are related to small changes dV in volume. This dV is exactly what appears in triple-integration.

For example, we know that small changes dr , $d\theta$, dz effect the change $dV = r \, dr \, d\theta \, dz$ in volume.

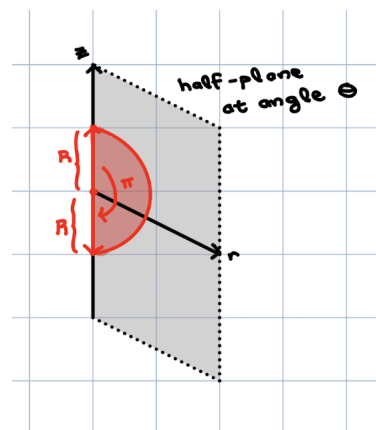
In spherical coordinates:

$$dV =$$

Example 2. Use spherical coordinates to derive the volume of a radius R sphere:

$$x^2 + y^2 + z^2 \leq R^2$$

To describe a solid sphere of radius R centered at the origin, the section of the sphere in each vertical half-plane at angle θ from 0 to 2π is traced out by rotating the line segment from $(0, 0, 0)$ to $(0, 0, R)$ down from angle 0 to angle π .



Example 3. Let E be the region outside the cone $z = \sqrt{x^2 + y^2}$ and inside the upper hemisphere of the sphere $x^2 + y^2 + z^2 \leq 1$. Find:

$$\iiint_E \frac{1}{\sqrt{x^2 + y^2 + z^2}} dV$$

The upper hemisphere of this sphere, which is centered at the origin, consists of the half of the sphere above the xy -plane.

