

Sum Formulas.

- $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
- $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
- if $|x| < 1$, then:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

Integration.

- integration by parts: $\int u \, dv = uv - \int v \, du$
- polar: $dx dy = r \, dr d\theta$, $r^2 = x^2 + y^2$, $\tan \theta = \frac{y}{x}$
- $\int \frac{1}{1+x^2} \, dx = \arctan x + C$

A1 Formulas.

- k-permutations: ${}_n P_k = \frac{n!}{(n-k)!}$
- k-combinations: ${}_n C_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$
- multiset permutations: $\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$
- binomial theorem: $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$
- multinomial theorem:

$$(x_1 + \cdots + x_r)^n = \sum_{\substack{n_1 + \cdots + n_r = n \\ n_1, \dots, n_r \geq 0}} \binom{n}{n_1, \dots, n_r} x_1^{n_1} \cdots x_r^{n_r}$$
- the # of integer solutions to $x_1 + \cdots + x_r = n$ is:
 - $\binom{n-1}{r-1}$ if we allow only, and all, positive solutions
 - $\binom{n+r-1}{r-1}$ if we allow only, and all, nonneg. solutions

A2 Formulas.

- inclusion-exclusion:

$$\mathbb{P}(E_1 \cup \cdots \cup E_n) = \sum_{r=1}^n (-1)^{r+1} \sum_{1 \leq i_1 < \cdots < i_r \leq n} \mathbb{P}(E_{i_1} \cdots E_{i_r})$$

A3 Formulas.

- conditional probability: $\mathbb{P}(E | F) = \frac{\mathbb{P}(EF)}{\mathbb{P}(F)}$
- independence: $E \perp F$ if and only if $\mathbb{P}(EF) = \mathbb{P}(E)\mathbb{P}(F)$
- law of total probability: if $F_1 \sqcup F_2 \sqcup \cdots \sqcup F_n = \Omega$, then:

$$\mathbb{P}(E) = \sum_{i=1}^n \mathbb{P}(E | F_i) \cdot \mathbb{P}(F_i)$$
- Bayes's formula: $\mathbb{P}(E | F) = \frac{\mathbb{P}(F|E)\mathbb{P}(E)}{\mathbb{P}(F)}$
- multiplication rule:

$$\mathbb{P}(E_1 E_2 \cdots E_n) = \mathbb{P}(E_1) \mathbb{P}(E_2 | E_1) \cdots \mathbb{P}(E_n | E_1 E_2 \cdots E_{n-1})$$

Discrete Random Variables.

- uniform random variable: $X \sim \text{Uniform}(\{1, \dots, n\})$
 - $\mathbb{P}(X = k) = \frac{1}{n}$ if $k \in \{1, \dots, n\}$
 - $\mathbb{E}[X] = \frac{n+1}{2}$ and $\text{Var}(X) = \frac{n^2-1}{12}$
- Bernoulli random variable: $X \sim \text{Bernoulli}(p)$
 - $\mathbb{P}(X = 1) = p$ and $\mathbb{P}(X = 0) = 1 - p$
 - $\mathbb{E}[X] = p$ and $\text{Var}(X) = p(1 - p)$
- binomial random variable: $X \sim \text{Binom}(n, p)$
 - $\mathbb{P}(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$
 - $\mathbb{E}[X] = np$ and $\text{Var}(X) = np(1 - p)$
- geometric random variable: $X \sim \text{Geom}(p)$
 - $\mathbb{P}(X = k) = (1 - p)^{k-1} p$
 - $\mathbb{E}[X] = \frac{1}{p}$ and $\text{Var}(X) = \frac{1}{p^2} - \frac{1}{p}$
- Poisson random variable: $X \sim \text{Poisson}(\lambda)$
 - $\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$
 - $\mathbb{E}[X] = \lambda$ and $\text{Var}(X) = \lambda$
- indicator function of an event E : $\mathbb{1}_E$
 - $\mathbb{E}[\mathbb{1}_E] = \mathbb{P}(E)$ and $\text{Var}(\mathbb{1}_E) = \mathbb{P}(E)\mathbb{P}(E^c)$

A5 Formulas.

- if X is discrete:

$$\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X = x)$$

$$\mathbb{E}[g(X)] = \sum_x g(x) \cdot \mathbb{P}(X = x)$$
- $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$
- $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
- if $X \perp Y$ then:
 - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
 - $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$
 - $\text{Cov}(X, Y) = 0$

B1 Formulas.

- Let X be a continuous random variable.
 - its cdf is $F(x) = \mathbb{P}(X \leq x)$
 - its pdf is $F'(x) = f(x)$
 - $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) \, dx$
 - $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) \, dx$
 - tail formula for expectation: if $X \geq 0$ then:

$$\mathbb{E}[X] = \int_0^{\infty} \mathbb{P}(X > x) \, dx$$

Continuous Random Variables.

- uniform random variable: $X \sim \text{Uniform}(a, b)$
 - pdf: $f(x) = \frac{1}{b-a}$ for $x \in (a, b)$
 - cdf: $\mathbb{P}(X \leq x) = \frac{x-a}{b-a}$ for $x \in (a, b)$
 - $\mathbb{E}[X] = \frac{b-a}{2}$ and $\text{Var}(X) = \frac{(b-a)^2}{12}$
- normal random variable: $X \sim \mathcal{N}(\mu, \sigma^2)$
 - pdf: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
 - cdf: $\mathbb{P}(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
 - $\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2$
 - normalizing factor: $\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sigma\sqrt{2\pi}$
- exponential random variable: $X \sim \text{Exp}(\lambda)$
 - pdf: $f(x) = \lambda e^{-\lambda x}$ for $x > 0$
 - tdf: $\mathbb{P}(X > x) = e^{-\lambda x}$ for $x > 0$
 - $\mathbb{E}[X] = \frac{1}{\lambda}$ and $\text{Var}(X) = \frac{1}{\lambda^2}$
- gamma random variable: $X \sim \text{Gamma}(\alpha, \lambda)$
 - pdf: $f(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha-1}}{\Gamma(\alpha)}$ for $x > 0$
 - $\mathbb{E}[X] = \frac{\alpha}{\lambda}$ and $\text{Var}(X) = \frac{\alpha}{\lambda^2}$
 - gamma function: $\Gamma(\alpha) = \int_0^{\infty} \lambda e^{-\lambda x} (\lambda x)^{\alpha-1} dx$
 - functional equation: $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$
 - for $n \geq 1 \in \mathbb{N}$ we have: $\Gamma(n) = (n-1)!$
- beta random variable: $X \sim \text{Beta}(\alpha, \beta)$
 - pdf: $f(x) = \frac{x^{\alpha-1} (1-x)^{\beta-1}}{B(\alpha, \beta)}$ for $0 < x < 1$
 - $\mathbb{E}[X] = \frac{\alpha}{\alpha+\beta}$ and $\text{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
 - normalizing factor: $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
- cauchy random variable: $X = \tan \Theta$ where Θ is uniform on any fixed interval with length an integer multiple of π

B4/B5 Formulas.

- If $f(x, y)$ is the jdf of X and Y :
 - $f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$ and $f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$
 - $f_{X|Y}(x | y) = \frac{f(x, y)}{f_Y(y)}$ and $f_{Y|X}(y | x) = \frac{f(x, y)}{f_X(x)}$
 - if $X \perp Y$ and $Z = X + Y$, then the pdf of Z is:
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy$$
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