Math 226 Final Exam

Fa24

Wed Dec 11

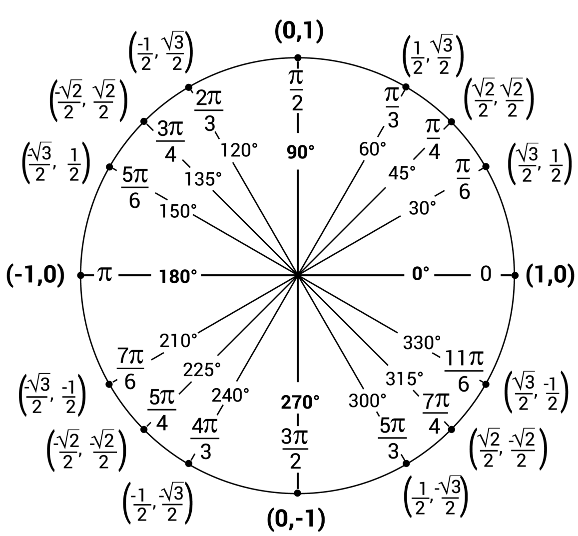
Firstname Lastname: ______________________________

UscID: ______________________________

Instructions

- This examination consists of 13 pages not including this cover page.
- Write your initials in the designated spot at the top–right of each page.
- This examination consists of 10 questions for a total of 100 points. You have 120 minutes to complete this examination.
- Do not use books, calculators, computers, tablets, or phones.
- You may use a single 8.5 in by 11 in page of notes, handwritten on both sides.
- Write legibly in the boxed area only. Cross out any work that you do not wish to have scored.
- Show all of your work and cite theorems you use. Unsupported answers may not earn credit.
- If you run out of space: there are two pages at the end where you can continue your work.
- All work you submit should represent your own thoughts and ideas. If the graders suspect otherwise: you can expect your instructor to file a report with USC's Office of Academic Integrity (OAI).

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	12	8	10	8	12	8	12	10	10	10	100



$$\sin^2\theta + \cos^2\theta = 1$$

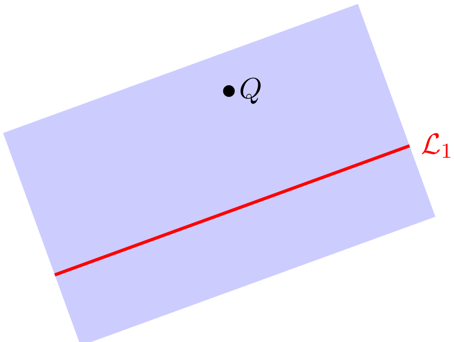
$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

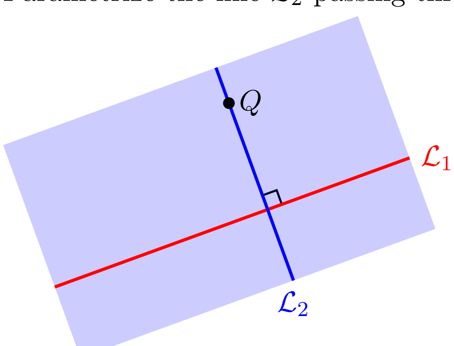
1. (12 points) Consider the line $\mathcal{L}_1$ parametrized by $\mathbf{r}(t) = \langle\, t-1, 2t, -t+2 \,\rangle$ and the point $Q(-1,-2,4)$.

 (a) Find an equation of the plane $\mathcal{P}$ that contains both the point Q and all points on the line $\mathcal{L}_1$.

 Note: picture not to scale.



 (b) Parametrize the line $\mathcal{L}_2$ passing through the point Q and intersecting the line $\mathcal{L}_1$ orthogonally.

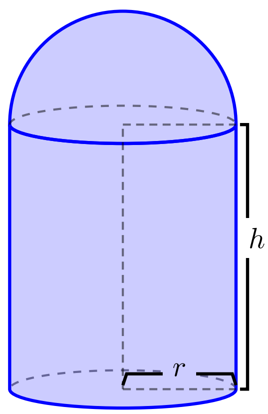


2. (8 points) The function $f(x, y) = 4xy - y^2 - 2x^2y$ has **three** critical points. Find them and classify them as local minimizers, local maximizers, or saddle points.

3. (10 points) A factory wishes to build cylindrical bins, with a hemispherical cap on the top, and a disk at the bottom. The radius and the height of the cylindrical part are labeled by r and h respectively. The cost of the material is:

- \$1 per square feet for for the base of the bin;
- \$0.5 per square feet for the cylindrical part;
- \$1 per square feet for the hemispherical cap.

Let $f(r,h)$ be the volume of the bin and $g(r,h)$ be the total cost. If the total cost of making one bin is 28π dollars, determine the maximal volume of a single bin. Note: you may assume a global maximum exists.



Hint: Here are the relevant formulas:

[Cylinder Sides Surface Area] $= 2\pi r h$

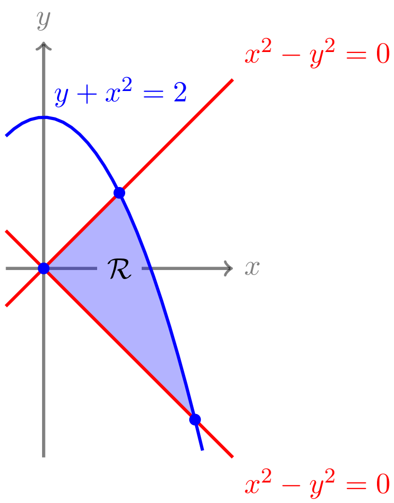
[Cylinder Volume] $= \pi r^2 h$

[Hemisphere Surface Area] $= 2\pi r^2$

[Hemisphere Volume] $= \frac{2}{3}\pi r^3$

[Disk Area] $= \pi r^2$

4. (8 points) Let $\mathcal{R}$ be the shaded region pictured below. Assume the straight lines are given by $x^2 - y^2 = 0$ and the parabola is given by $y + x^2 = 2$.



Set up **but do not evaluate** an integral (or a **sum** of integrals) over the region $\mathcal{R}$ in order $dydx$ **or** $dxdy$ (your choice: pick an order and stick with it) that **equals the area** of $\mathcal{R}$.

5. (12 points) Consider the surface $\mathcal{S}$ parametrized by:

$$\mathbf{r}(u,v) = \left\langle u\cos v, u\sin v, \frac{u^2}{2} \right\rangle$$

on the domain $\mathcal{D}$ defined by $0 \le u \le 2$ and $0 \le v \le \frac{\pi}{2}$.

(a) Compute the surface area of $\mathcal{S}$.

(b) Let $f(x,y,z)$ be a function such that:

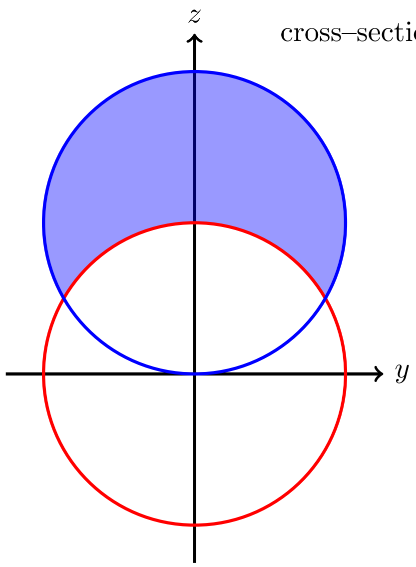
$$\begin{cases} f_x(0,2,2) = -2 \\ f_y(0,2,2) = 3 \\ f_z(0,2,2) = -1 \end{cases}$$

and let $(x,y,z) = \mathbf{r}(u,v)$ from the top of the page. Evaluate the following.

$$\left.\frac{\partial f}{\partial u}\right|_{(u,v)=(2,\frac{\pi}{2})}$$

6. (8 points) Consider the solid E inside the sphere $x^2 + y^2 + (z-2)^2 = 4$ and outside the sphere $x^2 + y^2 + z^2 = 4$.

   Use a triple integral in spherical coordinates to calculate the volume of E.



cross–section of E with yz–plane

7. (12 points) Consider the vector field $\mathbf{F}(x,y) = \left[\ln(y) + axy^3\right]\mathbf{i} + \left[(a+1)x^2y^2 + \frac{x}{y}\right]\mathbf{j}$

(a) Find the value of the constant a for which $\mathbf{F}$ is conservative in the open upper half-plane $\{y > 0\}$. You must justify that your answer is correct.

(b) For the value of a found above, find a potential f for $\mathbf{F}$. Note: recall that a potential for a vector field is a function whose gradient equals that vector field.

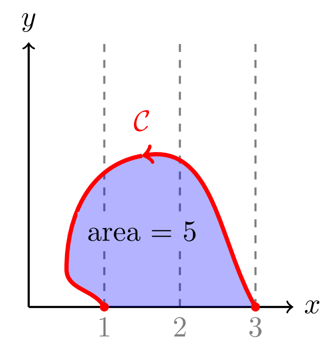
(c) Evaluate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathcal{C}$ is the curve parametrized by:

$$\mathbf{r}(t) = \langle\, 2 + \cos t, 1 + \sin t\,\rangle \quad \text{with } 0 \le t \le \pi$$

8. (10 points) Consider the 2–dimensional vector field:

$$\mathbf{F}(x,y) = (e^y + 3y)\,\mathbf{i} + (xe^y + 5x)\,\mathbf{j}$$

Let $\mathcal{C}$ be the oriented curve depicted below and assume the area of the shaded region is 5. Note: the shaded region is bounded by both $\mathcal{C}$ and the line segment from $(1,0)$ to $(3,0)$.



Find $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$. Hint: extend $\mathcal{C}$ to a closed curve and involve Green's Theorem.

9. (10 points) Let $\mathcal{S}$ be the **closed** surface consisting of the portion of the cone $z = \sqrt{x^2 + y^2}$ with $0 \leq z \leq 2$ along with its **top** at $z = 2$. Orient $\mathcal{S}$ with **inward** normals and evaluate the following integral by using the divergence theorem.

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \quad \text{where} \quad \mathbf{F}(x, y, z) = \left(\tfrac{5}{3}x^3 + ze^{y^3}\right)\mathbf{i} + \left(y + z^2\cos(x^2)\right)\mathbf{j} + \left(1 - z\right)\mathbf{k}$$

10. (10 points) Let $\mathcal{C}$ be the **boundary** curve of the portion of the **surface** $z = xy$ with $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Orient $\mathcal{C}$ **clockwise** as viewed from above. Evaluate the following **line** integral by using Stokes's Theorem to convert to an appropriate and simpler **surface** integral.

$$\int_{\mathcal{C}} \left[e^{\sin(e^x)} + z^2\right] dx + \left[\frac{1}{1+y^4}\right] dy + \left[1 + y^2\right] dz$$

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

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