Sum Formulas.

- $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
- $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
- if $|x| < 1$, then:
  $\sum_{k=0}^{\infty} x^n = \frac{1}{1-x}$
  $\sum_{k=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$

Integration.

- integration by parts: $\int u\, dv = uv - \int v\, du$
- polar: $dxdy = r\, dr d\theta$, $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$
- $\int \frac{1}{1+x^2}\, dx = \arctan x + C$

A1 Formulas.

- k–permutations: ${}_nP_k = \frac{n!}{(n-k)!}$
- k–combinations: ${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$
- multiset permutations: $\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$
- binomial theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$
- multinomial theorem:
  $(x_1 + \cdots + x_r)^n = \sum_{\substack{n_1+\cdots+n_r=n \\ n_1,\ldots,n_r \geqslant 0}} \binom{n}{n_1,\ldots,n_r} x_1^{n_1} \cdots x_r^{n_r}$
- the $\#$ of integer solutions to $x_1 + \cdots + x_r = n$ is:
  - $\binom{n-1}{r-1}$ if we allow only, and all, positive solutions
  - $\binom{n+r-1}{r-1}$ if we allow only, and all, nonneg. solutions

A2 Formulas.

- inclusion–exclusion:

$$\mathbb{P}(E_1 \cup \cdots \cup E_n) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} \mathbb{P}(E_{i_1} \cdots E_{i_r})$$

A3 Formulas.

- conditional probability: $\mathbb{P}(E \mid F) = \frac{\mathbb{P}(EF)}{\mathbb{P}(F)}$
- independence: $E \perp F$ if and only if $\mathbb{P}(EF) = \mathbb{P}(E)\mathbb{P}(F)$
- law of total probability: if $F_1 \sqcup F_2 \sqcup \cdots \sqcup F_n = \Omega$, then:
  $\mathbb{P}(E) = \sum_{i=1}^{n} \mathbb{P}(E \mid F_i) \cdot \mathbb{P}(F_i)$
- Bayes's formula: $\mathbb{P}(E \mid F) = \frac{\mathbb{P}(F|E)\mathbb{P}(E)}{\mathbb{P}(F)}$
- multiplication rule:
  $\mathbb{P}(E_1 E_2 \cdots E_n) = \mathbb{P}(E_1)\mathbb{P}(E_2 \mid E_1) \cdots \mathbb{P}(E_n \mid E_1 E_2 \cdots E_{n-1})$

Discrete Random Variables.

- uniform random variable: $X \sim \mathrm{Uniform}(\{1, \ldots, n\})$
  - $\mathbb{P}(X = k) = \frac{1}{n}$ if $k \in \{1, \ldots, n\}$
  - $\mathbb{E}[X] = \frac{n+1}{2}$ and $\mathrm{Var}(X) = \frac{n^2-1}{12}$
- Bernoulli random variable: $X \sim \mathrm{Bernoulli}(p)$
  - $\mathbb{P}(X = 1) = p$ and $\mathbb{P}(X = 0) = 1 - p$
  - $\mathbb{E}[X] = p$ and $\mathrm{Var}(X) = p(1-p)$
- binomial random variable: $X \sim \mathrm{Binom}(n, p)$
  - $\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$
  - $\mathbb{E}[X] = np$ and $\mathrm{Var}(X) = np(1-p)$
- geometric random variable: $X \sim \mathrm{Geom}(p)$
  - $\mathbb{P}(X = k) = (1-p)^{k-1} p$
  - $\mathbb{E}[X] = \frac{1}{p}$ and $\mathrm{Var}(X) = \frac{1}{p^2} - \frac{1}{p}$
- Poisson random variable: $X \sim \mathrm{Poisson}(\lambda)$
  - $\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$
  - $\mathbb{E}[X] = \lambda$ and $\mathrm{Var}(X) = \lambda$
- indicator function of an event E: $\mathbb{1}_E$
  - $\mathbb{E}[\mathbb{1}_E] = \mathbb{P}(E)$ and $\mathrm{Var}(\mathbb{1}_E) = \mathbb{P}(E)\mathbb{P}(E^c)$

A5 Formulas.

- if X is discrete:
  $\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X = x)$
  $\mathbb{E}[g(X)] = \sum_x g(x) \cdot \mathbb{P}(X = x)$
- $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$
- $\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
- if $X \perp Y$ then:
  - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
  - $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$
  - $\mathrm{Cov}(X, Y) = 0$

B1 Formulas.

- Let X be a continuous random variable.
  - its cdf is $F(x) = \mathbb{P}(X \leqslant x)$
  - its pdf is $F'(x) = f(x)$
  - $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\, dx$
  - $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$
  - tail formula for expectation: if $X \geqslant 0$ then:
    $\mathbb{E}[X] = \int_0^{\infty} \mathbb{P}(X > x)\, dx$

Continuous Random Variables.

- uniform random variable: $X \sim \text{Uniform}(a, b)$
  - pdf: $f(x) = \frac{1}{b-a}$ for $x \in (a, b)$
  - cdf: $F(x) = \frac{x}{b-a}$ for $x \in (a, b)$
  - $\mathbb{E}[X] = \frac{b-a}{2}$ and $\text{Var}(X) = \frac{(b-a)^2}{12}$
- normal random variable: $X \sim \mathcal{N}(\mu, \sigma^2)$
  - pdf: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
  - cdf: $\mathbb{P}(X \leqslant x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
  - $\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2$
  - normalizing factor: $\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \sigma\sqrt{2\pi}$
- exponential random variable: $X \sim \text{Exp}(\lambda)$
  - pdf: $f(x) = \lambda e^{-\lambda x}$ for $x > 0$
  - tdf: $\mathbb{P}(X > x) = e^{-\lambda x}$ for $x > 0$
  - $\mathbb{E}[X] = \frac{1}{\lambda}$ and $\text{Var}(X) = \frac{1}{\lambda^2}$
- gamma random variable: $X \sim \text{Gamma}(\alpha, \lambda)$
  - pdf: $f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}$
  - tdf: $\mathbb{P}(X > x) = e^{-x}$
  - $\mathbb{E}[X] = \frac{\alpha}{\lambda}$ and $\text{Var}(X) = \frac{\alpha}{\lambda^2}$
  - gamma function: $\Gamma(\alpha) = \int_0^{\infty} \lambda e^{-\lambda x}(\lambda x)^{\alpha-1}\, dx$
  - functional equation: $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$
  - for $n \geqslant 1 \in \mathbb{N}$ we have: $\Gamma(n) = (n-1)!$
- beta random variable: $X \sim \text{Beta}(\alpha, \beta)$
  - pdf: $f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$ for $0 < x < 1$
  - $\mathbb{E}[X] = \frac{\alpha}{\alpha+\beta}$ and $\text{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
  - normalizing factor: $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
- cauchy random variable: $X = \tan\Theta$ where Θ is uniform on any fixed interval with length an integer mulitple of π

B4/B5 Formulas.

- If $f(x, y)$ is the jdf of X and Y:
  - $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$ and $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$
  - $f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)}$ and $f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)}$
- if $X \perp Y$ and $Z = X + Y$, then the pdf of Z is:

  $f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\, dx = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)\, dy$