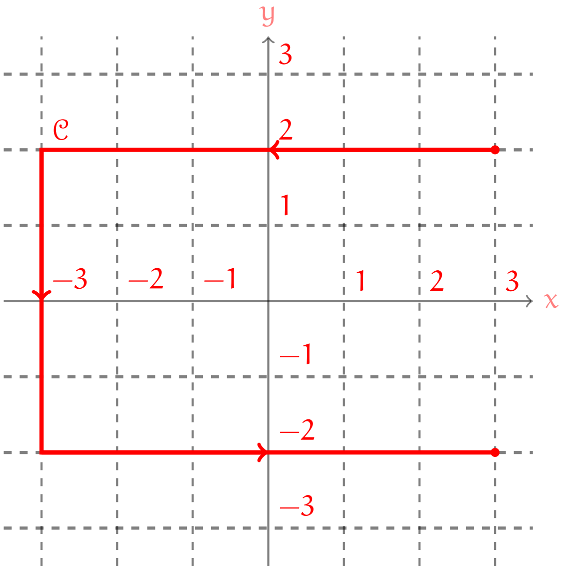
Example 1. Let:

$$\mathbf{F} = \langle x^2 + y^2, 2xy \rangle$$

and $\mathcal{C}$ be the curve depicted below, with counterclockwise orientation:

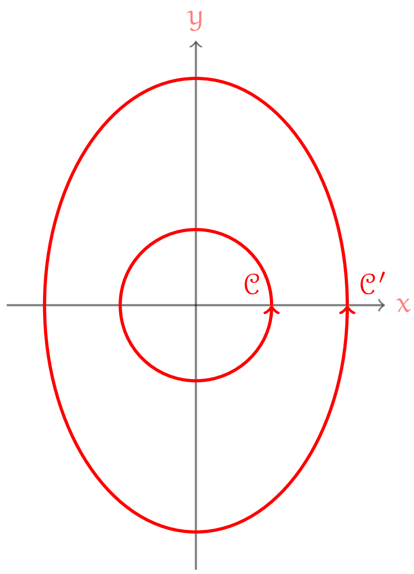


The primary difficulty in applying Green's Theorem is that the curve is not closed, and Green's Theorem requires closed (and simple) curves. The method in this case is to **close off the curve** by introducing a simple curve that, when combined with the given curve, is in fact closed. In this case, the curve we introduce is the vertical line segment that combines with $\mathcal{C}$ to form a rectangle.

Use Green's Theorem to find: $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$

Example 2. Let $\mathbf{F}(x, y)$ be vector field that is defined an irrotational everywhere except at the origin, and so that the integral of $\mathbf{F}$ over the counterclockwise unit circle $\mathcal{C}$ equals 2π. For the curve $\mathcal{C}'$ depicted below, find:

$$\oint_{\mathcal{C}'} \mathbf{F} \cdot \mathbf{dr}$$

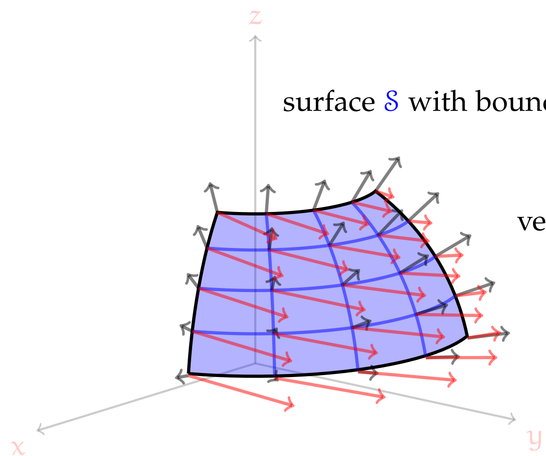


Remember that irrotational means $\operatorname{curl} \mathbf{F} = \mathbf{0}$. Also remember that when we are integrating the $Q_x - P_y$ from Green's Theorem was none other than $\operatorname{curl} \mathbf{F} \cdot \mathbf{k}$, from an earlier margin note.

We dealt with such a vector field before. It was a special one. Here it is my friends:

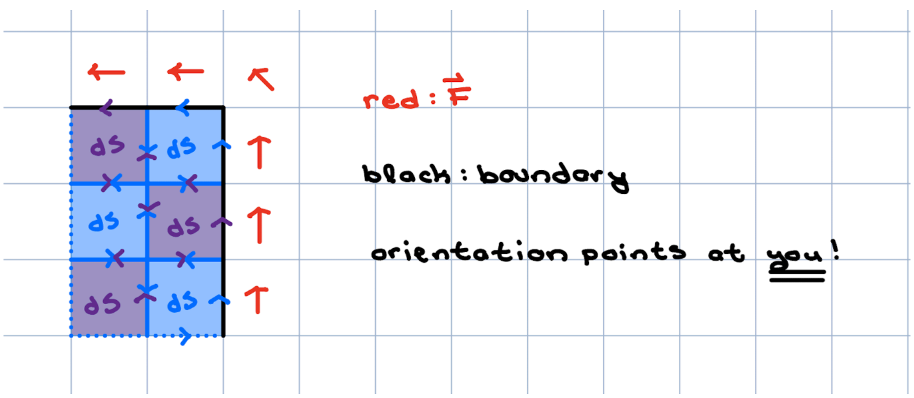
$$\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$$

A. **Stokes's Theorem.** The idea used to derive Green's Theorem does not only apply for surfaces in the xy-plane. Let's consider an oriented surface $\mathcal{S}$ in xyz–space and its **boundary** ∂S.



For a surface, the boundary can be understood as follows. Divide the surface into little rectangles, just like depicted to the left. The edges of these rectangles that do not overlap with the edges of any other rectangles form the boundary. The boundary is kind of like the paper–cut–edge of the surface, all in the presence of a vector field $\mathbf{F}(x, y, z)$.

We once again divide the region into infinitesimal bits of area, and calculate the circulation about each bit.



To calculate the circulation about the infinitesimal piece of surface area, which is orthogonal to the normal $d\mathbf{S}$, we calculate $\operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$.

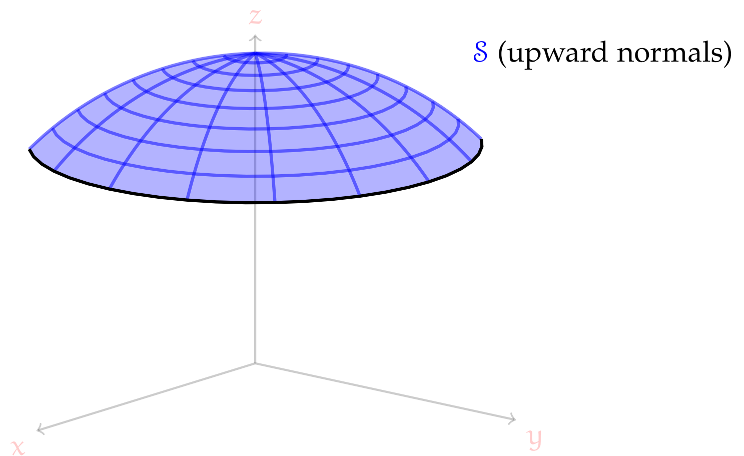
> **Stokes's Theorem.** If $\mathbf{F}$ is and defined and continuously differentiable throughout a compact oriented surface $\mathcal{S}$, and the boundary ∂S consists of simple, closed curves oriented so $\mathcal{S}$ is on the left when viewed from a perspective so the normals are pointing at you, then:

Another way to think about the orientation is using the righthand rule: flatten your right hand, then align it with the boundary so that your thumb is pointing in the direction of normals. If the surface is on the palm–side of your flattened hand, then your hand points in the correct orientation. Otherwise, your hand points opposite the correct orientation.

Example 3. Let:

$$\mathbf{F} = -y\mathbf{i} + \left(x + (z-1)xe^{x}\right)\mathbf{j} + \left(x^2 + y^2\right)\mathbf{k}$$

and $\mathcal{S}$ be the spherical cap $x^2 + y^2 + z^2 = 2$ with $z \geqslant 1$ and oriented with upward normals.



Use Stoke's Theorem to find:

$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} =$$