Classical mechanics

Classical mechanics describes the behaviour of objects in terms of two equations. One expresses the fact that the total energy is constant in the absence of external forces; the other expresses the response of particles to the forces acting on them.

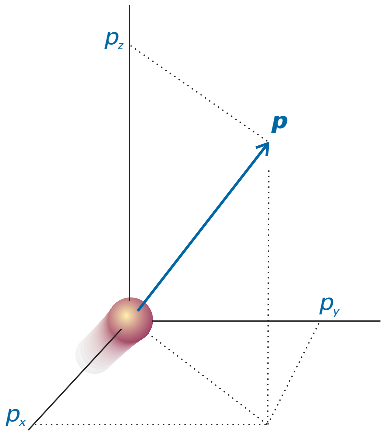

Fig. A3.1 The linear momentum of a particle is a vector property and points in the direction of motion.

A3.3 The trajectory in terms of the energy

The **velocity**, $\boldsymbol{v}$, of a particle is the rate of change of its position:

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \tag{A3.5}$$

The velocity is a vector, with both direction and magnitude. The magnitude of the velocity is the **speed**, v. The **linear momentum**, $\boldsymbol{p}$, of a particle of mass m is related to its velocity, $\boldsymbol{v}$, by

$$\boldsymbol{p} = m\boldsymbol{v} \tag{A3.6}$$

Like the velocity vector, the linear momentum vector points in the direction of travel of the particle (Fig. A3.1). In terms of the linear momentum, the total energy of a particle is

$$E = \frac{p^2}{2m} + V(x) \tag{A3.7}$$

This equation can be used to show that a particle will have a definite **trajectory**, or definite position and momentum at each instant. For example, consider a particle free to move in one direction (along the x-axis) in a region where $V = 0$ (so the energy is independent of position). Because $v = \mathrm{d}x/\mathrm{d}t$, it follows from eqns A3.6 and A3.7 that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(\frac{2E_\mathrm{K}}{m}\right)^{1/2} \tag{A3.8}$$

A solution of this differential equation is

$$x(t) = x(0) + m\left(\frac{2E_\mathrm{K}}{m}\right)^{1/2} t \tag{A3.9}$$

The linear momentum is a constant:

$$p(t) = mv(t) = m\frac{\mathrm{d}x}{\mathrm{d}t} = (2mE_\mathrm{K})^{1/2} \tag{A3.10}$$

Hence, if we know the initial position and momentum, we can predict all later positions and momenta exactly.

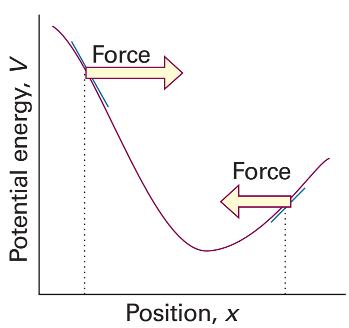

Fig. A3.2 The force acting on a particle is determined by the slope of the potential energy at each point. The force points in the direction of lower potential energy.

A3.4 Newton's second law

The **force**, F, experienced by a particle free to move in one dimension is related to its potential energy, V, by

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x} \tag{A3.11a}$$

This relation implies that the direction of the force is towards decreasing potential energy (Fig. A3.2). In three dimensions,

$$\boldsymbol{F} = -\nabla V \qquad \nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \tag{A3.11b}$$

Newton's second law of motion states that *the rate of change of momentum is equal to the force acting on the particle*. In one dimension:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = F \tag{A3.12a}$$

Because $p = m(\mathrm{d}x/\mathrm{d}t)$ in one dimension, it is sometimes more convenient to write this equation as

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F \tag{A3.12b}$$

The second derivative, $\mathrm{d}^2x/\mathrm{d}t^2$, is the **acceleration** of the particle, its rate of change of velocity (in this instance, along the x-axis). It follows that, if we know the force acting everywhere and at all times, then solving eqn A3.12 will also give the trajectory. This calculation is equivalent to the one based on E, but is more suitable in some applications. For example, it can be used to show that, if a particle of mass m is initially stationary and is subjected to a constant force F for a time τ, then its kinetic energy increases from zero to

$$E_\mathrm{K} = \frac{F^2\tau^2}{2m} \tag{A3.13}$$

and then remains at that energy after the force ceases to act. Because the applied force, F, and the time, τ, for which it acts may be varied at will, the solution implies that the energy of the particle may be increased to any value.

A3.5 Rotational motion

The rotational motion of a particle about a central point is described by its **angular momentum**, $\boldsymbol{J}$. The angular momentum is a vector: its magnitude gives the rate at which a particle circulates and its direction indicates the axis of rotation (Fig. A3.3). The magnitude of the angular momentum, J, is given by the expression

$$J = I\omega \tag{A3.14}$$

where ω is the **angular velocity** of the body, its rate of change of angular position (in radians per second), and I is the **moment of inertia**. The analogous roles of m and I, of v and ω, and of p and J in the translational and rotational cases, respectively, should be remembered, because they provide a ready way of constructing and recalling equations. For a point particle of mass m moving in a circle of radius r, the moment of inertia about the axis of rotation is given by the expression

$$I = mr^2 \tag{A3.15}$$

To accelerate a rotation it is necessary to apply a **torque**, $\boldsymbol{T}$, a twisting force. Newton's equation is then

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = \boldsymbol{T} \tag{A3.16}$$

If a constant torque is applied for a time τ, the rotational energy of an initially stationary body is increased to

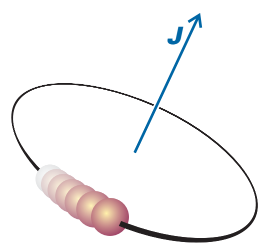

Fig. A3.3 The angular momentum of a particle is represented by a vector along the axis of rotation and perpendicular to the plane of rotation. The length of the vector denotes the magnitude of the angular momentum. The direction of motion is clockwise to an observer looking in the direction of the vector.

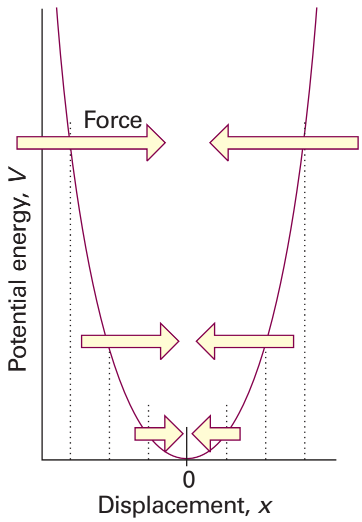

Fig. A3.4 The force acting on a particle that undergoes harmonic motion. The force is directed towards zero displacement and is proportional to the displacement. The corresponding potential energy is parabolic (proportional to x^2).

$$E_K = \frac{T^2\tau^2}{2I} \tag{A3.17}$$

The implication of this equation is that an appropriate torque and period for which it is applied can excite the rotation to an arbitrary energy.

A3.6 The harmonic oscillator

A **harmonic oscillator** consists of a particle that experiences a restoring force proportional to its displacement from its equilibrium position:

$$F = -kx \tag{A3.18}$$

An example is a particle joined to a rigid support by a spring. The constant of proportionality k is called the **force constant**, and the stiffer the spring the greater the force constant. The negative sign in F signifies that the direction of the force is opposite to that of the displacement (Fig. A3.4).

The motion of a particle that undergoes harmonic motion is found by substituting the expression for the force, eqn A3.18, into Newton's equation, eqn A3.12b. The resulting equation is

$$m\frac{d^2x}{dt^2} = -kx$$

A solution is

$$x(t) = A \sin \omega t \qquad p(t) = m\omega A \cos \omega t \qquad \omega = (k/m)^{1/2} \tag{A3.19}$$

These solutions show that the position of the particle varies **harmonically** (that is, as $\sin \omega t$) with a frequency $\nu = \omega/2\pi$. They also show that the particle is stationary ($p = 0$) when the displacement, x, has its maximum value, A, which is called the **amplitude** of the motion.

The total energy of a classical harmonic oscillator is proportional to the square of the amplitude of its motion. To confirm this remark we note that the kinetic energy is

$$E_K = \frac{p^2}{2m} = \frac{(m\omega A \cos \omega t)^2}{2m} = \tfrac{1}{2}m\omega^2 A^2 \cos 2\omega t \tag{A3.20}$$

Then, because $\omega = (k/m)^{1/2}$, this expression may be written

$$E_K = \tfrac{1}{2}kA^2 \cos^2\omega t \tag{A3.21}$$

The force on the oscillator is $F = -kx$, so it follows from the relation $F = -dV/dx$ that the potential energy of a harmonic oscillator is

$$V = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2 \sin^2\omega t \tag{A3.22}$$

The total energy is therefore

$$E = \tfrac{1}{2}kA^2 \cos^2\omega t + \tfrac{1}{2}kA^2 \sin^2\omega t = \tfrac{1}{2}kA^2 \tag{A3.23}$$

(We have used $\cos^2\omega t + \sin^2\omega t = 1$.) That is, the energy of the oscillator is constant and, for a given force constant, is determined by its maximum displacement. It follows that the energy of an oscillating particle can be raised to any value by stretching the spring to any desired amplitude A. Note that the frequency of the motion depends only on the inherent properties of the oscillator (as represented by k and m) and is independent of the energy; the amplitude governs the energy, through $E = \tfrac{1}{2}kA^2$, and is independent of the frequency. In other words, the particle will oscillate at the same frequency regardless of the amplitude of its motion.