2.1 解析函数的概念

定义:

设 $w=f(z)$ 为 D 内的单值函数, $z_0\in D$, 且 $w_0=f(z_0)$, 记 $\Delta z=z-z_0$, $\Delta w=f(z)-f(z_0)$, 若 $\lim\limits_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\lim\limits_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ 存在, 则称 $f(z)$ 在 z_0 处可导, 记 $\left.\frac{df(z)}{dz}\right|_{z=z_0}$, $f'(z_0)$

导数运算法则:

(i) $c'=0$

(ii) $[f(z)\pm g(z)]'=f'(z)\pm g'(z)$

(iii) $(f(z)g(z))'=f'(z)g(z)+f(z)g'(z)$

(iv) $\left(\frac{f(z)}{g(z)}\right)'=\frac{f'(z)g(z)-f(z)g'(z)}{g^2(z)}$

(v) $[f(g(z))]' = f'(g(z))\,g'(z)$

(vi) $[f^{-1}(w)]' = \dfrac{1}{f'(z)}$

定义:

若$f(z)$在z_0及$N(z_0,\delta)$内处处可导,则称$f(z)$在z_0解析,若$f(z)$在D每一点解析,则称$f(z)$在D解析.

故区域内,解析$\Leftrightarrow$可导,在一点处解析$\Rightarrow$可导

定义:

若$f(z)$在z_0处不解析,但在z_0的每一个邻域内,有若干个点使$f(z)$解析,则z_0称为$f(z)$的奇点.

定理:

(1) 在D内解析的两个单值函数 $f(z)$, $g(z)$ 的和、差、积、商(除去分母为零的点)在区域D内仍解析

(2) 设单值函数 $\zeta=g(z)$ 在D内解析,单值函数 $w=f(\zeta)$ 在区域 D' 内解析,且 $g(D)\subset D'$,则 $w=f[g(z)]$ 在D内解析

对于有理分式函数,$\frac{P(z)}{Q(z)}$ 在分母为零的点为其奇点、

2.2 函数解析的充要条件

Cauchy-Riemann方程: $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

定理:

设 $f(z)=u(x,y)+iv(x,y)$ 在区域D上的复变函数,则 $f(z)$ 在D内一点

$z=x+iy$处可导$\Leftrightarrow$ $u(x,y)$和$v(x,y)$在(x,y)可微且满足C-R方程

定理:

$f(z)=u(x,y)+iv(x,y)$在D内解析$\Leftrightarrow$ $u(x,y)$和$v(x,y)$在D内$z=x+iy$可微,

且满足C-R方程

系:

若$f'(z)$存在,则 $f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$

$$=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$$

例: $f(z)=x^2+axy+by^2+i(cx^2+dxy+y^2)$,试确定a,b,c,d的值,使f(z)

在z-平面内解析.

设 $u=x^2+axy+by^2$　$v=cx^2+dxy+y^2$，且

$\frac{\partial u}{\partial x}=2x+ay$　$\frac{\partial u}{\partial y}=ax+2by$　$\frac{\partial v}{\partial x}=2cx+dy$　$\frac{\partial v}{\partial y}=dx+2y$

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$，$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

$2x+ay=dx+2y$　$2cx+dy=-ax-2by$

当 $a=2$，$b=-1$，$c=-1$，$d=2$ 时，处处解析

例：证：$f(z)=e^x(\cos y+i\sin y)$ 在 z 平面解析，且 $f'(z)=f(z)$

$u=e^x\cos y$　$v=e^x\sin y$

$\frac{\partial u}{\partial x}=e^x\cos y$　$\frac{\partial u}{\partial y}=-e^x\sin y$

$\frac{\partial v}{\partial x}=e^x\sin y$　$\frac{\partial v}{\partial y}=e^x\cos y$

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$　$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$ 符合 C-R 方程，且偏导数在平面上处处连续.

$\therefore$ $f(z)$ 是 z-平面内的解析函数，同时

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=e^{x}(\cos y+i\sin y)=f(z)$$

满足C-R方程 $\not\Rightarrow$ 可导

例：$f(z)=\sqrt{|xy|}$

$u=\sqrt{|xy|}\quad v=0$

$$\left.\frac{\partial u}{\partial x}\right|_{z=0}=\lim_{\Delta x\to 0}\frac{u(\Delta x,0)-u(0,0)}{\Delta x}=0$$

$$\left.\frac{\partial u}{\partial y}\right|_{z=0}=\lim_{\Delta y\to 0}\frac{u(0,\Delta y)-u(0,0)}{\Delta y}=0$$

同理 $\left.\frac{\partial v}{\partial x}\right|_{z=0}=0\quad \left.\frac{\partial v}{\partial y}\right|_{z=0}=0$

$\therefore$ 满足C-R方程

但 $\lim\limits_{\Delta z\to 0}\frac{\sqrt{|\Delta x||\Delta y|}}{\Delta z}=\begin{cases}0 & \Delta y=0\quad \Delta x\to 0\\ \pm\frac{(1-i)}{2} & \Delta x=\Delta y\to 0\end{cases}$

$\therefore f(z)$ 在 $z=0$ 处 ~~可~~导

2.3 解析函数与调和函数

Laplace方程：$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$，$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$

定义：

凡在区域D内具有连续二阶偏导数而且满足Laplace方程的二元实函数$u(x,y)$，称在区域D内的调和函数

定理：

设$f(z)=u(x,y)+iv(x,y)$在区域D内解析，则$f(z)$的实部$u(x,y)$和虚部$v(x,y)$都是区域D内的调和函数

定义：

在区域D中满足C-R方程的两个调和函数u,v中，v称为共轭调和函数

几何上看，一对共轭调和函数的等值线 $u(x,y)=C_1$, $v(x,y)=C_2$，在其交点上永远相互正交的.

Actually，这两条曲线的法线方向向量为 $\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$ $\left(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right)$.

定理：

$f(z)=u(x,y)+iv(x,y)$ 在 D 内解析 $\Leftrightarrow$ 在 D 内，$f(z)$ 的虚部 $v(x,y)$ 是实部 $u(x,y)$ 的共轭调和函数

系：

给定单连通区域 D 内的调和函数 $\varphi(x,y)$ 总可作出一族解析函数使其实部（或虚部）为 $\varphi(x,y)$，这族解析函数中的任何两个至多相差一个纯

虚数(或实数)

例: $u(x,y)=e^x\cos y+x+y$，求满足 $f(0)=1$ 解析函数 $f(z)=u(x,y)+iv(x,y)$

$\frac{\partial u}{\partial x}=e^x\cos y+1$ $\frac{\partial u}{\partial y}=-e^x\sin y+1$

$\frac{\partial^2 u}{\partial x^2}=e^x\cos y$ $\frac{\partial^2 u}{\partial y^2}=-e^x\cos y$

满足 Laplace 方程，从而为调和函数

$\therefore v(x,y)=\int_{(0,0)}^{(x,y)}(e^x\sin y-1)dx+(e^x\cos y+1)dy+c$

$=\int_0^x(-1)dx+\int_0^y(e^x\cos y+1)dy+c$

$=e^x\sin y-x+y+c$

$\therefore f(z)=e^x\cos y+x+y+i(e^x\sin y-x+y+c)$

$=e^z+(1-i)z+ic$

$\because f(0)=1$ $\therefore c=0$ $\therefore f(z)=e^z+(1-i)z$

例：调和函数 $u(x,y)=x^2-y^2+xy$，求一满足条件 $f(0)=0$ 的解析函数

$f(z)=u+iv$

$\because \frac{\partial u}{\partial x}=2x+y \quad \frac{\partial u}{\partial y}=-2y+x$

由 C-R 方程，$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=2x+y$

$\therefore v=\int(2x+y)dy=2xy+\frac{1}{2}y^2+\varphi(x)$

又 $\because \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$

$\therefore 2y+\varphi'(x)=2y-x \Rightarrow \varphi'(x)=-x$

$\therefore \varphi(x)=-\frac{1}{2}x^2+C$

$v(x,y)=2xy+\frac{1}{2}y^2-\frac{1}{2}x^2+C$ 与 $u(x,y)=x^2-y^2+xy$ 的共轭调和函数

$f(z)=x^2-y^2+xy+i(2xy+\frac{1}{2}y^2-\frac{1}{2}x^2+C)$

$=(1-\frac{i}{2})z^2+iC$

$f(0)=0 \Rightarrow c=0$，则 $f(z)=\left(1-\frac{i}{2}\right)z^2$

例：$v(x,y)=\arctan\frac{y}{x}\ (x>0)$，$f(z)=u(x,y)+iv(x,y)$

$\frac{\partial u}{\partial x}=-\frac{y}{x^2+y^2}$ $\frac{\partial u}{\partial y}=\frac{x}{x^2+y^2}$ $\frac{\partial^2 u}{\partial x^2}=\frac{2xy}{(x^2+y^2)^2}$ $\frac{\partial^2 u}{\partial y^2}=\frac{-2xy}{(x^2+y^2)^2}$

$\therefore v(x,y)$ 右半部为调和函数.

$u(x,y)=\int\frac{\partial u}{\partial x}dx=\int\frac{\partial v}{\partial y}dx=\int\frac{x}{x^2+y^2}dx=\frac{1}{2}\ln(x^2+y^2)+\varphi(y)$

$\because \frac{\partial u}{\partial y}=\frac{y}{x^2+y^2}$，$\varphi'(y)=-\frac{\partial v}{\partial x}=\frac{y}{x^2+y^2}$

知 $\varphi'(y)=0$，故 $\varphi(c)=c$ $\therefore u(x,y)=\frac{1}{2}\ln(x^2+y^2)+c$

$\therefore f(z)=\frac{1}{2}\ln(x^2+y^2)+c+i\arctan\frac{y}{x}$

$=\ln|z|+i\arg z+c=\ln z+c$

例: $f(z)=u(x,y)+iv(x,y)$ 为解析函数,且 $u+v=x^3-y^3+3x^2y-3xy^2-2x-2y$,

求 $f(z)=u+iv$ 的表达式

$$\frac{\partial(u+v)}{\partial x}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=3x^2+6xy-3y^2-2$$

$$\frac{\partial(u+v)}{\partial y}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=3x^2-3y^2-6xy-2$$

$\because f(z)=u+iv$ 解析 $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial u}{\partial x}$

$$\therefore \frac{\partial u}{\partial x}=3x^2-3y^2-2 \qquad \frac{\partial v}{\partial x}=6xy$$

$$u(x,y)=\int\frac{\partial u}{\partial x}dx=\int(3x^2-3y^2-2)dy=x^3-3xy^2-2x+\varphi(y)$$

由C-R方程 $-6xy+\varphi'(y)=-6xy$

$\therefore \varphi'(y)=0 \Rightarrow \varphi(y)=c$

$\therefore u(x,y)=x^3-3xy^2-2x+c \qquad v(x,y)=-y^3+3x^2y-2y+c$

$\therefore f(z)=z^3-2z+(1-i)c \qquad c\in\mathbb{R}$

例：设调和函数 $u(x,y)=\varphi(\frac{y}{x})$，求其共轭调和函数 $v(x,y)$，$f(z)=u+iv$ 为解析函数

设 $t=\frac{y}{x}$

$\frac{\partial u}{\partial x}=\varphi'(t)(-\frac{y}{x^2})$　　$\frac{\partial u}{\partial y}=\varphi'(t)\cdot\frac{1}{x}$

$\frac{\partial^2 u}{\partial x^2}=\varphi''(t)\cdot\frac{y^2}{x^4}+\varphi'(t)\frac{2y}{x^3}$　$\frac{\partial^2 u}{\partial y^2}=\varphi''(t)\frac{1}{x^2}$

$\because u(x,y)$ 为调和函数

$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\varphi''(t)\frac{y^2}{x^4}+\varphi'(t)\frac{2y}{x^3}+\varphi''(t)\cdot\frac{1}{x^2}=0$

若 $\varphi'(t)=0$ 则 $\varphi(t)=C_1$　$u(x,y)=C_1$，此时 $v(x,y)=C_2$

$\therefore f(z)=u(x,y)+iv(x,y)=C_1+iC_2=\beta$，$\beta$ 为复常数，若 $\varphi'(t)$ 不恒为0，有

$$\frac{\varphi''(t)}{\varphi'(t)}=\frac{-2t}{1+t^2}$$

$\Rightarrow \ln\varphi'(t)=-\ln(1+t^2)+\ln C$

$\therefore \varphi(t) = \int \frac{c\,dt}{1+t^2} = c\arctan t + C_1$

$\therefore u(x,y) = c\arctan\frac{y}{x} + C_1$

由C-R方程：$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -c\frac{\frac{1}{x}}{1+(\frac{y}{x})^2} = -c\frac{x}{x^2+y^2}$

$v(x,y) = -c\int \frac{x}{x^2+y^2}dx = -\frac{c}{2}\ln(x^2+y^2) + g(y)$

又 $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = c\frac{-\frac{y}{x^2}}{1+(\frac{y}{x})^2} = -c\frac{y}{x^2+y^2} = -c\frac{y}{x^2+y^2} + g'(y)$

$\therefore g'(y) = 0 \quad g(y) = C_2$

$\therefore v(x,y) = -c\ln\sqrt{x^2+y^2} + C_2$

$\therefore f(z) = c\arctan\frac{y}{x} + C_1 + i[-c\ln\sqrt{x^2+y^2} + C_2]$

$= -ic(\ln\sqrt{x^2+y^2} + i\arctan\frac{y}{x}) + C_2 i + C_1$

$= \alpha(\ln|z| + i\arg z) + \beta$

$= \alpha\ln z + \beta \quad (\alpha = -ic, \beta$ 为复常数$)$

2.4 初等函数

一. 指数函数

$$e^{z}=e^{x+iy}=e^{x}(\cos y+i\sin y)$$

$|e^{z}|=e^{x}$　　y 为 e^{z} 的辐角

$e^{z_1+z_2}=e^{z_1}\cdot e^{z_2}$　　　$e^{z}=e^{z+2k\pi i}$　$k=0,\pm 1,\pm 2\cdots$

$$(e^{z})'=e^{z}$$

$w=e^{z}$ 将 $z\begin{cases}-\pi<\operatorname{Im}z<\pi\\ -\infty<\operatorname{Re}z<+\infty\end{cases}$ 映射到 $w\begin{cases}-\pi<\arg w<\pi\\ 0<|w|<+\infty\end{cases}$

Specially, 将 $z\begin{cases}-\pi<\operatorname{Im}z<\pi\\ -\infty<\operatorname{Re}z<0\end{cases}$ 映射到 $w\begin{cases}-\pi<\arg w<\pi\\ 0<|w|<1\end{cases}$

将 $z\begin{cases}-\pi<\operatorname{Im}z<\pi\\ 0<\operatorname{Re}z<+\infty\end{cases}$ 映射到 $w\begin{cases}-\pi<\arg w<\pi\\ 1<|w|<+\infty\end{cases}$

二. 三角函数与双曲函数

$\cos z=\frac{1}{2}(e^{iz}+e^{-iz})$ $\sin z=\frac{1}{2i}(e^{iz}-e^{-iz})$

$\cos(z+2k\pi)=\cos z$ $\sin(z+2k\pi)=\sin z$ $k=0,\pm1,\pm2,\cdots$

$\sin^2 z+\cos^2 z=1$

对实变数三角函数成立的一切恒等式,在复变数上自然成立.

但 $|\cos z|\leq 1$ 与 $|\sin z|\leq 1$ 不成立

例: $\cos(iy)=\frac{1}{2}(e^{-y}+e^{y})$ $\lim\limits_{y\to+\infty}\cos(iy)=+\infty$

$(\sin z)'=\cos z$ $(\cos z)'=-\sin z$

$\sin z=0$ 的根为 $z=k\pi$ $(k=0,\pm1,\pm2\cdots)$

$\cos z=0$ 的根为 $z=\frac{\pi}{2}+k\pi$ $(k=0,\pm1,\pm2,\cdots)$

$\tan' z=\sec^2 z$ $\cot' z=-\csc^2 z$

$\sec' z=\sec z\tan z$ $\csc' z=-\csc z\cot z$

$\tan(z+\pi) = \tan z$

定义：

$\operatorname{sh} z = \dfrac{e^{z}-e^{-z}}{2}$ $\quad \operatorname{ch} z = \dfrac{e^{z}+e^{-z}}{2}$ $\quad \operatorname{th} z = \dfrac{\operatorname{sh} z}{\operatorname{ch} z}$ $\quad \operatorname{cth} z = \dfrac{1}{\operatorname{th} z}$

$\operatorname{sech} z = \dfrac{1}{\operatorname{ch} z}$ $\quad \operatorname{csch} z = \dfrac{1}{\operatorname{sh} z}$

$\therefore \operatorname{sh} z = \operatorname{sh}(z+2k\pi i)$ $\quad \operatorname{ch} z = \operatorname{ch}(z+2k\pi i)$ $\quad k = 0, \pm 1, \pm 2, \cdots$

$\operatorname{ch} z = \operatorname{ch}(-z)$ $\quad \operatorname{sh} z = -\operatorname{sh}(-z)$

$(\operatorname{ch} z)' = \operatorname{sh} z$ $\quad (\operatorname{sh} z)' = \operatorname{ch} z$

且 $\operatorname{ch} iy = \cos y$ $\quad \operatorname{sh} iy = i \sin y$

及 $\begin{cases} \operatorname{ch}(x+iy) = \operatorname{ch} x \cos y + i \operatorname{sh} x \sin y \\ \operatorname{sh}(x+iy) = \operatorname{sh} x \cos y + i \operatorname{ch} x \sin y \end{cases}$

三.对数函数

$w=\mathrm{Ln}z$

$\mathrm{Ln}z=\ln|z|+i\mathrm{Arg}z$

$\mathrm{Ln}z$的主值 $\ln z=\ln|z|+i\arg z$

$\therefore$、 $\mathrm{Ln}z=\ln z+2k\pi i \quad (k=0,\pm1,\pm2,\cdots)$

对于每一个固定的k，上式为单值函数 称为$\mathrm{Ln}z$的一个分支

$\mathrm{Ln}z$有如下性质:

(1) $\mathrm{Ln}(z_1z_2)=\mathrm{Ln}z_1+\mathrm{Ln}z_2$

$\mathrm{Ln}\dfrac{z_1}{z_2}=\mathrm{Ln}z_1-\mathrm{Ln}z_2$

(2) $\mathrm{Ln}z$的主值$\ln z=\ln|z|+i\arg z$在$\mathbb{C}$-负实轴的单连通区域，

$\begin{cases}|z|>0\\-\pi<\arg z<\pi\end{cases}$ 内解析，且$(\ln z)'=\dfrac{1}{z}$

四.幂函数

设a为任意实数，对于$z\neq0$，$w=z^a=e^{a\ln z}=e^{a\ln z+i2ak\pi}=e^{a\ln z}e^{i2ak\pi}$ $(k=0,\pm1,\pm2\cdots)$

Specially:

(1) 当a为整数n时，有 $w=z^n=e^{n\ln z}$

(2) 当a为有理数$\frac{p}{q}$时有 $w=z^{\frac{p}{q}}=e^{\frac{p}{q}\ln z+\frac{p}{q}i2k\pi}$ $(k\in\mathbb{Z})$

$\because$ p与q互质 $\therefore$当k取$0,1,\cdots,q-1$时，有$e^{i2k\pi\frac{p}{q}}=(e^{i2kp\pi})^{\frac{1}{q}}$

是q个不同的值.

(3) 当a为其他值时，$w=z^{\alpha}$为无穷多值函数

$(z^{\alpha})'=\alpha z^{\alpha-1}$

五. 反三角函数和反双曲函数

$$\mathrm{Arccos}\, z = -i\,\mathrm{Ln}(z+\sqrt{z^2-1})$$

$$\mathrm{Arcsin}\, z = -i\,\mathrm{Ln}(iz+\sqrt{1-z^2})$$

$$\mathrm{Arctan}\, z = -\frac{i}{2}\,\mathrm{Ln}\frac{1+iz}{1-iz}$$

$$\mathrm{Arsh}\, z = \mathrm{Ln}(z+\sqrt{z^2+1})$$

$$\mathrm{Arch}\, z = \mathrm{Ln}(z+\sqrt{z^2-1})$$

$$\mathrm{Arth}\, z = \frac{1}{2}\,\mathrm{Ln}\frac{1+z}{1-z}$$

*2.5 解析函数的物理意义