Exercise

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November 29, 2013

1 Exercices

(1). Show that a face F of a polytope P is exactly the convex hull of all vertices of P contained in F. In particular, p jas only finitely many faces.

Solution

Observation 1.1. *The convex hull of a point set lying in a affine space lies in this affine space.*

Observation 1.2. *Let $F = \{x \mid ax = b\} \cap P$ be a face. Then the convex hull of its vertices lies in F.*

Proposition 1.3. *Let $F = \{x \mid ax = b\} \cap P$ be a face. Then F lies in the convex hull of their vertices.*

Proof. If $F = \emptyset$ it is clear.

Let be $p \in F$. Then, $p \in P$, so p can be written as $p_{i_1}\lambda_{i_1} + \cdots + p_{i_k}\lambda_{i_k}$ for some $k > 0$, where p_{i_j} are vertices of P, $\lambda_{i_j} > 0$ and $\sum_j \lambda_{i_j} = 1$.

By definition of face, we have all vertices of P in F p_i satisfy $ap_i = b$, and all vertices of P not in F satisfy $ap_i < b$. Then, by linearity we have that:

$$ap = a(p_{i_1}\lambda_{i_1} + \cdots + p_{i_k}\lambda_{i_k}) \leq (\sum_j \lambda_{i_j})b = b$$

with equality if and only if all p_{i_j} are vertices of P in F. As $p \in F$, the equality is required. Thus, p is in the convex hull of vertices of P in the face F.

□

(2). Let $P \subset \mathbb{R}^d$, $Q \subset \mathbb{R}^e$ be two non-empty polytopes. Prove that the set of faces of the cartesian product polytope $P \times Q = \{(p, q) \in \mathbb{R}^{d+e} : p \in P,\ q \in Q\}$ exactly equals $\{F \times G : F \text{ is face of } P,\ G \text{ is face of } Q\}$. Conclude that

$$f_k(P \times Q) \ = \sum_{i+j=k,\ i,j\geq 0} f_i(P)f_j(Q) \qquad \text{for } k \geq 0.$$

Solution Let $F = \{x \mid a_F x = b_F\} \cap P$ and $G = \{x \mid a_G x = b_G\} \cap Q$ be faces of P and Q respectively.

Then, by linearity, we have:

$$\begin{array}{rcl} F \times G & = & (\{x \mid a_F x = b_F\} \cap P) \times (\{x \mid a_G x = b_G\} \cap Q) \\ & = & (\{x \mid a_F x = b_F\} \times \{x \mid a_G x = b_G\}) \cap (P \times Q) \\ & = & \{x \mid a_F \oplus a_G x = b_F + b_G\} \cap P \times Q \end{array}$$

Using the same arguments for the inequalities we obtain:

$$\{x \mid a_F \oplus a_G x \leq b_F + b_G\} \supset P \times Q$$

thus $F \times Q$ is a face of $P \times Q$.

Conversely, if we have a face $H = \{x \mid ax = b\} \cap P \times Q$, let us define $a_F \in (\mathbb{R}^d)^*$ and $a_G \in (\mathbb{R}^e)^*$ as the only ones such that $a = a_F \oplus a_G$. Let us note in the same way the covectors $a_F \oplus 0$ and $0 \oplus a_G$.

If the face is empty (or total), we have it is the cartesian product of empty (total) faces of P and Q. Otherwise:

Then or a_F or a_G is different from zero. Let us suppose without loss of generality that $a_F \neq 0$.

As P is compact, it is bounded. Then, exists $b_F \in \mathbb{R}$ such that $F := \{x \mid a_F x = b_F\} \cap P$ is a non-empty face of P. Then let us call $b_G := b - b_F$.

Then, for all $x_F \oplus x_G \in H$, $x_F \in F$. Let us prove it:

If $x_F \notin F$, as $x_F \in P$, then $a_F x_F \neq b_F$. If $a_F x_F < b_F$, take a point $x'_F \in F$. Then $a(x'_F \oplus x_G) > b$, so H is not a face. Otherwise, if $a_F x_F > b_F$, then $a(x'_F \oplus x_G) < b$, but x'_F maximizes a_F in P, so H still being not a not-empty face. So $x_F \in F$. Visually, we are saying that a "tangent plane" must be "tangent in every dimension".

Finally, note that for all $x_F \oplus x_G \in H$, as $a_F x_F = b_F$, $a_G x_G = b_G$. The same fact is used to show that the set $G := \{x \mid a_G x_G = b_G\} \cap Q$ is a face of Q and the projection of all points of H in Q is in G. Observe that in this case no matters if $a_G = 0$ or not; if $a_G = 0$ then $G = Q$ since $H \neq \emptyset$.

Then we have seen that faces of the product are product of faces. By exercise one, it follows that dimension is sum of dimensions.