Discrete and Algorithmic Geometry

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Sheet 4

due on Monday, December 9, 2013

(1)
- The (d,k)-*hypersimplex* is the polytope $\Delta(d,k) = \Box_0^d \cap H_k$, where $\Box_0^d$ is the cube $\Box_0^d = \{x \in \mathbb{R}^d : 0 \le x_i \le 1 \text{ for all } i \in [d]\}$, and $H_k = \{x \in \mathbb{R}^d : \sum_{i=1}^d x_i = k\}$.
- $\Delta'(d,k) = \Box_0^d \cap S_k$, where S_k is the slab $\{x \in \mathbb{R}^d : k-1 \le \sum_{i=1}^d x_i \le k\}$.
- Analogously, define $\Sigma(d,k) = \Delta^d \cap H_k$, where $\Delta^d = \operatorname{conv}\{0, e_1, \dots, e_1 + \dots + e_d\}$, and $\Sigma'(d,k) = \Delta^d \cap S_k$.
- A polytope P is ℓ-*simplicial* if all ℓ-dimensional faces of P are simplices.
- P is ℓ-*simple* if all ℓ-dimensional faces of the polar polytope P^Δ are simplices.

(a) *"All faces of hypersimplices are hypersimplices".* True or false?

(b) *"All faces of a $\Sigma(d,k)$ are of the form $\Sigma(d',k')$".* True or false?

(c) Calculate f_0 and f_{d-1} for $\Delta(d,k)$, $\Delta'(d,k)$, $\Sigma(d,k)$ and $\Sigma'(d,k)$.

(d) Is there any relationship between $\Delta(d,k)$ and $\Delta'(d',k')$?

(e) Is there any relationship between $\Sigma(d,k)$ and $\Sigma'(d',k')$?

(f) *"The polytope $\Sigma(d,k)$ is the closure of the quotient of $\Delta(d,k)$ under an action of the symmetric group S_d".* Discuss.

(g) For each triple $(k,\ell,d) \in \mathbb{N}^3$ with $0 \le k, \ell \le d$, decide the truth of the following statements, where P is, in turn, $\Delta(d,k)$, $\Delta'(d,k)$, $\Sigma(d,k)$ and $\Sigma'(d,k)$:
(i) P is ℓ-simplicial; (ii) P is ℓ-simple; (iii) P is ℓ-neighborly.
Hint: Use `polymake` for some small cases, and extrapolate using (a), (b), if true.

(2) Let R be an integral rectangle whose edges are parallel to the coordinate axes in $\mathbb{R}^2$, and let T be a rectangular triangle two of whose edges are parallel to the coordinate axes. Show that Pick's Theorem holds for R and T.

(3) (a) For any $a,b,c,d \in \mathbb{N}$, consider the line segment $S = \operatorname{conv}\{(a,b),(c,d)\}$. Prove that the number of integer points on S is $\gcd(a-c, b-d)+1$.

(b) For any two fixed positive integers a,b, let T be the lattice triangle with vertices $(0,0)$, $(a,0)$, $(0,b)$.
(i) Compute $\mathsf{L}_T(t)$ and $\mathsf{Ehr}_T(z)$. (ii) Use (i) to derive the following formula:

$$\gcd(a,b) \;=\; 2\sum_{k=1}^{b-1} \left\lfloor \frac{ka}{b} \right\rfloor + a + b - ab.$$

Turning in your work

Put your answers into a .pdf file. To turn it in, use `gpg` and the public key `julian.gpg.pub` in the `github` repository to create an encrypted copy that is only readable by me. Then commit and push this encrypted file to the repository.