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A Zonotope Associated with Graphical Degree Sequences

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ABSTRACT. Let D_n denote the convex hull in $\mathbf{R}^n$ of all (ordered) degree sequences of simple n-vertex graphs. Using the fact that D_n is a zonotope, an explicit generating function is found for the number of these degree sequences. The f-vector of D_n is found using Zaslavsky's theory of signed graph colorings. Finally we give a generalization based on a result of Fulkerson, Hoffman, and MacAndrew.

1. Introduction

Let G be a simple graph (i.e., no loops or multiple edges) with vertex set $[n] := \{1, 2, \ldots, n\}$. Let $\deg(i)$ denote the degree of vertex i. We call the sequence $d(G) := (\deg(1), \ldots, \deg(n))$ the *degree sequence* of G. Note that our degree sequences are *ordered*; we do not require, as is often done, that the degrees are listed in descending order. Regard the vector $d(G)$ as a point in the vector space $\mathbf{R}^n$, and define

$$D_n = \operatorname{conv}\{d(G) : G \text{ is a simple graph with vertex set } [n]\},$$

where conv denotes convex hull. Thus D_n is a convex polytope, first considered by Koren [6] (suggested to him by M. Perles) and called the *polytope of degree sequences* (of length n). Koren determined the vertices of D_n and gave a system of (redundant) linear inequalities which determine D_n. Subsequently Peled and Srinivasan [8] obtained considerable further information about D_n, including a description of its facets and edges.

It follows from the Erdös-Gallai inequalities (explained in more detail below) that an integer point $(d_1, \ldots, d_n) \in D_n \cap \mathbf{Z}^n$ is the degree sequence of some graph G if and only if $d_1 + \cdots + d_n$ is even. Peled suggested to this writer that the number of distinct (ordered) degree sequences of n-vertex graphs might be closely approximated by half the volume of D_n, and

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asked whether it was possible to compute or estimate the volume of D_n. It turns out that D_n is a *zonotope*, a special kind of polytope (defined below) with a well-developed theory for computing the number of integer points, the volume, the f-vector, etc. Moreover, there is a slight modification of D_n, which we denote by $\tilde{D}_n$, which is a zonotope combinatorially equivalent to D_n, but whose integer points correspond *exactly* to the degree sequences of graphs. Using the theory of zonotopes we are able to determine exactly the number $f(n)$ of degree sequences of n-vertex graphs, viz.,

$$\sum_{n\geq 0} f(n)\frac{x^n}{n!} = \frac{1}{2}\left[\left(1+2\sum_{n\geq 1} n^n \frac{x^n}{n!}\right)^{1/2} \times \left(1-\sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}\right)+1\right] e^{\sum_{n\geq 1} n^{n-2}x^n/n!}.$$

Moreover, using the connection between the zonotope $\tilde{D}_n$ and Zaslavsky's theory of signed coloring [**14**], we are able to compute exactly the f-vector of $\tilde{D}_n$ (or of D_n).

2. The Ehrhart polynomial of an integer zonotope

A *zonotope* is by definition a (Minkowski) sum of closed line segments. In other words, if $L_1, \dots, L_r$ is a set of closed line segments in $\mathbf{R}^n$, then the corresponding zonotope is given by

$$Z(L_1, \dots, L_r) = \{\alpha_1 + \cdots + \alpha_r : \alpha_1 \in L_1, \dots, \alpha_r \in L_r\}. \tag{1}$$

If L_i joins the origin with the vector $\beta_i \in \mathbf{R}^n$, then we also write $Z(\beta_1, \dots, \beta_r)$ for $Z(L_1, \dots, L_r)$.

For any positive integer q and any convex polytope $\mathscr{P}$ with integer vertices, let $i(\mathscr{P}, q)$ denote the number of points $\alpha \in \mathscr{P}$ satisfying $q\alpha \in \mathbf{Z}^n$. Then $i(\mathscr{P}, q)$ is a polynomial function of q of degree $\dim \mathscr{P}$, called the *Ehrhart polynomial* of $\mathscr{P}$. Thus $i(\mathscr{P}, 1)$ is equal to the number of integer points in $\mathscr{P}$. Moreover, if $\mathscr{P} \subset \mathbf{R}^n$, then the coefficient of q^n in $i(\mathscr{P}, q)$ is equal to the n-dimensional volume of $\mathscr{P}$. For these and other basic facts concerning Ehrhart polynomials, see, e.g., [**11**, pp. 235–241]. The following lemma is a refinement of a result of Shephard [**9**, Theorem 54]. Define a *half-open cube* C to be a Minkowski sum of linearly independent half-open line segments $L_1, \dots, L_r$. In other words, if $\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_s \in \mathbf{R}^n$ and $\beta_1 - \alpha_1, \dots, \beta_s - \alpha_s$ are linearly independent, then C has the form

$$C = \{a_1\alpha_1 + (1-a_1)\beta_1 + a_2\alpha_2 + (1-a_2)\beta_2 + \cdots + a_s\alpha_s + (1-a_s)\beta_s : 0 \leq a_i < 1\}.$$

We call the vectors $\gamma_i := \beta_i - \alpha_i$ the *generating vectors* of C. (If $s = 0$ then C is a single point, whose set of generating vectors is empty.)

2.1. LEMMA. *Let L_i be the line segment connecting the origin to $\gamma_i \in \mathbf{R}^n$, $1 \le i \le r$. Then the zonotope $Z = Z(L_1, \dots, L_r) = Z(\gamma_1, \dots, \gamma_r)$ is a disjoint union $Z = \biguplus_X C_X$ of half-open cubes C_X, where* (a) *X ranges over all linearly independent subsets of $\{\gamma_1, \dots, \gamma_r\}$, and* (b) *if $X = \{\delta_1, \dots, \delta_s\}$ then there are signs $e_i \in \{-1, 1\}$ such that C_X is generated by $e_1\delta_1, \dots, e_s\delta_s$. Moreover, if each $\gamma_i \in \mathbf{Z}^n$ then the vertices of each C_X lie in $\mathbf{Z}^n$.*

SKETCH OF PROOF (suggested by G. Ziegler). The proof is a "Tutte-Grothendieck argument", in the sense of [**2**]. We proceed by induction on r. The lemma is clear for $r = 0$, when Z consists of a single point. Assume now the case for $r-1$. Let $Z' = Z(L_1, \dots, L_{r-1})$, so Z' has a desired decomposition $Z' = \biguplus C_X$, where X ranges over all linearly independent subsets of $\{\gamma_1, \dots, \gamma_{r-1}\}$. Now orthogonally project Z' to a hyperplane orthogonal to L_r. We obtain a zonotope $\overline{Z}$ generated by the projections $\overline{L}_1, \dots, \overline{L}_{r-1}$ of the line segments $L_1, \dots, L_{r-1}$. By induction decompose $\overline{Z} = \biguplus_{\overline{Y}} C_{\overline{Y}}$, where the sets $\overline{Y}$ are projections of subsets Y of $\{\gamma_1, \dots, \gamma_{r-1}\}$ for which $Y \cup \{\gamma_r\}$ is linearly independent. Now "lift" this decomposition back up to $\mathbf{R}^n$. Each half-open cube $C_{\overline{Y}}$ is lifted to a product $(O, \gamma_r] \times C_Y$, where $(O, \gamma_r]$ denotes the half-open interval from the origin O to γ_r which excludes O and includes γ_r. Then

$$Z = \left(\bigcup_X C_X\right) \cup \left(\bigcup_Y (O, \gamma_r] \times C_Y\right)$$

gives a desired decomposition of Z. $\square$

The next theorem is a basic result about the Ehrhart polynomial of an integer zonotope. It was stated without proof in [**10**, Example 3.1; **11**, Exercise 4.31]. The special case of the volume of a zonotope $Z \subset \mathbf{R}^n$ (the coefficient of q^n in $i(Z, q)$) is due to McMullen (see [**9**, (57)]) and also appears in [**7**].

2.2. THEOREM. *Let $\beta_1, \dots, \beta_r \in \mathbf{Z}^n$. Let Z denote the zonotope $Z(\beta_1, \dots, \beta_r)$. Then*

$$i(Z, q) = \sum_X h(X) q^{|X|},$$

where X ranges over all linearly independent subsets of $\{\beta_1, \dots, \beta_r\}$, and where $h(X)$ denotes the greatest common divisor of all minors of size $|X|$ of the matrix whose rows are the elements of X.

PROOF. By the preceding lemma, we have $Z = \biguplus C_X$, where X runs over all linearly independent subsets $\{\delta_1, \dots, \delta_s\}$ of $\{\beta_1, \dots, \beta_r\}$, and where C_X is a half-open cube with integer vertices generated by $\pm\delta_1, \dots, \pm\delta_s$ for some choice of signs. Suppose C is any half-open cube with integer vertices generated by vectors $\zeta_1, \dots, \zeta_s \in \mathbf{Z}^n$. Let $Y = \{\zeta_1, \dots, \zeta_s\}$. Let Λ be the (additive) abelian group of integer vectors in the subspace of $\mathbf{R}^n$ spanned by Y, so $\Lambda \cong \mathbf{Z}^s$. Let Γ be the subgroup of Λ generated by Y. By standard properties of determinants we have $h(Y) = [\Lambda : \Gamma]$, the index of Γ in Λ. On the other hand, it is easy to see that $C \cap \Lambda$ is a set of coset representatives for Γ in Λ. Hence $i(C, 1) = \#(C \cap \Lambda) = h(Y)$. By a simple scaling argument it follows that $i(C, q) = h(Y)q^{|Y|}$ for any integer $q \geq 1$. Hence

$$i(Z, q) = \sum_X i(C_X, q) = \sum_X h(X)q^{|X|}. \quad \square$$

3. Counting degree sequences

We will now apply Theorem 2.2 to a modification $\widetilde{D}_n$ of the polytope D_n. For a graph G on the vertex set $[n]$ with degree sequence $d(G) = (d_1, \dots, d_n)$, define the *extended degree sequence*

$$\tilde{d}(G) = (d_1, \dots, d_n, \tfrac{1}{2}(d_1 + \cdots + d_n)).$$

Then define the *polytope of extended degree sequences,*

$$\widetilde{D}_n = \operatorname{conv}\{\tilde{d}(G) : G \text{ is a simple graph on the vertex set } [n]\} \subset \mathbf{R}^{n+1}.$$

Since the last component of vectors in $\widetilde{D}_n$ is a fixed linear combination of the first n components, it follows that D_n and $\widetilde{D}_n$ are linearly equivalent and hence combinatorially equivalent. However (and this is the point of defining $\widetilde{D}_n$), they do not have the same Ehrhart polynomial.

Let us recall the Erdös-Gallai conditions (see any graph theory text) characterizing the degree sequences of simple graphs: A vector $(d_1, \dots, d_n)$ of positive integers d_i satisfying $d_1 \geq \cdots \geq d_n$ is the degree sequence of some simple graph G if and only if the following two conditions hold:

$$\text{(EG1)} \qquad \sum_{i=1}^{j} d_i - j(j-1) \leq \sum_{i=j+1}^{n} \min(j, d_i), \qquad 1 \leq j \leq n.$$

$$\text{(EG2)} \qquad d_1 + \cdots + d_n \text{ is even.}$$

From these conditions it is easy to deduce (see [**6**]) that the condition for an arbitrary vector $(d_1, \dots, d_n)$ of positive integers (i.e., not necessarily satisfying $d_1 \geq \cdots \geq d_n$) to be the degree sequence of a simple graph is given by a system of linear inequalities, together with (EG2). (These conditions in fact follow from an earlier result of Tutte [**12**] and are also obtained in a more general context in [**4**, Theorem 2.1].) It follows that $(d_1, \dots, d_n) \in \mathbf{Z}^n$ is a degree sequence if and only if $(d_1, \dots, d_n) \in D_n$ and $d_1 + \cdots + d_n$ is even. Hence from the definition of $\widetilde{D}_n$ we conclude:

3.1. Proposition. *There is a one-to-one correspondence between degree sequences $d(G)$ of simple graphs G on the vertex set $[n]$ and integer points in $\widetilde{D}_n$, given by $d(G) \leftrightarrow \tilde{d}(G)$.*

More generally, an analogue of the Erdös-Gallai condition for multigraphs (graphs with repeated edges, but no loops) with edge multiplicity bounded by q yields the following result (which is generalized further in Proposition 5.2).

3.2. PROPOSITION. *Let q be a positive integer. Then $i(\widetilde{D}_n, q)$ is equal to the number of distinct degree sequences of multigraphs on the vertex set $[n]$ such that no edge multiplicity can exceed q.*

Now let e_i denote the ith unit coordinate vector in $\mathbf{R}^n$, and write $e_{ij} = e_i + e_j$. Let $\tilde{e}_{ij}$ be the vector in $\mathbf{R}^{n+1}$ obtained by adjoining to e_{ij} an $(n+1)$st coordinate equal to 1. Clearly for a graph G on the vertex set $[n]$,

$$d(G) = \sum e_{ij}, \qquad \tilde{d}(G) = \sum \tilde{e}_{ij},$$

where both sums range over all edges $\{i, j\}$ of G. From this it follows easily that D_n and $\widetilde{D}_n$ are zonotopes, given explicitly by

$$\begin{aligned} D_n &= Z(e_{ij} : 1 \le i < j \le n), \\ \widetilde{D}_n &= Z(\tilde{e}_{ij} : 1 \le i < j \le n). \end{aligned} \tag{2}$$

(Of course, since D_n and $\widetilde{D}_n$ are linearly equivalent, it follows that one is a zonotope if and only if the other is.)

Now define a *quasitree* to be a connected graph which is either a tree or has exactly one cycle, which is of odd length. A *quasiforest* is a graph whose components are all quasitrees. We have come to the main result of this section.

3.3. THEOREM. *The Ehrhart polynomial $i(\tilde{D}_n, q)$ of $\tilde{D}_n$ is given by*

$$i(\widetilde{D}_n, q) = a_n(n)q^n + a_n(n-1)q^{n-1} + \cdots + a_n(0),$$

where

$$a_n(i) = \sum_X \max\{1, 2^{c(X)-1}\};$$

here X ranges over all quasiforests with i edges on the vertex set $[n]$, and $c(X)$ denotes the number of (odd) cycles of X.

PROOF. By Theorem 2.2 and (2), we have

$$a_n(i) = \sum_X h(X),$$

where X ranges over all i-element linearly independent subsets of $\widetilde{E}_n := \{\tilde{e}_{ij} : 1 \le i < j \le n\}$, and $h(X)$ has the meaning of Theorem 2.2. Let us identify a subset X of $\widetilde{E}_n$ with the graph on the vertex set $[n]$ which has an edge $\{i, j\}$ if and only if $\tilde{e}_{ij} \in X$. When is X linearly independent? We could appeal to Zaslavsky's extensive theory of signed graphs **[13]** to answer this question, but it is also easy to proceed directly. If X is linearly independent, then every component Y of X with i vertices may contain at most i edges, since any $i+1$ vectors in $\mathbf{R}^i$ are dependent. Thus every component Y of X is either a tree or contains one cycle. It is easily checked that a connected graph Y with at most one cycle is linearly independent if and only if Y is a tree or the unique cycle of Y has odd length. Thus we have that X is a quasiforest.

Assume then that X is a quasiforest, and set $m = |X|$. We need to compute $h(X)$, the gcd of the $m \times m$ minors of the matrix M whose rows are the vectors $\tilde{e}_{ij} \in X$. Thus the rows of M are indexed by the edges of X and the columns by the vertices, together with a last column of 1s. An $m \times m$ minor of M corresponds to choosing either m vertices of X (i.e., m numbers from $[n]$) or $m-1$ vertices of X and a final column of 1s. In the first case, in order for the columns to be linearly independent we must choose all vertices from any component of X with an odd cycle, and all but one vertex from any component of X which is a tree. Each odd cycle contributes to a factor ± 2 to the minor, and we obtain a value $\pm 2^{c(X)}$.

Now consider the submatrix N obtained by choosing $m-1$ vertices of X and a final column of 1s. Again, in order for the columns to be linearly independent we must choose our $m-1$ vertices in one of the following ways: (a) choose all but one vertex from every component Y which is a tree, choose all but one vertex from some component T_0 which has an odd cycle, and choose every vertex from the remaining components T with one odd cycle, or (b) choose all but two adjacent vertices i and j from some component which is a tree, choose all but one vertex from the remaining components which are trees, and choose every vertex from the components with one odd cycle.

In case (a), in the submatrix N subtract one-half of every column except the last from the last column. Every entry of the last column is now 0 or 1/2. Factor out 1/2 from the last column. The resulting matrix N' is the incidence matrix of a graph X' for which every component has exactly one cycle, which is of odd length. Moreover, X' has the same number of odd cycles as X. Hence $\det N' = \pm 2^{c(X)}$, so $\det N = \pm 2^{c(X)-1}$.

In case (b), there is a row which is all 0s except a 1 in the last column. Hence we may delete this row and the last column without affecting the value of the minor (except possibly for sign). Let X' denote the graph X with edge $\{i, j\}$ deleted (but retain all vertices). We now have a matrix obtained from the incidence matrix of X' by deleting a column (vertex) from every component which is a tree. Since X' still has $c(X)$ odd cycles, it follows that the value of the minor is $\pm 2^{c(X)}$.

Thus we have seen that all nonzero minors of M are equal to $\pm 2^{c(X)}$ or $\pm 2^{c(X)-1}$. The latter value is possible if and only if $c(X) \geq 1$. Hence the gcd of the minors of M is given by $\max\{1, 2^{c(X)-1}\}$, and the proof follows. □

3.4. COROLLARY. *The number $f(n)$ of distinct degree sequences of simple graphs on the vertex set $[n]$ is given by*

$$f(n) = \sum_X \max\{1, 2^{c(X)-1}\}, \tag{3}$$

where X ranges over all quasiforests on the vertex set $[n]$, and $c(X)$ denotes the number of (odd) cycles of X.

PROOF. Put $q = 1$ in Theorem 3.3. □

It would be interesting to find a direct combinatorial proof of (3). An analogous problem to determining $f(n)$ is to find the number $g(n)$ of distinct outdegree sequences of orientations of the complete graph K_n. Using the theory of zonotopes as we have done here, it was shown in [**10**, Example 3.1] (see also [**11**, Exercise 4.32]) that $g(n)$ is equal to the number of (labelled) forests on n vertices. A combinatorial proof of this result was subsequently given by Kleitman and Winston [**5**]. Perhaps their techniques can be used to establish (3). However, while the enumeration of outdegree sequences can be extended to orientations of *any* undirected graph (as discussed in the previous three references), an analogous generalization for degree sequences of subgraphs of a given graph G seems only to hold for a special class of graphs G (see §5 for further details).

It is a fairly routine exercise in enumerative combinatorics to convert Theorem 3.3 into a generating function for $i(\widetilde{D}_n, q)$.

3.5. PROPOSITION. *We have*

$$\sum_{n\geq 0} i(\widetilde{D}_n, q)\frac{x^n}{n!} = \frac{1}{2}\left[\left(1 + 2\sum_{n\geq 1} n^n \frac{q^n x^n}{n!}\right)^{1/2} \times \left(1 - \sum_{n\geq 1}(n-1)^{n-1}\frac{q^n x^n}{n!}\right) + 1\right] e^{\sum_{n\geq 1} n^{n-2}q^{n-1}x^n/n!}.$$

Here and below we set $0^0 = 1$ (which arises when $n = 1$ in the second sum on the right-hand side).

PROOF. We assume knowledge of the theory of exponential generating functions, in particular the exponential formula, as expounded for instance in [**3**, Theorem 4.3]. The number of rooted trees on n vertices is n^{n-1}, with exponential generating function

$$R(x) = \sum_{n\geq 1} n^{n-1}\frac{x^n}{n!}.$$

Hence the exponential generating function for k-tuples of rooted trees is $R(x)^k$ and so for undirected k-cycles of rooted trees (i.e., graphs with exactly one cycle, which is of length $k \geq 3$) is $R(x)^k/2k$.

Let $h(j, n)$ be the number of graphs on the vertex set $[n]$ such that every component has exactly one cycle, which is of odd length ≥ 3, and with j

cycles. (Such graphs have exactly n edges.) Then by the exponential formula we have

$$\begin{aligned}\sum_{j,n\geq 0} h(j,n)\frac{t^j x^n}{n!} &= \exp\sum_{k\geq 1}\frac{t}{2(2k+1)}R(x)^{2k+1}\\ &= \exp\frac{t}{2}\left[\frac{1}{2}(\log(1-R(x))^{-1}-\log(1+R(x))^{-1})-R(x)\right]\\ &= \left(\frac{1+R(x)}{1-R(x)}\right)^{t/4} e^{-tR(x)/2}.\end{aligned}$$

Thus,

$$\begin{aligned}1+\sum_{j,n\geq 1} 2^{j-1}h(j,n)\frac{x^n}{n!} &= 1+\frac{1}{2}\left[\left(\frac{1+R(x)}{1-R(x)}\right)^{1/2} e^{-R(x)}-1\right]\\ &= \frac{1}{2}\left[\left(-1+\frac{2}{1-R(x)}\right)^{1/2} e^{-R(x)}+1\right].\end{aligned}$$

It is well known (and easily deduced from $R(x)=xe^{R(x)}$) that

$$\frac{1}{1-R(x)}=\sum_{n\geq 0} n^n\frac{x^n}{n!}, \qquad e^{-R(x)}=1-\sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!},$$

so we get

$$\begin{aligned}&1+\sum_{j,n\geq 1} 2^{j-1}h(j,n)\frac{x^n}{n!}\\ &= \frac{1}{2}\left[\left(1+2\sum_{n\geq 1} n^n\frac{x^n}{n!}\right)^{1/2}\left(1-\sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}\right)+1\right].\end{aligned} \tag{4}$$

There are n^{n-2} free (i.e., unrooted) trees with n labelled vertices, each with $n-1$ edges. Hence if $r(i,n)$ denotes the number of forests with i edges on the vertex set $[n]$, then

$$\sum_{i,n\geq 0} r(i,n)\frac{q^i x^n}{n!}=\exp\sum_{n\geq 1} n^{n-2}\frac{q^{n-1}x^n}{n!}.$$

Since all graphs enumerated by $h(n,j)$ have exactly n edges, there follows

$$\sum_{n\geq 0} i(\widetilde{D}_n,q)\frac{x^n}{n!}=\left[1+\sum_{j,n\geq 1}2^{j-1}h(j,n)\frac{(qx)^n}{n!}\right]\cdot\exp\sum_{n\geq 1}n^{n-2}\frac{q^{n-1}x^n}{n!}.$$

Comparing with (4) completes the proof. □

Putting $q=1$ in Proposition 3.5 yields:

3.6. COROLLARY. *We have*

$$\sum_{n\geq 0} f(n)\frac{x^n}{n!} = \frac{1}{2}\left[\left(1+2\sum_{n\geq 1} n^n \frac{x^n}{n!}\right)^{1/2} \times \left(1-\sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}\right)+1\right] e^{\sum_{n\geq 1} n^{n-2}x^n/n!}.$$

Some small values of $i_n := i(\widetilde{D}_n, q)$ are given by:

$$\begin{aligned}
i_1 &= 1\\
i_2 &= 1+q\\
i_3 &= 1+3q+3q^2+q^3\\
i_4 &= 1+6q+15q^2+20q^3+12q^4\\
i_5 &= 1+10q+45q^2+120q^3+195q^4+162q^5\\
i_6 &= 1+15q+105q^2+455q^3+1320q^4+2508q^5+2540q^6\\
i_7 &= 1+21q+210q^2+1330q^3+5880q^4+18564q^5+39809q^6\\
&\quad +46035q^7\\
i_8 &= 1+28q+378q^2+3276q^3+20265q^4+93240q^5+317800q^6\\
&\quad +749200q^7+951552q^8.
\end{aligned}$$

Moreover,

$$\begin{aligned}
f(1) &= 1\\
f(2) &= 2\\
f(3) &= 8\\
f(4) &= 54\\
f(5) &= 533\\
f(6) &= 6944\\
f(7) &= 111850\\
f(8) &= 2135740\\
f(9) &= 47003045\\
f(10) &= 1168832808.
\end{aligned}$$

The coefficient of q^n in $i(\widetilde{D}_n, q)$ is the n-dimensional *relative volume* $V(\widetilde{D}_n)$ of $\widetilde{D}_n$ (see, e.g., [**10**, p. 335] or [**11**, pp. 238–239] for the definition). Putting $1/q$ for q and qx for x in Proposition 3.5 and then setting $q = 0$ yields:

3.7. COROLLARY. *We have*

$$\sum_{n\geq 0} V(\widetilde{D}_n)\frac{x^n}{n!} = \frac{1}{2}\left[\left(1+2\sum_{n\geq 1} n^n\frac{x^n}{n!}\right)^{1/2}\left(1-\sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}\right)+1\right].$$

We also have that

$$V(\widetilde{D}_n) = \sum_X 2^{c(X)-1}, \tag{5}$$

where X ranges over all graphs on the vertex set $[n]$ for which every component has exactly one cycle, which is of odd length, and where $c(X)$ denotes the number of (odd) cycles of X. Moreover, in regard to Peled's question about the volume of D_n, we have by similar reasoning that

$$V(D_n) = \sum_X 2^{c(X)},$$

summed over the same range as (5), so $V(D_n) = 2V(\widetilde{D}_n)$. Moreover, $V(D_n)$ is the ordinary volume of D_n for $n \geq 3$.

4. The f-vector of D_n and $\widetilde{D}_n$

We mentioned in §1 that Koren, Peled, and Srinivasan obtained a description of the vertices, edges, and facets of D_n (or $\widetilde{D}_n$). Beissinger and Peled [1] used the description of the vertices to obtain a formula for the number of vertices (see equation (7)). Here we extend this result to a determination of the entire f-vector $f(D_n) = (f_0(D_n), \dots, f_n(D_n)) = (f_0, \dots, f_n)$, where D_n has f_i i-dimensional faces. (Normally the f-vector of a d-polytope only includes f_i for $0 \leq i \leq d-1$, but it will be convenient for us to allow $0 \leq i \leq n$.) Thus, $f(D_1) = (1, 0)$, $f(D_2) = (2, 1, 0)$, and $f(D_3) = (8, 12, 6, 1)$, the latter since D_3 is a 3-cube.

Our derivation will be based on a result of Zaslavsky concerning the coloring of signed graphs. Let $-K_n$ denote the *negative complete signed graph*, i.e., the complete graph K_n with every edge labelled with a minus sign [**13**, §7D]. The matroid $M = M(-K_n)$ which Zaslavsky associates with $-K_n$ is just the linear matroid $\{e_{ij} : 1 \leq i < j \leq n\} \subset \mathbf{R}^n$. Hence by (2), Zaslavsky's zonotope $Z[-K_n]$ (defined in [**14**, p. 226]) is just D_n.

In [**14**, pp. 217–218], Zaslavsky defines a *signed coloring* of $-K_n$ (more generally, Zaslavsky deals with an arbitrary signed graph Σ) in $\mu \geq 0$ colors to be a function $k : N \to [-\mu, \mu] = \{-\mu, -\mu+1, \dots, 0, \dots, \mu-1, \mu\}$, where $N = [n]$ denotes the vertex set of $-K_n$. The *set of impropriety* $I(k)$ of k consists of all edges $\{i, j\}$ of $-K_n$ for which $k(i) = -k(j)$ [**14**, p. 218]. Call such edges *improper*. The *rank* of $I(k)$ is defined by $\operatorname{rk} I(k) = n - b(I(k))$, where $b(I(k))$ is the number of bipartite components of $I(k)$ (considered as a graph on the vertex set $[n]$) [**14**, p. 216]. The *Whitney* *polynomial* of $-K_n$ is defined by

$$w_{-K_n}(x, 2\mu+1) = \sum_k x^{\operatorname{rk} I(k)},$$

summed over all signed colorings in μ colors.

We can now state the result of Zaslavsky [**14**, Corollary 4.1″] (specialized to $\Sigma = -K_n$ and $Z[\Sigma] = D_n$), which will be our main tool for evaluating the f-vector $f(D_n)$.

4.1. Proposition. *We have*

$$\sum_{i=0}^{n} f_i(D_n)x^i = (-1)^n w_{-K_n}(-x, -1).$$

The main result of this section is the following:

4.2. Theorem. *We have*

$$\sum_{n\geq 0}\sum_{i=0}^{n} f_i(D_n)\frac{x^i t^n}{n!} = \frac{e^{xt} - (1+x)t - x(1+x)\frac{t^2}{2}}{1 + 2(e^{-t}-1) + \frac{2}{x}(e^{xt}-1) - \frac{1}{x}(e^{2xt}-1)}.$$

Proof. We may choose a signed coloring $k : N \to [-t, t]$ of $-K_n$ as follows. First choose the sets $A_i = k^{-1}\{-i, i\}$. Thus $(A_0, A_1, \dots, A_\mu)$ is a *weak ordered partition* of $[n]$ into $\mu + 1$ blocks, i.e., $\bigcup A_i = [n]$ and $A_i \cap A_j = \varnothing$ if $i \neq j$. ("Weak" means that we allow $A_i = \varnothing$.) Let $a_i = |A_i|$. Now for each block A_i with $1 \leq i \leq \mu$, choose a subset $B_i = k^{-1}(i)$. For two of these choices (when $A_i \neq \varnothing$), namely $B_i = A_i$ and $B_i = \varnothing$, there will be no improper edges incident to vertices in A_i, so A_i forms a_i bipartite components of $I(k)$, each consisting of a single vertex. For the remaining $2^{a_i} - 2$ choices of B_i (when $a_i \geq 1$), all edges from B_i to $A_i - B_i$ are improper; these edges form a single bipartite component of $I(k)$. There remains the case of A_0. All edges between vertices in A_0 are improper, so we get a component of $I(k)$ isomorphic to the complete graph K_{a_0}. This component will be bipartite if and only if $a_0 = 0, 1$, or 2. It follows that

$$(6)\quad w_{-K_n}(x, 2\mu+1) = x^n \sum_{(A_0, \dots, A_\mu)} v(a_0)\prod_{i=1}^{\mu}(2x^{a_i} + (2^{a_i}-2)\chi(a_i \neq 0)x^{-1}),$$

where: (a) the sum ranges over all weak ordered partitions of $[n]$ into $\mu+1$ blocks, (b) $\chi(a_i \neq 0) = 1$ if $a_i \neq 0$ and $= 0$ if $a_i = 0$, and (c) $v(a_0)$ is defined by

$$v(a) = \begin{cases} 1 & \text{if } a = 0 \text{ or } a \geq 3, \\ x^{-1} & \text{if } a = 1 \text{ or } a = 2. \end{cases}$$

Now suppose $g_0, \dots, g_\mu$ are any functions defined on $\mathbf{N}$, and define (using the notation of (6))

$$h(n) = \sum_{(A_0, \dots, A_\mu)} g_0(a_0)\cdots g_\mu(a_\mu).$$

It then follows from standard properties of exponential generating functions that

$$h(n) = \left[\frac{t^n}{n!}\right] \prod_{i=0}^{\mu} \left(\sum_{j\geq 0} g_i(j) \frac{t^j}{j!} \right),$$

where $[\frac{t^n}{n!}]F(t)$ denotes the coefficient of $t^n/n!$ in the generating function $F(t)$. Hence from (6) we have

$$\begin{aligned}
&W_{K_n}(x, 2\mu+1) \\
&\quad = x^n \left[\frac{t^n}{n!}\right] \left(x^{-1}t + x^{-1}\frac{t^2}{2} + \sum_{j\geq 3} \frac{t^j}{j!} \right) \left(1 + \sum_{j\geq 1} (2x^{-j} + (2^j - 2)x^{-1}) \frac{t^j}{j!} \right)^{\mu} \\
&\quad = \left[\frac{t^n}{n!}\right] \left(1 + \frac{xt^2}{2} + \sum_{j\geq 3} \frac{(xt)^j}{j!} \right) \left(1 + \sum_{j\geq 1} (2x^{-j} + (2^j - 2)x^{-1}) \frac{(xj)^j}{j!} \right)^{\mu} \\
&\quad = \left[\frac{t^n}{n!}\right] \left((1-x)t + x(1-x)\frac{t^2}{2} + e^{xt} \right) \\
&\qquad \times \left(1 + 2(e^t - 1) - \frac{2}{x}(e^{xt} - 1) + \frac{1}{x}(e^{2xt} - 1) \right)^{\mu}.
\end{aligned}$$

Substituting $-x$ for x, $-t$ for t, and -1 for μ yields

$$\begin{aligned}
(-1)^n w_{-K_n}(-x, -1) = \left[\frac{t^n}{n!}\right] &\left(e^{xt} - (1+x)t - x(1+x)\frac{t^2}{2} \right) \\
&\times \left(1 + 2(e^{-t} - 1) + \frac{2}{x}(e^{xt} - 1) - \frac{1}{x}(e^{2xt} - 1) \right)^{-1}.
\end{aligned}$$

The proof follows from Proposition 4.1. □

From Theorem 4.2 one can compute:

$$\begin{aligned}
f(D_1) &= (1, 0) \\
f(D_2) &= (2, 1, 0) \\
f(D_3) &= (8, 12, 6, 1) \\
f(D_4) &= (46, 108, 84, 22, 1) \\
f(D_5) &= (332, 1020, 1080, 450, 60, 1) \\
f(D_6) &= (2874, 10830, 14880, 9160, 2460, 224, 1) \\
f(D_7) &= (29874, 129486, 220920, 182770, 75670, 14238, 882, 1) \\
f(D_8) &= (334982, 1726648, 3529344, 3679872, 2074660, \\
&\qquad 610288, 81144, 3322, 1).
\end{aligned}$$

Putting $x = 0$ in Theorem 4.2 yields

$$\sum_{n\geq 0} f_0(D_n) \frac{t^n}{n!} = \frac{1-t}{-1+2e^{-t}} = \frac{(1-t)e^t}{2-e^t}, \tag{7}$$

agreeing with a result of Beissinger and Peled [**1**, p. 216] mentioned at the beginning of this section.

5. A generalization

We wish to generalize Theorem 3.3 by considering degree sequences of graphs which are contained in a fixed graph G. More precisely, let G be a multigraph (i.e., allowing multiple edges, but not loops) on the vertex set $[n]$. Define a convex polytope $\widetilde{D}(G) \subset \mathbf{R}^{n+1}$ by

$$\widetilde{D}(G) = \operatorname{conv}\{\tilde{d}(H) : H \text{ is a spanning subgraph of } G\}.$$

Thus $\widetilde{D}_n = \widetilde{D}(K_n)$, where K_n denotes the complete graph on $[n]$. It is easily seen that

$$\widetilde{D}_n = Z(\tilde{e}_{ij} : \{i, j\} \text{ is an edge of } G).$$

(Here we take $\tilde{e}_{ij}$ a total of m times if there are m edges between i and j.) The following generalization of Theorem 3.3 is proved in exactly the same way as Theorem 3.3.

5.1. THEOREM. *The Ehrhart polynomial* $i(\widetilde{D}(G), q)$ *of* $\widetilde{D}(G)$ *is given by*

$$i(\widetilde{D}(G), q) = a_G(n)q^n + a_G(n-1)q^{n-1} + \cdots + a_G(0),$$

where

$$a_G(i) = \sum_X \max\{1, 2^{c(X)-1}\};$$

here X *ranges over all spanning quasiforests of* G *with* i *edges, and* $c(X)$ *denotes the number of (odd) cycles of* X.

We would like to interpret $i(\widetilde{D}(G), q)$ in terms of degree sequences analogously to Proposition 3.2. We therefore need to know when Proposition 3.1 extends to $\widetilde{D}(G)$. Call G an *FHM-graph* (named after the authors of [**4**]) if the following property holds:

(FHM). No induced subgraph of G consists of two vertex-disjoint odd cycles (with no other edges). (Equivalently, every induced subgraph of G has at most one nonbipartite component.)

Given a multigraph G on the vertex set $[n]$, let

$$\Delta(G) = \{d(H) : H \text{ is a spanning subgraph of } G\}.$$

Define a map $\varphi_G : \Delta(G) \to \mathbf{Z}^{n+1}$ by $\varphi_G(d(H)) = \tilde{d}(H)$. Clearly φ_G is one-to-one. It is immediate from the definition of $\widetilde{D}(G)$ that

$$\varphi_G(\Delta(G)) \subseteq \widetilde{D}(G) \cap \mathbf{Z}^{n+1}. \tag{8}$$

5.2. PROPOSITION. *We have* $\varphi_G(\Delta(G)) = \widetilde{D}(G) \cap \mathbf{Z}^{n+1}$ *if and only if* G *is an FHM-graph.*

PROOF. The "if" part follows from [**4**, Theorem 2.1], while the "only if" part (which is easy) follows from the last paragraph on p. 170 of [**4**]. □[1]

[1] I am grateful to L. Lovász for bringing the paper [**4**] to my attention.

Now given any multigraph G on $[n]$ and any integer $q \geq 1$, define $G^{(q)}$ to be the multigraph obtained from G by replacing each edge by q edges, and let $f(G^{(q)})$ denote the number of distinct degree sequences of spanning subgraphs of $G^{(q)}$.

5.3. THEOREM. *For any multigraph G on $[n]$ and any integer $q \geq 1$, we have*

$$f(G^{(q)}) \leq i(\widetilde{D}(G), q).$$

Equality holds if and only if G is an FHM-graph.

PROOF. From the definition of FHM-graph we see that for any integer $q \geq 1$, G is an FHM-graph if and only if $G^{(q)}$ is. Moreover, by the definition of $\widetilde{D}(G)$ we have $\widetilde{D}(G^{(q)}) = q\widetilde{D}(G)$, where for any polytope $\mathscr{P}$ we define

$$q\mathscr{P} = \{q\alpha : \alpha \in \mathscr{P}\}.$$

Since

$$i(\widetilde{D}(G), q) = \#(q\widetilde{D}(G) \cap \mathbf{Z}^{n+1}) = \#(\widetilde{D}(G^{(q)}) \cap \mathbf{Z}^{n+1}),$$

the proof follows from (8) and Proposition 5.2 (and the fact that $\varphi_{G^{(q)}}$ is one-to-one). □

A special case of FHM-graphs is bipartite graphs. Theorems 5.1 and 5.3 yield for these graphs the following result.

5.4. COROLLARY. *Let G be a bipartite (multi)graph on the vertex set $[n]$. Then*

$$f(G^{(q)}) = b_G(n-1)q^{n-1} + b_G(n-2)q^{n-2} + \cdots + b_G(0),$$

where $b_G(i)$ is the number of spanning forests of G with i edges. In particular, $f(G)$ is equal to the number of spanning forests of G.

We mentioned after Corollary 3.4 that the number $g(G)$ of outdegree sequences of orientations of any graph (or multigraph) G is equal to the number of spanning forests of G, and that this result has a combinatorial proof. Hence by the preceding corollary we have $f(G) = g(G)$ when G is bipartite. It is easy to give a combinatorial proof of this fact. Namely, if G has vertex bipartition (A, B) and if D is an orientation of G, then let $\sigma(D)$ be the spanning subgraph of G consisting of those edges which are oriented from A to B in D. Then D and D' have the same outdegree sequence if and only if $\sigma(D)$ and $\sigma(D')$ have the same degree sequence, yielding the desired bijection. Hence the combinatorial proof **[5]** yields a combinatorial proof of Corollary 5.4 when $q = 1$, and this leads easily to a combinatorial proof for any q.

Combining Theorems 5.1 and 5.3 when $q = 1$ yields that for an FHM-graph G,

$$f(G) = \sum_X \max\{1, 2^{c(X)-1}\}, \tag{9}$$

where X ranges over all spanning quasiforests of G. It would be interesting to find a generalization of (9) valid for all graphs G. While we have been unable to do this, there is a more general class of graphs than FHM-graphs which we can handle. Recall that a *closed trail* T of length m in a graph G is a sequence $v_0e_1v_1e_2v_2\cdots e_mv_m$ such that each v_i is a vertex, each e_i is an edge, $v_0 = v_m$, any two adjacent terms are incident in G, and finally all the e_i's are distinct. Call the closed trail T *even* if m is even.

5.5. PROPOSITION. *Let e be an edge of the graph G. Then $f(G) = 2f(G-e)$ if and only if e does not belong to an even closed trail of G.*

PROOF. Note that $f(G) = 2f(G-e)$ if and only if there do not exist spanning subgraphs H, H' of G such that $d(H) = d(H')$, e is an edge of H, and e is not an edge of H'. Suppose e is an edge of the even closed trail $v_0e_1v_1\cdots e_{2r}v_{2r}$. Let H have edges $e_1, e_3, \ldots, e_{2r-1}$, and H' have edges $e_2, e_4, \ldots, e_{2r}$. Then $d(H) = d(H')$, and e is an edge of exactly one of H and H'. Thus $f(G) \neq 2f(G-e)$.

Conversely, suppose H and H' are spanning subgraphs of G such that $d(H) = d(H')$, and let e be an edge of H but not of H'. Let B be the set of edges of H which are not edges of H', and let R be the set of edges of H' which are not edges of H. Regard the edges in B as "blue" and in R as "red". Consider the graph G' whose edges are $B \cup R$. Every vertex G' is incident to the same number of blue edges as red edges. It follows easily that there is a closed trail T containing e whose edge alternate blue and red. Hence T is even, as desired. □

The point of Proposition 5.5 is that even though G may not be an FHM-graph, after we remove all edges not belonging to even closed trails the resulting graph may then be an FHM-graph. Thus equation (9) and Proposition 5.5 gives us a formula for $f(G)$. We also may assume G is connected, since clearly $f(G+H) = f(G)f(H)$, where $G+H$ denotes the disjoint union of G and H.

REFERENCES

1. J. S. Beissinger and U. N. Peled, *Enumeration of labelled threshold graphs and a theorem of Frobenius involving Eulerian polynomials*, Graphs Combin. **3** (1987), 213–219.
2. T. Brylawski, *A decomposition for combinatorial geometries*, Trans. Amer. Math. Soc. **171** (1972), 235–282.
3. D. Foata, *La série génératrice exponentielle dans les problèmes d'énumération*, Sém. Math. Sup., Presses Univ. Montréal, 1974.
4. D. R. Fulkerson, A. J. Hoffman, and M. H. McAndrew, *Some properties of graphs with multiple edges*, Canad. J. Math. **17** (1965), 166–177.
5. D. Kleitman and K. Winston, *Forests and score vectors*, Combinatorica **1** (1981), 49–54.
6. M. Koren, *Extreme degree sequences of simple graphs*, J. Combin. Theory Ser. B **15** (1973), 213–224.
7. H. L. Montgomery and J. D. Vaaler, E3301, American Math. Monthly **96** (1989), 54.
8. U. N. Peled and M. K. Srinivasan, *The polytope of degree sequences*, Linear Algebra Appl. **114/115** (1989), 349–377.

9. G. C. Shephard, *Combinatorial properties of associated zonotopes*, Canad. J. Math. **26** (1974), 302–321.
10. R. Stanley, *Decompositions of rational convex polytopes*, Ann. Discrete Math. **6** (1980), 333–342.
11. ——, *Enumerative combinatorics*, vol. 1, Wadsworth & Brooks/Cole, Pacific Grove, Cal., 1986.
12. W. T. Tutte, *The factors of graphs*, Canad. J. Math. **4** (1952), 314–329.
13. T. Zaslavsky, *Signed graphs*, Discrete Appl. Math. **4** (1982), 47–74.
14. ——, *Signed graph coloring*, Discrete Math. **39** (1982), 215–228.

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