# Discrete and Algorithmic Geometry 2011 (Part 2)

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This is the preliminary version of the lecture notes for the second part of *Discrete and Algorithmic Geometry* (Universitat Politècnica de Catalunya), held in the fall semester of 2011 by Vera Sacristan and Julian Pfeifle.

These notes are fruit of the collaborative effort of all participating students, who have taken turns in assembling this text. The name of each scribe figures in each corresponding section.

The main literature for this course consists of [**CS99**], [**CBGS08**] and [**Sen95**].

Suggestions for improvements will always be gladly received by `julian.pfeifle@upc.edu`.

# Contents

LECTURE 1

# Introduction to Packings

*Scribe: Ferran Dachs Cadefau*

The general content of the lectures.

## 1. Packings

**Definition 1.1.** A family $\{K_i\}_{i\in I}$ of compact convex sets $K_i \subseteq \mathbb{R}^d$ with non-empty interior (this implies that $K_i$ are full-dimensional) is a *packing* if:

$$\mathrm{int}(K_i \cap K_j) = \emptyset \qquad \text{for } i \neq j$$

It is possible that the boundaries of two different $K_i$ overlap, but not the interior. If we are working in a Hausdorff space, subsets are compact if and only if they are closed and bounded. More generally, we can work with non-convex packings, but they are harder to work with. For example the next example due to M.C. Escher:

FIGURE 1. M.C. Escher, Plane Filling II, Lithograph 1957

**Definition 1.2.** If there exists $C \in \mathbb{R}^d$ such that $\bigcup_{i\in I} K_i \subseteq C$ then $C$ is called a *container* of the packing. These always exist: take $C = \bigcup_{i\in I} K_i$. The *natural container* of the packing is

$$C_{nat} = \mathrm{conv} \bigcup_{i\in I} K_i$$

We will pack repetitions of the same figure, that is, $K_i$ for all $i \in I$ is the same set. Another thing that we can consider is a fixed container: For example, we can pack squares in squares as in [**Fri09**], or circles in squares, as in `http://hydra.nat.uni-magdeburg.de/packing/csq/csq.html`, or regular polyhedra [**Jea10**]. As we can see in the second example, if we have a fixed container it is hard to find a optimum solution, and moreover, the optimum solution can have no regularity!

**Definition 1.3.** We can speak about the quality of the packings using their *density*

$$\delta_{bin} = \frac{\sum_{i=1} V(K_i)}{V(C)}$$

and *natural density*

$$\delta_{Nat} = \frac{\sum_{i=1} V(K_i)}{V(C_{Nat})}.$$

From now on, the $K_i$ will be congruent spheres.

## 2. Density of disk packings in the plane

**Lemma 1.4** (Thue in 1892)**.**

$$\delta_{Nat}(n \textit{ disks in } \mathbb{R}^2) \underset{n\to\infty}{\longrightarrow} \delta_{Nat}(\textit{hexagonal packing})$$

$$\delta_{Nat}(n \textit{ thin disks in } \mathbb{R}^3) = 1$$

*where thin disks are: $D^2 \times \square^1$ and the ideal packing is a cylinder. For bigger dimensions (thin disks are: $D^2 \times \square^{d-1}$) the ideal packing is again a cylinder.*

## 3. Packings of Spheres

**Observation 1.5.** We defined: $S^d = \{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$. Except for $S^0$ all spheres are connected, and all $S^i$ for $i > 1$ are simply connected.

**Definition 1.6.** Let $Z = \operatorname{conv}\{\text{centers of } K_i : i \in I\}$. We say that the associated packing is a

(1) *Sausage* if $\dim Z = 1$;
(2) *Pizza* if $2 \leqslant \dim Z \leqslant d - 1$;
(3) *Pile* if $\dim Z = d$.

For example, in $\mathbb{R}^2$ a Sausage is composed of $n$ circles with their centers on a line. In $\mathbb{R}^3$, we get a Pizza for example by thinking of $n$ spheres with their centers on a plane.

$$\delta_{Nat}(\text{Sausage of } n \text{ spheres in } \mathbb{R}^d) = \frac{\sum_{i=1} V(K_i)}{V\left(\operatorname{conv}\bigcup_{i\in I} K_i\right)} = \frac{n\beta(d)}{\beta(d) + 2(n-1)\beta(d-1)},$$

where $\beta(d)$ are the volume of the unit ball in $\dim d$. To calculate the volume of $\operatorname{conv}\bigcup_{i\in I} K_i$ we have used that the convex hull is a cylinder of height $n-1$ and two halves of a sphere.

$$\delta_{Nat}(\text{Sausage of } n \text{ spheres in } \mathbb{R}^d) \underset{n\to\infty}{\longrightarrow} \frac{\beta(d)}{2\beta(d-1)}.$$

For example the case $n = 4$ and $d = 3$ the best packing is a Sausage instead of for example the Tetrahedral packing as we can see in The paper of J.M.Wills.

**Exercise 1.7.** *Calculate the $\delta_{Nat}$ of the tetrahedral packing.*

In dimension 3 the best packings are shown in Table 1.

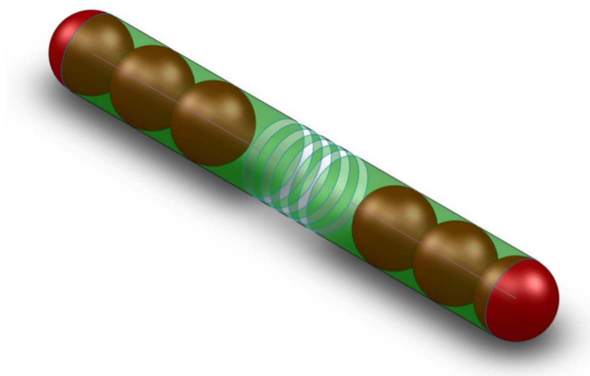


FIGURE 2. Sausage in $\mathbb{R}^3$ with its natural density $C_{Nat}$.

| $n$ (number of balls) | 4 | ... | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | $\geqslant 65$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Type of best packing | S | ... | S | P | S | S | P | P | P | P | S | S | P |
| Verified or Conjectured | V | C | C | V | C | C | V | V | V | V | C | C | V |

TABLE 1. Best packings in dimension 3. Here $S$ stands for Sausage, $P$ for Pile, $C$ is Conjectured and $V$ is Verified.

**Conjecture 1.8** (Sausage Conjecture (Lászlo Fejes Tóth))**.**

$$\delta(d,n) = \delta_{Nat}(W_n^d) \qquad \forall n \in \mathbb{N}, \quad d \geqslant 5$$

*Where $W_n^d$ is the sausage packing ("Wurst" in German).*

**Theorem 1.9** (Martin Henk, Jörg Wills, Ulrich Betke 1986; see [**BH98**])**.**

$$\delta(d,n) = \delta_{Nat}(W_n^d) \qquad \forall n \geqslant 42, \quad d \geqslant 5$$

## 4. The Unit cube

Now, we can consider $\square^d$, the unit cube in $\mathbb{R}^d$:

$$\square^d \ = \ \mathrm{conv}\{(a_1, ...a_d) | a_i = 0 \text{ or } 1, \text{ for } 0 \leqslant i \leqslant d\} \subseteq \mathbb{R}^d$$

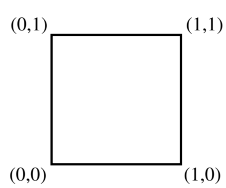



FIGURE 3. $\square^2$

**Observation 1.10.** The number of vertices of $\square^d$ is $2^d$.

**Definition 1.11.**

$$\square^d = \{(a_1, ...a_d) | 0 \leqslant a_i \leqslant 1 \quad \forall 0 \leqslant i \leqslant d\} \subseteq \mathbb{R}^d$$

We can consider the faces of $\square^d$, and his *dimension* are the dimension of his affine span.

- If dimension are $0$ we talk about *vertices*.
- If dimension are $1$ we talk about *edges*.
- If dimension are $d-1$ we talk about *facet*.

**Observation 1.12.** The number of facets of $\square^d$ is $2d$, one for each inequality.

**Exercise 1.13.** *Calculate all the number of dimension $i$ subspaces.*

**Observation 1.14.** The distance between a vertex and the barycenter is the radius of the *circumscribed sphere*. If $V$ is a vertex, and $B$ the barycenter, we have:

$$\|V_i - B\| = \|(0,...,0) - (1/2,...,1/2)\| = \|(1/2,...,1/2)\| = \sqrt{d}\frac{1}{2}$$

We can choose $V = (0,...,0)$ because all vertices are at the same distance from the barycenter.

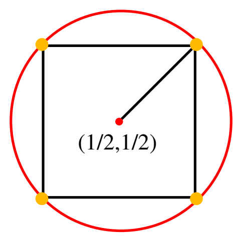



FIGURE 4. The distance between a vertex and the barycenter is the radius of the circumscribed sphere.

**Observation 1.15.** The distance between a facet and the barycenter is the radius of the *inscribed sphere*, $\frac{1}{2}$.

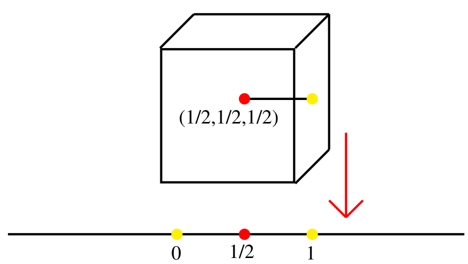



FIGURE 5. The distance between a facet and the barycenter is the radius of the inscribed sphere, $\frac{1}{2}$

Here we show the radii of the circumscribed and the inscribed spheres in some dimensions:

| $d$ | $1$ | $2$ | $100$ | $10^4$ |
|---|---|---|---|---|
| $\rho_{circ}$ | $\frac{1}{2}$ | $\frac{1}{2}\sqrt{2}$ | $5$ | $50$ |
| $\rho_{in}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

It's difficult to think in high dimensions. For more, see [**Bal97**].

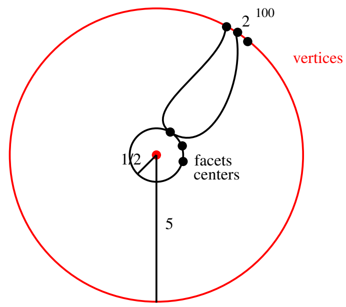



FIGURE 6. Representation of the vertices and the facets in dimension $100$.

**Observation 1.16.** If we draw $2^d$ spheres centered in the vertices with radius $\frac{1}{2}$. Which is the radius of the maximum sphere that we can draw centered in the barycenter tangent to the others (as we can see in Figure 7)? $\frac{1}{2}\left(\sqrt{d}-1\right)$

| $d$ | $2$ | $3$ | $4$ | $5$ | $100$ |
|---|---|---|---|---|---|
| $\frac{1}{2}\left(\sqrt{d}-1\right)$ | $0.2$ | $< \frac{1}{2}$ | $\frac{1}{2}$ | $> \frac{1}{2}$ | $\frac{9}{2}$ |

In the table we can see that in dimensions over $5$ the sphere goes out the facets!

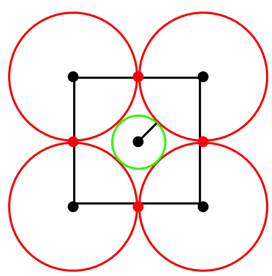


FIGURE 7. Representation of the vertices and the facets in dimension $100$.

LECTURE 2

# Volumes of balls and cubes; Lattice Polytopes

*Scribe: Victor Bravo*

## 1. Comparing the volumes of balls and cubes

Given an $n$-dimensional ball of radius $r$, we have that $\mathrm{vol}(B_r^n) = \dfrac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} r^n$, where $\Gamma$ is the Gamma Function, which is defined in the following way:

(1) $\Gamma(m+1) = m!$, for $m \in \mathbb{N}_0$.

(2) $\Gamma(m+\frac{1}{2}) = \dfrac{(2m)!}{m!4^m}\sqrt{\pi}$.

**Example 2.1.** $vol(B_r^1) = \dfrac{\sqrt{\pi}}{\Gamma(1+\frac{1}{2})} r = \dfrac{1!\sqrt{\pi}4^1}{2!\sqrt{\pi}} r = 2r.$

**Example 2.2.** $vol(B_r^2) = \dfrac{\pi}{1!} r^2 = \pi r^2.$

Now, we want to know the asymptotic behaviour, i.e., having a cube with a ball inside, we want to know how evolves the volume of the cube compared with the volume of the ball. Using the unit cube, in dimension $1$, we have the same volume for the cube and the ball because they are the same thing. In dimension $2$ (see figure 1), we have a square with every edge of length $1$ and then, the ball has radius $1/2$. In dimension $3$ (see figure 2), we have a cube with every edge of length $1$ and then, the ball also has radius $1/2$, etc.

Then, the general case is, $\dfrac{\mathrm{vol}(B^n_{1/2})}{\mathrm{vol}({}_0\square^n_1)} = \mathrm{vol}(B^n_{1/2}) =$ fraction of unit cube taken up by largest ball contained inside.

Now, using Stirling's approximation, $\Gamma(x+1) \approx \sqrt{2\pi x}\left(\dfrac{x}{e}\right)^x$, $x \in \mathbb{R}_{\geq 0}$, we have that asymptotically,

$$\mathrm{vol}(B^n_{1/2}) \overset{n\to\infty}{\longrightarrow} \frac{\pi^{\frac{n}{2}}(\frac{1}{2})^n}{\sqrt{2\pi\frac{n}{2}}(\frac{n/2}{e})^{\frac{n}{2}}} = \frac{\pi^{\frac{n}{2}}2^{\frac{n}{2}}e^{\frac{n}{2}}}{\sqrt{\pi n}n^{\frac{n}{2}}2^n} = \frac{1}{\sqrt{\pi n}}\left(\frac{\pi e}{2n}\right)^{\frac{n}{2}} \overset{n\to\infty}{\longrightarrow} 0.$$

**Example 2.3.** $\dfrac{vol(B^{100}_{1/2})}{vol(\square^{100})} \approx 10^{-67}.$

Then, we have bad news for numerical integration (for example in the case of Monte Carlo integration) when it is used in physics or in financial mathematics because, by the example above, we will be not able to count from $1$ to $10^{-67}$. This is too long. So, this works worst as the dimension increases. In conclusion, we will not use Monte Carlo Integration to calculate volumes in high dimensions because the volumes of the balls will be so tiny.

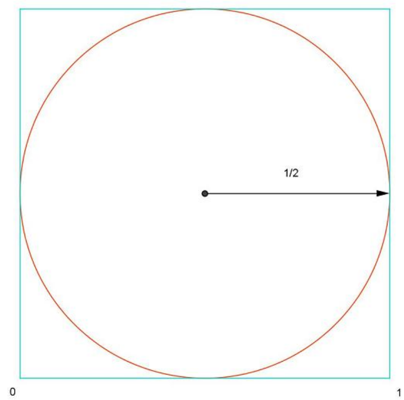



FIGURE 1. Example in dimension 2.

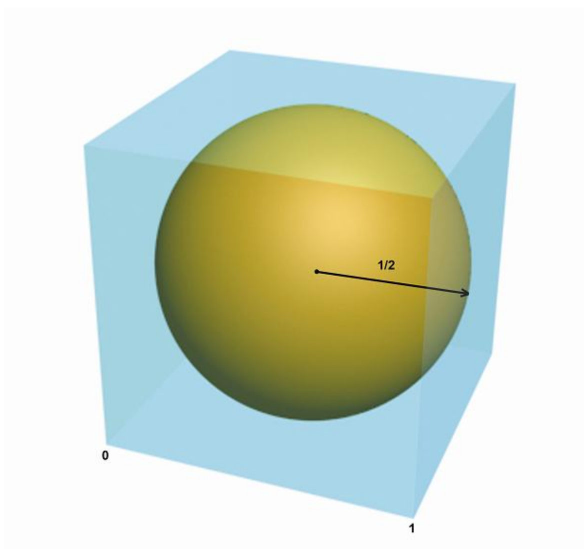



FIGURE 2. Example in dimension 3.

**Remark 2.4.** An example of Monte Carlo Integration in physics consists in throw random points into our space and count how many points fall inside and how many points fall outside. Then, do the fraction which divides the number of points inside and the number of points thrown and this fraction approximates the volume (it is used at CERN). In the other hand, Monte Carlo Integration is used in financial mathematics, for example if we have a portfolio with many variables and we have to integrate, one way to integrate by all this variables is using Monte Carlo Integration.

## 2. Lattices and lattice polytopes

Now, we will talk about lattice packings of spheres. A lattice has two different meanings in mathematics: a partially ordered set or a group. We are gonna talk about the group.

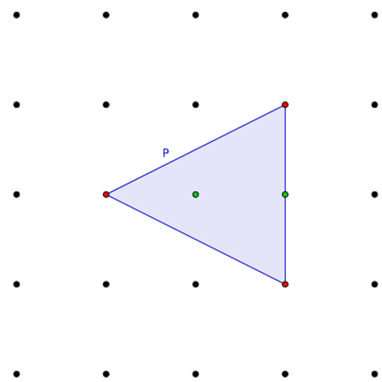



FIGURE 3. A lattice triangle.

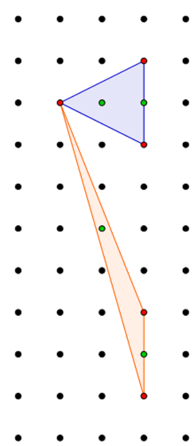


FIGURE 4. Two lattice triangles.

The most important lattice is $\mathbb{Z}^d$, and it's called the integer lattice. This is an abelian group with the sum: $x, y \in \mathbb{Z}^d \Rightarrow -x \in \mathbb{Z}^d, x + y \in \mathbb{Z}^d$, and the sum is commutative.

Now, if we have $v_1, \ldots, v_n \in \mathbb{Z}^d$, and we have a look to $P = \text{conv}\{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$, we define a lattice polytope as the convex hull of a finite set of points with integer coordinates.

Now, we can do the next question: When two lattice triangles "the same"? The first observation is that we have to answer is: When two polytopes are "the same"? In Figure 4, we can say that the two lattice triangles are "the same" because they share all properties respect to the lattice.

Now, forgetting lattices, the answer to the question for polytopes in general is: Klein's Erlangen Program. In this program, Klein identifies the geometry with the groups of automorphisms, i.e., what Klein makes is to say what the geometry is, by seeing which group of automorphisms leaves certain object invariant.

Some groups that we must have in mind are $O(n, \mathbb{R}) = \{A \in \mathbb{R}^{n \times n} : A^{-1} = A^T\}$ and $SO(n, \mathbb{R}) = O(n, \mathbb{R}) \cap \{A \in \mathbb{R}^{n \times n} : \det A = 1\}$. In the other hand, we can also have in mind the set of translations in $\mathbb{R}^{n\times n}$, $T(n,\mathbb{R}^{n\times n})$, which satisfies $SO(n,\mathbb{R}^{n\times n}) \rtimes T(n,\mathbb{R})$, where $\rtimes$ is the semi-direct product, which means: two subsets, $P, Q \subseteq \mathbb{R}^n$, are "the same" if $\exists A \in O(n)$ and $\exists t \in \mathbb{R}^n : Q = A \cdot P + t$, i.e., I can obtain $Q$ from $P$ through a rotation $A$ and a translation $t$ (i.e., $P$ and $Q$ are congruent), and this is what we know as Euclidean Geometry.

Now, remembering lattices, we have to change the euclidean geometry by lattice geometry, i.e., we want bijective homomorphisms that preserves the lattices. So, we want to determine $\mathrm{Aut}(\mathbb{Z}^d) = \{\text{affine transformations that leave } \mathbb{Z}^d \text{ invariant}\}$, and this is to find conditions on $A \in \mathbb{R}^{n\times n}$ and $t \in \mathbb{R}^n$ such that $Ax + t \in \mathbb{Z}^d$, $\forall x \in \mathbb{Z}^d$. These conditions are:

- $x = 0$, want $A \cdot 0 + t \in \mathbb{Z}^d \iff t \in \mathbb{Z}^d$.
- $x = e_i$, with $e_i$ a generating vector of our lattice, want $A \cdot e_i \in \mathbb{Z}^d \iff$ every column of $A \in \mathbb{Z}^d \iff A \in \mathbb{Z}^{d\times d}$.

Now, for $A$ to be an automorphism, it must be invertible, and $A^{-1}$ must belong to $\mathbb{Z}^{d\times d}$.

**Example 2.5.** *Suppose that $d = 2$. We have $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2}$. Then, $A^{-1} = \frac{1}{ad-bc}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) = \frac{1}{\det A}\left[(c_{ij})\right]$, where $(c_{ij})$ represents the cofactors of $A$. And $A^{-1} \in \mathbb{Z}^{2\times 2}$ because $ad - bc$ never divides the entries of $\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. (This has to be proved.)*

Then, $\mathrm{Aut}(\mathbb{Z}^d) = \left\{A \in \mathbb{Z}^{d\times d} : \det A = \pm 1\right\} \rtimes \mathbb{Z}$.

Observe that the set of orientation-preserving linear (not affine) automorphisms of $\mathbb{Z}^d$ is $\mathrm{Sl}_d(\mathbb{Z}) = \left\{A \in \mathbb{Z}^{d\times d} : \det A = 1\right\}$, the special linear group with integer coefficients. On the other hand, $\left\{A \in \mathbb{Z}^{d\times d} : \det A = -1\right\}$ is not a group.

Then, what lattice geometry means is that geometry with group automorphisms: $Sl_d(\mathbb{Z}) \rtimes \mathbb{Z}^d$, and this is mapping $x \longmapsto Ax + t$, with $t \in \mathbb{Z}^d$, $A \in \mathbb{Z}^{d\times d}$, and $\det A = \pm 1$. Then, any two lattice polytopes in correspondence by any of this automorphisms will be the same polytope.

Now, observe that in Figure 4, using that the image of the vectors are the same than the columns of the matrix $A$, we have that $A = \left[\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right]$, with $A \cdot e_1 = \left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right)$ and $A \cdot e_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Also, observe that the following transforms (called *shears*) are typical lattice transforms in $\mathbb{Z}^2$: $\left[\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right]$. This can be used in exercise 3 of list 1.

After seen this, we are going to see some definitions:

Let $P \subseteq \mathbb{R}^d$ be a polytope ($\sim$ convex hull of finitely many points). A linear inequality of the form $ax \leq b$ with $a \in (\mathbb{R}^d)^*$, $x \in \mathbb{R}^d$, $b \in \mathbb{R}$ is valid for $P$ if all points of $P$ satisfy it. (observe that $(\mathbb{R}^d)^*$ represents the dual space of $\mathbb{R}^d$).

A face of $P$ is $P \cap \{x \in \mathbb{R}^d : ax = b\}$, where $ax \leqslant b$ is a valid linear inequality for $P$. In particular, $\emptyset$ is always a face of $P$ (example: $0x \leqslant 1$), and $P$ is always a face of $P$ (example: $0x \leqslant 0$). This, bring us to a second meaning of lattice:

The face lattice of $P$ is the poset (partially ordered set) of faces of $P$ with the inclusion.

**Example 2.6.** *If we have the polytope of figure 5, this polytope will have the face lattice of figure 6 (where O represents the $\emptyset$).*

If anybody wants to read about this, then read "Lectures on Polytopes" by Ziegler.

This can be applied to cubes, for example, as follows:

$$(100011)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \leqslant k,$$

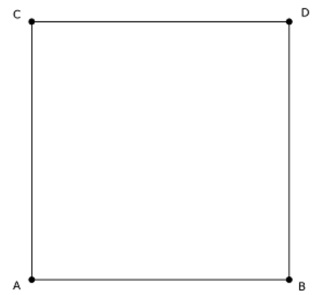



FIGURE 5. Square ABCD.

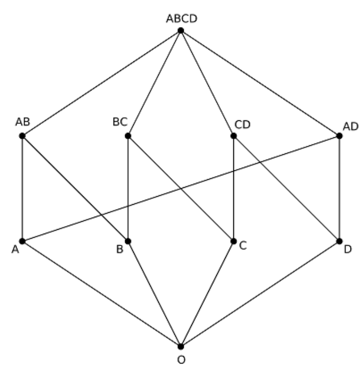



FIGURE 6. Its face lattice.

where $k$ represents the non-zero entries (in this case $k = 3$), and using this, we can calculate the barycenters.

LECTURE 3

# Pick's Theorem; Lattice packings of spheres

*Scribe: Miquel Raïch*

## 1. Pick's Theorem

**Theorem 3.1** (Pick). *Let $P$ be a lattice polygon in the plane ($P$ is closed, convex, simple and its vertices lie in $\mathbb{Z}^2$). The area of $P$ is*

$$A(P) = \mathrm{vol}_2\, P = I + \frac{1}{2}B - 1$$

*where:*
$I =$ *number of interior lattice points of* $P = \#\left\{(\mathit{int}\ P) \cap \mathbb{Z}^2\right\}$,
$B =$ *number of boundary lattice points of* $P = \#\left\{\partial P \cap \mathbb{Z}^2\right\}$

PROOF. [part]

(1) Show that Pick's formula is additive: if $P = P_1 \cup P_2$, then

$$I + \frac{1}{2}B - 1 = \left(I_1 + \frac{1}{2}b_1 - 1\right) + \left(I_2 + \frac{1}{2}B_2 - 1\right)$$

$$\begin{aligned} A(P) =& A(P_1) + A(P_2) \\ I = I(P) =& I_1 + I_2 + L - 2 \\ B = B(P) =& B_1 + B_2 - 2L + 2 \end{aligned}$$

(Principle of Inclusion-Exlusion $\to$ Möbius function)

$\lceil$ This proves:

(a) Pick$(P_1 \cup P_2) \Leftarrow$ Pick$(P_1)$, Pick$(P_2)$

(b) Pick$(P_1) \Leftarrow$ Pick$(P_1 \cup P_2)$, Pick$(P_2)$ $\rfloor$

(2) Prove it for lattice triangles.

$\square$

## 2. Lattice packings of spheres

A **lattice-packing** of congruent spheres ($\equiv$ same radius) in $\mathbb{R}^d$ is a packing such that the set $Z = \{$centers of the spheres$\}$ is a lattice $L$ (free abelian group).

Let $\{v_1, \ldots, v_n\} \in \mathbb{R}^d$ be a generating set for

$$M = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}, \text{ then } L = \{M\lambda \colon \lambda \in \mathbb{Z}^n\}$$

$$M = \overbrace{\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2}\sqrt{3} \end{bmatrix}}^{n} \Big\} d \qquad M\lambda = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \lambda_1 v_1 + \cdots + \lambda_n v_n$$

Sphere packing $B_0 + L = \{B_0 + v \colon v \in L\} = \{B_0 + M\lambda \colon \lambda \in \mathbb{Z}^n\}$

$\forall p, q \in P,\ \exists D_p, D_q \colon D_p \cap Dq = \emptyset$ $\qquad p \in P \subset \mathbb{R}^d$ discrete set of points

Voroni cell of $p$ w.r.t. $P$ is

$$\text{Vor}(P) = \{y \in \mathbb{R}^d \colon \|y - p\| \le \|y - q\| \ \forall q \in P\}$$

⌈ Georges Voronoi (s. XIX) ⌋

Voronoi cells are intersections of half spaces

$$\text{Vor}(P) = \bigcap_{q \in P} H_q \qquad \text{where } H_q = \{y \in \mathbb{R}^d \colon \|y - p\| \le \|y - q\|\}$$

**Definition 3.2.**
polyhedron $\equiv^{def}$ intersection of half-spaces
polytope $\equiv^{def}$ convex hull of a finite point set $\stackrel{\text{FTPT}}{\equiv}$ bounded polyhedron
(FTPT: Fundamental theorem of polytope theory)

(1) any convex hull of a finite point set is an intersection of half-spaces [easy by calculating convex hull].

(2) any <u>bounded</u> intersection of half-spaces is the convex hull of a finite set of points, unless the intersection is empty.

Any lattice is isomorphic to some $\mathbb{Z}^n$, as abelian groups, by the map $v \in L \leftrightarrow \lambda \in \mathbb{Z}^n \colon v = M\lambda$

[I will put images another day :P]

LECTURE 4

# The hexagonal lattice and laminated lattices

*Scribe: Ane Santos*

## 1. The hexagonal lattice

**Definition 4.1.** Let $v_1, \ldots, v_n \in \mathbb{Z}^d$ and the lattice $L = {}_{\mathbb{Z}}\langle v_1, \ldots, v_n\rangle = \{\sum \lambda_i v_i : \lambda_i \in \mathbb{Z}\} = \{M\lambda : \lambda \in \mathbb{Z}^n\}$ with $M = [v_1, \ldots, v_n] \in \mathbb{Z}^{d\times n}$. $M$ is called the *generator matrix* and ${}_{\mathbb{Z}}\langle v_1, \ldots, v_n\rangle$ the *integer hull* of the lattice.

We will study two variants of the hexagonal lattice:

$$A_{2,\mathbb{R}^2} = \left\{ \begin{bmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix} \lambda : \lambda \in \mathbb{Z}^2 \right\}, \qquad A_{2,\mathbb{R}^3} = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \lambda : \lambda \in \mathbb{Z}^2 \right\},$$

with respective generating matrices $M = \begin{bmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix}$ and $M' = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$.

Firstly, we study $A_{2,\mathbb{R}^3}$:

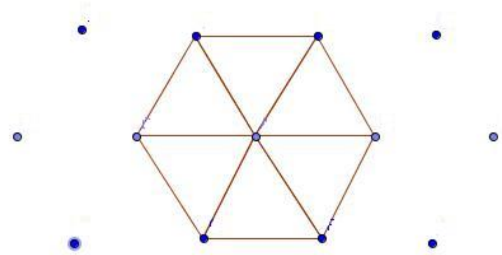


FIGURE 1. $A_{2,\mathbb{R}^3}$

$$A_{2,\mathbb{R}^3} = \left\{ M'\lambda : \lambda \in \mathbb{Z}^2 \right\} = \left\{ \begin{bmatrix} \lambda_1 \\ -\lambda_1 + \lambda_2 \\ -\lambda_2 \end{bmatrix} : \lambda_1, \lambda_2 \in \mathbb{Z} \right\}$$

We want to find a hyperplane that contains $A_{2,\mathbb{R}^3}$. We are in $\mathbb{R}^3$, so this hyperplane is of the form $\left\{x \in \mathbb{R}^3 : \langle a, x\rangle = a_0\right\}$. But we know 0 is in $A_{2,\mathbb{R}^3}$ so $a_0 = 0$ and

$$H = \left\{x \in \mathbb{R}^3 : \langle a, x\rangle = 0\right\} = \left\{\omega \in \mathbb{R}^3 : \langle \omega, x\rangle, \forall \in \operatorname{colspan} M\right\},$$

where colspan $M = {}_{\mathbb{R}}\langle v_1, \ldots, v_n \rangle = \operatorname{Im} M$ (it is an abelian group and it is also a vector space). So, $H = \{x \in \mathbb{R}^3 : \langle a, x \rangle = 0\} = \ker M$

$\dim \operatorname{Im} M = 2$ and $\dim A_{2,\mathbb{R}^3} = 3 \Longrightarrow \dim A_{2,\mathbb{R}^3} = \dim \ker M + \dim \operatorname{Im} M \Longrightarrow \dim \ker M = 1$

A generator of $\ker M$ will be $\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$

$[111] \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{smallmatrix}\right] = [00] \Rightarrow \ker M = {}_{\mathbb{R}}\langle [111] \rangle$

So $A_{2,\mathbb{R}^3} \subset \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\} = \{x \in \mathbb{R}^3 : \mathbb{1}x = 0\}$

**Definition 4.2.** The *Gram matrix* of a lattice $L$ with generator matrix $M$ is $G_L = M^T M$.

**Definition 4.3.** The *determinant of a lattice* $L$ with generator matrix $M$ is the determinant of the Gram matrix. $\det L = \det M^T \cdot \det M = (\det M)^2$

**Observation 4.4.** $G_L$ is always a symmetric matrix because $G_L^T = \left(M^T M\right)^T = M^T M$.

We calculate the determinants of $A_{2,\mathbb{R}^2}$ and $A_{2,\mathbb{R}^3}$:

$$\begin{aligned} \det A_{2,\mathbb{R}^2} &= \det \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2}\sqrt{3} \end{bmatrix} = (\tfrac{1}{2}\sqrt{3})(\tfrac{1}{2}\sqrt{3}) = \frac{3}{4}, \\ \det A_{2,\mathbb{R}^3} &= \det \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3. \end{aligned}$$

**Definition 4.5.** The *minimum norm* of a lattice $L$ is $\mu_L = \min \left\{ \|v\|^2 : v \in L \setminus \{0\} \right\}$

From the minimum norms $\mu_{A_{2,\mathbb{R}^2}} = 1$, $\mu_{A_{2,\mathbb{R}^3}} = \sqrt{2}$, we conclude that both the determinants and the minimum norms of $A_{2,\mathbb{R}^2}$ and $A_{2,\mathbb{R}^3}$ are different. However, we should not conclude that these lattices are really different:

**Definition 4.6.** Two lattices are *isomorphic* if one is obtained from the other by rotation, reflection, translation and scaling.

The most general map between isomorphic lattices is therefore of the form

$$x \longmapsto \alpha A + t, \qquad \text{where } t \in \mathbb{R}^n,\ A \in O(n),\ \alpha \in \mathbb{R}^*.$$

Note that negative $\alpha$ correspond to reflections.

**Definition 4.7** (Packing density of L)**.** $\Delta_L = \frac{\text{vol(sphere in packing)}}{\text{vol}(\Pi_L) = \sqrt{detL}}$, where $\Pi_L = \{\sum \lambda_i v_i : \lambda_i \in [0, 1)\}$ is the fundamental parallelopiped.

To calculate the packing density of $A_{2,\mathbb{R}^2}$ and $A_{2,\mathbb{R}^3}$, note that in $A_{2,\mathbb{R}^2}$ the radius of the sphere is $\frac{1}{2}$ so the volume is $\left(\frac{1}{2}\right)^2 \pi$. We obtain the same density, which is as it should be for isomorphic lattices:

$$\begin{aligned} \Delta_{A_{2,\mathbb{R}^2}} &= \frac{\left(\frac{1}{2}\right)^2 \pi}{\sqrt{3/4}} = \frac{\pi}{2\sqrt{3}}, \\ \Delta_{A_{2,\mathbb{R}^3}} &= \frac{\left(\frac{1}{2}\sqrt{2}\right)^2 \pi}{\sqrt{3}} = \frac{\pi}{2\sqrt{3}}. \end{aligned}$$

The connection between these two representations is via the map

$$\begin{bmatrix} 1 & \frac{-1}{\sqrt{3}} \\ -1 & \sqrt{3} \\ 0 & \frac{-2}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

So, we have two different ways to write the same lattice. The advantages of $M'$ over $M$ are that the coordinates are nicer and the symmetries of the lattice are more easily seen.

**Claim 4.8.** *Any permutation of the coordinate axes in $\mathbb{R}^3$ is a symmetry of $A_{2,\mathbb{R}^3}$.*

PROOF. Let $P : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a symmetry of $L = A_{2,\mathbb{R}^3}$, so that $P(L) = L$. This means that for all $x \in L$, we should have $P(x) \in L$, which is in turn equivalent to the condition that for all $\lambda \in \mathbb{Z}^2$, there must exist $\beta \in \mathbb{Z}^2$ such that

$$M\beta = PM\lambda. \tag{4.1}$$

(In particular, this coordinatizes $x \in L$ as $x = M\lambda$).

We want to prove that if $P$ is a permutation, then for any $\lambda \in \mathbb{Z}^2$ we can always find a $\beta \in \mathbb{Z}^2$ that makes equation (4.1) true. We know $P$, $M$ and $\lambda$, so we have to find $\beta$. This is a linear equation for $\beta$. We must show that the linear equation $M\beta = b$ has a unique solution for any $b = b_\lambda = PM\lambda$. The solution is unique if $\operatorname{rank} M$ is maximal, i.e. $\operatorname{rank} M = 2$. By inspection, $M$ really has rank 2, so we only have to see if it always has a solution. From the Fundamental Theorem of Linear Algebra (part 2) [**Str80**], [**Str93**], the system (4.1) has a solution if and only if

$$\begin{aligned} & b \in \operatorname{colspan} M = \operatorname{Im} M \\ \iff\ & b \perp (\operatorname{colspan} M)^\perp \\ \iff\ & b^T y = 0 \text{ whenever } y \perp \operatorname{colspan} M \\ \iff\ & b^T y = 0 \text{ whenever } y^T M = 0. \end{aligned}$$

Since $[y_1 y_2 y_3] \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{smallmatrix}\right] = [y_1 - y_2, y_2 - y_3]$, we conclude that $y^T M = 0$ if and only if

$$0y = \alpha \mathbb{1} b^T y = \lambda^T M^T P^T \alpha \mathbb{1} = \alpha \lambda^T M^T P^T \mathbb{1} = \alpha \lambda^T M^T \mathbb{1};$$

but $M^T \mathbb{1} = 0$ because $\mathbb{1}$ is in the $\ker$ of $M$. $\square$

## 2. Laminated lattices

Define $\mathbb{L}_0 = \{L^0\}$, $L^0 = \{0\} = \mathbb{R}^0$ the zero dimensional lattice and $m := 4$ (usually $m$ is 4 because then the spheres in the corresponding lattice packing have radius 1).

For $n > 0$, $\mathbb{L}_{n+1} = \{L_1^{n+1}, \ldots, L_{a_n}^{n+1}\}$ is the collection of $n+1$-dimensional lattices such that

(1) each $L_i^{n+1}$ has constant minimal norm $m$
(2) each $L_i^{n+1}$ contains at least one $L_j^n$ as a sublattice
(3) each $L_i^{n+1}$ has minimal determinant subject to (1), (2)

We will see which are these lattices:

**$\mathbb{L}_1$:** This lattice must be of the form $k\mathbb{Z}$. It needs minimal norm $m = 4$, so we must take $2\mathbb{Z}$, which satisfies (2) and (3). So the unique laminated lattice of rank 1 is $2\mathbb{Z}$.

$\mathbb{L}_2$**:** Taking $M = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right]$ satisfies (1),(2) and (3), and yields $2\mathbb{Z}^2$. However, it is not necessary that our laminated lattice contain only integer points, the only condition is that it must contain $2\mathbb{Z}$. Thus, we have the two candidates $2\mathbb{Z}^2$ and $A^2$. We decide between the two by calculating the determinant corresponding to the generator matrices $M_1 = \left[\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right]$ and $M_2 = \left[\begin{smallmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{smallmatrix}\right]$:

$$\det 2\mathbb{Z}^2 = \det M_1^T M_1 = \det \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 16;$$

$$\det A_2 = \det M_2^T M_2 = \det \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = 12.$$

We see that $\det A_2 < \det 2\mathbb{Z}^2$, which comes about because the area of the fundamental parallelopiped for $A_2$ is less than that of the square. So $\mathbb{L}_2$ is $A_2$.

We will see now how can we do the sphere packing:

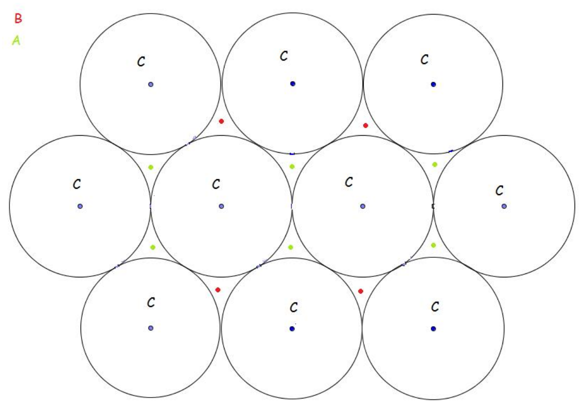



FIGURE 2. sphere packing

For the next layer we have two options, put them (the centers of the sphere) over the deep holes A or over the deep holes B. For the second layer we put, we will have the possibility of putting them over C (the centers of the spheres of the first layer). Each lattice obtained by snugly packing copies of $A_2$ is determied by the sequences ABAB.... (this is the hexagonal close packing(He attoms)) ABCABC....(this is the face-centered cubic lattice ($A_3$)) of equivalence classes of deep holes.

In each step there are two options to choose from, which makes uncountably many possibilities in total.

LECTURE 6

# Some basic concepts in geometry and software; Introduction to Packings

*Scribe: Ferran Dachs Cadefau*

For the first part see [**Pfe11**].

## 1. Big numbers

Using computers, you can represent big numbers, you can do:

| | | |
|---|---|---|
| $\mathbb{N}, \mathbb{Z}$ | `int, long int, long long int` or work with arbitrary precision (slower) | $-9.2 \cdot 10^{18} ... 9.2 \cdot 10^{18}$ `gmplib.org` |
| $\mathbb{Q}$ | $\{(n,d) : n \in \mathbb{Z}, d \in \mathbb{N}_{>0}\}/\sim$ where $(n,d) \sim (n',d')$ iff $nd' = n'd$ | `gmplib.org` |
| $\mathbb{R}$ | algebraic numbers ok ( $\sqrt{2}$, $\sqrt[4]{3}$), transcendental ones not ($\pi$, e) | `cgal.org, mpfr.org` |
| $\mathbb{C}, \mathbb{H}$ | pairs or quadruples of reals | `#include <complex>` |

But how to represent very big numbers? For example using 4 symbols what is the biggest number that you can represent?

$$10^{99} < 9^{999} < 9^{9^{9^9}} = 9 \uparrow 4 < 9 \uparrow 9 \uparrow 9 \uparrow 9 = 9 \uparrow\uparrow 4 < 9 \Uparrow 9 \Uparrow 9 \Uparrow 9$$

This is known as Knuth's up-arrow notation, to represent bigger numbers there are the Ackerman Function:

$$A(k,n) = n \Uparrow^{(k)} n$$
$$A(n,n) = n \Uparrow^{(n)} n$$

For example:

$$A(2) = 2 \Uparrow 2 = 2 \uparrow 2 \uparrow 2 = 2 \uparrow \left(2^2\right) = 2^{2^{2^2}}$$
$$A(3) = 3 \Uparrow^{(3)} 3 = 3 \Uparrow 3 \Uparrow 3 \Uparrow 3$$

Using this we can define $\alpha(x)$ as:

$$\alpha(x) = \max\{n \in \mathbb{N}_{\geqslant 1} : A(n) < x\}$$

It appears, for example, in the complexity of the lower envelope of a segment arrangement. In the plane is $\Omega(n\alpha(n))$

## 2. Projective geometry and Polarity

**2.1. Projective geometry.** We define the projective space as:

$$\begin{aligned} \mathbb{P}^n &= \{\text{lines through } 0 \in \mathbb{R}^{n+1}\} \\ &= S^n/\mathbb{Z}_2 \end{aligned}$$

We are identifying the antipodes: $x$ and $-x$. But we have a problem: $\mathbb{P}^1$, $\mathbb{P}^3$ are orientable, but for example $\mathbb{P}^2$ not: we can't define interior (inside/outside), therefore we can't define convex.

In $\mathbb{P}^2$ The line through points $p_1$ and $p_2$:

- $p_1$ lies on $l$: $p_1 \perp l$
- $p_2$ lies on $l$: $p_2 \perp l$

Therefore: $l = \lambda \cdot p_1 \times p_2$ ($\times$ is the cross-product of vectors in $\mathbb{R}^3$, in higher dimensions is the determinant).

The point $q$ on lines $l_1$ and $l_2$:

- $q$ lies on $l_1$: $q \perp l_1$
- $q$ lies on $l_2$: $q \perp l_2$

Therefore: $q = \lambda \cdot l_1 \times l_2$ (the cross-product of vectors in $\mathbb{R}^3$, in higher dimensions is the determinant).

Computationally, we can intersect lines and join points, in homogeneous coordinates or in Cartesian coordinates. In the first case we have:

```
void intersect_lines(const vec_t& l1, const vec_t& l2, vec_t& p)
{
  cross_product(l1, l2, p);
}
void join_points(const vec_t& p1, const vec_t& p2, vec_t& l)
{
  cross_product(p1, p2, l);
}
void cross_product(const vec_t& l1, const vec_t& l2, vec_t& p)
{
  p[0] = l1[1]*l2[2] - l1[2]*l2[1];
  p[1] = -l1[0]*l2[2] + l1[2]*l2[0];
  p[2] = l1[0]*l2[1] - l1[1]*l2[0];
}
```

This code is **correct** (calculates exactly what it should), **efficient** (No extraneous copying (&) and reuse of code), and **robust** (No influence of rounding errors and it handles all cases, even degenerate ones).

But, in Cartesian coordinates. A line is now $y = kx + d$, stored as a vector $(k, d)$.

```
bool intersect_lines(const vec_t& l1, const vec_t& l2, vec_t& p)
 {
if (l1[0]==l2[0]) {
if (l1[1]==l2[1]) {
return COINCIDENT_LINES;
}
return PARALLEL_LINES;
```

```
}
p[0] = (l2[1]-l1[1])/(l1[0]-l2[0]);
p[1] = l1[0]*p[0] + l1[1];
}
```

This code is *somewhat* **efficient** (Again, no copying, but: no reuse of code for join_points), **not robust** (It's unstable numerically (==, /)), and *not even* **correct** (It doesn't handle parallel lines, but that's Euclidean geometry's fault).

**2.2. Polarity.** It's related to Projective duality, points are dual to hyperplanes:

$$\begin{array}{ccc} \mathbb{P}^n(\mathbb{R}) & \xrightarrow{d} & (\mathbb{P}^n(\mathbb{R}))^\star \\ p & \longmapsto & p^\star \end{array}$$

A duality is an involutory order-anti-isomorphism (Involution: $d^2 = \mathrm{id}$, isomorphism: bijection collineation, order-anti: $p \in l$ implies $p \supset l$).

The polarity is the same except by convexs.

Polarity send: points to half-spaces. Given $a \in \mathbb{R}^n$:

$$a \mapsto H_a = \{x \in \mathbb{R}^d : \langle a, x \rangle \leqslant 1\} \leftrightarrow x \supset (\mathbb{R}^d)^\star$$

Equivalently, they identify points with hyperplanes via homogeneous coordinates.

For example, if we take $p = (a : b : 1) \in \overrightarrow{\mathbb{P}^2}$ a point, then $p^\star = (a : b : 1) \in \left(\overrightarrow{\mathbb{P}^2}\right)^\star$ a line, and $p^{\star\star} = p$.

$P \in l \Leftrightarrow p \perp l \Leftrightarrow p^\star \perp l^\star \Leftrightarrow l^\star \perp p^\star \Leftrightarrow l^\star \in p^\star$

## 3. Polymake

Polymake is a free program in perl. For example we can define a cube in dimension 3 as:

```
polytope > $p=cube(3);
```

Another thing is to know properties about a polytope. For example the coordinates of the vertices:

```
polytope > print $p->VERTICES;
1 -1 -1 -1
1 1 -1 -1
1 -1 1 -1
1 1 1 -1
1 -1 -1 1
1 1 -1 1
1 -1 1 1
1 1 1 1
```

Or the number of faces of each dimension:

```
polytope > print $p->F_VECTOR;
8 12 6
```

Or the vertices in each facet:

```
polytope > print $p->VERTICES_IN_FACETS;
{0 2 4 6}
{1 3 5 7}
{0 1 4 5}
{2 3 6 7}
{0 1 2 3}
```

```
{4 5 6 7}
```

To see a representation of the polytope you can execute:

```
polytope > $p-> VISUAL;
```

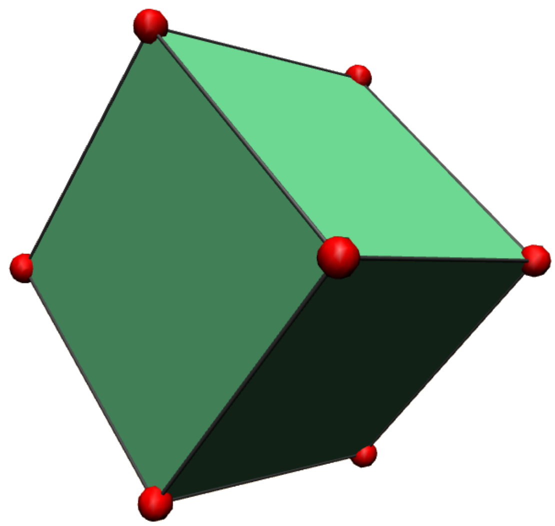


FIGURE 1. The representation of the cube in dimension 3 due with Polymake

To see a representation of graph of the polytope you can execute:

```
polytope > $p-> VISUAL_FACE_LATTICE;
```

To see the equations of the facets you can type:

```
polytope > print $p->FACETS;
1 1 0 0
1 -1 0 0
1 0 1 0
1 0 -1 0
1 0 0 1
1 0 0 -1
```

You must interpret as the following:

If you take $(1:1:0:0)$ gives you the facet:

$$\left\langle (1:1:0:0) \begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix} \right\rangle \geqslant 0$$

equivalently: $1+x \geqslant 0$ or $x \geqslant -1$. If you take $(1:-1:0:0)$ gives you the facet: $1-x \geqslant 0$ or $1 \geqslant x$.

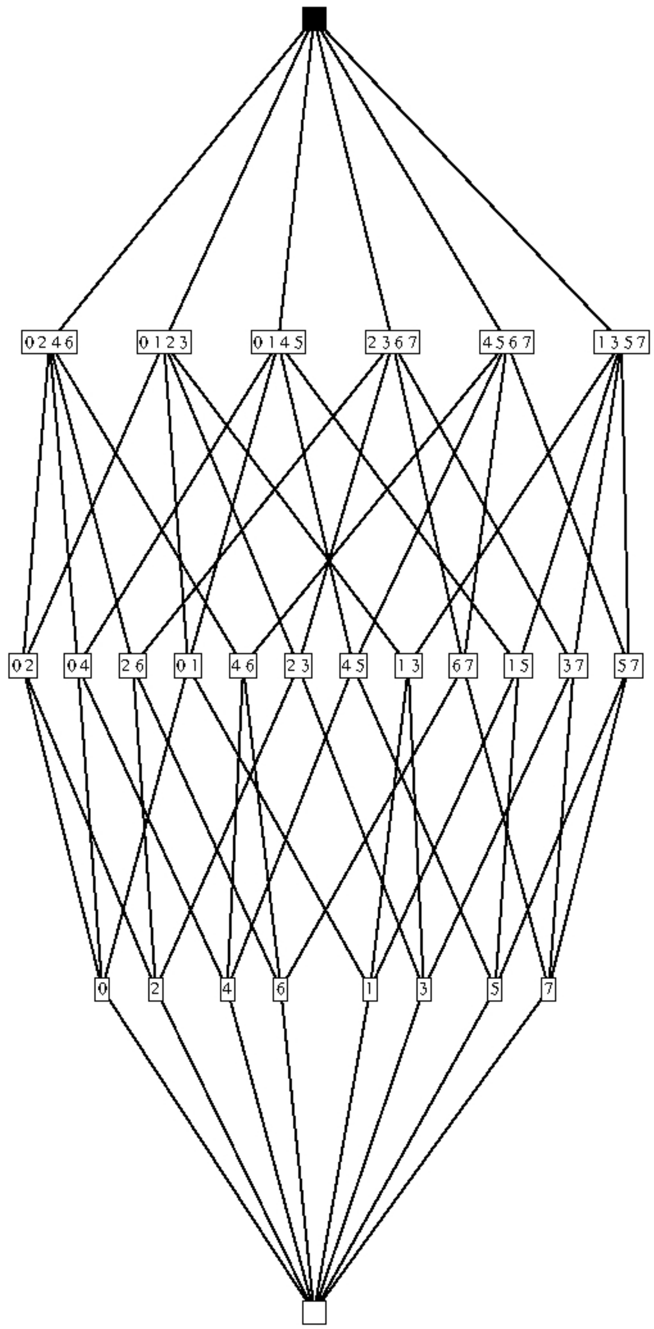



FIGURE 2. The representation of the graph of the cube due with Polymake.

We can polarize a polytope, for example:

```
polytope > $q=polarize($p);
```

doing this, if we want to print the vertices of this new polytope, as we expected, gives the facets of the cube:

```
polytope > print $q->VERTICES;
1 -1 0 0
1 1 0 0
1 0 -1 0
1 0 1 0
1 0 0 -1
1 0 0 1
```

Another thing that we can do is make Voronoi Diagrams. For example if we want to know the Voronoi Diagram of the points $(1,1),(0,1),(-1,1),(1,-1),(0,-1)$ and $(-1,-1)$:

```
$VD = new VoronoiDiagram(SITES=>[[1,1,1],[1,0,1],[1,-1,1],
[1,1,-1],[1,0,-1],[1,-1,-1]]);
```

If we want to see a graphical representation:

```
$VD->VISUAL_VORONOI;
```

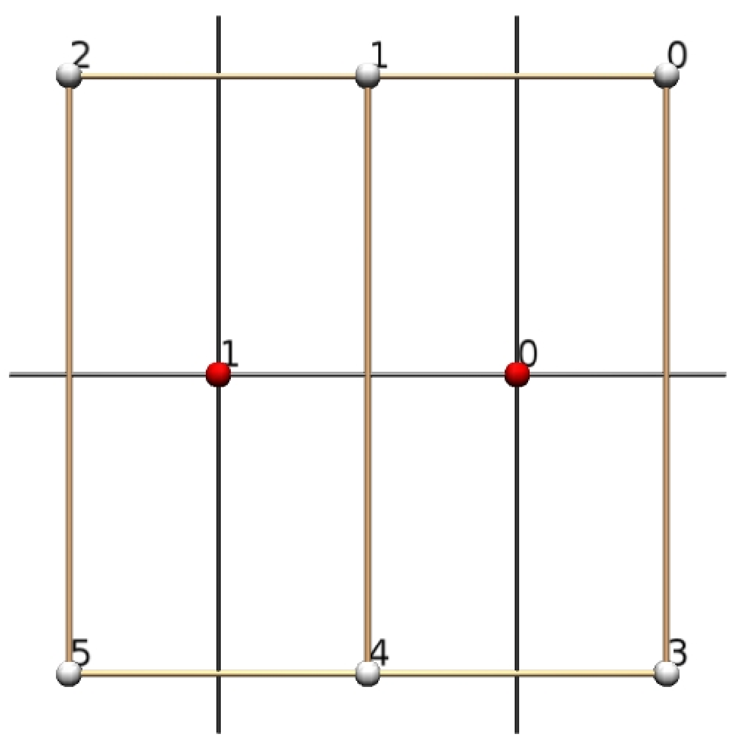



FIGURE 3. The representation of one Voronoi Diagram with Polymake.

Other properties are, for example the facets equation:

```
polytope > print $VD->FACETS;
2 -2 -2 1
1 0 -2 1
2 2 -2 1
2 -2 2 1
1 0 2 1
2 2 2 1
1 0 0 0
```

or the vertices of the diagram:

```
polytope > print $VD->VERTICES;
0 0 1 2
0 1 0 2
1 1/2 0 -1
0 -1 0 2
0 0 -1 2
1 -1/2 0 -1
```

## 4. Voronoi Cells in lattices

Given a lattice $L \subset \mathbb{R}^d$, find the facets of

$$\mathrm{Vor}(0) = \{x \in \mathbb{R}^d : \|x-0\|^2 \leqslant \|x-v\|^2, \forall v \in L\}$$

**Definition 6.1.** $v \in L$ is *relevant* if the bisector of $v$ and $0$ contains a full dimensional face (i.e. a facet) of $\mathrm{Vor}(0)$.

In $\mathbb{Z}^3$ we have a cube and the polytope defined by the relevant points is an octahedron.

**Observation 6.2.** If the relevant vectors are precisely the minimal vector then $\mathrm{Vor}(0)$ is polar dual to the vertex figure of 0 in $L$.

**Theorem 6.3** (Georges Voronoi, 1908)**.** *A non-zero vector $v \in L$ is relevant if and only if $\pm v$ are the only shortest vectors in the coset $v + 2L$.*

PROOF. $\Longrightarrow$
Suppose that $v, w \in L$ satisfy

$$w \in v + 2L \tag{6.1}$$

but

$$w \neq \pm v \tag{6.2}$$

and

$$\langle w, w\rangle \leqslant \langle v, v\rangle \tag{6.3}$$

Set:

$$t = \tfrac{1}{2}(v+w) \text{ and } u = \tfrac{1}{2}(v-w)$$

Using (6.1) we have that $t, u \neq 0$ and using (6.2), $t, u \in L$. We have:

$$H_v = \{x \in \mathbb{R}^d : \langle x, v\rangle \leqslant \frac{1}{2}\langle v, v\rangle\}$$

Let $x \in H_t \cap H_u$. Then:

$$\langle x, t\rangle \leqslant \frac{1}{2}\langle t, t\rangle$$

and

$$\langle x, u\rangle \leqslant \frac{1}{2}\langle u, u\rangle$$

Adding, we have:

$$\begin{aligned}\langle x, v\rangle &\leqslant \tfrac{1}{8}\left(\langle v, v\rangle + \langle w, w\rangle + 2\langle v, w\rangle + \langle v, v\rangle + \langle w, w\rangle - \langle v, w\rangle\right)\\ &= \tfrac{1}{4}\left(\langle v, v\rangle + \langle w, w\rangle\right)\\ &\leqslant \tfrac{1}{2}\langle v, v\rangle\end{aligned}$$

Were in the last one we have used (6.3). So $x \in H_v$, therefore $v$ is not relevant.

$\Longleftarrow$ Suppose $v \in L$ is not relevant. Then there exists some $w \in L$. $w \neq 0$, $w \neq \pm v$ with:

$$\left\langle \frac{1}{2}v, w\right\rangle \geqslant \frac{1}{2}\langle w, w\rangle \tag{6.4}$$

and $\frac{1}{2}v \notin H_w$.Then

$$\begin{aligned}\|v - 2w\|^2 &= \langle v - 2w, v - 2w\rangle\\ &= \langle v, v\rangle - 4\langle v, w\rangle + 4\langle w, w\rangle\\ &\leqslant \langle v, v\rangle - 4\langle v, w\rangle + 4\langle v, w\rangle\\ &= \langle v, v\rangle\end{aligned}$$

where in the inequality we have used (6.4), and is in $v + 2L$, not $0$, $w \in L$.

□

LECTURE 8

# Introduction to orbifolds

*Scribe: Ane Santos*

**Definition 8.1.** Informally, an *orbifold* is the quotient of a manifold (here, the Euclidean plane) by the action of a group.

| | | |
|---|---|---|
| torus | $\longleftrightarrow$ | o |
| holes | $\longleftrightarrow$ | * |
| non-orientability | $\longleftrightarrow$ | × |
| boundary singularity | $\longleftrightarrow$ | *n |
| cone point of order $n$ | $\longleftrightarrow$ | n |

**Theorem 8.2** (Magic theorem for the sphere)**.** *The total cost of the signature of any spherical group is* \$$2 - \frac{2}{g}$*, where $g$ denotes the total number of symmetries.*

The Magic theorem in the plane is a special case because the number of symmetries in a plane is infinite, so the cost is always 2.

There are 14 spherical symmetry groups: $m, n \geq 1$

| | | | | |
|---|---|---|---|---|
| *532 | *432 | *332 | *22n | *mn |
| | | 3*2 | 2*n | n* |
| | | | | n× |
| 532 | 432 | 332 | 22n | mn |

If $n \to \infty$ and $m \to \infty$ in *22n, *mn, 2*n, n*, n×, 22n and mn we get the 7 possible groups of friezes (cenefas).

We spent almost the entire lecture with scissors and tape, cutting out the orbifolds corresponding to the tesselations in [**CBGS08**].

LECTURE 9

# Orbifolds

*Scribe: Roger Ten*

## 1. Defining an orbifold

In this section $X$ denotes a topological space and $G$ denotes a topological group. So must begin by defining what is a topological group.

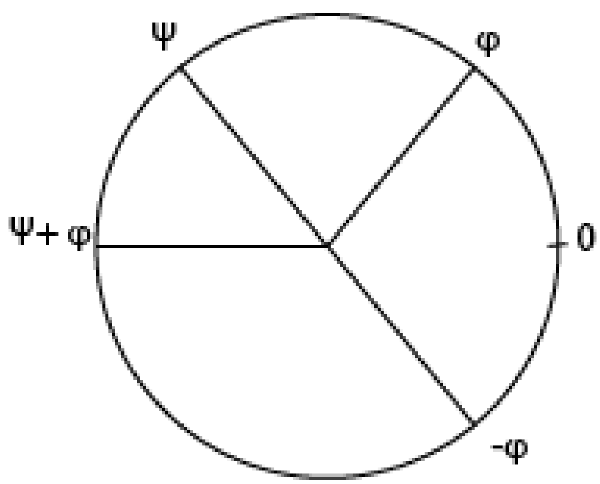
**Definition 9.1.** (1) A *topological group* is a topological space that is simultaneously a group such that, the group operations are continuous.

(2) A topological space $X$ is called *G-space* if a topological group $G$ acts on $X$ via a continuous map $G \times X \longrightarrow X$, $(g, x) \longmapsto gx$ such that $g(hx) = (gh)x$ and $1_G x = x$.

**Example 9.2.** • *Lie groups:* $O(n), SO(n)$

• $S^1$ *is a topological space and it is also a group via:*

$$\begin{array}{cccc} + : & S^1 \times S^1 & \to & S^1 \\ & (\varphi, \psi) & \to & \varphi + \psi \end{array}$$




Now is coming a list of definitions

**Definition 9.3.** Let $x \in X$ be a point.

(1) $G_x = \{g \in G : gx = x\} = Stab_G(x)$ is called the *stabilizer of* $x$ or the isotropy subgroup of $x$. $G_x$ is a subgroup of $G$ ($G_x \leqslant G$).
(2) $G(x) = \{gx : g \in G\} \subset X$ is the *orbit of* $x$.
(3) The action of $G$ on $X$ is *free* if $G_x = \{1\} \forall x \in X$.

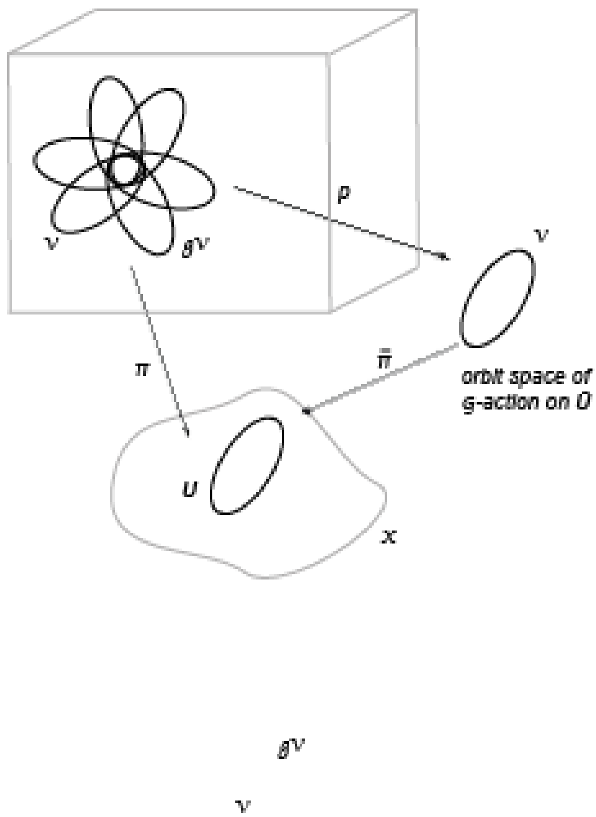
(4) The action of $G$ on $X$ is *transitive* if $G(x) = X \forall x \in X$, i.e., there exist only one orbit.
(5) The map $G/G_x \to G(x)$ is a continuous bijection
(6) The orbit space $X/G$ is the set of all orbits in $X$. (It is a topological space with quocient topology).
(7) A map $f : X \to Y$ of $G$-spaces is *$G$-equivariant* (or $G$-map) if $f(gx) = g(f(x)); g \in G$. so the following diagram commutes:

$$\begin{array}{ccc} X & \stackrel{f}{\to} & Y \\ \downarrow g & \circlearrowright & \downarrow g \\ X & \stackrel{f}{\to} & Y \end{array}$$

**Definition 9.4.** An *orbifold chart* on a topological space $X$ is tuple $(\tilde{\mathcal{U}}, G, \mathcal{U}, \pi)$, such that:

- $\mathcal{U} \subset X$ is an open subset (neighborhood).
- $\tilde{\mathcal{U}} \subset \mathbb{R}^n$ is an open subset.
- $G$ is a finite group of homomorphisms of $\tilde{\mathcal{U}}$.
- $\pi : \tilde{\mathcal{U}} \to \mathcal{U}$ can be factorized as $\pi = \tilde{\pi} \circ p$, where $p : \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}/G$ is the orbit map and $\tilde{\pi} : \tilde{\mathcal{U}}/G \to \tilde{\mathcal{U}}$ is an homomorphism.

$$\begin{tikzcd} G \arrow[r, phantom, "\underset{act\ on}{\circlearrowleft}"] & \tilde{\mathcal{U}} \arrow[dd, "\pi"] \arrow[dr, "p"] & \subset & \mathbb{R}^n \\ & & \tilde{\mathcal{U}}/G \arrow[dl, "\tilde{\pi}"] & \\ & \mathcal{U} & \subset & X \end{tikzcd}$$




**Definition 9.5.** Two orbifold charts are *compatible* if for all $\tilde{u_i} \in \tilde{\mathcal{U}}_i, i = 1, 2$ such that $\pi_1(\tilde{u_1}) = \pi_2(\tilde{u_2})$ there exists a homomorphism $h : \tilde{\mathcal{V}}_1 \to \tilde{\mathcal{V}}_2$, where $\tilde{\mathcal{V}}_i$ is a neighborhood of $\tilde{u_i}$ in $\tilde{\mathcal{U}}_i$, such that $\pi_1 = \pi_2 \circ h$ on $\tilde{\mathcal{V}}_1$

$$\begin{tikzcd}
\tilde{\mathcal{U}}_2 \arrow[d, "p_2"] & \supset & \tilde{\mathcal{V}}_2 \arrow[dr, "\pi_2"'] & & \tilde{\mathcal{V}}_1 \arrow[ll, "h"'] \arrow[dl, "\pi_1"] \arrow[d, phantom, "\circlearrowright"] & \subset & \tilde{\mathcal{U}}_1 \arrow[d, "p_1"] \\
\tilde{\mathcal{U}}_2/G \arrow[rrr, "\tilde{\pi}_2"] & & & \tilde{\mathcal{U}}_1 \cap \tilde{\mathcal{U}}_2 & & & \tilde{\mathcal{U}}_1/G0 \arrow[lll, "\tilde{\pi}_1"']
\end{tikzcd}$$

**Remark 9.6.** $G_i = \{1\}$ yields manifolds.

**Definition 9.7.** An *orbifold atlas* is a collection of compatible orbifold charts that cover $X$.
An *orbifold* $Q$ is an underlying Hausdorff topological space $|Q|$ with an orbifold atlas.

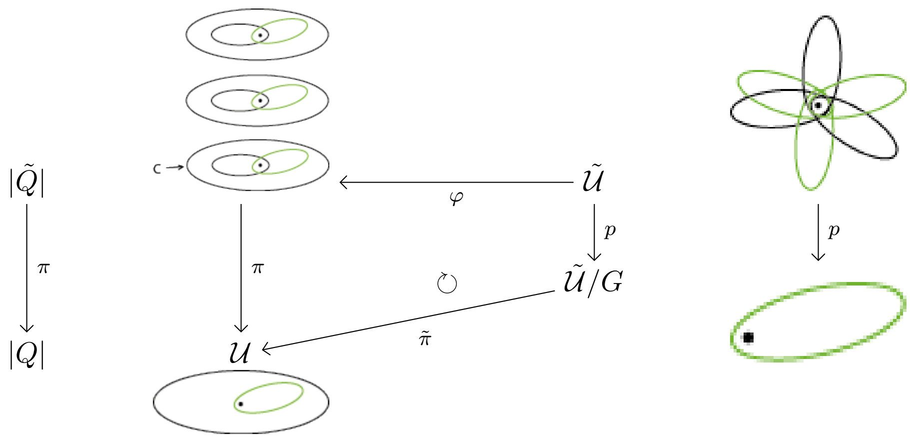
## 2. Covering orbifolds

A *covering* of a topological space $X$ is a topological space $Y$, together with a continuous surjective projection $\pi : Y \to X$ such that for every $x \in X$ exist $\mathcal{U}_x$ open neighborhood of $x$ such that the pre-image $\pi^{-1}(\mathcal{U}_x)$ is a disjoint union of copies of $\mathcal{U}_x$.

**Definition 9.8.** A *covering orbifold* of an orbifold $Q$ is an orbifold $\tilde{Q}$ with a projection $\pi : |\tilde{Q}| \to |Q|$ between the underlying spaces with the following property:

For any $x \in |Q|$ there exist a neighborhood $\mathcal{U} \cong \tilde{\mathcal{U}}/G$; $\mathcal{U} \subset \mathbb{R}^n$ such that each connected component $C$ of $\pi^{-1}(\mathcal{U})$ is homeomorphism to $\tilde{\mathcal{U}}/\Gamma_i$ for some subgroup $\Gamma_i \leqslant G$.

**Remark 9.9.** This homeomorphism $\varphi$ must respect both projections, namely $\pi$ and $p_i : \tilde{\mathcal{U}}/G \to \tilde{\mathcal{U}}/\Gamma_i$.




**Definition 9.10.** An orbifold is called *good* or developable if there exists some manifold that covers it. Otherwise, it is *bad* or not developable

LECTURE 10

# The Magic Theorem

*Scribe: Victor Bravo*

Today, we want to prove the Magic Theorem. To do this, we will introduce the language of the CW-Complexes:

CW-COMPLEXES:

**Observation 10.1.** The theory of CW-Complexes was developed around 1910 by J.H.C. Whitehead. Also, the 'C' do refence to 'closed' and the 'W' do reference to 'weak' in the sentence 'closed in the weak topology'.
We will use CW-Complexes to generalize triangulations of the space.

Let $X$ be a topological space.
We define $X^0 :=$ 0-skeleton of $X = \{\text{points in } X\}$.
Inductively, given the $(n-1)$-skeleton of $X$, take a bunch of $n$-dimensional closed disks ($\equiv$ balls) $e_1^n, \ldots, e_r^n$, and we can do the next attaching map $\varphi_i : \mathbb{S}^{n-1} = \partial e_i^n \longrightarrow X^{n-1}$.
$X^n := X^{n-1} \dot{\bigcup}_i e_i^n / \sim$, where $x \sim \varphi_i(x)$.
Since a 1-dimensional disk is an interval, taking a set of points in a plane, a 1-dimensional CW-Complex is a multigraph with loops.
In dimension 2, a good example can be to take only the point $p$ as the 0-skeleton, an empty number of 1-dimensional balls, and a 2-dimensional ball (which is a disk). So, we have that the attaching map goes from the boundary of this disk (which is $\partial e^2 = \mathbb{S}^1$) to the lower dimensional skeleton (which is the point $p$). Doing the $\sim$, we get the 2-dimensional sphere.
The same example works for any dimension $n$ using stereographic projection. So, we can calculate the number of cells that we need to decomposing $\mathbb{S}^n$ as a regular triangulation (regular in the sense of not having identifications on the boundary of the cell and in the sense of the intersection of two faces is empty or a face of both).
For dimension 0, we have two points. So, the $f$-vector of $\mathbb{S}^0$ is $(2)$ (which means: two 0-dimensional faces), and we need $(2)$ cells in the CW-complex.
For dimension 1, we have $\mathbb{S}^1$ and, to do a regular triangulation, we need three different distinguished points. So, the $f$-vector of $\mathbb{S}^1$ is $(3,3)$ (which means: three 0-dimensional faces and three 1-dimensional faces), and we need $(1,1)$ cells in the CW-complex.
For dimension 2, we have $\mathbb{S}^2$ and, to do a regular triangulation, we need a tetrahedron over the $\mathbb{S}^2$ surface. So, the $f$-vector of $\mathbb{S}^2$ is formed by 4 vertices, $\binom{4}{2}$ edges, and $\binom{4}{3}$ faces (triangles). This is $(4,6,4)$, and we need $(1,0,1)$ cells in the CW-complex.
For dimension $n$, we have the $n$-dimensional simplex. So, whose $f$-vector is equal to $(n+1, \binom{n+1}{2}, \binom{n+1}{3}, \ldots, \binom{n+1}{n})$ (that has $2^{n+1}-2$ elements), and we need $(1,0,\ldots,0,1)$ cells in the CW-complex.

To go towards the Magic Theorem, we need to remember the definition of the covering of an orbifold to explain:

THE ORBIFOLD EULER CHARACTERISTIC:

Let be $G$ a group that acts over an orbifold $Q$, $G \circlearrowleft Q$.
So, taking $x \in Q$, we have that $G_x \leq G$ is that we call the stabilizer of $x$.
What we want to do is to decompose $Q$ as a CW-complex into invariant cells, i.e., into a collection, $\mathcal{C}$, of cells such that the function $x \mapsto G_x$ is constant on each cell, $C \in \mathcal{C}$ (observe that $C^0 \subset C$).
An example of this is to take $|Q| = \mathbb{D}^2$ and $G$ as the abstract group $G = \{e = id, g, g^2\} = \mathbb{Z}_3$, where $g$ represents a rotation of $\frac{2\pi}{3}$ (this is the same that to think in $G$ as the spherical group formed by two rotation points of order three). Now, we want to decompose this as a CW-manifold into cells such that the stabilizing subgroup is constant. We have two differents types of points that we resume as points $y$, which are rotation points, and points $x$ which are not rotation points. Now, the stabilizing group of the points $x$ is $G_x = \{e\}$, and the stabilizing group of the points $y$ is $G_y = G$. So now, we will have to decompose this disk as a CW-complex made-up of invariant cells. To do this, we have to leave one of the rotation points in the interior of the disk, the other rotation point over the boundary of the disk, and add two new points over the boundary of the disk.
Now, we define the *Euler Characteristic of an orbifold* as

$$\chi(Q) = \sum_{C \in \mathcal{C}} \frac{(-1) \dim C}{\# G_C}.$$

To see that this definition works, we have to check two things.
The first thing is that this defition is an application of the original Euler Characte-ristic that works over orbifolds in the same sense of the original works over simplicial complexes, where the original Euler Characteristic is defined as the alternating sum of the $f$-vector of the simplicial complex.

**Example 10.2.** *Taking the simplicial complex formed by the three vertices, the three edges and the face of a triangle (i.e., a 2-dimensional ball), we have to sum $1(-1)^2$ for the face, $3(-1)^1$ for the edges, $3(-1)^0$ for the vertices and $1(-1)^{-1}$ for the $\emptyset$, which gives us $\chi = 0$.*
*In the same way, if we take the simplicial complex formed by the three vertices, the three edges of a triangle, this gives us $\chi = -1$.*

This works for any regular triangulation, so, if works for orbifolds.
The second thing to check is that this formula works for the group, but in the case of having a simplicial complex there is no group, or in other words, we have only the identity. So, it works for the group.
Now, what we really wants is apply this to coverings, so, remember:
In a $k$-fold, covering $\tilde{Q} \to Q$ of orbifolds, every point in $|Q|$ has $k$ pre-images.

**Theorem 10.3.** *If $\tilde{Q} \to Q$ is a $k$-fold covering of orbifolds, then $\chi(\tilde{Q}) = k\chi(Q)$.*

PROOF. Remembering session 9, let $y \in U$ be a non-singular point in an orbifold chart, $U$, which corresponds to a some number of points in $V$.
The number of pre-images of $y$ in each neighborhood of $\tilde{x}_i$ is $\dfrac{|G_x|}{|G_{\tilde{x}}|}$.

So, the total number of preimages of $y$ is

$$k = \sum_{\tilde{x} \in \Pi^{-1}(x)} \frac{|G_x|}{|G_{\tilde{x}}|},$$

which implies that

$$\frac{k}{|G_x|} = \sum_{\tilde{x} \in \Pi^{-1}(x)} \frac{1}{|G_{\tilde{x}}|}.$$

Now, using that the cells are invariant and $x \mapsto G_x$, we have that

$$\frac{k}{G_x} = \frac{k}{G_C}.$$

So, we have that

$$k\chi(Q) = \sum_{C \in \mathcal{C}} \frac{k(-1)^{\dim(C)}}{|G_C|} = \sum_{\tilde{C} \in Pi^{-1}(C)} \frac{(-1)^{\dim(C)}}{|G_{\tilde{C}}|} = \chi(\tilde{Q}),$$

where the last equality holds because $\tilde{C} \in Pi^{-1}(C), \forall C \in \mathcal{C}$, and this sums over all $\tilde{C} \in |\tilde{Q}|$. □

□

**Example 10.4.** *Let's take $|Q| = \mathbb{D}^2$ with $k$ mirrors and $k$ corner reflectors labeled $m_1, \ldots, m_k$. If we compute de signature of this, we have no blue part because there are not rotation points. Then, we have a $*$, and we can take, for example, the signature $*632$. So, we have one point, $A$, with $6$ mirrors through it, another point, $C$, with $3$ mirrors through it, and another point, $B$, with $2$ mirrors through it. Now, if we go down to the orbifold (taking one copy of every mirror), we have $3$ mirrors in the orbifold. So, $k = 3$. Now, we want to cellulate this CW-complex such that every cell in the CW-complex has the same stabilizer. Since, $A$ is stabilized by the subgroup $D_6$, $B$ is stabilized by the subgroup $D_2$, and $C$ is stabilized by the subgroup $D_3$, we can compute the Orbifold Euler Characteristic.*
*We have, for dimension $0$,*

$$\frac{(-1)^0}{|D_6|} + \frac{(-1)^0}{|D_2|} + \frac{(-1)^0}{|D_3|},$$

*for dimension $1$,*

$$\frac{(-1)^1}{|\mathbb{Z}_2|} + \frac{(-1)^1}{|\mathbb{Z}_2|} + \frac{(-1)^1}{|\mathbb{Z}_2|},$$

*and for dimension $2$,*

$$\frac{(-1)^2}{|\{e\}|}.$$

*So,*

$$\tilde{\chi} = \frac{1}{12} + \frac{1}{4} + \frac{1}{6} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{1} = 0.$$

**Example 10.5.** *Doing the same example with any $k$ corner reflectors using the notation $m_i$ to note the number of mirrors through the corner $i$, we will have*

$$\tilde{\chi} = \sum_{i=1}^{k} \frac{1}{2m_i} - \frac{k}{2} + 1 = \frac{1}{2} \sum_{i=1}^{k} \left(\frac{1}{m_i}\right) + 1 = 1 - \sum_{i=1}^{k} \frac{m_i - 1}{2m_i}.$$

*So, if we get a disk, it has this Euler Characteristic.*

**Example 10.6.** *Now, an example with true-blue signature (which means that we have only rotation points). Let's take $|Q| = \mathbb{S}^2$, and $l$ cone points labeled $n_1, \ldots, n_l$. Let's take, for example, $532$ over $\mathbb{S}^2$. So, we have a sphere and three cone points. The first thing we've got to do is to cellulate the sphere as a CW-complex such that each cell has constant stabilizer. So, it seems very clear that we are going to use $0$-dimensional vertices at the cone points. So, the stabilizing subgroup is $C_5$ for one point, $C_3$ for another point, and $C_2$ for another point. In the same way, the stabilizing subgroup of the edges that goes from one point to another is $\{e\}$, and also for the face, the subgroup is $\{e\}$. So, we have that*

$$\tilde{\chi} = \frac{(-1)^0}{5} + \frac{(-1)^0}{3} + \frac{(-1)^0}{2} - 1 - 1 - 1 + 1 + 1 = \sum_{i=1}^{l} \frac{1}{n_i} - l + 2 = 2 - \sum_{i=1}^{l} \frac{n_i - 1}{n_i}.$$

*What we have to observe here is that seems that we extract something from $2$. In the last example seems that happens the same thing, but there we had a $*$ and all was natural. What happens here is that using $\chi(\mathbb{S}^2) = (-1)^0 + (-1)^2 = 2$, we have that if we do a hole over the sphere, we are substracting $1$, because $\chi(\mathbb{S}^2$ with a hole$) = 1$, and we have the possible $*$'s covered. So, finally we have that $\tilde{\chi}(Q) = 2 - ($cost of the signature$)$.*

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