A Note on the Simplex Cosimplex problem
DRAFT

Karim Adiprasito*
Inst. Mathematics
FU Berlin
Arnimallee 2
14195 Berlin, Germany
adiprasito@math.fu-berlin.de

3.9.2011

Abstract

We construct, for every dimension $d \geq 3$, polytopal spheres S for which neither S nor its dual S^* contain 2-faces that are triangles. This answers the topological formulation of a problem of Kalai, Kleinschmidt and Meisinger.

1 Introduction

This note is to attack the topological variant of a question of Kalai, Kleinschmidt and Meisinger, concerning low-dimensional substructures that must appear in high-dimensional polytopes and spheres, cf. [6, 5]:

Problem 1. [5, Problem 4.4] *Does, for every k, exist a $d(k)$ such that for all polytopes of dimension at least $d(k)$, either P or P^* contain a k-simplex as a k-face.*

We will show that if one asks the question for polytopal spheres, the answer is negative even in the case $k = 2$. This is related to constructions of smooth metrics of nonpositive Ricci curvature on spheres and other manifolds, cf. [3, 4]. Not surprisingly, the method of construction used here is closely related to the one used in Lohkamps famous proof of the existence of nonpositive Ricci Curvature structures on all smooth manifolds [4].

The original question, whether the there exists polytopes P so that neither P nor its polar dual contain contain triangular 2-faces, is still open.

Main Theorem 1. *Let M be a triangulated manifold of dimension at least 3. Then there exists a cubical complex $\tilde{M}$ PL-homeomorphic to M such that $\tilde{M}$ and its dual $\tilde{M}^*$ contain no triangular 2-faces.*

In particular, we have as a corollary

Main Theorem 2. *For every $d \geq 3$, there exists a cubical complex S with $|S|$ homeomorphic to S^d such that neither S nor S^* contain triangular 2-faces.*

*Supported by Deutsche Forschungsgemeinschaft within the Research Training Group "Methods for Discrete Structures" (GRK1408).

2 Elementary Notions in Knot Theory and Dehn Surgery

We introduce the required notions from Knot Theory, taken from [2], slightly adapted to fit our purpose.

Definition 2.1. For us, a *knot* is a solid torus piecewise linearly embedded in a 3-sphere. Similarly, a *link* are several disjoint knots embedded in a 3-sphere. The single tori of a link will be called components of the link. A *knot-* or *link-complement* is the manifold that results from removing a knot (resp. a link) from the sphere.

Definition 2.2 (Definition and Lemma)**.** Let T be a solid torus. The *meridian* μ_T of T is (up to isotopy unique) a closed curve in ∂T, that bounds a disk in T but that is not null-homologous in ∂T. The following 2 properties are classical.
– 2 solid tori T, T‘ identified along their boundaries such that the meridians are μ_T, μ_T‘ identified are homeomorphic to $S^2 \times S^1$.
– The complement of an unknotted solid torus in the 3-sphere is a solid torus T'. We call this the *standard genus 0 splitting* of the 3-sphere, also known as *Heegard splitting.*
Consider again a Heegard splitting of the 3-sphere. The image of μ_T in T' is a curve transversal to the meridian μ'_T of T'. We call this curve the *longitude* of μ_T. More generally, the *longitude* of a knot is the (up to isotopy unique) incontractible loop in the boundary of the knot that bounds the Seifert surface in the complement of the knot.
Meridian and longitude of links are determined for each component of the link separately. The next lemma shows that we only have to care about the images of meridian or longitude when determining the result of a Dehn surgery.

Lemma 2.3. *Let M be a manifold whose boundary is a 2-torus t. Let γ be a noncontractible loop in t. Let T, T' be two solid tori, and let h, h' be two homeomeorphisms from ∂T (resp. $\partial T'$) to t such that $h(\mu_T) = h'(\mu'_T) = \gamma$. Then $M \sqcup_h T$, the gluing of T to M along the homeomorphism h, is homeomorphic to $M \sqcup_{h'} T'$.*

With this at hand, we can prove the simple lemma we will use in the construction:

Lemma 2.4. *Let T_1, T_2 be two unlinked unknotted solid tori embedded in S^3, and let γ be any curve connecting T_1 and T_2. Then, the result of removing T_1 and T_2 and identifying the remaining manifold with boundary along the boundary such that a meridian of T_1 gets identified with a longitude of T_2 is topologically a $S^2 \times S^1$. Furthermore, if γ is closed to a loop $\hat{\gamma}$ in the identification, then $\hat{\gamma}$ spans the fundamental group of the handlebody $S^2 \times S^1$.*

Proof. Let B be a topological ball in S^3 that contains T_1, but does not intersect T_2, and let B_2 be its complement in S^3. Remove T_1 from B_1 resp. T_2 from B_2. This leaves us with 2 solid tori minus a ball each (B_2 resp. B_1). By Lemma 2.2, prescribed identification yields a 3-sphere minus 2 balls. Identifying the boundaries $S_1 := \partial B_1$ and $S_2 := \partial B_2$ gives a $S^2 \times S^1$.

To see that $\hat{\gamma}$ spans the fundamental group of $S^2 \times S^1$, observe that any two curves γ, γ' that are connecting unlinked unknots are ambient isotopic. Thus, we only have to check the claim for a specific choice of γ, where it is trivial. □

2.1 Cell Decompositions of manifolds and Cubical Complexes

For the rest of the paper, when talking about cell complexes, we will talk about strongly regular CW-complexes. This means that the attaching maps are injections for each cell, and that two cells intersect in a (possibly empty) cell of the complex. More specially, a cubical complex is a cell complex in which all cells are combinatorial cubes.

Definition 2.5. A cell complex C that is topologically a d-manifold without boundary will be called *thick* if and only if every $d-2$-face of C is contained in at least 4 facets of C, and every 2-face of C has at least 4 vertices.

A cell complex C that is topologically a manifold with boundary will be called *thick* if and only if every $d-2$-face of C in the interior is contained in at least 4 facets of C, every $d-2$-face of C in the boundary is contained in at least 2 facets of C and every 2-face of C has at least 4 vertices.

Definition 2.6 (Cubical subdivision)**.** Let P be a d-polytope, or more generally, a d-ball whose boundary is subdivided as a strongly regular CW-complex. The d-cells of csd P, the *cubical subdivision* of P, are the union of cells in the barycentric subdivision of P that contain a common vertex of P. Thus, there is a cell for every vertex of P. If P is simple, then csd P is a complex of cubes.

If C is a polyhedral complex, then csd C, the cubical subdivision of C, is defined to be the complex that is the union of the cubical subdivision of each face of P. If C is thick, then so is csd C.

Definition 2.7 (Manipulating cubical complexes)**.** We describe some elementary manipulations of cubical complexes.

- *Adding hyperplanes* Let C be a cubical complex, and let e, e' be two edges. We say that e, e' are adjacent if there exists a quadrangle Q in C containing both e and e', and e and e' share no common vertex. The *cubical hyperplane* H_e in C orthogonal to an edge e is the closure of the adjacency relation over all edges of C.A cubical hyperplane H_e induces a natural subdivision of C: let P be any cube in C (of any dimension). If an edge f of P is in H_e, introduce a hyperplane to P separating the two disjoint facets of P joined by e.
- *Cushions of subcomplexes* Let C be a cubical complex (or more generally a cell complex), and let C_1, C_2 be two subcomplexes of C such that $C' = C_1 \cap C_2$ contains no facet of C, but $C_1 \cup C_2$ covers all of C. Then a cushion is just the prism over C'. Note that this prism has naturally two boundary components combinatorially isomorphic to C'. Glue C_1 to one of these boundaries along the natural isometry, and glue C_2 to the other one. We call the resulting complex the *cushioning of C at C'*.

Definition 2.8. Let B be a quadrilateral, or more generally, a $i \times j$ rectangle of quadrilaterals. Consider a prism over B, and stack an arbitrary number of these prisms along their bases, obtaining a tower of prisms over B. We call such a complex a *dig*. The 2 copies of B in a dig that do not lie in the relative interior of the dig will be called the *bases* of the dig. If we identify the bases of the dig to a solid torus, we call the result a *tunnel*. The number of prisms in the dig resp. the torus will be called the *length* of the dig resp. the tunnel, and the direction orthogonal to all copies of B will be called the direction of the dig resp. the tunnel. The union of the sides of the prisms will be called the *side* of the dig resp. the tunnel. With abuse of notation, a cubical subdivision of a dig resp. a tunnel, or more generally, the result of introducing a cubical hyperplane to a dig or a tunnel will also be called a dig resp. a tunnel. If the dig resp. tunnel is 3-dimensional, the circumference of a base (the combinatorial length) is called the *circumference* of the dig resp. the tunnel.

Let C is a pure complex of cubes, and let A, B be subcomplexes of C. We say that a dig Ψ *connects A and B* in C if Ψ intersects A resp. B in its bases, and its bases only. If Θ, Φ are disjoint digs or tunnels in C, and let Ψ be a dig such that Ψ intersects Θ resp. Φ in a base of Ψ resp. a facet in the side of Θ resp. Φ. We call Θ and Φ *parallel* along Ψ if the direction of Φ transported parallely along Ψ to Θ coincides with the direction of Θ. If that is not the case (i.e. if the directions are orthogonal after parallel transport along Ψ), we call Θ and Φ are *orthogonal* along Ψ.

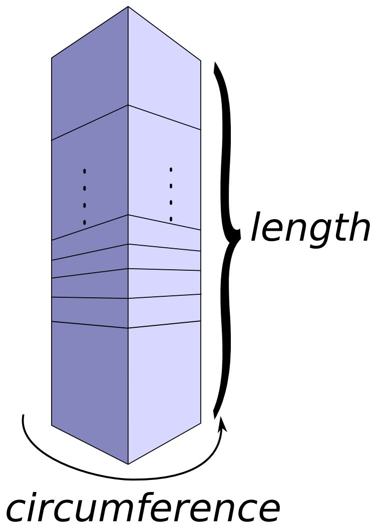

Figure 1: A dig with a square base

Described refinements serve a sole purpose, to give us enough space in the complex we would like to construct to perform surgery. It is obvious that cubic subdivision and introducing a cubic hyperplane preserve thickness, and it is equally easy to see that, if C_1 and C_2 are thick subcomplexes of a complex C fulfilling the requirements of cushioning, cushioning at their intersection yields also a thick complex. Furthermore, if C is a thick complex, and T is a tunnel or dig in C that lies strictly in the interior of T, then removing T from C yields a thick complex as well.

3 A 3-sphere without triangles and cotriangles

In this section we construct a 3-dimensional cubical sphere S that contains neither triangles or cotriangles. Building on this, we construct cell decompositions of (almost) arbitrary manifolds that contain neither triangles nor cotriangles.

Theorem 3.1. *There exists a polyhedral 3-sphere S that is thick, i.e. such that neither S nor its dual S^* contain triangles as 2-faces.*

Proof. Let $\{e_i\}$ be the canonical basis of $\mathbb{R}^4$. We will describe a certain cubical subdivision of the boundary of a truncated 4-cube. Let F_e^P denote the facet of a polytope P with outer normal e. If X is a refining subdivision of the boundary of a polytope P, let F_e^X denote the subcomplex of X of faces in the facet F_e^P. Let C_4 be the boundary of the regular 4-cube with vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$. Subdivide the boundary of C_4 cubically thrice, yielding a cubical complex X that is a subdivision of the boundary of C_4.

Truncate C_4 at the 2-complexes $F^X_{\pm e_1} \cap F^X_{\pm e_2}$ by introducing the additional linear constraints $< \pm e_1 \pm e_2, x > \leq 1 - \epsilon$, and truncate X with it by truncating the faces of X intersecting the faces $F^X_{\pm e_1} \cap F^X_{\pm e_2}$ of C_4. After one more cubical subdivision, the new subdivision of the truncated cube is a cubical complex again. Denote the truncated cube by P, and denote the subdivision of the boundary by D. We go through the necessary steps of the construction, and analyze the topology and combinatorics of the resulting complex after each step.

Step 1. Let T denote the subcomplex of D of facets with outer normal $\pm e_1 \pm e_2$. Find disjoint digs in a subdivision of D into 3-cubes with the following properties

- 2 digs $\Gamma_i,\ i\in\{1,2\}$ connecting the subcomplexes $F^D_{\pm e_i}$, and such that the circumference of Γ_1 coincides with the length of Γ_2 and *vice versa*, and Γ_i is symmetric with respect to reflection at the hyperplane $e_i^\perp$
- 1 dig Ψ connecting Γ_1 and Γ_2, and such that Γ_1 and Γ_2 are orthogonal along Ψ.
- 1 dig Θ that is parallel to T via a dig Ξ, such that Θ has the same length and circumference as T.
- Ψ and Ξ are orthogonal (via some dig) and the circumference of Ψ coincides with the length of Ξ and *vice versa*.

The conditions of the lengths and circumferences can easily be satisfied in the chosen subdivision of P, their relative positions, however, are not that easily found. Thus, we indicate their position in Figure 2. As it turns out, we can choose all the digs to lie in one facet of P, for example $F(e_3)^D$. Now, remove the subcomplexes F_{e_1} and F_{-e_1} and identify $\partial F_{e_1}{}^D$ in $D\setminus F^D_{\pm e_1}$ with

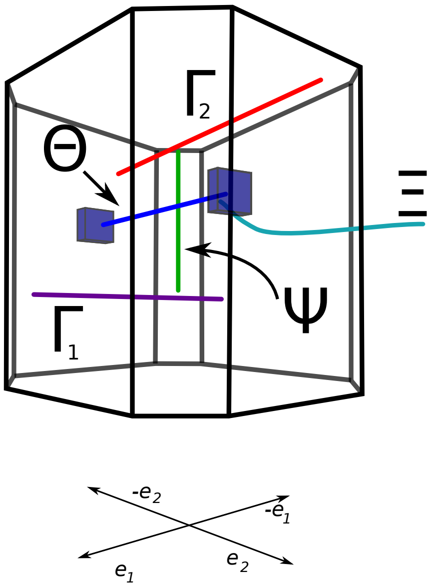

Figure 2: The relative position of the digs in the facet with outer normal e_3 (the latter is chosen arbitrarily)

$\partial F_{-e_1}{}^D$ by reflection at the hyperplane $e_1^\perp$. Similarly, remove the facets $F^D_{e_2}$ and $F^D_{-e_2}$, and identify $\partial F_{e_2}{}^D$ in $D\setminus(F^D_{\pm e_1}\cup F^D_{\pm e_2})$ with $\partial F^D_{-e_2}$ by reflection at the hyperplane $e_2^\perp$. Remove T from the resulting complex. We arrived at a complex E whose underlying manifold is a connected sum of two $S^2\times S^1$, minus a solid torus T.

- *Topology* The underlying space $|E|$ of the complex E is topologically $(S^2\times S^1\# S^2\times S^1)\setminus|T|$: When passing from C_4 to D, we ensured that $D\setminus(F^D_{\pm e_1}\cup F^D_{\pm e_2})$ is a sphere minus 4 balls whose boundaries *do not* intersect. By identifying these boundary components in pairs, we result in a $S^2\times S^1\# S^2\times S^1$, and removing the solid torus T yields the desired manifold.
- *Combinatorics* D is not thick, but after removal of $F^D_{\pm e_1}$ and $F^D_{\pm e_2}$ and their mutual identification, the edges which lie in less than 4 3-cells are edges in the side of T. Thus, removal of T yields a thick complex.

Step 2. When passing from D to E, the digs $\Gamma_i,\ i\in\{1,2\}$ are identified to tunnels, spanning the fundamental groups of each handle of $|E|$ respectively. Both Γ_i are unlinked, i.e. neglecting the removal of T, there exists a decomposition of $|E|$ into two handlebodies (i.e. $S^2\times S^1$) that contain one of the Γ_i and not the other. Remove the Γ_i and identify the boundaries in $E\setminus(\Gamma_1\cup\Gamma_2)$ such that the meridian of Γ_1 maps to a curve transversal to the meridian of Γ_2 and *vice versa*. Denote the resulting complex by F. Since Ψ connects the Γ_i, and the Γ_i are chosen to be orthogonal, the identification can be chosen in such a way that Ψ closes to a tunnel. The resulting complex is a thick complex, denote it by F.

- *Topology* Disregard the removal of T. Then the resulting complex is topologically a $S^2 \times S^1$. To make this clear, note that $|E|$ is topologically a $S^2 \times S^1 \# S^2 \times S^1$. Removing the Γ_i yields a manifold that is topologically a sphere minus 2 unlinked unknots, connected by Ψ. By Lemma 2.4, the result of the surgery is a $S^2 \times S^1$ handlebody whose fundamental group is spanned by the tunnel Ψ.
- *Combinatorics* E is thick, and removing tunnels from a thick complex leaves a thick complex. Thus, removing $\Gamma_i,\ i \in \{1, 2\}$ leaves the complex thick. Identifying thick complexes along their boundaries yields a thick complex, thus, F is thick.

Step 3. The identification of $\partial\Gamma_1$ and $\partial\Gamma_2$ in $E \setminus \Gamma_{1,2}$ was chosen in such a way that Ψ closes up to a tunnel. Remove the resulting tunnel Ψ from the complex. The result is denoted by G.

- *Topology* $|F \cup T|$ is a $S^2 \times S^1$-handlebody whose fundamental group is spanned by Ψ. The removal of Ψ yields a solid torus, and subsequent removal of T yields a 3-sphere minus two linked tori. Ψ is, on its own, an unknot. One can see (although it is not necessary here) that both tori are unknotted in the sphere, but linked.
- *Combinatorics* Removing Ψ from F leaves the complex thick, with the exception of the boundary of T in G.

With one more surgery at Ψ, we could get back to a S^3, but we have to fill up the hole left by T as well. It is, however, not possible to glue the boundaries of Ψ and T, and apply Lemma 2.4 directly, since Ψ and T might be linked. Thus, we fill in the hole at ∂T first, and then reduce to a 3-sphere.

Step 4. We attach handles to G, and use it to fill the hole left by removing T: Remove the facets of G adjacent to the bases of Θ from F and identify their boundaries in F to close Θ to a tunnel, attaching a $S^2 \times S^1$-handle to G. Remove Θ and T, and identify the resulting complex along $\partial\Theta$ and ∂T such that meridian of T maps to the meridian of Θ. Denote the resulting complex by H. We can choose the identification in such a way that Ξ closes up to a tunnel, because T and Θ are parallel along Ξ. The resulting complex is a $S^2 \times S^1$, minus a tunnel Ψ that is null-homotopic in $S^2 \times S^1 - \Xi$, and Ξ spans the fundamental group of the handle.

- *Topology* Assume for simplicity of the argumentation that the torus boundary $\partial\Psi$ in G is filled. This leaves a 3-sphere minus a (possibly knotted) torus T. Attaching the handle leaves a $S^2 \times S^1$-handlebody minus a solid torus that can be contracted by an ambient isotopy on the handle. Thus, removing Θ from the handlebody leaves a 3-sphere minus 2 unlinked tori. The same argumentation as in *Step 2.* shows that the result of the prescribed gluing is a $S^2 \times S^1$ handlebody, by Lemma 2.4. Now, we can remove the filling of $\partial\Psi$ again.
- *Combinatorics* Again, the complex H is thick, by just the same argumentation as in the previous steps: Removing T and Θ makes the complex thick, and identifying their boundaries does not change this.

Step 5. Remove the tunnel Ξ from H, and identify its boundary with the boundary at $\partial\Psi$ such that the meridian of the first maps to a curve transversal to the meridian of the second and such some dig connecting. The result called is J. Since Ψ and Ξ are orthogonal, we can choose the identification such that the dig connecting Ψ and Ξ closes to a tunnel Ω.

- *Topology* Repeating the argumentation of *Step 2.* with Lemma 2.4, the result is a $S^2 \times S^1$ handlebody.
- *Combinatorics* J is a thick complex, as in the previous argumentations.

Step 6. Remove Ω from J spanning the fundamental group of the handlebody. The result is a thick solid torus I. Consider 2 copies of I, and glue them to a sphere as in the standard genus 0 splitting. The result is also thick, as desired. This finishes the construction. □

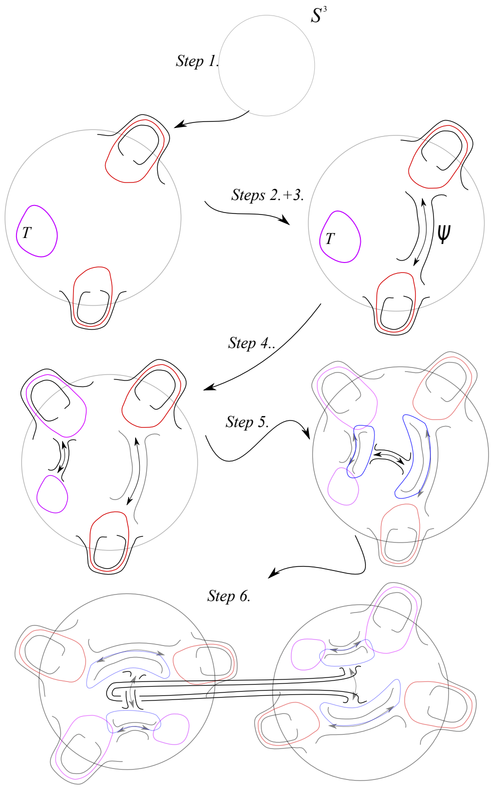

Figure 3: The surgery steps, one by one (with some simplifications). *Step 1.* attaches 2 handles, and removes a torus T from the interior (T is actually interlocked with the 2 handles, but for simplicity, we draw it separate). *Step 2.* performs Heegard surgery on the handles, reducing their number by 1, and *Step 3.* removes the tunnel Ψ. *Step 4.* attaches a handle, and uses them to fill the holes left by T. Using Heegard splits twice *Step 4.* and *Step 5.*, we reduce the manifold to a 3-sphere.

4 Thickness for general manifolds and in arbitrary dimensions

Recall that a cell complex is called *thick* if every 2-face is of degree at least 4, every interior ridge is of degree at least 4, and every boundary ridge is of degree at least 2.

Theorem 4.1. *Every d-sphere, $d \geq 3$ does allow a representation as a cubical complex that is thick.*

Proof. First, note the following canonical handle decompositon of a unit regular n-cube C^n: Choose $\lfloor \frac{n}{2} \rfloor$ pairs of vectors of the canonical basis, i.e. let e_i; e_j be two such base vectors. The facets of C^n whose normal direction is either $\pm e_i$ or $\pm e_j$ form a $B^{n-1} \times S^1$ immersed in the boundary of the n-cube. If n is even, these handles decompose the boundary, if n is an odd number, there is one more copy of $B^n \times S^0$ needed to decompose the the cube.

We assume now we have constructed a thick n-sphere Σ^n, and construct a thick $n+1$-sphere. We will furthermore assume that the n-sphere from the induction assumption contains a n-cube, such that there is a thick n-ball whose boundary is a cube and whose every boundary ridge is of degree 3 or more.
Case 1. n is an even number If n is even, consider $n = 2$ copies of $B^n \times Q$, where B^n is a thick n-ball (for example, Σ_n minus an interior cube) and Q is the boundary of a quadrangle, and glue them to a $n+1$-sphere as in the decomposition of the boundary of the $n+2$-cube into copies of $B^{n-1} \times S^1$.Since the $B^n \times Q$ are thick and a gluing of thick polyhedral complexes is

thick, the resulting $n+1$-sphere is thick.
Case 2. n is an odd number In this case, we have to take a small detour, as a decomposition of S^{n+1} into $B^n \times S^1$ in the manner above can not work for genus reasons. So, we will make an additional assumption on the thick n-sphere Σ^n in the induction assumption, namely, that there exists a n-cube in Σ^n and n tunnels in Σ^n crossing the cube transversally in every direction. By cubical subdivision, we can deduct that there is a n-cube C in some thick n-sphere $\widetilde{\Sigma^n}$ that is passed tangentially by tunnels T_i in each of its facets (amounting to $2n$ tunnels in total). Consider the polyhedral complex $(\widetilde{\Sigma^n} \setminus C) \times D_l$, where D_l is the boundary of a planar l-gon. The T_i lift to $2ln$ tunnels T'_j in $(\widetilde{\Sigma^n} \setminus C) \times D_l$. Take any tunnel T of these tunnels T'_j, cushion the complex $(\widetilde{\Sigma^n} \setminus C) \times D_l$ at the boundary of T, remove the tunnel T and glue in a copy $(\widetilde{\Sigma^n} \setminus C) \times D_{L(T)}$ where $L(T)$ denotes the length of the tunnel T. Repeating this with every single one of the tunnels T'_j we get a polyhedral decomposition of $B^n \times S^l$ which is thick with the additional property that every boundary ridge is of degree at least 3. Consider the case $l = 4$, and glue the handles to a sphere as described. Because the boundary ridges are of degree 3, we can glue in $B^n \times S^0$ without losing thickness of the decomposition, as desired.

It remains to make a remark on the assumption we made in this construction.
Appendix: The existence of a cube passed transversely by tunnels Let Σ^n be a thick n-sphere and let C be a 3-cube in Σ^n. Cushion Σ^n at ∂C, remove C, and double the resulting complex at the boundary. Every n-cube in the cushion at ∂C now has $n-1$ tunnels passing transversely. To get 1 more tunnel, choose 1 of the tunnels and cushion once more at it's boundary. □

As a trivial corollary we achieve:

Corollary 4.2. *Every n-manifold, $n \geq 3$ that allows a triangulation also allows a decomposition into polyhedra that is thick.*

Remark 4.3. Let $\mathcal{P}$ be an PL-equivalence class of cell decompositions of a manifold M of dimension at least 4. Then our construction shows that, should this class not be empty, it contains a cell decomposition without triangles and cotriangles.

Remark 4.4. Our construction implies that there exists a 3-polytope P whose boundary can be assigned with a piecewise flat metric (that is flat on every boundary cell of P) such that the angle defect around each edge is larger than 2π, since some subdivision of the cubical 3-sphere constructed in Theorem 3.1 is polytopal.
One could hope that by iterated cubical subdivision of a PL sphere without triangles or cotriangles, or by using cushions or cubical hyperplanes, one could reach a sphere that has a realization as a polytope itself. This is, unfortunately, not true in general.

References

[1] G. Blind and R. Blind, Convex polytopes without triangular faces, Israel Journal of Mathematics, Vol 71 (1990), 129–134.
[2] W. B. R. Lickorish, Introduction to Knot Theory, Grad. Texts in Math., Springer 1997.
[3] L. Z. Gao and S. T. Yau, The existence of negatively Ricci curved metrics on three manifolds, Inventiones Mathematicae, Vol. 85 (1986), 637–652.
[4] J. Lohkamp, Metrics of Negative Ricci Curvature, The Annals of Mathematics, Second Series, Vol. 140 (1994), 655–683.
[5] G. Kalai, On low-dimensional faces that high-dimensional polytopes must have (1990), Combinatorica, Vol. 10, 271–280.

[6] G. Kalai, P. Kleinschmidt and G. Meisinger, Flag Numbers and FLAGTOOL, in Polytopes ; Combinatorics and Computation, DMV Seminar 29 (1999) 75–103.
[7] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, Annals of Math. 169 (2004), 903–991.
[8] W. P. Thurston, Three-dimensional geometry and topology, Princeton University Press, 1997.