Topological obstructions for vertex numbers of Minkowski sums

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Abstract

We show that for polytopes $P_1, P_2, \ldots, P_r \subset \mathbb{R}^d$, each having $n_i \geq d+1$ vertices, the Minkowski sum $P_1 + P_2 + \cdots + P_r$ cannot achieve the maximum of $\prod_i n_i$ vertices if $r \geq d$. This complements a recent result of Fukuda & Weibel (2006), who show that this is possible for up to d-1 summands. The result is obtained by combining methods from discrete geometry (Gale transforms) and topological combinatorics (van Kampen-type obstructions)

1 Introduction

For two polytopes $P, Q \subset \mathbb{R}^d$ their *Minkowski sum* is the (innocent-looking) convex polytope

$$P+Q=\{p+q:p\in P,\,q\in Q\}\subset\mathbb{R}^d.$$

Minkowski sums have starred in applied areas, such as robot motion planning [6] and computer aided design [3], as well as in fields of pure mathematics, among them commutative algebra and tropical geometry [13]. In applications it is essential to understand the facial structure of P+Q. But, even with quite detailed knowledge of P and Q, it is in general difficult to determine the combinatorics of P+Q. Even for special cases, the knowledge of complete face lattices is meager. The best understood Minkowski sums are zonotopes [14] and sums of perfectly centered polytopes with their polar duals [4].

So it is natural (and vital) to investigate the combinatorial structure of Minkowski sums. From the standpoint of combinatorial geometry, a less ambitious goal one can settle for is the question of f-vector shapes. This includes different kinds of upper and lower bounds for the f-vector entries with respect to the corresponding entries of the summands. Starting with the first entry of an f-vector, the number of vertices, Fukuda & Weibel [4] studied the maximal number of vertices of a Minkowski sum. Their starting point was the following upper bound on the number of vertices.

Proposition 1.1 (Trivial Upper Bound; cf. [4]). Let $P_1, \ldots, P_r \subset \mathbb{R}^d$ be polytopes. Then

$$f_0(P_1+\cdots+P_r) \leq \prod_{i=1}^r f_0(P_i).$$

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Fukuda and Weibel gave a construction that showed that the trivial upper bound can be attained independent of the dimension but with a restricted number of summands.

Theorem 1.2 (Fukuda & Weibel [4]). For every r < d there are d-polytopes $P_1, \ldots, P_r \subset \mathbb{R}^d$, each with arbitrarily large number of vertices, such that the Minkowski sum $P_1 + \cdots + P_r$ attains the trivial upper bound on the number of vertices.

This result was our point of departure. In this paper, we set out to prove the following result that asserts that the restriction to the number of summands is best possible.

Theorem 1.3. Let $r \geq d$ and let $P_1, \ldots, P_r \subset \mathbb{R}^d$ be polytopes with $f_0(P_i) \geq d+1$ vertices for all $i = 1, \ldots, r$. Then

$$f_0(P_1 + \dots + P_r) \le \left(1 - \frac{1}{(d+1)^r}\right) \prod_{i=1}^r f_0(P_i).$$

However, based on a preliminary version of this paper, C. Weibel (personal communication) noted that Theorem 1.3 is not optimal with respect to the number of vertices of the summands.

The paper is organized in the following manner: In Section 2 we gather some observations that will reduce the statement to one special case (per dimension) whose validity has to be checked. In particular, we give a reformulation of the problem that casts it into a stronger question concerning projections of polytopes. The very structure of the problem at hand allows for the utilization of the tools devised in Rörig, Sanyal & Ziegler [12]. In Section 3 we give a review to the terminology and techniques.

The punchline will be that a realization of a polytope with certain properties under projection gives rise to, first, a polytope associated to the projection and, secondly, to a simplicial complex that is embedded in the boundary of this polytope. To prove the non-existence of a certain realization it will suffice to show that this complex is not embeddable into a sphere of the prescribed dimension. To show the reader that we did not deal one difficult problem for another one, we give, in Section 4, a short account of Matoušek's book [10], in which he presents means for dealing with embeddability questions in a combinatorial fashion. In Section 5, we finally put together the pieces gathered to prove a stronger version of Theorem 1.3 and we close with some remarks in Section 6.

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2 The problem, some reductions & a reformulation

Forming Minkowski sums is not a purely combinatorial construction, i.e. in contrast to basic polytope constructions such as products, direct sums, joins, etc. the resulting face lattice is not determined by the face lattices of the polytopes involved. For a sum P+Q of two polytopes P and Q it is easy to see that if $F \subseteq P+Q$ is a proper face, then F is of the form F=G+H with $G \subseteq P$ and $H \subseteq Q$ being faces. This, in particular, sheds new light on the "Trivial Upper Bound" in the last section: It states that the set of vertices of a Minkowski sum is a subset of the pairwise sums of vertices of the polytopes involved.

As a guiding example let us consider the first non-trivial case: Are there two triangles P and Q in the plane whose sum is a 9-gon? An ad-hoc argument for this case, that uses notation

and terminology presented in [14], is the following: Clearly, the polytope P+Q is a 9-gon if its normal fan $\mathcal{N}(P+Q)$ has nine extremal rays. The normal fan $\mathcal{N}(P+Q)$ equals $\mathcal{N}(P) \wedge \mathcal{N}(Q)$, the common refinement of the fans $\mathcal{N}(P)$ and $\mathcal{N}(Q)$. Thus, the cones in $\mathcal{N}(P+Q)$ are pairwise intersections of cones of $\mathcal{N}(P)$ and $\mathcal{N}(Q)$. It follows that the extremal rays, i.e. the 1-dimensional cones, of $\mathcal{N}(P+Q)$ are just the extremal rays of P and of Q. If P and Q are triangles, then each one has only three extremal rays. Therefore, $\mathcal{N}(P+Q)$ has at most six extremal rays and falls short of being a 9-gon. The same reasoning yields the following result.

Proposition 2.1. Let P and Q be two polygons in the plane. Then

$$f_0(P+Q) \le f_0(P) + f_0(Q).$$

However, this elementary geometrical reasoning fails in higher dimensions and we will employ topological machinery for the general case. But for now let us give some observations that will simplify the general case.

The first observation concerns the dimensions of the polytopes involved in the sum.

Observation 1 (Dimension of summands). Let $P_1, P_2, \ldots, P_r \subset \mathbb{R}^d$ be polytopes each having at least d+1 vertices. Then there are full-dimensional polytopes P'_1, P'_2, \ldots, P'_r with $f_0(P'_i) = f_0(P_i)$ and $f_0(P'_1 + P'_2 + \cdots + P'_r) \geq f_0(P_1 + P_2 + \cdots + P_r)$

Clearly, if one of the summands, say P_1 , is not full-dimensional, then the number of vertices prevents P_1 from being a lower dimensional simplex. Choosing a vertex v of P_1 that is not a cone point and pulling v in a direction perpendicular to its affine hull yields a polytope P'_1 with $f_0(P_1) = f_0(P'_1)$ with $\dim P'_1 = \dim P + 1$. Exchanging P_1 for P'_1 possibly increases the number of vertices of the Minkowski sum.

Observation 2 (Number of summands). Let $P_1, P_2, \ldots, P_r \subset \mathbb{R}^d$ be d-polytopes such that $P_1 + P_2 + \cdots + P_r$ attains the trivial upper bound, then so does every subsum $P_{i_1} + P_{i_2} + \cdots + P_{i_k}$ with $\{i_1, \ldots, i_k\} \subseteq [r]$.

Thus we can restrict to the situation of sums with d summands. The next observation turns out to be even more valuable. It states that we can even assume that every summand is a simplex.

Observation 3 (Combinatorial type of summands). Let $P_1, P_2, \ldots, P_r \subset \mathbb{R}^d$ be d-polytopes such that $P_1 + P_2 + \cdots + P_r$ attains the trivial upper bound. For every $i \in [r]$ let $P_i' \subset P_i$ be a vertex induced, full-dimensional subpolytope, then $P_1' + P_2' + \cdots + P_r'$ attains the trivial upper bound.

The last observation casts the problem into the realm of polytope projections.

Observation 4. The Minkowski sum P+Q is the projection of the product $P\times Q$ under the map $\pi: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ with $\pi(x,y) = x+y$.

We will derive Theorem 1.3 from the following stronger statement.

Theorem 2.2. Let P be a polytope combinatorially equivalent to a d-fold product of d-simplices and let $\pi: P \to \mathbb{R}^d$ be a linear projection. Then

$$f_0(\pi P) \le f_0(P) - 1 = (d+1)^d - 1.$$

Before we give a proof of Theorem 1.3 let us remark on a few things concerning the previous theorem.

Special emphasis should be put on the phrase "combinatorially equivalent to a product." The standard product $P \times Q$ of two polytopes $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ is obtained by taking the Cartesian product of P and Q, that is taking the convex hull of vert $P \times \text{vert } Q \subset \mathbb{R}^{d+e}$. One feature of this construction is that if $P' \subset P$ and $Q' \subset Q$ are vertex induced subpolytopes, then $P \times Q$ contains $P' \times Q'$ again as a vertex induced subpolytope. This no longer holds for combinatorial products: The Goldfarb cube G_4 is a 4-polytope combinatorially equivalent to a 4-cube with the property that a projection to 2-space retains all 16 vertices (cf. [5, 1]). The 4-cube itself is combinatorially equivalent to a product of two quadrilaterals which, in the standard product, contains a vertex induced product of two triangles. The subpolytope on the corresponding vertices of G_4 is not a combinatorial product of two triangles; indeed, this would contradict Theorem 2.2 for d = 2.

The bound of Theorem 2.2 seems to be tight: In Section 6 we give a realization of a product of two triangles such that a projection to 2-space has 8 vertices. Therefore, the bound given in Theorem 1.3 is *not* tight for d = 2: The sum of two triangles in the plane has at most 6 vertices.

Theorem 2.2 touches upon properties of the realization space of products of simplices. While, in general, realization spaces are rather delicate objects, the statement at hand is on par with the fact that positively spanning vector configurations with prescribed sign patterns of linear dependencies do not exist. Methods for treating such problems were developed in [12]; the next two sections give a made-to-measure introduction.

Proof of Theorem 1.3. We can assume that all polytopes P_i have their vertices V_i in general position. For every choice $V_i' \subseteq V_i$ of d+1 vertices from each P_i the polytopes $P_i' = \operatorname{conv} V_i'$ are d-simplices. The Minkowski sum $P_1' + P_2' + \cdots + P_r'$ is a projection of a product of $r \geq d$ simplices and, by Theorem 2.2, does not attain the trivial upper bound. There are exactly $\prod_{i=1}^r \binom{f_0(P_i)}{d+1}$ choices for the P_i' . On the other hand, every sum of vertices $v_1 + v_2 + \cdots + v_r$ occurs in only $\prod_{i=1}^r \binom{f_0(P_i)-1}{d}$ different subsums. Thus, by the pigeonhole principle, there are at least

$$\prod_{i=1}^{r} \frac{\binom{f_0(P_i)}{d+1}}{\binom{f_0(P_i)-1}{d}} = \prod_{i=1}^{r} \frac{f_0(P_i)}{d+1}$$

sums of vertices that fail to be a vertex in at least one subsum.

3 Geometric and Combinatorial Properties of Projections

In polytope projections faces can collapse or get mapped to the interior. Therefore, it is difficult to predict the (combinatorial) outcome of a projection. There is, however, a class of faces that behave nicely under projection and whose properties we will exploit in the following.

Definition 3.1 (Strictly preserved faces, cf. [15]). Let P be a polytope, $F \subseteq P$ a proper face and $\pi: P \to \pi(P)$ a linear projection of polytopes. The face F is strictly preserved under π if

- i) $H = \pi(F)$ is a face of $\pi(P)$,
- ii) F and H are combinatorially isomorphic, and
- iii) $\pi^{-1}(H)$ is equal to F.

The first two conditions should trigger an agreeing nod since they model the intuition behind "preserved faces." The third condition is a little less clear. In order to talk about *distinct* faces of a projection we have to rule out that two preserved faces come to lie on top of each other and this situation is dealt with in condition iii). Figure 1 shows instances of non-preserved, preserved, and strictly preserved faces. We will generally drop the "strictly" whenever we talk about strictly preserved faces.

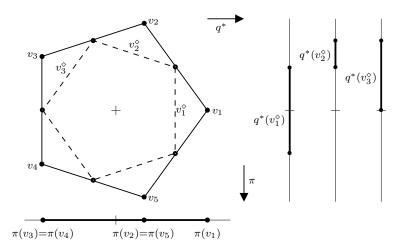


Figure 1: Projection of pentagon to the real line: v_1 is strictly preserved, v_2 is not preserved, and v_3 is preserved but not strictly. In accordance with the Projection Lemma: $q^*(v_1^{\diamond})$ contains 0 in interior, $q^*(v_2^{\diamond})$ does not, and $q^*(v_3^{\diamond})$ has 0 in its boundary.

What makes strictly preserved faces so nice is the fact that the above conditions can be checked prior to the projection by purely (linear) algebraic means. The key to that is the Projection Lemma (cf. [15, Proposition 3.2]). We will give a variant of it which requires a few more notions.

Let $P \subset \mathbb{R}^n$ be a full-dimensional polytope with 0 in the interior. The dual polytope is

$$P^{\Delta} \ = \ \{\ell \in (\mathbb{R}^n)^* : \ell(x) \le 1 \ \text{ for all } x \in P\} = \operatorname{conv} \{\ell_1, \dots, \ell_m\} \ \subset \ (\mathbb{R}^n)^*$$

and for every face $F \subseteq P$ we denote by $F^{\diamond} = \{\ell \in P^{\Delta} : \ell|_F = 1\}$ the corresponding face of P^{Δ} . Furthermore, we define $I : \{\text{faces of } P\} \to 2^{[m]}$ to be the map satisfying

$$\operatorname{conv}\left\{\ell_i:i\in I(F)\right\}=F^\diamond$$

for every face $F \subseteq P$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^d$ be a linear projection and let q be a map fitting into the short exact sequence

$$0 \longrightarrow \mathbb{R}^{n-d} \stackrel{q}{\longrightarrow} \mathbb{R}^n \stackrel{\pi}{\longrightarrow} \mathbb{R}^d \longrightarrow 0.$$

Dualizing gives rise to a (dual) exact sequence

$$0 \longleftarrow (\mathbb{R}^{n-d})^* \xleftarrow{q^*} (\mathbb{R}^n)^* \xleftarrow{\pi^*} (\mathbb{R}^d)^* \longleftarrow 0.$$

The characterization of strictly preserved faces will be in terms of the dual map q^* and the dual to the face under consideration.

Lemma 3.2 (Projection Lemma). Let P be a polytope and $F \subset P$ a face. Then F is strictly preserved iff $0 \in \text{int } q^*(F^{\diamond}) = \text{int conv } \{q^*(\ell_i) : i \in I(F)\}.$

We first sort out the situation for (non-strictly) preserved faces.

Proposition 3.3. Let $F \subset P$ be a face. Then $\pi(F)$ is a face of $Q = \pi(P)$ iff $0 \in q^*(F^{\diamond})$.

Proof. p(F) is a face of Q iff there is an $\ell \in (\mathbb{R}^d)^*$ such that

$$\ell \circ \pi(x) < 1$$
 for all $x \in P \setminus F$ and $\ell \circ \pi(x) = 1$ for all $x \in F$

This is the case iff $\pi^*(\ell) \in F^{\diamond}$ which in turn holds iff $0 = q^* \circ \pi^*(\ell) \in q^*(F^{\diamond})$.

Proposition 3.4. Let $F \subset P$ be a face. Then $\pi(F)$ is combinatorially equivalent to F iff $q^*(F^{\diamond})$ is full-dimensional.

Proof. Let aff $F = x_0 + L$ with L being a linear space. Then aff $F^{\diamond} = \ell_0 + L^0$ with $L^0 = \ker((\mathbb{R}^n)^* \to L^*)$. The result follows if we show that π is injective on L iff q^* restricted to L^0 is surjective. Now $\pi|_L$ is injective iff $\operatorname{im} q \cap L = 0$. This holds iff q^* is surjective when restricted to $L^0 = (\mathbb{R}^n/L)^*$.

Proof of Projection Lemma. Let $F \subset P$ be a preserved face, i.e. $\pi(F)$ is a face of $Q = \pi(P)$ combinatorially equivalent to F, and let $G = \pi^{-1}(\pi(F))$. Clearly, G is a face of P and $F \subseteq G$.

The inclusion is strict iff G^{\diamond} is a proper face of F^{\diamond} . Now, $\pi(G)$ is a face iff $0 \in q^*(G^{\diamond})$ by Proposition 3.3. This, in turn, is the case iff $0 \notin \operatorname{int} q^*(F^{\diamond})$.

The geometric side

We made use of the fact that $q^*(F^{\diamond}) = \text{conv}\{q^*(w) : w \in \text{vert } F^{\diamond}\}$ and for later reference we denote by $G = \{g_i = q^*(\ell_i) : i = 1, ..., m\}$ the projection of the vertices of P^{Δ} . Note that we will not treat G as the set of vertices of a polytope (especially since not all would be vertices) but as a configuration of vectors. In case that all vertices survive the projection, this vector configuration has some strong properties: it is a *Gale transform*. Gale transforms are a well-known notion from discrete geometry; we refer the reader to Matoušek [9] and Ziegler [14] for full treatments (from different perspectives) and McMullen [11] for an extensive survey.

A set of vectors $W = \{w_1, \ldots, w_k\} \subset \mathbb{R}^d$ is positively spanning if every point in \mathbb{R}^d is a non-negative combination of the vectors w_i , that is, if $\operatorname{cone} W = \mathbb{R}^d$. Equivalently, U is positively spanning if $\operatorname{conv} W$ is a full dimensional polytope with 0 in its interior. We also need the weaker notion of positively dependent which holds if $0 \in \operatorname{relint} \operatorname{conv} W$.

Definition 3.5 (Gale transform). A finite vector configuration $G = \{g_1, \ldots, g_m\}$ is a *Gale transform* if for every $i = 1, \ldots, m$ the subconfiguration $G \setminus g_i$ is positively spanning.

The main reason why Gale transforms are useful is that they are yet another way to represent polytopes.

Theorem 3.6 (Gale duality). Let $G = \{g_1, \ldots, g_m\}$ be a Gale transform in (m - d - 1)-dimensional space, then there is a d-polytope Q with vertices $V = \{v_1, \ldots, v_m\}$ such that for every $I \subset [m]$

conv $\{v_i : i \in [m] \setminus I\}$ is a face of $Q \iff \{g_j : j \in I\}$ are positively dependent.

Furthermore, Q is unique up to affine isomorphisms.

The condition given by the Projection Lemma can be rephrased in terms of positive spans. The key observation is that the set G is actually a Gale transform if all vertices survive the projection.

Proposition 3.7. Let $P \subset \mathbb{R}^n$ be a n-polytope with m facets and $\pi : \mathbb{R}^n \to \mathbb{R}^d$ a linear projection. The set G is a Gale transform if all vertices of P are strictly preserved under π .

Proof. Since P is full dimensional for every $i \in [m]$ there is a vertex $v \in \text{vert } P$ such that $i \notin I(v)$. Since v is preserved under projection the set $\{g_j : j \in I(v)\} \subset G \setminus g_i$ is positively spanning. \square

Thus, by Theorem 3.6, there is a combinatorially unique polytope $\mathcal{A}(P,\pi)$ associated to (P,π) which we simply call the associated polytope. So far, the main property of $\mathcal{A}(P,\pi)$ is that it sort of witnesses the survival of the vertices.

For the case that interests us, namely P being a product of simplices, we can even assume that $\mathcal{A}(P,\pi)$ is a simplicial polytope, the reason being the following: If P^{Δ} is simplicial we can wiggle the vertices without changing the combinatorial type. For strictly preserved faces of P, the condition as dictated by the Projection Lemma is open, i.e. stable under small perturbations. Thus perturbing the bounding hyperplanes of P does neither alter the type nor the fact that all vertices survive the projection. The effect on the Gale transform G is that the vectors are in general position with respect to hyperplanes containing the origin. On the other hand, this is yet another characterization of the fact that $\mathcal{A}(P,\pi)$ is a polytope with vertices in general positions and hence simplicial.

Summing up so far, we have the following.

Corollary 3.8. Let $P \subset \mathbb{R}^{d^2}$ be a polytope combinatorially equivalent to a d-fold product of d-simplices such that a projection to d-space preserves all the vertices. Then there is a (2d-1)-dimensional, simplicial polytope $\mathcal{A}(P,\pi)$ with d(d+1) vertices associated to the projection.

The combinatorial side

Given that all vertices of a polytope P survive the projection we obtain an associated polytope $\mathcal{A} = \mathcal{A}(P,\pi)$. Furthermore, for every vertex $v \in P$ the polytope $q^*(v^{\diamond})$ has zero in its interior which, in particular, implies that its vertices are positively dependent. By Gale duality (Theorem 3.6), this induces a face in \mathcal{A} . The collection of all the induced faces is a polytopal complex in the boundary of \mathcal{A} . If P is a simple polytope, we argued that \mathcal{A} can be assumed to be simplicial. Thus, the polytopal complex is a simplicial complex whose combinatorics is determined by the sole knowledge of the combinatorics of P. We give a rather general description of this complex since it seems that it is the first occurrence in the literature. For background on simplicial complexes as well as notation, we refer to the first chapter of [10].

Definition 3.9 (Complement complex). Let $K \subseteq 2^V$ be a simplicial complex on vertices V. The *complement complex* K^c of K is the closure of

$$\{V \mid \tau : \tau \in \mathsf{K}, \ \tau \text{ a facet}\}.$$

From the bare definition of the complement complex we deduce the following simple properties whose proofs we omit.

Proposition 3.10. Let K and L be simplicial complexes. Then the following statements hold

1.
$$(K^c)^c = K$$

- 2. $\dim K^c = n \dim K 1$.
- 3. $(K * L)^{c} = K^{c} * L^{c}$.

In particular, the first property states that no information is lost in the passage from K to its complement complex. To the best of our knowledge there has been no work on this construction of simplicial complexes. A possible reason for that is the lack of topological plausibility. For a simplicial complex K there seem to be no obvious relation between K and K^c concerning homotopy type and/or (co)homology. For a complex being a *matroid*, the complement complex is just the matroid dual which is well understood combinatorially and topologically. But matroids are rare among simplicial complexes.

For a simplicial polytope P^{Δ} we denote by $\mathcal{B}(P^{\Delta})$ the simplicial complex of all proper faces of P^{Δ} . For a fixed numbering of the facets of P, the vertex set of $\mathcal{B}(P^{\Delta})$ can be identified with $[m] = \{1, \ldots, m\}$.

Theorem 3.11. Let P be a simple polytope whose vertices are preserved under π . Then the complex $\mathcal{B}(P^{\Delta})^{\mathsf{c}}$ is realized in the boundary of the associated polytope $\mathcal{A}(P,\pi)$.

Proof. Since P is simple, the associated polytope $\mathcal{A}(P,\pi)$ is simplicial. Therefore, it is sufficient to show that all facets of $\mathcal{B}(P^{\Delta})^{\mathsf{c}}$ are part of the boundary. For now, let q_1, \ldots, q_m be the vertices of $\mathcal{A}(P,\pi)$ labelled in accordance with the elements of its Gale transform.

A facet of $\mathcal{B}(P^{\Delta})^{c}$ is of the form $[m]\setminus I(v)$ for some vertex $v \in \text{vert } P$. Since $\pi(v)$ is a vertex of the projection, we have $0 \in \text{int conv } \{g_i : i \in I(v)\}$ by the Projection Lemma. By Gale duality, this corresponds to the fact that $\text{conv } \{q_i : i \in [m]\setminus I(v)\}$ is a face of $\mathcal{A}(P,\pi)$.

For a full-dimensional polytope $P \times Q$ with 0 in its interior the polar dual is $P^{\Delta} \oplus Q^{\Delta}$ whose proper faces are the joins of proper faces of P^{Δ} and of Q^{Δ} . Thus for the complement of the boundary complex we have

$$\mathcal{B}\left((P\times Q)^\Delta\right)^{\mathsf{c}} = \mathcal{B}(P^\Delta)^{\mathsf{c}} * \mathcal{B}(Q^\Delta)^{\mathsf{c}}.$$

Focusing again on the polytopes in question, we want to consider the complement complex for the dual of a d-fold product of simplices. Since a simplex is self-dual, we have that $\mathcal{B}(\Delta_d) \cong \binom{[d+1]}{\leq d}$. Thus, for a d-fold product of d-simplices the corresponding complement complex is equivalent to

$$\binom{[d+1]}{\leq 1}^{*d}$$
,

that is, the d-fold join of a complex consisting of d+1 isolated points.

Corollary 3.12. If there is a realization of a d-fold product of d-simplices such that a projection to d-space retains all vertices, then the complex $\binom{[d+1]}{\leq 1}^{*d}$ is embeddable into a sphere of dimension 2d-2.

This will be our punchline: We will show that the embedding claimed by Corollary 3.12 does not exist. Let us rest for a moment and reconsider the example from Section 2.

Let P be a realization of a product of two triangles such that a projection π to the plane preserves all (nine) vertices. By Corollary 3.8, the associated polytope $\mathcal{A}(P,\pi)$ is a 3-dimensional simplicial polytope with 6 vertices. The complement complex is $\binom{[3]}{\leq 1}^{*2}$, can be thought of as $\mathsf{K}_{3,3}$, the complete bipartite graph on 6 vertices. By Corollary 3.12, this complex is embedded in the

boundary of $\mathcal{A}(P,\pi)$, which is a 2-sphere. This, however, is impossible: Graphs embeddable into the 2-sphere are planar while the $\mathsf{K}_{3,3}$ is minimal non-planar.

For a 3-fold product of 3-simplices the boundary of the associated polytope is a 4-sphere and the complement complex is 2-dimensional. So, again, there is no (obvious) elementary reasoning neither are there off-the-shelf results showing non-embeddability. We therefore have to resort to more sophisticated machinery, as presented in the following section.

4 Interlude: Embeddability of simplicial complexes

We only give an *executive summary* of the techniques and results needed for the following; see [10].

The category of free \mathbb{Z}_2 -spaces consists of topological spaces X together with a free action of the group \mathbb{Z}_2 , i.e. a fixed point free involution on X. Morphisms in this category are continuous maps that commute with the respective \mathbb{Z}_2 -actions. The foremost examples of \mathbb{Z}_2 -spaces are spheres \mathbb{S}^d with the antipodal action. For a \mathbb{Z}_2 -space X a numerical invariant is the \mathbb{Z}_2 -index ind \mathbb{Z}_2 X which is the smallest integer X such that there is a \mathbb{Z}_2 -equivariant map $X \to \mathbb{Z}_2$ \mathbb{S}^d . For example ind \mathbb{Z}_2 \mathbb{S}^d which is a statement equivalent to the Borsuk–Ulam theorem.

For a simplicial complex K we define the deleted join of K to be the complex

$$\mathsf{K}^{*2}_{\Delta} = \{ \sigma \uplus \tau : \sigma, \tau \in \mathsf{K}, \sigma \cap \tau = \emptyset \}$$

The deleted join turns an arbitrary simplicial complex into a free \mathbb{Z}_2 -complex by means of $\sigma \uplus \tau \mapsto \tau \uplus \sigma$.

Theorem 4.1 ([10], Theorem 5.5.5). Let K be a simplicial complex. If

$$\operatorname{ind}_{\mathbb{Z}_2} \mathsf{K}_{\Delta}^{*2} > d,$$

then K is not embeddable into the d-sphere.

The \mathbb{Z}_2 -index is rather difficult to calculate for general spaces. Luckily, for the situation in which we will apply Theorem 4.1 there is a beautiful theorem due to Karanbir Sarkaria (see [10]). In order to state it properly we need some more definitions.

Minimal non-faces. Let $K \subset 2^V$ be a simplicial complex. A set $F \subset V$ is called a *non-face* if $F \not\in K$ and its a *minimal non-face* if every proper subset of F is in K. We will denote by $\mathcal{F}(K)$ the set of minimal non-faces.

Generalized Kneser graphs. For a collection of sets $\mathcal{F} = \{F_1, \dots, F_k\}$ we denote by $\mathsf{KG}(\mathcal{F})$ the (abstract) graph with vertex set \mathcal{F} . Two vertices F_i, F_j share an edge iff $F_i \cap F_j = \emptyset$. Such a graph is called a *generalized Kneser graph*.

Finally, for a graph G we denote by $\chi(G)$ its *chromatic number*, i.e. the minimal number of colors to properly color the graph.

Theorem 4.2 (Sarkaria's coloring/embedding theorem). Let K be a simplicial complex with n vertices and let $\mathcal{F} = \mathcal{F}(K)$ be the set of minimal non-faces. Then

$$\operatorname{ind}_{\mathbb{Z}_2} \mathsf{K}^{*2}_{\Delta} \geq n - \chi(\mathsf{KG}(\mathcal{F})) - 1.$$

Taking up the example of triangle times triangle for the last time, let us use Theorem 4.2 to show that $K = {3 \choose \le 1}^{*2}$ does not embed into the 2-sphere. This complex has 6 vertices and, for reasons we will give in the next section, the Kneser graph of its non-faces is, again, the complete bipartite graph $K_{3,3}$. Thus, using Sarkaria's theorem, we get

$$\operatorname{ind}_{\mathbb{Z}_2} \mathsf{K}_{\Delta}^{*2} \ge 6 - 2 - 1 = 3,$$

which shows that $K_{3,3}$ is not planar.

5 Analysis of the complement complex

Although determining upper bounds on the chromatic number of graphs is easier than finding equivariant maps, it is, in general, still hard enough. The key property that enables us to calculate chromatic numbers for the Kneser graphs we will encounter is that the complexes are made up of (possibly) simpler ones, that is they are joins of complexes. The following results will show that this continues to hold if we pass from complexes to non-faces and then to Kneser graphs.

Lemma 5.1. Let K and L be simplicial complexes. Then

$$\mathcal{F}(\mathsf{K} * \mathsf{L}) = \{ F \uplus \emptyset : F \in \mathcal{F}(\mathsf{K}) \} \cup \{ \emptyset \uplus G : G \in \mathcal{F}(\mathsf{L}) \}.$$

Proof. Let $F \uplus G \in \mathcal{F}(\mathsf{K} * \mathsf{L})$ and $i \in F$ and $j \in G$. Since $F \uplus G$ is a minimal non-face, it follows that $F \setminus i \uplus G$ and $F \uplus G \setminus j$ are both in $\mathsf{K} * \mathsf{L}$. This, however, implies that $F \in \mathsf{K}$ and $G \in \mathsf{L}$. \square

On the level of Kneser graphs this fact results in a bipartite sum of the respective Kneser graphs. Let G and H be graphs with disjoint vertex sets U and V. The bipartite sum of G and H is the graph $G \bowtie H$ with vertex set $U \cup V$ and edges $E(G) \cup E(H) \cup (U \times V)$.

Proposition 5.2. Let G and H be graphs. Then

$$\chi(G \bowtie H) = \chi(G) + \chi(H).$$

Proof. The edges $U \times V \subset E(G \bowtie H)$ force the set of colors on U and V to be disjoint. Thus a coloring on $G \bowtie H$ is minimal iff it is minimal on the subgraphs G and H.

The complex to which we want to apply the result is $K = L^{*d}$ with $L = \binom{[d+1]}{\leq 1}$. We will analyze the chromatic number of $KG(\mathcal{F})$ for $\mathcal{F} = \mathcal{F}(L)$ and use Proposition 5.2 to get an obstruction to the embeddability of K into some sphere.

From the definition of L we see that the minimal non-faces are exactly the two element subsets of [d+1], that is $\mathcal{F} = {[d+1] \choose 2}$. The resulting Kneser graph $\mathsf{KG}(\mathcal{F})$ is an instance of a famous family of graphs, the *ordinary* Kneser graphs $\mathsf{KG}_{n,k} := \mathsf{KG}{[n] \choose k}$. The determination of their chromatic numbers is one of the first success stories of topological combinatorics.

Theorem 5.3 (Lovász–Kneser theorem [8]). For $0 < 2k - 1 \le n$ the chromatic number of the Kneser graph $\mathsf{KG}_{n,k}$ is $\chi(\mathsf{KG}_{n,k}) = n - 2k + 2$.

With that last bit of information we can finally complete the proof of Theorem 2.2.

Proof of Theorem 2.2. Assume that there is a realization of a d-fold product of d-simplices whose projection to d-space preserves all the vertices. This implies, by Corollary 3.12, that the complex $\mathsf{K} = \binom{[d+1]}{<1}^{*d}$ is embeddable into a (2d-2)-sphere.

Let $\mathsf{L} = \binom{[d+1]}{\leq 1}$ and let $\mathcal{F}(\mathsf{L}) = \binom{[d+1]}{2}$ be the minimal non-faces of L . For the associated Kneser graph $\mathsf{KG}(\mathcal{F}(\mathsf{L})) = \mathsf{KG}_{d+1,2}$ we have $\chi(\mathsf{KG}_{d+1,2}) = d-1$.

By Lemma 5.1 and Proposition 5.2 we have for $\mathcal{F} = \mathcal{F}(\mathsf{K})$

$$\operatorname{ind}_{\mathbb{Z}_2} \mathsf{K}^{*2}_{\Delta} \geq d(d+1) - \chi(\mathsf{KG}(\mathcal{F})) - 1 = d(d+1) - d\chi(\mathsf{KG}_{d+1}, 2) - 1 = 2d - 1 > 2d - 2$$

By Theorem 4.1, this contradicts the claim that K is embeddable into a (2d-2)-sphere.

It is true that any upper bound on the chromatic number of $\mathsf{KG}_{d+1,2}$ would have sufficed and it was thus unnecessary to invoke the Lovász–Kneser theorem. We, nevertheless, wish to argue that the application of the Lovász–Kneser is justified. Sarkaria's theorem can be used with any upper bound on the chromatic number of the Kneser graph. This, however, results in a weaker bound on the \mathbb{Z}_2 -index of K_Δ^{*2} . Using the actual chromatic number shows that Theorem 2.2 is sharp concerning the number of factors, i.e. there are no topological obstruction for a product of less than d simplices. On the other hand, by Proposition 5.3.2 in [10, p. 96], we have that $2d-1 \geq \mathsf{ind}_{\mathbb{Z}_2} \, \mathsf{K}_\Delta^{*2}$ and, therefore, the calculation in the preceding proof gives the actual \mathbb{Z}_2 -index. Thus, Theorem 2.2 is also sharp with respect to the dimension of the target space, i.e. projecting to a space of dimension $\geq d$. This, in particular, stands in favor for the result of Fukuda & Weibel.

6 Remarks

At the Oberwolfach-Workshop "Geometric and Topological Combinatorics" in January 2007 Rade Živaljević suggested a different argument involving Lovász' colored Helly theorem.

Theorem 6.1 (Colored Helly Theorem; cf. [7]). Let C_1, \ldots, C_r be collections of convex sets in \mathbb{R}^d with $r \geq d+1$. If $\bigcap_{i=1}^r C_i \neq \emptyset$ for every choice $C_i \in C_i$ then there is a $j \in [r]$ such that $\bigcap_{C \in C_i} C \neq \emptyset$.

The following proof was supplied by Imre Bárány (personal communication): Let $P_1, \ldots, P_r \subset \mathbb{R}^d$ be d-polytopes and let $C_i = \{C_v \subset (\mathbb{R}^d)^* : v \in \text{vert } P_i\}$. The C_v are defined by the condition that $\ell \in C_v$ if and only if ℓ attains its unique maximum over P_i in v. It is clear that the C_v are pairwise disjoint. Now, if $P_1 + \cdots + P_r$ attains the trivial upper bound then for every choice of vertices $v_i \in \text{vert } P_i$ the intersection $C_{v_1} \cap \cdots \cap C_{v_r}$ is non-empty and, thus, contradicts the colored Helly theorem. In the colored Helly theorem the bound $r \geq d+1$ is tight and, thus, yields Theorem 1.3 in a slightly weaker version with at least d+1 summands.

In Section 2 we claimed the existence of a combinatorial product of two triangles that projects to a plane 8-gon. Consider the polytope $P \subset \mathbb{R}^4$ given as the set of solutions to the following system of inequalities

The numbers to the left label the facets of P. For $\varepsilon = 0$ this is just a Cartesian product of two triangles and, since this is a simple polytope, we can choose $\varepsilon > 0$ sufficiently small without changing the combinatorial type. Taking $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ to be the projection to the first and last coordinate and identifying $(\mathbb{R}^4)^*$ with \mathbb{R}^4 via the standard scalar product we get the ordered set

$$G=q^*(\operatorname{vert} P^\Delta)=\left(\begin{array}{ccccc} 1 & 1 & -1 & -\varepsilon & & \\ & & -\varepsilon & -1 & 1 & 1 \end{array}\right)$$

For $0 < \varepsilon \le 1$ the set G is the Gale transform of a polytope \mathcal{A} combinatorially equivalent to an octahedron (e.g. set $\varepsilon = 1$ and observe that G is a Gale transform of a regular octahedron.) As intersections of facets the set of vertices is given by $S = \{[6] \setminus \{i,j\} : \{i,j\} \in \mathsf{K}\}$ where the complement complex K is the complete bipartite graph on the partition $\{1,2,3\}$ and $\{4,5,6\}$. We show that the only vertex v_0 that fails to survive the projection is given by the intersection of the facets $[6] \setminus \{1,4\}$. By the Projection Lemma and Gale duality this is case if and only if $\mathsf{K} - \{1,4\}$ is a subcomplex of the 1-skeleton of \mathcal{A} . Figure 2 shows \mathcal{A} and the embedding of $\mathsf{K} - \{1,4\}$, thus finishing the proof. The missing edge between the vertices 1 and 4 shows that v_0 fails short of being a vertex of $\pi(P)$.

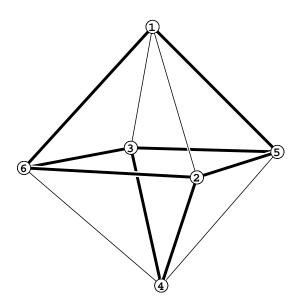


Figure 2: The polytope \mathcal{A} associated to P. The marked edges correspond to the embedding of $K - \{1, 4\}$, that is $K_{3,3}$ minus an edge.

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