CHAPTER 2

Mathematical Models of Systems

PREVIEW

Mathematical models of physical systems are key elements in the design and analysis of control systems. The dynamic behavior is generally described by ordinary differential equations. We will consider a wide range of systems. Since most physical systems are nonlinear, we will discuss linearization approximations which allow us to use Laplace transform methods. We will then proceed to obtain the input–output relationship in the form of transfer functions. The transfer functions can be organized into block diagrams or signal-flow graphs to graphically depict the interconnections. Block diagrams and signal-flow graphs are very convenient and natural tools for designing and analyzing complicated control systems. We conclude the chapter by developing transfer function models for the various components of the Sequential Design Example: Disk Drive Read System.

DESIRED OUTCOMES

Upon completion of Chapter 2, students should:

- ❑ Recognize that differential equations can describe the dynamic behavior of physical systems.
- ❑ Be able to utilize linearization approximations through the use of Taylor series.
- ❑ Understand the application of Laplace transforms and their role in obtaining transfer functions.
- ❑ Be aware of block diagrams and signal-flow graphs and their role in analyzing control systems.
- ❑ Understand the important role of modeling in the control system design process.

2.1 INTRODUCTION

To understand and control complex systems, one must obtain quantitative **mathematical models** of these systems. It is necessary therefore to analyze the relationships between the system variables and to obtain a mathematical model. Because the systems under consideration are dynamic in nature, the descriptive equations are usually **differential equations**. Furthermore, if these equations can be **linearized**, then the **Laplace transform** can be used to simplify the method of solution. In practice, the complexity of systems and our ignorance of all the relevant factors necessitate the introduction of **assumptions** concerning the system operation. Therefore we will often find it useful to consider the physical system, express any necessary assumptions, and linearize the system. Then, by using the physical laws describing the linear equivalent system, we can obtain a set of time-invariant, ordinary linear differential equations. Finally, using mathematical tools, such as the Laplace transform, we obtain a solution describing the operation of the system. In summary, the approach to dynamic system modeling can be listed as follows:

1. Define the system and its components.
2. Formulate the mathematical model and fundamental necessary assumptions based on basic principles.
3. Obtain the differential equations representing the mathematical model.
4. Solve the equations for the desired output variables.
5. Examine the solutions and the assumptions.
6. If necessary, reanalyze or redesign the system.

2.2 DIFFERENTIAL EQUATIONS OF PHYSICAL SYSTEMS

The differential equations describing the dynamic performance of a physical system are obtained by utilizing the physical laws of the process [1–4]. Consider the torsional spring–mass system in Figure 2.1 with applied torque $T_a(t)$. Assume the torsional spring element is massless. Suppose we want to measure the torque $T_s(t)$ transmitted to the mass m. Since the spring is massless, the sum of the torques acting on the spring itself must be zero, or

$$T_a(t) - T_s(t) = 0,$$

which implies that $T_s(t) = T_a(t)$. We see immediately that the external torque $T_a(t)$ applied at the end of the spring is transmitted *through* the torsional spring. Because of this, we refer to the torque as a **through-variable**. In a similar manner, the angular rate difference associated with the torsional spring element is

$$\omega(t) = \omega_s(t) - \omega_a(t).$$

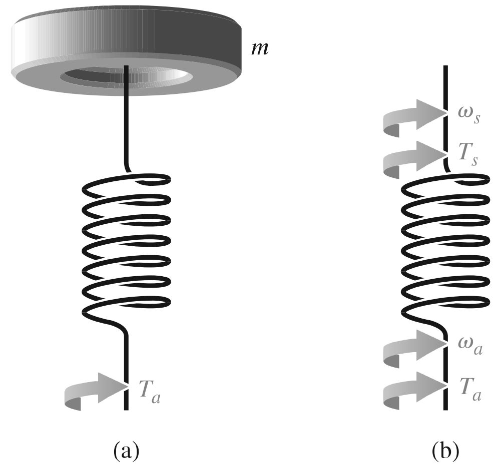


FIGURE 2.1 (a) Torsional spring–mass system. (b) Spring element.

Thus, the angular rate difference is measured across the torsional spring element and is referred to as an **across-variable**. These same types of arguments can be made for most common physical variables (such as force, current, volume, flow rate, etc.). A more complete discussion on through- and across-variables can be found in [26, 27]. A summary of the through- and across-variables of dynamic systems is given in Table 2.1 [5]. Information concerning the International System (SI) of units associated with the various variables discussed in this section can be found online, as well in many handy references, such as the MCS website.† For example, variables that measure temperature are degrees Kelvin in SI units, and variables that measure length are meters. A summary of the describing equations for lumped,

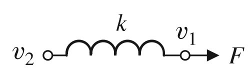
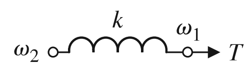
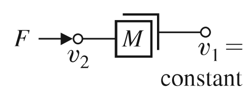
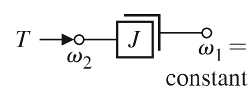
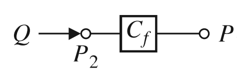
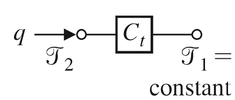
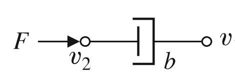
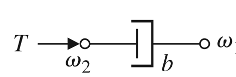
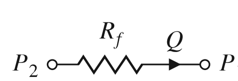
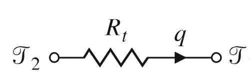
Table 2.1 Summary of Through- and Across-Variables for Physical Systems

System	Variable Through Element	Integrated Through-Variable	Variable Across Element	Integrated Across-Variable
Electrical	Current, i	Charge, q	Voltage difference, v_{21}	Flux linkage, λ_{21}
Mechanical translational	Force, F	Translational momentum, P	Velocity difference, v_{21}	Displacement difference, y_{21}
Mechanical rotational	Torque, T	Angular momentum, h	Angular velocity difference, ω_{21}	Angular displacement difference, θ_{21}
Fluid	Fluid volumetric rate of flow, Q	Volume, V	Pressure difference, P_{21}	Pressure momentum, γ_{21}
Thermal	Heat flow rate, q	Heat energy, H	Temperature difference, $\mathcal{T}_{21}$	

†The companion website is found at www.pearsonhighered.com/dorf.

linear, dynamic elements is given in Table 2.2 [5]. The equations in Table 2.2 are idealized descriptions and only approximate the actual conditions (for example, when a linear, lumped approximation is used for a distributed element).

Table 2.2 Summary of Governing Differential Equations for Ideal Elements

Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Inductive storage	Electrical inductance	$v_{21} = L\dfrac{di}{dt}$	$E = \dfrac{1}{2}Li^2$	
	Translational spring	$v_{21} = \dfrac{1}{k}\dfrac{dF}{dt}$	$E = \dfrac{1}{2}\dfrac{F^2}{k}$	
	Rotational spring	$\omega_{21} = \dfrac{1}{k}\dfrac{dT}{dt}$	$E = \dfrac{1}{2}\dfrac{T^2}{k}$	
	Fluid inertia	$P_{21} = I\dfrac{dQ}{dt}$	$E = \dfrac{1}{2}IQ^2$	
Capacitive storage	Electrical capacitance	$i = C\dfrac{dv_{21}}{dt}$	$E = \dfrac{1}{2}Cv_{21}{}^2$	
	Translational mass	$F = M\dfrac{dv_2}{dt}$	$E = \dfrac{1}{2}Mv_2{}^2$	
	Rotational mass	$T = J\dfrac{d\omega_2}{dt}$	$E = \dfrac{1}{2}J\omega_2{}^2$	
	Fluid capacitance	$Q = C_f\dfrac{dP_{21}}{dt}$	$E = \dfrac{1}{2}C_fP_{21}{}^2$	
	Thermal capacitance	$q = C_t\dfrac{d\mathcal{T}_2}{dt}$	$E = C_t\mathcal{T}_2$	
Energy dissipators	Electrical resistance	$i = \dfrac{1}{R}v_{21}$	$\mathcal{P} = \dfrac{1}{R}v_{21}{}^2$	
	Translational damper	$F = bv_{21}$	$\mathcal{P} = bv_{21}{}^2$	
	Rotational damper	$T = b\omega_{21}$	$\mathcal{P} = b\omega_{21}{}^2$	
	Fluid resistance	$Q = \dfrac{1}{R_f}P_{21}$	$\mathcal{P} = \dfrac{1}{R_f}P_{21}{}^2$	
	Thermal resistance	$q = \dfrac{1}{R_t}\mathcal{T}_{21}$	$\mathcal{P} = \dfrac{1}{R_t}\mathcal{T}_{21}$	

Nomenclature

- *Through-variable:* F = force, T = torque, i = current, Q = fluid volumetric flow rate, q = heat flow rate.
- *Across-variable:* v = translational velocity, ω = angular velocity, v = voltage, P = pressure, $\mathcal{T}$ = temperature.
- *Inductive storage:* L = inductance, $1/k$ = reciprocal translational or rotational stiffness, I = fluid inertance.
- *Capacitive storage:* C = capacitance, M = mass, J = moment of inertia, C_f = fluid capacitance, C_t = thermal capacitance.
- *Energy dissipators:* R = resistance, b = viscous friction, R_f = fluid resistance, R_t = thermal resistance.

The symbol v is used for both voltage in electrical circuits and velocity in translational mechanical systems and is distinguished within the context of each differential equation. For mechanical systems, one uses Newton's laws; for electrical systems, Kirchhoff's voltage laws. For example, the simple spring-mass-damper mechanical system shown in Figure 2.2(a) is described by Newton's second law of motion. The free-body diagram of the mass M is shown in Figure 2.2(b). In this spring-mass-damper example, we model the wall friction as a **viscous damper**, that is, the friction force is linearly proportional to the velocity of the mass. In reality the friction force may behave in a more complicated fashion. For example, the wall friction may behave as a **Coulomb damper**. Coulomb friction, also known as dry friction, is a nonlinear function of the mass velocity and possesses a discontinuity around zero velocity. For a well-lubricated, sliding surface, the viscous friction is appropriate and will be used here and in subsequent spring-mass-damper examples. Summing the forces acting on M and utilizing Newton's second law yields

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t), \tag{2.1}$$

where k is the spring constant of the ideal spring and b is the friction constant. Equation (2.1) is a second-order linear constant-coefficient (time-invariant) differential equation.

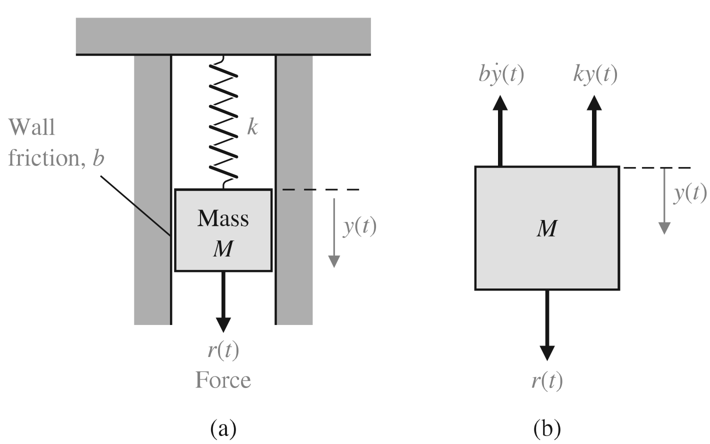


FIGURE 2.2 (a) Spring-mass-damper system. (b) Free-body diagram.

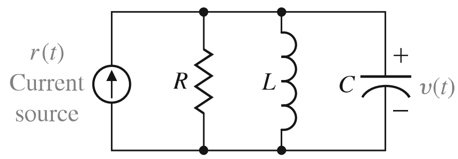


FIGURE 2.3
RLC circuit.

Alternatively, one may describe the electrical *RLC* circuit of Figure 2.3 by utilizing Kirchhoff's current law. Then we obtain the following integrodifferential equation:

$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t)\,dt = r(t). \tag{2.2}$$

The solution of the differential equation describing the process may be obtained by classical methods such as the use of integrating factors and the method of undetermined coefficients [1]. For example, when the mass is initially displaced a distance $y(0) = y_0$ and released, the dynamic response of the system can be represented by an equation of the form

$$y(t) = K_1 e^{-\alpha_1 t}\sin(\beta_1 t + \theta_1). \tag{2.3}$$

A similar solution is obtained for the voltage of the *RLC* circuit when the circuit is subjected to a constant current $r(t) = I$. Then the voltage is

$$v(t) = K_2 e^{-\alpha_2 t}\cos(\beta_2 t + \theta_2). \tag{2.4}$$

A voltage curve typical of an *RLC* circuit is shown in Figure 2.4.

To reveal further the close similarity between the differential equations for the mechanical and electrical systems, we shall rewrite Equation (2.1) in terms of velocity:

$$v(t) = \frac{dy(t)}{dt}.$$

Then we have

$$M\frac{dv(t)}{dt} + bv(t) + k\int_0^t v(t)\,dt = r(t). \tag{2.5}$$

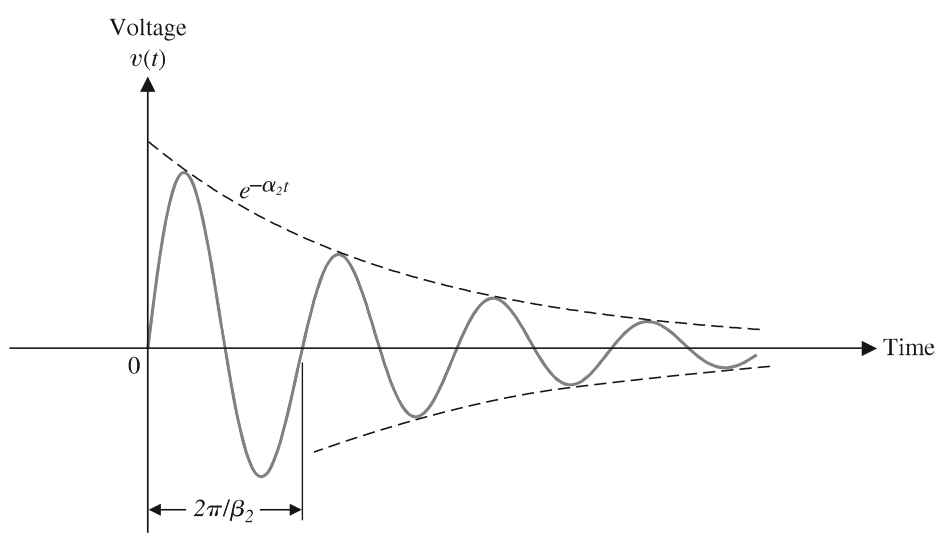


FIGURE 2.4
Typical voltage response for an *RLC* circuit.

One immediately notes the equivalence of Equations (2.5) and (2.2) where velocity $v(t)$ and voltage $v(t)$ are equivalent variables, usually called **analogous variables**, and the systems are analogous systems. Therefore the solution for velocity is similar to Equation (2.4), and the response for an underdamped system is shown in Figure 2.4. The concept of analogous systems is a very useful and powerful technique for system modeling. The voltage–velocity analogy, often called the force–current analogy, is a natural one because it relates the analogous through- and across-variables of the electrical and mechanical systems. Another analogy that relates the velocity and current variables is often used and is called the force–voltage analogy [21, 23].

Analogous systems with similar solutions exist for electrical, mechanical, thermal, and fluid systems. The existence of analogous systems and solutions provides the analyst with the ability to extend the solution of one system to all analogous systems with the same describing differential equations. Therefore what one learns about the analysis and design of electrical systems is immediately extended to an understanding of fluid, thermal, and mechanical systems.

2.3 LINEAR APPROXIMATIONS OF PHYSICAL SYSTEMS

A great majority of physical systems are linear within some range of the variables. In general, systems ultimately become nonlinear as the variables are increased without limit. For example, the spring-mass-damper system of Figure 2.2 is linear and described by Equation (2.1) as long as the mass is subjected to small deflections $y(t)$. However, if $y(t)$ were continually increased, eventually the spring would be overextended and break. Therefore the question of linearity and the range of applicability must be considered for each system.

A system is defined as linear in terms of the system excitation and response. In the case of the electrical network, the excitation is the input current $r(t)$ and the response is the voltage $v(t)$. In general, a **necessary condition** for a linear system can be determined in terms of an excitation $x(t)$ and a response $y(t)$. When the system at rest is subjected to an excitation $x_1(t)$, it provides a response $y_1(t)$. Furthermore, when the system is subjected to an excitation $x_2(t)$, it provides a corresponding response $y_2(t)$. For a linear system, it is necessary that the excitation $x_1(t) + x_2(t)$ result in a response $y_1(t) + y_2(t)$. This is the **principle of superposition**.

Furthermore, the magnitude scale factor must be preserved in a **linear system**. Again, consider a system with an input $x(t)$ that results in an output $y(t)$. Then the response of a linear system to a constant multiple β of an input x must be equal to the response to the input multiplied by the same constant so that the output is equal to $\beta y(t)$. This is the property of **homogeneity**.

A linear system satisfies the properties of superposition and homogeneity.

A system characterized by the relation $y(t) = x^2(t)$ is not linear, because the superposition property is not satisfied. A system represented by the relation $y(t) = mx(t) + b$ is not linear, because it does not satisfy the homogeneity property. However, this second system may be considered linear about an operating point x_0, y_0 for small changes Δx and Δy. When $x(t) = x_0 + \Delta x(t)$ and $y(t) = y_0 + \Delta y(t)$, we have

$$y(t) = mx(t) + b$$

or

$$y_0 + \Delta y(t) = mx_0 + m\Delta x(t) + b.$$

Therefore, $\Delta y(t) = m\Delta x(t)$, which satisfies the necessary conditions.

The linearity of many mechanical and electrical elements can be assumed over a reasonably large range of the variables [7]. This is not usually the case for thermal and fluid elements, which are more frequently nonlinear in character. Fortunately, however, one can often linearize nonlinear elements assuming small-signal conditions. This is the normal approach used to obtain a linear equivalent circuit for electronic circuits and transistors. Consider a general element with an excitation (through-) variable $x(t)$ and a response (across-) variable $y(t)$. Several examples of dynamic system variables are given in Table 2.1. The relationship of the two variables is written as

$$y(t) = g(x(t)), \tag{2.6}$$

where $g(x(t))$ indicates $y(t)$ is a function of $x(t)$. The normal operating point is designated by x_0. Because the curve (function) is continuous over the range of interest, a **Taylor series** expansion about the operating point may be utilized [7]. Then we have

$$y(t) = g(x(t)) = g(x_0) + \left.\frac{dg}{dx}\right|_{x(t)=x_0} \frac{(x(t) - x_0)}{1!} + \left.\frac{d^2g}{dx^2}\right|_{x(t)=x_0} \frac{(x(t) - x_0)^2}{2!} + \cdots. \tag{2.7}$$

The slope at the operating point,

$$m = \left.\frac{dg}{dx}\right|_{x(t)=x_0},$$

is a good approximation to the curve over a small range of $x(t) - x_0$, the deviation from the operating point. Then, as a reasonable approximation, Equation (2.7) becomes

$$y(t) = g(x_0) + \left.\frac{dg}{dx}\right|_{x(t)=x_0} (x(t) - x_0) = y_0 + m(x(t) - x_0). \tag{2.8}$$

Finally, Equation (2.8) can be rewritten as the linear equation

$$y(t) - y_0 = m(x(t) - x_0)$$

or

$$\Delta y(t) = m\Delta x(t). \tag{2.9}$$

Consider the case of a mass, M, sitting on a nonlinear spring, as shown in Figure 2.5(a). The normal operating point is the equilibrium position that occurs when the spring force balances the gravitational force Mg, where g is the gravitational constant. Thus, we obtain $f_0 = Mg$, as shown. For the nonlinear spring with $f(t) = y^2(t)$, the equilibrium position is $y_0 = (Mg)^{1/2}$. The linear model for small deviation is

$$\Delta f(t) = m\Delta y(t),$$

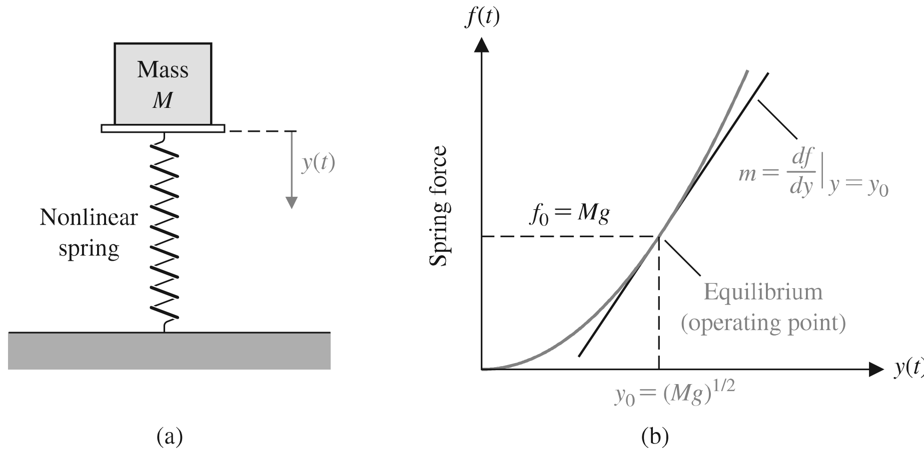


FIGURE 2.5
(a) A mass sitting on a nonlinear spring.
(b) The spring force versus $y(t)$.

where

$$m = \left.\frac{df}{dy}\right|_{y(t)=y_0},$$

as shown in Figure 2.5(b). Thus, $m = 2y_0$. A **linear approximation** is as accurate as the assumption of small signals is applicable to the specific problem.

If the dependent variable $y(t)$ depends upon several excitation variables, $x_1(t), x_2(t), \ldots, x_n(t)$, then the functional relationship is written as

$$y(t) = g(x_1(t), x_2(t), \ldots, x_n(t)). \tag{2.10}$$

The Taylor series expansion about the operating point $x_{1_0}, x_{2_0}, \ldots, x_{n_0}$ is useful for a linear approximation to the nonlinear function. When the higher-order terms are neglected, the linear approximation is written as

$$\begin{aligned} y(t) = g(x_{1_0}, x_{2_0}, \ldots, x_{n_0}) &+ \left.\frac{\partial g}{\partial x_1}\right|_{x(t)=x_0}(x_1(t) - x_{1_0}) + \left.\frac{\partial g}{\partial x_2}\right|_{x(t)=x_0}(x_2(t) - x_{2_0}) \\ &+ \cdots + \left.\frac{\partial g}{\partial x_n}\right|_{x(t)=x_0}(x_n(t) - x_{n_0}), \end{aligned} \tag{2.11}$$

where x_0 is the operating point. Example 2.1 will clearly illustrate the utility of this method.

EXAMPLE 2.1 **Pendulum oscillator model**

Consider the pendulum oscillator shown in Figure 2.6(a). The torque on the mass is

$$T(t) = MgL \sin \theta(t), \tag{2.12}$$

where g is the gravity constant. The equilibrium condition for the mass is $\theta_0 = 0°$. The nonlinear relation between $T(t)$ and $\theta(t)$ is shown graphically in Figure 2.6(b). The first derivative evaluated at equilibrium provides the linear approximation, which is

$$T(t) - T_0 \cong MgL \left.\frac{\partial \sin \theta}{\partial \theta}\right|_{\theta(t)=\theta_0}(\theta(t) - \theta_0),$$

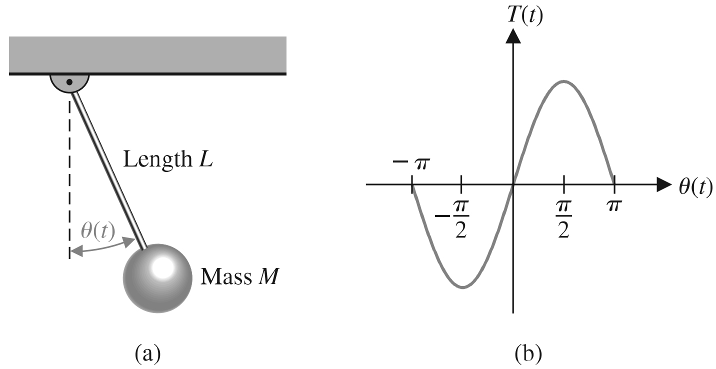


FIGURE 2.6 Pendulum oscillator.

where $T_0 = 0$. Then, we have

$$T(t) = MgL\theta(t). \tag{2.13}$$

This approximation is reasonably accurate for $-\pi/4 \le \theta \le \pi/4$. For example, the response of the linear model for the swing through $\pm 30°$ is within 5% of the actual nonlinear pendulum response. ■

2.4 THE LAPLACE TRANSFORM

The ability to obtain linear time-invariant approximations of physical systems allows the analyst to consider the use of the **Laplace transformation**. The Laplace transform method substitutes relatively easily solved algebraic equations for the more difficult differential equations [1, 3]. The time-response solution is obtained by the following operations:

1. Obtain the linearized differential equations.
2. Obtain the Laplace transformation of the differential equations.
3. Solve the resulting algebraic equation for the transform of the variable of interest.

The Laplace transform exists for linear differential equations for which the transformation integral converges. Therefore, for $f(t)$ to be transformable, it is sufficient that

$$\int_{0^-}^{\infty} |f(t)| e^{-\sigma_1 t}\, dt < \infty,$$

for some real, positive σ_1 [1]. The 0^- indicates that the integral should include any discontinuity, such as a delta function at $t = 0$. If the magnitude of $f(t)$ is $|f(t)| < Me^{\alpha t}$ for all positive t, the integral will converge for $\sigma_1 > \alpha$. The region of convergence is therefore given by $\infty > \sigma_1 > \alpha$, and σ_1 is known as the abscissa of absolute convergence. Signals that are physically realizable always have a Laplace transform. The Laplace transformation for a function of time, $f(t)$, is

$$\boxed{F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\, dt = \mathcal{L}\{f(t)\}.} \tag{2.14}$$

The **inverse Laplace transform** is written as

$$\boxed{f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{+st}\, ds.} \tag{2.15}$$

The transformation integrals have been employed to derive tables of Laplace transforms that are used for the great majority of problems. A table of important Laplace transform pairs is given in Table 2.3. A more complete list of Laplace transform pairs can be found in many references, including at the MCS website.

Table 2.3 Important Laplace Transform Pairs

f(t)	*F(s)*
Step function, $u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s + a}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$f^{(k)}(t) = \frac{d^k f(t)}{dt^k}$	$s^k F(s) - s^{k-1} f(0^-) - s^{k-2} f'(0^-) - \ldots - f^{(k-1)}(0^-)$
$\int_{-\infty}^{t} f(t)\, dt$	$\frac{F(s)}{s} + \frac{1}{s} \int_{-\infty}^{0} f(t)\, dt$
Impulse function $\delta(t)$	1
$e^{-at} \sin \omega t$	$\frac{\omega}{(s + a)^2 + \omega^2}$
$e^{-at} \cos \omega t$	$\frac{s + a}{(s + a)^2 + \omega^2}$
$\frac{1}{\omega}[(\alpha - a)^2 + \omega^2]^{1/2} e^{-at} \sin(\omega t + \phi)$, $\phi = \tan^{-1} \frac{\omega}{\alpha - a}$	$\frac{s + \alpha}{(s + a)^2 + \omega^2}$
$\frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \omega_n \sqrt{1 - \zeta^2} t, \zeta < 1$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
$\frac{1}{a^2 + \omega^2} + \frac{1}{\omega\sqrt{a^2 + \omega^2}} e^{-at} \sin(\omega t - \phi)$, $\phi = \tan^{-1} \frac{\omega}{-a}$	$\frac{1}{s[(s + a)^2 + \omega^2]}$
$1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1 - \zeta^2} t + \phi\right)$, $\phi = \cos^{-1} \zeta, \zeta < 1$	$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
$\frac{\alpha}{a^2 + \omega^2} + \frac{1}{\omega}\left[\frac{(\alpha - a)^2 + \omega^2}{a^2 + \omega^2}\right]^{1/2} e^{-at} \sin(\omega t + \phi)$. $\phi = \tan^{-1} \frac{\omega}{\alpha - a} - \tan^{-1} \frac{\omega}{-a}$	$\frac{s + \alpha}{s[(s + a)^2 + \omega^2]}$

Alternatively, the Laplace variable s can be considered to be the differential operator so that

$$s \equiv \frac{d}{dt}. \tag{2.16}$$

Then we also have the integral operator

$$\frac{1}{s} \equiv \int_{0^-}^{t} dt. \tag{2.17}$$

The inverse Laplace transformation is usually obtained by using the Heaviside partial fraction expansion. This approach is particularly useful for systems analysis and design because the effect of each characteristic root or eigenvalue can be clearly observed.

To illustrate the usefulness of the Laplace transformation and the steps involved in the system analysis, reconsider the spring-mass-damper system described by Equation (2.1), which is

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t). \tag{2.18}$$

We wish to obtain the response, $y(t)$, as a function of time. The Laplace transform of Equation (2.18) is

$$M\left(s^2Y(s) - sy(0^-) - \frac{dy}{dt}(0^-)\right) + b(sY(s) - y(0^-)) + kY(s) = R(s). \tag{2.19}$$

When

$$r(t) = 0, \quad \text{and} \quad y(0^-) = y_0, \quad \text{and} \quad \left.\frac{dy}{dt}\right|_{t=0^-} = 0,$$

we have

$$Ms^2Y(s) - Msy_0 + bsY(s) - by_0 + kY(s) = 0. \tag{2.20}$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{(Ms + b)y_0}{Ms^2 + bs + k} = \frac{p(s)}{q(s)}. \tag{2.21}$$

The denominator polynomial $q(s)$, when set equal to zero, is called the **characteristic equation** because the roots of this equation determine the character of the time response. The roots of this characteristic equation are also called the **poles** of the system. The roots of the numerator polynomial $p(s)$ are called the **zeros** of the system; for example, $s = -b/M$ is a zero of Equation (2.21). Poles and zeros are critical frequencies. At the poles, the function $Y(s)$ becomes infinite, whereas at the zeros, the function becomes zero. The complex frequency **s-plane** plot of the poles and zeros graphically portrays the character of the natural transient response of the system.

For a specific case, consider the system when $k/M = 2$ and $b/M = 3$. Then Equation (2.21) becomes

$$Y(s) = \frac{(s + 3)y_0}{(s + 1)(s + 2)}. \tag{2.22}$$

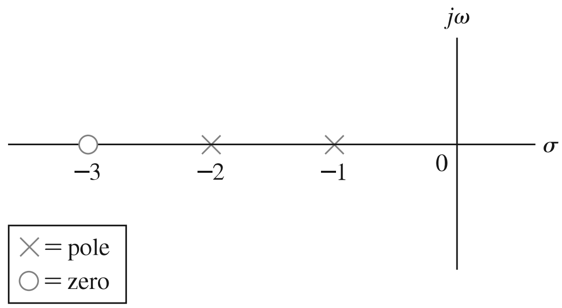


FIGURE 2.7 An s-plane pole and zero plot.

The poles and zeros of $Y(s)$ are shown on the s-plane in Figure 2.7.

Expanding Equation (2.22) in a partial fraction expansion, we obtain

$$Y(s) = \frac{k_1}{s+1} + \frac{k_2}{s+2}, \tag{2.23}$$

where k_1 and k_2 are the coefficients of the expansion. The coefficients k_i are called **residues** and are evaluated by multiplying through by the denominator factor of Equation (2.22) corresponding to k_i and setting s equal to the root. Evaluating k_1 when $y_0 = 1$, we have

$$k_1 = \left.\frac{(s - s_1)p(s)}{q(s)}\right|_{s=s_1} \tag{2.24}$$

$$= \left.\frac{(s+1)(s+3)}{(s+1)(s+2)}\right|_{s_1=-1} = 2$$

and $k_2 = -1$. Alternatively, the residues of $Y(s)$ at the respective poles may be evaluated graphically on the s-plane plot, since Equation (2.24) may be written as

$$k_1 = \left.\frac{s+3}{s+2}\right|_{s=s_1=-1} \tag{2.25}$$

$$= \left.\frac{s_1+3}{s_1+2}\right|_{s_1=-1} = 2.$$

The graphical representation of Equation (2.25) is shown in Figure 2.8. The graphical method of evaluating the residues is particularly valuable when the order of the characteristic equation is high and several poles are complex conjugate pairs.

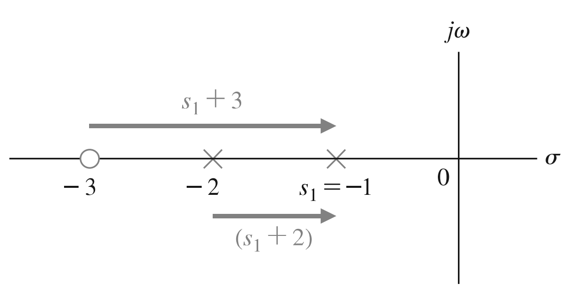


FIGURE 2.8 Graphical evaluation of the residues.

The inverse Laplace transform of Equation (2.22) is then

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-1}{s+2}\right\}. \tag{2.26}$$

Using Table 2.3, we find that

$$y(t) = 2e^{-t} - 1e^{-2t}. \tag{2.27}$$

Finally, it is usually desired to determine the **steady-state** or **final value** of the response of $y(t)$. For example, the final or steady-state rest position of the spring-mass-damper system may be calculated. The **final value theorem** states that

$$\boxed{\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s),} \tag{2.28}$$

where a simple pole of $Y(s)$ at the origin is permitted, but poles on the imaginary axis and in the right half-plane and repeated poles at the origin are excluded. Therefore, for the specific case of the spring-mass-damper, we find that

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = 0. \tag{2.29}$$

Hence the final position for the mass is the normal equilibrium position $y = 0$.

Reconsider the spring-mass-damper system. The equation for $Y(s)$ may be written as

$$Y(s) = \frac{(s + b/M)y_0}{s^2 + (b/M)s + k/M} = \frac{(s + 2\zeta\omega_n)y_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{2.30}$$

where ζ is the dimensionless **damping ratio**, and ω_n is the **natural frequency** of the system. The roots of the characteristic equation are

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}, \tag{2.31}$$

where, in this case, $\omega_n = \sqrt{k/M}$ and $\zeta = b/(2\sqrt{kM})$. When $\zeta > 1$, the roots are real and the system is **overdamped**; when $\zeta < 1$, the roots are complex and the system is **underdamped**. When $\zeta = 1$, the roots are repeated and real, and the condition is called **critical damping**.

When $\zeta < 1$, the response is underdamped, and

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}. \tag{2.32}$$

The s-plane plot of the poles and zeros of $Y(s)$ is shown in Figure 2.9, where $\theta = \cos^{-1}\zeta$. As ζ varies with ω_n constant, the complex conjugate roots follow a

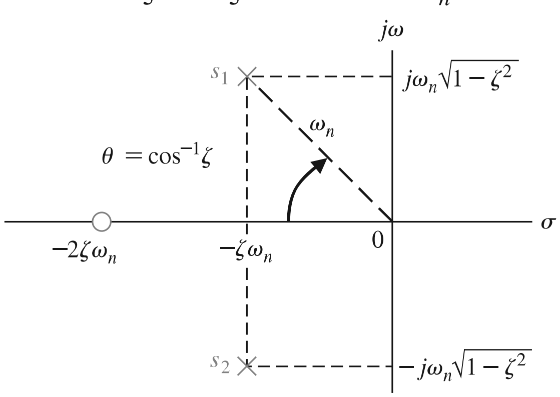


FIGURE 2.9 An s-plane plot of the poles and zeros of $Y(s)$.

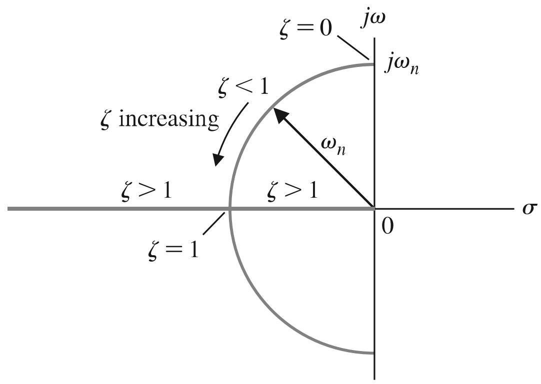


FIGURE 2.10 The locus of roots as ζ varies with ω_n constant.

circular locus, as shown in Figure 2.10. The transient response is increasingly oscillatory as the roots approach the imaginary axis when ζ approaches zero.

The inverse Laplace transform can be evaluated using the graphical residue evaluation. The partial fraction expansion of Equation (2.30) is

$$Y(s) = \frac{k_1}{s - s_1} + \frac{k_2}{s - s_2}. \tag{2.33}$$

Since s_2 is the complex conjugate of s_1, the residue k_2 is the complex conjugate of k_1 so that we obtain

$$Y(s) = \frac{k_1}{s - s_1} + \frac{\hat{k}_1}{s - \hat{s}_1}$$

where the hat indicates the conjugate relation. The residue k_1 is evaluated from Figure 2.11 as

$$k_1 = \frac{y_0(s_1 + 2\zeta\omega_n)}{s_1 - \hat{s}_1} = \frac{y_0 M_1 e^{j\theta}}{M_2 e^{j\pi/2}}, \tag{2.34}$$

where M_1 is the magnitude of $s_1 + 2\zeta\omega_n$, and M_2 is the magnitude of $s_1 - \hat{s}_1$. A review of complex numbers can be found in many online references, as well as on the MCS website. In this case, we obtain

$$k_1 = \frac{y_0(\omega_n e^{j\theta})}{2\omega_n\sqrt{1 - \zeta^2}e^{j\pi/2}} = \frac{y_0}{2\sqrt{1 - \zeta^2}e^{j(\pi/2-\theta)}}, \tag{2.35}$$

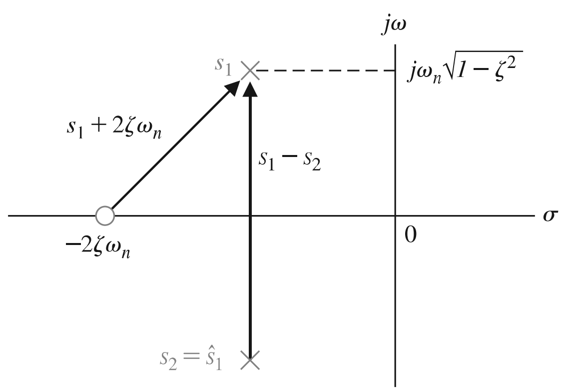


FIGURE 2.11 Evaluation of the residue k_1.

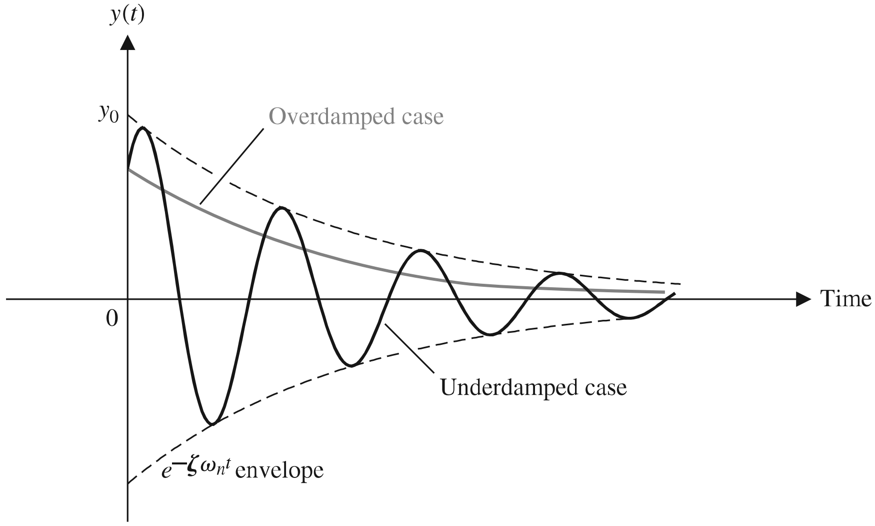


FIGURE 2.12 Response of the spring-mass-damper system.

where $\theta = \cos^{-1} \zeta$. Therefore,

$$k_2 = \frac{y_0}{2\sqrt{1 - \zeta^2}} e^{j(\pi/2 - \theta)}. \tag{2.36}$$

Finally, letting $\beta = \sqrt{1 - \zeta^2}$, we find that

$$\begin{aligned} y(t) &= k_1 e^{s_1 t} + k_2 e^{s_2 t} \\ &= \frac{y_0}{2\sqrt{1 - \zeta^2}} \left(e^{j(\theta - \pi/2)} e^{-\zeta\omega_n t} e^{j\omega_n \beta t} + e^{j(\pi/2 - \theta)} e^{-\zeta\omega_n t} e^{-j\omega_n \beta t}\right) \\ &= \frac{y_0}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2}\, t + \theta). \end{aligned} \tag{2.37}$$

The solution, Equation (2.37), can also be obtained using item 11 of Table 2.3. The transient responses of the overdamped ($\zeta > 1$) and underdamped ($\zeta < 1$) cases are shown in Figure 2.12. The transient response that occurs when $\zeta < 1$ exhibits an oscillation in which the amplitude decreases with time, and it is called a **damped oscillation**.

The relationship between the s-plane location of the poles and zeros and the form of the transient response can be interpreted from the s-plane pole–zero plots. For example, as seen in Equation (2.37), adjusting the value of $\zeta\omega_n$ varies the $e^{-\zeta\omega_n t}$ envelope, hence the response $y(t)$ shown in Figure 2.12. The larger the value of $\zeta\omega_n$, the faster the damping of the response, $y(t)$. In Figure 2.9, we see that the location of the complex pole s_1 is given by $s_1 = -\zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}$. So, making $\zeta\omega_n$ larger moves the pole further to the left in the s-plane. Thus, the connection between the location of the pole in the s-plane and the step response is apparent—moving the pole s_1 farther in the left half-plane leads to a faster damping of the transient step response. Of course, most control systems will have more than one complex pair of poles, so the transient response will be the result of the contributions of all the poles. In fact, the magnitude of the response of each pole, represented by the residue, can be visualized by examining the graphical residues on the s-plane. We will discuss the connection between the pole and zero locations and the transient and steady-state response more in subsequent chapters. We will find that the Laplace transformation and the s-plane approach are very useful techniques for system analysis and design where emphasis is placed on the transient and steady-state performance. In fact, because the study of control systems is concerned primarily with the transient and steady-state performance of dynamic systems, we have real cause to appreciate the value of the Laplace transform techniques.

2.5 THE TRANSFER FUNCTION OF LINEAR SYSTEMS

The **transfer function** of a linear system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero. The transfer function of a system (or element) represents the relationship describing the dynamics of the system under consideration.

A transfer function may be defined only for a linear, stationary (constant parameter) system. A nonstationary system, often called a time-varying system, has one or more time-varying parameters, and the Laplace transformation may not be utilized. Furthermore, a transfer function is an input–output description of the behavior of a system. Thus, the transfer function description does not include any information concerning the internal structure of the system and its behavior.

The transfer function of the spring-mass-damper system is obtained from the original Equation (2.19), rewritten with zero initial conditions as follows:

$$Ms^2Y(s) + bsY(s) + kY(s) = R(s). \tag{2.38}$$

Then the transfer function is the ratio of the output to the input, or

$$G(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + bs + k}. \tag{2.39}$$

The transfer function of the RC network shown in Figure 2.13 is obtained by writing the Kirchhoff voltage equation, yielding

$$V_1(s) = \left(R + \frac{1}{Cs}\right)I(s), \tag{2.40}$$

expressed in terms of transform variables. We shall frequently refer to variables and their transforms interchangeably. The transform variable will be distinguishable by the use of an uppercase letter or the argument (s).

The output voltage is

$$V_2(s) = I(s)\left(\frac{1}{Cs}\right). \tag{2.41}$$

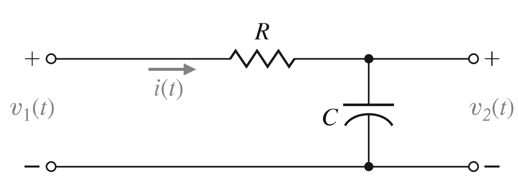


FIGURE 2.13
An *RC* network.

Therefore, solving Equation (2.40) for $I(s)$ and substituting in Equation (2.41), we have

$$V_2(s) = \frac{(1/Cs)V_1(s)}{R + 1/Cs}.$$

Then the transfer function is obtained as the ratio $V_2(s)/V_1(s)$,

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{RCs + 1} = \frac{1}{\tau s + 1} = \frac{1/\tau}{s + 1/\tau}, \tag{2.42}$$

where $\tau = RC$, the **time constant** of the network. The single pole of $G(s)$ is $s = -1/\tau$. Equation (2.42) could be immediately obtained if one observes that the circuit is a voltage divider, where

$$\frac{V_2(s)}{V_1(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}, \tag{2.43}$$

and $Z_1(s) = R$, $Z_2 = 1/Cs$.

A multiloop electrical circuit or an analogous multiple-mass mechanical system results in a set of simultaneous equations in the Laplace variable. It is usually more convenient to solve the simultaneous equations by using matrices and determinants [1, 3, 15]. An introduction to matrices and determinants can be found in many references online, as well as on the MCS website.

Let us consider the long-term behavior of a system and determine the response to certain inputs that remain after the transients fade away. Consider the dynamic system represented by the differential equation

$$\frac{d^n y(t)}{dt^n} + q_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + q_0 y(t) = p_{n-1}\frac{d^{n-1}r(t)}{dt^{n-1}} + p_{n-2}\frac{d^{n-2}r(t)}{dt^{n-2}} + \cdots + p_0 r(t), \tag{2.44}$$

where $y(t)$ is the response, and $r(t)$ is the input or forcing function. If the initial conditions are all zero, then the transfer function is the coefficient of $R(s)$ in

$$Y(s) = G(s)R(s) = \frac{p(s)}{q(s)}R(s) = \frac{p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_0}{s^n + q_{n-1}s^{n-1} + \cdots + q_0}R(s). \tag{2.45}$$

The output response consists of a natural response (determined by the initial conditions) plus a forced response determined by the input. We now have

$$Y(s) = \frac{m(s)}{q(s)} + \frac{p(s)}{q(s)}R(s),$$

where $q(s) = 0$ is the characteristic equation. If the input has the rational form

$$R(s) = \frac{n(s)}{d(s)},$$

then

$$Y(s) = \frac{m(s)}{q(s)} + \frac{p(s)}{q(s)}\frac{n(s)}{d(s)} = Y_1(s) + Y_2(s) + Y_3(s), \tag{2.46}$$

where $Y_1(s)$ is the partial fraction expansion of the natural response, $Y_2(s)$ is the partial fraction expansion of the terms involving factors of $q(s)$, and $Y_3(s)$ is the partial fraction expansion of terms involving factors of $d(s)$.

Taking the inverse Laplace transform yields

$$y(t) = y_1(t) + y_2(t) + y_3(t).$$

The transient response consists of $y_1(t) + y_2(t)$, and the steady-state response is $y_3(t)$.

EXAMPLE 2.2 **Solution of a differential equation**

Consider a system represented by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2r(t),$$

where the initial conditions are $y(0) = 1, \frac{dy}{dt}(0) = 0$, and $r(t) = 1, t \geq 0$.

The Laplace transform yields

$$[s^2Y(s) - sy(0)] + 4[sY(s) - y(0)] + 3Y(s) = 2R(s).$$

Since $R(s) = 1/s$ and $y(0) = 1$, we obtain

$$Y(s) = \frac{s+4}{s^2+4s+3} + \frac{2}{s(s^2+4s+3)},$$

where $q(s) = s^2 + 4s + 3 = (s+1)(s+3) = 0$ is the characteristic equation, and $d(s) = s$. Then the partial fraction expansion yields

$$Y(s) = \left[\frac{3/2}{s+1} + \frac{-1/2}{s+3}\right] + \left[\frac{-1}{s+1} + \frac{1/3}{s+3}\right] + \frac{2/3}{s} = Y_1(s) + Y_2(s) + Y_3(s).$$

Hence, the response is

$$y(t) = \left[\frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}\right] + \left[-1e^{-t} + \frac{1}{3}e^{-3t}\right] + \frac{2}{3},$$

and the steady-state response is

$$\lim_{t\to\infty} y(t) = \frac{2}{3}. \blacksquare$$

EXAMPLE 2.3 **Transfer function of an op-amp circuit**

The operational amplifier (op-amp) belongs to an important class of analog integrated circuits commonly used as building blocks in the implementation of control systems and in many other important applications. Op-amps are active elements (that is, they have external power sources) with a high gain when operating in their linear regions. A model of an ideal op-amp is shown in Figure 2.14.

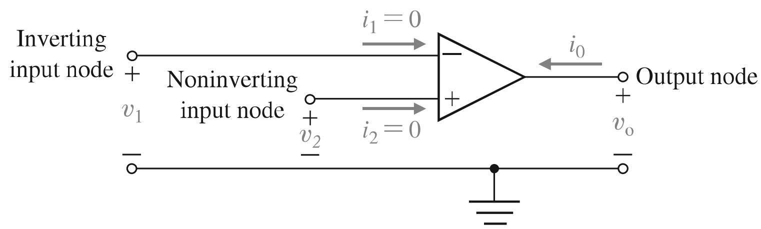


FIGURE 2.14
The ideal op-amp.

The operating conditions for the ideal op-amp are (1) $i_1 = 0$ and $i_2 = 0$, thus implying that the input impedance is infinite, and (2) $v_2 - v_1 = 0$ (or $v_1 = v_2$). The input–output relationship for an ideal op-amp is

$$v_0 = K(v_2 - v_1) = -K(v_1 - v_2),$$

where the gain K approaches infinity. In our analysis, we will assume that the linear op-amps are operating with high gain and under idealized conditions.

Consider the inverting amplifier shown in Figure 2.15. Under ideal conditions, we have $i_1 = 0$, so that writing the node equation at v_1 yields

$$\frac{v_1 - v_{in}}{R_1} + \frac{v_1 - v_0}{R_2} = 0.$$

Since $v_2 = v_1$ (under ideal conditions) and $v_2 = 0$ (see Figure 2.15 and compare it with Figure 2.14), it follows that $v_1 = 0$. Therefore,

$$-\frac{v_{in}}{R_1} - \frac{v_0}{R_2} = 0,$$

and rearranging terms, we obtain

$$\frac{v_0}{v_{in}} = -\frac{R_2}{R_1}.$$

We see that when $R_2 = R_1$, the ideal op-amp circuit inverts the sign of the input, that is, $v_0 = -v_{in}$ when $R_2 = R_1$. ■

EXAMPLE 2.4 **Transfer function of a system**

Consider the mechanical system shown in Figure 2.16 and its electrical circuit analog shown in Figure 2.17. The electrical circuit analog is a force–current analog as outlined in Table 2.1. The velocities $v_1(t)$ and $v_2(t)$ of the mechanical system are

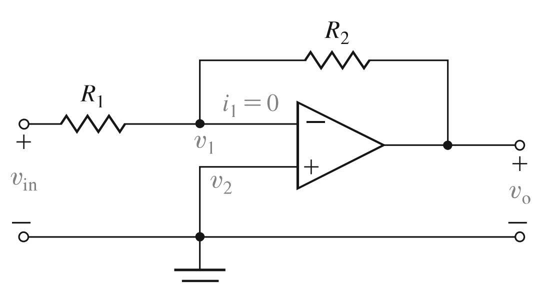


FIGURE 2.15
An Inverting amplifier operating with ideal conditions.

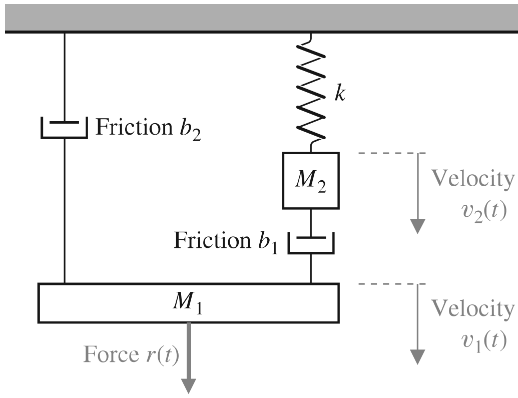


FIGURE 2.16 Two-mass mechanical system.

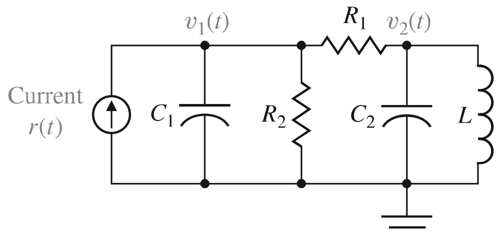


FIGURE 2.17 Two-node electric circuit analog $C_1 = M_1, C_2 = M_2, L = 1/k, R_1 = 1/b_1, R_2 = 1/b_2$.

directly analogous to the node voltages $v_1(t)$ and $v_2(t)$ of the electrical circuit. The simultaneous equations, assuming that the initial conditions are zero, are

$$M_1 s V_1(s) + (b_1 + b_2)V_1(s) - b_1 V_2(s) = R(s), \tag{2.47}$$

and

$$M_2 s V_2(s) + b_1(V_2(s) - V_1(s)) + k\frac{V_2(s)}{s} = 0. \tag{2.48}$$

These equations are obtained using the force equations for the mechanical system of Figure 2.16. Rearranging Equations (2.47) and (2.48), we obtain

$$(M_1 s + (b_1 + b_2))V_1(s) + (-b_1)V_2(s) = R(s),$$

$$(-b_1)V_1(s) + \left(M_2 s + b_1 + \frac{k}{s}\right)V_2(s) = 0,$$

or, in matrix form,

$$\begin{bmatrix} M_1 s + b_1 + b_2 & -b_1 \\ -b_1 & M_2 s + b_1 + \dfrac{k}{s} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix}. \tag{2.49}$$

Assuming that the velocity of M_1 is the output variable, we solve for $V_1(s)$ by matrix inversion or Cramer's rule to obtain [1, 3]

$$V_1(s) = \frac{(M_2 s + b_1 + k/s)R(s)}{(M_1 s + b_1 + b_2)(M_2 s + b_1 + k/s) - b_1^{\,2}}. \tag{2.50}$$

Then the transfer function of the mechanical (or electrical) system is

$$\begin{aligned} G(s) = \frac{V_1(s)}{R(s)} &= \frac{(M_2 s + b_1 + k/s)}{(M_1 s + b_1 + b_2)(M_2 s + b_1 + k/s) - b_1^{\,2}} \\ &= \frac{(M_2 s^2 + b_1 s + k)}{(M_1 s + b_1 + b_2)(M_2 s^2 + b_1 s + k) - b_1^{\,2} s}. \end{aligned} \tag{2.51}$$

If the transfer function in terms of the position $x_1(t)$ is desired, then we have

$$\frac{X_1(s)}{R(s)} = \frac{V_1(s)}{sR(s)} = \frac{G(s)}{s}. \tag{2.52}$$ ■

As an example, let us obtain the transfer function of an important electrical control component, the **DC motor** [8]. A DC motor is used to move loads and is called an **actuator**.

An actuator is a device that provides the motive power to the process.

EXAMPLE 2.5 **Transfer function of the DC motor**

The DC motor is a power actuator device that delivers energy to a load, as shown in Figure 2.18(a); a sketch of a DC motor is shown in Figure 2.18(b). The DC motor converts direct current (DC) electrical energy into rotational mechanical energy. A major fraction of the torque generated in the rotor (armature) of the motor is available to drive an external load. Because of features such as high torque, speed controllability over a wide range, portability, well-behaved speed–torque characteristics, and adaptability to various types of control methods, DC motors are widely used in numerous control applications, including robotic manipulators, tape transport mechanisms, disk drives, machine tools, and servovalve actuators.

The transfer function of the DC motor will be developed for a linear approximation to an actual motor, and second-order effects, such as hysteresis and the voltage drop across the brushes, will be neglected. The input voltage may be applied to the field or armature terminals. The air-gap flux $\phi(t)$ of the motor is proportional to the field current, provided the field is unsaturated, so that

$$\phi(t) = K_f i_f(t). \tag{2.53}$$

The torque developed by the motor is assumed to be related linearly to $\phi(t)$ and the armature current as follows:

$$T_m(t) = K_1 \phi(t) i_a(t) = K_1 K_f i_f(t) i_a(t). \tag{2.54}$$

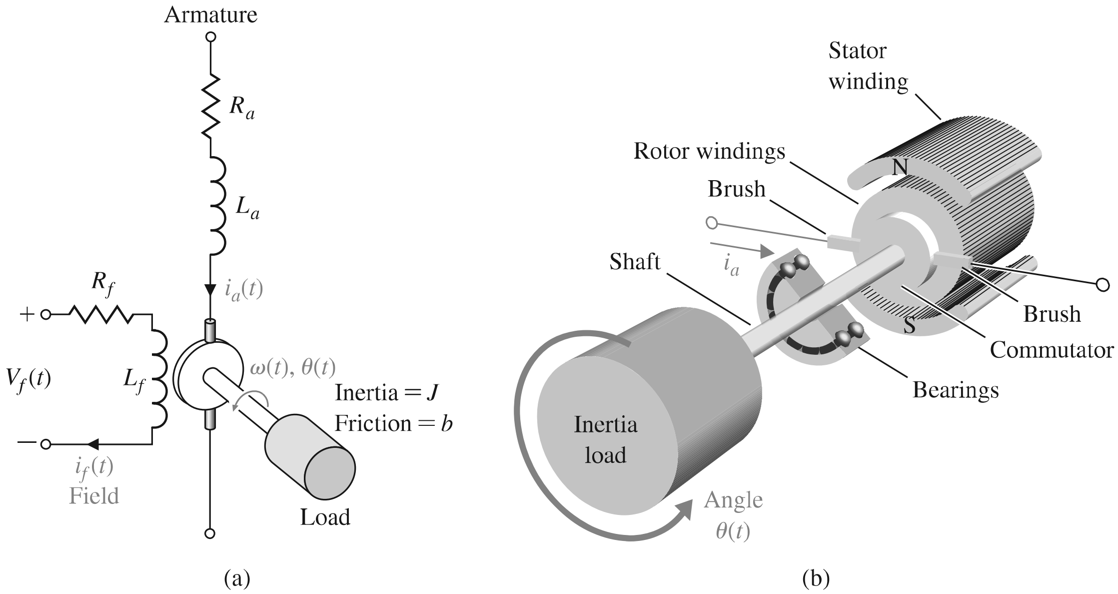


FIGURE 2.18 A DC motor (a) electrical diagram and (b) sketch.

It is clear from Equation (2.54) that, to have a linear system, one current must be maintained constant while the other current becomes the input current. First, we shall consider the field current controlled motor, which provides a substantial power amplification. Then we have, in Laplace transform notation,

$$T_m(s) = (K_1 K_f I_a) I_f(s) = K_m I_f(s), \tag{2.55}$$

where $i_a = I_a$ is a constant armature current, and K_m is defined as the motor constant. The field current is related to the field voltage as

$$V_f(s) = (R_f + L_f s) I_f(s). \tag{2.56}$$

The motor torque $T_m(s)$ is equal to the torque delivered to the load. This relation may be expressed as

$$T_m(s) = T_L(s) + T_d(s), \tag{2.57}$$

where $T_L(s)$ is the load torque and $T_d(s)$ is the disturbance torque, which is often negligible. However, the disturbance torque often must be considered in systems subjected to external forces such as antenna wind-gust forces. The load torque for rotating inertia, as shown in Figure 2.18, is written as

$$T_L(s) = Js^2\theta(s) + bs\theta(s). \tag{2.58}$$

Rearranging Equations (2.55)–(2.57), we have

$$T_L(s) = T_m(s) - T_d(s), \tag{2.59}$$

$$T_m(s) = K_m I_f(s), \tag{2.60}$$

$$I_f(s) = \frac{V_f(s)}{R_f + L_f s}. \tag{2.61}$$

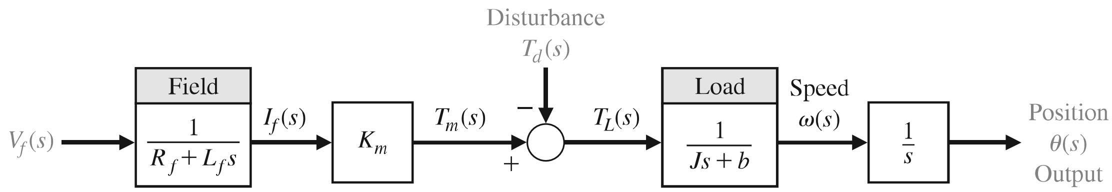


FIGURE 2.19 Block diagram model of field-controlled DC motor.

Therefore, the transfer function of the motor–load combination, with $T_d(s) = 0$, is

$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js + b)(L_f s + R_f)} = \frac{K_m/(JL_f)}{s(s + b/J)(s + R_f/L_f)}. \tag{2.62}$$

The block diagram model of the field-controlled DC motor is shown in Figure 2.19. Alternatively, the transfer function may be written in terms of the time constants of the motor as

$$\frac{\theta(s)}{V_f(s)} = G(s) = \frac{K_m/(bR_f)}{s(\tau_f s + 1)(\tau_L s + 1)}, \tag{2.63}$$

where $\tau_f = L_f/R_f$ and $\tau_L = J/b$. Typically, one finds that $\tau_L > \tau_f$ and often the field time constant may be neglected.

The armature-controlled DC motor uses the armature current i_a as the control variable. The stator field can be established by a field coil and current or a permanent magnet. When a constant field current is established in a field coil, the motor torque is

$$T_m(s) = (K_1 K_f I_f) I_a(s) = K_m I_a(s). \tag{2.64}$$

When a permanent magnet is used, we have

$$T_m(s) = K_m I_a(s),$$

where K_m is a function of the permeability of the magnetic material.

The armature current is related to the input voltage applied to the armature by

$$V_a(s) = (R_a + L_a s) I_a(s) + V_b(s), \tag{2.65}$$

where $V_b(s)$ is the back electromotive-force voltage proportional to the motor speed. Therefore, we have

$$V_b(s) = K_b \omega(s), \tag{2.66}$$

where $\omega(s) = s\theta(s)$ is the transform of the angular speed and the armature current is

$$I_a(s) = \frac{V_a(s) - K_b \omega(s)}{R_a + L_a s}. \tag{2.67}$$

Equations (2.58) and (2.59) represent the load torque, so that

$$T_L(s) = Js^2\theta(s) + bs\theta(s) = T_m(s) - T_d(s). \tag{2.68}$$

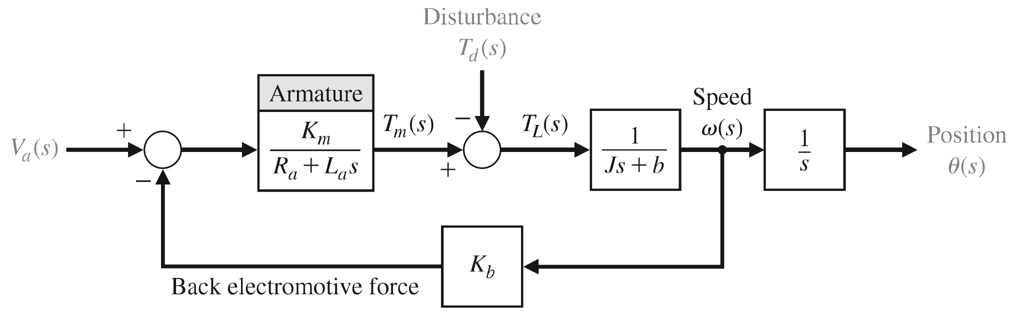


FIGURE 2.20 Armature-controlled DC motor.

The relations for the armature-controlled DC motor are shown schematically in Figure 2.20. Using Equations (2.64), (2.67), and (2.68) or the block diagram, and letting $T_d(s) = 0$, we solve to obtain the transfer function

$$G(s) = \frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[(R_a + L_a s)(Js + b) + K_b K_m]}$$
$$= \frac{K_m}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}. \tag{2.69}$$

However, for many DC motors, the time constant of the armature, $\tau_a = L_a/R_a$, is negligible; therefore,

$$G(s) = \frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[R_a(Js + b) + K_b K_m]} = \frac{K_m/(R_a b + K_b K_m)}{s(\tau_1 s + 1)}, \tag{2.70}$$

where the equivalent time constant $\tau_1 = R_a J/(R_a b + K_b K_m)$.

Note that K_m is equal to K_b. This equality may be shown by considering the steady-state motor operation and the power balance when the rotor resistance is neglected. The power input to the rotor is $K_b\omega(t)i_a(t)$, and the power delivered to the shaft is $T(t)\omega(t)$. In the steady-state condition, the power input is equal to the power delivered to the shaft so that $K_b\omega(t)i_a(t) = T(t)\omega(t)$; since $T(t) = K_m i_a(t)$ (Equation 2.64), we find that $K_b = K_m$. ■

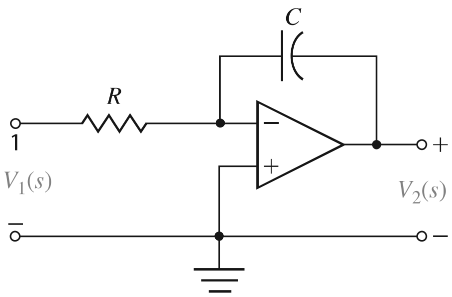
The transfer function concept and approach is very important because it provides the analyst and designer with a useful mathematical model of the system elements. We shall find the transfer function to be a continually valuable aid in the attempt to model dynamic systems. The approach is particularly useful because the s-plane poles and zeros of the transfer function represent the transient response of the system. The transfer functions of several dynamic elements are given in Table 2.4.

In many situations in engineering, the transmission of rotary motion from one shaft to another is a fundamental requirement. For example, the output power of an automobile engine is transferred to the driving wheels by means of the gearbox and differential. The gearbox allows the driver to select different gear ratios depending on the traffic situation, whereas the differential has a fixed ratio. The speed of the engine in this case is not constant, since it is under the control of the driver. Another example is a set of gears that transfer the power at the shaft of an electric motor to the shaft of a rotating antenna. Examples of mechanical converters are gears, chain drives, and belt drives. A commonly used electric converter is the electric transformer. An example of a device that converts rotational motion to linear motion is the rack-and-pinion gear shown in Table 2.4, item 17.

Table 2.4 Transfer Functions of Dynamic Elements and Networks

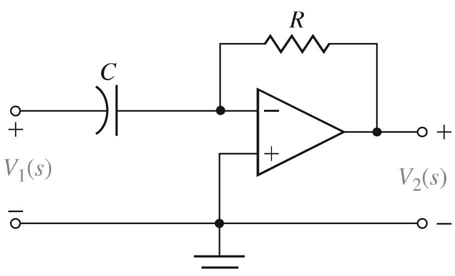
Element or System	$G(s)$

1. Integrating circuit, filter



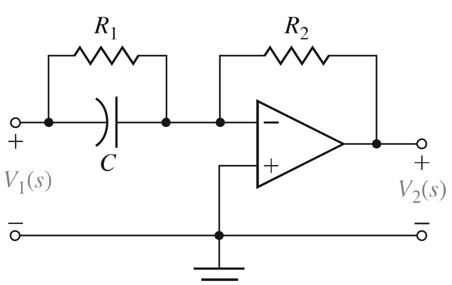
$$\frac{V_2(s)}{V_1(s)} = -\frac{1}{RCs}$$

2. Differentiating circuit



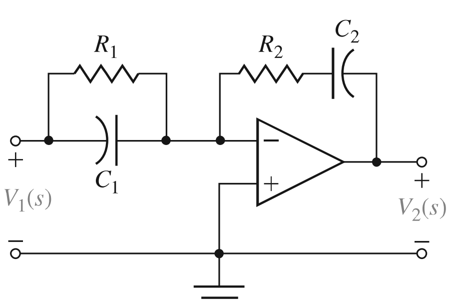
$$\frac{V_2(s)}{V_1(s)} = -RCs$$

3. Differentiating circuit



$$\frac{V_2(s)}{V_1(s)} = -\frac{R_2(R_1Cs + 1)}{R_1}$$

4. Integrating filter



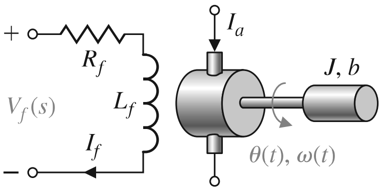
$$\frac{V_2(s)}{V_1(s)} = -\frac{(R_1C_1s + 1)(R_2C_2s + 1)}{R_1C_2s}$$

(*continued*)

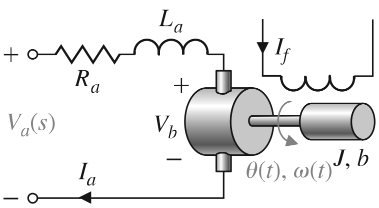
Table 2.4 *(continued)*

Element or System	$G(s)$
5. DC motor, field-controlled, rotational actuator	$\dfrac{\theta(s)}{V_f(s)} = \dfrac{K_m}{s(Js + b)(L_f s + R_f)}$
6. DC motor, armature-controlled, rotational actuator	$\dfrac{\theta(s)}{V_a(s)} = \dfrac{K_m}{s[(R_a + L_a s)(Js + b) + K_b K_m]}$
7. AC motor, two-phase control field, rotational actuator	$\dfrac{\theta(s)}{V_c(s)} = \dfrac{K_m}{s(\tau s + 1)}$ $\tau = J/(b - m)$ $m =$ slope of linearized torque-speed curve (normally negative)
8. Rotary Amplifier (Amplidyne)	$\dfrac{V_o(s)}{V_c(s)} = \dfrac{K/(R_c R_q)}{(s\tau_c + 1)(s\tau_q + 1)}$ $\tau_c = L_c/R_c, \quad \tau_q = L_q/R_q$ for the unloaded case, $i_d \approx 0$, $\tau_c \approx \tau_q$, $0.05\ \text{s} < \tau_c < 0.5\ \text{s}$ $V_q, V_{34} = V_d$
9. Hydraulic actuator [9, 10]	$\dfrac{Y(s)}{X(s)} = \dfrac{K}{s(Ms + B)}$ $K = \dfrac{Ak_x}{k_p}, \quad B = \left(b + \dfrac{A^2}{k_p}\right)$ $k_x = \left.\dfrac{\partial g}{\partial x}\right\vert_{x_0, P_0}, \quad k_p = \left.\dfrac{\partial g}{\partial P}\right\vert_{x_0, P_0},$ $g = g(x, P) =$ flow $A =$ area of piston $M =$ load mass $b =$ load friction

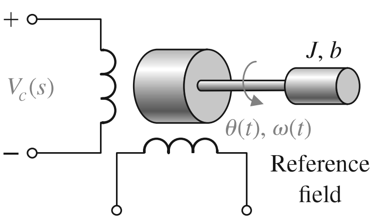
5. DC motor, field-controlled, rotational actuator



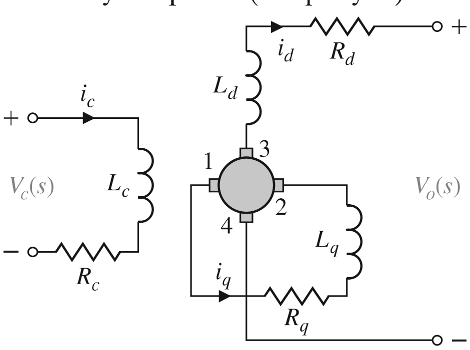
6. DC motor, armature-controlled, rotational actuator



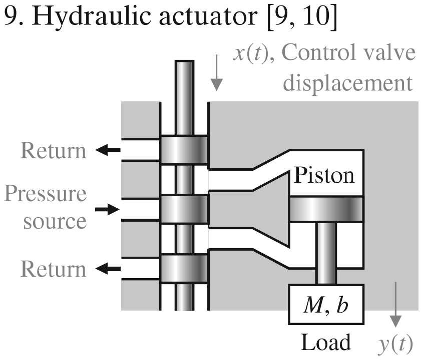
7. AC motor, two-phase control field, rotational actuator



8. Rotary Amplifier (Amplidyne)



9. Hydraulic actuator [9, 10]

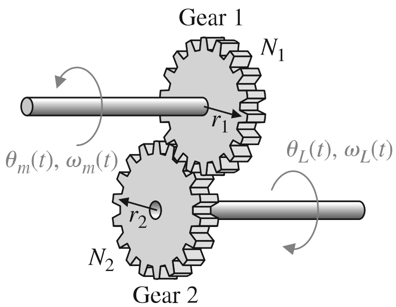


(*continued*)

Table 2.4 (continued)

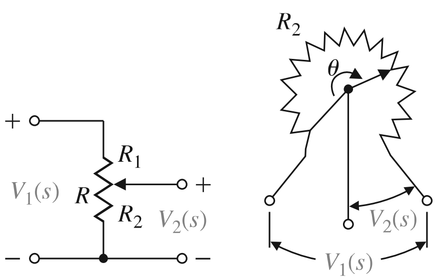
Element or System	$G(s)$

10. Gear train, rotational transformer



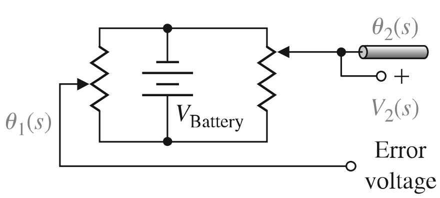
$$\text{Gear ratio} = n = \frac{N_1}{N_2}$$
$$N_2\theta_L(t) = N_1\theta_m(t), \qquad \theta_L(t) = n\theta_m(t)$$
$$\omega_L(t) = n\omega_m(t)$$

11. Potentiometer, voltage control



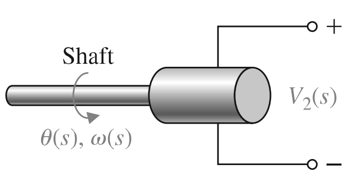
$$\frac{V_2(s)}{V_1(s)} = \frac{R_2}{R} = \frac{R_2}{R_1 + R_2}$$
$$\frac{R_2}{R} = \frac{\theta}{\theta_{max}}$$

12. Potentiometer, error detector bridge



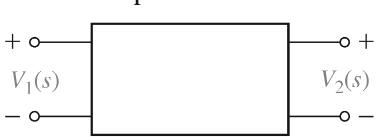
$$V_2(s) = k_s(\theta_1(s) - \theta_2(s))$$
$$V_2(s) = k_s\theta_{error}(s)$$
$$k_s = \frac{V_{Battery}}{\theta_{max}}$$

13. Tachometer, velocity sensor



$$V_2(s) = K_t\omega(s) = K_t s\theta(s)$$
$$K_t = \text{constant}$$

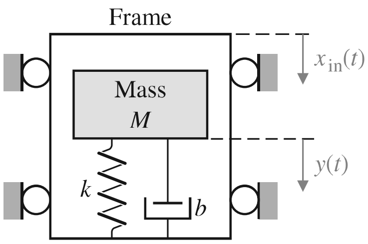
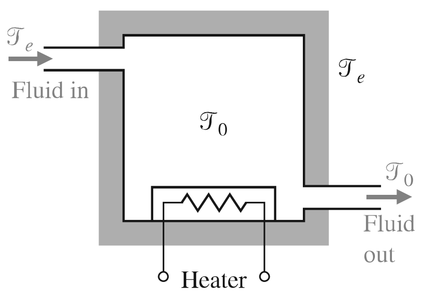
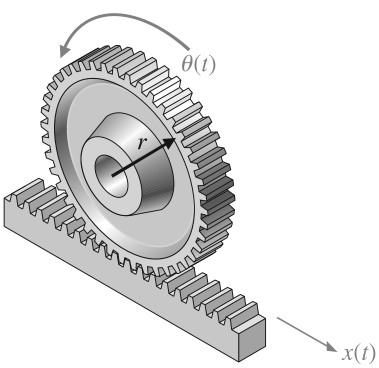
14. DC amplifier



$$\frac{V_2(s)}{V_1(s)} = \frac{k_a}{s\tau + 1}$$
$$R_o = \text{output resistance}$$
$$C_o = \text{output capacitance}$$
$$\tau = R_oC_o, \tau \ll 1\text{s}$$
and is often negligible for controller amplifier

(continued)

Table 2.4 ***(continued)***

Element or System	$G(s)$
15. Accelerometer, acceleration sensor  	$x_o(t) = y(t) - x_{in}(t)$, $\dfrac{X_o(s)}{X_{in}(s)} = \dfrac{-s^2}{s^2 + (b/M)s + k/M}$ For low-frequency oscillations, where $\omega < \omega_n$, $\dfrac{X_o(j\omega)}{X_{in}(j\omega)} \simeq \dfrac{\omega^2}{k/M}$
16. Thermal heating system  	$\dfrac{\mathcal{T}(s)}{q(s)} = \dfrac{1}{C_t s + (QS + 1/R_t)}$, where $\mathcal{T} = \mathcal{T}_o - \mathcal{T}_e =$ temperature difference due to thermal process $C_t =$ thermal capacitance $Q =$ fluid flow rate $=$ constant $S =$ specific heat of water $R_t =$ thermal resistance of insulation $q(s) =$ transform of rate of heat flow of heating element
17. Rack and pinion  	$x(t) = r\theta(t)$ converts radial motion to linear motion

2.6 BLOCK DIAGRAM MODELS

The dynamic systems that comprise feedback control systems are typically represented mathematically by a set of simultaneous differential equations. As we have noted in the previous sections, the Laplace transformation reduces the problem to the solution of a set of linear algebraic equations. Since control systems are concerned with the control of specific variables, the controlled variables must relate to the controlling variables. This relationship is typically represented by the transfer function of the subsystem relating the input and output variables. Therefore, one can correctly assume that the transfer function is an important relation for control engineering.