CHAPTER

3 State Variable Models

PREVIEW

In this chapter, we consider system modeling using time-domain methods. We consider physical systems described by an nth-order ordinary differential equation. Utilizing a (nonunique) set of variables, known as state variables, we can obtain a set of first-order differential equations. We group these first-order equations using a compact matrix notation in a model known as the state variable model. The relationship between signal-flow graph models and state variable models will be investigated. Several interesting physical systems, including a space station and a printer belt drive, are presented and analyzed. The chapter concludes with the development of a state variable model for the Sequential Design Example: Disk Drive Read System.

DESIRED OUTCOMES

Upon completion of Chapter 3, students should:

- Understand state variables, state differential equations, and output equations.
- Recognize that state variable models can describe the dynamic behavior of physical systems and can be represented by block diagrams and signal flow graphs.
- Know how to obtain the transfer function model from a state variable model, and vice versa.
- Be aware of solution methods for state variable models and the role of the state transition matrix in obtaining the time responses.
- Understand the important role of state variable modeling in control system design.

3.1 INTRODUCTION

In the preceding chapter, we developed and studied several useful approaches to the analysis and design of feedback systems. The Laplace transform was used to transform the differential equations representing the system to an algebraic equation expressed in terms of the complex variable s. Using this algebraic equation, we were able to obtain a transfer function representation of the input–output relationship.

In this chapter, we represent system models utilizing a set of ordinary differential equations in a convenient matrix-vector form. The **time domain** is the mathematical domain that incorporates the description of the system, including the inputs, outputs, and response, in terms of time, t. Linear time-invariant single-input, single-output models, can be represented via state variable models. Powerful mathematical concepts from linear algebra and matrix-vector analysis, as well as effective computational tools, can be utilized in the design and analysis of control systems in the time domain. Also, these time domain design and analysis methods are readily extended to nonlinear, time-varying, and multiple input–output systems. As we shall see, mathematical models of linear time-invariant physical systems can be represented in either the frequency domain or the time domain. The time domain design techniques are another tool in the designer's toolbox.

A time-varying control system is a system in which one or more of the parameters of the system may vary as a function of time.

For example, the mass of an airplane varies as a function of time as the fuel is expended during flight. A multivariable system is a system with several input and output signals.

The time-domain representation of control systems is an essential basis for modern control theory and system optimization. In later chapters, we will have an opportunity to design optimum control systems by utilizing time-domain methods. In this chapter, we develop the time-domain representation of control systems and illustrate several methods for the solution of the system time response.

3.2 THE STATE VARIABLES OF A DYNAMIC SYSTEM

The time-domain analysis and design of control systems uses the concept of the state of a system [1–3, 5].

The state of a system is a set of variables whose values, together with the input signals and the equations describing the dynamics, will provide the future state and output of the system.

For a dynamic system, the state of a system is described in terms of a set of **state variables** $\boldsymbol{x}(t) = (x_1(t), x_2(t),..., x_n(t))$. The state variables are those variables that determine the future behavior of a system when the present state of the

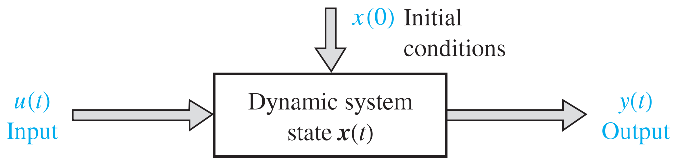

FIGURE 3.1
Dynamic system.

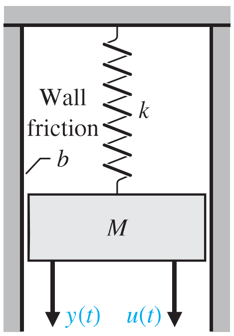

FIGURE 3.2
A spring-mass-damper system.

system and the inputs are known. Consider the system shown in Figure 3.1, where $y(t)$ is the output signal and $u(t)$ is the input signal. A set of state variables $\boldsymbol{x}(t) = (x_1(t), x_2(t),\ldots, x_n(t))$ for the system shown in the figure is a set such that knowledge of the initial values of the state variables $\boldsymbol{x}(t_0) = (x_1(t_0), x_2(t_0),\ldots, x_n(t_0))$ at the initial time t_0, and of the input signal $u(t)$ for $t \geq t_0$, suffices to determine the future values of the outputs and state variables [2].

The concept of a set of state variables that represent a dynamic system can be illustrated in terms of the spring-mass-damper system shown in Figure 3.2. The number of state variables chosen to represent this system should be as small as possible in order to avoid redundant state variables. A set of state variables sufficient to describe this system includes the position and the velocity of the mass. Therefore, we will define a set of state variables as $\boldsymbol{x}(t) = (x_1(t), x_2(t))$, where

$$x_1(t) = y(t) \quad \text{and} \quad x_2(t) = \frac{dy(t)}{dt}.$$

The differential equation describes the behavior of the system and can be written as

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = u(t). \tag{3.1}$$

To write Equation (3.1) in terms of the state variables, we substitute the state variables as already defined and obtain

$$M\frac{dx_2(t)}{dt} + bx_2(t) + kx_1(t) = u(t). \tag{3.2}$$

Therefore, we can write the equations that describe the behavior of the spring-mass-damper system as the set of two first-order differential equations

$$\frac{dx_1(t)}{dt} = x_2(t) \tag{3.3}$$

and

$$\frac{dx_2(t)}{dt} = \frac{-b}{M}x_2(t) - \frac{k}{M}x_1(t) + \frac{1}{M}u(t). \tag{3.4}$$

This set of differential equations describes the behavior of the state of the system in terms of the rate of change of each state variable.

As another example of the state variable characterization of a system, consider the *RLC* circuit shown in Figure 3.3. The state of this system can be described by a set of state variables $\boldsymbol{x}(t) = (x_1(t), x_2(t))$, where $x_1(t)$ is the capacitor voltage $v_c(t)$

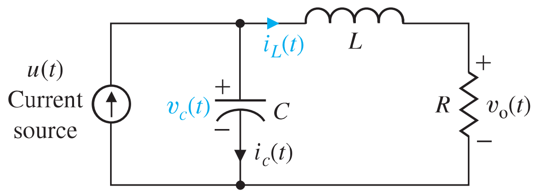

FIGURE 3.3
An *RLC* circuit.

and $x_2(t)$ is the inductor current $i_L(t)$. This choice of state variables is intuitively satisfactory because the stored energy of the network can be described in terms of these variables as

$$\mathscr{E} = \frac{1}{2}Li_L^2(t) + \frac{1}{2}Cv_c^2(t). \tag{3.5}$$

Therefore $x_1(t_0)$ and $x_2(t_0)$ provide the total initial energy of the network and the state of the system at $t = t_0$. For a passive *RLC* network, the number of state variables required is equal to the number of independent energy-storage elements. Utilizing Kirchhoff's current law at the junction, we obtain a first-order differential equation by describing the rate of change of capacitor voltage as

$$i_c(t) = C\frac{dv_c(t)}{dt} = +u(t) - i_L(t). \tag{3.6}$$

Kirchhoff's voltage law for the right-hand loop provides the equation describing the rate of change of inductor current as

$$L\frac{di_L(t)}{dt} = -Ri_L(t) + v_c(t). \tag{3.7}$$

The output of this system is represented by the linear algebraic equation

$$v_o(t) = Ri_L(t).$$

We can rewrite Equations (3.6) and (3.7) as a set of two first-order differential equations in terms of the state variables $x_1(t)$ and $x_2(t)$ as

$$\frac{dx_1(t)}{dt} = -\frac{1}{C}x_2(t) + \frac{1}{C}u(t), \tag{3.8}$$

and

$$\frac{dx_2(t)}{dt} = +\frac{1}{L}x_1(t) - \frac{R}{L}x_2(t). \tag{3.9}$$

The output signal is then

$$y_1(t) = v_o(t) = Rx_2(t). \tag{3.10}$$

Utilizing Equations (3.8) and (3.9) and the initial conditions of the network represented by $\boldsymbol{x}(t) = (x_1(t_0), x_2(t_0))$, we can determine the future behavior.

The state variables that describe a system are not a unique set, and several alternative sets of state variables can be chosen. For example, for a second-order system, such as the spring-mass-damper or *RLC* circuit, the state variables may be any two independent linear combinations of $x_1(t)$ and $x_2(t)$. For the *RLC* circuit, we might choose the set of state variables as the two voltages, $v_c(t)$ and $v_L(t)$, where $v_L(t)$ is the voltage drop across the inductor. Then the new state variables, $x_1^*(t)$ and $x_2^*(t)$, are related to the old state variables, $x_1(t)$ and $x_2(t)$, as

$$x_1^*(t) = v_c(t) = x_1(t), \tag{3.11}$$

and

$$x_2^*(t) = v_L(t) = v_c(t) - Ri_L(t) = x_1(t) - Rx_2(t). \tag{3.12}$$

Equation (3.12) represents the relation between the inductor voltage and the former state variables $v_c(t)$ and $i_L(t)$. In a typical system, there are several choices of a set of state variables that specify the energy stored in a system and therefore adequately describe the dynamics of the system. It is usual to choose a set of state variables that can be readily measured.

An alternative approach to developing a model of a device is the use of the bond graph. Bond graphs can be used for electrical, mechanical, hydraulic, and thermal devices or systems as well as for combinations of various types of elements. Bond graphs produce a set of equations in the state variable form [7].

The state variables of a system characterize the dynamic behavior of a system. The engineer's interest is primarily in physical systems, where the variables typically are voltages, currents, velocities, positions, pressures, temperatures, and similar physical variables. However, the concept of system state is also useful in analyzing biological, social, and economic systems. For these systems, the concept of state is extended beyond the concept of the current configuration of a physical system to the broader viewpoint of variables that will be capable of describing the future behavior of the system.

3.3 THE STATE DIFFERENTIAL EQUATION

The response of a system is described by the set of first-order differential equations written in terms of the state variables $(x_1(t), x_2(t), \ldots, x_n(t))$ and the inputs $(u_1(t), u_2(t), \ldots, u_m(t))$. A set of linear first-order differential equations can be written in general form as

$$\begin{aligned}
\dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_{11}u_1(t) + \cdots + b_{1m}u_m(t),\\
\dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_{21}u_1(t) + \cdots + b_{2m}u_m(t),\\
&\vdots\\
\dot{x}_n(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + b_{n1}u_1(t) + \cdots + b_{nm}u_m(t),
\end{aligned} \tag{3.13}$$

where $\dot{x}(t) = dx(t)/dt$. Thus, this set of simultaneous differential equations can be written in matrix form as follows [2, 5]:

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \vdots & \cdots & \vdots \\ a_{n1} & a_{n2}\cdots & a_{nn} \end{bmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{bmatrix} b_{11}\cdots & b_{1m} \\ \vdots & \vdots \\ b_{n1}\cdots & b_{nm} \end{bmatrix}\begin{pmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{pmatrix}. \tag{3.14}$$

The column matrix consisting of the state variables is called the **state vector** and is written as

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \tag{3.15}$$

where the boldface indicates a vector. The vector of input signals is defined as $\mathbf{u}(t)$. Then the system can be represented by the compact notation of the **state differential equation** as

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t).} \tag{3.16}$$

Equation (3.16) is also commonly called the state equation.

The matrix $\mathbf{A}$ is an $n \times n$ square matrix, and $\mathbf{B}$ is an $n \times m$ matrix.† The state differential equation relates the rate of change of the state of the system to the state of the system and the input signals. In general, the outputs of a linear system can be related to the state variables and the input signals by the **output equation**

$$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),} \tag{3.17}$$

where $y(t)$ is the set of output signals expressed in column vector form. The **state-space representation** (or state-variable representation) comprises the state differential equation and the output equation.

We use Equations (3.8) and (3.9) to obtain the state variable differential equation for the RLC of Figure 3.3 as

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & \dfrac{-1}{C} \\ \dfrac{1}{L} & \dfrac{-R}{L} \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} \dfrac{1}{C} \\ 0 \end{bmatrix}u(t) \tag{3.18}$$

†Boldfaced lowercase letters denote vector quantities and boldfaced uppercase letters denote matrices. For an introduction to matrices and elementary matrix operations, refer to the MCS website and references [1] and [2].

and the output as

$$y(t) = [0 \quad R]\mathbf{x}(t). \tag{3.19}$$

When $R = 3$, $L = 1$, and $C = 1/2$, we have

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix}u(t)$$

and

$$y(t) = [0 \quad 3]\mathbf{x}(t).$$

The solution of the state differential equation can be obtained in a manner similar to the method for solving a first-order differential equation. Consider the first-order differential equation

$$\dot{x}(t) = ax(t) + bu(t), \tag{3.20}$$

where $x(t)$ and $u(t)$ are scalar functions of time. We expect an exponential solution of the form e^{at}. Taking the Laplace transform of Equation (3.20), we have

$$sX(s) - x(0) = aX(s) + bU(s);$$

therefore,

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s). \tag{3.21}$$

The inverse Laplace transform of Equation (3.21) is

$$x(t) = e^{at}x(0) + \int_0^t e^{+a(t-\tau)}bu(\tau)d\tau. \tag{3.22}$$

We expect the solution of the general state differential equation to be similar to Equation (3.22) and to be of exponential form. The **matrix exponential function** is defined as

$$\boxed{e^{\mathbf{A}t} = \exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots + \frac{\mathbf{A}^kt^k}{k!} + \cdots,} \tag{3.23}$$

which converges for all finite t and any $\mathbf{A}$[2]. Then the solution of the state differential equation is found to be

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0) + \int_0^t \exp[\mathbf{A}(t-\tau)]\mathbf{B}\mathbf{u}(\tau)d\tau. \tag{3.24}$$

Equation (3.24) may be verified by taking the Laplace transform of Equation (3.16) and rearranging to obtain

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s), \tag{3.25}$$

where we note that $[s\mathbf{I} - \mathbf{A}]^{-1} = \mathbf{\Phi}(s)$ is the Laplace transform of $\mathbf{\Phi}(t)= \exp(\mathbf{A}t)$. Taking the inverse Laplace transform of Equation (3.25) and noting that the second term on the right-hand side involves the product $\mathbf{\Phi}(s)\mathbf{B}\mathbf{U}(s)$, we obtain Equation (3.24). The matrix exponential function describes the unforced response of the system and is called the **fundamental** or **state transition matrix** $\mathbf{\Phi}(t)$. Thus, Equation (3.24) can be written as

$$\boxed{\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau.} \tag{3.26}$$

The solution to the unforced system (that is, when $\mathbf{u}(t) = 0$) is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{bmatrix} \phi_{11}(t) & \cdots & \phi_{1n}(t) \\ \phi_{21}(t) & \cdots & \phi_{2n}(t) \\ \vdots & & \vdots \\ \phi_{n1}(t) & \cdots & \phi_{nn}(t) \end{bmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix}. \tag{3.27}$$

We note that to determine the state transition matrix, all initial conditions are set to 0 except for one state variable, and the output of each state variable is evaluated. That is, the term $\phi_{ij}(t)$ is the response of the ith state variable due to an initial condition on the jth state variable when there are zero initial conditions on all the other variables. We shall use this relationship between the initial conditions and the state variables to evaluate the coefficients of the transition matrix in a later section. However, first we shall develop several suitable signal-flow state models of systems and investigate the stability of the systems by utilizing these flow graphs.

EXAMPLE 3.1 Two rolling carts

Consider the system shown in Figure 3.4. The variables of interest are noted on the figure and defined as: M_1, M_2 = mass of carts, $p(t), q(t)$ = position of carts, $u(t)$ = external force acting on system, k_1, k_2 = spring constants, and

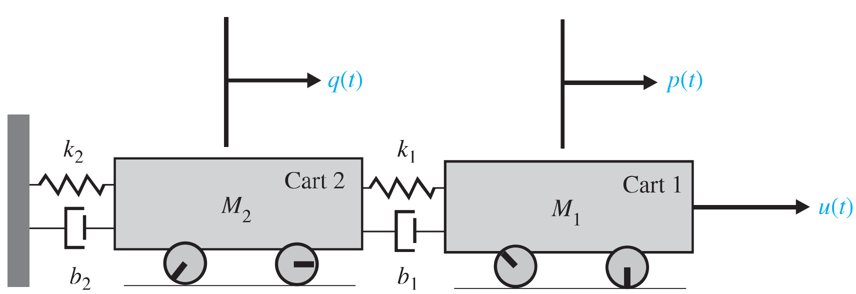

FIGURE 3.4 Two rolling carts attached with springs and dampers.

b_1, b_2 = damping coefficients. The free-body diagram of mass M_1 is shown in Figure 3.5(b), where $\dot{p}(t), \dot{q}(t)$ = velocity of M_1 and M_2, respectively. We assume that the carts have negligible rolling friction. We consider any existing rolling friction to be lumped into the damping coefficients, b_1 and b_2.

Now, given the free-body diagram with forces and directions appropriately applied, we use Newton's second law (sum of the forces equals mass of the object multiplied by its acceleration) to obtain the equations of motion—one equation for each mass. For mass M_1 we have

$$M_1\ddot{p}(t) + b_1\dot{p}(t) + k_1 p(t) = u(t) + k_1 q(t) + b_1\dot{q}(t), \tag{3.28}$$

where

$$\ddot{p}(t), \ddot{q}(t) = \text{acceleration of } M_1 \text{ and } M_2, \text{ respectively.}$$

Similarly, for mass M_2 in Figure 3.5(a), we have

$$M_2\ddot{q}(t) + (k_1 + k_2)q(t) + (b_1 + b_2)\dot{q}(t) = k_1 p(t) + b_1\dot{p}(t). \tag{3.29}$$

We now have a model given by the two second-order ordinary differential equations in Equations (3.28) and (3.29). We can start developing a state-space model by defining

$$x_1(t) = p(t),$$
$$x_2(t) = q(t).$$

We could have alternatively defined $x_1(t) = q(t)$ and $x_2(t) = p(t)$. The state-space model is not unique. Denoting the derivatives of $x_1(t)$ and $x_2(t)$ as $x_3(t)$ and $x_4(t)$, respectively, it follows that

$$x_3(t) = \dot{x}_1(t) = \dot{p}(t), \tag{3.30}$$
$$x_4(t) = \dot{x}_2(t) = \dot{q}(t). \tag{3.31}$$

Taking the derivative of $x_3(t)$ and $x_4(t)$ yields, respectively,

$$\dot{x}_3(t) = \ddot{p}(t) = -\frac{b_1}{M_1}\dot{p}(t) - \frac{k_1}{M_1}p(t) + \frac{1}{M_1}u(t) + \frac{k_1}{M_1}q(t) + \frac{b_1}{M_1}\dot{q}(t), \tag{3.32}$$

$$\dot{x}_4(t) = \ddot{q}(t) = -\frac{k_1 + k_2}{M_2}q(t) - \frac{b_1 + b_2}{M_2}\dot{q}(t) + \frac{k_1}{M_2}p(t) + \frac{b_1}{M_2}\dot{p}(t), \tag{3.33}$$

where we use the relationship for $\ddot{p}(t)$ given in Equation (3.28) and the relationship for $\ddot{q}(t)$ given in Equation (3.29). But $\dot{p}(t) = x_3(t)$ and $\dot{q}(t) = x_4(t)$, so Equation (3.32) can be written as

$$\dot{x}_3(t) = -\frac{k_1}{M_1}x_1(t) + \frac{k_1}{M_1}x_2(t) - \frac{b_1}{M_1}x_3(t) + \frac{b_1}{M_1}x_4(t) + \frac{1}{M_1}u(t) \tag{3.34}$$

and Equation (3.33) as

$$\dot{x}_4(t) = \frac{k_1}{M_2}x_1(t) - \frac{k_1 + k_2}{M_2}x_2(t) + \frac{b_1}{M_2}x_3(t) - \frac{b_1 + b_2}{M_2}x_4(t). \tag{3.35}$$

In matrix form, Equations (3.30), (3.31), (3.34), and (3.35) can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = \begin{pmatrix} p(t) \\ q(t) \\ \dot{p}(t) \\ \dot{q}(t) \end{pmatrix},$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{M_1} & \frac{k_1}{M_1} & -\frac{b_1}{M_1} & \frac{b_1}{M_1} \\ \frac{k_1}{M_2} & -\frac{k_1 + k_2}{M_2} & \frac{b_1}{M_2} & -\frac{b_1 + b_2}{M_2} \end{bmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ 0 \end{bmatrix},$$

and $u(t)$ is the external force acting on the system. If we choose $p(t)$ as the output, then

$$y(t) = [1 \quad 0 \quad 0 \quad 0]\mathbf{x}(t) = \mathbf{C}\mathbf{x}(t).$$

Suppose that the two rolling carts have the following parameter values: $k_1 = 150$ N/m; $k_2 = 700$ N/m; $b_1 = 15$ N s/m; $b_2 = 30$ N s/m; $M_1 = 5$ kg; and $M_2 = 20$ kg. The response of the two rolling cart system is shown in Figure 3.6 when the initial conditions are $p(0) = 10$ cm, $q(0) = 0$, and $\dot{p}(0) = \dot{q}(0) = 0$ and there is no input driving force, that is, $u(t) = 0$.

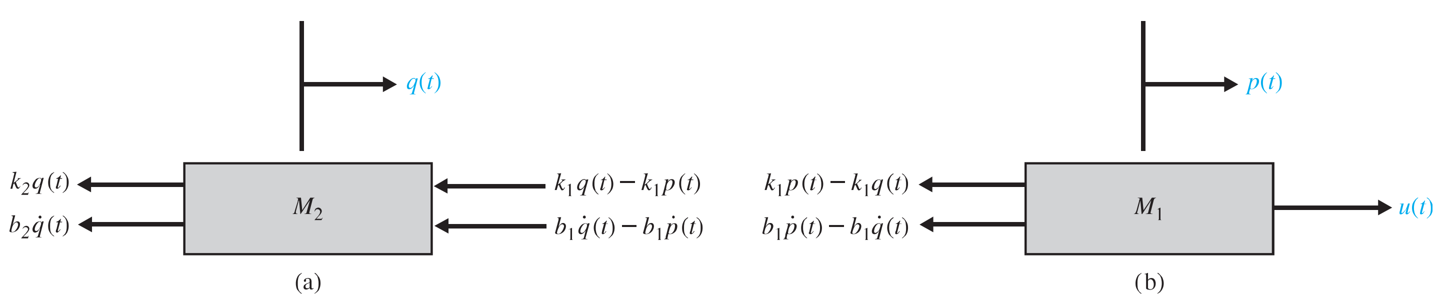

FIGURE 3.5 Free-body diagrams of the two rolling carts. (a) Cart 2; (b) Cart 1.

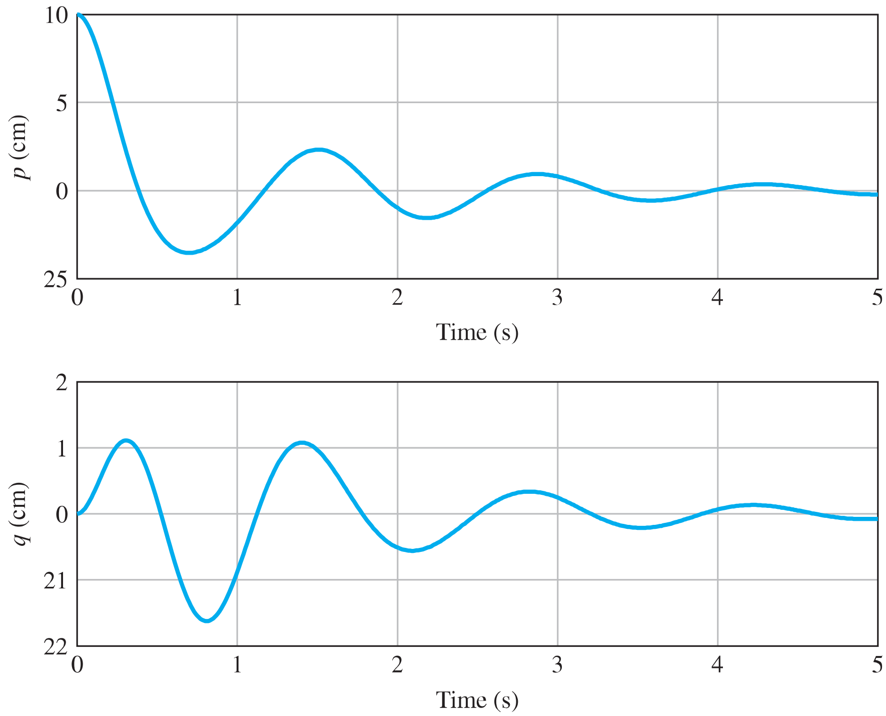

FIGURE 3.6 Initial condition response of the two cart system.

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3.4 SIGNAL-FLOW GRAPH AND BLOCK DIAGRAM MODELS

The state of a system describes the dynamic behavior where the dynamics of the system are represented by a set of first-order differential equations. Alternatively, the dynamics of the system can be represented by a state differential equation as in Equation (3.16). In either case, it is useful to develop a graphical model of the system and use this model to relate the state variable concept to the familiar transfer function representation. The graphical model can be represented via signal-flow graphs or block diagrams.

As we have learned in previous chapters, a system can be meaningfully described by an input–output relationship, the transfer function $G(s)$. For example, if we are interested in the relation between the output voltage and the input voltage of the network of Figure 3.3, we can obtain the transfer function

$$G(s) = \frac{V_0(s)}{U(s)}.$$

The transfer function for the RLC network of Figure 3.3 is of the form

$$G(s) = \frac{V_0(s)}{U(s)} = \frac{\alpha}{s^2 + \beta s + \gamma}, \tag{3.36}$$