Non-backtracking Random Walks

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SCR 2020
Provo, UT
February 29,2020

Problem

Is the mixing rate of a non-backtracking random walk faster than that of a simple random walk on a graph?

Mixing Rate

Definition

Let P be the transition probability matrix of a graph G and let π be the stationary distribution of a random walk on G. Then the **mixing rate** of a random walk on a graph G is defined pointwise by

$$\rho = \limsup_{t\to\infty} \max_{u,v} |P^t(u,v) - \pi(v)|^{1/t}.$$

Mixing Rate and Eigenvalues

Theorem (Lovasz 1993)

Let G be a connected non-bipartite graph with transition probability matrix P. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the eigenvalues of P. Then the mixing rate of G is $\rho = \max\{|\mu_2|, |\mu_n|\}$.

Hashimoto Matrix

Definition

The **Hashimoto Matrix** of a graph G is

$$B((u,v),(x,y)) = \begin{cases} 1 & v = x \text{ and } u \neq y \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Ihara's Theorem

Theorem (Ihara)

Given a graph G with n vertices and m edges, define B to be the Hashimoto matrix of G. Let A be the adjacency matrix of G and let D be the diagonal degree matrix. Then

$$\text{det}(I - \mu B) = (1 - u^2)^{m-n}\text{det}(I - \mu A + \mu^2(D - I)).$$

Thus, the eigenvalues of B are μ.

The Matrix K

From Krzakala et. al. (2013), the matrix K of a graph G

$$K = \begin{pmatrix} A & D - I \\ -I & \mathbf{0} \end{pmatrix}$$

is a invariant subspace of B.

The Matrix K

Define the following matrices:

$$S((u,v),x) = \begin{cases} 1 & v = x \\ 0 & \text{otherwise} \end{cases}$$

$$T(x,(u,v)) = \begin{cases} 1 & u = x \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$B \begin{pmatrix} S & T^T \end{pmatrix} v = \begin{pmatrix} S & T^T \end{pmatrix} K v.$$

If v is an eigenvector of K with eigenvalue μ, then

$$B \begin{pmatrix} S & T^T \end{pmatrix} v = \mu \begin{pmatrix} S & T^T \end{pmatrix} v.$$

Eigenvalues of K

If $\begin{pmatrix} x & y \end{pmatrix}^T$ is an eigenvector of K,

$$\begin{pmatrix} A & D-I \\ -I & \mathbf{0} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mu \begin{pmatrix} x \\ y \end{pmatrix},$$

then $x = -\mu y$. This means that $\mu^2 y - \mu A y + (D-I)y = \mathbf{0}$ for all $\mu \in \sigma(K)$.

Ihara's Theorem

Theorem (Ihara, 1966)

Given a graph G and the associated Hasimoto matrix B, adjacency matrix A, and diagonal degree matrix D, the following is true:

$$\textit{det}(I - \mu B) = (1 - u^2)^{m-n}\textit{det}(I - uA - u^2(D - I)).$$

So the eigenvalues of B are the $\frac{1}{u}$.

Relationship Between A and K

Let $\mathbf{x}$ be the eigenvector associated with λ_2 (the second largest eigenvalue) of A. Let μ_2 be the second largest eigenvalue of K such that $K\begin{pmatrix}-\mu_2\mathbf{y} & \mathbf{y}\end{pmatrix}^T = \mu_2\begin{pmatrix}-\mu_2\mathbf{y} & \mathbf{y}\end{pmatrix}^T$. Scale $\mathbf{x}$ such that $\mathbf{x}^T\mathbf{y} = 1$.

Then,

$$\begin{aligned} \mu_2^2\mathbf{x}^T\mathbf{y} - \mu_2\mathbf{x}^TA\mathbf{y} + \mathbf{x}^T(D-I)\mathbf{y} &= 0, \\ \mu_2^2 - \mu_2\lambda_2 + \mathbf{x}^T(D-I)\mathbf{y} &= 0. \end{aligned}$$

Solving for μ_2,

$$\mu_2 = \frac{\lambda_2 \pm \sqrt{\lambda_2^2 - 4\mathbf{x}^T(D-I)\mathbf{y}}}{2}.$$

Questions That Arise

$$\mu = \frac{\lambda \pm \sqrt{\lambda^2 - 4\mathbf{x}^T(D-I)\mathbf{y}}}{2}$$

- How are μ and λ related?
- Is it $+$ or $-$?
- Can we get all eigenvalues μ?
- How to convert μ to ρ?

Relating μ and λ

Assume that $\lambda \geq 2\sqrt{\mathbf{x}^T(D-I)\mathbf{y}}$:

$$\mu = \frac{\lambda + \sqrt{\lambda^2 - 4\mathbf{x}^T(D-I)\mathbf{y}}}{2} \leq \frac{\lambda + \lambda}{2} = \lambda$$
$$\mu = \frac{\lambda - \sqrt{\lambda^2 - 4\mathbf{x}^T(D-I)\mathbf{y}}}{2} \leq \frac{\lambda}{2}.$$

In either case $\mu \leq \lambda$.

Bipartite Graphs and K

INSERT PICTURE OF BIPARTITE GRAPH HERE

Bipartite Graphs and K

Theorem

The following are equivalent for a graph G:

1. G *is bipartite,*
2. *The spectrum of* B *is symmetric,*
3. *The spectrum of* K *is symmetric,*
4. -1 *is an eigenvalue of* K.

Bipartite $\Rightarrow \sigma(B)$ Is Symmetric

If G is bipartite, then

$$B = \begin{pmatrix} 0 & B_2 \\ B_1 & 0 \end{pmatrix}.$$

Let $\lambda \in \sigma(B)$. Then,

$$B \begin{pmatrix} x & y \end{pmatrix}^T = \lambda \begin{pmatrix} x & y \end{pmatrix}^T \tag{2}$$

$$\begin{pmatrix} B_2 y & B_1 x \end{pmatrix}^T = \lambda \begin{pmatrix} x & y \end{pmatrix}^T. \tag{3}$$

This means that

$$B \begin{pmatrix} x & -y \end{pmatrix}^T = \begin{pmatrix} -B_2 y & B_1 x \end{pmatrix}^T \tag{4}$$

$$= \begin{pmatrix} -\lambda x & \lambda y \end{pmatrix}^T \tag{5}$$

$$= -\lambda \begin{pmatrix} x & -y \end{pmatrix}^T. \tag{6}$$

So $-\lambda \in \sigma(B)$.

$\sigma(B)$ Is Symmetric $\Rightarrow \sigma(K)$ Is Symmetric

Theorem (Ihara, 1966)

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$\sigma(K)$ is symmetric $\Rightarrow \sigma(B)$ is symmetric.

Theorem (Ihara, 1966)

Given a graph G and the associated Hasimoto matrix B, adjacency matrix A, and diagonal degree matrix D, the following is true:

$$\det(I - \mu B) = (1 - u^2)^{m-n} \det(I - uA - u^2(D - I)).$$

$\sigma(B)$ is symmetric $\Rightarrow G$ is bipartite

Consider B as the adjacency matrix of its own graph.

Paths: Cospectrality and Fractional Cospectrality

When can two vertices of a path be cospectral or fractionally cospectral?

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Lemma

Let u, v be vertices of a path on n vertices P_n

- *u and v are cospectral if and only if $\{u, v\} = \{i, n-i\}$.*

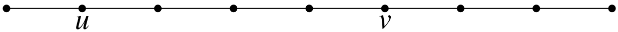
Paths: Cospectrality and Fractional Cospectrality

When can two vertices of a path be cospectral or fractionally cospectral?

Lemma

Let u, v be vertices of a path on n vertices P_n

- *u and v are cospectral if and only if $\{u, v\} = \{i, n - i\}$.*
- *u and v are fractionally cospectral (and not cospectral) if and only if $n = 5d - 1$ for some positive integer d, and $\{u, v\} = \{d, 3d\}$ or (by symmetry) $\{u, v\} = \{2d, 4d\}$.*



Fractional Cospectrality in Paths

We prove this by counting walks in paths

$$A^{2d+2j}(u,u) = \binom{2d+2j}{d+j} - \binom{2d+2j}{j}$$
$$A^{2d+2j}(v,v) = \binom{2d+2j}{d+j}$$
$$A^{2d+2j}(u,v) = \binom{2d+2j}{j}.$$

Paths: Eigenvalue Condition

The eigenvalues of P_{5d-1} are

$$\lambda_j = 2\cos\frac{\pi j}{5d}, \quad j = 1, ..., 5d-1$$

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Let m be an odd integer, $0 \leq a < k$ integers. Then

$$\sum_{i=0}^{m-1} (-1)^i \cos\left(\frac{(a+ik)\pi}{km}\right) = 0.$$

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Eigenvalue condition satisfied if and only if $d = 2^k$ for some k.

PGFR in Paths

Theorem

Vertices u and v of P_n exhibit PGFR if and only if we are in one of the following cases:

- *There is PGST from u to v (characterized previously).*
- $n = 5 \cdot 2^k - 1$ *and* $\{u, v\} = \{2^k, 3 \cdot 2^k\}$ *or* $\{2 \cdot 2^k, 4 \cdot 2^k\}$.

Future Work

- What other graphs have fractionally cospectral pairs of nodes? Which of these exhibit FR/PGFR?

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- What other graphs have fractionally cospectral pairs of nodes? Which of these exhibit FR/PGFR?
- Weighted paths?
- FR or PGFR among more than 2 nodes in paths?

Thank You!