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Bipartite NBRW

Theorem 1. *A graph G is bipartite if and only if the spectrum of B is symmetric.*

Proof. Assume that G is bipartite. If G is bipartite, then the adjacency matrix A can be written as $\begin{pmatrix} 0 & A_2 \\ A_1 & 0 \end{pmatrix}$ (see *Spectra of Graphs* by Brouwer and Haemers). We know that $A = ST$. Thus for a bipartite graph, we define the following matrices:

$$T_1 \colon = \begin{cases} 1 & i_1 \mapsto (i_1, i_2) \\ 0 & \text{otherwise} \end{cases}, \tag{1}$$

$$T_2 \colon = \begin{cases} 1 & i_2 \mapsto (i_2, i_1) \\ 0 & \text{otherwise} \end{cases}, \tag{2}$$

$$S_1 \colon = \begin{cases} 1 & (i_2, i_1) \mapsto i_1 \\ 0 & \text{otherwise} \end{cases}, \text{ and} \tag{3}$$

$$S_2 \colon = \begin{cases} 1 & (i_1, i_2) \mapsto i_2 \\ 0 & \text{otherwise} \end{cases}. \tag{4}$$

In these matrices, i_j represents a node in partition j and (i_j, i_k) represents an edge from partition j to partition k. Thus by simple computation, we see that $A = \begin{pmatrix} 0 & T_1S_2 \\ T_2S_1 \end{pmatrix}$ and the matrix $C = \begin{pmatrix} 0 & S_2T_2 \\ S_1T_1 \end{pmatrix}$. Hence, the edges are also divided into two edge partitions.

In order to compute B, we also define a matrix

$$\tau_1 \colon = \begin{cases} 1 & (w, x)_1 \mapsto (y, z)_2 \\ 0 & \text{otherwise} \end{cases} \text{ and} \tag{5}$$

$$\tau_2 \colon = \begin{cases} 1 & (w, x)_2 \mapsto (y, z)_1 \\ 0 & \text{otherwise} \end{cases}, \tag{6}$$

where $(w, x)_j$ represents an edge in edge partition j. We then see that $\tau = \begin{pmatrix} 0 & \tau_2 \\ \tau_1 & 0 \end{pmatrix}$. So the matrix $B = \begin{pmatrix} 0 & S_2T_2 - \tau_2 \\ S_1T_1 - \tau_1 \end{pmatrix}$. Defining $B_j = S_jT_j - \tau_j$, we get that $B = \begin{pmatrix} 0 & B_2 \\ B_1 & 0 \end{pmatrix}$.

Let $\begin{pmatrix} x & y \end{pmatrix}^T$ be an eigenvector of B with corresponding eigenvalue μ. Then

$$B \begin{pmatrix} x & y \end{pmatrix}^T = \mu \begin{pmatrix} x & y \end{pmatrix}^T \tag{7}$$

$$\begin{pmatrix} B_2y & B_1x \end{pmatrix}^T = \mu \begin{pmatrix} x & y \end{pmatrix}^T. \tag{8}$$

Consider the vector $\begin{pmatrix} x & -y \end{pmatrix}^T$. We see that

$$B \begin{pmatrix} x & -y \end{pmatrix}^T = \begin{pmatrix} -B_2y & B_1x \end{pmatrix}^T \tag{9}$$

$$= \begin{pmatrix} -\mu x & \mu y \end{pmatrix}^T \tag{10}$$

$$= -\mu \begin{pmatrix} x & -y \end{pmatrix}^T. \tag{11}$$

So $-\mu$ is an eigenvalue of B with eigenvector $\begin{pmatrix} x & -y \end{pmatrix}^T$. Hence the spectrum of B is symmetric around 0.

Now assume that the spectrum of B is symmetric around 0. Consider B as its own graph. The number of closed walks of length k on B can be represented by $tr(B^k)$. Recall that if $\mu \in \sigma(B)$, then $\mu^k \in \sigma(B^k)$ for any k. Since $tr(B^k)$ is the sum of its eigenvalues, the number of closed walks of length k on B is $\sum_{i=1}^n \mu_i^k$. If k is odd, the $\sum_{i=1}^n \mu_i^k = 0$. So every closed walk on B must have even length, so B must be bipartite (see https://www.epfl.ch/labs/dcg/wp-content/uploads/2018/10/ADM-Eigenvalues-v3.pdf). Thus B can be written as $\begin{pmatrix} 0 & B_2 \\ B_1 & 0 \end{pmatrix}$. Then by similar argument as above, $A = \begin{pmatrix} 0 & A_2 \\ A_1 & 0 \end{pmatrix}$. So G is bipartite. $\square$

Corollary 1.1. *A graph G is bipartite if and only if $-\rho$ is an eigenvalue of B.*

Remark. *If μ is a complex eigenvalue, it is known that its conjugate $\overline{\mu}$ is also an eigenvalue. Here, we find that if μ is an eigenvalue, then its mirror image over the real axis is also a complex eigenvalue (i.e. if μ is an eigenvalue, $\overline{-\mu}$ is also an eigenvalue).*