Non-Backtracking Spectrum of Graphs

Cory Glover

Brigham Young University
Department of Mathematics

Joint work with
Mark Kempton, Brigham Young University

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Non-backtracking Random Walks

Definition (Random Walk)

Let $G = (V, E)$ be a graph. A *random walk* across G is a walk across a graph where v_{i+1} is chosen uniformly at random from the set of neighbors of v_i.

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Definition (Non-backtracking Random Walk)

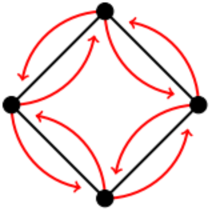
Let $G = (V, E)$ be a graph. A *non-backtracking random walk* across G is a walk across a graph where v_{i+1} is chosen uniformly at random from the set of neighbors of v_i excluding v_{i-1}.

Non-backtracking Matrix

Definition (NB Matrix)

Let $B \in M_{2m}$ be the non-backtracking matrix. Then

$$B((u,v),(x,y)) = \begin{cases} 1 & v = x \text{ and } u \neq y \\ 0 & \text{otherwise} \end{cases}.$$



Spectrum of B

Question: What do we know about the spectrum of B, $\sigma(B)$, for a given graph G?

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Theorem (Ihara's Theorem)

Let G be a graph with adjacency matrix A, degree matrix D, and non-backtracking matrix B. Then

$$\det(\mu I - B) = (1-\mu^2)^{m-n}\det(\mu^2 I - \mu A + (D - I)).$$

d-regular Graphs

Let G be a d-regular graph with adjacency matrix A and non-backtracking matrix B. Then

$$\pm 1, \frac{\lambda_i \pm \sqrt{\lambda_i^2 - 4(d-1)}}{2}$$

are the eigenvalues of B where $\lambda_i \in \sigma(A)$ and ± 1 each have multiplicity $m - n$.

The Matrix K

From Krzakala et. al. (2013), the matrix K of a graph G

$$K = \begin{pmatrix} A & D - I \\ -I & \mathbf{0} \end{pmatrix}$$

has characteristic polynomial

$$\det(\mu I - K) = \det(\mu^2 I - \mu A + (D - I)).$$

Decomposition of B

$$S((u,v),x) = \begin{cases} 1 & v = x \\ 0 & \text{otherwise} \end{cases} \qquad T(x,(u,v)) = \begin{cases} 1 & u = x \\ 0 & \text{otherwise} \end{cases}$$

$$\tau((i,j),(k,l)) = \begin{cases} 1 & i = l \text{ and } j = k \\ 0 & \text{otherwise} \end{cases}$$

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$$B \begin{bmatrix} S & T^T \end{bmatrix} = \begin{bmatrix} S & T^T \end{bmatrix} K$$

Decomposition of B

Theorem (Lubetzky and Peres, 2010)

Let G be a connected d-regular graph ($d \geq 3$) on n vertices. Let $N = dn$ and let $\lambda_i \in \sigma(A)$, with $\lambda_1 = d$. Then the operator B is unitarily similar to

$$\Lambda = \mathit{diag}\left(d-1, \begin{bmatrix} \theta_2 & \alpha_2 \\ 0 & \theta_2' \end{bmatrix}, ..., \begin{bmatrix} \theta_n & \alpha_n \\ 0 & \theta_n' \end{bmatrix}, -1, ... -1, 1, ..., 1 \right)$$

where $|\alpha_i| < 2(d-1)$ for all i, θ_i and θ_i' are defined as the solutions of

$$\theta^2 - \lambda_i \theta + d - 1 = 0$$

and -1 has multiplicity $N/2 - n$ and 1 has multiplicity $N/2 - n + 1$.

Decomposition of B

Theorem

Let G be a connected graph and B its non-backtracking matrix. Define $\mathscr{E}_i$ to be the eigenspace of i for τ $(i = \pm)$. Let $R \in M_{2m\times 2(m-n)}$ where the columns of R are linearly independent and the first $m-n$ columns are taken from $\mathscr{E}_{-1} \cap$ Null(ST) and the second $m-n$ columns are taken from $\mathscr{E}_1 \cap$ Null(ST). Then

$$BX = X \begin{bmatrix} K & 0 & 0 \\ 0 & I_{m-n} & 0 \\ 0 & 0 & -I_{m-n} \end{bmatrix}$$

where $X = \begin{bmatrix} S & T^T & R \end{bmatrix}$.

The Matrix K

Proposition

Let G be a graph and K as defined previously. Then the following are true:

1. *Every eigenvalue-eigenvector of K is of the form $(\mu, \begin{bmatrix} -\mu y & y \end{bmatrix}^T)$*
2. *$1 \in \sigma(K)$ with algebraic multiplicity equal to the number of connected components of G,*
3. *the nullity of K is the number of degree one vertices, and*
4. *$K^{-1} = \begin{bmatrix} 0 & -I \\ (D-I)^{-1} & (D-I)^{-1}A \end{bmatrix}$ when $d \geq 2$.*

Eigenvalues of K

Let $\mu \in \sigma(K)$. Then

$$\mu \begin{bmatrix} -\mu y \\ y \end{bmatrix} = \begin{bmatrix} A & D - I \\ -I & 0 \end{bmatrix} \begin{bmatrix} -\mu y \\ y \end{bmatrix}$$

$$0 = \mu^2 y - \mu A y + (D - I) y$$

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Let $\lambda \in \sigma(A)$ with eigenvector x. If $x^T y \neq 0$, scale x such that $x^T y = 1$. Then

$$0 = \mu^2 - \mu\lambda + x^T (D - I) y$$
$$\mu = \frac{\lambda \pm \sqrt{\lambda^2 - 4x^T (D - I) y}}{2}.$$

Spectral Radius of K

Proposition

Let G be a connected graph that is not a cycle and $d \geq 2$. Then B is irreducible.

Lemma

Let G be a connected graph that is not a cycle and $d \geq 2$. Then $\rho(K) > 1$ and there is a positive vector y such that $K \begin{bmatrix} -\rho(K)y & y \end{bmatrix}^T = \rho(K) \begin{bmatrix} -\rho(K)y & y \end{bmatrix}^T$.

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Sketch:

- Break into regular and non-regular case for $\rho(K) > 1$

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- By Perron-Frobenius and since $B \begin{bmatrix} S & T^T \end{bmatrix} = \begin{bmatrix} S & T^T \end{bmatrix} K$, we know that $T^T y \succ \rho(K)Sy$ or $\rho(K)Sy \succ T^T y$

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- BWOC, show $\rho(K) S y \succ T^T y$.

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- BWOC, show $\rho(K)Sy \succ T^T y$.
- Use the fact that S and T^T have one nonzero entry in each row.

Spectral Radius of K

Theorem

Let G be a connected graph, A its adjacency matrix, D the degree matrix, and B the non-backtracking matrix. If $\rho(A) \geq 2x^T(D-I)y$, then

$$\rho(B) \leq \frac{\rho(A) + \sqrt{\rho(A)^2 - 4(d_{min} - 1)}}{2}.$$

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Sketch:

- If $Ax = \rho(A)x$ and $K \begin{bmatrix} -\rho(K)y & y \end{bmatrix}^T = \rho(K) \begin{bmatrix} -\rho(K)y & y \end{bmatrix}^T$, we know $x^T y \neq 0$ from previous lemma
- Apply $\mu = \frac{\lambda \pm \sqrt{\lambda^2 - 4x^T(D-I)y}}{2}$

Spectral Radius of K

From a bound by Das and Kumar (2004), we get

Corollary

Let G be a connected graph and B its non-backtracking matrix. If $\rho(A) \geq 2\sqrt{x^T(D-I)y}$,

$$\rho(B) \leq \frac{\sqrt{2m-n-1}+\sqrt{2m-n-4d_{min}+1}}{2}.$$

Smallest Eigenvalue of K

Proposition

Let G be a connected graph with $d \geq 2$. Then $|\mu| \geq 1$ for all $\mu \in \sigma(K)$.

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Sketch:

- Build equation $\mu = \frac{y^T Ay \pm \sqrt{(y^T Ay)^2 - 4y^T (D-I)y}}{2}$

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Sketch:

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- When $(y^T A y)^2 \leq 4y^T (D - I)y$, use $d \geq 2$

Smallest Eigenvalue of K

Proposition

Let G be a connected graph with $d_{\min} \geq 2$. Then $|\mu| \geq 1$ for all $\mu \in \sigma(K)$.

Sketch:

- Build equation $\mu = \frac{y^T Ay \pm \sqrt{(y^T Ay)^2 - 4y^T (D-I)y}}{2}$
- When $(y^T Ay)^2 \leq 4y^T(D-I)y$, use $d_{\min} \geq 2$
- When $(y^T Ay)^2 > 4y^T(D-I)y$, use $d_{\min} \geq 2$ and the positive semi-definiteness of the Laplacian

Bipartite Graphs

Theorem

Let G be a connected graph and B its non-backtracking matrix. The following are equivalent:

1. *G is a bipartite graph,*
2. *$\sigma(K)$ is symmetric,*
3. *$\sigma(B)$ is symmetric,*
4. *$-1 \in \sigma(K)$,*
5. *$\lambda_n = -\lambda_1$ for $\lambda_i \in \sigma(K)$, and*
6. *$\mu_n = -\mu_1$ for $\mu_i \in \sigma(B)$.*

Conclusion

What we know:

- We can use K to understand the spectrum of B
- B can be related to a block diagonal matrix showing more explicitly the spectrum of B
- We can use K to bound the spectral radius of B
- The spectrum of B and K can indicate whether G is bipartite

Conclusion

Future Work:

- Can we identify the spectral gap of B?
- Can we relate the transition probability matrix of G to the non-backtracking transition probability matrix?

The End

Thank You!