Research Plan 05-2020

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This is a layout to plan progress on thesis during Summer of 2020. It will be divided by project. Proofs will be found in appendix as needed.

1 Mixing Rates of Non-backtracking Random Walks

1.1 Background

Let A be the adjacency matrix of a connected graph G. Let P be the probability matrix associated with G. Let S be the in-matrix of G and T be the out-matrix. Let $\hat{A}$ be the edge-adjacency matrix of G. Let D be the degree matrix of G. Let B be the non-backtracking edge adjacency matrix of G. Let τ be the reversal operator.

The following properties and relationships hold:

1. $A = TS$

2. $\hat{A} = ST$

3. $B = ST - \tau$

4. $D = T\tau S$

Additionally we define the matrix $K = \begin{bmatrix} A & D-I \\ -I & \mathbf{0} \end{bmatrix}$. Let $\mathbf{v} = \begin{bmatrix} x & y \end{bmatrix}^T$ be an eigenvector of K with eigenvalue μ. Note then that,

$$\begin{bmatrix} A & D-I \\ -I & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \mu \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

implies that $\mathbf{x} = -\mu\mathbf{y}$. So $\mathbf{v} = \begin{bmatrix} -\mu\mathbf{y} & \mathbf{y} \end{bmatrix}^T$.

K is an invariant subspace of B as follows:

$$B \begin{bmatrix} S & T^T \end{bmatrix} \mathbf{v} = \begin{bmatrix} S & T^T \end{bmatrix} K\mathbf{v} = \mu \begin{bmatrix} S & T^T \end{bmatrix} \mathbf{v} \tag{1}$$

So all the eigenvalues of K are also eigenvalues of B.

We also note the following property:

$$-\mu A\mathbf{y} + (D - I)\mathbf{y} = -\mu^2\mathbf{y} \tag{2}$$

$$\mu^2\mathbf{y} - \mu A\mathbf{y} + (D - I)\mathbf{y} = 0 \tag{3}$$

$$(\mu^2 - \mu A + (D - I))\mathbf{y} = 0 \tag{4}$$

1.2 Regular Case

In the case that the graph is regular, it is known that $\mu = \frac{\lambda \pm \sqrt{\lambda^2 - 4(d-1)}}{2(d-1)}$ where d is the degree of the graph. If $2\sqrt{d-1} \leq \lambda \leq d$, then

$$\mu = \frac{\lambda + \sqrt{\lambda^2 - 4(d-1)}}{2(d-1)} \leq \frac{\lambda}{2} \tag{5}$$

$$\mu = \frac{\lambda - \sqrt{\lambda^2 - 4(d-1)}}{2(d-1)} \leq \frac{\lambda}{2(d-1)} \leq \frac{\lambda}{2}. \tag{6}$$

If $\lambda < 2\sqrt{d-1}$, then

$$\mu = \frac{\lambda \pm \sqrt{\lambda^2 - 4(d-1)}}{2(d-1)} = \frac{\lambda \pm i\sqrt{4(d-1) - \lambda^2}}{2(d-1)}. \tag{7}$$

Thus,

$$|\mu|^2 = \frac{1}{d-1} \Rightarrow |\mu| = \frac{1}{\sqrt{d-1}}.$$

1.3 Solving for μ

We left multiply Equation 4 by some vector $\mathbf{u}^T$, where $\mathbf{u}^T\mathbf{y} \neq 0$. Thus,

$$\mu = \frac{\mathbf{u^T} A\mathbf{y} \pm \sqrt{(\mathbf{u}^T A\mathbf{y})^2 - 4\mathbf{u}^T\mathbf{y}\mathbf{u}^T(D - I)\mathbf{y}}}{2\mathbf{u}^T\mathbf{y}} \tag{8}$$

1.3.1 $\mathbf{u} = \mathbf{x}$

If $\mathbf{u} = \mathbf{x}$ where $A\mathbf{x} = \lambda\mathbf{x}$. Then,

$$\mu = \frac{\mathbf{x}^T A\mathbf{y} \pm \sqrt{(\mathbf{x}^T A\mathbf{y})^2 - 4\mathbf{x}^T\mathbf{y}\mathbf{x}^T(D - I)\mathbf{y}}}{2\mathbf{x}^T\mathbf{y}} \tag{9}$$

$$= \frac{\lambda\mathbf{x}^T\mathbf{y} \pm \sqrt{\lambda^2(\mathbf{x}^T\mathbf{y})^2 - 4(\mathbf{x}^T\mathbf{y})(\mathbf{x}^T(D - I)\mathbf{y}}}{2\mathbf{x}^T\mathbf{y})} \tag{10}$$

Since $\mathbf{x}$ is an eigenvector and $\mathbf{y}$ is a part of an eigenvector, we scale them such that $\mathbf{x}^T\mathbf{y} = 1$. So,

$$\mu = \frac{\lambda \pm \sqrt{\lambda^2 - 4\mathbf{x}^T(D - I)\mathbf{y}}}{2} \tag{11}$$

However, if λ_2 is the second largest eigenvalue in magnitude of A, it is not clear whether μ_1 or μ_2 is the second largest eigenvalue in magnitude of B, where

$$\mu_1 = \frac{\lambda_2+\sqrt{\lambda_2^2-4\mathbf{x}^T(D-I)\mathbf{y}}}{2}, \quad \mu_2 = \frac{\lambda_2-\sqrt{\lambda_2^2-4\mathbf{x}^T(D-I)\mathbf{y}}}{2}.$$

In the case that the second largest eigenvalue is μ_2, then

$$\mu_2 = \frac{\lambda_2 - \sqrt{\lambda_2^2 - 4\mathbf{x}^T(D-I)\mathbf{y}}}{2} \leq \frac{\lambda_2}{2} \leq \lambda_2.$$

1.3.2 Various Algebraic Manipulations

New Manipulation:

Assume $\lambda \geq 2\sqrt{\mathbf{x}^T(D-I)\mathbf{y}}$, then

$$\mu = \frac{\lambda + \sqrt{\lambda^2 - 4\mathbf{x}^T(D-I)\mathbf{y}}}{2} \leq \frac{\lambda+\lambda}{2} = \lambda \tag{12}$$

$$\mu = \frac{\lambda - \sqrt{\lambda^2 - 4\mathbf{x}^T(D-I)\mathbf{y}}}{2} \leq \frac{\lambda}{2} \tag{13}$$

Assume $\lambda < 2\sqrt{\mathbf{x}^T(D-I)\mathbf{y}}$.

$$\mu = \frac{\lambda \pm \sqrt{\lambda^2 - 4\mathbf{x}^T(D-I)\mathbf{y}}}{2} = \frac{\lambda \pm i\sqrt{4\mathbf{x}^T(D-I)\mathbf{y} - \lambda^2}}{2} \tag{14}$$

$$|\mu|^2 = \frac{\lambda^2}{4} + \frac{4\mathbf{x}^T(D-I)\mathbf{y} - \lambda^2}{4} = \mathbf{x}^T(D-I)\mathbf{y} \tag{15}$$

$$|\mu| = \sqrt{\mathbf{x}^T(D-I)\mathbf{y}} \tag{16}$$

New Manipulation:

$$\mu_1^2 = \frac{\lambda^2}{4} + \frac{\lambda\sqrt{\lambda - 4\mathbf{x}^T(D-I)\mathbf{y}}}{2} + \frac{\lambda^2}{4} - \mathbf{x}^T(D-I)\mathbf{y} \tag{17}$$

$$= \lambda\mu_1 - \mathbf{x}^T(D-I)\mathbf{y} \tag{18}$$

New Manipulation:

$$|\mathbf{x}^T(D-I)\mathbf{y}|^2 = |\sum_{i=1}^{n}(d_i-1)x_iy_i|^2 \tag{19}$$

$$\leq \sum_{i=1}^{n}(d_i-1)^2\sum_{j=1}^{n}|x_j|^2\sum_{k=1}|y_k|^2 \tag{20}$$

$$= \sum_{i=1}^{n}(d_i^2+2d_i+1)\sum_{j=1}^{n}|x_j|\sum_{k=1}^{n}|y_k| \tag{21}$$

$$\leq \sum_{i=1}^{n}(d_i^2+2d_i+d_i)\sum_{j=1}^{n}|x_j|\sum_{k=1}^{n}|y_k| \tag{22}$$

$$\leq \sum_{i=1}^{n}(d_i)(d_i-1)\sum_{j=1}^{n}|x_j|\sum_{k=1}^{n}|y_k| \tag{23}$$

$$= D(D-I)\mathbf{x}^T\mathbf{x}\mathbf{y}^T\mathbf{y} \tag{24}$$

1.3.3 $\mathbf{u} = \nVdash$

Let $\mathbf{u} = \nVdash$. Then,

$$\mu = \frac{\lambda \nVdash^T \mathbf{y} \pm \sqrt{\lambda^2 (\nVdash^T \mathbf{y})^2 - 4 \nVdash^T \mathbf{y} \nVdash^T (D - I)\mathbf{y}}}{2 \nVdash^T \mathbf{y}} \tag{25}$$

$$= \frac{\lambda \sum_i y_i \pm \sqrt{\lambda^2 (\sum_i y_i)^2 - 4 \sum_i y_i \sum_j (d_j - 1) y_j}}{2 \sum_i y_i} \tag{26}$$

We scale $\mathbf{y}$ such that $\nVdash^T \mathbf{y} = 1$. Then,

$$\mu = \frac{\lambda \pm \sqrt{\lambda^2 - 4 \sum_i (d_i - 1) y_i}}{2}. \tag{27}$$

1.3.4 Algebraic Manipulations

Assume $\lambda \geq 2\sqrt{\sum_i (d_i - 1) y_i}$. Then,

$$\mu = \frac{\lambda + \sqrt{\lambda^2 - 4 \sum_i (d_i - 1) y_i}}{2} \tag{28}$$

$$\leq \frac{\lambda + \lambda}{2} \tag{29}$$

$$= \lambda \tag{30}$$

$$\mu = \frac{\lambda - \sqrt{\lambda^2 - 4 \sum_i (d_i - 1) y_i}}{2} \tag{31}$$

$$\leq \frac{\lambda}{2} \tag{32}$$

If $\lambda < 2\sqrt{\sum_i (d_i - 1) y_i}$, then

$$\mu = \frac{\lambda \pm \sqrt{\lambda^2 - 4 \sum_i (d_i - 1) y_i}}{2} \tag{33}$$

$$= \frac{\lambda \pm i\sqrt{4 \sum_i (d_i - 1) y_i - \lambda^2}}{2} \tag{34}$$

$$|\mu|^2 = \frac{\lambda^2}{4} + \sum_i (d_i - 1) y_i - \frac{\lambda^2}{4} \tag{35}$$

$$= \sum_i (d_i - 1) y_i \tag{36}$$

$$|\mu| = \sqrt{\sum_i (d_i - 1) y_i} \tag{37}$$

1.4 Diagonalizing K

In order to ensure we can get all eigenvalues $\mu \in \sigma(K)$, we need to know that K itself is diagonalizable. Thus far, it seems that K is diagonalizable if the markov chain along the non-backtracking random walk converges to some stationary distribution.

Ideas tried thus far:

1. Using the fact that if $(K - \mu I)^2 x = 0$ implies $(K - \mu I)x = 0$, then K is diagonalizable.

2. Using that the NBRW is reversible (this is not true).

3. Looking for patterns in the diagonalization.

1.5 Finding an invariant subspace of P

In order to find an invariant subspace of P, we first need to write $\hat{D}$ in terms of S, T and τ. Thus far, we have noticed that $(C\tau)^2$ gives the correct diagonal entries but has entries corresponding to the degree of the row in the locations of $C\tau - I$.

1.6 Work to be done

The following needs to be done:

1. Proving K is diagonalizable.

2. Determining bound on relationship between μ and λ.

3. Convert μ to ρ.

4. Find invariant subspace of P.

2 NBRW PageRank

Open questions still to look at are when PageRank gives the same values as a simple random walk and whether it can be calculated faster than a simple random walk.

2.1 Bipartite Graphs

We have shown that the a bipartite biregular graph gives the same pagerank values in both a NBRW and SRW.

2.2 Bahmini Alg

Bahmini algorithm seems to work faster for NBRW but as the number of nodes approaches infinity, the difference is insignificant. This happens fairly quickly.

2.3 Work To Be Done

The following needs to be worked on:

1. Observe patterns in Borgs et. al. algorithm.

2. Prove that ranking is the same regardless of SRW and NBRW

3 Summer Plan

The following are action items for the summer:

1. Prove K is diagonalizable (or die trying...)
2. Identify useful bounds that relate μ and λ
3. Find a relationship between μ and ρ
4. Understand and apply Borg algorithm to NBRW
5. Prove ranking of NBRW is the same as SRW
6. Bipartite K
7. Spectrum of K and B when G is a tree
8. Spectrum of K and B when G is a unicycle