Non-backtracking Random Walks

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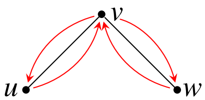
Problem

- When the Pagerank vector of a non-backtracking random walk give the same values as a simple random walk?
- Can the Pagerank vector of a non-backtracking random walk be calculated faster than a simple random walk?

Review

Definition (Hashimoto Matrix)

$$B((u,v),(x,y)) = \begin{cases} 1 & v = x, u \neq y \\ 0 & \text{otherwise} \end{cases}$$



Regular Graphs

Theorem (Arrigo et. al., 2018)

Let A be the adjacency matrix of an undirected k-regular graph, with $k \geq 2$. For any $\epsilon \in [0,1]$, define

$$\hat{P} = \epsilon \hat{D}^{-1} B + \frac{1-\epsilon}{n} \mathbf{1}\mathbf{v}^T$$

and $P = \epsilon D^{-1} A + \frac{1-\epsilon}{n} \mathbf{1}\mathbf{1}^T$, where $\mathbf{v} = T^T D^{-1} \mathbf{1}$. If $\hat{P}^T \hat{y} = \hat{y}$ and $P^T x = x$, with $\|x\|_1 = \|\hat{y}\|_1 = 1$, then

$$T\hat{y} = x.$$

Bipartite Graphs

Theorem

Let A be the adjacency matrix of a bipartite biregular graph, with $d_1, d_2 \geq 2$. For any $\epsilon \in [0,1]$, define $\hat{P} = \epsilon \hat{D}^{-1} B + \frac{1-\epsilon}{n} \mathbf{1}\mathbf{v}^T$ and $P = \epsilon D^{-1} A + \frac{1-\epsilon}{n} \mathbf{1}\mathbf{1}^T$, where $\mathbf{v} = T^T D^{-1} \mathbf{1}$. If $\hat{P}^T \hat{y} = \hat{y}$ and $P^T x = x$, with $\|x\|_1 = \|\hat{y}\|_1 = 1$, then

$$T\hat{y} = x.$$

Estimating Pagerank vs Nonbacktracking Pagerank

Conjecture

In most cases, NBRW mix faster than simple random walks.

Alon et. al. (2007) showed that in most cases, NBRW mix faster on k-regular graphs, with $k \geq 2$.

Mixing Rate

Theorem (Lovasz 1993)

Let G be a connected non-bipartite graph with transition probability matrix P. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the eigenvalues of P. Then the mixing rate of G is $\rho = \max\{|\lambda_2|, |\lambda_n|\}$.

Ihara's Theorem

Theorem (Ihara)

Given a graph G with n vertices and m edges, define B to be the Hashimoto matrix of G. Let A be the adjacency matrix of G and let D be the diagonal degree matrix. Then

$$\textit{det}(I - uB) = (1 - u^2)^{m-n}\textit{det}(I - uA + u^2(D - I)).$$

Thus, the eigenvalues of B are $\mu = \frac{1}{u}$.

Comparing Eigenvalue

From Krzakala et. al. (2013), the matrix K of a graph G

$$K = \begin{pmatrix} A & D - I \\ -I & \mathbf{0} \end{pmatrix}$$

is a invariant subspace of B.

Eigenvalues of K

$$\begin{pmatrix} A & D-I \\ -I & \mathbf{0} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mu \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\begin{gathered} x = -\mu y \\ \mu^2 y - \mu A y + (D-I)y = \mathbf{0} \end{gathered}$$

for all $\mu \in \sigma(K)$.

Relationship Between A and K

Let $\mathbf{x}$ be the eigenvector associated with $\lambda \in \sigma(A)$.

Then,

$$\begin{aligned} \mu^2 \mathbf{x}^T \mathbf{y} - \mu \mathbf{x}^T A \mathbf{y} + \mathbf{x}^T (D - I)\mathbf{y} &= 0, \\ \mu^2 - \mu\lambda + \mathbf{x}^T (D - I)\mathbf{y} &= 0. \end{aligned}$$

Solving for μ,

$$\mu = \frac{\lambda \pm \sqrt{\lambda^2 - 4\mathbf{x}^T (D - I)\mathbf{y}}}{2}.$$

Open Questions

$$\mu = \frac{\lambda \pm \sqrt{\lambda^2 - 4\mathbf{x}^T(D-I)\mathbf{y}}}{2}$$

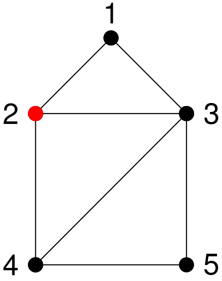
- How are μ and λ related?
- Is it $+$ or $-$?
- Can we get all eigenvalues μ?
- How to convert μ to eigenvalue of P?

Current Pagerank Algorithms

- Borgs et. al. (2016) created algorithm to solve significant Pagerank problem, i.e. find all nodes with Pagerank value greater than Δ, in $O(\frac{n}{\Delta})$
- Bahmini, Chowdhury, and Goel (2010) created algorithm to estimate personalized pagerank and global pagerank in $O(\frac{n\ln(m)}{\epsilon^2})$.

Bahmini et. al. Algorithm

$R = 2$

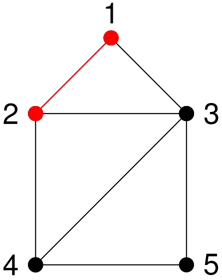


Current: $[2]$

Paths: $[]$

Bahmini et. al. Algorithm

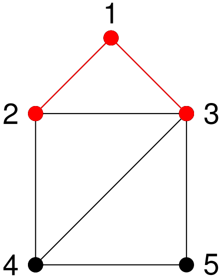
$$R = 2$$



Current: $[2, 1]$
Paths: $[]$

Bahmini et. al. Algorithm

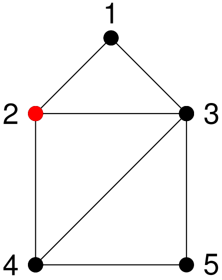
$R = 2$



Current: $[2, 1, 3]$
Paths: $[]$

Bahmini et. al. Algorithm

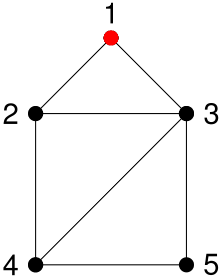
$R = 2$



Current: $[2]$
Paths: $[[2, 1, 3]]$

Bahmini et. al. Algorithm

$$R = 2$$



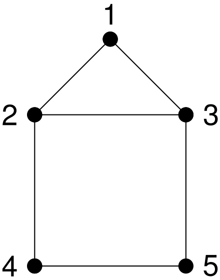
Current: $[1]$
Paths: $[[2,1,3],[2,4]]$

Bahmini et al Algorithm

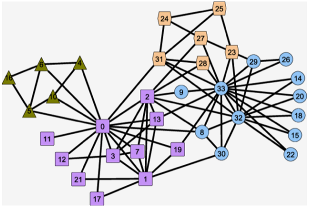
$$\tilde{\pi}_v = \frac{X_v}{nR/\epsilon}$$

where X_v is the number of paths containing v.

Computational Results



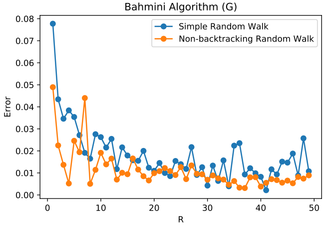
(a) Simple Example



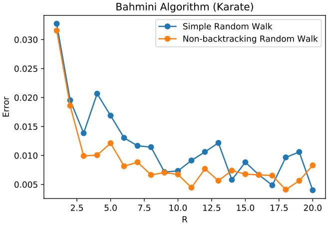
(b) Karate

Computational Results

$$y = Cx^{-\alpha}$$

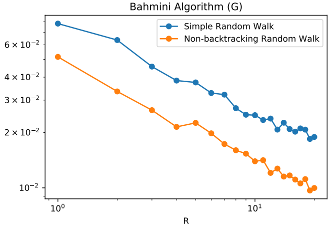


(a) Simple Example

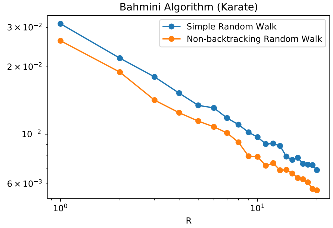


(b) Karate

Computational Results



(a) SRW: $\alpha = .54$, NBRW: $\alpha = .56$



(b) SRW: $\alpha = .49$, NBRW: $\alpha = .52$

Figure: Simple Example: $|C - C_{NB}| = .034$
Karate: $|C - C_{NB}| = .0052$

Conclusions

Thank you!