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1/16/20
Math 540

HW 8
2.3.3,8,10
2.4.2,4,6

2.3.3. Let X be l^∞ and let Y be the subset of all sequences with only finitely many nonzero terms. We first show that Y is a subspace of l^∞. We first note that the zero sequence has no nonzero terms, and thus is in Y. So Y is nonempty. Let $x = (x_i)$ and $y = (y_i)$ be elements of Y. Let x_n be the last nonzero term of x and let y_m be the last nonzero term of y. Then $\alpha x + \beta y$ has a last nonzero term at either n or m (αx_n or βy_m respectively). Thus, $x + y \in Y$. So Y is a subspace.

However consider the sequence $x_n = (1+\frac{1}{n}, 1+\frac{1}{n}, ..., 1+\frac{1}{n}, 0, ...)$ where the first n entries of each tuple are $1+\frac{1}{n}$ and the rest are zero. Then for $\epsilon > 0$, there exists an N such that when $n > N$, $\|x_n - (1,1,1,...)\|\ leq\epsilon$. Since $(1,1,1,...) \notin Y$, we see that Y is not closed, and thus is not a closed subspace.

2.3.8. Let X be a normed space where absolute convergence implies convergence. Let $(x_n) \in X$ be a Cauchy sequence. Then for $\epsilon^* > 0$, there exists an N such that when $m, n > N$, then $\|x_n - x_m\| < \epsilon^*$. We then define $y_r = x_r - x_{r-1}$. Since (x_n) is Cauchy, for all $r > N+1$, $\|y_n\| < \epsilon^*$. We choose a subsequence (y_{r_k}) such that $\|y_{r_k}\| < \frac{1}{k^2}$. We can do this by redefining ϵ at each term y_r and only choosing elements such that $\|y_{r_k}\| < \frac{1}{k^2}$ for our sequence (this will be sequential since the distance between elements is shrinking). Then we know that $\sum_{i=1}^\infty \|y_{r_i}\| < \sum_{i=1}^\infty \frac{1}{k^2}$. Since $\sum_{i=1}^\infty \frac{1}{k^2}$ converges, we know that $\sum_{i=1}^\infty \|y_{r_i}\|$ converges. Then since every absolutely convergent sequence converges in X, we know that $\sum_{i=1}^\infty y_{r_k}$ converges to some s. Note that $\sum_{i=1}^\infty y_{r_i}$ is a telescoping sum, such that $\sum_{i=1}^\infty y_{r_i} = \lim_{k\to\infty} x_{r_k} - x_{r_1}$. Thus, $\lim_{k\to\infty} x_{r_k} - x_{r_1} = s$. Thus, $\lim_{k\to\infty} x_{r_k} = s + x_{r_1}$. We note that $x_{r_1} \in X$ since $x_{r_1} \in (x_r)$. Further, we know that $s \in X$ since $\sum_{i=1}^\infty y_{r_i}$ converges to s. Since X is a vector space, $s + x_{r_1} \in X$. Thus, (x_{r_k}) converges. Since (x_r) is Cauchy, and it has a convergent subsequence, (x_r) must converge. Thus, X is complete.

2.3.10. Assume that X is a normed space with a Schauder basis. Thus, for all $x \in X$, $x = \sum_{k=1}^\infty \alpha_k e_k$ where (e_n) is the Schauder basis for X. Consider the set $X' \subset X$ where each x' can be represented as $\sum_{k=1}^\infty \beta_k e_k$ where $\beta_k \in \mathbb{Q}$. Let $x \in X$ such that $x = \sum_{i=1}^\infty \alpha_i x_i$. Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$. Choose $x' = \sum_{i=1}^\infty \beta_i e_i \in X'$ such that for every α_i and β_i, $\|\alpha_i - \beta_i\| < \frac{\epsilon_i}{\|\sum_{i=1}^\infty e_i\|}$ for every $\epsilon_i > 0$. Let $\epsilon = \max_i \frac{\epsilon_i}{\|\sum_{i=1}^\infty e_i\|}$. Then,

$$\|x' - x\| = \|\sum_{i=1}^\infty \beta_i e_i - \sum_{i=1}^\infty \alpha_i e_i\| = \|\sum_{i=1}^\infty (\beta_i - \alpha_i) e_i\| \le \|\sum_{i=1}^\infty \|\beta_i - \alpha_i\| e_i\| < \|\epsilon \sum_{i=1}^\infty e_i\| = \epsilon \|\sum_{i=1}^\infty e_i\| = \max_i \epsilon_i.$$

So $x' \in B(x; \max_i \epsilon_i)$. Thus, $x \in \overline{X'}$. Since x was arbitrary, $\overline{X'} = X$. Further, since X' is countable, X is separable.

2.4.2. Let $X = \mathbb{R}^2$ and $x_1 = (1,0)$ and $x_2 = (0,1)$. Note that for α_1 and α_2,

$$\|\alpha_1 x_1 + \alpha_2 x_2\| = \|\alpha_1(1,0) + \alpha_2(0,1)\| = \|(\alpha_1, \alpha_2)\| = \sqrt{\alpha_1^2 + \alpha_2^2}.$$

Note that the largest c in equation (1) will bring equality. So we solve for c as follows:

$$\sqrt{\alpha_1^2 + \alpha_2^2} = c|\alpha_1| + c|\alpha_2| \tag{1}$$

$$c = \frac{\sqrt{\alpha_1^2 + \alpha_2^2}}{|\alpha_1| + |\alpha_2|}. \tag{2}$$

So the largest c could be is $c = \frac{\sqrt{\alpha_1^2 + \alpha_2^2}}{|\alpha_1| + |\alpha_2|}$.

Now let $X = \mathbb{R}^3$ and $x_1 = (1,0,0), x_2 = (0,1,0)$ and $x_3 = (0,0,1)$. Following the same procedure as above, we solve for c.

$$\|\alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1)\| = \|(\alpha_1, \alpha_2, \alpha_3)\| \tag{3}$$
$$= \left(\alpha_1^3 + \alpha_2^3 + \alpha_3^3\right)^{1/3} \tag{4}$$
$$= c(|\alpha_1| + |\alpha_2| + |\alpha_3|). \tag{5}$$

So the largest c could be is $c = \frac{\left(\alpha_1^3+\alpha_2^3+\alpha_3^3\right)^{1/3}}{|\alpha_1|+|\alpha_2|+|\alpha_3|}$.

2.4.4. Let X be a vector space and let $\|\cdot\|$ and $\|\cdot\|_*$ be equivalent norms on X. This means that there exists positive a and b such that $a\|x\|_* \leq \|x\| \leq b\|x\|_*$ for all $x \in X$. Define $\mathscr{T}_1$ as the open subsets of X with respect to $\|\cdot\|$ and $\mathscr{T}_2$ the open subsets of X with respect to $\|\cdot\|_*$.. Now let $X_2 \in \mathscr{T}_2$ be an open subset with respect to $\|\cdot\|_*$. We want to show that X_2 is an open subset with respect to $\|\cdot\|$. Let $x_0 \in B(x; a\epsilon)$ where this is a ball with respect to $\|\cdot\|$, $x \in X_2$ and $\epsilon > 0$. Then

$$a\|x - x_0\|_* \leq \|x - x_0\| < a\epsilon.$$

So $x_0 \in B^*(x;\epsilon)$. Thus, $B(x; a\epsilon) \subseteq B^*(x;\epsilon)$. So every open ball with respect to $\|\cdot\|$ around $x \in X_2$ is in X_2 since it is a subset of a ball with respect to $\|\cdot\|_*$. So X_2 is an open subset with respect to $\|\cdot\|$. Thus, $X_2 \in \mathscr{T}_1$. So $\mathscr{T}_2 \subseteq \mathscr{T}_1$.

Now let $X_1 \in \mathscr{T}_1$ be an open subset with respect to $\|\cdot\|$. Recall that since $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent, there exists a' and b' that are positive such that $a'\|x\| \leq \|x\|_* \leq b'\|x\|$ for all $x \in X$. Let $x \in X_1$, $\epsilon > 0$ and $x_0 \in B^*(x; a\epsilon)$ where B^* is a ball with respect to $\|\cdot\|_*$. Then we see that

$$a'\|x - x_0\| \leq \|x - x_0\|_* < a\epsilon.$$

So $x_0 \in B(x;\epsilon)$ where B is a ball with respect to $\|\cdot\|$. So $B^*(x; a\epsilon) \subseteq B(x;\epsilon)$. Thus every open ball with respect to $\|\cdot\|_*$ around $x \in X_1$ is in X_1. So X_1 is an open subset with respect to $\|\cdot\|_*$. So $X_1 \in \mathscr{T}_2$. Thus, $\mathscr{T}_1 \subseteq \mathscr{T}_2$.

Hence, $\mathscr{T}_1 = \mathscr{T}_2$.

2.4.6. Let X be the vector space of ordered n-tuples of numbers. Let $x = (x_1, ..., x_n) \in X$. Then $\|x\|_\infty^2 = \max_i |x_i|^2 \leq \sum_{i=1}^n |x_i|^p = \|x\|_2^2$. Thus,

$$\|x\|_\infty^2 \leq \|x\|_2^2 \tag{6}$$
$$\|x\|_\infty\| \leq \|x\|_2. \tag{7}$$

Further, $n^2 \max_i |x_i|^2 \geq n \max_i |x_i|^2 \geq \sum_{i=1}^n |x_i|^2$. Thus,

$$n^2\|x\|_\infty^2 \geq \|x\|_2^2 \tag{8}$$
$$n\|x\|_\infty \geq \|x\|_2. \tag{9}$$

So $\|x\|_\infty \leq \|x\|_2 \leq n\|x\|_\infty$. Since $1, n > 0$, the $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are equivalent by definition 2.4-4.