Cory Glover
1/27/20
Math 540

HW 11
2.7.2,6,10,14
2.8.2,4,11

2.7.2. Let X and Y be normed spaces. Assume that a linear operator $T: X \to Y$ is bounded. Let $A \subseteq X$ be a set that is bounded. So for all $a \in A$, $\|a\| \leq M$ for some $M \in \mathbb{R}$ (or $\mathbb{C}$. The same proof follow either way). Let $b \in B$ such that $T(A) = B$. Thus, there exists an $a \in A$ such that $T(a) = b$. We know that $\|T(a)\| \leq c\|a\|$ for some c. Thus, $\|b\| \leq c\|a\|$. Since A is bounded, then $\|b\| \leq c\|a\| \leq cM$. Since $cM \in \mathbb{R}$ and is constant, then $\|b\|$ is bounded. Since b was arbitrary, B must be bounded. So T maps bounded sets into bounded sets.

Now assume that $T: X \to Y$ maps bounded sets in X to bounded sets in Y. Let $x \in X$. Then $\|Tx\| = \|y\|$ where $y \in Y$. If x is in some bounded set of X, then $\|Tx\| \leq c$ for some c since y must be in a bounded set. Let $d = \frac{c}{\|x\|}$. Then $\|Tx\| \leq d\|x\|$.

Now assme that x is not in some bounded set of X.

2.7.6. Let $T: l^\infty \to l^\infty$ defined on $y = (y_i) = Tx$, $y_i = \frac{x_i}{i}$ and $x = (x_i)$. We first show that T is linear and bounded.

1. (Linear): Let $a, b \in l^\infty$ be denoted $a = (a_i), b = (b_i)$. Let $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} T(\alpha a + \beta b) &= T((\alpha a_i) + (\beta b_i)) && (1)\\ &= T((\alpha a_i + \beta b_i)) && (2)\\ &= (\frac{\alpha a_i + \beta b_i}{i}) && (3)\\ &= (\frac{\alpha a_i}{i}) + (\frac{\beta b_i}{i}) && (4)\\ &= \alpha(\frac{a_i}{i}) + \beta(\frac{b_i}{i}) && (5)\\ &= \alpha T(a) + \beta T(b). && (6)\end{aligned}$$

   So T is linear.

2. (Bounded): Note that $\|Tx\| = \|(\frac{x_i}{i})\|$ Since $(\frac{x_i}{i}) \in l^\infty$, each $\frac{x_i}{i}$ is bounded by some M. Thus, $\|(\frac{x_i}{i})\| = \sup_i \|\frac{x_i}{i}\| \leq M$. Let $c = \frac{M}{\|x\|}$. Then $\|Tx\| = \|(\frac{x_i}{i})\| = \sup_i |\frac{x_i}{i}| \leq M = c\|x\|$. So $\|Tx\| \leq c\|x\|$. So T is bounded.

Thus we know that T is a bounded linear operator. We now consider $R(T)$. We know every sequence in $R(T)$ is of the form $(\frac{x_i}{i})$. Consider the sequence of sequences (x_i) where $(x_i) = ()$