

HW 5  
1.5.1,2,3,8,10

**1.5.1.** Let  $a, b \in \mathbb{R}$  and  $a < b$ . Consider the open interval  $(a, b)$ . The sequence  $(a + \frac{b-a}{n}) \in (a, b)$ . Let  $\epsilon > 0$ . Choose  $N = \frac{(m-n)(b-a)}{\epsilon}$  such that  $m, n > N$ . Then,

$$|a + \frac{b-a}{m} - a - \frac{b-a}{n}| = |\frac{m(b-a) - n(b-a)}{mn}| = |\frac{(m-n)(b-a)}{mn}| < |\frac{(m-n)(b-a)}{N}| = \epsilon.$$

So the sequence is Cauchy. However as  $n \rightarrow \infty$ ,  $(a + \frac{b-a}{n}) \rightarrow a \notin (a, b)$ . So  $(a, b)$  is not complete.

Now let  $(x_n)$  be a Cauchy sequence on  $[a, b]$ . Then for every  $\epsilon > 0$ , there exists  $N$  such that for  $n, m > N$ ,  $|x_n - x_m| < \epsilon$ . Since  $\mathbb{R}$  is complete, we know that  $(x_n)$  converges. Let's say  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . So there exists  $\epsilon^* > 0$  such that for some  $N$ ,  $|x_n - x| < \epsilon^*$  for  $n > N$ . Thus, there exists an open ball  $B(x, \epsilon^*)$  such that  $x$  is an element of this ball.

**1.5.2.** Let  $X$  be the space of all ordered  $n$ -tuples  $x = (x_1, \dots, x_n)$  for real numbers and  $d(x, y) = \max_j |x_j - y_j|$  where  $y = (y_i)$ . Let  $(x_k)$  be a Cauchy sequence in  $X$ . Then for  $\epsilon > 0$ , there exists  $N$  with  $m, k > N$  such that  $d(x_k, x_m) = \max_j |x_{k_j} - x_{m_j}| < \epsilon$ . Since this is true,  $d(x_{k_i}, x_{m_i}) < \epsilon$  where  $x_{k_i}$  is the  $i^{th}$  entry of the  $k^{th}$  tuple. Since each entry is from  $\mathbb{R}$  and  $\mathbb{R}$  is complete, then for  $\frac{\epsilon}{n} > 0$ , there exists  $N$  such that when  $i > N$ ,  $|x_{k_i} - x_i^*| < \frac{\epsilon}{n}$ . Let  $x^* = (x_1^*, \dots, x_n^*)$ . Then  $d(x_k, x^*) = \max_j |x_{k_j} - x_j^*| < \epsilon$ . Since  $x^*$  is an ordered  $n$ -tuple of real numbers,  $x^* \in X$ . So  $(x_k)$  converges.

**1.5.3.** Let  $M \subset l^\infty$  be the subspace consisting of all sequences  $x = (x_i)$  with at most finitely many nonzero terms. Consider the sequence of sequences  $(x_n) = (1 + \frac{1}{n}, \dots, 1 + \frac{1}{n}, 0, 0, \dots)$  where the first  $n$  entries are  $1 + \frac{1}{n}$ . Thus, for  $\epsilon > 0$ , there exists  $N = \frac{1}{\epsilon}$  such that when  $m > n > N$ ,

$$d((x_n), (x_m)) = \sup_i |(x_n)_i - (x_m)_i| = |\frac{1}{m}| < \frac{1}{N} = \epsilon$$

. So  $(x_n)$  is Cauchy. However each entry is made up of entries in  $\mathbb{R}$  which is complete. So  $1 + \frac{1}{n}$  converges. Namely, as  $n \rightarrow \infty$ ,  $1 + \frac{1}{n} \rightarrow 1$ . So as  $n \rightarrow \infty$ ,  $(x_n) \rightarrow (1, 1, \dots)$ . This is not a sequence with finitely many nonzero terms, in fact it has infinite. Thus,  $(x_n)$  converges to a sequence not in  $M$ , so it does not converge. So  $M$  is not complete.

**1.5.8.** Consider  $Y \subset C[a, b]$  the space of all  $x \in C[a, b]$  where  $x(a) = x(b)$ . Let  $(x_n)$  be a Cauchy sequence in  $Y$ . Then for  $\epsilon > 0$ , there exists  $N$  such that when  $m, n > N$ ,  $d(x_n, x_m) = \max_{t \in [a, b]} |x_n(t) - x_m(t)| < \epsilon$ . Since  $C[a, b]$  is complete, we know that  $(x_n)$  converges to  $x$ . Thus, for  $\epsilon^* > 0$ , there exist  $M$  such that when  $n > M$ ,  $d(x_n, x) = \max_t |x_n(t) - x(t)| < \epsilon^*$ . Thus,  $|x_n(a) - x(a)| < \epsilon^*$  and  $|x_n(b) - x(b)| < \epsilon^*$ . Since  $x_n(a) = x_n(b)$ , we know that  $|x_n(b) - x(a)| < \epsilon^*$  and  $|x_n(a) - x(b)| < \epsilon^*$ . So  $|x(b) - x(a)| \leq |x(b) - x_n(a)| + |x_n(a) - x(a)| < \epsilon^* + \epsilon^* = 2\epsilon^*$ . Thus,  $x \in Y$ . So  $Y$  is complete.

**1.5.10.** Consider the discrete metric space  $X = (X, d)$ . Let  $(x_n)$  be a Cauchy sequence. Then for  $\epsilon > 0$ , there exist  $N$  such that for  $m, n > N$ ,  $d(x_n, x_m) < \epsilon$ . So  $d(x_n, x_m) = 0$ . Let  $x = x_m$ . Then  $d(x_n, x) = 0 < \epsilon$ . So  $(x_n)$  converges. Since  $x_m \in (x_n)$ ,  $x_m \in X$ . So  $X$  is complete.