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Math 540
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HW 5
1.5.1,2,3,8,10

1.5.1. Let $a, b \in \mathbb{R}$ and $a < b$. Consider the open interval (a, b). The sequence $(a + \frac{b-a}{n}) \in (a, b)$. Let $\epsilon > 0$. Choose $N = \frac{(m-n)(b-a)}{\epsilon}$ such that $m, n > N$. Then,

$$|a + \frac{b-a}{m} - a - \frac{b-a}{n}| = |\frac{m(b-a) - n(b-a)}{mn}| = |\frac{(m-n)(b-a)}{mn}| < |\frac{(m-n)(b-a)}{N}| = \epsilon.$$

So the sequence is Cauchy. However as $n \to \infty$, $(a + \frac{b-a}{n}) \to a \notin (a, b)$. So (a, b) is not complete.

Now let (x_n) be a Cauchy sequence on $[a, b]$. Then for every $\epsilon > 0$, there exists N such that for $n, m > N$, $|x_n - x_m| < \epsilon$. Since $\mathbb{R}$ is complete, we know that (x_n) converges. Let's say $x_n \to x$ as $n \to \infty$. So there exists $\epsilon^* > 0$ such that for some N, $|x_n - x| < \epsilon^*$ for $n > N$. By way of contradiction, assume that $x \notin [a, b]$. Since $[a, b]$ is closed, we know that $B(x, \epsilon^*) \notin [a, b]$ (i.e., $[a, b]^c$ is open). However $x_n \in [a, b]$ and $x_n \in B(x, \epsilon^*)$. This is a contradiction. So $x \in [a, b]$. Hence (x_n) converges in $[a, b]$. So $[a, b]$ is complete.

1.5.2. Let X be the space of all ordered n-tuples $x = (x_1, ..., x_n)$ for real numbers and $d(x, y) = \max_j |x_j - y_j|$ where $y = (y_i)$. Let (x_k) be a Cauchy sequence in X. Then for $\epsilon > 0$, there exists N with $m, k > N$ such that $d(x_k, x_m) = \max_j |x_{k_j} - x_{m_j}| < \epsilon$. Since this is true, $d(x_{k_i}, x_{m_i}) < \epsilon$ where x_{k_i} is the i^{th} entry of the k^{th} tuple. Since each entry is from $\mathbb{R}$ and $\mathbb{R}$ is complete, then for $\frac{\epsilon}{n} > 0$, there exists N such that when $i > N$, $|x_{k_i} - x_i^*| < \frac{\epsilon}{n}$. Let $x^* = (x_1^*, ..., x_n^*)$. Then $d(x_k, x^*) = \max_j |x_{k_j} - x_j^*| < \epsilon$. Since x^* is an ordered n-tuple of real numbers, $x^* \in X$. So (x_k) converges.

1.5.3. Let $M \subset l^\infty$ be the subspace consisting of all sequences $x = (x_i)$ with at most finitely many nonzero terms. Consider the sequence of sequences $(x_n) = (1 + \frac{1}{n}, ..., 1 + \frac{1}{n}, 0, 0, ...)$ where the first n entries are $1 + \frac{1}{n}$. Thus, for $\epsilon > 0$, there exists $N = \frac{1}{\epsilon}$ such that when $m > n > N$,

$$d((x_n), (x_m)) = \sup_i |(x_{n_i} - x_{m-i}| = |\frac{1}{n}| < \frac{1}{N} = \epsilon$$

. So (x_n) is Cauchy. However each entry is made up of entries in $\mathbb{R}$ which is complete. So $1 + \frac{1}{n}$ converges. Namely, as $n \to \infty$, $1 + \frac{1}{n} \to 1$. So as $n \to \infty$, $(x_n) \to (1, 1, ...)$. This is not a sequence with finitely many nonzero terms, in fact it has infinite. Thus, (x_n) converges to a sequence not in M, so it does not converge. So M is not complete.

1.5.8. Consider $Y \subset C[a, b]$ the space of all $x \in C[a, b]$ where $x(a) = x(b)$. Let (x_n) be a Cauchy sequence in Y. Then for $\epsilon > 0$, there exists N such that when $m, n > N$, $d(x_n, x_m) = \max_{t \in [a,b]} |x_n(t) - x_m(t)| < \epsilon$. Since $C[a, b]$ is complete, we know that (x_n) converges to x. Thus, for $\epsilon^* > 0$, there exist M such that when $n > M$, $d(x_n, x) = \max_t |x_n(t) - x(t)| < \epsilon^*$. Thus, $|x_n(a) - x(a)| < \epsilon^*$ and $|x_n(b) - x(b)| < \epsilon^*$. Since $x_n(a) = x_n(b)$, we know that $|x_n(b) - x(a)| < \epsilon^*$ and $|x_n(a) - x(b)| < \epsilon^*$. So $|x(b) - x(a)| \leq |x(b) - x_n(a)| + |x_n(a) - x_n(a)| < \epsilon^* + \epsilon^* = 2\epsilon^*$. Thus, $x \in Y$. So Y is complete.

1.5.10. Consider the discrete metric space $X = (X, d)$. Let (x_n) be a Cauchy sequence. Then for $\epsilon > 0$, there exist N such that for $m, n > N$, $d(x_n, x_m) < \epsilon$. So $d(x_n, x_m) = 0$. Let $x = x_m$. Then $d(x_n, x) = 0 < \epsilon$. So (x_n) converges. Since $x_m \in (x_n)$, $x_m \in X$. So X is complete.