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1/27/20
Math 540

HW 12
2.9.1,4,5,9,10

2.9.1. We solve for the null space as follows:

$$\begin{pmatrix} 1 & 3 & 2 \\ -2 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 2 \\ 0 & 7 & 4 \end{pmatrix}. \tag{1}$$

So $x_1 + 3x_2 + 2x_3 = 0$ and $7x_2 + 4x_3 = 0$. Thu,s $x_1 + 3x_2 = -2x_3$. Substituting we get $7x_2 - 2x_1 - 6x_2 = x_2 - 2x_1 = 0$. So $x_2 = 2x_1$. Substituting back into the first equation, we get that $x_1 + 3x_2 + 2x_3 = x_1 + 2x_1 + 3x_3 = 3x_1 + 3x_3 = 0$. So $x_3 = -x_1$. Thus the null space is spanned by $\begin{pmatrix} x_1 & 2x_1 & -x_1 \end{pmatrix}^T$ where $x_1 \in \mathbb{R}$.

2.9.4. Let $\{f_1, f_2, f_3\}$ be the dual basis of $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$ where $e_1 = (1,1,1), e_2 = (1,1,-1)$, and $e_3 = (1,-1,-1)$. Let $x = (1,0,0)$. So $x = e_1 + e_3$. Thus, $f_1(x) = \frac{1}{2}, f_2(x) = 0$, and $f_3(x) = \frac{1}{2}$.

2.9.5. Let f be a linear functional on an n-dimensional vector space X. Assume that f is the zero function, then $\mathscr{N}(f) = X$. So the null space is n-dimensional. Now assume that f is not the zero function. We know that the f maps to the real numbers (or complex numbers) which each have dimension one. Then by the rank-nullity theorem, we know that $n = \mathscr{N}(f) + 1$. So the null space must have dimension $n-1$.

2.9.9. Let X be the vector space of all real polynomials of a real variable and of degree less than a given n, together with $x = 0$. Let $f(x) = x^{(k)}(a)$, the value of the k^{th} derivative (k fixed) of $x \in X$ at a fixed $a \in \mathbb{R}$. We show that this is a linear functional. It is clear that f maps to the reals, so it suffices to show that it is linear. Let $x, y \in X$ and $\gamma, \lambda \in \mathbb{R}$. Then let $x = \alpha_0 + \alpha_1 r + \cdots + \alpha_n r^n$ and $y = \beta_0 + \beta_1 r + \cdots + \beta_n r^n$ where α_j and/or $\beta_j = 0$ for all $j > l$ if x (or y) has degree l. Then

$$\begin{aligned} f(\gamma x + \lambda y) &= f(\gamma(\alpha_0 + \alpha_1 r + \cdots + \alpha_n r^n) + \lambda(\beta_0 + \beta_1 r + \cdots + \beta_n r^n)) && (2) \\ &= f((\gamma\alpha_0 + \lambda\beta_0) + (\gamma\alpha_1 + \lambda\beta_1)r + \cdots + (\gamma\alpha_n + \lambda\beta_n)r^n) && (3) \\ &= ((\gamma\alpha_1 + \lambda\beta_1) + (\gamma\alpha_2 + \lambda\beta_2)a + \cdots + (\gamma\alpha_n + \lambda\beta_n)a^{n-1} && (4) \\ &= \gamma\alpha_1 + \gamma\alpha_2 a + \cdots + \gamma\alpha_n a^{n-1} + \lambda\beta_1 + \lambda\beta_2 a + \cdots + \lambda\beta_n a^{n-1} && (5) \\ &= \gamma(\alpha_1 + \alpha_2 a + \cdots + \alpha_n a^{n-1}) + \lambda(\beta_1 + \beta_2 a + \cdots + \beta_n a^{n-1}) && (6) \\ &= \gamma f(x) + \lambda f(y). && (7) \end{aligned}$$

So f is linear and thus a linear functional.

2.9.10. Let Z be a proper subspace of an n-dimensional vector space X, and let $x_0 \in X - Z$. We define a basis of Z as $\{z_1, z_2, ..., z_r\}$. Since $x_0 \notin X - Z$, we extend the basis of Z to $\{z_1, ..., z_r, x_0\}$. Then we extend this basis to be a basis of X by $\{z_1, ..., z_r, x_0, x_1, ..., x_k\}$. So every $x \in X$ can be written as $x = \sum_{i=1}^{r} \alpha_i z_i + \beta_0 x_0 + \sum_{i=1}^{k} \beta_i x_i$. Then we define the dual basis functions as $f_v(x) = \delta_{vx}$ as described in the chapter. From the chapter, we know that these are linear functionals. Then the basis functional, $f_{x_0}(x_0) = 1$ but $f_{x_0}(z) = 0$ for all $z \in Z$.