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1/25/20
Math 525

1. C
2. B
3.

1. 7.3

   (a) Consider Katz centrality $\mathbf{x} = \alpha A\mathbf{x} + \mathbf{1}$. Solving we get $(I - \alpha A)\mathbf{x} = \mathbf{1}$. If we can take an inverse (which we can with probability 1), we get that $\mathbf{x} = (I - \alpha A)^{-1}\mathbf{1}$. We then see we have a geometric series. So

$$\mathbf{x} = \Big(\sum_{i=0}^{\infty}(\alpha A)^r\Big)\mathbf{1} \tag{1}$$
$$= \mathbf{1} + \alpha A\mathbf{1} + \alpha^2 A^2\mathbf{1} + \cdots . \tag{2}$$

   (b) Recall that $A\mathbf{1} = \mathbf{k}$. So we have that $\mathbf{x} = \mathbf{1} + \alpha\mathbf{k} + \alpha^2 A\mathbf{k} + \cdots$. Thus, we get that $\mathbf{x} = \mathbf{1} + (\sum_{i=0}^{\infty}\alpha^{i+1}A^i)\mathbf{k} = \mathbf{1} + \alpha(\sum_{i=0}^{\infty}\alpha^i A^i)\mathbf{k}$. Thus, for small α, we get that $\mathbf{x} \approx \mathbf{1} + \alpha\mathbf{k}$ since the terms $\alpha^r\mathbf{k}$ for $r \geq 2$ go to zero much faster. Adding $\mathbf{1}$ to $\mathbf{k}$ does not change the ranking of $\mathbf{k}$ and multiplying each entry of $\mathbf{k}$ by the same constant α does not change the ranking. So $\mathbf{x}$ gives the same ranking at $\mathbf{k}$ (i.e. degree centrality) as $\alpha \to 0$.

   (c) Consider when $\alpha \to \frac{1}{\kappa_1}$. Then by part (a) we have that $\mathbf{x}$ is approaching

$$\mathbf{x} \to \mathbf{1} + \frac{1}{\kappa_1}A\mathbf{1} + \frac{1}{\kappa_2^2}A^2\mathbf{1} + \cdots \tag{3}$$

   This implies that $\kappa_1\mathbf{x} \to \kappa_1\mathbf{1} + A\mathbf{1} + \frac{1}{\kappa_1}A^2\mathbf{1} + \cdots$. Again, evaluating on the dominant terms gives that $\kappa_1\mathbf{x} \approx \kappa_1\mathbf{1} + A\mathbf{1} = (\kappa_1 I + A)\mathbf{1}$.
   This implies that $(I - \alpha A)\mathbf{x} = \mathbf{1} \to (I - \frac{1}{\kappa_1}A)\mathbf{x} = \mathbf{1}$.

2. 7.5: Let the center node be defined as x_1. Then

$$x_1 = \alpha\sum_j A_{1j}\frac{x_j}{k_j^{out}} + \beta.$$

   Since we are looking at a tree, we know that $k_j^{out} = 1$ for all nodes x_j. So we get that

$$x_1 = \alpha\sum_j A_{1j}x_j + \beta.$$

   This is just Katz centrality. Thus by problem 7.3, we see that

$$x_1 = \beta(1 + \alpha\sum_j A_{1j}\mathbf{1} + \alpha^2\sum_j A_{1j}^2\mathbf{1} + \cdots).$$

   Recall that $\sum_j A_{1j}^r\mathbf{1}$ counts the number of walks of length r to 1. Since G is a tree, we know that the number of walks of length r to 1 is the number of nodes with $d_i = r$ (where d_i is the distance from i to the center node). This means that $x_1 = \beta(1 + \sum_i \alpha^{d_i})$.