Cory Glover
Math 540
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HW 3

1.3.1,5,8,14,15

1.3.1. Let $x \in B(x_0;\epsilon)$. Then $d(x,x_0) < \epsilon$. Let Y be a set such that for all $y \in Y$, $d(y,x_0) = \epsilon$. Let $y_0 = \text{argmin}_{y\in Y} d(y,x)$. Define $\delta = d(y_0,x)$. Then there exists a ball $B(x;\delta/2)$ such that $x \in B(x;\delta/2)$. Since all the points $z \in B(x;\delta/2)$, and $d(x,y_0) > \delta/2$, then $z \in B(x_0;\epsilon)$. So there exists an open ball around every $x \in B(x_0;\epsilon)$. So an open ball is an open set.

Consider the closed ball $\overline{B(x_0;\epsilon)}$. Let x be an element of the complement of the closed ball. Let Y be a set such that for all $y \in Y$, $d(y,x_0) = \epsilon$. Let $y_0 = \text{argmin}_{y\in Y} d(y,x)$. Define $\delta = d(y_0,x)$. Then there exists a ball $B(x;\delta/2)$ such that $x \in B(x;\delta/2)$. Since all the points $z \in B(x;\delta/2)$, and $d(x,y_0) > \delta/2$, then $z \notin \overline{B(x_0;\epsilon)}$. So the complement of the closed ball $\overline{B(x_0;\epsilon)}$ is open. So every closed ball is a closed set.

1.3.5. Consider the set X. From the properties of a topology (T1), X is an open set. Note that $X^C = \emptyset$. By the properties of a topology (T1), $\emptyset$ is an open set. So X is also a closed set.

Similarly, by the properties of a topology (T1), $\emptyset$ is an open set, but since X is open and $\emptyset^C = X$, $\emptyset$ is also a closed set.

Consider a metric space $X = (X,d)$ where d is the discrete metric. Let $Y \subset X$ and let $y \in Y$. Then $B(y;1) = \{y\}$ so $B(y;1) \in Y$ since $y \in Y$. So every subset of X is open.

Now consider Y^c and let $z \in Y^C$. Then $B(z;1) = \{z\}$. So $B(z;1) \in Y^C$. So Y^C is open. Thus, Y is closed.

So all subsets of X are both open and closed.

1.3.8. Consider the space $X = (X,d)$ where d is the discrete metric. Then the

1.3.14. Assume that $T\colon X \to Y$ is continuous. Let $M \subset Y$ and $T^{-1}(M)$ be the inverse image of M. Let M be closed. Then M^C is open. Since T is continuous, $T^{-1}(M^C)$ is also open. Let $m \in T^{-1}(M)^C$. Then $T(m) \in M^c$. So $T^{-1}(M)^C \subset T^{-1}(M^C)$. Since $T^{-1}(M^C)$ is open, then $T^{-1}(M)^C$ is open. So $T^{-1}(M)$ is closed.

Let $T\colon X \to Y$ be a mapping from X to Y. Let M be a closed set in Y. Then M^c is open. Assume that $T^{-1}(M)$ is also closed. Let $m \in M^C$. Since M is closed, then m has an open ball around it still contained in M^C. Then $T^{-1}(m) \notin T^{-1}(M)$. So $T^{-1}(m) \in T^{-1}(M)^c$. Thus, $T^{-1}(m)$ has open ball around it still contained in $T^{-1}(M)^c$ since $T^{-1}(M)$ is closed. Since m is arbitrary, $T^{-1}(M)^c$ is open. So the inverse image of every open set in Y is an open set in X. Thus, T is continuous.

1.3.15. Let $X = (\mathbb{R},d)$ where d is the discrete metric and $Y = (\mathbb{R},\overline{d})$ where $\overline{d}$ is the Euclidean metric. Define $T(A) = A$ where A is a subset of the real line. Then for every open set $A \subset Y$, $T^{-1}(Y)$ is open (since every subset in X is open. However, consider the subset $[0,1] \in X$. This set is open, however $T(X)$ is not open.