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Math 522
1/9/20

HW 1
2.1,2a,2b,2c,6,12,13,14

2.1.

1. Let Ω be the entire set. Note that $\Omega \cap \emptyset = \emptyset$. Then
$$P(\Omega) = P(\Omega \cup \emptyset) = P(\Omega) + P(\emptyset).$$
So $1 = 1 + P(\emptyset)$. Thus, $P(\emptyset) = 0$.

2. Assume that $\alpha \subseteq \beta$. Note that $\beta = (\beta \cap \alpha^c) \cup \alpha$. Note that $(\beta \cap \alpha^c) \cap \alpha = \emptyset$. Thus,
$$\begin{aligned} P(\beta) &= P((\beta \cap \alpha^c) \cup \alpha) && (1)\\ &= P(\beta \cap \alpha^c) + P(\alpha) && (2)\\ &\geq P(\alpha) \qquad \text{since } P(\beta \cap \alpha^c) \geq 0 && (3) \end{aligned}$$

3. Note the following $\alpha \cup \beta = (\alpha \cup \beta) \cap (\beta \cup \beta^c)$ since $(\beta \cup \beta^c)$. This means that $\alpha \cup \beta = \beta \cup (\alpha \cap \beta^c)$. Then $P(\alpha \cup \beta) = P(\beta) + P(\alpha \cap \beta^c)$. Further note that $P(\alpha) = P((\beta \cap \alpha) \cup (\beta^c \cap \alpha)) = P(\beta \cap \alpha) + P(\beta^c \cap \alpha)$. This tells us that $P(\alpha) - P(\beta \cap \alpha) = P(\beta^c \cap \alpha)$. Plugging into our first equation we have
$$P(\alpha \cup \beta) = P(\beta) + P(\beta) - P(\beta \cap \alpha).$$

2.2.

1. Let X and Y be binary random variables. Assume that $x^0 \perp y^0$. We want to show that $x^0 \perp y^1, x^1 \perp y^0, x^1 \perp y^1$.
   (a) Note that $P(x^0 \mid y^0) = P(x^0)$. So $1 - P(x^0 \mid y^0) = 1 - P(x^0)$. So $P(x^1 \mid y^0) = P(x^1)$. So $x^1 \perp y^0$.
   (b) Note that $P(y^0 \mid x^0) = P(y^0)$. So $1 - P(y^0 \mid x^0) = 1 - P(y^0)$. Then $P(y^1 \mid x^0) = P(y^1)$. So $y^1 \perp x^0$.
   (c) Then $P(y^1 \mid x^1) = 1 - P(y^0 \mid x^1) = 1 - P(y^0) = P(y^1)$. So $x^1 \perp y^1$.

2. Define X where $P(x^0) = .2, P(x^1) = .4$, and $P(x^2) = .4$. Define Y where $P(y^0) = .3$, $P(y^1 \mid x^0) = .1$, $P(y^1 \mid x^1) = .4$, $P(y^1 \mid x^2) = 0$, $P(y^3) = .2$.

3. This is not the case. Let Z be a binary valued random variable. Let X and Y be random variables, each with three possible events. Assume that $x^0 \perp y^0$ always. Assume that if z^0 occurs, $P(x^2) = 0$ and $P(y^2) = 0$. Then X and Y are independent by part (i). However assume that $P(x^i \mid z^1) \neq 0$ and $P(y^i \mid z^1) \neq 0$ for all i. Then X and Y are not necessarily independent by part ii. So $(X \perp Y \mid z^0) \not\Rightarrow (X \perp Y \mid Z)$ since X and Y are not necessarily independent given z^1.

6. Let X, Y and Z be random variables. Then
$$\begin{aligned} \sum_z P(X, z \mid Y) &= \sum_z P(X \mid Y) P(z \mid X, Y) \qquad \text{by the chain rule} && (4)\\ &= P(X \mid Y) \sum_z P(z \mid X, Y) && (5)\\ &= P(X \mid Y) \qquad \text{since } \sum_z P(z \mid X, Y) = 1. && (6) \end{aligned}$$

So $\sum_z P(X,z \mid Y) = P(X \mid Y)$.

12. Let X be a random variable such that $P(X \geq 0) = 1$, then for any $t \geq 0$

$$\begin{aligned}
\mathbb{E}(X) &= \sum_x xP(X = x) && (7)\\
&\geq \sum_{x \geq t} xP(X = x) && (8)\\
&\geq \sum_{x \geq t} tP(X = x) && (9)\\
&= t\sum_{x \geq t} P(X = x) && (10)\\
&= tP(X \geq t). && (11)
\end{aligned}$$

So $\frac{\mathbb{E}(X)}{t} \geq P(X \geq t)$. (I used a hint online as I couldn't figure out the proof and didn't want to just look at the proof in the ACME textbook. Due to this, I probably shouldn't receive credit for the problem).

13. Let X be a random variable. Then

$$\begin{aligned}
P(|X - \mathbb{E}(X)| \geq t) &= P((X - E(X))^2 \geq t^2) && (12)\\
&\leq \frac{\mathbb{E}((X - \mathbb{E}(X))^2)}{t^2} \quad \text{by Markov's inequality} && (13)\\
&= \frac{\mathrm{Var}(X)}{t^2} && (14)
\end{aligned}$$

14. Let $X \sim \mathscr{N}(\mu;\sigma^2)$. Let $Y = aX+b$. Then $\mathbb{E}(Y) = \mathbb{E}(aX+b) = \mathbb{E}(aX)+\mathbb{E}(b) = a\mathbb{E}(X)+b = a\mu+b$. Further, $\mathrm{Var}(Y) = \mathrm{Var}(aX+b) = a^2\mathrm{Var}(X) = a^2\sigma^2$. Further we note that if $a = 0$, then Y gives probability b to all events (since the expected value is b and the variance is 0). This is equivalent to $Y \sim \mathscr{N}(b, 0)$. If a is not 0, then

$$\begin{aligned}
P(Y \geq y) &= P(aX + b \geq y) && (15)\\
&= P(aX \geq y - b) && (16)\\
& && (17)
\end{aligned}$$

If $a > 0$, then we get that

$$P(Y \geq y) = P(X \geq \frac{y-b}{a}). \tag{18}$$

If $a < 0$, then we get that

$$\begin{aligned}
P(Y \geq y) &= P(X \leq \frac{y-b}{a}) && (19)\\
&= 1 - P(X \geq \frac{y-b}{a}). && (20)
\end{aligned}$$

Since X is distributed normally, and Y can be written in terms of of the probability of X, Y must also be normal. So $Y \sim \mathscr{N}(a\mu + b; a^2\sigma^2)$.