Cory Glover
1/27/20
Math 540

HW 11
2.7.2,6,10,14
2.8.2,4,11

**2.7.2.** Let $X$ and $Y$ be normed spaces. Assume that a linear operator $T\colon X \to Y$ is bounded. Let $A \subseteq X$ be a set that is bounded. So for all $a \in A$, $\|a\| \leq M$ for some $M \in \mathbb{R}$ (or $\mathbb{C}$. The same proof follow either way). Let $b \in B$ such that $T(A) = B$. Thus, there exists an $a \in A$ such that $T(a) = b$. We know that $\|T(a)\| \leq c\|a\|$ for some $c$. Thus, $\|b\| \leq c\|a\|$. Since $A$ is bounded, then $\|b\| \leq c\|a\| \leq cM$. Since $cM \in \mathbb{R}$ and is constant, then $\|b\|$ is bounded. Since $b$ was arbitrary, $B$ must be bounded. So $T$ maps bounded sets into bounded sets.

Now assume that $T\colon X \to Y$ maps bounded sets in $X$ to bounded sets in $Y$. Let $x \in X$. Then $\|Tx\| = \|y\|$ where $y \in Y$. If $x$ is in some bounded set of $X$, then $\|Tx\| \leq c$ for some $c$ since $y$ must be in a bounded set. Let $d = \frac{c}{\|x\|}$. Then $\|Tx\| \leq d\|x\|$.

Now assume that $x$ is not in some bounded set of $X$. So $\{x\}$ is unbounded. This means that $\|x\| = \infty$. This is a contradiction. So all $x$ are in some bounded set. Thus by the first part of the proof, $T$ is a bounded operator.

**2.7.6.** Let $T\colon l^\infty \to l^\infty$ defined on $y = (y_i) = Tx$, $y_i = \frac{x_i}{i}$ and $x = (x_i)$. We first show that $T$ is linear and bounded.

1. (Linear): Let $a, b \in l^\infty$ be denoted $a = (a_i), b = (b_i)$. Let $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}
T(\alpha a + \beta b) &= T((\alpha a_i) + (\beta b_i)) && (1)\\
&= T((\alpha a_i + \beta b_i)) && (2)\\
&= (\frac{\alpha a_i + \beta b_i}{i}) && (3)\\
&= (\frac{\alpha a_i}{i}) + (\frac{\beta b_i}{i}) && (4)\\
&= \alpha(\frac{a_i}{i}) + \beta(\frac{b_i}{i}) && (5)\\
&= \alpha T(a) + \beta T(b). && (6)
\end{aligned}$$

   So $T$ is linear.

2. (Bounded): Note that $\|Tx\| = \|(\frac{x_i}{i})\|$ Since $(\frac{x_i}{i}) \in l^\infty$, each $\frac{x_i}{i}$ is bounded by some $M$. Thus, $\|(\frac{x_i}{i})\| = \sup_i \|\frac{x_i}{i}\| \leq M$. Let $c = \frac{M}{\|x\|}$. Then $\|Tx\| = \|(\frac{x_i}{i})\| = \sup_i |\frac{x_i}{i}| \leq M = c\|x\|$. So $\|Tx\| \leq c\|x\|$. So $T$ is bounded.

Thus we know that $T$ is a bounded linear operator. We now consider $R(T)$. We know every sequence in $R(T)$ is of the form $(\frac{x_i}{i})$. Consider the sequence of sequences $(x_i)$ where $(x_n) = (1, 2, 3, ..., n, 0, 0, ...)$. Then $Tx_n = (1, 1, ..., 1, 0, ...)$. Note that as $n \to \infty$, $Tx_n \to (1, 1, ...)$. Since $(1, 1, 1, ...)$ is bounded by 1, $(1, 1, 1, ...) \in l^\infty$. However $T^{-1}(1, 1, 1, ...) = (1, 2, 3, ....) \notin X$. So $(1, 1, 1, ...) \notin R(T)$. Thus, $R(T)$ is not closed.

**2.7.10.** Let $X$ be $C[0, 1]$ and define $S\colon = y(s) = s\int_0^1 x(t)dt$ and $T\colon = y(s) = sx(s)$. First note that

$$TSx(s) = T(s\int_0^1 x(t)dt) = s^2 \int_0^1 x(t)dt \tag{7}$$

$$STx(s) = S(sx(s)) = s\int_0^1 tx(t)dt. \tag{8}$$

So $ST \neq TS$.

Then we see that

$$\|S\| = \sup_{x\in\mathscr{D},\|x\|=1} \|Sx\| \tag{9}$$

$$= \sup_{x\in\mathscr{D},\|x\|=1} \|s\int_0^1 x(t)dt\| \tag{10}$$

$$= \max_{s\in[0,1]} |s\int_0^1 x(t)dt| \tag{11}$$

$$= |\int_0^1 x(t)|dt \tag{12}$$

$$= \|x\|. \tag{13}$$

$$\|T\| = \sup_{x\in\mathscr{D}(T),\|x\|=1} \|Tx\| \tag{14}$$

$$= \max_{s\in[0,1]} |sx(s)|. \tag{15}$$

$$\|ST\| = \max_{s\in[0,1]} |s\int_0^1 tx(t)dt| \tag{16}$$

$$= |\int_0^1 tx(t)dt| \tag{17}$$

$$= \|tx(t)\| \tag{18}$$

$$\|TS\| = \max_{s\in[0,1]} |s^2\int_0^1 x(t)dt| \tag{19}$$

$$= |\int_0^1 x(t)dt| \tag{20}$$

$$= \|x\|. \tag{21}$$

**2.7.14.** Let $\|x\|_1 = \sum_{k=1}^n |x_k|$ and $\|y\|_2 = \sum_{j=1}^r |y_j|$. Let $A$ be a $r \times n$ matrix $A = (\alpha_{jk})$ which defines a linear operator from the vector space $X$ of all ordered $n$-tuples of numbers into the vector space $Y$ of all ordered $r$-tuples of numbers. Then we see that

$$\|Ax\|_2 = \sum_{i=1}^r |\sum_{j=1}^n \alpha_{ij}x_j| \tag{22}$$

$$\leq \sum_{i=1}^r \sum_{j=1}^n |\alpha_{ij}x_j| \tag{23}$$

$$\leq \max_k \sum_{i=1}^r \sum_{j=1}^n |\alpha_{ik}x_j| \tag{24}$$

$$\leq \max_k \sum_{i=1}^r |\alpha_{ik}| \sum_{j=1}^n |x_j| \tag{25}$$

$$= \|A\|\|x\|_1. \tag{26}$$

So $\|A\|$ is compatible.

**2.8.2.** Let $y_0 \in C[a, b]$. Then we see that for $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$), that

$$
\begin{aligned}
f_1(\alpha x + \beta y) &= \int_a^b (\alpha x + \beta y)(t) y_0(t) dt && (27)\\
&= \int_a^b ((\alpha x)(t) + \beta y(t)) y_0(t) dt && (28)\\
&= \int_a^b (\alpha x(t) + \beta y(t)) y_0(t) dt && (29)\\
&= \int_a^b \alpha x(t) y_0(t) + \beta y(t) y_0(t) dt && (30)\\
&= \int_a^b \alpha x(t) y_0(t) dt + \int_a^b \beta y(t) y_0(t) dt && (31)\\
&= \alpha \int_a^b x(t) y_0(t) + \beta \int_a^b y(t) y_0(t) dt && (32)\\
&= \alpha f_1(x) + \beta f_2(y). && (33)
\end{aligned}
$$

So $f_1$ is linear. We now note that the continuity of $y_0(t)$ on a closed square implies that $y_0$ is bounded, say $|y_0(t)| \leq c$. So,

$$
\begin{aligned}
\|f_1(x)\| &= |\int_a^b x(t) y_0(t) dt| && (34)\\
&\leq c|\int_a^b x(t)| && (35)\\
&\leq c(b-a) \max_{t \in [a,b]} |x(t)| && (36)\\
&= c(b-a)\|x\|. && (37)
\end{aligned}
$$

Thus, $\|f_1(x)\| \leq c\|x\|$. So $f_1$ is bounded.

Now consider $f_2(x) = \alpha x(a) + \beta x(b)$ for some fixed $\alpha, \beta$. Let $r, s$ be scalars. Then we see that

$$
\begin{aligned}
f_2(rx + sy) &= \alpha((rx + sy)(a)) + \beta((rx + sy)(b)) && (38)\\
&= \alpha((rx)(a) + (sy)(a)) + \beta((rx)(b) + (sy)(b)) && (39)\\
&= \alpha(rx)(a) + \alpha(sy)(a) + \beta(rx)(b) + \beta(sy)(b) && (40)\\
&= r\alpha x(a) + r\beta x(b) + s\alpha y(a) + s\beta y(b) && (41)\\
&= r(\alpha x(a) + \beta x(b)) + s(\alpha y(a) + \beta y(b)) && (42)\\
&= r f_2(x) + s f_2(y). && (43)
\end{aligned}
$$

So $f_2$ is linear.

We now show that $f_2$ is bounded.

$$
\|f_2(x)\| = |\int_a^b \alpha x(a) + \beta \qquad (44)
$$