Definition

Metric

Definition

Metric Space

Definition

Supspace (of a metric space)

The function d is a *metric* on X, defined on $X \times X$ such that for all $x, y, z \in X$ we have:

1. d is finite, real-valued, and non-negative,
2. $d(x, y) = 0$ if and only if $x = y$,
3. $d(x, z) \leq d(x, y) + d(y, z)$,
4. $d(x, y) = d(y, x)$.

A *metric space* is a set X with an associated metric d. Denoted (X, d).

A subset $Y \subset X$ of a metric space (X, d) where the associated metric is $d|_{Y \times Y}$, the metric on X restricted to $Y \times Y$. This metric is said to be induced.

EXAMPLE

Metric on $\mathbb{R}^n$ *(*$\mathbb{C}^n$*)*

EXAMPLE

Metric on l^∞

DEFINITION

l^∞

$$d(\curvearrowleft, \curvearrowright) = \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2}$$

Let $x = (x_i)$ and $y = (y_i)$. Then $d(x, y) = \sup_i |x_i - y_i|$.

The set of all bounded sequences of complex numbers.

Example

Metric on $C[a,b]$

Example

Discrete Metric

Example

Metric on General Sequence Space

Let $x(t), y(t) \in C[a,b]$. Then $d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|$.

Let $x, y \in X$. Then $d(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$.

Let X be a set of sequences (not necessarily bounded). Then on metric is

$$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|}.$$

Example

Metric on space of bounded functions $B(A)$

Example

Metric on l^p for $p \geq 1$

Theorem

Hölder Inequality

Let $x(t), y(t) \in B(A)$. Then

$$d(x,y) = \sup_{t \in A} |x(t) - y(t)|.$$

Let $(x), (y) \in l^p$. Then

$$d(x,y) = \Big(\sum_{j=1}^{\infty} |x_j - y_j|^p \Big)^{1/p}.$$

Let $p > 1$ and define q by $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{j=1}^{\infty} |x_j y_j| \leq \Big(\sum_{j=1}^{\infty} |x_i|^p \Big)^{1/p} \Big(\sum_{j=1}^{\infty} |y_j|^q \Big)^{1/q}.$$

Theorem

Cauchy-Schwarz Inequality

Theorem

Minkowski's Inequality

Definition

Open Ball

Hölder Inequality for $p = 2$:

$$\sum_{j=1}^{\infty} |x_j y_j| \leq \sqrt{\sum_{j=1}^{\infty} |x_i|^2} \sqrt{\sum_{j=1}^{\infty} |y_j|^2}.$$

Let $p \geq 1$. Then

$$\left(\sum_{j=1}^{\infty} |x_i + y_i|^p\right)^{1/p} = \left(\sum_{j=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{j=1}^{\infty} |y_i|^p\right)^{1/p}.$$

Given a point $x_0 \in X$ and a real number $r > 0$, then a ball $B(x_0; r) = \{x \in X : d(x, x_0) < r\}$.

Definition

Sphere

Definition

Closed Ball

Definition

Open Set

Given a point $x_0 \in X$ and a real number $r > 0$, then a sphere $S(x_0; r) = \{x \in X : d(x, x_0) = r\}$.

Given a point $x_0 \in X$ and a real number $r > 0$, then a closed ball is $\widetilde{B}(x_0; r) = \{x \in X : d(x, x_0) \leq r\}$.

Let $M \subseteq X$ be a subset of X. Then M is an open set if for every $x \in M$, there exists an open ball $B(x; r) \in M$ for some $r > 0$. (If $r = \epsilon$, then this is an ϵ-neighborhood).

Definition

Closed Set

Math 540: Linear Analysis (Midterm 1)

Definition

Interior of a set M

Math 540: Linear Analysis (Midterm 1)

Definition

Topology

Math 540: Linear Analysis (Midterm 1)

Let $M \subseteq X$ be a subset of X. Then M is a closed set if M^c is open.

An interior point x of M is a point x where M is a neighborhood containing x. The interior of M, M^0, is the set of all interior points of M. This is the largest open set contained in M.

A collection $\mathscr{T}$ of open subsets of X. It satisfies the following properties:

1. $\emptyset \in \mathscr{T}, X \in \mathscr{T}$
2. The union of any members of $\mathscr{T}$ is a member of $\mathscr{T}$
3. The intersection of finitely many members of $\mathscr{T}$ is a member of $\mathscr{T}$.

Definition

Continuous Mapping

Theorem

Continuous Mapping Theorem

Math 540: Linear Analysis (Midterm 1)

Definition

Accumulation Points and Closure

Math 540: Linear Analysis (Midterm 1)

Let $X = (X, d)$ and $Y = (Y, \widetilde{d})$ be metrics spaces. A mapping $T \colon X \to Y$ is continuous at x_0 if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\widetilde{}(Tx, Tx_0) < \epsilon$ for all $x \in X$ satisfying $d(x, x_0) < \delta$. T is continuous if it is continuous for all $x \in X$.

Let $X = (X, d)$ and $Y = (Y, \widetilde{d})$ be a metric spaces and let $T \colon X \to Y$. Then T is continuous if and only if the inverse image of any open set in Y is an open set in X.

A point $x_0 \in X$ is an accumulation point of M is for every neighborhood of x_0, there is at least one point $y \in M$ distinct from x_0. The set consisting of all the points in M and the accumulation points of M is the closure of M, denoted $\overline{M}$.

Definition

Dense Set and Separable Space

Example

Examples of Separable Sets

Example

Unexample of separable sets

A subset of M of a metric space is dense in X if $\overline{M} = X$. X is said to be separable if it has a countable subset which is dense in X.

1. $\mathbb{R}^n$
2. $\mathbb{C}^n$
3. A discrete metric space X is separable if and only if X is countable.
4. The space l^p for $1 \leq p < \infty$ is separable.

l^∞

Definition

Convergence of a sequence, limit

Definition

Bounded Set (in a metric space)

Theorem

Boundedness, limit on metric spaces

A sequence $(x_n) \in X$ converges if there exists some $x \in X$ such that for $\epsilon > 0$, there exists $N > 0$ such that when $n > N$, then $d(x_n, x) < \epsilon$. The limit of (x_n) is said to be x.

A set M where $\sup_{x,y \in M} d(x, y) < \infty$.

Let $X = (X, d)$ be a metric space. Then

1. A convergent sequence in X is bounded and its limit is unique.
2. If $x_n \to x$ and $y_n \to y$, then $d(x_n, y_n) \to d(x, y)$.

Definition

Cauchy Sequences and Completeness

Example

Examples of Complete Metric Space

Theorem

Convergent Sequences and Cauchy Sequences

A sequence (x_n) in a metric space (X, d) is Cauchy if for every $\epsilon > 0$, there exists an N such that $d(x_n, x_m) < \epsilon$ for $n, m > N$.
The metric space X is said to be complete if every Cauchy sequence converges.

1. Both $\mathbb{R}$ and $\mathbb{C}$.
2. l^∞
3. The space c of all convergent sequences of complex numbers, with the metric induced from l^∞
4. l^p
5. $C[a, b]$

Every convergent sequence in a metric space $X = (X, d)$ is Cauchy in X.

Theorem

Closure and Closed Sets

Theorem

Complete Subspace

Theorem

Continuous mapping (convergence)

Let M be a nonempty subset of a metric space (X, d) and $\overline{M}$ its closure. Then

1. $x \in \overline{M}$ if and only if there is a sequence (x_n) in M such that $x_n \to x$,
2. M is closed if and only if having $x_n \in M$, then $x_n \to x$ implies that $x \in M$.

A subspace M of a complete metric space X is complete if and only if the set M is closed in X.

A mapping $T\colon X \to Y$ of a metric space $X = (X, d)$ into a metric space $Y = (Y, \widetilde{d})$ is continuous at a point $x_0 \in X$ if and only if $x_n \to x_0$ implies $Tx_n \to Tx_0$.

Theorem

Uniform Convergence

Example

Unexamples of Complete Metric Spaces

Defintion

Isometric Mapping, Isometric Spaces

Convergence $x_m \to x$ in the space $C[a,b]$ is uniform convergence.

1. $\mathbb{Q}$
2. Polynomials
3. Continuous functions

Let $X = (X, d)$ and $\widetilde{X} = (\widetilde{X}, \widetilde{d})$ be metric spaces. Then

1. A mapping T of X into $\widetilde{X}$ is said to be isometric or an isometry if T preserves distances, that is, if for all $x, y \in X$, $\widetilde{d}(Tx, Ty) = d(x, y)$,
2. The space X is said to be isometric with the space $\widetilde{X}$ if there exists a bijective isometry of X onto $\widetilde{X}$.

Theorem

Metric Space Completion Theorem

Definition

Vector Space

Example

Examples of Vector Spaces

For a metric space $X = (X, d)$ there exists a complete metric space $\hat{X} = (\hat{X}, \hat{d})$ which has a subspace W that is isometric with X and is dense in $\hat{X}$. This space $\hat{X}$ is unique up to isometries.

A nonempty set X over a field K with addition and multiplication.

1. $\mathbb{R}^n$
2. $\mathbb{C}^n$
3. $C[a, b]$
4. l^2

Definition

Hamel Basis

Definition

Normed space, Banach Space

Definition

Norm

Let X be a vector space (not necessarily finite dimensional). Then a linearly independent subset of X which spans X is called a Hamel basis.

A normed space X is a vector space with a norm. A Banach space is a complete normed space.

Let X be a vector space. Then $\|\cdot\|$ is a norm on X if it is a real-valued function such that

1. $\|x\| \geq 0$
2. $\|x+y\| \leq \|x\| + \|y\|$
3. $\|x\| = 0$ if and only if $x = 0$
4. $\|\alpha x\| = |\alpha|\|x\|$.

Definition

Metric Induced by a norm

Definition

Reverse triangle inequality

Example

Norm on $\mathbb{R}^n$ and $\mathbb{C}^n$

$$d(x, y) = \|x - y\|$$

$$|\|y\| - \|x\|| \leq \|y - x\|$$

$$\|x\| = \left(\sum_{j=1}^{\infty} |x_j|^2\right)^{1/2}.$$

Example

Norm on l^p

Example

Norm on l^∞

Example

Norm on $C[a,b]$

$$\|x\| = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}$$

$$\|x\| = \sup_i |x_i|$$

$$\|x\| = \max_{t\in[a,b]} |x(t)|$$

Example

L^p

Theorem

Translation-Invariance

Theorem

Subspace of a Banach Space

Completion of the normal space of all continuous real-value functions with norm $\|x\|_p = \left(\int_a^b |x(t)|^p dt\right)^{1/p}$

A metric d induced by a norm on a normed space X satisfies:

$$d(x + a, y + a) = d(x, y)$$

$$d(\alpha x, \alpha y) = |\alpha| d(x, y)$$

for all $x, y, a \in X$ and every scalar α.

A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.

Definition

Schauder Basis

Theorem

Completion of Normed Spaces

Theorem

Linear combinations

If a normed space X contains a sequence $(e - n)$ with the property that for every $x \in X$ there is a unique sequence of scalars (α_n) such that $\|x - (\alpha_1 e_1 + \cdots + \alpha_n e_n)\| \to 0$ as $n \to \infty$, then (e_n) is a Schauder basis for X.

Let $X = (X, \|\cdot\|)$ be a normed space. Then there is a Banach space $\hat{X}$ and an isometry A from X onto a subspace W of $\hat{X}$ which is dense in $\hat{X}$. The space $\hat{X}$ is unique, except for isometries.

Let $\{x_1, ..., x_n\}$ be a linearly independent set of vectors in a normed space X (of any dimension). Then there is a number $c > 0$ such that for every choice of scalars $\alpha_1, ..., \alpha_n$ we have

$$\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|).$$

THEOREM

Finite Dimensional subspaces and completeness

THEOREM

Finite Dimensional and Closed

DEFINITION

Equivalent Norms

Every finite dimensional subspace Y of a normed space X is complete. In particular, every finite dimensional normed space is complete.

Every finite dimensional subspace Y of a normed space X is closed in X.

A norm $\|\cdot\|$ on a vector space X is said to be equivalent to a norm $\|\cdot\|_0$ on X if there are positive numbers a and b such that for all $x \in X$, we have $a\|x\|_0 \leq \|x\| \leq b\|x\|_0$.

THEOREM

Finite Dimensional: Equivalent Norms

THEOREM

Topology: Equivalent norms

DEFINITION

Compactness

For every pair of norms on a finite dimensional subspace of a normed space X, these norms are equivalent.

Equivalent norms on X define the same topology on X.

A metric space is said to be compact if every sequence in X has a convergent subsequence.

Theorem

Compactness and Subspaces

Theorem

Compactness and Finite Dimensions

Theorem

Riesz's Lemma

A compact subset M of a metric space is closed and bounded.

In a finite dimensional normed space X, any subset $M \subset X$ is compact if and only if it is closed and bounded.

Let Y and Z be subspaces of a normed space X (of any dimension), and suppose that Y is closed and is a proper subset of Z. Then for every real number θ in the interval $(0, 1)$ there is a $z \in Z$ such that $\|z| = 1$ and $\|z - y\| \geq \theta$ for all $y \in Y$.

THEOREM

Compact unit ball

THEOREM

Continuous mapping - compact

THEOREM

Maximum and Minimum

If a normed space X has the property that the closed unit ball $M - \{x \colon \|x\| \leq 1\}$ is compact, then X is finite dimensional.

Let X and Y be metric spaces and $T \colon X \to Y$ a continuous mapping. Then the image of a compact subset M of X under T is compact.

A continuous mapping T of a compact subset M of a metric space X into $\mathbb{R}$ assumes a maximum and a minimum at some points of M.

Example

Examples of Linear Operators

Theorem

Range and Null Space

Theorem

Inverse Operator

1. Identity operator
2. Zero operator
3. Differentiation
4. Integration
5. Multiplication by t
6. Elementary vector algebra (cross and dot product)
7. Matrices

Let T be a linear operator. Then

1. $\mathscr{R}(T)$ is a vector space
2. $\mathscr{N}(t)$ is a vector space
3. If $dim(\mathscr{D}(T)) = n < \infty$, then $\mathscr{R}(T) \leq n$.

Let T be a linear operator. Then

1. T^{-1} exists if and only if $Tx = 0$ implies that $x = 0$.
2. If T^{-1} exists, it is a linear operator.
3. If $\dim \mathscr{D}(T) = n < \infty$ and T^{-1} exists, then $dim(\mathscr{R}(T)) = dim(\mathscr{D}(T))$.

THEOREM

Inverse of product

DEFINITION

Bounded Linear Operator

DEFINITION

Norm of an operator

Let $T\colon X \to Y$ and $S\colon Y \to Z$ be bijective linera operators, where X, Y and Z are vector spaces. Then the inverse $(ST)^{-1}\colon Z \to X$ of the product ST exists and is $(ST)^{-1} = T^{-1}S^{-1}$.

Let X and Y be normed spaces and $T\colon \mathscr{D}(T) \to Y$ a linear operator, where $\mathscr{D}(T) \subset X$. The operator T is said to be bounded if there is a real number c such that

$$\|Tx\| \leq c\|x\|.$$

$\|T\| = \sup_{x \in \mathscr{D}(T), x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x \in \mathscr{D}(T), \|x\|=1} \|Tx\|.$

Example

Examples of Bounded Linear Operators

Theorem

Finite dimensional bounded oeprators

Theorem

Continuity and boundedness of operators

1. Identity operator
2. Zero operator
3. Differentiation operator
4. Integral operator
5. Matrix

If a normed space is finite dimensional, then every linear operator on X is bounded.

Let T be a linear operator where X, Y are normed spaces. Then

1. T is continuous if and only if T is bounded
2. If T is continuous at a single point, it is continuous.

Theorem

Continuity, null space

Theorem

Bounded Linear Extension

Definition

Linear Functional

Let T be a bounded linear operator. Then,

1. $x_n \to x$ implies that $Tx_n \to Tx$.
2. The null space $\mathscr{N}(T)$ is closed.

Let $T \colon \mathscr{D}(T) \to T$ be a bounded linear operator, where $\mathscr{D}(T)$ lies in a normed space X and Y is a Banach space. Then T has an extension

$$\widetilde{T} \colon \overline{\mathscr{D}(T)} \to Y$$

where $\widetilde{T}$ is a bounded linear operator of norm $\|\widetilde{T}\| = \|T\|$.

A linear functional f is a linear operator with domain in a vector space X and range in the scalar field K of X.

DEFINITION

Bounded Linear Functional

DEFINITION

Norm of a Linear Functional

THEOREM

Continuity and boundedness - Linear Functional

A bounded linear functional f is a bounded linear operator with range in the scalar field of the normed space X in which the domain $\mathscr{D}(f)$ lies. Thus there exists a real number c such that for all $x \in \mathscr{D}(f)$,

$$|f(x)| \leq c\|x\|.$$

$\|f\| = \sup_{x \in \mathscr{D}(f), \|x\|=1} |f(x)|.$

A linear functional f with domain $\mathscr{D}(f)$ in a normed space is continuous if and only if f is bounded.

Example

Linear Functional Examples

Definition

Algebraic Dual Space

Definition

Second Algebraic Dual Space

1. Norm
2. Dot product
3. Definite integral

The set of all linear functionals defined on a vector space X. Denoted X^*.

The set of linear functionals on X^*.

Definition

Canonical Mapping

Definition

Algebraically Reflexive

Mapping $C\colon X \to X^{*}*$ by $x \mapsto g_x$.

If C is surjective (and hence bijective), then X is algebraically reflexive.