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1/16/20
Math 540

HW 1.6.1,4,10,12

1.6.1. Let $M \subset Y$ where Y is a metric space. Assume that M has finitely many points. Let $d^* = \min_{x,y \in M, x \neq y} d(x,y)$. Let (x_n) be a Cauchy sequence in M. Then for $\epsilon > 0$, there exists an N such that $d(x_n, x_m) < \epsilon$ for $m, n > N$. Choose $\epsilon = \frac{d^*}{2}$. Since the minimum distance between any distinct points is d^*, and $d(x_n, x_m) < \frac{d^*}{2}$ for all $m, n > N$, this means that $x_n = x_m$ for all $m, n > N$. Thus, (x_n) converges to x_m. So M is complete.

1.6.4. Let X_1 and X_2 be isometric metric spaces and let X_1 be complete. Let $T \colon X_2 \to X_1$ be an isometry. Let $(x_n) \in X_2$ be a Cauchy sequence. Then for $\epsilon > 0$, there exists an N such that $d(x_n, x_m) < \epsilon$ for all $m, n > N$. Since T is an isometry, then $d(Tx_n, Tx_m) < \epsilon$. So (Tx_n) is a Cauchy sequence in X_1. Since X_1 is complete, there exists a $Tx \in X_1$ such that (Tx_n) converges to Tx (where $x \in X_2$). We note that every element in X_1 can be written as Ty where $y \in X_2$ since T is bijective. So there exists an ϵ^* such that $d(Tx_n, Tx) < \epsilon^*$. Since T is an isometry, $d(x_n, x) < \epsilon^*$. So (x_n) converges to x. So X_2 is complete.

1.6.10. Let (x_n) and (x'_n) be convergent sequences in a metric space in a metric space (X, d) and have the same limit l. Thus for $\epsilon > 0$, there exists an N such that when $n > N$, $d(x_n, x) < \frac{\epsilon}{2}$. There exists an N' such that when $n' > N'$, $d(x'_n, x) < \frac{\epsilon}{2}$. Then we see that $d(x_n, x'_n) \leq d(x_n, x) + d(x, x'_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ by the triangle inequality for $n, n' > \max(N, N')$. Thus, $\lim_{n\to\infty} d(x_n, x'_n) = 0$.

1.6.12. Let (x_n) be a Cauchy sequence in (X, d) and let (x'_n) be such that $\lim_{n\to\infty} d(x_n, x'_n) = 0$. Then for $\epsilon > 0$, there exists N such that $n, m > N$ $d(x_n, x_m) < \frac{\epsilon}{3}$. There exists an N' such that when $n, n' > N'$, $d(x_n, x'_n) < \frac{\epsilon}{3}$. And there exists an N'' such that when $m, m' > N''$, $d(x_m, x'_m) < \frac{\epsilon}{3}$. So for $m, m', n, n' > \max(N, N'N'')$,

$$d(x'_n, x'_m) \leq d(x'_n, x_n) + d(x_n, x_m) + d(x_m, x'_m) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$