Cory Glover
1/27/20
Math 540

HW 11
2.7.2,6,10,14
2.8.2,4,11

2.7.2. Let X and Y be normed spaces. Assume that a linear operator $T\colon X \to Y$ is bounded. Let $A \subseteq X$ be a set that is bounded. So for all $a \in A$, $\|a\| \leq M$ for some $M \in \mathbb{R}$ (or $\mathbb{C}$. The same proof follow either way). Let $b \in B$ such that $T(A) = B$. Thus, there exists an $a \in A$ such that $T(a) = b$. We know that $\|T(a)\| \leq c\|a\|$ for some c. Thus, $\|b\| \leq c\|a\|$. Since A is bounded, then $\|b\| \leq c\|a\| \leq cM$. Since $cM \in \mathbb{R}$ and is constant, then $\|b\|$ is bounded. Since b was arbitrary, B must be bounded. So T maps bounded sets into bounded sets.

Now assume that $T\colon X \to Y$ maps bounded sets in X to bounded sets in Y. Let $x \in X$. Then $\|Tx\| = \|y\|$ where $y \in Y$. If x is in some bounded set of X, then $\|Tx\| \leq c$ for some c since y must be in a bounded set. Let $d = \frac{c}{\|x\|}$. Then $\|Tx\| \leq d\|x\|$.

Now assume that x is not in some bounded set of X. So $\{x\}$ is unbounded. This means that $\|x\| = \infty$. This is a contradiction. So all x are in some bounded set. Thus by the first part of the proof, T is a bounded operator.

2.7.6. Let $T\colon l^\infty \to l^\infty$ defined on $y = (y_i) = Tx$, $y_i = \frac{x_i}{i}$ and $x = (x_i)$. We first show that T is linear and bounded.

1. (Linear): Let $a, b \in l^\infty$ be denoted $a = (a_i), b = (b_i)$. Let $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} T(\alpha a + \beta b) &= T((\alpha a_i) + (\beta b_i)) && (1)\\ &= T((\alpha a_i + \beta b_i)) && (2)\\ &= (\frac{\alpha a_i + \beta b_i}{i}) && (3)\\ &= (\frac{\alpha a_i}{i}) + (\frac{\beta b_i}{i}) && (4)\\ &= \alpha(\frac{a_i}{i}) + \beta(\frac{b_i}{i}) && (5)\\ &= \alpha T(a) + \beta T(b). && (6) \end{aligned}$$

   So T is linear.

2. (Bounded): Note that $\|Tx\| = \|(\frac{x_i}{i})\|$ Since $(\frac{x_i}{i}) \in l^\infty$, each $\frac{x_i}{i}$ is bounded by some M. Thus, $\|(\frac{x_i}{i})\| = \sup_i \|\frac{x_i}{i}\| \leq M$. Let $c = \frac{M}{\|x\|}$. Then $\|Tx\| = \|(\frac{x_i}{i})\| = \sup_i |\frac{x_i}{i}| \leq M = c\|x\|$. So $\|Tx\| \leq c\|x\|$. So T is bounded.

Thus we know that T is a bounded linear operator. We now consider $R(T)$. We know every sequence in $R(T)$ is of the form $(\frac{x_i}{i})$. Consider the sequence of sequences (x_i) where $(x_n) = (1, 2, 3, ..., n, 0, 0, ...)$. Then $Tx_n = (1, 1, ..., 1, 0, ...)$. Note that as $n \to \infty$, $Tx_n \to (1, 1, ...)$. Since $(1, 1, 1, ...)$ is bounded by 1, $(1, 1, 1, ...) \in l^\infty$. However $T^{-1}(1, 1, 1, ...) = (1, 2, 3, ...) \notin X$. So $(1, 1, 1, ...) \notin R(T)$. Thus, $R(T)$ is not closed.

2.7.10. Let X be $C[0, 1]$ and define $S\colon = y(s) = s\int_0^1 x(t)dt$ and $T\colon = y(s) = sx(s)$. First note that

$$TSx(s) = T(s\int_0^1 x(t)dt) = s^2\int_0^1 x(t)dt \tag{7}$$

$$STx(s) = S(sx(s)) = s\int_0^1 tx(t)dt. \tag{8}$$

So $ST \neq TS$.

Then we see that

$$
\begin{aligned}
\|S\| &= \sup_{x\in\mathscr{D},\|x\|=1} \|Sx\| && (9)\\
&= \sup_{x\in\mathscr{D},\|x\|=1} \|s\int_0^1 x(t)dt\| && (10)\\
&= \max_{s\in[0,1]} |s\int_0^1 x(t)dt| && (11)\\
&= |\int_0^1 x(t)|dt && (12)\\
&= \|x\|. && (13)\\
\|T\| &= \sup_{x\in\mathscr{D}(T),\|x\|=1} \|Tx\| && (14)\\
&= \max_{s\in[0,1]} |sx(s)|. && (15)\\
\|ST\| &= \max_{s\in[0,1]} |s\int_0^1 tx(t)dt| && (16)\\
&= |\int_0^1 tx(t)dt| && (17)\\
&= \|tx(t)\| && (18)\\
\|TS\| &= \max_{s\in[0,1]} |s^2\int_0^1 x(t)dt| && (19)\\
&= |\int_0^1 x(t)dt| && (20)\\
&= \|x\|. && (21)
\end{aligned}
$$

2.7.14. Let $\|x\|_1 = \sum_{k=1}^n |x_k|$ and $\|y\|_2 = \sum_{j=1}^r |y_j|$. Let A be a $r \times n$ matrix $A = (\alpha_{jk})$ which defines a linear operator from the vector space X of all ordered n-tuples of numbers into the vector space Y of all ordered r-tuples of numbers. Then we see that

$$
\begin{aligned}
\|Ax\|_2 &= \sum_{i=1}^r |\sum_{j=1}^n \alpha_{ij}x_j| && (22)\\
&\leq \sum_{i=1}^r \sum_{j=1}^n |\alpha_{ij}x_j| && (23)\\
&\leq \max_k \sum_{i=1}^r \sum_{j=1}^n |\alpha_{ik}x_j| && (24)\\
&\leq \max_k \sum_{i=1}^r |\alpha_{ik}| \sum_{j=1}^n |x_j| && (25)\\
&= \|A\|\|x\|_1. && (26)
\end{aligned}
$$

So $\|A\|$ is compatible.

2.8.2. Let $y_0 \in C[a,b]$. Then we see that for $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$), that

$$\begin{aligned}
f_1(\alpha x + \beta y) &= \int_a^b (\alpha x + \beta y)(t) y_0(t) dt && (27)\\
&= \int_a^b ((\alpha x)(t) + \beta y(t)) y_0(t) dt && (28)\\
&= \int_a^b (\alpha x(t) + \beta y(t)) y_0(t) dt && (29)\\
&= \int_a^b \alpha x(t) y_0(t) + \beta y(t) y_0(t) dt && (30)\\
&= \int_a^b \alpha x(t) y_0(t) dt + \int_a^b \beta y(t) y_0(t) dt && (31)\\
&= \alpha \int_a^b x(t) y_0(t) + \beta \int_a^b y(t) y_0(t) dt && (32)\\
&= \alpha f_1(x) + \beta f_2(y). && (33)
\end{aligned}$$

So f_1 is linear. We now note that the continuity of $y_0(t)$ on a closed square implies that y_0 is bounded, say $|y_0(t)| \leq c$. So,

$$\begin{aligned}
|f_1(x)| &= |\int_a^b x(t) y_0(t) dt| && (34)\\
&\leq c| \int_a^b x(t)| && (35)\\
&\leq c(b-a) \max_{t \in [a,b]} |x(t)| && (36)\\
&= c(b-a)\|x\|. && (37)
\end{aligned}$$

Thus, $\|f_1(x)\| \leq c\|x\|$. So f_1 is bounded.

Now consider $f_2(x) = \alpha x(a) + \beta x(b)$ for some fixed α, β. Let r, s be scalars. Then we see that

$$\begin{aligned}
f_2(rx + sy) &= \alpha((rx + sy)(a)) + \beta((rx + sy)(b)) && (38)\\
&= \alpha((rx)(a) + (sy)(a)) + \beta((rx)(b) + (sy)(b)) && (39)\\
&= \alpha(rx)(a) + \alpha(sy)(a) + \beta(rx)(b) + \beta(sy)(b) && (40)\\
&= r\alpha x(a) + r\beta x(b) + s\alpha y(a) + s\beta y(b) && (41)\\
&= r(\alpha x(a) + \beta x(b)) + s(\alpha y(a) + \beta y(b)) && (42)\\
&= r f_2(x) + s f_2(y). && (43)
\end{aligned}$$

So f_2 is linear.

We now show that f_2 is bounded.

$$\begin{aligned}
|f_2(x)| &= |\alpha x(a) + \beta x(b)| && (44)\\
&\leq |\alpha||x(a)| + |\beta||x(b)|| && (45)\\
&\leq |\alpha|\|x\| + |\beta|\|x\| && (46)\\
&= (|\alpha| + |\beta|)\|x\| && (47)
\end{aligned}$$

Since these values are all constant, then f_2 is bounded.

2.8.4. First consider $f_1(x) = \max_{t \in J} x(t)$ where $x \in C[a,b]$. Assume that $x(t) = y(t)$. Then $f_1(x) = \max_{t \in J} x(t) = \max_{t \in J} y(t) = f_1(y)$. So f_1 is well-defined. Further, since x is a function, we see that

$f_1(x) \in \mathbb{R}$ or $\mathbb{C}$. So f_1 is a functional. However, consider $x(t) = t^2$ and $y(t) = -t^2$ on the interval $[-1, \frac{1}{2}]$. Then $f_1(x) = 1$ and $f_1(y) = 0$. However $f_1(x+y) = \max_{t \in [-1, \frac{1/2}{]}} 0 = 0$, which is not $f_1(x) + f_1(y) = 1$. So f_1 is not linear on every interval $[a, b]$. Yet, we do see that

$$f_1(x) = \max_{t \in J} x(t) \tag{48}$$

$$\leq |\int_a^b x(t)dt| \tag{49}$$

$$= \|x\|. \tag{50}$$

So f_1 is bounded.

Now consider $f_2(x) \min_{t \in J}$ where $x \in C[a, b]$. Assume that $x(t) = y(t)$. Then $f_2(x) = \min_{t \in J} x(t) = \min_{t \in J} y(t) = f_2(y)$. So f_2 is well-defined. Further, since x is a function, we see that $f_2(x) \in \mathbb{R}$ or $\mathbb{C}$. So f_2 is a functional. However, consider $x(t) = t^2$ and $y(t) = -t^2$ on the interval $[-1, \frac{1}{2}]$. So $f_2(x) = 0$ and $f_2(y) = 1$. However, $f_2(x+y) = \min_{t \in [-1, \frac{1}{2}]} 0 = 0$, which is not $f_2(x) + f_2(y) = 1$. So f_2 is not linear. Yet, we do see that

$$f_2(x) = \min_{t \in J} x(t) \tag{51}$$

$$\leq |\int_a^b x(t)dt| \tag{52}$$

$$= \|x\|. \tag{53}$$

So f_2 is bounded.

2.8.11. Let $f_1 \neq 0$ and $f_2 \neq 0$ be two linear functionals defined on the same vector space X. Assume that they have the same null space. Define a map $T \colon \mathscr{R}(f_1) \to \mathscr{R}(f_2)$ such that $Tf_1(x) = f_2(x)$. Given that the function is based on x, we can clearly see that T is well-defined. We show that T is injective. Assume that $Tf_1(x) = Tf_1(y)$. Then, $T(f_1(x) - f_1(y)) = 0$ so $T(f_1(x-y)) = 0$. So $f_2(x-y) = 0$. Since f_2 and f_1 have the same null space, then $f_1(x-y) = 0$. So $f_1(x) = f_1(y)$. Hence, T is injective. Now let $f_2(x) \in \mathscr{R}(f_2)$. Then $f_1(x) \in \mathscr{R}(f_2)$ maps to $Tf_1(x) = f_2(x)$. So T is bijective. Hence, $\mathscr{R}(f_1) \cong \mathscr{R}(f_2)$.

Now since $\mathscr{R}(f_1) \subseteq \mathbb{R}$, we know that T can be represented by a 1x1 matrix (since T is a linear operator and bijective) which is just some scalar α. This means that $Tf_1(x) = \alpha f_1(x) = f_2(x)$.