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# HW 3

1.3.1,5,8,14,15

**1.3.1.** Let $x \in B(x_0; \epsilon)$. Then $d(x, x_0) < \epsilon$. Let $Y$ be a set such that for all $y \in Y$, $d(y, x_0) = \epsilon$. Let $y_0 = \text{argmin}_{y \in Y} d(y, x)$. Define $\delta = d(y_0, x)$. Then there exists a ball $B(x; \delta/2)$ such that $x \in B(x; \delta/2)$. Since all the points $z \in B(x; \delta/2)$, and $d(x, y_0) > \delta/2$, then $z \in B(x_0; \epsilon)$. So there exists an open ball around every $x \in B(x_0; \epsilon)$. So an open ball is an open set.

Consider the closed ball $\overline{B}(x_0; \epsilon)$. Let $x$ be an element of the complement of the closed ball. Let $Y$ be a set such that for all $y \in Y$, $d(y, x_0) = \epsilon$. Let $y_0 = \text{argmin}_{y \in Y} d(y, x)$. Define $\delta = d(y_0, x)$. Then there exists a ball $B(x; \delta/2)$ such that $x \in B(x; \delta/2)$. Since all the points $z \in B(x; \delta/2)$, and $d(x, y_0) > \delta/2$, then $z \notin \overline{B}(x_0; \epsilon)$. So the complement of the closed ball $\overline{B}(x_0; \epsilon)$ is open. So every closed ball is a closed set.

**1.3.5.** Consider the set $X$. From the properties of a topology (T1), $X$ is an open set. Note that $X^C = \emptyset$. By the properties of a topology (T1), $\emptyset$ is an open set. So $X$ is also a closed set.

Similarly, by the properties of a topology (T1), $\emptyset$ is an open set, but since $X$ is open and $\emptyset^C = X$, $\emptyset$ is also a closed set.

Consider a metric space $X = (X, d)$ where $d$ is the discrete metric. Let $Y \subset X$ and let $y \in Y$. Then $B(y; 1) = \{y\}$ so $B(y; 1) \in Y$ since $y \in Y$. So every subset of $X$ is open.

Now consider $Y^c$ and let $z \in Y^C$. Then $B(z; 1) = \{z\}$. So $B(z; 1) \in Y^C$. So $Y^C$ is open. Thus, $Y$ is closed.

So all subsets of $X$ are both open and closed.

**1.3.8.** Consider the space $X = ([0, 1] \cup \{2\}, d)$ where $d$ is the Euclidean metrix. Then $\overline{B}(1, 1) = [0, 1] \cup \{2\}$ since $d(1, 2) = 1$. However, $2 \notin \overline{B(1, 1)}$. So the closure of a ball and a closed ball are not necessarily equal.

**1.3.14.** Assume that $T\colon X \to Y$ is continuous. Let $M \subset Y$ and $T^{-1}(M)$ be the inverse image of $M$. Let $M$ be closed. Then $M^C$ is open. Since $T$ is continuous, $T^{-1}(M^C)$ is also open by the continuous mapping theorem.. Let $m \in T^{-1}(M)^C$. Then $T(m) \in M^c$. So $T^{-1}(M)^C \subset T^{-1}(M^C)$. Now let $x \in T^{-1}(M^C)$. Then $Tx \in M^C$. So $Tx \notin M$. Thus, $x \notin T^{-1}(M)$. So $x \in T^{-1}(M)^c$. So $T^{-1}(M)^c = T^{-1}(M^c)$. Since $T^{-1}(M^C)$ is open, then $T^{-1}(M)^C$ is open. So $T^{-1}(M)$ is closed.

Let $T\colon X \to Y$ be a mapping from $X$ to $Y$. Let $M$ be a closed set in $Y$. Then $M^c$ is open. Assume that $T^{-1}(M)$ is also closed. Let $m \in M^C$. Since $M$ is closed, then $m$ has an open ball around it still contained in $M^C$. Then $T^{-1}(m) \notin T^{-1}(M)$. So $T^{-1}(m) \in T^{-1}(M)^c$. Thus, $T^{-1}(m)$ has open ball around it still contained in $T^{-1}(M)^c$ since $T^{-1}(M)$ is closed. Since $m$ is arbitrary, $T^{-1}(M)^c$ is open. So the inverse image of every open set in $Y$ is an open set in $X$. Thus, $T$ is continuous.

**1.3.15.** Let $X = (\mathbb{R}, d)$ where $d$ is the discrete metric and $Y = (\mathbb{R}, \overline{d})$ where $\overline{d}$ is the Euclidean metric. Define $T(A) = A$ where $A$ is a subset of the real line. Then for every open set $A \subset Y$, $T^{-1}(Y)$ is open (since every subset in $X$ is open). However, consider the subset $[0, 1] \in X$. This set is open, however $T(X)$ is not open.