

HW 12
2.9.1,4,5,9,10

2.9.1. We solve for the null space as follows:

$$\begin{pmatrix} 1 & 3 & 2 \\ -2 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & 7 & 4 \end{pmatrix}. \quad (1)$$

So $x_1 + 3x_2 + 2x_3 = 0$ and $7x_2 + 4x_3 = 0$. Thus, $x_1 + 3x_2 = -2x_3$. Substituting we get $7x_2 - 2x_1 - 6x_2 = x_2 - 2x_1 = 0$. So $x_2 = 2x_1$. Substituting back into the first equation, we get that $x_1 + 3x_2 + 2x_3 = x_1 + 2x_1 + 3x_3 = 3x_1 + 3x_3 = 0$. So $x_3 = -x_1$. Thus the null space is spanned by $(x_1 \ 2x_1 \ -x_1)^T$ where $x_1 \in \mathbb{R}$.

2.9.4. Let $\{f_1, f_2, f_3\}$ be the dual basis of $\{e_1, e_2, e_3\}$ for \mathbb{R}^3 where $e_1 = (1, 1, 1)$, $e_2 = (1, 1, -1)$, and $e_3 = (1, -1, -1)$. Let $x = (1, 0, 0)$. So $x = e_1 + e_3$. Thus, $f_1(x) = f_1(e_1 + e_3) = 1$, $f_2(x) = f_2(e_1 + e_3) = 0$, and $f_3(x) = f_3(e_1 + e_3) = 1$.

2.9.5. Let f be a linear functional on an n -dimensional vector space X . Assume that f is the zero function, then $\mathcal{N}(f) = X$. So the null space is n -dimensional. Now assume that f is not the zero function. Let $x \in \mathcal{N}(f)$. Since $x \in X$, we can rewrite x as $x = \sum_{i=1}^n \alpha_i x_i$. So, $f(x) = \sum_{i=1}^n \alpha_i f(x_i) = 0$. Thus, $\hat{\alpha}_1 f(x_1) = \sum_{i=2}^n \alpha_i f(x_i)$ where $\hat{\alpha} = -\alpha$.

2.9.9. Let X be the vector space of all real polynomials of a real variable and of degree less than a given n , together with $x = 0$. Let $f(x) = x^{(k)}(a)$, the value of the k^{th} derivative (k fixed) of $x \in X$ at a fixed $a \in \mathbb{R}$. We show that this is a linear functional. It is clear that f maps to the reals, so it suffices to show that it is linear. Let $x, y \in X$ and $\gamma, \lambda \in \mathbb{R}$. Then let $x = \alpha_0 + \alpha_1 r + \cdots + \alpha_n r^n$ and $y = \beta_0 + \beta_1 r + \cdots + \beta_n r^n$ where α_j and/or $\beta_j = 0$ for all $j > l$ if x (or y) has degree l . Then

$$f(\gamma x + \lambda y) = f(\gamma(\alpha_0 + \alpha_1 r + \cdots + \alpha_n r^n) + \lambda(\beta_0 + \beta_1 r + \cdots + \beta_n r^n)) \quad (2)$$

$$= f((\gamma\alpha_0 + \lambda\beta_0) + (\gamma\alpha_1 + \lambda\beta_1)r + \cdots + (\gamma\alpha_n + \lambda\beta_n)r^n) \quad (3)$$

$$= ((\gamma\alpha_1 + \lambda\beta_1) + (\gamma\alpha_2 + \lambda\beta_2)a + \cdots + (\gamma\alpha_n + \lambda\beta_n)a^{n-1}) \quad (4)$$

$$= \gamma\alpha_1 + \gamma\alpha_2 a + \cdots + \gamma\alpha_n a^{n-1} + \lambda\beta_1 + \lambda\beta_2 a + \cdots + \lambda\beta_n a^{n-1} \quad (5)$$

$$= \gamma(\alpha_1 + \alpha_2 a + \cdots + \alpha_n a^{n-1}) + \lambda(\beta_1 + \beta_2 a + \cdots + \beta_n a^{n-1}) \quad (6)$$

$$= \gamma f(x) + \lambda f(y). \quad (7)$$

So f is linear and thus a linear functional.

2.9.10. Let Z be a proper subspace of an n -dimensional vector space X , and let $x_0 \in X - Z$. Let f be a functional defined by

$$f(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0 \end{cases}.$$

We show that f is linear. Let $\alpha, \beta \in \mathbb{R}$. We work by cases:

1. Let $x \in X$ and $x \neq x_0$. Then $f(x + x_0) = 0$ and $f(x) + f(x_0) = 1$.