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1/27/20
Math 540

HW 10
2.6.2,3,14

2.6.2. For all the operations, the range and domain are both over the reals. It suffices to show that the second part of the definition is true for all operators.

1. Consider the operator $T(x_1, x_2) = (x_1)$. Then for some $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}
T(\alpha(x_1, x_2) + \beta(y_1, y_2)) &= T((\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2)) && (1)\\
&= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) && (2)\\
&= (\alpha x_1 + \beta y_1, 0) && (3)\\
&= (\alpha x_1, 0) + (\beta y_1, 0) && (4)\\
&= \alpha(x_1, 0) + \beta(y_1, 0) && (5)\\
&= \alpha T(x_1, x_2) + \beta T(y_1, y_2). && (6)
\end{aligned}$$

So T is linear.

2. Consider the operator $T(x_1, x_2) = (0, x_2)$. Then for some $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}
T(\alpha(x_1, x_2) + \beta(y_1, y_2)) &= T((\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2)) && (7)\\
&= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) && (8)\\
&= (0, \alpha x_2 + \beta y_2) && (9)\\
&= (0, \alpha x_2) + (0, \beta y_2) && (10)\\
&= \alpha(0, x_2) + \beta(0, y_2) && (11)\\
&= \alpha T(x_1, x_2) + \beta T(y_1, y_2). && (12)
\end{aligned}$$

So T is linear.

3. Consider the operator $T(x_1, x_2) = (x_2, x_1)$. Then for some $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}
T(\alpha(x_1, x_2) + \beta(y_1, y_2)) &= T((\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2)) && (13)\\
&= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) && (14)\\
&= (\alpha x_2 + \beta y_2, \alpha x_1 + \beta y_1) && (15)\\
&= (\alpha x_2, \alpha x_1) + (\beta y_2, \beta y_1) && (16)\\
&= \alpha(x_2, x_1) + \beta(y_2, y_1) && (17)\\
&= \alpha T(x_1, x_2) + \beta T(y_1, y_2). && (18)
\end{aligned}$$

So T is linear.

4. Consider the operator $T(x_1, x_2) = (\gamma x_1, \gamma x_2)$. Then for some $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}
T(\alpha(x_1, x_2) + \beta(y_1, y_2)) &= T((\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2)) && (19)\\
&= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) && (20)\\
&= (\gamma(\alpha x_1 + \beta y_1), \gamma(\alpha x_2 + \beta y_2)) && (21)\\
&= (\gamma\alpha x_1 + \gamma\beta y_1, \gamma\alpha x_2 + \gamma\beta y_2) && (22)\\
&= (\gamma\alpha x_1, \gamma\alpha x_2) + (\gamma\beta y_1, \gamma\beta y_2) && (23)\\
&= \alpha(\gamma x_1, \gamma x_2) + \beta(\gamma y_1, \gamma y_2) && (24)\\
&= \alpha T(x_1, x_2) + \beta T(y_1, y_2). && (25)
\end{aligned}$$

So T is linear.

2.6.3.

1. $D(T_1) = \mathbb{R}^2$, $R(T_1) = \{(x, 0) \colon x \in \mathbb{R}\}$, $N(T_1) = \{(0, x) \colon x \in \mathbb{R}\}$.
2. $D(T_2) = \mathbb{R}^2$, $R(T_2) = \{(0, x) \colon x \in \mathbb{R}\}, N(T_2) = \{(x, 0) \colon x \in \mathbb{R}\}$.
3. $D(T_3) = R(T_3) = \mathbb{R}^2$, $N(T_3) = \{(0, 0)\}$.

2.6.14. Let $T \colon X \to Y$ be a linear operator and $dim(X) = dim(Y) = n < \infty$. Assume that $R(T) = Y$. Thus, T is onto. So for every $y \in Y$, there exists $x \in X$ such that $T(x) = y$. We now show that T is injective. Assume that $y = y^*$. Since Y has dimension n, we know there exists a basis such that

$$y = \alpha_1 y_1 + \cdots + \alpha_n x_n.$$

We also know that y^* has the same decomposition. So

$$y = y^* \tag{26}$$
$$\alpha_1 y_1 + \cdots + \alpha_n y_n = \alpha_1 y_1 + \cdots + \alpha_n y_n. \tag{27}$$

We then know each y_1 can be written as $T(x_1)$. So

$$\alpha_1 T(x_1) + \cdots \alpha_n T(x_n) = \alpha_1 T(x_1) + \cdots \alpha_n T(x_n) \tag{28}$$
$$T(\alpha_1 x_1 + \cdots + \alpha_n x_n) = T(\alpha_1 x_1 + \cdots + \alpha_n x_n). \tag{29}$$

Since the decompostion of the domain values mapping to y and y^* are the same, we know that $x = x^*$ where $T(x) = y$ and $T(x^*) = y^*$. So T is injective. Thus, T is bijective and an inverse exists.

Now assume that T^{-1} exists. Then $dim(D(T^{-1}) = n < \infty$. Clearly T exists and $(T^{-1})^{-1} = T$. Then by Theorem 2.6-10, we know that $dim(D(T^{-1})) = n$. Since T^{-1} takes on values from Y, and $dim(Y) = n$, then $D(T^{-1}) = Y$. Note that $R(T) = D(T^{-1})$. So $R(T) = Y$.