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1/16/20
Math 540

HW 7
2.1.2,4,5,10
2.2.3,7,10,13

**2.1.2.** Let $x$ be a vector in a given vector space $X$. Then $0x = (\alpha - \alpha)x = \alpha x - \alpha x = \theta$ where $\alpha$ is a scalar. Further, $\alpha(\theta) = \alpha(0x) = (\alpha 0)x = 0x = \theta$.

Lastly, $(-1)x = (1-2)x = x - 2x = -x$.

**2.1.4.** Which of the following subsets of $\mathbb{R}^3$ constitutes a subspace? (Here $x = (\xi_1, \xi_2, \xi_3)))$.

1. All $x$ with $\xi_1 = \xi_2$ and $\xi_3 = 0$. Note that $\theta$ is in this space as it satisfies all properties therein. So the set is non-empty. Let $x, y$ be elements of this space where $x = (x_1, x_1, 0)$ and $y = (y_1, y_2, 0)$. Then for scalars $\alpha$ and $\beta$
$$\alpha x + \beta y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, 0).$$
So this is a subspace.

2. All $x$ with $\xi_1 = \xi_2 + 1$. Note that the zero vector is not in this space so it cannot be a subspace.

3. All $x$ with positive $\xi_1, \xi_2, \xi_3$. This cannot be a subspace since $-x$ is not in the space.

4. All $x$ with $\xi_1 - \xi_2 + \xi_3 = k$ where $k$ is some constant. Then we see $\theta$ is in this space since $0 - 0 + 0 = 0$. So $k = 0$. Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be in this space. Then for some $\alpha$ and $\beta$ scalars,
$$\alpha x + \beta y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + betay_3).$$
We see that $\alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2 + \alpha x_3 + \beta y_3 = \alpha(x_1 - x_2 + x_3) + \beta(y_1 - y_2 + y_3) = \alpha(0) + \beta(0) = 0$. So this is a subspace when $k = 0$.

**2.1.5.** Consider $\{x_1, ..., x_n\}$ where $x_j(t) = t^j$ in the space $C[a, b]$. Assume that
$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 0.$$
Then applying $t$ to the function finds
$$(\alpha_1 x_1 + \cdots + \alpha_n x_n)(t) = 0(t) \tag{1}$$
$$\alpha_1 x_1(t) + \cdots + \alpha_n x_n(t) = 0 \tag{2}$$
$$\alpha_1 t + \alpha_2 t^2 + \cdots \alpha_n t^n = 0. \tag{3}$$
By way of contradiction, assume that not all $\alpha_i$ are 0 for some $t^*$. Then
$$\alpha_1 t^* + \alpha_2 (t^*)^2 + \cdots + \alpha_n (t^*)^n = 0.$$
Solving for $\alpha_1$, we get that $\alpha_1 = -(\sum_{i=1}^{n-1} \alpha_{i+1}(t^*)^i$. However, then $\alpha_1$ has a different value for a different $t$. Since the constants $\alpha_i$ cannot change, this is a contradiction. Hence, $\{x_1, ..., x_n\}$ is linearly independent.

**2.1.10.** Let $Y$ and $Z$ be subspaces of a vector space $X$. Thus, $0 \in Y$ and $0 \in Z$. So $Y \cap Z$ is non-empty. Let $y, z \in Y \cap Z$ and $\alpha$ and $\beta$ be scalars. Then $\alpha y + \beta z \in Y$ since both $\alpha y$ and $\beta z$ are in $Y$ and $Y$ is a subspace. Similarly, $\alpha y + \beta z \in Z$. So $\alpha y + \beta z \in Y \cap Z$. So $Y \cap Z$ is a subspace.

Let $Y$ be the subspace of the form $x = (x_1, 2x_1)$ in $\mathbb{R}^2$ and let $Z$ be the subspace of the form $y = (3y_2, y_2)$. Then both $x$ and $y$ as defined are in the union of $Y$ and $Z$. However, $x + y = (x_1 + 3y_2, 2x_1 + y_2)$ which is neither $Y$ or $Z$ and thus not in $Y \cup Z$. So $Y \cup Z$ is not a vector space.

However, considering $Y \cap Z$ in the example above, we get that $Y \cap Z = \{\theta\}$ which is a vector space.

**2.2.3.** Let $x, y$ be vectors in a vector space $X$. Then

$$\begin{aligned} |\|y\| - \|x\|| &= \|y - x + x\| - \|x - y + y\| && (4)\\ &\leq \|y - x\| + \|x\| - \|x\| - \|y - y\| && (5)\\ &= \|y - x\|, && (6) \end{aligned}$$

by the triangle inequality.

**2.2.7.** We verify that $\|x\|\left(\sum_{j=1}^{\infty} |\xi_j|^p\right)^{1/p}$.

1. Note that $|\xi_j|$ is non-negative since it is an absolute value. Non-negative numbers raised to a power are still non-negative. The sum of non-negative numbers is non-negative. The the $p^{th}$ root of a non-negative number is non-negative. Hence $\|x\| \geq 0$.

2. Assume that $\|x\| = 0$. So

$$\begin{aligned} 0 &= \|x\| && (7)\\ &= \left(\sum_{j=0}^{\infty} |\xi_j|^p\right)^{1/p} && (8)\\ 0^p &= 0 && (9)\\ &= \sum_{j=0}^{\infty} |\xi_j|^p && (10) \end{aligned}$$

   Then since all the numbers in the sum are non-negative, the only way the summation is zero is if every $\xi_j$ is 0. Hence, $x = 0$.

   Now assume that $x = 0$. Then

$$\begin{aligned} \|x\| &= \left(\sum_{j=1}^{\infty} |\xi_j|^p\right)^{1/p} && (11)\\ &= \left(\sum_{j=1}^{\infty} 0\right)^{1/p} && (12)\\ &= 0. && (13) \end{aligned}$$

   So condition 2 is satisfied.

3.

$$\begin{aligned} \|\alpha x\| &= \left(\sum_{j=0}^{\infty} |\alpha \xi_j|^p\right)^{1/p} && (14)\\ &= \left(\sum_{j=0}^{\infty} |\alpha|^p |\xi_j|^p\right)^{1/p} && (15)\\ &= \left(|\alpha|^p \sum_{j=0}^{\infty} |\xi_j|^p\right)^{1/p} && (16)\\ &= \left(|\alpha|^p\right)^{1/p} \left(\sum_{j=0}^{\infty} |\xi_j|^p\right)^{1/p} && (17)\\ &= |\alpha| \left(\sum_{j=0}^{\infty} |\xi_j|^p\right)^{1/p} && (18)\\ &= |\alpha| \|x\|. && (19) \end{aligned}$$

4. Let $x = (x_i)$ and $y = (y_i)$ be element of the vector space. Then

$$\|x+y\| = \Big(\sum_{j=0}^{\infty} |x_j + y_j|^p\Big)^{1/p} \tag{20}$$

$$\leq \Big(\sum_{j=0}^{\infty} |x_j|^p + |y_j|^p\Big)^{1/p} \tag{21}$$

$$\leq \Big(\sum_{j=0}^{\infty} |x_j|^p + \sum_{j=0}^{\infty} |y_j|^p\Big)^{1/p} \tag{22}$$

$$\leq \Big(\sum_{j=0}^{\infty} |x_j|^p\Big)^{1/p} + \Big(\sum_{j=0}^{\infty} |y_j|^p\Big)^{1/p} \tag{23}$$

$$= \|x\| + \|y\|. \tag{24}$$

So $\|x\|$ is a norm.

**2.2.10.** The sphere

$$S(0;1) = \{x \in X \mid \|x\| = 1\}$$

in a normed space is called the unit sphere.

1. If $\|x\| = \|x\|_1$, then $x \in S(0;1)$ when $|x_1| + |x_2| = 1$. This happens on the lines $x_2 = -x_1 + 1, x_2 = x_1 + 1, x_2 = -x_1 - 1, x_2 = x_1 - 1$. These are the lines drawn in the picture.

2. If $\|x\| = \|x\|_2$, then $x \in S(0;1)$ when $(|x_1|^2 + |x_2|^2)^{1/2} = 1$. This happens when $|x_1|^2 + |x_2|^2 = 1$. This is true for the unit circle (that is the lines $x_2 = -\sqrt{x_1} + 1, x_2 = \sqrt{x_1} - 1$.

3. If $|x\| = \|x\|_\infty$, the $x \in S(0;1)$ if $\max\{|x_1|, |x_2|\} = 1$. This is true for the lines $x_2 = \pm 1$ and $x_1 = \pm 1$.

4. If $\|x\| = \|x\|_4$, then $x \in S(0;1)$ if $x_1^4 + x_2^4 = 1$. This is true for the equations $x_2 = -x_1^{1/4} + 1$ and $x_2 = x_1^{1/4} + 1$. This is the circle shown below.

**2.2.13.** Let $x, y$ be in a discrete metric space $X$ where $x \neq y$. Then $2x \neq 2y$. Thus, $d(2x, 2y) = 1$ but $|2|d(x,y) = 2$. So $d(\alpha x, \alpha y) \neq |\alpha| d(x,y)$. So by the translation invariance lemma, the discrete metric is not induced by a norm.