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Math 571
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HW 1.1
1.1.2,4,7,14

1.1.2. Consider the set of real numbers $\mathbb{R}$ and the function $d(x,y) = (x-y)^2$ where $x, y \in \mathbb{R}$. We show that d is a metric using definition 1.1-1.

1. Since x, y are real-valued and finite, we know that $x - y$ is real-valued and finite. We also know that $(x-y)^2$ is real valued and finite. Thus, $d(x,y)$ is real-valued and finite. Further, the square of any real number is nonnegative. Thus, $d(x,y)$ is nonnegative.
2. Assume that $d(x,y) = 0$. Then $(x-y)^2 = 0$. Taking the square root of both sides we find that $x - y = 0$. Thus, $x = y$.

   Now assume that $x = y$. Then $d(x,y) = (x-y)^2 = (x-x)^2 = 0^2 = 0$. So $d(x,y) = 0$.
3. Note that $x-y = -(y-x)$. So squaring both sides we get that $(x-y)^2 = (-(y-x))^2 = (-1)^2(y-x)^2 = (y-x)^2$. So $(x-y)^2 = (y-x)^2$. Hence, $d(x,y)$ is symmetric.
4. Let $z \in \mathbb{R}$. Then we see that

$$\begin{aligned} (x-y)^2 &= x^2 - 2xy + y^2 && (1) \\ &\leq x^2 - 2xy + y^2 + 2z^2 && (2) \end{aligned}$$

1.1.4. Let the set X be a set with 2 points x, y. Let r be a positive real-number. We define a function d_r on X by

$$d_r(x,y) = \begin{cases} 0 & x = y \\ r & x \neq y \end{cases}.$$

Thus, d_r is clearly real-valued since $r, 0 \in \mathbb{R}$, clearly nonnegative since $0 = 0$ and $r > 0$, and is clearly finite since $r, 0 \in \mathbb{R}$. Note that if $d(x,y) = 0$, then $x = y$ by definition. Further, if $x = y$, then $d(x,y) = 0$. Next, if $x = y$, then $d(x,y) = 0 = d(y,x)$ and if $x \neq y$, then $d(x,y) = r = d(y,x)$. So d is symmetric. Lastly, we see that $d(x,y) = 0 + d(x,y) = d(x,x) + d(x,y)$ and $d(x,y) = d(x,y) + 0 = d(x,y) + d(y,y)$. So the triangle inequality holds. Thus d_r is a metric.

It suffices to show that d_r is the only possible metric on X. Assume another metric exists on X that is not d_r. We know that $d(x,y) = 0$ if $x = y$. Then $d(x,y) = r$ for some positive real-valued number r when $x \neq y$. This is because $d(x,y) = 0$ if and only if $x = y$ and d is nonnegative. Thus, $d = d_r$. This is a contradiction. So every metric on X can be expressed as a metric d_r where r is any positive real-valued number.

Now let Y be the set with 1 point. We denote this point as α. Let d be a metric defined on $Y \times Y$. So d is only defined for $d(\alpha, \alpha)$. Since d is a metric, $d(\alpha,\alpha) = 0$. So any metric on Y is the 0 metric (i.e., $d(x,y) = 0$ for all x, y).

1.1.7. Let A be a subspace of l^∞ consisting of all sequences of zeros and ones.

1.1.14. Let d be a metric on X. We want to show that $d(x_1, x_n) \leq d(x_1,x_2) + d(x_2,x_3) + \cdots + d(x_{n-1},x_n)$ for all $n\mathbb{Z}$ where $n \geq 2$ (since if $n = 1$, this pattern does not make sense). Let $n = 2$. Then $d(x_1,x_2) \leq d(x_1,x_2)$ (in fact they are equal). Then for $n = 3$, let $x_1, x_2, x_3 \in X$. We get that $d(x_1,x_3) \leq d(x_1,x_2) + d(x_2,x_3)$ since d is a metric.

Now assume by induction that $d(x_1,x_n) \leq d(x_1,x_2) + d(x_2,x_3) + \cdots d(x_{n-1},x_n)$ for some $n \geq 3$. Then we see that $d(x_1,x_{n+1}) \leq d(x_1,x_n) + d(x_n,x_{n+1})$ by the triangle inequality since d is a metric. Then by the inductive hypothesis,

$$d(x_1,x_{n+1}) \leq d(x_1,x_2) + \cdots + d(x_{n-1},x_n) + d(x_n,x_{n+1}).$$

Hence for all $n \in \mathbb{Z}$ where $n \geq 2$, $d(x_1,x_n) \leq d(x_1,x_2) + d(x_2,x_3) + \cdots + d(x_{n-1},x_n)$.