

HW 9// 2.5.2,9.10

2.5.2. Let X be a discrete metric space with infinitely many points. Consider the sequence $(x_1, x_2, x_3, \dots) \in X$. Note that for any x_i and x_{i+1} , then $d(x_i, x_{i+1}) = 1$. Thus, we choose an arbitrary subsequence $(x_i, x_{i+1}, x_{i+2}, \dots)$ which may or may not be infinite. Then for $\epsilon = \frac{1}{2}$, then for every k with x_k an element of the subsequence, $d(x_k, x_{k+1}) = 1 > \epsilon$. So the subsequence does not converge. Since the subsequence was arbitrary, X is not compact.

2.5.9. Let X be a compact metric space and that $M \subset X$ is closed. Let $(x_n) \in M$. Since X is compact, we know that (x_n) has a subsequence (x_{n_k}) which converges to some x . Since (x_{n_k}) is a subsequence of (x_n) , each element of (x_{n_k}) is in M . Further, since M is closed, $x \in M$. Thus, since (x_n) was arbitrary, M must be compact.

2.5.10. Let X and Y be metric spaces. Let X be compact and let $T: X \rightarrow Y$ be bijective and continuous. We want to show that T is a homeomorphism (that is T has a continuous inverse). Consider the function T^{-1} . Let $x, x_0 \in X$ such that for every $\epsilon > 0$, $d(x, x_0) < \epsilon$. Since X is compact, there exists some sequence (x_n) where $x, x_0 \in (x_n)$ which has a convergent subsequence (namely (x, x_0)). We write $x = T^{-1}(y)$ and $x_0 = T^{-1}(y_0)$ for some $y, y_0 \in Y$, since T is bijective. Since T is continuous, $T(T^{-1}(y), T^{-1}(y_0))$ maps to some subsequence (y_n, y_0) which converges. So $d(y_n, y_0) < \delta$ for some δ . Thus, T^{-1} is continuous.