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1/16/20
Math 540

HW 8
2.3.3,8,10
2.4.2,4,6

2.3.3. Let X be l^∞ and let Y be the subset of all sequences with only finitely many nonzero terms. We first show that Y is a subspace of l^∞. We first note that the zero sequence has no nonzero terms, and thus is in Y. So Y is nonempty. Let $x = (x_i)$ and $y = (y_i)$ be elements of Y. Let x_n be the last nonzero term of x and let y_m be the last nonzero term of y. Then $\alpha x + \beta y$ has a last nonzero term at either n or m (αx_n or βy_m respectively). Thus, $x + y \in Y$. So Y is a subspace.

However consider the sequence $x_n = (1, 1, 0, ...)$ where the first n entries of each tuple is 1 and the rest are zero. Then for $\epsilon > 0$, there exists an N such that when $n > N$, $\|x_n - (1, 1, 1, ...)\| \leq \epsilon$. Since $(1, 1, 1, ...) \notin Y$, we see that Y is not closed, and thus is not a closed subspace.

2.3.8. Let X be a normed space where absolute convergence implies convergence. Let $(x_n) \in X$ be a Cauchy sequence. Then for $\epsilon^* > 0$, there exists an N such that when $m, n > N$, then $\|x_n - x_m\| < \epsilon^*$. We then define $s_k = \sum_{i=1}^k \|x_n - x_{i+m}\|$. Let $\epsilon^* = \frac{\epsilon}{k}$ where $\epsilon > 0$. Then for some K, when $k > K$, $\|s_k - 0\| = \|\sum_{i=1}^k \|x_n - x_{i+m}\|\| < \|\sum_{i=1}^k \epsilon^*\| = \epsilon$. So s_k converges to 0. This means that the sequence $\hat{s}_k = \sum_{i=1}^k x_n - x_{i+m}$ converges to some x. This means that for some $\hat{\epsilon} > 0$, there exists M such that when $k > M$,

$$\|\sum_{i=1}^k x_n - x_{i+m} - x\| = \|x_n - x - \sum_{i=1}^k x_{i+m}\| \tag{1}$$
$$< \hat{\epsilon}. \tag{2}$$

2.3.10. Assume that X is a normed space with a Schauder basis. Thus, for all $x \in X$, $x = \sum_{k=1}^\infty \alpha_k e_k$ where (e_n) is the Schauder basis for X. Consider the set $X' \subset X$ where each x' can be represented as $\sum_{k=1}^\infty \beta_k e_k$ where $\beta_k \in \mathbb{Q}$. Let $x \in X$ such that $x = \sum_{i=1}^\infty \alpha_i x_i$. Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$. Choose $x' = \sum_{i=1}^\infty \beta_i e_i \in X'$ such that for every α_i and β_i, $\|\alpha_i - \beta_i\| < \frac{\epsilon_i}{\|\sum_{i=1}^\infty e_i\|}$ for every $\epsilon_i > 0$. Let $\epsilon = \max_i \frac{\epsilon_i}{\|\sum_{i=1}^\infty e_i\|}$. Then,

$$\|x' - x\| = \|\sum_{i=1}^\infty \beta_i e_i - \sum_{i=1}^\infty \alpha_i e_i\| = \|\sum_{i=1}^\infty (\beta_i - \alpha_i) e_i\| \leq \|\sum_{i=1}^\infty \|\beta_i - \alpha_i\| e_i\| < \|\epsilon \sum_{i=1}^\infty e_i\| = \epsilon \|\sum_{i=1}^\infty e_i\| = \max_i \epsilon_i.$$

So $x' \in B(x; \max_i \epsilon_i)$. Thus, $x \in \overline{X'}$. Since x was arbitrary, $\overline{X'} = X$. Further, since X' is countable, X is separable.

2.4.2. Let $X = \mathbb{R}^2$ and $x_1 = (1, 0)$ and $x_2 = (0, 1)$. Note that for α_1 and α_2,

$$\|\alpha_1 x_1 + \alpha_2 x_2\| = \|\alpha_1(1, 0) + \alpha_2(0, 1)\| = \|(\alpha_1, \alpha_2)\| = \sqrt{\alpha_1^2 + \alpha_2^2}.$$

Note that the largest c in equation (1) will bring equality. So we solve for c as follows:

$$\sqrt{\alpha_1^2 + \alpha_2^2} = c|\alpha_1| + c|\alpha_2| \tag{3}$$
$$c = \frac{\sqrt{\alpha_1^2 + \alpha_2^2}}{|\alpha_1| + |\alpha_2|}. \tag{4}$$

So the largest c could be is $c = \frac{\sqrt{\alpha_1^2 + \alpha_2^2}}{|\alpha_1| + |\alpha_2|}$.

Now let $X = \mathbb{R}^3$ and $x_1 = (1,0,0), x_2 = (0,1,0)$ and $x_3 = (0,0,1)$. Following the same procedure as above, we solve for c.

$$\begin{aligned}
\|\alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1)\| &= \|(\alpha_1, \alpha_2, \alpha_3)\| && (5)\\
&= \left(\alpha_1^3 + \alpha_2^3 + \alpha_3^3\right)^{1/3} && (6)\\
&= c(|\alpha_1| + |\alpha_2| + |\alpha_3|). && (7)
\end{aligned}$$

So the largest c could be is $c = \frac{\left(\alpha_1^3+\alpha_2^3+\alpha_3^3\right)^{1/3}}{|\alpha_1|+|\alpha_2|+|\alpha_3|}$.

2.4.4. Let X be a vector space and let $\|\cdot\|$ and $\|\cdot\|_*$ be equivalent norms on X. This means that there exists positive a and b such that $a\|x\|_* \leq \|x\| \leq b\|x\|_*$ for all $x \in X$. Define $\mathscr{T}_1$ as the open subsets of X with respect to $\|\cdot\|$ and $\mathscr{T}_2$ the open subsets of X with respect to $\|\cdot\|_*$.. Now let $X_2 \in \mathscr{T}_2$ be an open subset with respect to $\|\cdot\|_*$. We want to show that X_2 is an open subset with respect to $\|\cdot\|$. Let $x_0 \in B(x; a\epsilon)$ where this is a ball with respect to $\|\cdot\|$, $x \in X_2$ and $\epsilon > 0$. Then

$$a\|x - x_0\|_* \leq \|x - x_0\| < a\epsilon.$$

So $x_0 \in B^*(x;\epsilon)$. Thus, $B(x; a\epsilon) \subseteq B^*(x;\epsilon)$. So every open ball with respect to $\|\cdot\|$ around $x \in X_2$ is in X_2 since it is a subset of a ball with respect to $\|\cdot\|_*$. So X_2 is an open subset with respect to $\|\cdot\|$. Thus, $X_2 \in \mathscr{T}_1$. So $\mathscr{T}_2 \subseteq \mathscr{T}_1$.

Now let $X_1 \in \mathscr{T}_1$ be an open subset with respect to $\|\cdot\|$. Recall that since $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent, there exists a' and b' that are positive such that $a'\|x\| \leq \|x\|_* \leq b'\|x\|$ for all $x \in X$. Let $x \in X_1$, $\epsilon > 0$ and $x_0 \in B^*(x; a\epsilon)$ where B^* is a ball with respect to $\|\cdot\|_*$. Then we see that

$$a'\|x - x_0\| \leq \|x - x_0\|_* < a\epsilon.$$

So $x_0 \in B(x;\epsilon)$ where B is a ball with respect to $\|\cdot\|$. So $B^*(x; a\epsilon) \subseteq B(x;\epsilon)$. Thus every open ball with respect to $\|\cdot\|_*$ around $x \in X_1$ is in X_1. So X_1 is an open subset with respect to $\|\cdot\|_*$. So $X_1 \in \mathscr{T}_2$. Thus, $\mathscr{T}_1 \subseteq \mathscr{T}_2$.

Hence, $\mathscr{T}_1 = \mathscr{T}_2$.

2.4.6. Let X be the vector space of ordered n-tuples of numbers. Let $x = (x_1, ..., x_n) \in X$. Then $\|x\|_\infty^2 = \max_i |x_i|^2 \leq \sum_{i=1}^n |x_i|^p = \|x\|_2^2$. Thus,

$$\begin{aligned}
\|x\|_\infty^2 &\leq \|x\|_2^2 && (8)\\
\|x\|_\infty\| &\leq \|x\|_2. && (9)
\end{aligned}$$

Further, $n^2 \max_i |x_i|^2 \geq n \max_i |x_i|^2 \geq \sum_{i=1}^n |x_i|^2$. Thus,

$$\begin{aligned}
n^2\|x\|_\infty^2 &\geq \|x\|_2^2 && (10)\\
n\|x\|_\infty &\geq \|x\|_2. && (11)
\end{aligned}$$

So $\|x\|_\infty \leq \|x\|_2 \leq n\|x\|_\infty$. Since $1, n > 0$, the $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are equivalent by definition 2.4-4.