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Math 540

HW 13
2.10.2,4,6,10

2.10.2. Let f and g be bounded linear functionals with domains in a normed space X. Let α, β be nonzero scalars. We define a function $h = \alpha f + \beta g$. Since h is defined in terms of f, $\mathscr{D}(h) \subset \mathscr{D}(f)$. Similarly, $\mathscr{D}(h) \subset \mathscr{D}(g)$. Thus, $\mathscr{D}(h) \subset \mathscr{D}(f) \cap \mathscr{D}(g)$. Further, let $x \in \mathscr{D}(h)$. So $h(x) = \alpha f(x) + \beta g(x)$ is defined. So $x \in \mathscr{D}(f) \cap \mathscr{D}(g)$. Thus, $\mathscr{D}(h) = \mathscr{D}(f) \cap \mathscr{D}(g)$.

Since h takes on values of X and maps them to real-valued numbers (since it is a linear combination of real-valued numbers), h is indeed a functional. It suffices to show it is bounded. Note that for all $x \in \mathscr{D}(h)$,

$$\begin{aligned}
|h(x)| &= |\alpha f(x) + \beta g(x)| && (1)\\
&\leq |\alpha f(x)| + |\beta g(x)| && (2)\\
&\leq |\alpha||f(x)| + |\beta||g(x)| && (3)\\
&\leq |\alpha| c\|x\| + |\beta| d\|x\| && (4)\\
&= (|\alpha| c + |\beta| d)\|x\|, && (5)
\end{aligned}$$

where $c, d \in \mathbb{R}$ since f and g are bounded linear functionals. Thus, h is a bounded linear functional.

2.10.4. Let X and Y be normed spaces and $T_n \colon X \to Y$ ($n = 1, 2, \cdots$) bounded linear operators. Assume that $T_n \to R$. Let $\epsilon > 0$ and choose N. Then when $n > N$, we see that $\|T_n - T\| \leq \epsilon$. Let $x \in \overline{B(x_0; \epsilon^*)}$. So $\|x - x_0\| \leq \epsilon^*$. Then we see that

$$\begin{aligned}
\|T_n x - Tx\| &= \|T_n x - Tx + Tnx_0 - Tnx_0 + Tx_0 - Tx_0\| && (6)\\
&\leq \|T_n x - Tx - T_n x_0 + Tx\| + \|T_n x_0 - Tx_0\| && (7)\\
&= \|(T_n - T)(x - x_0)\| + \|(T_n - T)x_0\| && (8)\\
&\leq \|T_n - T\|\|x - x_0\| + \|T_n - T\|\|x_0\| && (9)\\
&< \epsilon(\epsilon^* + \|x_0\|). && (10)
\end{aligned}$$

We choose $\epsilon_{x_0} = \frac{\epsilon}{\epsilon^* + \|x_0\|}$ such that $\epsilon_{x_0} < \epsilon'$. Since ϵ is arbitrary, then $\|T_n x - Tx\| < \epsilon'$.

2.10.6. Let X be the space of ordered n-tuples of real numbers and $\|x\| = \max_i |x_i|$ where $x = (x_1, ..., x_n)$. We first define the basis on X by $e_i = (0, 0, ..., 0, 1, 0, ..., 0)$ where the only nonzero entry is the i^{th} entry. Then for $x \in X$, $x = \sum_{i=1}^n \alpha_i e_i$.

Now let f be in the dual space of X. So f is a bounded linear functional. Thus,

$$f(x) = f(\sum_{i=1}^n \alpha_i e_i) = \sum_{i=1}^n \alpha_i f(e_i).$$

Define $f(e_i) = \gamma_i$. Note that

$$|\gamma_i| = |f(e_i)| \leq \|f\|\|e_i\| = \|f\|$$

since f is bounded and since $\|e_i\| = 1$. Thus we see that $\max_i |\gamma_i| \leq \|f\|$.

Now note that

$$|f(x)| = |f(\sum_{i=1}^n \alpha_i e_i)| = |\sum_{i=1}^n \alpha_i f(e_i)| \leq \sum_{i=1}^n |\alpha_i||f(e_i)| \leq n \max_{i,j} |\alpha_i||\gamma_j| = \max_i \|nx\||\gamma_j|.$$

Taking the maximum over all x of norm $\frac{1}{n}$, we get that $\|f\| \leq \max_j |\gamma_j|$. So $\|f\| = \max_j |\gamma_j|$.

2.10.10. Let X and $Y \neq \{0\}$ be normed vector spaces, with dim $X = \infty$. Let $B = (x_1, x_2, ...)$ be a Hamel basis for X. Define a linear operator $T \colon X \to Y$. We work by cases:

1. Let $\dim Y = \infty$. Then there exists a Hamel basis for Y, $(y_1, y_2, ...)$. Then define $T(x_i) = y_i$. So consider $x = x_1 + 2x_2 + 3x_3 + \cdots$. Assume by way of contradiction that there exists $c \in \mathbb{R}$ such that $\|Tx\| \leq c\|x\|$. Then

$$\|Tx\| = \|\sum_{i=1}^{\infty} iTx_i\| \tag{11}$$
$$\leq c\|x\|. \tag{12}$$

However also note that

$$\|Tx\| = \|\sum_{i=1}^{\infty} iTx_i\| \tag{13}$$
$$\leq \sum_{i=1}^{\infty} i\|Tx_i\| \tag{14}$$