Cory Glover
Math 571
12/11/19

HW 1.1
1.1.2,4,7,14

1.1.2. Consider the set of real numbers $\mathbb{R}$ and the function $d(x,y) = (x-y)^2$ where $x, y \in \mathbb{R}$. We show that d is not a metric on $\mathbb{R}$. Assume that d is a metric on $\mathbb{R}$. Then for all $x, y, z \in \mathbb{R}$, $d(x,y) \leq d(x,z) + d(z,y)$ by the triangle inequality property of metrics. Let $x = -1, y = 1$, and $z = 0$. Then

$$\begin{aligned} d(-1,1) &\leq d(-1,0) + d(0,1) && (1)\\ 4 &\leq 1+1 && (2)\\ 4 &\leq 2. && (3) \end{aligned}$$

This is a contradiction. Hence, d is not a metric on $\mathbb{R}$.

1.1.4. Let the set X be a set with 2 points α, β. Let r be a positive real-number. We define a function d_r on X by

$$d_r(x,y) = \begin{cases} 0 & x = y \\ r & x \neq y \end{cases}.$$

Thus, d_r is clearly real-valued since $r, 0 \in \mathbb{R}$, clearly nonnegative since $0 = 0$ and $r > 0$, and is clearly finite since $r, 0 \in \mathbb{R}$. Note that if $d(x,y) = 0$, then $x = y$ by definition. Further, if $x = y$, then $d(x,y) = 0$. Next, if $x = y$, then $d(x,y) = 0 = d(y,x)$ and if $x \neq y$, then $d(x,y) = r = d(y,x)$. So d is symmetric. Lastly, we see that $d(x,y) = 0 + d(x,y) = d(x,x) + d(x,y)$ and $d(x,y) = d(x,y) + 0 = d(x,y) + d(y,y)$. So the triangle inequality holds. Thus d_r is a metric.

It suffices to show that d_r is the only possible metric on X. Assume another metric exists on X that is not d_r. Then $d(x,y) = r$ for some positive real-valued number r when $x \neq y$. This is because $d(x,y) = 0$ if and only if $x = y$ and d is nonnegative. Thus, $d = d_r$ since there are only two elements in X, and so d can only take on the values 0 and r. This is a contradiction. So every metric on X can be expressed as a metric d_r where r is any positive real-valued number.

Now let Y be the set with 1 point. We denote this point as α. Let d be a metric defined on Y. So d is only defined for $d(\alpha, \alpha)$. Since d is a metric, $d(\alpha, \alpha) = 0$. So any metric on Y is the 0 metric (i.e., $d(x,y) = 0$ for all $x, y \in Y$).

1.1.7. Let A be a subspace of l^∞ consisting of all sequences of zeros and ones. Recall the metric defined in example 6 (i.e., $d(x,y) = \sup_{j \in \mathbb{N}} |\xi_i - \eta_i|$). Now let $x, y \in A$ where $x = (\xi_1, \xi_2, ...)$ and $y = (\eta_1, \eta_2, ...)$. If x and y distinct, then for all j where $\xi_j \neq \eta_j$, $|\xi_j - \eta_j| = 1$. For all k where $\xi_k = \eta_k$, $|\xi_k - \eta_k| = 0$. So $d(x,y) = 1$. If $x = y$, then $d(x,y) = 0$ since d is a metric. So the induced metric on A is $d\mid_A(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$. This is the discrete metric.

1.1.14. Let d be a metric on X. We want to show that $d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$ for all $n \in \mathbb{Z}$ where $n \geq 2$ (since if $n = 1$, this pattern does not make sense). Let $n = 2$. Then $d(x_1, x_2) \leq d(x_1, x_2)$ (in fact they are equal). Then for $n = 3$, let $x_1, x_2, x_3 \in X$. We get that $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$ since d is a metric.

Now assume by induction that $d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots d(x_{n-1}, x_n)$ for some $n \geq 3$. Then we see that $d(x_1, x_{n+1}) \leq d(x_1, x_n) + d(x_n, x_{n+1})$ by the triangle inequality since d is a metric. Then by the inductive hypothesis,

$$d(x_1, x_{n+1}) \leq d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) + d(x_n, x_{n+1}).$$

Hence for all $n \in \mathbb{Z}$ where $n \geq 2$, $d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$.