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Math 540
1/14/19

HW 1.4.2,4,5

1.4.2. Let (x_n) be a Cauchy sequence with a convergent subsequence $x_{n_k} \to x$. So $\epsilon > 0$, there exists a K such that for $k > K$, $d(x_{n_k}, x) < \frac{\epsilon}{2}$. Further, since (x_n) is Cauchy, we know that for some N, there exists $n, n_k > N$ such that $d(x_{n_k}, x_n) < \frac{\epsilon}{2}$. So for $\max(N, K)$, we see by the triangle inequality that

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So $x_n \to x$.

1.4.4. Let (x_n) be a Cauchy sequence. Thus, for every $\epsilon > 0$, there exists N such that for $m, n > N$, $d(x_n, x_m) < \epsilon$. Now consider all x_k such that $k \leq N$. Since k is finite, we are looking at a finite number of elements. Let $M = \sup_{x_k, x_j \,:\, k,j \leq N} d(x_k, d_j)$. We can do this because there are finite elements. Let $B = \max(M, \epsilon)$. Then $\sup_{x_r, x_s \in (x_n)} d(x, y) \leq B < B + 1$. Since $B + 1$ is finite, (x_n) is bounded. So every Cauchy sequence is bounded.

1.4.5. Boundedness is not necessary for a sequence to be either Cauchy or convergent. Consider the sequence on the real line $(0, 1, 0, 1, 0, 1, 0, 1, \ldots)$. This sequence is bounded by 1 but it does not converge to any number and the distance between elements does not approach $\epsilon > 0$.