Problem 2.15 Find the indicated entry of the matrix, if it is defined.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & 4 \end{pmatrix}$$

(a) $a_{2,1} = 2$

(b) $a_{1,2} = 3$

(c) $a_{2,2} = -1$

(d) $a_{3,1} = undefined$

Problem 2.16 Give the size of each matrix.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & 4 \end{pmatrix}$$

(a) $\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 5 \end{pmatrix}$ is a 2×3 matrix.

(b) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 3 & -1 \end{pmatrix}$ is a 3×2 matrix.

(c) $\begin{pmatrix} 5 & 10 \\ 10 & 5 \end{pmatrix}$ is a 2×2 matrix.

Problem 2.17 Perform the indicated vector operation, if it is defined.

(a) $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}$

(b) $5 \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 20 \\ -1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}$

(d) $7 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 9 \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 14 \\ 7 \end{pmatrix} + \begin{pmatrix} 21 \\ 45 \end{pmatrix} = \begin{pmatrix} 35 \\ 52 \end{pmatrix}$

(e) $\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}1\\2\\3\end{pmatrix}=undefined$

(f) $6\begin{pmatrix}3\\1\\1\end{pmatrix}-4\begin{pmatrix}2\\0\\3\end{pmatrix}+2\begin{pmatrix}1\\1\\5\end{pmatrix}=\begin{pmatrix}18\\6\\6\end{pmatrix}-\begin{pmatrix}8\\0\\12\end{pmatrix}+\begin{pmatrix}2\\2\\10\end{pmatrix}=\begin{pmatrix}10\\6\\-6\end{pmatrix}+\begin{pmatrix}2\\2\\10\end{pmatrix}=\begin{pmatrix}12\\8\\4\end{pmatrix}$

Problem 2.18 Solve each system using matrix notation. Express the solution using vectors.

(a)

$$\begin{aligned}3x+6y&=18\\x+2y&=6\end{aligned}\tag{1}$$

As a matrix:

$$\left(\begin{array}{cc|c}3&6&18\\1&2&6\end{array}\right)\tag{2}$$

Which can be reduced as follows:

$$\left(\begin{array}{cc|c}3&6&18\\1&2&6\end{array}\right)\xrightarrow{r_1-3r_2}\left(\begin{array}{cc|c}0&0&0\\1&2&6\end{array}\right)\xrightarrow{r_1\leftrightarrow r_2}\left(\begin{array}{cc|c}1&2&6\\0&0&0\end{array}\right)\tag{3}$$

The solution set for this set of linear equations is:

$$\{(x,y)\mid x+2y=6\}\tag{4}$$

As we only have one equation, the first variable x is leading, and the second variable y is free. Therefore we can define x in terms of y $x=6-2y$, giving us the solutions set:

$$\{((6)-(2)y,y)\mid y\in\mathbb{R}\}\tag{5}$$

Which can be rewritten using vectors by grouping together coefficients:

$$\{\begin{pmatrix}-2\\1\end{pmatrix}y+\begin{pmatrix}6\\0\end{pmatrix}\mid y\in\mathbb{R}\}\tag{6}$$

An example solution from this set, when $y=2$:

$$\vec{s}=\begin{pmatrix}-2\\1\end{pmatrix}2+\begin{pmatrix}6\\0\end{pmatrix}=\begin{pmatrix}-4\\2\end{pmatrix}+\begin{pmatrix}6\\0\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix}\tag{7}$$

(b)

$$\begin{aligned} x + y &= 1 \\ x - y &= -1 \end{aligned} \tag{8}$$

As a matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & -1 \end{array}\right) \tag{9}$$

Which can be reduced as follows:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & -1 \end{array}\right) \xrightarrow{r_1 - r_2} \left(\begin{array}{cc|c} 0 & 2 & 2 \\ 1 & -1 & -1 \end{array}\right) \xrightarrow{r_1 \leftrightarrow \frac{1}{2} r_2} \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 1 \end{array}\right) \tag{10}$$

Which gives us $y = 1$ and $x = 0$, so the solution set is:

$$\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\} \tag{11}$$

(c)

$$\begin{aligned} x_1 \quad\quad + \ \ x_3 &= 4 \\ x_1 - x_2 + 2x_3 &= 5 \\ 4x_1 - x_2 + 5x_3 &= 17 \end{aligned} \tag{12}$$

As a matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 1 & -1 & 2 & 5 \\ 4 & -1 & 5 & 17 \end{array}\right) \tag{13}$$

Which can be reduced as follows:

$$\begin{aligned} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 1 & -1 & 2 & 5 \\ 4 & -1 & 5 & 17 \end{array}\right) &\xrightarrow{r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & -1 & 1 & 1 \\ 4 & -1 & 5 & 17 \end{array}\right) \\ &\xrightarrow{r_3 - 4r_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right) \\ &\xrightarrow{r_3 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{aligned} \tag{14}$$

This leaves us with x_1 and x_2 as leading variables, and x_3 as a free variable. We can rewrite equation r_1 as $x_1 = 4 - x_3$ and r_2 as $x_2 = x_3 - 1$, giving the solution set:

$$\{(4 - x_3, x_3 - 1, x_3) \mid x_3 \in \mathbb{R}\} \tag{15}$$

Written using vectors:

$$\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} x_3 + \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} \mid x_3 \in \mathbb{R}\} \tag{16}$$

(d)

$$\begin{aligned} 2a + b - c &= 2 \\ 2a \quad + c &= 3 \\ a - b \quad &= 0 \end{aligned} \tag{17}$$

As a matrix:

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 2 & 0 & 1 & 3 \\ 1 & -1 & 0 & 0 \end{array}\right) \tag{18}$$

Which can be reduced as follows:

$$\begin{aligned} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 2 & 0 & 1 & 3 \\ 1 & -1 & 0 & 0 \end{array}\right) &\xrightarrow{r_2 - 2r_3} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & 2 & 1 & 3 \\ 1 & -1 & 0 & 0 \end{array}\right) \\ &\xrightarrow{r_1 - 2r_3} \left(\begin{array}{ccc|c} 0 & 3 & -1 & 2 \\ 0 & 2 & 1 & 3 \\ 1 & -1 & 0 & 0 \end{array}\right) \\ &\xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 3 & -1 & 2 \end{array}\right) \\ &\xrightarrow{2r_3 - 3r_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & -5 & -5 \end{array}\right) \\ &\xrightarrow{-1r_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \end{array}\right) \end{aligned} \tag{19}$$

This gives us $c = 1$, $b = 1$ and $c = 1$. The solution set for this is:

$$\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\} \tag{20}$$

Problem 2.21 The vector is in the set. What value of the paramaters produce that vector?

(a)

$$\begin{pmatrix} 5 \\ -5 \end{pmatrix} \in \{\begin{pmatrix} 1 \\ -1 \end{pmatrix} k \mid k \in \mathbb{R}\} \tag{1}$$

Using a system of linear equations:

$$\begin{array}{rcl} k & = & 5 \\ -k & = & -5 \end{array} \tag{2}$$

We can see $k = 5$ results in $\begin{pmatrix} 5 \\ -5 \end{pmatrix}$.

(b)

$$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \in \{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} i + \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} j \mid i, j \in \mathbb{R} \} \tag{3}$$

Using a system of linear equations:

$$\begin{array}{rcrcr} -2i & + & 3j & = & -1 \\ i & & & = & 2 \\ & & j & = & 1 \end{array} \tag{4}$$

We can see $i = 2$ and $j = 1$ result in $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$.

(c)

$$\begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix} \in \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} m + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} n \mid m, n \in \mathbb{R} \} \tag{5}$$

Using a system of linear equations:

$$\begin{array}{rcrcr} m & + & 2n & = & 0 \\ m & & & = & -4 \\ & & n & = & 2 \end{array} \tag{6}$$

We can see $m = -4$ and $n = 2$ result in $\begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix}$.

Problem 2.22 Decide if the vector is in the set.

(a)

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} \in \{ \begin{pmatrix} -6 \\ 2 \end{pmatrix} k \mid k \in \mathbb{R} \} \tag{1}$$

Using a system of linear equations:

$$\begin{aligned} -6k &= 3 \\ 2k &= -1 \end{aligned} \tag{2}$$

Representing that system as a matrix:

$$\left(\begin{array}{c|c} -6 & 3 \\ 2 & -1 \end{array}\right) \xrightarrow{r_1+3r_2} \left(\begin{array}{c|c} 0 & 0 \\ 2 & -1 \end{array}\right) \xrightarrow{\frac{1}{2}r_2} \left(\begin{array}{c|c} 0 & 0 \\ 1 & -\frac{1}{2} \end{array}\right) \xrightarrow{r_1 \overset{r}{\longleftrightarrow} r_2} \left(\begin{array}{c|c} 1 & -\frac{1}{2} \\ 0 & 0 \end{array}\right) \tag{3}$$

So this vector is part of the set when $k = -\frac{1}{2}$.

(b)

$$\begin{pmatrix} 5 \\ 4 \end{pmatrix} \in \{\begin{pmatrix} 5 \\ -4 \end{pmatrix} j \mid j \in \mathbb{R}\} \tag{4}$$

Intuitively we can see that $\begin{pmatrix} 5 & 4 \end{pmatrix}^T$ is not a member of this set, as there is no single real value j that would invert -4 but not invert 5.

(c)

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \in \{\begin{pmatrix} 0 \\ 3 \\ -7 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} r \mid r \in \mathbb{R}\} \tag{5}$$

Using a system of linear equations:

$$\begin{aligned} r & & &= 2 \\ -r &+ &3 &= 1 \\ 3r &+ &-7 &= -1 \end{aligned} \tag{6}$$

Which can be simplified to:

$$\begin{aligned} r &= 2 \\ r &= 2 \\ r &= 2 \end{aligned} \tag{7}$$

So this vector is a member the set, corresponding to $r = 2$.

(d)

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in \{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} j + \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} k \mid j, k \in \mathbb{R}\} \tag{8}$$

Using a system of linear equations:

$$\begin{array}{rcrcl} 2j & - & 3k & = & 1 \\ & - & k & = & 0 \\ j & + & k & = & 1 \end{array} \tag{9}$$

Representing this system as a matrix:

$$\begin{aligned} \left(\begin{array}{rr|r} 2 & -3 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{array}\right) \xrightarrow{-1r_2} & \left(\begin{array}{rr|r} 2 & -3 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right) \\ \xrightarrow{r_1-2r_3} & \left(\begin{array}{rr|r} 0 & -5 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right) \\ \xrightarrow{-\frac{1}{5}r_1} & \left(\begin{array}{rr|r} 0 & 1 & \frac{1}{5} \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right) \\ \xrightarrow{r_3-r_2} & \left(\begin{array}{rr|r} 0 & 1 & \frac{1}{5} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right) \\ \xrightarrow{r_3 \overset{r}{\longleftrightarrow}_1} & \left(\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & \frac{1}{5} \end{array}\right) \end{aligned} \tag{10}$$

So this vector is **not** a member the set, as the system of linear equations is inconsistent for $\begin{pmatrix}1 & 0 & 1\end{pmatrix}^T$.

Problem 2.23 A famer with a 1200 acre farm is considering planting three different crops.

- Corn costs \$20 per acre.
- Soybeans cost \$50 per acre.
- Oat cost \$12 per acre.

The famer has \$40,000 and intends to spend it all.

(a) We can represent this problem as a system of linear equations:

$$\begin{array}{rcrcrcl} 20x_1 & + & 50x_2 & + & 12x_3 & = & 40000 \\ x_1 & + & x_2 & + & x_3 & = & 1200 \end{array} \tag{1}$$

(b) We can represent (1) in matrix form:

$$\left(\begin{array}{ccc|c} 20 & 50 & 12 & 40000 \\ 1 & 1 & 1 & 1200 \end{array}\right) \xrightarrow{r_1-20r_2} \left(\begin{array}{ccc|c} 0 & 30 & -8 & 16000 \\ 1 & 1 & 1 & 1200 \end{array}\right) \xrightarrow{\frac{1}{30}r_1} \left(\begin{array}{ccc|c} 0 & 1 & -\frac{4}{15} & \frac{1600}{3} \\ 1 & 1 & 1 & 1200 \end{array}\right) \tag{2}$$

Which makes x_3 our free variable, so we can rewrite x_2 as:

$$x_2 = \frac{1600}{3} + \frac{4}{15}x_3 \tag{3}$$

And rewrite x_1 as:

$$\begin{aligned} x_1 &= 1200 - x_2 - x_3 \\ &= 1200 - \frac{1600}{3} - \frac{4}{15}x_3 - x_3 \\ &= \frac{2000}{3} - \frac{19}{15}x_3 \end{aligned} \tag{4}$$

Giving the solution set:

$$\begin{aligned} &\{(\frac{2000}{3} - \frac{19}{15}x_3, \frac{1600}{3} + \frac{4}{15}x_3, x_3) \mid x_3 \in \mathbb{R}\} \\ =&\{\begin{pmatrix} -1\frac{19}{15} \\ \frac{4}{15} \\ 1 \end{pmatrix} x_3 + \begin{pmatrix} \frac{2000}{3} \\ \frac{1600}{3} \\ 0 \end{pmatrix} \mid x_3 \in \mathbb{R}\} \end{aligned} \tag{5}$$

This gives us many possible solutions such as when $x_3 = 0$:

$$\vec{s}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{2000}{3} \\ \frac{1600}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2000}{3} \\ \frac{1600}{3} \\ 0 \end{pmatrix} \tag{6}$$

Or when $x_3 = 526$:

$$\vec{s}_2 = \begin{pmatrix} \frac{2}{5} \\ \frac{33688}{5} \\ 526 \end{pmatrix} \tag{7}$$

(c) Supposing the farmer can bring in:

- \$100 per acre for Corn.
- \$300 per acre for Soybeans.
- \$80 per acre for Oats.

We can check which of the results from **(b)** is most profitable by plugging the solution sets into:

$$100x_1 + 300x_2 + 80x_3 \tag{8}$$

For $\vec{s}_1$, the farmer's revenue is $\sim$ \$226666.

For $\vec{s}_2$, the farmer's revenue is \$244200.

Problem 2.25 Using Gauss's Method

(a) Solve the left-hand side of:

$$\begin{aligned} w + 2x \quad - \quad z &= a \\ 2w \quad + y \quad &= b \\ w + \ x \quad + 2z &= -2 \end{aligned} \tag{1}$$

As a matrix:

$$\begin{aligned} &\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & a \\ 2 & 0 & 1 & 0 & b \\ 1 & 1 & 0 & 2 & c \end{array}\right) \xrightarrow{r_1 - r_3} \left(\begin{array}{cccc|r} 0 & 1 & 0 & -3 & a-c \\ 2 & 0 & 1 & 0 & b \\ 1 & 1 & 0 & 2 & c \end{array}\right) \\ &\xrightarrow{r_2 - 2r_3} \left(\begin{array}{cccc|r} 0 & 1 & 0 & -3 & a-c \\ 0 & -2 & 1 & -4 & b-2c \\ 1 & 1 & 0 & 2 & c \end{array}\right) \\ &\xrightarrow{r_2 + 2r_1} \left(\begin{array}{cccc|r} 0 & 1 & 0 & -3 & a-c \\ 0 & 0 & 1 & -10 & 2a+b-4c \\ 1 & 1 & 0 & 2 & c \end{array}\right) \\ &\xrightarrow{r_3 - r_1} \left(\begin{array}{cccc|r} 0 & 1 & 0 & -3 & a-c \\ 0 & 0 & 1 & -10 & 2a+b-4c \\ 1 & 0 & 0 & 5 & -a+2c \end{array}\right) \end{aligned} \tag{2}$$

Which gives us:

$$\begin{aligned} w &= -a + 2c + -5z \\ x &= a - c + 3z \\ y &= 2a + b - 4c + 10z \end{aligned} \tag{3}$$

Therefore our solution set is:

$$\{ \begin{pmatrix} -a+2c \\ a-c \\ 2a+b-4c \\ 0 \end{pmatrix} + \begin{pmatrix} -5 \\ 3 \\ 10 \\ 1 \end{pmatrix} w \mid w \in \mathbb{R} \} \tag{4}$$