Problem 1.20 Which of these subsets of the vector space of 2×2 matrices are subspaces dunder the inherited operations? For each one that is a subspace, parametrize its desciprtion. For each that is not, ive a condition that fails.

(a) Let $S = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R}\}$.

First we parametrize S:

$$S = \{a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{R}\} \tag{1}$$

Which shows S can be described as a linear combination of 2×2 matrices.

First we show $\mathbf{0} \in S$:

$$\begin{aligned} a &= 0 \\ b &= 0 \\ 0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0} \end{aligned} \tag{2}$$

Then we show S is closed under scalar multiplication:

$$\begin{aligned} \mathbf{s} &= a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ r\mathbf{s} &= r \left(a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \\ &= ra \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + rb \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ ra, rb &\in \mathbb{R} \\ \therefore r\mathbf{s} &\in S \end{aligned} \tag{3}$$

Finally we show S is closed under addition:

$$\begin{aligned} \mathbf{s_1} &= a_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \mathbf{s_2} &= a_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \mathbf{s_1} + \mathbf{s_2} &= a_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + b_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= (a_1 + a_2) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (b_1 + b_2) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ a_1 + a_2, b_1 + b_2 &\in \mathbb{R} \\ \therefore \mathbf{s_1} + \mathbf{s_2} &\in S \end{aligned} \tag{4}$$

Therefore S is a subspace.

Alternativley, using (1) and the definition of a spanning set

$$span(V) = \{c_1\mathbf{v_1} + \ldots c_n\mathbf{v_n} \mid c_1, \ldots, c_n \in \mathbb{R} \text{ and } \mathbf{v_1}, \ldots, \mathbf{v_n} \in V\} \tag{5}$$

We can see that $S = span\left(\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}\right)$, and as all spanning sets are subspaces, this is another proof S is a subspace.

(b) Let $S = \{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} \mid a + b = 0\}$.

Paramaterizing S:

$$S = \{\begin{pmatrix}-b & 0\\ 0 & b\end{pmatrix} \mid b \in \mathbb{R}\} = \{b\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \mid b \in \mathbb{R}\} \tag{1}$$

Showing $S = span\left(\left\{\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right\}\right)$, and therefore S is a subspace.

(c) Let $S = \{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} \mid a + b = 5\}$.

By paramaterizing S:

$$S = \{\begin{pmatrix}-b+5 & 0\\ 0 & b\end{pmatrix} \mid b \in \mathbb{R}\} \tag{1}$$

We can see S is not a vector space as $\mathbf{0} \notin S$, which means S is not closed under scalar multiplcation.

(d) Let $S = \{\begin{pmatrix}a & c\\ 0 & b\end{pmatrix} \mid a + b = 0, c \in \mathbb{R}\}$.

By paramaterizing S:

$$\begin{aligned} S &= \{\begin{pmatrix}-b & c\\ 0 & b\end{pmatrix} \mid b, c \in \mathbb{R}\} \\ &= \{b\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} + c\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \mid b, c \in \mathbb{R}\} \end{aligned} \tag{1}$$

Showing $S = span\left(\left\{\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\}\right)$, and therefore S is a subspace.