Problem 2.15 Find the indicated entry of the matrix, if it is defined.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & 4 \end{pmatrix}$$

(a) $a_{2,1} = 2$

(b) $a_{1,2} = 3$

(c) $a_{2,2} = -1$

(d) $a_{3,1} = undefined$

Problem 2.16 Give the size of each matrix.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & 4 \end{pmatrix}$$

(a) $\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 5 \end{pmatrix}$ is a 2×3 matrix.

(b) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 3 & -1 \end{pmatrix}$ is a 3×2 matrix.

(c) $\begin{pmatrix} 5 & 10 \\ 10 & 5 \end{pmatrix}$ is a 2×2 matrix.

Problem 2.17 Perform the indicated vector operation, if it is defined.

(a) $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}$

(b) $5\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 20 \\ -1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}$

(d) $7\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 9\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 14 \\ 7 \end{pmatrix} + \begin{pmatrix} 21 \\ 45 \end{pmatrix} = \begin{pmatrix} 35 \\ 52 \end{pmatrix}$

(e) $\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}1\\2\\3\end{pmatrix} = undefined$

(f) $6\begin{pmatrix}3\\1\\1\end{pmatrix} - 4\begin{pmatrix}2\\0\\3\end{pmatrix} + 2\begin{pmatrix}1\\1\\5\end{pmatrix} = \begin{pmatrix}18\\6\\6\end{pmatrix} - \begin{pmatrix}8\\0\\12\end{pmatrix} + \begin{pmatrix}2\\2\\10\end{pmatrix} = \begin{pmatrix}10\\6\\-6\end{pmatrix} + \begin{pmatrix}2\\2\\10\end{pmatrix} = \begin{pmatrix}12\\8\\4\end{pmatrix}$

Problem 2.18 Solve each system using matrix notation. Express the solution using vectors.

(a)

$$\begin{aligned} 3x + 6y &= 18 \\ x + 2y &= 6 \end{aligned} \tag{1}$$

As a matrix:

$$\left(\begin{array}{cc|c} 3 & 6 & 18 \\ 1 & 2 & 6 \end{array}\right) \tag{2}$$

Which can be reduced as follows:

$$\left(\begin{array}{cc|c} 3 & 6 & 18 \\ 1 & 2 & 6 \end{array}\right) \xrightarrow{r_1 - 3r_2} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 2 & 6 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & 0 & 0 \end{array}\right) \tag{3}$$

The solution set for this set of linear equations is:

$$\{(x, y) \mid x + 2y = 6\} \tag{4}$$

As we only have one equation, the first variable x is leading, and the second variable y is free. Therefore we can define x in terms of y $x = 6 - 2y$, giving us the solutions set:

$$\{((6) - (2)y, y) \mid y \in \mathbb{R}\} \tag{5}$$

Which can be rewritten using vectors by grouping together coefficients:

$$\{\begin{pmatrix}-2\\1\end{pmatrix} y + \begin{pmatrix}6\\0\end{pmatrix} \mid y \in \mathbb{R}\} \tag{6}$$

An example solution from this set, when $y = 2$:

$$\vec{s} = \begin{pmatrix}-2\\1\end{pmatrix} 2 + \begin{pmatrix}6\\0\end{pmatrix} = \begin{pmatrix}-4\\2\end{pmatrix} + \begin{pmatrix}6\\0\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} \tag{7}$$

(b)

$$\begin{aligned} x + y &= 1 \\ x - y &= -1 \end{aligned} \tag{8}$$

As a matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & -1 \end{array}\right) \tag{9}$$

Which can be reduced as follows:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & -1 \end{array}\right) \xrightarrow{r_1 - r_2} \left(\begin{array}{cc|c} 0 & 2 & 2 \\ 1 & -1 & -1 \end{array}\right) \xrightarrow{r_1 \leftrightarrow \frac{1}{2} r_2} \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 1 \end{array}\right) \tag{10}$$

Which gives us $y = 1$ and $x = 0$, so the solution set is:

$$\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\} \tag{11}$$

(c)

$$\begin{aligned} x_1 + x_3 &= 4 \\ x_1 - x_2 + 2x_3 &= 5 \\ 4x_1 - x_2 + 5x_3 &= 17 \end{aligned} \tag{12}$$

As a matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 1 & -1 & 2 & 5 \\ 4 & -1 & 5 & 17 \end{array}\right) \tag{13}$$

Which can be reduced as follows:

$$\begin{aligned} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 1 & -1 & 2 & 5 \\ 4 & -1 & 5 & 17 \end{array}\right) &\xrightarrow{r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & -1 & 1 & 1 \\ 4 & -1 & 5 & 17 \end{array}\right) \\ &\xrightarrow{r_3 - 4r_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right) \\ &\xrightarrow{r_3 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{aligned} \tag{14}$$

This leaves us with x_1 and x_2 as leading variables, and x_3 as a free variable. We can rewrite equation r_1 as $x_1 = 4 - x_3$ and r_2 as $x_2 = x_3 - 1$, giving the solution set:

$$\{(4 - x_3, x_3 - 1, x_3) \mid x_3 \in \mathbb{R}\} \tag{15}$$

Written using vectors:

$$\{\begin{pmatrix}-1\\1\\1\end{pmatrix}x_3+\begin{pmatrix}4\\-1\\0\end{pmatrix}\mid x_3\in\mathbb{R}\}\tag{16}$$

(d)

$$\begin{aligned}2a+b-c&=2\\2a\qquad+c&=3\\a-b\qquad&=0\end{aligned}\tag{17}$$

As a matrix:

$$\left(\begin{array}{ccc|c}2&1&-1&2\\2&0&1&3\\1&-1&0&0\end{array}\right)\tag{18}$$

Which can be reduced as follows:

$$\begin{aligned}\left(\begin{array}{ccc|c}2&1&-1&2\\2&0&1&3\\1&-1&0&0\end{array}\right)\xrightarrow{r_2-2r_3}&\left(\begin{array}{ccc|c}2&1&-1&2\\0&2&1&3\\1&-1&0&0\end{array}\right)\\\xrightarrow{r_1-2r_3}&\left(\begin{array}{ccc|c}0&3&-1&2\\0&2&1&3\\1&-1&0&0\end{array}\right)\\\xrightarrow{r_1\leftrightarrow r_3}&\left(\begin{array}{ccc|c}1&-1&0&0\\0&2&1&3\\0&3&-1&2\end{array}\right)\\\xrightarrow{2r_3-3r_2}&\left(\begin{array}{ccc|c}1&-1&0&0\\0&2&1&3\\0&0&-5&-5\end{array}\right)\\\xrightarrow{-1r_3}&\left(\begin{array}{ccc|c}1&-1&0&0\\0&2&1&3\\0&0&5&5\end{array}\right)\end{aligned}\tag{19}$$

This gives us $c=1$, $b=1$ and $c=1$. The solution set for this is:

$$\{\begin{pmatrix}1\\1\\1\end{pmatrix}\}\tag{20}$$

Problem 2.21 The vector is in the set. What value of the paramaters produce that vector?

(a)

$$\begin{pmatrix}5\\-5\end{pmatrix}\in\{\begin{pmatrix}1\\-1\end{pmatrix}k\mid k\in\mathbb{R}\}\tag{1}$$

Using a system of linear equations:

$$\begin{array}{rcl} k & = & 5 \\ -k & = & -5 \end{array} \tag{2}$$

We can see $k = 5$ results in $\begin{pmatrix} 5 \\ -5 \end{pmatrix}$.

(b)

$$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \in \{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} i + \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} j \mid i, j \in \mathbb{R} \} \tag{3}$$

Using a system of linear equations:

$$\begin{array}{rcrcl} -2i & + & 3j & = & -1 \\ i & & & = & 2 \\ & & j & = & 1 \end{array} \tag{4}$$

We can see $i = 2$ and $j = 1$ result in $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$.

(c)

$$\begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix} \in \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} m + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} n \mid m, n \in \mathbb{R} \} \tag{5}$$

Using a system of linear equations:

$$\begin{array}{rcrcl} m & + & 2n & = & 0 \\ m & & & = & -4 \\ & & n & = & 2 \end{array} \tag{6}$$

We can see $m = -4$ and $n = 2$ result in $\begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix}$.

Problem 2.22 Decide if the vector is in the set.

(a)

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} \in \{ \begin{pmatrix} -6 \\ 2 \end{pmatrix} k \mid k \in \mathbb{R} \} \tag{1}$$

Using a system of linear equations:

$$\begin{aligned} -6k &= 3 \\ 2k &= -1 \end{aligned} \tag{2}$$

Representing that system as a matrix:

$$\left(\begin{array}{c|c} -6 & 3 \\ 2 & -1 \end{array}\right) \xrightarrow{r_1+3r_2} \left(\begin{array}{c|c} 0 & 0 \\ 2 & -1 \end{array}\right) \xrightarrow{\frac{1}{2}r_2} \left(\begin{array}{c|c} 0 & 0 \\ 1 & -\frac{1}{2} \end{array}\right) \xrightarrow{r_1 \overset{r}{\longleftrightarrow} r_2} \left(\begin{array}{c|c} 1 & -\frac{1}{2} \\ 0 & 0 \end{array}\right) \tag{3}$$

So this vector is part of the set when $k = -\frac{1}{2}$.

(b)

$$\begin{pmatrix} 5 \\ 4 \end{pmatrix} \in \{\begin{pmatrix} 5 \\ -4 \end{pmatrix} j \mid j \in \mathbb{R}\} \tag{4}$$

Intuitively we can see that $\begin{pmatrix} 5 & 4 \end{pmatrix}^T$ is not a member of this set, as there is no single real value j that would invert -4 but not invert 5.

(c)

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \in \{\begin{pmatrix} 0 \\ 3 \\ -7 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} r \mid r \in \mathbb{R}\} \tag{5}$$

Using a system of linear equations:

$$\begin{array}{rcrcr} r & & & = & 2 \\ -r & + & 3 & = & 1 \\ 3r & + & -7 & = & -1 \end{array} \tag{6}$$

Which can be simplified to:

$$\begin{aligned} r &= 2 \\ r &= 2 \\ r &= 2 \end{aligned} \tag{7}$$

So this vector is a member the set, corresponding to $r = 2$.

(d)

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in \{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} j + \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} k \mid j, k \in \mathbb{R}\} \tag{8}$$

Using a system of linear equations:

$$\begin{array}{rcrcl} 2j & - & 3k & = & 1 \\ & - & k & = & 0 \\ j & + & k & = & 1 \end{array} \tag{9}$$

Representing this system as a matrix:

$$\begin{aligned} \left(\begin{array}{rr|r} 2 & -3 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{array}\right) &\xrightarrow{-1r_2} \left(\begin{array}{rr|r} 2 & -3 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right) \\ &\xrightarrow{r_1-2r_3} \left(\begin{array}{rr|r} 0 & -5 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right) \\ &\xrightarrow{-\frac{1}{5}r_1} \left(\begin{array}{rr|r} 0 & 1 & \frac{1}{5} \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right) \\ &\xrightarrow{r_3-r_2} \left(\begin{array}{rr|r} 0 & 1 & \frac{1}{5} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right) \\ &\xrightarrow{r_3 \overset{r}{\leftrightarrow}_1} \left(\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & \frac{1}{5} \end{array}\right) \end{aligned} \tag{10}$$

So this vector is **not** a member the set, as the system of linear equations is inconsistent for $\begin{pmatrix}1 & 0 & 1\end{pmatrix}^T$.

Problem 2.23 A famer with a 1200 acre farm is considering planting three different crops.

- Corn costs \$20 per acre.
- Soybeans cost \$50 per acre.
- Oat cost \$12 per acre.

The famer has \$40,000 and intends to spend it all.

(a) We can represent this problem as a system of linear equations:

$$\begin{array}{rcrcrcl} 20x_1 & + & 50x_2 & + & 12x_3 & = & 40000 \\ x_1 & + & x_2 & + & x_3 & = & 1200 \end{array} \tag{1}$$

(b) We can represent (1) in matrix form:

$$\left(\begin{array}{ccc|c} 20 & 50 & 12 & 40000 \\ 1 & 1 & 1 & 1200 \end{array}\right) \xrightarrow{r_1-20r_2} \left(\begin{array}{ccc|c} 0 & 30 & -8 & 16000 \\ 1 & 1 & 1 & 1200 \end{array}\right) \xrightarrow{\frac{1}{30}r_1} \left(\begin{array}{ccc|c} 0 & 1 & -\frac{4}{15} & \frac{1600}{3} \\ 1 & 1 & 1 & 1200 \end{array}\right) \tag{2}$$

Which makes x_3 our free variable, so we can rewrite x_2 as:

$$x_2 = \frac{1600}{3} + \frac{4}{15}x_3 \tag{3}$$

And rewrite x_1 as:

$$\begin{aligned} x_1 &= 1200 - x_2 - x_3 \\ &= 1200 - \frac{1600}{3} - \frac{4}{15}x_3 - x_3 \\ &= \frac{2000}{3} - \frac{19}{15}x_3 \end{aligned} \tag{4}$$

Giving the solution set:

$$\begin{aligned} &\{(\frac{2000}{3} - \frac{19}{15}x_3, \frac{1600}{3} + \frac{4}{15}x_3, x_3) \mid x_3 \in \mathbb{R}\} \\ =&\{\begin{pmatrix} -1\frac{19}{15} \\ \frac{4}{15} \\ 1 \end{pmatrix} x_3 + \begin{pmatrix} \frac{2000}{3} \\ \frac{1600}{3} \\ 0 \end{pmatrix} \mid x_3 \in \mathbb{R}\} \end{aligned} \tag{5}$$

This gives us many possible solutions such as when $x_3 = 0$:

$$\vec{s}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{2000}{3} \\ \frac{1600}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2000}{3} \\ \frac{1600}{3} \\ 0 \end{pmatrix} \tag{6}$$

Or when $x_3 = 526$:

$$\vec{s}_2 = \begin{pmatrix} \frac{2}{5} \\ \frac{3368}{5} \\ 526 \end{pmatrix} \tag{7}$$

(c) Supposing the farmer can bring in:

- \$100 per acre for Corn.
- \$300 per acre for Soybeans.
- \$80 per acre for Oats.

We can check which of the results from **(b)** is most profitable by plugging the solution sets into:

$$100x_1 + 300x_2 + 80x_3 \tag{8}$$

For $\vec{s}_1$, the farmer's revenue is $\sim$ \$226666.

For $\vec{s}_2$, the farmer's revenue is \$244200.

Problem 2.25 Solve the following using Gauss' method