Problem 2.15 Find the indicated entry of the matrix, if it is defined.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & 4 \end{pmatrix}$$

(a) $a_{2,1} = 2$

(b) $a_{1,2} = 3$

(c) $a_{2,2} = -1$

(d) $a_{3,1} = undefined$

Problem 2.16 Give the size of each matrix.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & 4 \end{pmatrix}$$

(a) $\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 5 \end{pmatrix}$ is a 2×3 matrix.

(b) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 3 & -1 \end{pmatrix}$ is a 3×2 matrix.

(c) $\begin{pmatrix} 5 & 10 \\ 10 & 5 \end{pmatrix}$ is a 2×2 matrix.

Problem 2.17 Perform the indicated vector operation, if it is defined.

(a) $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}$

(b) $5 \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 20 \\ -1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}$

(d) $7 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 9 \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 14 \\ 7 \end{pmatrix} + \begin{pmatrix} 21 \\ 45 \end{pmatrix} = \begin{pmatrix} 35 \\ 52 \end{pmatrix}$

(e) $\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}1\\2\\3\end{pmatrix} = undefined$

(f) $6\begin{pmatrix}3\\1\\1\end{pmatrix} - 4\begin{pmatrix}2\\0\\3\end{pmatrix} + 2\begin{pmatrix}1\\1\\5\end{pmatrix} = \begin{pmatrix}18\\6\\6\end{pmatrix} - \begin{pmatrix}8\\0\\12\end{pmatrix} + \begin{pmatrix}2\\2\\10\end{pmatrix} = \begin{pmatrix}10\\6\\-6\end{pmatrix} + \begin{pmatrix}2\\2\\10\end{pmatrix} = \begin{pmatrix}12\\8\\4\end{pmatrix}$

Problem 2.18 Solve each system using matrix notation. Express the solution using vectors.

(a)

$$\begin{aligned} 3x + 6y &= 18 \\ x + 2y &= 6 \end{aligned} \tag{1}$$

As a matrix:

$$\left(\begin{array}{cc|c} 3 & 6 & 18 \\ 1 & 2 & 6 \end{array}\right) \tag{2}$$

Which can be reduced as follows:

$$\left(\begin{array}{cc|c} 3 & 6 & 18 \\ 1 & 2 & 6 \end{array}\right) \xrightarrow{r_1 - 3r_2} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 2 & 6 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & 0 & 0 \end{array}\right) \tag{3}$$

The solution set for this set of linear equations is:

$$\{(x, y) \mid x + 2y = 6\} \tag{4}$$

As we only have one equation, the first variable x is leading, and the second variable y is free. Therefore we can define x in terms of y $x = 6 - 2y$, giving us the solutions set:

$$\{((6) - (2)y, y) \mid y \in \mathbb{R}\} \tag{5}$$

Which can be rewritten using vectors by grouping together coefficients:

$$\{\begin{pmatrix}-2\\1\end{pmatrix} y + \begin{pmatrix}6\\0\end{pmatrix} \mid y \in \mathbb{R}\} \tag{6}$$

An example solution from this set, when $y = 2$:

$$\vec{s} = \begin{pmatrix}-2\\1\end{pmatrix} 2 + \begin{pmatrix}6\\0\end{pmatrix} = \begin{pmatrix}-4\\2\end{pmatrix} + \begin{pmatrix}6\\0\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} \tag{7}$$

(b)

$$\begin{aligned} x + y &= 1 \\ x - y &= -1 \end{aligned} \tag{1}$$

As a matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & -1 \end{array}\right) \tag{2}$$

Which can be reduced as follows:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & -1 \end{array}\right) \xrightarrow{r_1 - r_2} \left(\begin{array}{cc|c} 0 & 2 & 2 \\ 1 & -1 & -1 \end{array}\right) \xrightarrow{r_1 \leftrightarrow \frac{1}{2} r_2} \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 1 \end{array}\right) \tag{3}$$

Which gives us $y = 1$ and $x = 0$, so the solution set is:

$$\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \} \tag{4}$$

(c)

$$\begin{aligned} x_1 \qquad + \; x_3 &= 4 \\ x_1 - x_2 + 2x_3 &= 5 \\ 4x_1 - x_2 + 5x_3 &= 17 \end{aligned} \tag{1}$$

As a matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 1 & -1 & 2 & 5 \\ 4 & -1 & 5 & 17 \end{array}\right) \tag{2}$$

Which can be reduced as follows:

$$\begin{aligned} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 1 & -1 & 2 & 5 \\ 4 & -1 & 5 & 17 \end{array}\right) &\xrightarrow{r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & -1 & 1 & 1 \\ 4 & -1 & 5 & 17 \end{array}\right) \\ &\xrightarrow{r_3 - 4r_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right) \\ &\xrightarrow{r_3 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{aligned}$$

This leaves us with x_1 and x_2 as leading variables, and x_3 as a free variable. We can rewrite equation r_1 as $x_1 = 4 - x_3$ and r_2 as $x_2 = x_3 - 1$, giving the solution set:

$$\{(4 - x_3, x_3 - 1, x_3) \mid x_3 \in \mathbb{R}\} \tag{3}$$

Written using vectors:

$$\{\begin{pmatrix}-1\\1\\1\end{pmatrix}x_3+\begin{pmatrix}4\\-1\\0\end{pmatrix}\mid x_3\in\mathbb{R}\}\tag{4}$$

(d)

$$\begin{aligned}2a+b-c&=2\\2a\qquad+c&=3\\a-b\qquad&=0\end{aligned}\tag{1}$$

As a matrix:

$$\left(\begin{array}{rrr|r}2&1&-1&2\\2&0&1&3\\1&-1&0&0\end{array}\right)\tag{2}$$

Which can be reduced as follows:

$$\begin{aligned}\left(\begin{array}{rrr|r}2&1&-1&2\\2&0&1&3\\1&-1&0&0\end{array}\right)&\xrightarrow{r_2-2r_3}\left(\begin{array}{rrr|r}2&1&-1&2\\0&2&1&3\\1&-1&0&0\end{array}\right)\\&\xrightarrow{r_1-2r_3}\left(\begin{array}{rrr|r}0&3&-1&2\\0&2&1&3\\1&-1&0&0\end{array}\right)\\&\xrightarrow{r_1\leftrightarrow r_3}\left(\begin{array}{rrr|r}1&-1&0&0\\0&2&1&3\\0&3&-1&2\end{array}\right)\\&\xrightarrow{2r_3-3r_2}\left(\begin{array}{rrr|r}1&-1&0&0\\0&2&1&3\\0&0&-5&-5\end{array}\right)\\&\xrightarrow{-1r_3}\left(\begin{array}{rrr|r}1&-1&0&0\\0&2&1&3\\0&0&5&5\end{array}\right)\end{aligned}$$

This gives us $c=1$, $b=1$ and $c=1$. The solution set for this is:

$$\{\begin{pmatrix}1\\1\\1\end{pmatrix}\}\tag{3}$$