Problem 1.1 Find the canonical name for each vector.

(a) The vector from $\begin{pmatrix}2\\1\end{pmatrix}$ to $\begin{pmatrix}4\\2\end{pmatrix}$ in $\mathbb{R}^2$ is $\begin{pmatrix}4\\2\end{pmatrix} - \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$.

(b) The vector from $\begin{pmatrix}3\\3\end{pmatrix}$ to $\begin{pmatrix}2\\5\end{pmatrix}$ in $\mathbb{R}^2$ is $\begin{pmatrix}2\\5\end{pmatrix} - \begin{pmatrix}3\\3\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$.

(c) The vector from $\begin{pmatrix}1\\0\\6\end{pmatrix}$ to $\begin{pmatrix}5\\0\\3\end{pmatrix}$ in $\mathbb{R}^3$ is $\begin{pmatrix}1\\0\\6\end{pmatrix} - \begin{pmatrix}5\\0\\3\end{pmatrix} = \begin{pmatrix}-4\\0\\3\end{pmatrix}$.

(d) The vector from $\begin{pmatrix}6\\8\\8\end{pmatrix}$ to $\begin{pmatrix}6\\8\\8\end{pmatrix}$ in $\mathbb{R}^3$ is $\begin{pmatrix}0\\0\\0\end{pmatrix}$

Problem 1.2 Decide if the two vectors are equal.

(a) The vector $\vec{a}$ from $\begin{pmatrix}5\\3\end{pmatrix}$ to $\begin{pmatrix}6\\2\end{pmatrix}$ and the vector $\vec{b}$ from $\begin{pmatrix}1\\-2\end{pmatrix}$ to $\begin{pmatrix}1\\1\end{pmatrix}$

$$\vec{a} = \begin{pmatrix}6\\2\end{pmatrix} - \begin{pmatrix}5\\3\end{pmatrix} = \begin{pmatrix}6-5\\2-3\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$$
$$\vec{b} = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}1-1\\1+2\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix}$$

$\therefore \vec{a} \neq \vec{b}$.

(b) The vector $\vec{a}$ from $\begin{pmatrix}2\\1\\1\end{pmatrix}$ to $\begin{pmatrix}3\\0\\4\end{pmatrix}$ and the vector $\vec{b}$ from $\begin{pmatrix}5\\1\\4\end{pmatrix}$ to $\begin{pmatrix}6\\0\\7\end{pmatrix}$

$$\vec{a} = \begin{pmatrix}3\\0\\4\end{pmatrix} - \begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}3-2\\0-1\\4-1\end{pmatrix} = \begin{pmatrix}1\\-1\\3\end{pmatrix}$$
$$\vec{b} = \begin{pmatrix}6\\0\\7\end{pmatrix} - \begin{pmatrix}5\\1\\4\end{pmatrix} = \begin{pmatrix}6-5\\0-1\\7-4\end{pmatrix} = \begin{pmatrix}1\\-1\\3\end{pmatrix}$$

$\therefore \vec{a} = \vec{b}$.

Problem 1.3 Does the point $\begin{pmatrix}1\\0\\2\\1\end{pmatrix}$ lie on the line through $\begin{pmatrix}-2\\1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}5\\10\\-1\\4\end{pmatrix}$?

First find $\vec{a}$ which describes the displacement vector from $\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 5 \\ 10 \\ -1 \\ 4 \end{pmatrix}$:

$$\vec{a} = \begin{pmatrix} 5 \\ 10 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ -2 \\ 4 \end{pmatrix} \tag{1}$$

$\vec{a}$ describes the direction of the line, so the solution set for the line is:

$$\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 7 \\ 9 \\ -2 \\ 4 \end{pmatrix} x \mid x \in \mathbb{R} \} \tag{2}$$

The displacement vector from $\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$ is:

$$\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \\ 1 \end{pmatrix} \tag{3}$$

There is no real scalar multiplier that would transform $\begin{pmatrix} 7 \\ 9 \\ -2 \\ 4 \end{pmatrix}$ to $\vec{b}$.

$\therefore$ the given point does not lie on the given line.

Problem 1.4

(a) Describe the plane through $\begin{pmatrix} 1 \\ 1 \\ 5 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 0 \\ 4 \end{pmatrix}$

(b) Does this plane pass through the origin?

[1.4]

(a) The first directional vector $\vec{t}$ of the plane is:

$$\vec{s} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix} \tag{4}$$

The second directional vector $\vec{s}$ of the plane is:

$$\vec{t} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -5 \\ 5 \end{pmatrix} \tag{5}$$

Using $\vec{s}$ and $\vec{t}$ we can describe the plane using vectors:

$$\{ \begin{pmatrix} 1 \\ 1 \\ 5 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix} s + \begin{pmatrix} 2 \\ 0 \\ -5 \\ 5 \end{pmatrix} t \mid s, t \in \mathbb{R} \} \tag{6}$$

(b) To figure out if this plane passes through the origin we need to solve:

$$\begin{aligned} \begin{pmatrix} 1 \\ 1 \\ 5 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix} s + \begin{pmatrix} 2 \\ 0 \\ -5 \\ 5 \end{pmatrix} t &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \equiv \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix} s + \begin{pmatrix} 2 \\ 0 \\ -5 \\ 5 \end{pmatrix} t &= \begin{pmatrix} -1 \\ -1 \\ -5 \\ 1 \end{pmatrix} \end{aligned} \tag{7}$$

Convert to reduced row echelon matrix form:

$$\begin{aligned} \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 1 & 0 & -1 \\ -3 & -5 & -5 \\ 1 & 5 & 1 \end{array}\right) &\xrightarrow{r_1 - r_2} \left(\begin{array}{cc|c} 0 & 2 & 0 \\ 1 & 0 & -1 \\ -3 & -5 & -5 \\ 1 & 5 & 1 \end{array}\right) \\ &\xrightarrow{r_3 + 3r_2} \left(\begin{array}{cc|c} 0 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & -5 & -8 \\ 1 & 5 & 1 \end{array}\right) \\ &\xrightarrow{r_3 + \frac{5}{2} r_1} \left(\begin{array}{cc|c} 0 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & -8 \\ 1 & 5 & 1 \end{array}\right) \end{aligned} \tag{8}$$

We can stop here, as r_3 contains a contradiction, we know this system has no solutions. Therefore the plane does not pass through the origin.

Problem 1.5 Give a vector description of each of the following.

(a) The plane subset of $\mathbb{R}^3$ with equation $x - 2y + z = 4$.

Here x is the leading variable, and (y, z) are the free variables.

Parametizing x we get $x = 4 + 2y - z$.

In vector form:

$$\{\begin{pmatrix}4\\0\\0\end{pmatrix} + \begin{pmatrix}2\\1\\0\end{pmatrix} y + \begin{pmatrix}-1\\0\\1\end{pmatrix} z \mid y, z \in \mathbb{R}\}$$

(b) The plane in $\mathbb{R}^3$ with equation $2x + y + 4z = -1$.

Here x is the leading variable, and (y, z) are the free variables.

Parametizing x we get $x = -\frac{1}{2} - \frac{1}{2}y - 2z$.

In vector form:

$$\{\begin{pmatrix}\frac{1}{2}\\0\\0\end{pmatrix} + \begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix} y + \begin{pmatrix}-2\\0\\1\end{pmatrix} z \mid y, z \in \mathbb{R}\}$$

(c) The hyperplane in $\mathbb{R}^4$ with equation $x + y + z + w = 10$.

Here x is the leading variable, and (y, z, w) are the free variables.

Parametizing x we get $x = 10 - y - z - w$.

In vector form:

$$\{\begin{pmatrix}10\\0\\0\\0\end{pmatrix} + \begin{pmatrix}-1\\1\\0\\0\end{pmatrix} y + \begin{pmatrix}-1\\0\\1\\0\end{pmatrix} y + \begin{pmatrix}-1\\0\\0\\1\end{pmatrix} w \mid y, z, w \in \mathbb{R}\}$$

Problem 1.6 Describe the plane that contains this point and this line:

$$\begin{pmatrix}2\\0\\3\end{pmatrix} \qquad \{\begin{pmatrix}-1\\0\\-4\end{pmatrix} + \begin{pmatrix}1\\1\\2\end{pmatrix} t \mid t \in \mathbb{R}\}$$

When $t = 0$, our solution set produces the vector $\begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix}$.

The direction vector $\vec{s}$ from this point to the point we wish to include is:

$$\vec{s} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 7 \end{pmatrix} \tag{9}$$

Adding this as a direction the original line:

$$\{ \begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 7 \end{pmatrix} s + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} t \mid s, t \in \mathbb{R} \} \tag{10}$$