Selmer Schemes I

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Disclaimer

These lecture slides come with a bibliography at the end. However, there has been no attempt at accurate attribution of mathematical results. Rather, the list mostly contains works the lecturer has consulted during preparation, which he hopes will be helpful for users.

I. Background: Arithmetic of Algebraic Curves

Arithmetic of algebraic curves

X: a smooth algebraic curve of genus g defined over $\mathbb{Q}$.

For example, given by a polynomial equation

$$f(x, y) = 0$$

of degree d with rational coefficients, where

$$g = (d-1)(d-2)/2.$$

Diophantine geometry is concerned with the set $X(\mathbb{Q})$ of rational solutions.

Structure is quite different in the three cases:

$g = 0$, spherical geometry (positive curvature);
$g = 1$, flat geometry (zero curvature);
$g \geq 2$, hyperbolic geometry (negative curvature).

Arithmetic of algebraic curves: $g = 0, d \leq 2$

Even now (after millennia of studying these problems), $g = 0$ is the only case that is completely understood.

For $g = 0$, techniques reduce to class field theory and algebraic geometry: local-to-global methods, generation of solutions via sweeping lines, etc.

Idea is to study $\mathbb{Q}$-solutions by considering the geometry of solutions in various completions, the local fields

$$\mathbb{R}, \mathbb{Q}_2, \mathbb{Q}_3, \ldots, \mathbb{Q}_{691}, \ldots,$$

Local-to-global methods sometimes allow us to 'globalise'. For example [Serre],

$$37x^2 + 59y^2 - 67 = 0$$

has a $\mathbb{Q}$-solution if and only if it has a solution in each of $\mathbb{R}, \mathbb{Q}_2, \mathbb{Q}_{37}, \mathbb{Q}_{59}, \mathbb{Q}_{67}$, (Hasse principle) a criterion that can be effectively implemented.

If the existence of a solution is guaranteed, it can be found by an exhaustive search. From one solution, there is a method for parametrising all others: From $(0, -1)$, generate solutions

$$\left(\frac{t^2 - 1}{t^2 + 1}, \frac{2t}{t^2 + 1}\right)$$

to $x^2 + y^2 = 1$.

Arithmetic of algebraic curves: $g = 0$

In other words, there is a successful study of the inclusion

$$X(\mathbb{Q}) \subset X(\mathbb{A}_{\mathbb{Q}}) = \prod' X(\mathbb{Q}_p)$$

coming from **reciprocity laws**.

Arithmetic of algebraic curves: $g = 1$ ($d = 3$)

$X(\mathbb{Q}) = \phi$, non-empty finite, infinite, all are possible.

Hasse principle fails:

$$3x^3 + 4y^3 + 5 = 0$$

has points in $\mathbb{Q}_v$ for all v, but no rational points.

If $X(\mathbb{Q}) \neq \phi$, then fixing an origin $O \in X(\mathbb{Q})$ gives $X(\mathbb{Q})$ the structure of a finitely-generated abelian group via the chord-and-tangent method:

$$X(\mathbb{Q}) \simeq X(\mathbb{Q})_{tor} \times \mathbb{Z}^r.$$

Here, r is called the rank of the curve and $X(\mathbb{Q})_{tor}$ is a finite effectively computable abelian group.

To compute $X(\mathbb{Q})_{tor}$, write

$$X := \{y^2 = x^3 + ax + b\} \cup \{\infty\}$$

$(a, b \in \mathbb{Z})$.

Then $(x, y) \in X(\mathbb{Q})_{tor} \Rightarrow x, y$ are integral and

$$y^2 | (4a^3 + 27b^2).$$

Arithmetic of algebraic curves: $g = 1$

However, the algorithmic computation of the rank and a full set of generators for $X(\mathbb{Q})$ is very difficult, and is the subject of the conjecture of Birch and Swinnerton-Dyer.

In practice, it is often possible to compute these. For example, for

$$y^2 = x^3 - 2,$$

Sage will give you $r = 1$ and the point $(3, 5)$ as generator.

Note that

$$2(3, 5) = (129/100, -383/1000)$$

$$3(3, 5) = (164323/29241, -66234835/5000211)$$

$$4(3, 5) = (2340922881/58675600, 113259286337279/449455096000)$$

Arithmetic of algebraic curves: $g = 1$

Figure: Denominators of $N(3, 5)$

Arithmetic of algebraic curves: $g \geq 2$ ($d \geq 4$)

$X(\mathbb{Q})$ is always finite (Mordell conjecture [Faltings])

However, *very* difficult to compute: consider

$$x^n + y^n = 1$$

for $n \geq 4$.

Sometime easy, such as

$$x^4 + y^4 = -1.$$

However, when there isn't an obvious reason for non-existence, e.g., there already is one solution, then it's hard to know when you have the full list. For example,

$$y^3 = x^6 + 23x^5 + 37x^4 + 691x^3 - 631204x^2 + 5168743489$$

obviously has the solution $(0, 1729)$, but are there any others?

Arithmetic of algebraic curves: $g \geq 2$ ($d \geq 4$)

Effective Mordell problem:

> **Find a terminating algorithm:** $X \mapsto X(\mathbb{Q})$

The **Effective Mordell conjecture** (Szpiro, Vojta, ABC, ...) makes this precise using height inequalities.

Will describe an approach to this problem using the arithmetic geometry of principal bundles.

II. Arithmetic Principal Bundles

Arithmetic principal bundles: (G_K, R, P)

K: field of characteristic zero.
$G_K = \mathrm{Gal}(\bar{K}/K)$: absolute Galois group of K.

A group over K is a topological group R with a continuous action of G_K by group automorphisms:

$$G_K \times R \longrightarrow R.$$

A principal R-bundle over K is a topological space P with compatible continuous actions of G_K (left) and R (right, simply transitive):

$$P \times R \longrightarrow P;$$
$$G_K \times P \longrightarrow P;$$
$$g(zr) = g(z)g(r)$$

for $g \in G_K$, $z \in P$, $r \in R$.

Arithmetic principal bundles: (G_K, R, P)

Note that P is *trivial*, i.e., $\cong R$, exactly when there is a fixed point $z \in P^{G_K}$:

$$R \cong z \times R \cong P.$$

Arithmetic principal bundles

Example:

$$R = \mathbb{Z}_p(1) := \varprojlim \mu_{p^n},$$

where $\mu_{p^n} \subset \bar{K}$ is the group of p^n-th roots of 1.

Thus,

$$\mathbb{Z}_p(1) = \{(\zeta_n)_n\},$$

where

$$\zeta_n^{p^n} = 1; \quad \zeta_{nm}^{p^m} = \zeta_n.$$

As a group,

$$\mathbb{Z}_p(1) \simeq \mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n,$$

but there is a continuous action of G_K.

Arithmetic principal bundles

Given any $x \in K^*$, get principal $\mathbb{Z}_p(1)$-bundle

$$P(x) := \{(y_n)_n \mid y_n^{p^n} = x, \ y_{nm}^{p^m} = y_n.\}$$

over K.

$P(x)$ is trivial iff x admits a p^n-th root in K for all n.

For example, when $K = \mathbb{C}$, $P(x)$ is always trivial.

When $K = \mathbb{Q}$, $P(x)$ is trivial iff $x = 1$ or p is odd and $x = -1$.

For $K = \mathbb{R}$, and p odd, $P(x)$ is trivial for all x.

For $K = \mathbb{R}$ and $p = 2$, $P(x)$ is trivial iff $x > 0$.

Arithmetic principal bundles: moduli spaces

Given a principal R-bundle P over K, choose $z \in P$. This determines a continuous function $c_P : G_K \longrightarrow R$ via

$$g(z) = zc_P(g).$$

It satisfies the condition

$$c_P(g_1 g_2) = c_P(g_1)g_1(c_P(g_2)),$$

defining the set $Z^1(G, R)$. If we choose $z' = zr^{-1}$, then

$$g(z') = g(z)g(r^{-1}) = zc_P(g)g(r^{-1}) = z'rc_P(g)g(r^{-1}).$$

Thus, we get a well-defined class

$$[c_P] \in R \backslash Z^1(G_K, R) =: H^1(G_K, R)$$

where the R-action is defined by

$$c^r(g) = rc(g)g(r^{-1}).$$

Arithmetic principal bundles: moduli spaces

This induces a bijection

{Isomorphism classes of principal R-bundles over K} $\cong H^1(G_K, R)$.

> **Our main concern is the geometry of non-abelian cohomology spaces in various forms.**

For these lectures, R will mostly be a unipotent fundamental group of an algebraic curve with a very complicated K-structure.

Two more classes of important examples:

$-R$ is the monodromy group of a specific local system on a curve. (Lawrence and Venkatesh [LV])

$-R$ is a reductive group with a trivial K-structure:

$$H^1(G_K, R) = R\backslash\text{Hom}(G_K, R).$$

These are analytic moduli spaces of Galois representations [Chenevier].

Arithmetic principal bundles: moduli spaces

When K is a number field, there are completions K_v and injections

$$G_v = \mathrm{Gal}(\bar{K}_v / K_v) \hookrightarrow G = \mathrm{Gal}(\bar{K}/K).$$

giving rise to the localisation map

$$loc : H^1(K, R) \longrightarrow \prod_v H^1(K_v, R).$$

A wide range of problems in number theory rely on the study of its image. The general principle is that the local-to-global problem is easier to study for principal bundles than for points.

III. Diophantine principal bundles: elliptic curves

Diophantine principal bundles: elliptic curves

[Silverman]

E: elliptic curve over $\mathbb{Q}$.

We let $G = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ act on the exact sequence

$$0 \longrightarrow E[p](\bar{\mathbb{Q}}) \longrightarrow E(\bar{\mathbb{Q}}) \xrightarrow{\;p\;} E(\bar{\mathbb{Q}}) \longrightarrow 0$$

to generate the long exact sequence

$$0 \longrightarrow E(\mathbb{Q})[p] \longrightarrow E(\mathbb{Q}) \xrightarrow{\;p\;} E(\mathbb{Q})$$

$$\longrightarrow H^1(\mathbb{Q}, E[p]) \longrightarrow H^1(\mathbb{Q}, E) \xrightarrow{\;p\;} H^1(\mathbb{Q}, E),$$

from which we get the inclusion (Kummer map)

$$0 \longrightarrow E(\mathbb{Q})/pE(\mathbb{Q}) \hookrightarrow H^1(\mathbb{Q}, E[p])$$

Diophantine principal bundles: elliptic curves

The central problem in the theory of elliptic curves is the identification of the image

$$Im(E(\mathbb{Q})/pE(\mathbb{Q})) \subset H^1(G, E[p]).$$

To this end, define the p-Selmer group

$$Sel(\mathbb{Q}, E[p]) \subset H^1(\mathbb{Q}, E[p])$$

to be the classes in $H^1(\mathbb{Q}, E[p])$ that locally come from points. This is useful because the local version of this problem can be solved.

Diophantine principal bundles: elliptic curves

$$
\begin{array}{ccccc}
0 & \longrightarrow & E(\mathbb{Q})/pE(\mathbb{Q}) & \hookrightarrow & H^1(\mathbb{Q}, E[p]) \\
& & \downarrow loc_v & & \downarrow loc_v \\
0 & \longrightarrow & E(\mathbb{Q}_v)/pE(\mathbb{Q}_v) & \hookrightarrow & H^1(\mathbb{Q}_v, E[p])
\end{array}
$$

Then
$$
Sel(\mathbb{Q}, E[p]) := \cap_v loc_v^{-1}(Im(E(\mathbb{Q}_v)/pE(\mathbb{Q}_v))).
$$

Diophantine principal bundles: elliptic curves

The key point is that **the p-Selmer group is a finite-dimensional $\mathbb{F}_p$-vector space that is effectively computable** and this already gives us a bound on the Mordell-Weil group of E:

$$E(\mathbb{Q})/pE(\mathbb{Q}) \subset Sel(\mathbb{Q}, E[p]).$$

This is then refined by way of the diagram

$$
\begin{array}{ccccc}
0 & \longrightarrow & E(\mathbb{Q})/p^n E(\mathbb{Q}) & \longrightarrow & Sel(\mathbb{Q}, E[p^n]) \\
& & \downarrow & & \downarrow \\
& & & & \\
0 & \longrightarrow & E(\mathbb{Q})/pE(\mathbb{Q}) & \longrightarrow & Sel(\mathbb{Q}, E[p])
\end{array}
$$

for increasing values of n.

Diophantine principal bundles: elliptic curves

Conjecture: (BSD, Tate-Shafarevich)

$$Im(E(\mathbb{Q})/pE(\mathbb{Q})) = \cap_{n=1}^{\infty} Im[Sel(\mathbb{Q}, E[p^n]] \subset Sel(\mathbb{Q}, E[p]).$$

Of course this implies that

$$Im(E(\mathbb{Q})/pE(\mathbb{Q})) = Im[Sel(\mathbb{Q}, E[p^N]] \subset Sel(\mathbb{Q}, E[p])$$

at some finite level p^N, and there is a conditional algorithm for verifying this. A main goal of BSD is to remove the conditional aspect.

IV. Diophantine principal bundles II: The non-abelian case

Diophantine principal bundles II: The non-abelian case

To generalise, focus on the sequence of maps

$$\cdots \longrightarrow E[p^3] \xrightarrow{\ p\ } E[p^2] \xrightarrow{\ p\ } E[p]$$

of which we take the inverse limit to get the p-adic Tate module of E:

$$T_p E := \varprojlim E[p^n].$$

This is a free $\mathbb{Z}_p$-module of rank 2.

The previous finite boundary maps can be packaged into

$$j : E(\mathbb{Q}) \longrightarrow \varprojlim H^1(\mathbb{Q}, E[p^n]) = H^1(\mathbb{Q}, T_p E).$$

The key point is that

$$T_p E \simeq \pi_1^p(\bar{E}, O),$$

where $\pi_1^p(\bar{X}, b)$ refers to the pro-p completion of the fundamental group $\pi_1(X(\mathbb{C}), b)$ of a variety X.

The map j can be thought of as

$$x \mapsto \pi^p(\bar{E}; O, x).$$

Diophantine principal bundles II: The non-abelian case

Fundamental fact of arithmetic homotopy:

> **If X is a variety defined over $\mathbb{Q}$ and $b, x \in X(\mathbb{Q})$, then**
>
> $$\pi_1^p(\bar{X}, b), \quad \pi_1^p(\bar{X}; b, x)$$
>
> **admit compatible actions of $G = \mathbf{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.**

The triples

$$(G_{\mathbb{Q}}, \pi_1^p(\bar{X}, b), \pi_1^p(\bar{X}; b, x))$$

are important concrete examples of (G_K, R, P) from the general definitions.

Diophantine principal bundles II: The non-abelian case

This formulation then extends to general X, whereby we get a map

$$j : X(\mathbb{Q}) \longrightarrow H^1(G, \pi_1^p(\bar{X}, b))$$

given by

$$x \mapsto [\pi_1^p(\bar{X}; b, x)]$$

For each prime v, have local versions

$$j_v : X(\mathbb{Q}_v) \longrightarrow H^1(G_v, \pi_1^p(\bar{X}, b))$$

given by

$$x \mapsto [\pi_1^p(\bar{X}; b, x)]$$

which turn out to be far more computable than the global map.

Diophantine principal bundles II: The non-abelian case

Localization diagram:

$$
\begin{array}{ccc}
X(\mathbb{Q}) & \longrightarrow & \prod_v X(\mathbb{Q}_v) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \prod_v j_v} \\
H^1(G, \pi_1^p(\bar{X}, b)) & \xrightarrow{loc} & \prod_v H^1(G_v, \pi_1^p(\bar{X}, b))
\end{array}
$$

As in the elliptic curve case, our interest is in the interaction between the images of loc and $\prod_v j_v$.

Diophantine principal bundles II: The non-abelian case

Actual applications use

$$
\begin{array}{ccc}
X(\mathbb{Q}) & \longrightarrow & \prod_v X(\mathbb{Q}_v) \\
\Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle \prod_v j_v} \\
H^1(G, U(\bar{X}, b)) & \longrightarrow & \prod_v H^1(G_v, U(\bar{X}, b))
\end{array}
$$

where

$$
U(\bar{X}, b) = {}^{\backprime}\pi_1^p(X, b) \otimes \mathbb{Q}_p{}^{\backprime}
$$

is the $\mathbb{Q}_p$-pro-unipotent completion of $\pi_1^p(\bar{X}, b)$ [Deligne].

The effect is that the moduli spaces become pro-algebraic schemes over $\mathbb{Q}_p$ and the lower row of this diagram an algebraic map.

Diophantine principal bundles II: The non-abelian case

That is, the key object of study is

$$H^1_f(G, U(\bar{X}, b))$$

the **Selmer scheme** of X, defined to be the subfunctor of $H^1(G, U(\bar{X}, b))$ satisfying local conditions at all (or most) v.

These are conditions like 'unramified at most primes', 'crystalline at p', and often a few extra conditions [BD, Fontaine].

Diophantine principal bundles II: The non-abelian case

$$
\begin{array}{ccc}
X(\mathbb{Q}) & \longrightarrow & \prod_v X(\mathbb{Q}_v) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \prod_v j_v} \\
H_f^1(G, U(\bar{X}, b)) & \longrightarrow \prod_v H_f^1(G_v, U(\bar{X}, b)) & \xrightarrow{\alpha} \mathbb{Q}_p
\end{array}
$$

If α is an algebraic function vanishing on the image, then

$$
\alpha \circ \prod_v j_v
$$

gives a defining equation for $X(\mathbb{Q})$ inside $\prod_v X(\mathbb{Q}_v)$.

Diophantine principal bundles II: The non-abelian case

To make this concretely computable, we take the projection

$$pr_p : \prod_v X(\mathbb{Q}_v) \longrightarrow X(\mathbb{Q}_p)$$

and try to compute

$$\cap_\alpha pr_p(Z(\alpha \circ \prod_v j_v)) \subset X(\mathbb{Q}_p).$$

Non-Archimedean effective Mordell Conjecture:

I. $\boxed{\cap_\alpha pr_p(Z(\alpha \circ \prod_v j_v)) = X(\mathbb{Q})}$

II. $\boxed{\text{This set is effectively computable.}}$

Diophantine principal bundles II: The non-abelian case

Two remarks:

1. As soon as there is one α with α_p non-trivial, $pr_p(Z(\alpha \circ \prod_v j_v))$ is finite.

2. There is a (highly reliable) conjectural mechanism for producing infinitely many algebraically independent α.

V. Computing Rational Points

Computing rational points

[DW1, DW2]

For $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$,

$$X(\mathbb{Z}[1/2]) = \{2, -1, 1/2\} \subset \{D_2(z) = 0\} \cap \{D_4(z) = 0\},$$

where

$$D_2(z) = \ell_2(z) + (1/2) \log(z) \log(1 - z),$$

$$D_4(z) = \zeta(3)\ell_4(z) + (8/7)[\log^3 2/24 + \ell_4(1/2)/\log 2] \log(z)\ell_3(z)$$

$$+ [(4/21)(\log^3 2/24 + \ell_4(1/2)/\log 2) + \zeta(3)/24] \log^3(z) \log(1 - z),$$

and

$$\ell_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

Numerically, the inclusion appears to be an equality.

Computing rational points

Some qualitative results:

(Coates and Kim [CK])

$$ax^n + by^n = c$$

for $n \geq 4$ has only finitely many rational points.

Standard structural conjectures on mixed motives (generalised BSD)
$\Rightarrow$ There exist (many) non-zero α as above
$\Rightarrow$ Faltings's theorem over $\mathbb{Q}$.

Recently, Lawrence and Venkatesh [LV] have succeeded in giving an unconditional new proof of Faltings's theorem using the same diagram with U replaced by (the vector bundle associated to a representation of) the monodromy group of π_1 with respect to a semi-simple representation

$$\rho : \pi_1 \longrightarrow Aut(V).$$

Computing rational points

A recent result on modular curves by Balakrishnan, Dogra, Mueller, Tuitmann, Vonk [BDMTV]

$$X_s^+(N) = X(N)/C_s^+(N),$$

where $X(N)$ the the compactification of the moduli space of pairs

$$(E, \phi : E[N] \simeq (\mathbb{Z}/N)^2),$$

and $C_s^+(N) \subset GL_2(\mathbb{Z}/N)$ is the normaliser of a split Cartan subgroup.

Bilu-Parent-Rebolledo [BP, BPR] had shown that $X_s^+(p)(\mathbb{Q})$ consists entirely of cusps and CM points for all primes $p > 7$, $p \neq 13$. They called $p = 13$ the 'cursed level'.

Computing rational points

Theorem (BDMTV)

The modular curve

$$X_s^+(13)$$

has exactly 7 rational points, consisting of the cusp and 6 CM points.

This concludes an important chapter of a conjecture of Serre:

There is an absolute constant A such that

$$G \longrightarrow \mathrm{Aut}(E[p])$$

is surjective for all non-CM elliptic curves $E/\mathbb{Q}$ and primes $p > A$.

Computing rational points

[Burcu Baran]

$$y^4 + 5x^4 - 6x^2y^2 + 6x^3z + 26x^2yz + 10xy^2z - 10y^3z$$

$$-32x^2z^2 - 40xyz^2 + 24y^2z^2 + 32xz^3 - 16yz^3 = 0$$
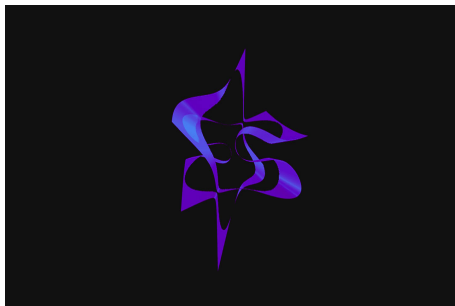
Figure: The cursed curve

$\{(1{:}1{:}1), \ (1{:}1{:}2), \ (0{:}0{:}1), \ (\text{-}3{:}3{:}2), \ (1{:}1{:}0), \ (0{,}2{:}1), \ (\text{-}1{:}1{:}0) \ \}$

VI. Some speculations on rational points and critical points

Some speculations on rational points and critical points

Would like to think of

$$H^1(G, U(\bar{X}, b)) \longrightarrow \prod_v H^1(G_v, U(\bar{X}, b))$$

as being like

$$\mathbb{S}(M, G) \subset \mathcal{A}(M, G)$$

the space of solutions to a set of Euler-Lagrange equations on a space of connections.

In particular, functions cutting out the image of localisation should be thought of as 'classical equations of motion' for gauge fields.

Some speculations on rational points and critical points

When X is smooth and projective, $X(\mathbb{Q}) = X(\mathbb{Z})$, and we are actually interested in

$$Im(H^1(G_S, U)) \cap \prod_{v \in S} H^1_f(G_v, U) \subset \prod_{v \in S} H^1(G_v, U),$$

where

$$H^1_f(G_v, U) \subset H^1(G_v, U)$$

is a subvariety defined by some integral or Hodge-theoretic conditions.

In order to apply symplectic techniques, replace U by

$$T^*(1)U := (LieU)^*(1) \rtimes U.$$

Some speculations on rational points and critical points

Then

$$\prod_{v \in S} H^1(G_v, T^*(1)U)$$

is a symplectic variety and

$$Im(H^1(G_S, T^*(1)U)), \quad \prod_{v \in S} H^1_f(G_v, T^*(1)U)$$

are Lagrangian subvarieties.

Thus, the (derived) intersection

$$\mathcal{D}_S(X) := Im(H^1(G_S, T^*(1)U)) \cap \prod_{v \in S} H^1_f(G_v, T^*(1)U)$$

has a $[-1]$-shifted symplectic structure.

Zariski-locally the critical set of a function. (Brav, Bussi, Joyce [BBJ])

Some speculations on rational points and critical points

$$
\begin{array}{ccc}
X(\mathbb{Z}) \longrightarrow j_S^{-1}(\mathcal{D}_S(X)) & \hookrightarrow & \displaystyle\prod_{v\in S} X(\mathbb{Q}_v) \\
\downarrow{\scriptstyle j^g} \qquad\qquad \downarrow{\scriptstyle j_S} & & \downarrow{\scriptstyle j_S} \\
H_f^1(G_S, T^*(1)U) \xrightarrow{\ loc_S\ } \mathcal{D}_S(X) & \hookrightarrow & \displaystyle\prod_{v\in S} H^1(G_v, T^*(1)U_n)
\end{array}
$$

From this view, the global points can be obtained by pulling back 'Euler-Lagrange equations' via a period map.

Some speculations on rational points and critical points
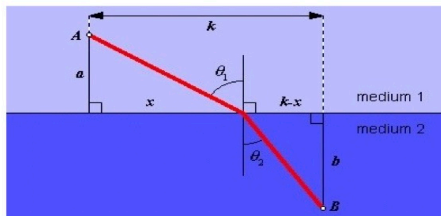
For integers $n > 2$ the equation

$$a^n + b^n = c^n$$

cannot be solved with positive integers a, b, c.

Figure: Pierre de Fermat (1607-1665)

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