

# Geometric quadratic Chabauty

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## Abstract

Determining all rational points on a curve of genus at least 2 can be difficult. Chabauty's method (1941) is to intersect, for a prime number  $p$ , in the  $p$ -adic Lie group of  $p$ -adic points of the jacobian, the closure of the Mordell-Weil group with the  $p$ -adic points of the curve. If the Mordell-Weil rank is less than the genus then this method has never failed.

Minhyong Kim's non-abelian Chabauty programme aims to remove the condition on the rank. The simplest case, called quadratic Chabauty, was developed by Balakrishnan, Dogra, Müller, Tuitman and Vonk, and applied in a tour de force to the so-called cursed curve (rank and genus both 3).

This article aims to make the quadratic Chabauty method *small* and *geometric* again, by describing it in terms of only 'simple algebraic geometry' (line bundles over the jacobian and models over the integers).

Keywords: rational points, curves, Chabauty, non-abelian, quadratic, geometric.

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# 1 Introduction

Determining all rational points on a curve of genus  $g \geq 2$  can be a difficult problem. Chabauty’s method (1941) is to intersect, for a prime number  $p$ , in the  $p$ -adic Lie group of  $p$ -adic points of the jacobian, the closure of the Mordell-Weil group with the  $p$ -adic points of the curve. If the Mordell-Weil rank  $r$  satisfies  $r < g$  then this method has never failed.

Minhyong Kim’s non-abelian Chabauty programme aims to remove the condition that  $r < g$ . The ‘non-abelian’ refers to fundamental groups; the fundamental group of the jacobian of a curve is the abelianised fundamental group of the curve. The most striking result in this direction is the so-called quadratic Chabauty method, applied in [2], a technical tour de force, to the so-called cursed curve ( $r = g = 3$ ). For more details we recommend the introduction of [2].

This article aims to make the quadratic Chabauty method *small* and *geometric* again, by describing it in terms of only ‘simple algebraic geometry’ (line bundles over the jacobian and models over the integers). Section 2 describes the geometric method in less than 3 pages, sections 3–5 give the necessary theory, sections 6–7 give descriptions that are suitable for computer calculations, and section 8 treats an example with  $r = g = 2$  and 14 rational points. Theorem 4.12 gives a criterion for a given list of rational points to be complete, in terms of

points with values in  $\mathbb{Z}/p^2\mathbb{Z}$ . We expect that this approach will make it possible to treat many more curves.

The biextensions in our approach correspond well with the name ‘quadratic’, and this quadratic nature also manifests itself in the equations over  $\mathbb{F}_p$  in Theorem 4.12. The fundamental group playing a role here is a subgroup of a higher dimensional Heisenberg group, where the commutator pairing is the intersection pairing of the fundamental group of the curve (see [3], Section 4). The passage from our approach to that of [2] is given by  $p$ -adic local systems on our geometric objects. Because of the technicalities involved we do not go into this (less is more).

## 2 Algebraic geometry

Let  $C$  be a scheme over  $\mathbb{Z}$ , proper, flat, regular, with  $C_{\mathbb{Q}}$  of dimension one and geometrically connected. Let  $n$  be in  $\mathbb{Z}_{\geq 1}$  such that the restriction of  $C$  to  $\mathbb{Z}[1/n]$  is smooth. Let  $g$  be the genus of  $C_{\mathbb{Q}}$ . We assume that  $g \geq 2$  and that we have a rational point  $b \in C(\mathbb{Q})$ ; it extends uniquely to a  $b \in C(\mathbb{Z})$ . We let  $J$  be the Néron model over  $\mathbb{Z}$  of the jacobian  $\text{Pic}_{C_{\mathbb{Q}}/\mathbb{Q}}^0$ . We denote by  $J^{\vee}$  the Néron model over  $\mathbb{Z}$  of the dual  $J_{\mathbb{Q}}^{\vee}$  of  $J_{\mathbb{Q}}$ , and  $\lambda: J \rightarrow J^{\vee}$  the isomorphism extending the canonical principal polarisation of  $J_{\mathbb{Q}}$ . We let  $P_{\mathbb{Q}}$  be the Poincaré *line bundle* on  $J_{\mathbb{Q}} \times J_{\mathbb{Q}}^{\vee}$ , trivialised on the union of  $\{0\} \times J_{\mathbb{Q}}^{\vee}$  and  $J_{\mathbb{Q}} \times \{0\}$ . Then the Poincaré *torsor* is the  $\mathbb{G}_m$ -torsor on  $J_{\mathbb{Q}} \times J_{\mathbb{Q}}^{\vee}$  defined as

$$(2.1) \quad P_{\mathbb{Q}}^{\times} = \mathbf{Isom}_{J_{\mathbb{Q}} \times J_{\mathbb{Q}}^{\vee}}(\mathcal{O}_{J_{\mathbb{Q}} \times J_{\mathbb{Q}}^{\vee}}, P_{\mathbb{Q}}).$$

For every scheme  $S$  over  $J_{\mathbb{Q}} \times J_{\mathbb{Q}}^{\vee}$ ,  $P_{\mathbb{Q}}^{\times}(S)$  is the set of isomorphisms from  $\mathcal{O}_S$  to  $(P_{\mathbb{Q}})_S$ , with a free and transitive action of  $\mathcal{O}_S(S)^{\times}$ . Locally on  $S$  for the Zariski topology,  $(P_{\mathbb{Q}}^{\times})_S$  is trivial, and  $P_{\mathbb{Q}}^{\times}$  is represented by a scheme over  $J_{\mathbb{Q}} \times J_{\mathbb{Q}}^{\vee}$ .

The theorem of the cube gives  $P_{\mathbb{Q}}^{\times}$  the structure of a *biextension* of  $J_{\mathbb{Q}}$  and  $J_{\mathbb{Q}}^{\vee}$  by  $\mathbb{G}_m$ , a notion for the details of which we recommend Section I.2.5 of [9], Grothendieck’s Exposés VII and VIII [11], and references therein. This means the following. For  $S$  a  $\mathbb{Q}$ -scheme,  $x_1$  and  $x_2$  in  $J_{\mathbb{Q}}(S)$ , and  $y$  in  $J_{\mathbb{Q}}^{\vee}(S)$ , the theorem of the cube gives a canonical isomorphism of  $\mathcal{O}_S$ -modules

$$(2.2) \quad (x_1, y)^* P_{\mathbb{Q}} \otimes_{\mathcal{O}_S} (x_2, y)^* P_{\mathbb{Q}} = (x_1 + x_2, y)^* P_{\mathbb{Q}}.$$

This induces a morphism of schemes

$$(2.3) \quad (x_1, y)^* P_{\mathbb{Q}}^{\times} \times_S (x_2, y)^* P_{\mathbb{Q}}^{\times} \longrightarrow (x_1 + x_2, y)^* P_{\mathbb{Q}}^{\times}.$$

as follows. For any  $S$ -scheme  $T$ , and  $z_1$  in  $((x_1, y)^* P_{\mathbb{Q}}^{\times})(T)$  and  $z_2$  in  $((x_2, y)^* P_{\mathbb{Q}}^{\times})(T)$ , we view  $z_1$  and  $z_2$  as nowhere vanishing sections of the invertible  $\mathcal{O}_T$ -modules  $(x_1, y)^* P_{\mathbb{Q}}$  and  $(x_2, y)^* P_{\mathbb{Q}}$ . The tensor product of these two then gives an element of  $((x_1 + x_2, y)^* P_{\mathbb{Q}}^{\times})(T)$ . This gives  $P_{\mathbb{Q}}^{\times} \rightarrow J_{\mathbb{Q}}^{\vee}$  the structure of a commutative group scheme, which is an extension of  $J_{\mathbb{Q}}$  by  $\mathbb{G}_m$ ,

over the base  $J_{\mathbb{Q}}^{\vee}$ . We denote this group law, and the one on  $J_{\mathbb{Q}} \times J_{\mathbb{Q}}^{\vee}$ , as

$$(2.4) \quad \begin{array}{ccc} (z_1, z_2) & \xrightarrow{\quad} & z_1 +_1 z_2 \\ \downarrow & & \downarrow \\ ((x_1, y), (x_2, y)) & \xrightarrow{\quad} & (x_1, y) +_1 (x_2, y) \equiv (x_1 + x_2, y). \end{array}$$

In the same way,  $P_{\mathbb{Q}}^{\times} \rightarrow J_{\mathbb{Q}}$  has a group law  $+_2$  that makes it an extension of  $J_{\mathbb{Q}}^{\vee}$  by  $\mathbb{G}_m$  over the base  $J_{\mathbb{Q}}$ . In this way,  $P_{\mathbb{Q}}^{\times}$  is both the universal extension of  $J_{\mathbb{Q}}$  by  $\mathbb{G}_m$  as the universal extension of  $J_{\mathbb{Q}}^{\vee}$  by  $\mathbb{G}_m$ . The final ingredient of the notion of biextension is that the two partial group laws are compatible in the following sense. For any  $\mathbb{Q}$ -scheme  $S$ , for  $x_1$  and  $x_2$  in  $J_{\mathbb{Q}}(S)$ ,  $y_1$  and  $y_2$  in  $J_{\mathbb{Q}}^{\vee}(S)$ , and, for all  $i$  and  $j$  in  $\{1, 2\}$ ,  $z_{i,j}$  in  $((x_i, y_j)^* P_{\mathbb{Q}}^{\times})(S)$ , we have

$$(2.5) \quad \begin{array}{ccc} (z_{1,1} +_1 z_{2,1}) +_2 (z_{1,2} +_1 z_{2,2}) & \equiv & (z_{1,1} +_2 z_{1,2}) +_1 (z_{2,1} +_2 z_{2,2}) \\ \downarrow & & \downarrow \\ (x_1 + x_2, y_1) +_2 (x_1 + x_2, y_2) & \equiv & (x_1, y_1 + y_2) +_1 (x_2, y_1 + y_2) \end{array}$$

with the equality in the upper line taking place in  $((x_1 + x_2, y_1 + y_2)^* P_{\mathbb{Q}}^{\times})(S)$ .

Now we extend the geometry above over  $\mathbb{Z}$ . We denote by  $J^0$  the fibrewise connected component of 0 in  $J$ , which is an open subgroup scheme of  $J$ , and by  $\Phi$  the quotient  $J/J^0$ , which is an étale (not necessarily separated) group scheme over  $\mathbb{Z}$ , with finite fibres, supported on  $\mathbb{Z}/n\mathbb{Z}$ . Similarly, we let  $J^{\vee 0}$  be the fibrewise connected component of  $J^{\vee}$ . Theorem 7.1, in Exposé VIII of [11] gives that  $P_{\mathbb{Q}}^{\times}$  extends uniquely to a  $\mathbb{G}_m$ -biextension

$$(2.6) \quad P^{\times} \longrightarrow J \times J^{\vee 0}$$

(Grothendieck's pairing on component groups is the obstruction to the existence of such an extension). Note that in this case the existence and the uniqueness follow directly from the requirement of extending the rigidification on  $J_{\mathbb{Q}} \times \{0\}$ . For details see Section 6.7.

Our base point  $b \in C(\mathbb{Z})$  gives an embedding  $j_b: C_{\mathbb{Q}} \rightarrow J_{\mathbb{Q}}$ , which sends, functorially in  $\mathbb{Q}$ -schemes  $S$ , an element  $c \in C_{\mathbb{Q}}(S)$  to the class of the invertible  $\mathcal{O}_{C_S}$ -module  $\mathcal{O}_{C_S}(c - b)$ . Then  $j_b$  extends uniquely to a morphism

$$(2.7) \quad j_b: C^{\text{sm}} \longrightarrow J$$

where  $C^{\text{sm}}$  is the open subscheme of  $C$  consisting of points at which  $C$  is smooth over  $\mathbb{Z}$ . Note that  $C_{\mathbb{Q}}(\mathbb{Q}) = C(\mathbb{Z}) = C^{\text{sm}}(\mathbb{Z})$ .

Our next step is to lift  $j_b$ , at least on certain opens of  $C^{\text{sm}}$ , to a morphism to a  $\mathbb{G}_m^{\rho-1}$ -torsor over  $J$ , where  $\rho$  is the rank of the free  $\mathbb{Z}$ -module  $\text{Hom}(J_{\mathbb{Q}}, J_{\mathbb{Q}}^{\vee})^+$ , the  $\mathbb{Z}$ -module of self-dual morphisms from  $J_{\mathbb{Q}}$  to  $J_{\mathbb{Q}}^{\vee}$ . This torsor will be the product of pullbacks of  $P^{\times}$  via morphisms

$$(2.8) \quad (\text{id}, m \cdot \circ \text{tr}_c \circ f): J \rightarrow J \times J^{\vee 0},$$

with  $f: J \rightarrow J^{\vee}$  a morphism of group schemes,  $c \in J^{\vee}(\mathbb{Z})$ ,  $\text{tr}_c$  the translation by  $c$ ,  $m$  the least common multiple of the exponents of all  $\Phi(\overline{\mathbb{F}}_p)$  with  $p$  ranging over all primes, and  $m \cdot$  the multiplication by  $m$  map on  $J^{\vee}$ . For such a map  $m \cdot \circ \text{tr}_c \circ f$ ,  $j_b: C_{\mathbb{Q}} \rightarrow J_{\mathbb{Q}}$  can be lifted to

$(\text{id}, m \cdot \circ \text{tr}_c \circ f)^* P_{\mathbb{Q}}^{\times}$  if and only if  $j_b^*(\text{id}, m \cdot \circ \text{tr}_c \circ f)^* P_{\mathbb{Q}}^{\times}$  is trivial. The degree of this  $\mathbb{G}_m$ -torsor on  $C_{\mathbb{Q}}$  is minus the trace of  $\lambda^{-1} \circ m \cdot \circ (f + f^{\vee})$  acting on  $H_1(J(\mathbb{C}), \mathbb{Z})$ . For example, for  $f = \lambda$  the degree is  $-4mg$ . Note that  $j_b: C_{\mathbb{Q}} \rightarrow J_{\mathbb{Q}}$  induces

$$(2.9) \quad j_b^* = -\lambda^{-1}: J_{\mathbb{Q}}^{\vee} \rightarrow J_{\mathbb{Q}},$$

(see [8], Propositions 2.7.9 and 2.7.10). This implies that for  $f$  such that this degree is zero, there is a unique  $c$  such that  $j_b^*(\text{id}, \text{tr}_c \circ f)^* P_{\mathbb{Q}}^{\times}$  is trivial on  $C_{\mathbb{Q}}$ , and hence also its  $m$ th power  $j_b^*(\text{id}, m \cdot \circ \text{tr}_c \circ f)^* P_{\mathbb{Q}}^{\times}$ .

The map

$$(2.10) \quad \text{Hom}(J_{\mathbb{Q}}, J_{\mathbb{Q}}^{\vee}) \longrightarrow \text{Pic}(J_{\mathbb{Q}}) \longrightarrow \text{NS}_{J_{\mathbb{Q}}/\mathbb{Q}}(\mathbb{Q}) = \text{Hom}(J_{\mathbb{Q}}, J_{\mathbb{Q}}^{\vee})^+$$

sending  $f$  to the class of  $(\text{id}, f)^* P_{\mathbb{Q}}$  sends  $f$  to  $f + f^{\vee}$ , hence its kernel is  $\text{Hom}(J_{\mathbb{Q}}, J_{\mathbb{Q}}^{\vee})^-$ , the group of antisymmetric morphisms. But actually, for  $f$  antisymmetric, its image in  $\text{Pic}(J_{\mathbb{Q}})$  is already zero (see for example [3] and the references therein). Hence the image of  $\text{Hom}(J_{\mathbb{Q}}, J_{\mathbb{Q}}^{\vee})$  in  $\text{Pic}(J_{\mathbb{Q}})$  is free of rank  $\rho$ , and its subgroup of classes with degree zero on  $C_{\mathbb{Q}}$  is free of rank  $\rho-1$ . Let  $f_1, \dots, f_{\rho-1}$  be elements of  $\text{Hom}(J_{\mathbb{Q}}, J_{\mathbb{Q}}^{\vee})$  whose images in  $\text{Pic}(J_{\mathbb{Q}})$  form a basis of this subgroup, and let  $c_1, \dots, c_{\rho-1}$  be the corresponding elements of  $J^{\vee}(\mathbb{Z})$ .

By construction, for each  $i$ , the morphism  $j_b: C_{\mathbb{Q}} \rightarrow J_{\mathbb{Q}}$  lifts to  $(\text{id}, m \cdot \circ \text{tr}_{c_i} \circ f_i)^* P_{\mathbb{Q}}^{\times}$ , unique up to  $\mathbb{Q}^{\times}$ . Now we spread this out over  $\mathbb{Z}$ , to open subschemes  $U$  of  $C^{\text{sm}}$  obtained by removing, for each  $q$  dividing  $n$ , all but one irreducible components of  $C_{\mathbb{F}_q}^{\text{sm}}$ , with the remaining irreducible component geometrically irreducible. For such a  $U$ , the morphism  $\text{Pic}(U) \rightarrow \text{Pic}(C_{\mathbb{Q}})$  is an isomorphism, and  $\mathcal{O}_C(U) = \mathbb{Z}$ , thus, for each  $i$ , there is a lift

$$(2.11) \quad \begin{array}{ccc} & (\text{id}, m \cdot \circ \text{tr}_{c_i} \circ f_i)^* P^{\times} & \\ & \downarrow & \\ U & \xrightarrow{j_b} & J \end{array}$$

unique up to  $\mathbb{Z}^{\times} = \{1, -1\}$ .

At this point we can explain the strategy of our approach to the quadratic Chabauty method. Let  $T$  be the  $\mathbb{G}_m^{\rho-1}$ -torsor on  $J$  obtained by taking the product of all  $(\text{id}, m \cdot \circ \text{tr}_{c_i} \circ f_i)^* P^{\times}$ :

$$(2.12) \quad \begin{array}{ccccc} & T & \xrightarrow{\quad} & P^{\times, \rho-1} & \\ & \downarrow & & \downarrow & \\ U & \xrightarrow{j_b} & J & \xrightarrow{(\text{id}, m \cdot \circ \text{tr}_{c_i} \circ f_i)_i} & J \times (J^{\vee 0})^{\rho-1}. \end{array}$$

Then each  $c \in C_{\mathbb{Q}}(\mathbb{Q}) = C^{\text{sm}}(\mathbb{Z})$  lies in one of the finitely many  $U(\mathbb{Z})$ 's. For each  $U$ , we have a lift  $\tilde{j}_b: U \rightarrow T$ , and, for each prime number  $p$ ,  $\tilde{j}_b(U(\mathbb{Z}))$  is contained in the intersection, in  $T(\mathbb{Z}_p)$ , of  $\tilde{j}_b(U(\mathbb{Z}_p))$  and the closure  $\overline{T(\mathbb{Z})}$  of  $T(\mathbb{Z})$  in  $T(\mathbb{Z}_p)$  with the  $p$ -adic topology. Of course, one expects this closure to be of dimension at most  $r := \text{rank}(J(\mathbb{Q}))$ , and therefore one expects this method to be successful if  $r < g + \rho - 1$ , the dimension of  $T(\mathbb{Z}_p)$ . The next two sections make this strategy precise, giving first the necessary  $p$ -adic formal and analytic geometry, and then the description of  $\overline{T(\mathbb{Z})}$  as a finite disjoint union of images of  $\mathbb{Z}_p^r$  under maps constructed from the biextension structure.

### 3 From algebraic geometry to formal geometry

Let  $p$  be a prime number. Given  $X$  a smooth scheme of relative dimension  $d$  over  $\mathbb{Z}_p$  and  $x \in X(\mathbb{F}_p)$  let us describe the set  $X(\mathbb{Z}_p)_x$  of elements of  $X(\mathbb{Z}_p)$  whose image in  $X(\mathbb{F}_p)$  is  $x$ . The smoothness implies that the maximal ideal of  $\mathcal{O}_{X,x}$  is generated by  $p$  together with  $d$  other elements  $t_1, \dots, t_d$ . In this case we call  $p, t_1, \dots, t_d$  *parameters at  $x$* ; if moreover  $x_l \in X(\mathbb{Z}_p)_x$  is a lift of  $x$  such that  $t_1(x_l) = \dots = t_d(x_l) = 0$  then we say that the  $t_i$ 's are *parameters at  $x_l$* . The  $t_i$  can be evaluated on all the points in  $X(\mathbb{Z}_p)_x$ , inducing a bijection  $t := (t_1, \dots, t_d): X(\mathbb{Z}_p)_x \rightarrow (p\mathbb{Z}_p)^d$ . We get a bijection

$$(3.1) \quad \tilde{t} := (\tilde{t}_1, \dots, \tilde{t}_d) = \left( \frac{t_1}{p}, \dots, \frac{t_d}{p} \right) : X(\mathbb{Z}_p)_x \xrightarrow{\sim} \mathbb{Z}_p^d.$$

This bijection can be geometrically interpreted as follows. Let  $\pi: \tilde{X}_x \rightarrow X$  denote the blow up of  $X$  in  $x$ . By shrinking  $X$ ,  $X$  is affine and the  $t_i$  are regular on  $X$ , and  $t: X \rightarrow \mathbb{A}_{\mathbb{Z}_p}^d$  is etale. Therefore  $\pi: \tilde{X}_x \rightarrow X$  is the pull back of the blow up of  $\mathbb{A}_{\mathbb{Z}_p}^d$  at the origin over  $\mathbb{F}_p$ . The affine open part  $\tilde{X}_x^p$  of  $\tilde{X}_x$  where  $p$  generates the image of the ideal  $m_x$  of  $x$  is the pullback of the corresponding open part of the blow up of  $\mathbb{A}_{\mathbb{Z}_p}^d$ , which is the multiplication by  $p$  morphism  $\mathbb{A}_{\mathbb{Z}_p}^d \rightarrow \mathbb{A}_{\mathbb{Z}_p}^d$  that corresponds to  $\mathbb{Z}_p[t_1, \dots, t_d] \rightarrow \mathbb{Z}_p[\tilde{t}_1, \dots, \tilde{t}_d]$  with  $t_i \mapsto p\tilde{t}_i$ . It follows that the  $p$ -adic completion  $\mathcal{O}(\tilde{X}_x^p)^{\wedge p}$  of  $\mathcal{O}(\tilde{X}_x^p)$  is the  $p$ -adic completion  $\mathbb{Z}_p\langle \tilde{t}_1, \dots, \tilde{t}_d \rangle$  of  $\mathbb{Z}_p[\tilde{t}_1, \dots, \tilde{t}_d]$ . Explicitly, we have

$$(3.2) \quad \mathbb{Z}_p\langle \tilde{t}_1, \dots, \tilde{t}_d \rangle = \left\{ \sum_{I \in \mathbb{N}^d} a_I \tilde{t}^I \in \mathbb{Z}_p[[\tilde{t}_1, \dots, \tilde{t}_d]] : \forall n \geq 0, \quad \forall^{\text{almost}} I, v_p(a_I) \geq n \right\}.$$

With these definitions, we have

$$(3.3) \quad \begin{aligned} X(\mathbb{Z}_p)_x &= \tilde{X}_x^p(\mathbb{Z}_p) = \text{Hom}(\mathbb{Z}_p\langle \tilde{t}_1, \dots, \tilde{t}_d \rangle, \mathbb{Z}_p) = \mathbb{A}^d(\mathbb{Z}_p), \\ (\tilde{X}_x^p)_{\mathbb{F}_p} &= \text{Spec}(\mathbb{F}_p[\tilde{t}_1, \dots, \tilde{t}_d]). \end{aligned}$$

The affine space  $(\tilde{X}_x^p)_{\mathbb{F}_p}$  is canonically a torsor under the tangent space of  $X_{\mathbb{F}_p}$  at  $x$ .

This construction is functorial. Let  $Y$  be a smooth  $\mathbb{Z}_p$ -scheme,  $f: X \rightarrow Y$  a morphism over  $\mathbb{Z}_p$ , and  $y := f(x) \in Y(\mathbb{F}_p)$ . Then the ideal in  $\mathcal{O}_{\tilde{X}_x^p}$  generated by the image of  $m_{f(x)}$  is generated by  $p$ . That gives us a morphism  $\tilde{X}_x^p \rightarrow \tilde{Y}_{f(x)}^p$ , and then a morphism from  $\mathcal{O}(\tilde{Y}_{f(x)}^p)^{\wedge p}$  to  $\mathcal{O}(\tilde{X}_x^p)^{\wedge p}$ . Reduction mod  $p$  then gives a morphism  $(\tilde{X}_x^p)_{\mathbb{F}_p} \rightarrow (\tilde{Y}_{f(x)}^p)_{\mathbb{F}_p}$ , the tangent map of  $f$  at  $x$ , up to a translation.

If this tangent map is injective, and  $d_x$  and  $d_y$  denote the dimensions of  $X_{\mathbb{F}_p}$  at  $x$  and of  $Y_{\mathbb{F}_p}$  at  $y$ , then there are  $t_1, \dots, t_{d_x}$  in  $\mathcal{O}_{Y,y}$  such that  $p, t_1, \dots, t_{d_y}$  are parameters at  $y$ , and such that  $t_{d_x+1}, \dots, t_{d_y}$  generate the kernel of  $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ . Then the images in  $\mathcal{O}_{X,x}$  of  $p, t_1, \dots, t_{d_x}$  are parameters at  $x$ , and  $\mathcal{O}(\tilde{Y}_{f(x)}^p)^{\wedge p} \rightarrow \mathcal{O}(\tilde{X}_x^p)^{\wedge p}$  is  $\mathbb{Z}_p\langle \tilde{t}_1, \dots, \tilde{t}_{d_y} \rangle \rightarrow \mathbb{Z}_p\langle \tilde{t}_1, \dots, \tilde{t}_{d_x} \rangle$ , with kernel generated by  $\tilde{t}_{d_x+1}, \dots, \tilde{t}_{d_y}$ .

### 4 Integral points, closure and finiteness

Let us now return to our original problem. The notation  $U, J, T, j_b, \tilde{j}_b, r, \rho$  etc., is as at the end of Section 2. We assume moreover that  $p$  does not divide  $n$  ( $n$  as in the start of Section 2)

and that  $p > 2$  (for  $p = 2$  everything that follows can probably be adapted by working with residue polydiscs modulo 4).

Let  $u$  be in  $U(\mathbb{F}_p)$ , and  $t := \tilde{j}_b(u)$ . We want a description of the closure  $\overline{T(\mathbb{Z})}_t$  of  $T(\mathbb{Z})_t$  in  $T(\mathbb{Z}_p)_t$ . Using the biextension structure of  $P^\times$ , we will produce, for each element of  $J(\mathbb{Z})_{j_b(u)}$ , an element of  $T(\mathbb{Z})$  over it. Not all of these points are in  $T(\mathbb{Z})_t$ , but we will then produce a subset of  $T(\mathbb{Z})_t$  whose closure is  $\overline{T(\mathbb{Z})}_t$ .

If  $T(\mathbb{Z})_t$  is empty then  $\overline{T(\mathbb{Z})}_t$  is empty, too. So we assume that we have an element  $\tilde{t} \in T(\mathbb{Z})_t$  and we define  $x_{\tilde{t}} \in J(\mathbb{Z})$  to be the projection of  $\tilde{t}$ . Let  $f = (f_1, \dots, f_{\rho-1}): J \rightarrow J^{\vee, \rho-1}$ , let  $c = (c_1, \dots, c_{\rho-1}) \in J^{\vee, \rho-1}(\mathbb{Z})$ . We denote by  $P^{\times, \rho-1}$  the product over  $J \times (J^{\vee 0})^{\rho-1}$  of the  $\rho-1$   $\mathbb{G}_m$ -torsors obtained by pullback of  $P^\times$  via the projections to  $J \times J^{\vee 0}$ ; it is a biextension of  $J$  and  $(J^{\vee 0})^{\rho-1}$  by  $\mathbb{G}_m^{\rho-1}$ , and  $T = (\text{id}, m \cdot \circ \text{tr}_c \circ f)^* P^{\times, \rho-1}$ . We choose a basis  $x_1, \dots, x_r$  of the free  $\mathbb{Z}$ -module  $J(\mathbb{Z})_0$ , the kernel of  $J(\mathbb{Z}) \rightarrow J(\mathbb{F}_p)$ . For each  $i, j \in \{1, \dots, r\}$  we choose  $P_{i,j}$ ,  $R_{i,\tilde{t}}$ , and  $S_{\tilde{t},j}$  in  $P^{\times, \rho-1}(\mathbb{Z})$  whose images in  $(J \times (J^{\vee 0})^{\rho-1})(\mathbb{Z})$  are  $(x_i, f(mx_j))$ ,  $(x_i, (m \cdot \circ \text{tr}_c \circ f)(x_{\tilde{t}}))$  and  $(x_{\tilde{t}}, f(mx_j))$ :

$$(4.1) \quad \begin{array}{cccc} P_{i,j} & R_{i,\tilde{t}} & S_{\tilde{t},j} & P^{\times, \rho-1} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ (x_i, f(mx_j)) & (x_i, (m \cdot \circ \text{tr}_c \circ f)(x_{\tilde{t}})) & (x_{\tilde{t}}, f(mx_j)) & J \times (J^{\vee 0})^{\rho-1}. \end{array}$$

For each such choice there are  $2^{\rho-1}$  possibilities.

For each  $n \in \mathbb{Z}^r$  we use the biextension structure on  $P^{\times, \rho-1} \rightarrow J \times (J^{\vee 0})^{\rho-1}$  to define the following points in  $P^{\times, \rho-1}(\mathbb{Z})$ , with specified images in  $(J \times (J^{\vee 0})^{\rho-1})(\mathbb{Z})$ :

$$(4.2) \quad \begin{array}{cc} A_{\tilde{t}}(n) = \sum_{j=1}^r n_j \cdot_2 S_{\tilde{t},j} & B_{\tilde{t}}(n) = \sum_{i=1}^r n_i \cdot_1 R_{i,\tilde{t}} \\ \downarrow & \downarrow \\ \left( x_{\tilde{t}}, \sum_{i=1}^r n_i f(mx_i) \right) & \left( \sum_{i=1}^r n_i x_i, (m \cdot \circ \text{tr}_c \circ f)(x_{\tilde{t}}) \right) \end{array}$$

$$(4.3) \quad \begin{array}{c} C(n) = \sum_{i=1}^r n_i \cdot_1 \left( \sum_{j=1}^r n_j \cdot_2 P_{i,j} \right) \\ \downarrow \\ \left( \sum_{i=1}^r n_i x_i, \sum_{i=1}^r n_i f(mx_i) \right) \end{array}$$

where  $\sum_1$  and  $\cdot_1$  denote iterations of the first partial group law  $+_1$  as in (2.4), and analogously for the second group law. We define, for all  $n \in \mathbb{Z}^r$ ,

$$(4.4) \quad D_{\tilde{t}}(n) := (C(n) +_2 B_{\tilde{t}}(n)) +_1 (A_{\tilde{t}}(n) +_2 \tilde{t}) \in P^{\times, \rho-1}(\mathbb{Z}),$$

which is mapped to

$$(4.5) \quad \left( x_{\tilde{t}} + \sum_{i=1}^r n_i x_i, (m \cdot \circ \text{tr}_c \circ f) \left( x_{\tilde{t}} + \sum_{i=1}^r n_i x_i \right) \right) \in (J \times (J^{\vee 0})^{\rho-1})(\mathbb{Z}).$$

Hence  $D_{\tilde{t}}(n)$  is in  $T(\mathbb{Z})$ , and its image in  $J(\mathbb{F}_p)$  is  $j_b(u)$ . We do not know its image in  $T(\mathbb{F}_p)$ .

We claim that for  $n$  in  $(p-1)\mathbb{Z}^r$ ,  $D_{\tilde{t}}(n)$  is in  $T(\mathbb{Z})_t$ . Let  $n'$  be in  $\mathbb{Z}^r$  and let  $n = (p-1)n'$ . Then, in the trivial  $\mathbb{F}_p^{\times, \rho-1}$ -torsor  $P^{\times, \rho-1}(j_b(u), 0)$ , on which  $+_2$  is the group law, we have:

$$(4.6) \quad A_{\tilde{t}}(n) = (p-1) \cdot_2 A_{\tilde{t}}(n') = 1 \quad \text{in } \mathbb{F}_p^{\times, \rho-1}.$$

Similarly, in  $P^{\times, \rho-1}(0, (m \cdot \circ \text{tr}_c \circ f)(j_b(u))) = \mathbb{F}_p^{\times, \rho-1}$ , we have  $B_{\tilde{t}}(n) = 1$ , and, similarly, in  $P^{\times, \rho-1}(0, 0) = \mathbb{F}_p^{\times, \rho-1}$ , we have  $C(n) = 1$ . So, with apologies for the mix of additive and multiplicative notations, in  $P^{\times, \rho-1}(\mathbb{F}_p)$  we have

$$(4.7) \quad D_{\tilde{t}}(n) = (1 +_2 1) +_1 (1 +_2 t) = t,$$

mapping to the following element in  $(J \times J^{\vee 0, \rho-1})(\mathbb{F}_p)$ :

$$(4.8) \quad \begin{aligned} & ((0, 0) +_2 ((0, (m \cdot \circ \text{tr}_c \circ f)(j_b(u)))) +_1 ((j_b(u), 0) +_2 (j_b(u), (m \cdot \circ \text{tr}_c \circ f)(j_b(u)))) \\ & = (j_b(u), (m \cdot \circ \text{tr}_c \circ f)(j_b(u))). \end{aligned}$$

We have proved our claim that  $D_{\tilde{t}}(n) \in T(\mathbb{Z})_t$ .

So we now have the map

$$(4.9) \quad \kappa_{\mathbb{Z}}: \mathbb{Z}^r \rightarrow T(\mathbb{Z})_t, \quad n \mapsto D_{\tilde{t}}((p-1)n).$$

The following theorem will be proved in Section 5.

**4.10 Theorem** *Let  $x_1, \dots, x_g$  be in  $\mathcal{O}_{J, j_b(u)}$  such that together with  $p$  they form a system of parameters of  $\mathcal{O}_{J, j_b(u)}$ , and let  $v_1, \dots, v_{\rho-1}$  be in  $\mathcal{O}_{T, t}$  such that  $p, x_1, \dots, x_g, v_1, \dots, v_{\rho-1}$  are parameters of  $\mathcal{O}_{T, t}$ . As in Section 3 these parameters, divided by  $p$ , give a bijection*

$$(4.10.1) \quad T(\mathbb{Z}_p)_t \longrightarrow \mathbb{Z}_p^{g+\rho-1}.$$

*The composition of  $\kappa_{\mathbb{Z}}$  with the map (4.10.1) is given by uniquely determined  $\kappa_1, \dots, \kappa_{g+\rho-1}$  in  $\mathcal{O}(\mathbb{A}_{\mathbb{Z}_p}^r)^{\wedge p} = \mathbb{Z}_p \langle z_1, \dots, z_r \rangle$ . The images in  $\mathbb{F}_p[z_1, \dots, z_r]$  of  $\kappa_1, \dots, \kappa_g$  are of degree at most 1, and the images of  $\kappa_{g+1}, \dots, \kappa_{g+\rho-1}$  are of degree at most 2. The map  $\kappa_{\mathbb{Z}}$  extends uniquely to the continuous map*

$$(4.10.2) \quad \kappa = (\kappa_1, \dots, \kappa_{g+\rho-1}): \mathbb{A}^r(\mathbb{Z}_p) = \mathbb{Z}_p^r \longrightarrow T(\mathbb{Z}_p)_t.$$

*and the image of  $\kappa$  is  $\overline{T(\mathbb{Z})}_t$ .*

Now the moment has come to confront  $U(\mathbb{Z}_p)_u$  with  $\overline{T(\mathbb{Z})}_t$ . We have  $\tilde{j}_b: U \rightarrow T$ , whose tangent map (mod  $p$ ) at  $u$  is injective (here we use that  $C_{\mathbb{F}_p}$  is smooth over  $\mathbb{F}_p$ ). Then, as at the end of Section 3,  $\tilde{j}_b: \tilde{U}_u^p \rightarrow \tilde{T}_t^p$  is, after reduction mod  $p$ , an affine linear embedding of codimension  $g+\rho-2$ ,  $\tilde{j}_b^*: \mathcal{O}(\tilde{T}_t^p)^{\wedge p} \rightarrow \mathcal{O}(\tilde{U}_u^p)^{\wedge p}$  is surjective and its kernel is generated by



elements  $f_1, \dots, f_{g+\rho-2}$ , whose images in  $\mathbb{F}_p \otimes \mathcal{O}(\tilde{T}_t^p)$  are of degree at most 1, and such that  $f_1, \dots, f_{g-1}$  are in  $\mathcal{O}(\tilde{J}_{j_b(u)}^p)^{\wedge p}$ . The pullbacks  $\kappa^* f_i$  are in  $\mathbb{Z}_p \langle z_1, \dots, z_r \rangle$ ; let  $I$  be the ideal in  $\mathbb{Z}_p \langle z_1, \dots, z_r \rangle$  generated by them, and let

$$(4.11) \quad A := \mathbb{Z}_p \langle z_1, \dots, z_r \rangle / I.$$

Then the elements of  $\mathbb{Z}_p^r$  whose image is in  $U(\mathbb{Z}_p)_u$  are zeros of  $I$ , hence morphisms of rings from  $A$  to  $\mathbb{Z}_p$ , and hence from the reduced quotient  $A_{\text{red}}$  to  $\mathbb{Z}_p$ .

**4.12 Theorem** *For  $i \in \{1, \dots, g+\rho-2\}$ , let  $\kappa^* \overline{f_i}$  be the image of  $\kappa^* f_i$  in  $\mathbb{F}_p[z_1, \dots, z_r]$ , and let  $\overline{I}$  be the ideal of  $\mathbb{F}_p[z_1, \dots, z_r]$  generated by them. Then  $\kappa^* \overline{f_1}, \dots, \kappa^* \overline{f_{g-1}}$  are of degree at most 1, and  $\kappa^* \overline{f_g}, \dots, \kappa^* \overline{f_{g+\rho-2}}$  are of degree at most 2. Assume that  $\overline{A} := A/pA = \mathbb{F}_p[z_1, \dots, z_r]/\overline{I}$  is finite. Then  $\overline{A}$  is the product of its localisations  $\overline{A}_m$  at its finitely many maximal ideals  $m$ . The sum of the  $\dim_{\mathbb{F}_p} \overline{A}_m$  over the  $m$  such that  $\overline{A}/m = \mathbb{F}_p$  is an upper bound for the number of elements of  $\mathbb{Z}_p^r$  whose image under  $\kappa$  is in  $U(\mathbb{Z}_p)_u$ , and also an upper bound for the number of elements of  $U(\mathbb{Z})$  with image  $u$  in  $U(\mathbb{F}_p)$ .*

**Proof** As every  $\overline{f_i}$  is of degree at most 1 in  $x_1, \dots, x_g, v_1, \dots, v_{\rho-1}$ , every  $\kappa^* \overline{f_i}$  is an  $\mathbb{F}_p$ -linear combination of  $\kappa_1, \dots, \kappa_{g+\rho-1}$ , hence of degree at most 2. For  $i < g$ ,  $\overline{f_i}$  is a linear combination of  $x_1, \dots, x_g$ , and therefore  $\kappa^* \overline{f_i}$  is of degree at most 1.

We claim that  $A$  is  $p$ -adically complete. More generally, let  $R$  be a noetherian ring that is  $J$ -adically complete for an ideal  $J$ , and let  $I$  be an ideal in  $R$ . The map from  $R/I$  to its  $J$ -adic completion  $(R/I)^\wedge$  is injective ([1, Thm.10.17]). As  $J$ -adic completion is exact on finitely generated  $R$ -modules ([1, Prop.10.12]), it sends the surjection  $R \rightarrow R/I$  to a surjection  $R = R^\wedge \rightarrow (R/I)^\wedge$  (see [1, Prop.10.5] for the equality  $R = R^\wedge$ ). It follows that  $R/I \rightarrow (R/I)^\wedge$  is surjective.

Now we assume that  $\overline{A}$  is finite. As  $A$  is  $p$ -adically complete,  $A$  is the limit of the system of its quotients by powers of  $p$ . These quotients are finite: for every  $m \in \mathbb{Z}_{\geq 1}$ ,  $A/p^{m+1}A$  is, as abelian group, an extension of  $A/pA$  by a quotient of  $A/p^m A$ . As  $\mathbb{Z}_p$ -module,  $A$  is generated by any lift of an  $\mathbb{F}_p$ -basis of  $\overline{A}$ . Hence  $A$  is finitely generated as  $\mathbb{Z}_p$ -module.

The set of elements of  $\mathbb{Z}_p^r$  whose image under  $\kappa$  is in  $U(\mathbb{Z}_p)$  is in bijection with the set of  $\mathbb{Z}_p$ -algebra morphisms  $\text{Hom}(A, \mathbb{Z}_p)$ . As  $A$  is the product of its localisations  $A_m$  at its maximal ideals,  $\text{Hom}(A, \mathbb{Z}_p)$  is the disjoint union of the  $\text{Hom}(A_m, \mathbb{Z}_p)$ . For each  $m$ ,  $\text{Hom}(A_m, \mathbb{Z}_p)$  has at most  $\text{rank}_{\mathbb{Z}_p}(A_m)$  elements, and is empty if  $\mathbb{F}_p \rightarrow A/m$  is not an isomorphism. This establishes the upper bound for the number of elements of  $\mathbb{Z}_p^r$  whose image under  $\kappa$  is in  $U(\mathbb{Z}_p)$ . By Theorem 4.10, the elements of  $U(\mathbb{Z})$  with image  $u$  in  $U(\mathbb{F}_p)$  are in  $\overline{T(\mathbb{Z})}_t$ , and therefore of the form  $\kappa(x)$  with  $x \in \mathbb{Z}_p^r$  such that  $\kappa(x)$  is in  $U(\mathbb{Z}_p)_u$ . This establishes the upper bound for the number of elements of  $U(\mathbb{Z})$  with image  $u$  in  $U(\mathbb{F}_p)$ .  $\square$

We include some remarks to explain how Theorem 4.12 can be used, and what we hope that it can do.

**4.13 Remark** The  $\kappa^* \overline{f_i}$  in Theorem 4.12 can be computed from the reduction  $\mathbb{F}_p^r \rightarrow T(\mathbb{Z}/p^2\mathbb{Z})$  of  $\kappa_{\mathbb{Z}}$  and (to get the  $\overline{f_i}$ ) from  $\tilde{j}_b: U(\mathbb{Z}/p^2\mathbb{Z})_u \rightarrow T(\mathbb{Z}/p^2\mathbb{Z})_t$ . For this, one does not need to

treat  $T$  and  $J$  as schemes, one just computes with  $\mathbb{Z}/p^2\mathbb{Z}$ -valued points. Now assume that  $r \leq g + \rho - 2$ . If, for some prime  $p$ , the criterion in Theorem 4.12 fails (that is,  $\overline{A}$  is not finite), then one can try the next prime. We hope (but also expect) that one quickly finds a prime  $p$  such that  $\overline{A}$  is finite for every  $U$  and for every  $u$  in  $U(\mathbb{F}_p)$  such that  $\tilde{j}_b(u)$  is in the image of  $T(\mathbb{Z}) \rightarrow T(\mathbb{F}_p)$ . By the way, note that our notation in Theorem 4.12 does not show the dependence on  $U$  and  $u$  of  $\tilde{j}_b$ ,  $\kappa_{\mathbb{Z}}$ ,  $\kappa$  and the  $\overline{f}_i$ .

Already for the case of classical Chabauty (working with  $J$  instead of  $T$ , and under the assumption that  $r < g$ ), where everything is linear, the criterion of Theorem 4.12 can be useful; this is currently being worked out and implemented by Pim Spelier in his MSc thesis.

**4.14 Remark** If  $r < g + \rho - 2$  then we think that it is likely (when varying  $p$ ), for dimension reasons, that, for all  $u \in U(\mathbb{F}_p)$ , the upper bound in Theorem 4.12 for the number of elements of  $U(\mathbb{Z})$  with image  $u$  in  $U(\mathbb{F}_p)$  is sharp.

**4.15 Remark** Suppose that  $r = g + \rho - 2$ . Then we expect, for dimension reasons, that it is likely (when varying  $p$ ) that, for some  $u \in U(\mathbb{F}_p)$ , the upper bound in Theorem 4.12 for the number of elements of  $U(\mathbb{Z})$  with image  $u$  in  $U(\mathbb{F}_p)$  is not sharp. Then, as in the classical Chabauty method, one must combine the information gotten from several primes, analogous to ‘Mordell-Weil sieving’. Suppose that we are given a subset  $B$  of  $U(\mathbb{Z})$  that we want to prove to be equal to  $U(\mathbb{Z})$ . Let  $B'$  be the complement in  $U(\mathbb{Z})$  of  $B$ . For every prime  $p > 2$  not dividing  $n$ , Theorem 4.12 gives, interpreting  $\overline{A}$  as in the end of the proof of Theorem 4.12, a subset  $O_p$  of  $J(\mathbb{Z})$ , that is a union of cosets for the subgroup  $p \cdot \ker(J(\mathbb{Z}) \rightarrow J(\mathbb{F}_p))$ , that contains  $j_b(B')$ . Then one hopes that, taking a large enough finite set  $S$  of primes, that the intersection of the  $O_p$  for  $p$  in  $S$  is empty.

## 5 Parametrisation of integral points, and power series

In this section we give a proof of Theorem 4.10. The main tools here are the formal logarithm and formal exponential of a commutative smooth group scheme over a  $\mathbb{Q}$ -algebra ([5], Theorem 1): they give us identities as  $n \cdot g = \exp(n \cdot \log g)$  that allow us to extend the multiplication to elements  $n$  of  $\mathbb{Z}_p$ .

The evaluation map from  $\mathbb{Z}_p\langle z_1, \dots, z_n \rangle$  to the set of maps  $\mathbb{Z}_p^n \rightarrow \mathbb{Z}_p$  is injective (induction on  $n$ , non-zero elements of  $\mathbb{Z}_p\langle z \rangle$  have only finitely many zeros in  $\mathbb{Z}_p$ ).

We say that a map  $f: \mathbb{Z}_p^n \rightarrow \mathbb{Z}_p^m$  is *given by integral convergent power series* if its coordinate functions are in  $\mathbb{Z}_p\langle z_1, \dots, z_n \rangle = \mathcal{O}(\mathbb{A}_{\mathbb{Z}_p}^n)^{\wedge^p}$ . This property is stable under composition: composition of polynomials over  $\mathbb{Z}/p^k\mathbb{Z}$  gives polynomials.

### 5.1 Logarithm and exponential

Let  $p$  be a prime number, and let  $G$  be a commutative group scheme, smooth of relative dimension  $d$  over a scheme  $S$  smooth over  $\mathbb{Z}_p$ , with unit section  $e$  in  $G(S)$ . For any  $s$  in  $S(\mathbb{F}_p)$ ,  $G(\mathbb{Z}_p)_{e(s)}$  is a group fibred over  $S(\mathbb{Z}_p)_s$ . The fibres have a natural  $\mathbb{Z}_p$ -module structure:  $G(\mathbb{Z}_p)_{e(s)}$  is the limit of the  $G(\mathbb{Z}/p^n\mathbb{Z})_{e(s)}$  ( $n \geq 1$ ),  $S(\mathbb{Z}_p)_s$  is the limit of the  $S(\mathbb{Z}/p^n\mathbb{Z})_s$ , and for each

$n \geq 1$ , the fibres of  $G(\mathbb{Z}/p^n\mathbb{Z})_{e(s)} \rightarrow S(\mathbb{Z}/p^n\mathbb{Z})_s$  are commutative groups annihilated by  $p^{n-1}$ . Let  $T_{G/S}$  be the relative (geometric) tangent bundle of  $G$  over  $S$ . Then its pullback  $T_{G/S}(e)$  by  $e$  is a vector bundle on  $S$  of rank  $d$ .

**5.1.1 Lemma** *In this situation, and with  $n$  the relative dimension of  $S$  over  $\mathbb{Z}_p$ , the formal logarithm and exponential of  $G$  base changed to  $\mathbb{Q} \otimes \mathcal{O}_{S,s}$  converge to maps*

$$\begin{aligned} \log: \tilde{G}_{e(s)}^p(\mathbb{Z}_p) = G(\mathbb{Z}_p)_{e(s)} &\rightarrow (T_{G/S}(e))(\mathbb{Z}_p)_{0(s)} \\ \exp: \tilde{T}_{G/S}(e)_{0(s)}^p(\mathbb{Z}_p) = (T_{G/S}(e))(\mathbb{Z}_p)_{0(s)} &\rightarrow G(\mathbb{Z}_p)_{e(s)}, \end{aligned}$$

that are each others inverse and, after a choice of parameters for  $G \rightarrow S$  at  $e(s)$  as in (3.1), are given by  $n + d$  elements of  $\mathcal{O}(\tilde{G}_{e(s)}^p)^{\wedge p}$  and  $n + d$  elements of  $\mathcal{O}(\tilde{T}_{G/S}(e)_{0(s)}^p)^{\wedge p}$ .

For  $a$  in  $\mathbb{Z}_p$  and  $g$  in  $G(\mathbb{Z}_p)_{e(s)}$  we have  $a \cdot g = \exp(a \cdot \log g)$ , and, after a choice of parameters for  $G \rightarrow S$  at  $e(s)$ , this map  $\mathbb{Z}_p \times G(\mathbb{Z}_p)_{e(s)} \rightarrow G(\mathbb{Z}_p)_{e(s)}$  is given by  $n + d$  elements of  $\mathcal{O}(\mathbb{A}_{\mathbb{Z}_p}^1 \times_{\mathbb{Z}_p} \tilde{G}_{e(s)}^p)^{\wedge p}$ . The induced morphism  $\mathbb{A}_{\mathbb{F}_p}^1 \times (\tilde{G}_{e(s)}^p)_{\mathbb{F}_p} \rightarrow (\tilde{G}_{e(s)}^p)_{\mathbb{F}_p}$ , where  $(\tilde{G}_{e(s)}^p)_{\mathbb{F}_p}$  is viewed as the product  $T_{S_{\mathbb{F}_p}}(s)$  and  $T_{G/S}(e(s))$ , is a morphism over  $T_{S_{\mathbb{F}_p}}(s)$ , bilinear in  $\mathbb{A}_{\mathbb{F}_p}^1$  and  $T_{G/S}(e(s))$ .

**Proof** Let  $t_1, \dots, t_n$  be in  $\mathcal{O}_{S,s}$  such that  $p, t_1, \dots, t_n$  are parameters at  $s$ . Then we have a bijection

$$(5.1.2) \quad \tilde{t}: S(\mathbb{Z}_p)_s \rightarrow \mathbb{Z}_p^n, \quad a \mapsto p^{-1} \cdot (t_1(a), \dots, t_n(a)).$$

Similarly, let  $x_1, \dots, x_d$  be generators for the ideal  $I_{e(s)}$  of  $e$  in  $\mathcal{O}_{G,e(s)}$ . Then  $p$ , the  $t_i$  and the  $x_j$  together are parameters for  $\mathcal{O}_{G,e(s)}$ , and give the bijection

$$(5.1.3) \quad (t, x)^\sim: G(\mathbb{Z}_p)_{e(s)} \rightarrow \mathbb{Z}_p^{n+d}, \quad b \mapsto p^{-1} \cdot (t_1(b), \dots, x_d(b)).$$

The  $dx_i$  form an  $\mathcal{O}_{S,s}$ -basis of  $\Omega_{G/S}^1(e)_s$ , and so give translation invariant differentials  $\omega_i$  on  $G_{\mathcal{O}_{S,s}}$ . As  $G$  is commutative, for all  $i$ ,  $d\omega_i = 0$  ([5], Proposition 1.3). We also have the dual  $\mathcal{O}_{S,s}$ -basis  $\partial_i$  of  $T_{G/S}(e)$  and the bijection

$$(5.1.4) \quad (t, x)^\sim: (T_{G/S}(e))(\mathbb{Z}_p)_{0(s)} \rightarrow \mathbb{Z}_p^{n+d}, \quad (a, \sum_i v_i \partial_i) \mapsto p^{-1} \cdot (t_1(a), \dots, t_n(a), v_1, \dots, v_d).$$

Then  $\log$  is given by elements  $\log_i$  in  $(\mathbb{Q} \otimes \mathcal{O}_{S,s})[[x_1, \dots, x_d]]$  whose constant term is 0, uniquely determined (Proposition 1.1 in [5]) by the equality

$$(5.1.5) \quad d \log_i = \omega_i, \quad \text{in } \oplus_j \mathcal{O}_{S,s}[[x_1, \dots, x_d]] \cdot dx_j.$$

Hence the formula from calculus,  $\log_i(x) - \log_i(0) = \int_0^1 (t \mapsto tx)^* \omega_i$ , gives us that, with

$$(5.1.6) \quad \log_i = \sum_{J \neq 0} \log_{i,J} x^J \quad \text{and} \quad \log_{i,J} \in (\mathbb{Q} \otimes \mathcal{O}_{S,s}),$$

we have, for all  $i$  and  $J$ , with  $|J|$  denoting the total degree of  $x^J$ ,

$$(5.1.7) \quad |J| \cdot \log_{i,J} \in \mathcal{O}_{S,s}.$$

The claim about convergence and definition of  $\log: G(\mathbb{Z}_p)_{e(s)} \rightarrow (T_{G/S}(e))(\mathbb{Z}_p)_{0(s)}$ , is now equivalent to having an analytic bijection  $\mathbb{Z}_p^{n+d} \rightarrow \mathbb{Z}_p^{n+d}$  given by

$$(5.1.8) \quad \begin{array}{ccc} G(\mathbb{Z}_p)_{e(s)} & \xrightarrow{\quad ? \quad} & (T_{G/S}(e))(\mathbb{Z}_p)_{0(s)} \\ \downarrow (t,x)^\sim & & \downarrow (t,x)^\sim \\ \mathbb{Z}_p^{n+d} & \xrightarrow{\quad ? \quad} & \mathbb{Z}_p^{n+d} \end{array}$$

$$(a, b) \longmapsto \left( a, p^{-1} \cdot \left( \sum_{J \neq 0} \log_{i,J}(\tilde{t}^{-1}(a))(pb)^J \right)_i \right).$$

We have, for each  $i$ ,

$$(5.1.9) \quad p^{-1} \cdot \sum_{J \neq 0} \log_{i,J}(\tilde{t}^{-1}(a))(pb)^J = \sum_{J \neq 0} \frac{p^{|J|-1}}{|J|} (|J| \log_{i,J}(\tilde{t}^{-1}(a)) b^J).$$

For each  $i$ , this expression is an element of  $\mathbb{Z}_p \langle \tilde{t}_1, \dots, \tilde{t}_n, \tilde{x}_1, \dots, \tilde{x}_d \rangle = \mathcal{O}(\tilde{G}_{e(s)}^p)^{\wedge p}$ , even when  $p = 2$ , because for each  $J$ ,  $|J| \log_{i,J}$  is in  $\mathcal{O}_{S,s}$ , which is contained in  $\mathbb{Z}_p \langle \tilde{t}_1, \dots, \tilde{t}_n \rangle$ , and the function  $\mathbb{Z}_{\geq 1} \rightarrow \mathbb{Q}_p$ ,  $r \mapsto p^{r-1}/r$  has values in  $\mathbb{Z}_p$  and converges to 0. The existence and analyticity of  $\log$  is now proved (even for  $p = 2$ ). As  $p > 2$ , the image of (5.1.9) in  $\mathbb{F}_p \otimes \mathcal{O}(\tilde{G}_{e(s)}^p)^{\wedge p}$  is  $\tilde{x}_i$ , and on the first  $n$  coordinates,  $\log$  is the identity, so, by applying Hensel modulo powers of  $p$ ,  $\log$  is invertible, and the inverse is also given by  $n + d$  elements of  $\mathcal{O}(\tilde{T}_{G/S}(e)_{0(s)}^p)^{\wedge p}$ .

The function  $\mathbb{Z}_p \times G(\mathbb{Z}_p)_{e(s)} \rightarrow G(\mathbb{Z}_p)_{e(s)}$ ,  $(a, g) \mapsto \exp(a \cdot \log g)$  is a composition of maps given by integral convergent power series, hence it is also of that form.  $\square$

## 5.2 Parametrisation by power series

The assumptions are as in the beginning of Section 4, in particular,  $p > 2$ . We have a  $t$  in  $T(\mathbb{F}_p)$ , with image  $j_b(u)$  in  $J(\mathbb{F}_p)$ , and a  $\tilde{t}$  in  $T(\mathbb{Z})$  lifting  $t$ . For every  $Q$  in  $T(\mathbb{Z})$  mapping to  $j_b(u)$  in  $J(\mathbb{F}_p)$  there are unique  $\varepsilon \in \mathbb{Z}^{\times, \rho-1}$  and  $n \in \mathbb{Z}^r$  such that  $Q = \varepsilon \cdot D_{\tilde{t}}(n)$ : the image of  $Q$  in  $J(\mathbb{Z})$  is in  $J(\mathbb{Z})_{j_b(u)}$ , hence differs from the image  $x_{\tilde{t}}$  in  $J(\mathbb{Z})$  of  $\tilde{t}$  by an element of  $J(\mathbb{Z})_0$  (with here  $0 \in J(\mathbb{F}_p)$ ),  $\sum_i n_i x_i$  for a unique  $n \in \mathbb{Z}^r$ , hence  $D_{\tilde{t}}(n)$  and  $Q$  are in  $T(\mathbb{Z})$  and have the same image in  $J(\mathbb{Z})$ , and that gives the unique  $\varepsilon$ . So we have a bijection

$$(5.2.1) \quad \mathbb{Z}^{\times, \rho-1} \times \mathbb{Z}^r \longrightarrow T(\mathbb{Z})_{j_b(u)} = \{Q \in T(\mathbb{Z}) : Q \mapsto j_b(u) \in J(\mathbb{F}_p)\}, \quad (\varepsilon, n) \mapsto \varepsilon \cdot D_{\tilde{t}}(n).$$

But a problem that we are facing is that the map  $\mathbb{Z}^r \rightarrow T(\mathbb{F}_p)_{j_b(u)}$  sending  $n$  to the image of  $D_{\tilde{t}}(n)$  depends on the (unkown) images of the  $P_{i,j}$ ,  $R_{i,\tilde{t}}$  and  $S_{i,j}$  from (4.1) in  $P^{\times, \rho-1}(\mathbb{F}_p)$ , and so we do not know for which  $n$  and  $\varepsilon$  the point  $\varepsilon \cdot D_{\tilde{t}}(n)$  is in  $T(\mathbb{Z})_t$ . Luckily we have the  $\mathbb{Z}_p^{\times, \rho-1}$ -action on  $T(\mathbb{Z}_p)$ . Using that  $\mathbb{Z}_p^\times = \mathbb{F}_p^\times \times (1 + p\mathbb{Z}_p)$  we have  $\mathbb{F}_p^{\times, \rho-1}$  acting on  $T(\mathbb{Z}_p)_{j_b(u)}$ , compatibly with the torsor structure on  $T(\mathbb{F}_p)_{j_b(u)}$ . So, for every  $n$  in  $\mathbb{Z}^r$  there is a unique  $\xi(n)$  in  $\mathbb{F}_p^{\times, \rho-1}$  such that  $\xi(n) \cdot D_{\tilde{t}}(n)$  is in  $T(\mathbb{Z}_p)_t$ . We define

$$(5.2.2) \quad D'(n) := \xi(n) \cdot D_{\tilde{t}}(n).$$

Then for all  $n$  in  $\mathbb{Z}^r$ ,

$$(5.2.3) \quad \kappa_{\mathbb{Z}}(n) = D_{\tilde{t}}((p-1) \cdot n) = D'((p-1) \cdot n),$$

because  $D_{\tilde{t}}((p-1) \cdot n)$  maps to  $t$  in  $T(\mathbb{F}_p)$ . Moreover for every  $Q$  in  $T(\mathbb{Z})_t$  there is a unique  $n \in \mathbb{Z}^r$  and a unique  $\varepsilon \in \mathbb{Z}^{\times, \rho-1}$  such that  $Q = \varepsilon \cdot D_{\tilde{t}}(n) = \xi(n) \cdot D_{\tilde{t}}(n) = D'(n)$ . Hence

$$(5.2.4) \quad T(\mathbb{Z})_t \subset D'(\mathbb{Z}^r).$$

We will now show that, after any choice of parameters of  $\mathcal{O}_{T,t}$  as in Theorem 4.10,  $D'$  is given by elements  $\kappa'_1, \dots, \kappa'_{g+\rho-1}$  of  $\mathcal{O}(\mathbb{A}_{\mathbb{Z}_p}^r)^{\wedge p}$ , and then  $\kappa_{\mathbb{Z}}$  is given by  $\kappa_1, \dots, \kappa_{g+\rho-1}$  with, for all  $i \in \{1, \dots, g+\rho-1\}$  and all  $a \in \mathbb{Z}_p^r$ ,  $\kappa_i(a) = \kappa'_i((p-1)a)$ .

We want a formula for  $D'(n)$ , so we introduce variants of the  $P_{i,j}$ ,  $R_{i,\tilde{t}}$ , and  $S_{\tilde{t},j}$  as follows. The images in  $(J \times (J^{\vee 0})^{\rho-1})(\mathbb{F}_p)$  of these points are of the form  $(0, *)$ ,  $(0, *)$ , and  $(*, 0)$ , respectively. Hence the fibers over them of  $P^{\times, \rho-1}$  are rigidified, that is, equal to  $\mathbb{F}_p^{\times, \rho-1}$ . We define their variants  $P'_{i,j}$ ,  $R'_{i,\tilde{t}}$ , and  $S'_{\tilde{t},j}$  in  $P^{\times, \rho-1}(\mathbb{Z}_p)$  to be the unique elements in their orbits under  $\mathbb{F}_p^{\times, \rho-1}$  whose images in  $P^{\times, \rho-1}(\mathbb{F}_p)$  are equal to the element 1 in  $\mathbb{F}_p^{\times, \rho-1}$ . Replacing, in (4.2) and (4.3), these  $P_{i,j}$ ,  $R_{i,\tilde{t}}$ , and  $S_{\tilde{t},j}$  by  $P'_{i,j}$ ,  $R'_{i,\tilde{t}}$ , and  $S'_{\tilde{t},j}$  gives variants  $A'$ ,  $B'$  and  $C'$ , and using these in (4.4) gives a variant  $D'_t(n)$  of 5.2.2. Then, for all  $n$  in  $\mathbb{Z}^r$ ,  $D'_t(n)$  and  $D'(n)$  (as in (5.2.2)) are equal, because both are in  $P^{\times, \rho-1}(\mathbb{Z}_p)_t$ , and in the same  $\mathbb{F}_p^{\times, \rho-1}$ -orbit. Hence we have, for all  $n$  in  $\mathbb{Z}^r$ :

$$(5.2.5) \quad \begin{aligned} A'(n) &= \sum_{j=1}^r n_j \cdot_2 S'_{\tilde{t},j}, & B'(n) &= \sum_{i=1}^r n_i \cdot_1 R'_{i,\tilde{t}}, \\ C'(n) &= \sum_{i=1}^r n_i \cdot_1 \left( \sum_{j=1}^r n_j \cdot_2 P'_{i,j} \right), \\ D'(n) &= (C'(n) +_2 B'(n)) +_1 (A'(n) +_2 \tilde{t}). \end{aligned}$$

This shows how the map  $n \mapsto D'(n)$  is built up from the two partial group laws  $+_1$  and  $+_2$  on  $P^{\times, \rho-1}$ , and the iterations  $\cdot_1$  and  $\cdot_2$ . Lemma 5.1.1 gives that the iterations are given by integral convergent power series. The functoriality in Section 3 gives that the maps induced by  $+_1$  and  $+_2$  on residue polydisks are given by integral convergent power series. Stability under composition then gives that  $n \mapsto D'(n)$  is given by elements  $\kappa'_1, \dots, \kappa'_{g+\rho-1}$  of  $\mathbb{Z}_p \langle z_1, \dots, z_r \rangle$ .

We call the  $\kappa'_i$  the coordinate functions of the extension  $D': \mathbb{Z}_p^r \rightarrow T(\mathbb{Z}_p)_t = \mathbb{Z}_p^{g+\rho-1}$ , and their images  $\bar{\kappa}'_1, \dots, \bar{\kappa}'_{g+\rho-1}$  in  $\mathbb{F}_p[z_1, \dots, z_r]$  the mod  $p$  coordinate functions, viewed as a morphism  $\bar{D}'_{\mathbb{F}_p}: \mathbb{A}_{\mathbb{F}_p}^r \rightarrow \mathbb{A}_{\mathbb{F}_p}^{g+\rho-1}$ .

The mod  $p$  coordinate functions of  $A': \mathbb{Z}_p^r \rightarrow P^{\times, \rho-1}(\mathbb{Z}_p) = \mathbb{Z}_p^{\rho g + \rho - 1}$  (after choosing the necessary parameters) are all of degree at most 1. The same holds for  $B'$ . We define

$$(5.2.6) \quad C'_2: \mathbb{Z}^r \times \mathbb{Z}^r \longrightarrow P^{\times, \rho-1}(\mathbb{Z}_p), \quad C'_2(n, m) = \sum_{i=1}^r n_i \cdot_1 \left( \sum_{j=1}^r m_j \cdot_2 P'_{i,j} \right).$$

Then the mod  $p$  coordinate functions of  $C'_2$ , elements of  $\mathbb{F}_p[x_1, \dots, x_r, y_1, \dots, y_r]$ , are linear in the  $x_i$ , and in the  $y_j$ . Hence of degree at most 2, and the same follows for the mod  $p$  coordinate functions of  $C'$ . However, as the first  $\rho g$  parameters for  $P^{\times, \rho-1}$  come from  $J \times J^{\vee \rho-1}$ , and the 1st and 2nd partial group laws there act on different factors, the first  $\rho g$  mod  $p$  coordinate functions of  $C'$  are in fact linear. As  $D'$  is obtained by summing, using the partial group laws, the results of  $A'$ ,  $B$  and  $C'$ , we conclude that  $\bar{\kappa}'_1, \dots, \bar{\kappa}'_g$  are of degree at most 1, and the remaining  $\bar{\kappa}_j$  are of degree at most 2. The same holds then for all  $\bar{\kappa}_j$ .

### 5.3 The $p$ -adic closure

We know from (5.2.3) that  $\kappa_{\mathbb{Z}}(\mathbb{Z}^r) = D'((p-1)\mathbb{Z}^r)$ . From (4.9) we know that  $\kappa_{\mathbb{Z}}(\mathbb{Z}^r) \subset T(\mathbb{Z})_t$ . From (5.2.4) we know that  $T(\mathbb{Z})_t \subset D'(\mathbb{Z}^r)$ . So together we have:

$$(5.3.1) \quad D'((p-1)\mathbb{Z}^r) = \kappa_{\mathbb{Z}}(\mathbb{Z}^r) \subset T(\mathbb{Z})_t \subset D'(\mathbb{Z}^r).$$

We have extended  $D'$  to a continuous map  $\mathbb{Z}_p^r \rightarrow T(\mathbb{Z}_p)_t$ . As  $\mathbb{Z}_p^r$  is compact,  $D'(\mathbb{Z}_p^r)$  is closed in  $T(\mathbb{Z}_p)_t$ . As  $\mathbb{Z}^r$  and  $(p-1)\mathbb{Z}^r$  are dense in  $\mathbb{Z}_p^r$ , the closures of their images under  $D'$  are both equal to  $D'(\mathbb{Z}_p^r)$ , and equal to  $\kappa(\mathbb{Z}_p^r)$ . This finishes the proof of Theorem 4.10.

## 6 Explicit description of the Poincaré torsor

We give an explicit description of the Poincaré torsor on  $J \times J^{\vee,0}$  and its partial group laws, in terms of divisors and invertible  $\mathcal{O}$ -modules on  $C$ , first over  $\mathbb{Z}[1/n]$ , where  $C$  is smooth, and then over  $\mathbb{Z}$ .

### 6.1 Norms

Let  $S$  be a scheme,  $f: S' \rightarrow S$  be finite and locally free, say of rank  $n$ . Then  $\mathcal{O}_{S'} = f_*\mathcal{O}_{S'}$  (we view  $\mathcal{O}_{S'}$  as a sheaf on  $S$ ) is an  $\mathcal{O}_S$ -algebra, locally free as  $\mathcal{O}_S$ -module of rank  $n$ , and  $\mathcal{O}_{S'}^\times$  is a subsheaf of groups of  $\mathrm{GL}_{\mathcal{O}_S}(\mathcal{O}_{S'})$ . Then the norm morphism is the composition

$$(6.1.1) \quad \begin{array}{ccc} & \text{Norm}_{S'/S} & \\ & \curvearrowright & \\ \mathcal{O}_{S'}^\times & \hookrightarrow \mathrm{GL}_{\mathcal{O}_S}(\mathcal{O}_{S'}) & \xrightarrow{\det} \mathcal{O}_S^\times \end{array}$$

For  $T$  an  $\mathcal{O}_{S'}^\times$ -torsor (triviality locally on  $S$  and  $S'$  are equivalent, from the equivalence with invertible  $\mathcal{O}_{S'}$ -modules), we let  $\mathrm{Norm}_{S'/S}(T)$  be the  $\mathcal{O}_S^\times$ -torsor

$$(6.1.2) \quad \mathrm{Norm}_{S'/S}(T) := \mathcal{O}_S^\times \otimes_{\mathcal{O}_{S'}^\times} T = (\mathcal{O}_S^\times \times T) / \mathcal{O}_{S'}^\times,$$

with, for every open  $U$  of  $S$ , and every  $u \in \mathcal{O}_{S'}^\times(U)$ ,  $u$  acting as  $(v, t) \mapsto (v \cdot \mathrm{Norm}_{S'/S}(u), u^{-1} \cdot t)$ . This is functorial in  $T$ : a morphism  $\varphi: T_1 \rightarrow T_2$  induces an isomorphism  $\mathrm{Norm}_{S'/S}(\varphi)$ . It is also functorial for cartesian diagrams  $(S'_2 \rightarrow S_2) \rightarrow (S'_1 \rightarrow S_1)$ .

For  $U \subset S$  open,  $T$  an  $\mathcal{O}_{S'}^\times$ -torsor, and  $t \in T(U)$ , we have the isomorphism of  $\mathcal{O}_{S'}^\times|_U$ -torsors  $\mathcal{O}_{S'}^\times|_U \rightarrow T|_U$  sending 1 to  $t$ . Functoriality gives  $\mathrm{Norm}_{S'/S}(t)$  in  $(\mathrm{Norm}_{S'/S}(T))(U)$ , also denoted  $1 \otimes t$ .

The norm functor (6.1.2) is multiplicative:

$$(6.1.3) \quad \mathrm{Norm}_{S'/S}(T_1 \otimes_{\mathcal{O}_{S'}^\times} T_2) = \mathrm{Norm}_{S'/S}(T_1) \otimes_{\mathcal{O}_S^\times} \mathrm{Norm}_{S'/S}(T_2),$$

such that, if  $U \subset S$  is open and  $t_1$  and  $t_2$  are in  $T_1(U)$  and  $T_2(U)$ , then

$$(6.1.4) \quad \mathrm{Norm}_{S'/S}(t_1 \otimes t_2) \mapsto \mathrm{Norm}_{S'/S}(t_1) \otimes \mathrm{Norm}_{S'/S}(t_2).$$

Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_{S'}$ -module; locally on  $S$ , it is free of rank 1 as  $\mathcal{O}_{S'}$ -module. This gives us the  $\mathcal{O}_{S'}^\times$ -torsor (on  $S$ )  $\text{Isom}_{\mathcal{O}_{S'}}(\mathcal{O}_{S'}, \mathcal{L})$ , which gives back  $\mathcal{L}$  as  $\mathcal{L} = \mathcal{O}_{S'} \otimes_{\mathcal{O}_S^\times} \text{Isom}_{\mathcal{O}_{S'}}(\mathcal{O}_{S'}, \mathcal{L})$ . The norm of  $\mathcal{L}$  via  $f: S' \rightarrow S$  is then defined as

$$(6.1.5) \quad \text{Norm}_{S'/S}(\mathcal{L}) := \mathcal{O}_S \otimes_{\mathcal{O}_S^\times} \text{Norm}_{S'/S}(\text{Isom}_{\mathcal{O}_{S'}}(\mathcal{O}_{S'}, \mathcal{L})).$$

This construction is functorial for isomorphisms of invertible  $\mathcal{O}_{S'}$ -modules.

## 6.2 Norms along finite relative Cartier divisors

This part is inspired by [6], section 1.1. Let  $S$  be a scheme, let  $f: X \rightarrow S$  be an  $S$ -scheme of finite presentation. A finite effective relative Cartier divisor on  $f: X \rightarrow S$  is a closed subscheme  $D$  of  $X$  that is finite and locally free over  $S$ , and whose ideal sheaf  $I_D$  is locally generated by a non-zero divisor (equivalently,  $I_D$  is locally free of rank 1 as  $\mathcal{O}_X$ -module). For such a  $D$  and an invertible  $\mathcal{O}_X$ -module  $\mathcal{L}$ , the norm of  $\mathcal{L}$  along  $D$  is defined, using (6.1.5), as

$$(6.2.1) \quad \text{Norm}_{D/S}(\mathcal{L}) := \text{Norm}_{D/S}(\mathcal{L}|_D).$$

Then  $\text{Norm}_{D/S}(\mathcal{L})$  is functorial for cartesian diagrams  $(X' \rightarrow S', \mathcal{L}') \rightarrow (X \rightarrow S, \mathcal{L})$ .

**6.2.2 Lemma** *Let  $f: X \rightarrow S$  be a morphism of schemes that is of finite presentation. For  $D$  a finite effective relative Cartier divisor on  $f$ , the norm functor  $\text{Norm}_{D/S}$  in (6.2.1) is multiplicative in  $\mathcal{L}$ :*

$$(6.2.3) \quad \text{Norm}_{D/S}(\mathcal{L}_1 \otimes \mathcal{L}_2) = \text{Norm}_{D/S}(\mathcal{L}_1) \otimes_{\mathcal{O}_S} \text{Norm}_{D/S}(\mathcal{L}_2),$$

with, for  $U \subset S$  open,  $V \subset X$  open, containing  $f^{-1}U \cap D$  and  $l_i \in \mathcal{L}_i(V)$  generating  $\mathcal{L}_i|_V$ ,

$$(6.2.4) \quad \text{Norm}_{D/S}(l_1 \otimes l_2) = \text{Norm}_{D/S}(l_1) \otimes \text{Norm}_{D/S}(l_2).$$

Let  $D_1$  and  $D_2$  be finite effective relative Cartier divisors on  $f$ . Then the ideal sheaf  $I_{D_1}I_{D_2} \subset \mathcal{O}_X$  is locally free of rank 1, the closed subscheme  $D_1 + D_2$  defined by it is a finite effective relative Cartier divisor on  $f$ . The norm functor in (6.2.1) is additive in  $D$ :

$$(6.2.5) \quad \text{Norm}_{(D_1+D_2)/S}(\mathcal{L}) = \text{Norm}_{D_1/S}(\mathcal{L}) \otimes_{\mathcal{O}_S} \text{Norm}_{D_2/S}(\mathcal{L}),$$

with, for  $U \subset S$  open,  $V \subset X$  open, containing  $f^{-1}U \cap (D_1 + D_2)$  and  $l \in \mathcal{L}(V)$  generating  $\mathcal{L}|_{D_1+D_2}$ ,

$$(6.2.6) \quad \text{Norm}_{(D_1+D_2)/S}(l) = \text{Norm}_{D_1/S}(l) \otimes \text{Norm}_{D_2/S}(l).$$

**Proof** Let  $D_1$  and  $D_2$  be as stated. If  $V \subset X$  is open, and  $f_i$  generates  $I_{D_i}|_V$ , then  $f_1 f_2$  generates  $(I_{D_1} I_{D_2})|_V$ , and this element of  $\mathcal{O}_X(V)$  is not a zero-divisor because  $f_1$  and  $f_2$  are not. To show that  $D_1 + D_2$  is affine over  $S$ , we replace  $S$  by an affine open of it, and then reduce to the noetherian case, using the assumption that  $f$  is of finite presentation. Then,

$(D_1 + D_2)_{\text{red}}$  is the image of  $D_{1,\text{red}} \amalg D_{2,\text{red}} \rightarrow X$ , and therefore is proper. Hence  $D_1 + D_2$  is proper over  $S$ , and quasi-finite over  $S$ , hence finite over  $S$ . The short exact sequence

$$(6.2.7) \quad \begin{array}{ccccc} I_{D_2}/I_{D_1+D_2} & \hookrightarrow & \mathcal{O}_{D_1+D_2} & \twoheadrightarrow & \mathcal{O}_{D_2} \\ \parallel & & & & \\ (I_{D_2})|_{D_1} & & & & \end{array}$$

shows that  $\mathcal{O}_{D_1+D_2}$  is locally free as  $\mathcal{O}_S$ -module, of rank the sum of the ranks of the  $\mathcal{O}_{D_i}$ . So  $D_1 + D_2$  is a finite effective relative Cartier divisor on  $X \rightarrow S$ .

We prove (6.2.5), by proving the required statement about sheaves of groups. The diagram

$$(6.2.8) \quad \begin{array}{c} \text{Norm}_{(D_1+D_2)/S} \\ \curvearrowright \\ \mathcal{O}_{D_1+D_2}^\times \longrightarrow \mathcal{O}_{D_1}^\times \times \mathcal{O}_{D_2}^\times \xrightarrow{\text{Norm}_{D_1/S} \times \text{Norm}_{D_2/S}} \mathcal{O}_S^\times \times \mathcal{O}_S^\times \longrightarrow \mathcal{O}_S^\times \\ \text{u} \longmapsto \text{Norm}_{D_1/S}(u) \text{Norm}_{D_2/S}(u) . \end{array}$$

commutes, because multiplication by  $u$  on  $\mathcal{O}_{D_1+D_2}$  preserves the short exact sequence (6.2.7), multiplying on the sub and quotient by its images in  $\mathcal{O}_{D_1}^\times$  and in  $\mathcal{O}_{D_2}^\times$ ; note that the sub is an invertible  $\mathcal{O}_{D_1}$ -module.  $\square$

### 6.3 Explicit description of the Poincaré torsor of a smooth curve

Let  $g$  be in  $\mathbb{Z}_{\geq 1}$ , let  $S$  be a scheme, and  $\pi: C \rightarrow S$  be a proper smooth curve, with geometrically connected fibres of genus  $g$ , with a section  $b \in C(S)$ . Let  $J \rightarrow S$  be its jacobian. On  $C \times_S J$  we have  $\mathcal{L}^{\text{univ}}$ , the universal invertible  $\mathcal{O}$ -module of degree zero on  $C$ , rigidified at  $b$ .

Let  $d \geq 0$ , and  $C^{(d)}$  the  $d$ th symmetric power of  $C \rightarrow S$  (we note that the quotient  $C^d \rightarrow C^{(d)}$  is finite, locally free of rank  $d!$ , and commutes with base change on  $S$ ). Then on  $C \times_S C^{(d)}$  we have  $D$ , the universal effective relative Cartier divisor on  $C$  of degree  $d$ . Hence, on  $C \times_S J \times_S C^{(d)}$  we have their pullbacks  $D_J$  and  $\mathcal{L}_{C^{(d)}}^{\text{univ}}$ , giving us

$$(6.3.1) \quad \mathcal{N}_d := \text{Norm}_{D_J/(J \times_S C^{(d)})}(\mathcal{L}_{C^{(d)}}^{\text{univ}}).$$

This invertible  $\mathcal{O}$ -module  $\mathcal{N}_d$  on  $J \times_S C^{(d)}$ , rigidified at the zero-section of  $J$ , gives us a morphism of  $S$ -schemes  $C^{(d)}$  to  $\text{Pic}_{J/S}$ . The point  $db$  (the divisor  $d$  times the base point  $b$ ) in  $C^{(d)}(S)$  is mapped to 0, precisely because  $\mathcal{L}^{\text{univ}}$  is rigidified at  $b$ , and 6.2.5. Hence there is a unique morphism  $\square: C^{(d)} \rightarrow J^\vee = \text{Pic}_{J/S}^0$  such that the pullback of the Poincaré bundle  $P$  on  $J \times J^\vee$  by  $(\text{id}, \square): J \times C^{(d)} \rightarrow J \times J^\vee$ , with its rigidifications, is the same as  $\mathcal{N}_d$ . The following proposition tells us what the morphism  $\square$  is, and the next section tells us what the induced isomorphism is between the fibres of  $\mathcal{N}_d$  at points of  $J \times C^{(d)}$  with the same image in  $J \times_S J$ .



**6.3.2 Proposition** *The pullback of  $P$  by  $(j_b, j_b^{*, -1}): C \times_S J \rightarrow J \times_S J^\vee$  together with its rigidifications at  $b$  and  $0$ , is equal to  $\mathcal{L}^{\text{univ}}$ .*

Let  $d$  be in  $\mathbb{Z}_{\geq 0}$ . The morphism  $\square: C^{(d)} \rightarrow J^\vee = \text{Pic}_{J/S}^0$  is the composition of first  $\Sigma: C^{(d)} \rightarrow J$ , sending, for every  $S$ -scheme  $T$  and any  $D$  in  $C^{(d)}(T)$  to the class of  $\mathcal{O}_{C_T}(D - db)$  twisted by the pullback from  $T$  that makes it rigidified at  $b$ , followed by  $j_b^{*, -1}: J \rightarrow J^\vee$ . Summarised in a diagram, with  $\mathcal{M} := (\text{id} \times j_b^{*, -1})^* P$ :

$$(6.3.3) \quad \begin{array}{ccccccc} \mathcal{L}^{\text{univ}} & \longleftarrow & P & \longrightarrow & \mathcal{M} & \xrightarrow{\widetilde{\text{id} \times \Sigma}} & \mathcal{N}_d \\ C \times_S J & \xrightarrow{j_b \times j_b^{*, -1}} & J \times_S J^\vee & \xleftarrow{\text{id} \times j_b^{*, -1}} & J \times_S J & \xleftarrow{\text{id} \times \Sigma} & J \times_S C^{(d)}. \end{array}$$

Then  $\mathcal{M}$ , with its rigidifications at  $\{0\} \times_S J$  and  $J \times_S \{0\}$ , is symmetric. For  $T \rightarrow S$ ,  $x$  in  $J(T)$  given by an invertible  $\mathcal{O}$ -module  $\mathcal{L}$  on  $C_T$  rigidified at  $b$ , and  $y = \Sigma(D)$  in  $J(T)$  given by an effective relative divisor  $D$  of degree  $d$  on  $C_T$  we have

$$(6.3.4) \quad P(x, j_b^{*, -1}(y)) = \mathcal{M}(x, y) = \text{Norm}_{D/T}(\mathcal{L}).$$

For  $c_1$  and  $c_2$  in  $C(S)$ , we have

$$(6.3.5) \quad \mathcal{M}(j_b(c_1), j_b(c_2)) = c_2^*(\mathcal{O}_C(c_1 - b)) \otimes b^*(\mathcal{O}_C(b - c_1)),$$

and, as invertible  $\mathcal{O}$ -modules on  $C \times_S C$ , with  $\Delta$  the diagonal and  $\text{pr}_0: C \times_S C \rightarrow S$  the structure morphism, we have

$$(6.3.6) \quad (j_b \times j_b)^* \mathcal{M} = \mathcal{O}(\Delta) \otimes \text{pr}_1^* \mathcal{O}(-b) \otimes \text{pr}_2^* \mathcal{O}(-b) \otimes \text{pr}_0^* b^* T_{C/S}.$$

For  $d > 2g - 2$ ,  $\widetilde{\text{id} \times \Sigma}$  gives  $\mathcal{N}_d$  a descent datum along  $\text{id} \times \Sigma$  that gives  $\mathcal{M}$  on  $J \times_S J$ . For  $T$  an  $S$ -scheme,  $x \in J(S)$  given by  $\mathcal{L}$  on  $C_T$ , rigidified at  $b$ ,  $D_1$  and  $D_2$  in  $C^{(d_1)}(S)$  and  $C^{(d_2)}(S)$ , the isomorphism

$$(6.3.7) \quad \mathcal{M}(x, \Sigma(D_1 + D_2)) = \mathcal{M}(x, \Sigma(D_1)) \otimes \mathcal{M}(x, \Sigma(D_2))$$

corresponds, via  $\widetilde{\text{id} \times \Sigma}$ , to

$$(6.3.8) \quad \begin{aligned} \mathcal{N}_{d_1+d_2}(x, D_1 + D_2) &= \text{Norm}_{(D_1+D_2)/T}(\mathcal{L}) = \text{Norm}_{D_1/T}(\mathcal{L}) \otimes \text{Norm}_{D_2/T}(\mathcal{L}) \\ &= \mathcal{N}_{d_1}(x, D_1) \otimes \mathcal{N}_{d_2}(x, D_2), \end{aligned}$$

using Lemma 6.2.2.

For  $T$  an  $S$ -scheme and  $x_1$  and  $x_2$  in  $J(T)$  given by  $\mathcal{O}$ -modules  $\mathcal{L}_1$  and  $\mathcal{L}_2$  on  $C_T$ , rigidified at  $b$ , and  $D$  in  $C^{(d)}(T)$ , the isomorphism

$$(6.3.9) \quad \mathcal{M}(x_1 + x_2, \Sigma(D)) = \mathcal{M}(x_1, \Sigma(D)) \otimes \mathcal{M}(x_2, \Sigma(D))$$

corresponds, via  $\widetilde{\text{id} \times \Sigma}$ , to

$$(6.3.10) \quad \begin{aligned} \mathcal{N}_d(x_1 + x_2, D) &= \text{Norm}_{D/T}(\mathcal{L}_1 \otimes \mathcal{L}_2) = \text{Norm}_{D/T}(\mathcal{L}_1) \otimes \text{Norm}_{D/T}(\mathcal{L}_2) \\ &= \mathcal{N}_d(x_1, D) \otimes \mathcal{N}_d(x_2, D), \end{aligned}$$

using Lemma 6.2.2.

**Proof** Let  $T$  be an  $S$ -scheme, and  $x$  be in  $J(T)$ . Then  $x$  corresponds to the invertible  $\mathcal{O}$ -module  $(\text{id} \times x)^* \mathcal{L}^{\text{univ}}$  on  $C_T$ , rigidified at  $b$ . Let  $z := j_b^{*, -1}(x)$  in  $J^\vee(T)$ . Then  $j_b^*(z) = x$ , meaning that the pullback of  $(\text{id} \times z)^* P$  on  $J_T$  rigidified at 0 by  $j_b$  equals  $(\text{id} \times x)^* \mathcal{L}^{\text{univ}}$  on  $C_T$  rigidified at  $b$ . Taking  $T := J$  and  $x$  the tautological point gives the first claim of the proposition.

The symmetry of  $\mathcal{M}$  with its rigidifications follows from [8], (2.7.1) and Lemma 2.7.5, and (2.7.7), using 2.9.

Now we prove (6.3.4). So let  $T$  and  $x$  be as above, and  $y = \Sigma(D)$  in  $J(T)$  given by a relative divisor  $D$  of degree  $d$  on  $C_T$ . As  $C^d \rightarrow C^{(d)}$  is finite and locally free of rank  $d!$ , we may and do suppose that  $D$  is a sum sections, say  $D = \sum_{i=1}^d (c_i)$ , with  $c_i \in C(T)$ . Then we have, functorially:

$$\begin{aligned}
 (6.3.11) \quad P(x, j_b^{*, -1}(y)) &= P(y, j_b^{*, -1}(x)) = P(\Sigma(D), j_b^{*, -1}(x)) \\
 &= P\left(\sum_i j_b(c_i), j_b^{*, -1}(x)\right) = \bigotimes_i P(j_b(c_i), j_b^{*, -1}(x)) \\
 &= \bigotimes_i \mathcal{L}^{\text{univ}}(c_i, x) = \bigotimes_i \mathcal{L}(c_i) = \text{Norm}_{D/T}(\mathcal{L}).
 \end{aligned}$$

Identities (6.3.5) and (6.3.6) follow directly from (6.3.4).

Now we prove the claimed compatibility between (6.3.9) and (6.3.10). We do this by considering the case where  $\mathcal{L}$  is universal, that is, base changing to  $J_T$  and  $x$  the universal point. Then, on  $J_T$ , we have 2 isomorphisms from  $\text{Norm}_{(D_1+D_2)/J_T}(\mathcal{L})$  to  $\text{Norm}_{D_1/J_T}(\mathcal{L}) \otimes \text{Norm}_{D_2/J_T}(\mathcal{L})$ . These differ by an element of  $\mathcal{O}(J_T)^\times = \mathcal{O}(T)^\times$ . Hence it suffices to check that this element equals 1 at  $0 \in J(T)$ . This amounts to checking that the 2 isomorphisms are equal for  $\mathcal{L} = \mathcal{O}_{C_T}$  with the standard rigidification at  $b$ . Then, both isomorphisms are the multiplication map  $\mathcal{O}_T \otimes_{\mathcal{O}_T} \mathcal{O}_T \rightarrow \mathcal{O}_T$ .

The compatibility between (6.3.7) and (6.3.8) is proved analogously.  $\square$

**6.3.12 Remark** From Proposition 6.3.2 one easily deduces, in that situation, for  $T$  an  $S$ -scheme,  $x$  in  $J(T)$  given by an invertible  $\mathcal{O}$ -module  $\mathcal{L}$  on  $C_T$ , and  $D_1$  and  $D_2$  effective relative Cartier divisors on  $C_T$ , of the same degree, a canonical isomorphism

$$(6.3.13) \quad \mathcal{M}(x, \Sigma(D_1) - \Sigma(D_2)) = \text{Norm}_{D_1/T}(\mathcal{L}) \otimes \text{Norm}_{D_2/T}(\mathcal{L})^{-1},$$

satisfying the analogous compatibilities as in Proposition 6.3.2. No rigidification of  $\mathcal{L}$  at  $b$  is needed. In fact, for  $\mathcal{L}_0$  an invertible  $\mathcal{O}_T$ -module, we have  $\text{Norm}_{D_1/T}(\pi^* \mathcal{L}_0) = \mathcal{L}_0^{\otimes d}$ , where  $\pi: C_T \rightarrow T$  is the structure morphism and  $d$  is the degree of  $D_1$ . Hence the right hand side of (6.3.13) is independent of the choice of  $\mathcal{L}$ , given  $x$ .

## 6.4 Explicit isomorphism for norms along equivalent divisors

Let  $g$  be in  $\mathbb{Z}_{\geq 1}$ , let  $S$  be a scheme, and  $p: C \rightarrow S$  be a proper smooth curve, with geometrically connected fibres of genus  $g$ , with a section  $b \in C(S)$ . Let  $D_1, D_2$  be effective relative Cartier divisors of degree  $d$  on  $C$ , that we also view as elements of  $C^{(d)}(S)$ . Recall from Proposition 6.3.2 the morphism  $\Sigma: C^{(d)} \rightarrow J$ . Then  $\Sigma(D_1) = \Sigma(D_2)$  if and only if  $D_1, D_2$  are linearly equivalent

in the following sense: locally on  $S$ , there exists an  $f$  in  $\mathcal{O}_C(U)^\times$ , with  $U := C \setminus (D_1 \cup D_2)$ , such that  $f \cdot: \mathcal{O}_U \rightarrow \mathcal{O}_U$  extends to an isomorphism  $f \cdot: \mathcal{O}_C(D_1) \rightarrow \mathcal{O}_C(D_2)$ . In this case, we define  $\text{div}(f) = D_2 - D_1$ . Proposition 6.3.2 gives us, for each invertible  $\mathcal{O}$ -module  $\mathcal{L}$  of degree 0 on  $C$  rigidified at  $b$  (viewed as an element of  $J(S)$ ) specific isomorphisms

$$(6.4.1) \quad \begin{aligned} \text{Norm}_{D_1/S}(\mathcal{L}) &= \mathcal{N}_d(\mathcal{L}, D_1) = \mathcal{M}(\mathcal{L}, \Sigma(D_1)) = \mathcal{M}(\mathcal{L}, \Sigma(D_2)) = \mathcal{N}_d(\mathcal{L}, D_2) \\ &= \text{Norm}_{D_2/S}(\mathcal{L}). \end{aligned}$$

Now we describe explicitly this isomorphism  $\text{Norm}_{D_1/S}(\mathcal{L}) \rightarrow \text{Norm}_{D_2/S}(\mathcal{L})$ . To do so we first describe *an* isomorphism

$$(6.4.2) \quad \varphi_{\mathcal{L}, D_1, D_2}: \text{Norm}_{D_1/S}(\mathcal{L}) \longrightarrow \text{Norm}_{D_2/S}(\mathcal{L})$$

that is functorial for Cartesian diagrams  $(C' \rightarrow S', \mathcal{L}', D'_1, D'_2) \rightarrow (C \rightarrow S, \mathcal{L}, D_1, D_2)$  and then we prove that *this* isomorphism is the one in (6.4.1).

We construct  $\varphi_{\mathcal{L}, D_1, D_2}$  locally on  $S$  and the functoriality of the construction takes care of making it global. So, suppose that  $f$  is as above:  $f \in \mathcal{O}_C(U)^\times$ , and  $f \cdot: \mathcal{O}_U \rightarrow \mathcal{O}_U$  extends to an isomorphism  $f \cdot: \mathcal{O}_C(D_1) \rightarrow \mathcal{O}_C(D_2)$ . Let  $n \in \mathbb{Z}$  with  $n > 2g - 2 + 2d$ . Then  $p_*(\mathcal{L}(nb)) \rightarrow p_*\mathcal{L}(nb)|_{D_1+D_2}$  and  $p_*(\mathcal{O}_C(nb)) \rightarrow p_*\mathcal{O}_C(nb)|_{D_1+D_2}$  are surjective, and (still localising on  $S$ )  $p_*(\mathcal{L}(nb))$  and  $p_*(\mathcal{O}_C(nb))$  are free  $\mathcal{O}_S$ -modules and  $\mathcal{L}(nb)|_{D_1+D_2}$  and  $\mathcal{O}_C(nb)|_{D_1+D_2}$  are free  $\mathcal{O}_{D_1+D_2}$ -modules of rank 1. Then we have  $l_0$  in  $(\mathcal{L}(nb))(C)$  and  $l_1$  in  $(\mathcal{O}_C(nb))(C)$  restricting to generators on  $D_1 + D_2$ . Let  $D^- := \text{div}(l_1)$  and  $D^+ := \text{div}(l_0)$ , and let  $V := C \setminus (D^+ + D^-)$ . Note that  $V$  contains  $D_1 + D_2$  and that  $U$  contains  $D^+ + D^-$ . Then, on  $V$ ,  $l := l_0/l_1$  is in  $\mathcal{L}(V)$ , generates  $\mathcal{L}|_{D_1+D_2}$ , and multiplication by  $l$  is an isomorphism  $\cdot l: \mathcal{O}_C(D^+ - D^-) \rightarrow \mathcal{L}$ , that is,  $\text{div}(l) = D^+ - D^-$ . Let

$$(6.4.3) \quad f(\text{div}(l)) = f(D^+ - D^-) := \text{Norm}_{D^+/S}(f|_{D^+}) \cdot \text{Norm}_{D^-/S}(f|_{D^-})^{-1} \in \mathcal{O}_S(S)^\times,$$

and let  $\varphi_{\mathcal{L}, l, f}$  be the isomorphism, given in terms of generators

$$(6.4.4) \quad \begin{aligned} \varphi_{\mathcal{L}, l, f}: \text{Norm}_{D_1/S}(\mathcal{L}) &\longrightarrow \text{Norm}_{D_2/S}(\mathcal{L}) \\ \text{Norm}_{D_1/S}(l) &\longmapsto f(\text{div}(l))^{-1} \cdot \text{Norm}_{D_2/S}(l). \end{aligned}$$

Now suppose that we made other choices  $n', l'_0, l'_1$ . Then we get  $D'^-, D'^+, V', l'$  and  $\varphi_{\mathcal{L}, l', f}$ . Then there is a unique function  $g \in \mathcal{O}_C(V \cap V')^\times$  such that  $l' = gl$  in  $\mathcal{L}(V \cap V')$ . Then

$$(6.4.5) \quad \begin{aligned} \varphi_{\mathcal{L}, l', f}(\text{Norm}_{D_1/S}(l)) &= \varphi_{\mathcal{L}, l', f}(\text{Norm}_{D_1/S}(g^{-1}l')) \\ &= \varphi_{\mathcal{L}, l', f}(g^{-1}(D_1)\text{Norm}_{D_1/S}(l')) \\ &= g^{-1}(D_1) \cdot \varphi_{\mathcal{L}, l', f}(\text{Norm}_{D_1/S}(l')) \\ &= g^{-1}(D_1) \cdot f(\text{div}(l'))^{-1} \cdot \text{Norm}_{D_2/S}(l') \\ &= g^{-1}(D_1) \cdot f(\text{div}(gl))^{-1} \cdot \text{Norm}_{D_2/S}(gl) \\ &= g^{-1}(D_1) \cdot f(\text{div}(g) + \text{div}(l))^{-1} \cdot g(D_2) \cdot \text{Norm}_{D_2/S}(l) \\ &= g^{-1}(D_1) \cdot f(\text{div}(g))^{-1} \cdot g(D_2) \cdot f(\text{div}(l))^{-1} \cdot \text{Norm}_{D_2/S}(l) \\ &= g(\text{div}(f)) \cdot f(\text{div}(g))^{-1} \cdot \varphi_{\mathcal{L}, l, f}(\text{Norm}_{D_1/S}(l)) \\ &= \varphi_{\mathcal{L}, l, f}(\text{Norm}_{D_1/S}(l)), \end{aligned}$$

where, in the last step, we used Weil reciprocity, in a generality for which we do not know a reference. The truth in this generality is clear from the classical case by reduction to the universal case, in which the base scheme is integral: take a suitable level structure on  $J$ , then consider the universal curve with this level structure, and the universal 4-tuple of effective divisors with the necessary conditions. We conclude that  $\varphi_{\mathcal{L},l,f} = \varphi_{\mathcal{L},l',f}$ .

Now suppose that  $f'$  is in  $\mathcal{O}_C(U)^\times$  with  $\text{div}(f') = \text{div}(f)$ . Then there is a unique  $u \in \mathcal{O}_S(S)^\times$  such that  $f' = u \cdot f$ , and since  $\mathcal{L}$  has degree 0 on  $C$

$$\begin{aligned}
 \varphi_{\mathcal{L},l,f'}(\text{Norm}_{D_1/S}(l)) &= (u \cdot f)(\text{div}(l))^{-1} \cdot \text{Norm}_{D_2/S}(l) \\
 (6.4.6) \quad &= u^{-\deg(\text{div}(l))} f(\text{div}(l))^{-1} \cdot \text{Norm}_{D_2/S}(l) \\
 &= f(\text{div}(l))^{-1} \cdot \text{Norm}_{D_2/S}(l) = \varphi_{\mathcal{L},l,f}(\text{Norm}_{D_1/S}(l)).
 \end{aligned}$$

Hence  $\varphi_{\mathcal{L},l,f'} = \varphi_{\mathcal{L},l,f}$ . We define

$$(6.4.7) \quad \varphi_{D_1,D_2,\mathcal{L}}: \text{Norm}_{D_1/S}(\mathcal{L}) \longrightarrow \text{Norm}_{D_2/S}(\mathcal{L})$$

as the isomorphism  $\varphi_{\mathcal{L},l,f}$  in (6.4.4) for any local choice of  $f$  and  $l$ .

We now prove that  $\varphi_{\mathcal{L},D_1,D_2}$  is the same as the isomorphism in (6.4.1). We do this, as in the proof of Proposition 6.3.2, by considering the case of the universal  $\mathcal{L}$ , that is, we base change via  $J \rightarrow S$ , and then restricting to  $0 \in J(S)$ . This amounts to checking that the 2 isomorphisms are equal for  $\mathcal{L} = \mathcal{O}_C$  with the standard rigidification at  $b$ . In this case,  $\text{Norm}_{D_i/S}(\mathcal{O}_C) = \mathcal{O}_S$ , with  $\text{Norm}_{D_i/S}(1) = 1$ . Hence  $\varphi_{D_1,D_2,\mathcal{O}_C} = \varphi_{\mathcal{O}_C,1,f}$  is the identity on  $\mathcal{O}_S$  (use (6.4.4)). The other isomorphism is the identity on  $\mathcal{O}_S$  because of the rigidifications of  $\mathcal{M}$  and  $\mathcal{N}_d$  on  $0 \times J$  and  $0 \times C^{(d)}$ .

## 6.5 Symmetry of the Norm for divisors on smooth curves

Let  $C \rightarrow S$  be a proper and smooth curve with geometrically connected fibres. For  $D_1, D_2$  effective relative Cartier divisors on  $C$  we define an isomorphism

$$(6.5.1) \quad \varphi_{D_1,D_2}: \text{Norm}_{D_1/S}(\mathcal{O}_C(D_2)) \longrightarrow \text{Norm}_{D_2/S}(\mathcal{O}_C(D_1))$$

that is functorial for cartesian diagrams  $(C'/S', D'_1, D'_2) \rightarrow (C/S, D_1, D_2)$ .

It suffices to define this isomorphism in the universal case, that is, over the scheme that parametrises all  $D_1$  and  $D_2$ . Let  $d_1$  and  $d_2$  be in  $\mathbb{Z}_{\geq 0}$ , and let  $U := C^{(d_1)} \times_S C^{(d_2)}$ , and let  $D_1$  and  $D_2$  be the universal divisors on  $C_U$ . Then we have the invertible  $\mathcal{O}_U$ -modules  $\text{Norm}_{D_1/U}(\mathcal{O}_C(D_2))$  and  $\text{Norm}_{D_2/U}(\mathcal{O}_C(D_1))$ . The image of  $D_1 \cap D_2$  in  $U$  is closed, let  $U^0$  be its complement. Then, over  $U^0$ ,  $D_1$  and  $D_2$  are disjoint, and the restrictions of  $\text{Norm}_{D_1/U}(\mathcal{O}_C(D_2))$  and  $\text{Norm}_{D_2/U}(\mathcal{O}_C(D_1))$  are generated by  $\text{Norm}_{D_1/U}(1)$  and  $\text{Norm}_{D_2/U}(1)$ , and there is a unique isomorphism  $(\varphi_{D_1,D_2})_{U^0}$  that sends  $\text{Norm}_{D_1/U}(1)$  to  $\text{Norm}_{D_2/U}(1)$ .

We claim that this isomorphism extends to an isomorphism over  $U$ . To see it, we base change by  $U' \rightarrow U$ , where  $U' = C^{d_1} \times_S C^{d_2}$ , then  $U' \rightarrow U$  is finite, locally free of rank  $d_1! \cdot d_2!$ . Then  $D_1 = P_1 + \dots + P_{d_1}$  and  $D_2 = Q_1 + \dots + Q_{d_2}$  with the  $P_i$  and  $Q_j$  in  $C(U')$ . The complement of the inverse image  $U'^0$  in  $U'$  of  $U^0$  is the union of the pullbacks  $D_{i,j}$  under  $\text{pr}_{i,j}: U' \rightarrow C \times_S C$

of the diagonal, that is, the locus where  $P_i = Q_j$ . Each  $D_{i,j}$  is an effective relative Cartier divisor on  $U'$ , isomorphic as  $S$ -scheme to  $C^{d_1+d_2-1}$ , hence smooth over  $S$ . Now

$$(6.5.2) \quad \text{Norm}_{D_1/U'}(\mathcal{O}(D_2)) = \bigotimes_{i,j} P_i^* \mathcal{O}(Q_j), \quad \text{Norm}_{D_2/U'}(\mathcal{O}(D_1)) = \bigotimes_{i,j} Q_j^* \mathcal{O}(P_i),$$

and, on  $U'^0$ ,

$$(6.5.3) \quad \text{Norm}_{D_1/U'}(1) = \bigotimes_{i,j} 1, \quad \text{Norm}_{D_2/U'}(1) = \bigotimes_{i,j} 1, \quad \text{in } \mathcal{O}(U'^0).$$

The divisor on  $U'$  of the tensor-factor 1 at  $(i, j)$ , both in  $\text{Norm}_{D_1/U'}(1)$  and in  $\text{Norm}_{D_2/U'}(1)$ , is  $D_{i,j}$ . Therefore, the isomorphism  $(\varphi_{D_1, D_2})_{U'^0}$  extends, uniquely, to an isomorphism  $\varphi_{D_1, D_2}$  over  $U'$ , which descends uniquely to  $U$ .

Our description of  $\varphi_{D_1, D_2}$  allows us to compute it in the trivial case where  $D_1$  and  $D_2$  are disjoint. One should be a bit careful in other cases. For example, when  $d_1 = d_2 = 1$  and  $P = Q$ , we have  $P^* \mathcal{O}_C(Q) = P^* \mathcal{O}_C(P)$  is the tangent space of  $C \rightarrow S$  at  $P$ , and hence also at  $Q$ , but  $\varphi_{P, Q}$  is multiplication by  $-1$  on that tangent space. The reason for that is that the switch automorphism on  $C \times_S C$  induces  $-1$  on the normal bundle of the diagonal.

If  $b$  is an  $S$ -point on  $C$ , because of the symmetry in Proposition 6.3.2, using (6.3.13), for  $D_1, D_2$  relative effective divisors on  $C$  of degree  $d_1, d_2$  over  $S$  we have the following diagram of isomorphisms defining  $\psi_{D_1, D_2}$

$$(6.5.4) \quad \begin{array}{ccc} \mathcal{M}(\Sigma(D_2), \Sigma(D_1)) & \xlongequal{\quad} & \text{Norm}_{D_1/S}(\mathcal{O}_C(D_2 - d_2 b)) \otimes b^* \mathcal{O}_C(D_2 - d_2 b)^{-d_1} \\ \parallel & & \downarrow \psi_{D_1, D_2} \\ \mathcal{M}(\Sigma(D_1), \Sigma(D_2)) & \xlongequal{\quad} & \text{Norm}_{D_2/S}(\mathcal{O}_C(D_1 - d_1 b)) \otimes b^* \mathcal{O}_C(D_1 - d_1 b)^{-d_2}. \end{array}$$

Then

$$(6.5.5) \quad \psi_{D_1, D_2} = \varphi_{D_1, D_2} \otimes \varphi_{D_1, d_2 b}^{-1} \otimes \varphi_{d_1 b, D_2}^{-1} \otimes \varphi_{d_1 b, d_2 b}.$$

It is enough to prove it in the universal case, that is when  $D_1$  and  $D_2$  are the universal divisors on  $C_U$ , and there we know that there exists a  $u$  in  $\mathcal{O}_U(U)^\times = \mathcal{O}_S(S)^\times$  such that

$$(6.5.6) \quad u \cdot \psi_{D_1, D_2} = \varphi_{D_1, D_2} \otimes \varphi_{D_1, d_2 b}^{-1} \otimes \varphi_{d_1 b, D_2}^{-1} \otimes \varphi_{d_1 b, d_2 b}.$$

Since the symmetry in Proposition 6.3.2 is compatible with the rigidification at  $(0, 0) \in (J \times J)(S)$  then  $\psi_{d_1 b, d_2 b}$  is the identity on  $\mathcal{O}_U$ , as well as the right hand side of (6.5.5) when  $D_i = d_i b$ . Hence  $u = u(d_1 b, d_2 b) = 1$ , proving (6.5.5).

Moreover the isomorphisms  $\varphi_{D_1, D_2}$ , and consequently  $\psi_{D_1, D_2}$ , are compatible with addition of divisors, that is, under (6.3.10) and (6.3.8), for every triple  $D_1, D_2, D_3$  of relative Cartier divisors on  $C$  we have

$$(6.5.7) \quad \varphi_{D_1+D_2, D_3} = \varphi_{D_1, D_3} \otimes \varphi_{D_2, D_3}, \quad \varphi_{D_1, D_2+D_3} = \varphi_{D_1, D_2} \otimes \varphi_{D_1, D_3}.$$

As for (6.5.5), it is enough to prove it in the universal case and then we can reduce to the case where  $D_1 = d_1 b$ ,  $D_2 = d_2 b$  and  $D_3 = d_3 b$  for  $d_i$  positive integers where we have

$$(6.5.8) \quad \begin{aligned} \varphi_{d_1 b + d_2 b, d_3 b} &= \varphi_{d_1 b, d_3 b} \otimes \varphi_{d_2 b, d_3 b} = (-1)^{(d_1 + d_2) d_3}, \\ \varphi_{d_1 b, d_2 b + d_3 b} &= \varphi_{d_1 b, d_2 b} \otimes \varphi_{d_1 b, d_3 b} = (-1)^{d_1 (d_2 + d_3)}. \end{aligned}$$

## 6.6 Explicit residue disks and partial group laws

Let  $C$  be a smooth curve over  $\mathbb{Z}/p^2$ , let  $b \in C(\mathbb{Z}/p^2)$ , and let  $\mathcal{M}$  be as in Proposition 6.3.2. Let  $D = D^+ - D^-$  and  $E = E^+ - E^-$  be relative Cartier divisors of degree 0 on  $C$ . For each  $\alpha$  in  $\mathcal{M}^\times(\mathbb{F}_p)$  whose image in  $(J \times J)(\mathbb{F}_p)$  is given by  $(D, E)$  we parametrise  $\mathcal{M}^\times(\mathbb{Z}/p^2)_\alpha$ , under the assumption that there exists a non-special split reduced divisor of degree  $g$  on  $C_{\mathbb{F}_p}$ .

Let  $b_1, \dots, b_g$  in  $C(\mathbb{Z}/p^2)$  have distinct images  $\bar{b}_i$  in  $C(\mathbb{F}_p)$  such that  $h^0(C_{\mathbb{F}_p}, \bar{b}_1 + \dots + \bar{b}_g) = 1$ , and let  $b_{g+1}, \dots, b_{2g}$  in  $C(\mathbb{Z}/p^2)$  be such that the  $\bar{b}_{g+i}$  are distinct and  $h^0(C_{\mathbb{F}_p}, \bar{b}_{g+1} + \dots + \bar{b}_{2g}) = 1$ . Then the maps

$$(6.6.1) \quad \begin{aligned} f_1: C^g &\longrightarrow J, & (c_1, \dots, c_g) &\longmapsto [\mathcal{O}_C(c_1 + \dots + c_g - (b_1 + \dots + b_g) + D)] \\ f_2: C^g &\longrightarrow J, & (c_1, \dots, c_g) &\longmapsto [\mathcal{O}_C(c_1 + \dots + c_g - (b_{g+1} + \dots + b_{2g}) + E)] \end{aligned}$$

are étale respectively in the points  $(b_1, \dots, b_g) \in C^g(\mathbb{F}_p)$  and  $(b_{g+1}, \dots, b_{2g}) \in C^g(\mathbb{F}_p)$  and consequently give bijections  $C^g(\mathbb{Z}/p^2)_{(b_1, \dots, b_g)} \rightarrow J(\mathbb{Z}/p^2)_{\bar{D}}$  and  $C^g(\mathbb{Z}/p^2)_{(b_{g+1}, \dots, b_{2g})} \rightarrow J(\mathbb{Z}/p^2)_{\bar{E}}$ . For each point  $c \in C(\mathbb{F}_p)$  we choose  $x_{D,c}$  a generator of  $\mathcal{O}_C(-D)_c$  and  $x_c \in \mathcal{O}_{C_{\mathbb{Z}_p}, c}$  so that  $p, x_c$  are parameters in  $c$ . For each  $i = 1, \dots, 2g$  we choose  $x_{b_i}$  so that  $x_{b_i}(b_i) = 0$ . For each  $(\mathbb{Z}/p^2)$ -point  $c \in C(\mathbb{Z}/p^2)$  with image  $\bar{c}$  in  $C(\mathbb{F}_p)$  and for each  $\lambda \in \mathbb{F}_p$  let  $c_\lambda$  be the unique point in  $C(\mathbb{Z}/p^2)_{\bar{c}}$  with  $x_{\bar{c}}(c_\lambda) = \lambda p$ . Then the map  $\lambda \mapsto c_\lambda$  is a bijection  $\mathbb{F}_p \rightarrow C(\mathbb{Z}/p^2)_{\bar{c}}$  hence the maps  $f_1, f_2$  induce bijections

$$(6.6.2) \quad \begin{aligned} \mathbb{F}_p^g &\longrightarrow J(\mathbb{Z}/p^2)_{\bar{D}}, & \lambda &\longmapsto D_\lambda := D + (b_{1, \lambda_1} - b_1) + \dots + (b_{g, \lambda_g} - b_g) \\ \mathbb{F}_p^g &\longrightarrow J(\mathbb{Z}/p^2)_{\bar{E}}, & \mu &\longmapsto E_\mu := E + (b_{g+1, \mu_1} - b_{g+1}) + \dots + (b_{2g, \mu_g} - b_{2g}). \end{aligned}$$

Hence  $\mathcal{M}^\times(\mathbb{Z}/p^2)_{\bar{D}, \bar{E}}$  is the union of  $\mathcal{M}^\times(D_\lambda, E_\mu)$  as  $\lambda$  and  $\mu$  vary in  $\mathbb{F}_p^g$  and by Proposition 6.3.2 and Remark 6.3.12 we have

$$(6.6.3) \quad \begin{aligned} \mathcal{M}(D_\lambda, E_\mu) &= \text{Norm}_{E^+/\mathbb{Z}/p^2}(\mathcal{O}_C(D_\lambda)) \otimes \text{Norm}_{E^-/\mathbb{Z}/p^2}(\mathcal{O}_C(D_\lambda))^{-1} \otimes \\ &\otimes \bigotimes_{i=1}^g (b_{g+i, \mu_i}^* \mathcal{O}_C(D_\lambda) \otimes b_{g+i}^* \mathcal{O}_C(D_\lambda)^{-1}). \end{aligned}$$

For each  $i \in \{1, \dots, g\}$ ,  $c \in C(\mathbb{Z}/p^2)$  and  $\lambda \in \mathbb{F}_p$  we define  $x_i(c, \lambda) := 1$  if  $\bar{c} \neq \bar{b}_i$  and  $x_i(c, \lambda) := x_{b_i} - \lambda p$  if  $\bar{c} = \bar{b}_i$ , so that  $c^* x_i(c, \lambda)^{-1}$  generates  $c^* \mathcal{O}(b_{i, \lambda})$ . Then, for each  $c \in C(\mathbb{Z}/p^2)$  and each  $\lambda \in \mathbb{F}_p^g$ ,

$$(6.6.4) \quad c^* \left( x_{D, c}^{-1} \cdot \prod_{i=1}^g \frac{x_i(c, 0)}{x_i(c, \lambda_i)} \right) \quad \text{generates } c^* \mathcal{O}_C(D_\lambda).$$

We write  $E^\pm = E^{0, \pm} + \dots + E^{g, \pm}$  so that  $E^{0, \pm}$  is disjoint from  $\{\bar{b}_1, \dots, \bar{b}_g\}$ , and  $E^{i, \pm}$ , restricted to  $C_{\mathbb{F}_p}$ , is supported on  $\bar{b}_i$ . Let  $x_{D, E}$  be a generator of  $\mathcal{O}_C(-D)$  in a neighborhood of  $E^+ \cup E^-$ . Then, for each  $\lambda$  in  $\mathbb{F}_p^g$ ,

$$(6.6.5) \quad \text{Norm}_{E^{0, \pm}/\mathbb{Z}/p^2}(x_{D, E}^{-1}) \otimes \bigotimes_{i=1}^g \text{Norm}_{E^{i, \pm}/\mathbb{Z}/p^2} \left( x_{D, E}^{-1} \cdot \frac{x_{b_i}}{x_{b_i} - \lambda_i p} \right)$$

generates  $\text{Norm}_{E^\pm/(\mathbb{Z}/p^2)}(\mathcal{O}_C(D_\lambda))$ . By (6.6.3), (6.6.4) and (6.6.5) we see that, for  $\lambda$  and  $\mu$  in  $\mathbb{F}_p^g$ ,

(6.6.6)

$$\begin{aligned} s_{D,E}(\lambda, \mu) := & \text{Norm}_{E^{0,+}/(\mathbb{Z}/p^2)}(x_{D,E}^{-1}) \otimes \bigotimes_{i=1}^g \text{Norm}_{E^{i,+}/(\mathbb{Z}/p^2)} \left( x_{D,E}^{-1} \cdot \frac{x_{b_i}}{x_{b_i} - \lambda_i p} \right) \otimes \\ & \otimes \text{Norm}_{E^{0,-}/(\mathbb{Z}/p^2)}(x_{D,E}^{-1})^{-1} \otimes \bigotimes_{i=1}^g \text{Norm}_{E^{i,-}/(\mathbb{Z}/p^2)} \left( x_{D,E}^{-1} \cdot \frac{x_{b_i}}{x_{b_i} - \lambda_i p} \right)^{-1} \otimes \\ & \otimes \bigotimes_{i=1}^g \left( b_{g+i, \mu_i}^* \left( x_{D, b_{g+i}}^{-1} \cdot \prod_{j=1}^g \frac{x_j(b_{g+i, \mu_i}, 0)}{x_j(b_{g+i, \mu_i}, \lambda_j)} \right) \otimes b_{g+i}^* \left( x_{D, b_{g+i}}^{-1} \cdot \prod_{j=1}^g \frac{x_j(b_{g+i}, 0)}{x_j(b_{g+i}, \lambda_j)} \right)^{-1} \right) \end{aligned}$$

generates the free rank one  $\mathbb{Z}/p^2$ -module  $\mathcal{M}(D_\lambda, E_\mu)$ . The fibre  $\mathcal{M}^\times(\overline{D}, \overline{E})$  over  $(\overline{D}, \overline{E})$  in  $(J \times J)(\mathbb{F}_p)$  is an  $\mathbb{F}_p^\times$ -torsor, containing  $\overline{s_{D,E}(0,0)}$  are in bijection with the elements  $\xi$  in  $\mathbb{F}_p^\times$  and are exactly the points  $\xi \cdot s_{D,E}(0,0)$ . Using that  $(\mathbb{Z}/p^2)^\times = \mathbb{F}_p^\times \times (1 + p\mathbb{F}_p)$ , we parametrise, for each  $\xi \in \mathbb{F}_p^\times$ , the residue disk of  $\xi \cdot \overline{s_{D,E}(0,0)}$  by the bijection

$$(6.6.7) \quad \mathbb{F}_p^g \times \mathbb{F}_p^g \times \mathbb{F}_p \longrightarrow \mathcal{M}^\times(\mathbb{Z}/p^2)_{\xi \cdot \overline{s_{D,E}(0,0)}}, \quad (\lambda, \mu, \tau) \longmapsto (1 + p\tau) \cdot \xi \cdot s_{D,E}(\lambda, \mu).$$

Using this parametrization it easy to describe the two partial group laws on  $\mathcal{M}^\times(\mathbb{Z}/p^2)$  when one of the two points we are summing lies over  $(\overline{D}, \overline{E})$  and the other lies over  $(\overline{D}, 0)$  or  $(0, \overline{E})$ . To compute the group law in  $J(\mathbb{Z}/p^2)$  we notice that for each  $c \in C(\mathbb{Z}/p^2)$  such that  $x_c(c) = 0$  and for each  $\lambda, \mu \in \mathbb{F}_p$  we have

$$(6.6.8) \quad \frac{x_c^2}{(x_c - \lambda p)(x_c - \mu p)} = \frac{x_c^2}{x_c^2 - \lambda p x_c - \mu p x_c} = \frac{x_c}{x_c - (\lambda + \mu)p}$$

and since these rational functions generate  $\mathcal{O}_C(c_\lambda - c + c_\mu - c)$  and  $\mathcal{O}_C(c_{\lambda+\mu} - c)$  in a neighborhood of  $c$ , we have the *equality* of relative Cartier divisors on  $C$

$$(6.6.9) \quad (c_\lambda - c) + (c_\mu - c) = c_{\lambda+\mu} - c.$$

Hence, under the definition for  $\lambda \in \mathbb{F}_p^g$  of

$$(6.6.10) \quad D_\lambda^0 := (b_{1, \lambda_1} - b_1) + \cdots + (b_{g, \lambda_g} - b_g), \quad E_\lambda^0 := (b_{g+1, \lambda_1} - b_{g+1}) + \cdots + (b_{2g, \lambda_g} - b_{2g}),$$

we have, for all  $\lambda, \mu \in \mathbb{F}_p^g$ , that  $D_\lambda + D_\mu^0 = D_{\lambda+\mu}$  and  $E_\lambda + E_\mu^0 = E_{\lambda+\mu}$ . Definition 6.6.6, applied with  $(D, 0)$  and  $(0, E)$ , with  $x_{0,E} = 1$  and, for every  $c \in C(\mathbb{F}_p)$ , with  $x_{0,c} = 1$ , gives, for all  $\lambda, \mu$  in  $\mathbb{F}_p^g$ , the elements

$$(6.6.11) \quad s_{D,0}(\lambda, \mu) \in \mathcal{M}^\times(D_\lambda, E_\mu^0), \quad s_{0,E}(\lambda, \mu) \in \mathcal{M}^\times(D_\lambda^0, E_\mu).$$

Then (6.6.8) and (6.6.9), together with the equivalence of (6.3.7) and (6.3.8) and the equivalence of (6.3.9) and (6.3.10) in Proposition 6.3.2, give that for all  $\lambda, \lambda_1, \lambda_2, \mu, \mu_1, \mu_2$  in  $\mathbb{F}_p^g$

$$(6.6.12) \quad \begin{aligned} s_{D,0}(\lambda, \mu_1) +_2 s_{D,E}(\lambda, \mu_2) &= s_{D,0}(\lambda, \mu_1) \otimes s_{D,E}(\lambda, \mu_2) = s_{D,E}(\lambda, \mu_1 + \mu_2) \\ s_{0,E}(\lambda_1, \mu) +_1 s_{D,E}(\lambda_2, \mu) &= s_{D,0}(\lambda_1, \mu) \otimes s_{D,E}(\lambda_2, \mu) = s_{D,E}(\lambda_1 + \lambda_2, \mu), \end{aligned}$$

and, consequently, for all  $\tau_1, \tau_2 \in \mathbb{F}_p$  and  $\xi_1, \xi_2 \in \mathbb{F}_p^\times$ , that

$$\begin{aligned}
(6.6.13) \quad & \xi_1(1+\tau_1 p) \cdot s_{D,0}(\lambda, \mu_1) + \xi_2(1+\tau_2 p) \cdot s_{D,E}(\lambda, \mu_2) = \xi_1(1+\tau_1 p) \xi_2(1+\tau_2 p) \cdot s_{D,E}(\lambda, \mu_1 + \mu_2) \\
& = \xi_1 \xi_2 (1 + (\tau_1 + \tau_2) p) \cdot s_{D,E}(\lambda, \mu_1 + \mu_2), \\
& \xi_1(1+\tau_1 p) \cdot s_{0,E}(\lambda_1, \mu) + \xi_2(1+\tau_2 p) \cdot s_{D,E}(\lambda_2, \mu) = \xi_1 \xi_2 (1 + (\tau_1 + \tau_2) p) \cdot s_{D,E}(\lambda_1 + \lambda_2, \mu).
\end{aligned}$$

Let us now prove that the parametrization (6.6.7) is the inverse of a bijection given by parameters on  $\mathcal{M}^\times$  analogously to (3.1). Let  $\mathcal{Q}$  be the pullback of  $\mathcal{M}$  by  $f_1 \times f_2$  with  $f_1$  and  $f_2$  as in (6.6.1). Then the lift  $\widetilde{f_1 \times f_2}: \mathcal{Q}^\times \rightarrow \mathcal{M}^\times$  is étale at any point  $\beta \in \mathcal{Q}(\mathbb{F}_p)$  lying over  $\bar{b} = (b_1, \dots, b_{2g}) \in (C^{2g})(\mathbb{F}_p)$  and induces a bijection between  $\mathcal{Q}^\times(\mathbb{Z}/p^2)_{\bar{b}}$  and  $\mathcal{M}^\times(\mathbb{Z}/p^2)_{(\bar{D}, \bar{E})}$ . In particular we can interpret  $s_{D,E}(\lambda, \mu)$  as a section of  $\mathcal{Q}(b_{1,\lambda_1}, \dots, b_{2g,\mu_g})$  and we can interpret (6.6.7) as a parametrization of  $\mathcal{Q}^\times(\mathbb{Z}/p^2)_{\xi s_{D,E}(0,0)}$ . It is then enough to prove that (6.6.7) is the inverse of a bijection given by parameters on  $\mathcal{Q}^\times$ . It comes from the definition of  $c_\nu$  for  $c \in C(\mathbb{Z}/p^2)$  and  $\nu \in \mathbb{F}_p$ , that the maps  $\lambda_i \mu_i: C^{2g}(\mathbb{Z}/p^2)_{\bar{b}}$  are given by parameters in  $\mathcal{O}_{C^{2g}, \bar{b}}$  divided by  $p$ . In order to see that also the coordinate  $\tau: \mathcal{Q}^\times(\mathbb{Z}/p^2)_{\xi s_{D,E}(0)} \rightarrow \mathbb{F}_p$  is given by a parameter divided by  $p$  it is enough to prove that there is an open subset  $U \subset C^{2g}$  containing  $\bar{b}$  and a section  $s$  trivializing  $\mathcal{Q}|_U$  such that  $s_{D,E}(\lambda, \mu) = s(b_{1,\lambda_1}, \dots, b_{2g,\mu_g})$ . Remark 6.3.12 and (6.5.1) give that

$$\begin{aligned}
(6.6.14) \quad & \mathcal{Q} = \bigotimes_{i,j=1}^g \left( (\pi_i, \pi_{g+j})^* \mathcal{O}_{C \times C}(\Delta) \right) \\
& \otimes \bigotimes_{i=1}^g \left( \pi_i^* \mathcal{O}_C(E - (b_{g+1} + \dots + b_{2g})) \otimes \pi_{g+i}^* \mathcal{O}_C(D - (b_1 + \dots + b_g)) \right) \\
& \otimes \text{Norm}_{E/\mathbb{Z}/p^2}(\mathcal{O}_C(D - (b_1 + \dots + b_g))) \otimes \bigotimes_{i=1}^g b_{g+i}^* \mathcal{O}_C(D - (b_1 + \dots + b_g))^{-1}
\end{aligned}$$

where  $\Delta \subset C \times C$  is the diagonal and  $\pi_i$  is the  $i$ -th projection  $C^g \times C^g \rightarrow C$ . We can prove that there is an open subset  $U \subset C^g \times C^g$  containing  $b$  and a section  $s$  trivializing  $\mathcal{Q}|_U$  such that  $s_{D,E}(\lambda, \mu) = s(b_{1,\lambda_1}, \dots, b_{2g,\mu_g})$ , by trivializing each factor of the above tensor product in a neighborhood of  $b$ . Let us see it, for example, for the pieces of the form  $(\pi_i, \pi_{g+j})^* \mathcal{O}_{C \times C}(\Delta)$ . Let  $\pi_1, \pi_2$  be the two projections  $C \times C \rightarrow C$  and let us consider the divisor  $\Delta$ : for each pair of points  $c_1, c_2 \in C(\mathbb{F}_p)$  the invertible  $\mathcal{O}$ -module  $\mathcal{O}_{C \times C}(-\Delta)$  is generated by the section  $x_{\Delta, c_1, c_2} := 1$  in a neighborhood of  $(c_1, c_2)$  if  $c_1 \neq c_2$ , while it is generated by the section  $x_{\Delta, c_1, c_2} := \pi_1^* x_{c_1} - \pi_2^* x_{c_2}$  in a neighborhood of  $(c_1, c_2)$  if  $c_1 = c_2$ . If we now take  $c_1 = b_i, c_2 = b_{g+j} \in C(\mathbb{F}_p)$  we deduce there is a neighborhood  $U$  of  $(b_i, b_{g+j})$  such that  $x_{\Delta, b_i, b_{g+j}}^{-1}$  generates  $\mathcal{O}_{C \times C}(\Delta)|_U$ . For each  $\lambda, \mu \in \mathbb{F}_p^g$  the point  $(b_{i,\lambda_i}, b_{g+j,\mu_j})$  lies in  $U(\mathbb{Z}/p^2)$  and the canonical isomorphism  $(b_{i,\lambda_i}, b_{g+j,\mu_j})^* \mathcal{O}_{C \times C}(\Delta) = b_{g+j,\mu_j}^* \mathcal{O}(b_{i,\lambda_i})$  sends the generating section  $(b_{i,\lambda_i}, b_{j,\mu_j})^* x_{\Delta, c_1, c_2}^{-1}$  to  $b_{j,\mu_j}^* x_i(b_{g+j}, \lambda_i)^{-1}$ , which is a factor in (6.6.6). This gives a section  $s_{i,j}$  trivializing  $\left( (\pi_i, \pi_{g+j})^* \mathcal{O}_{C \times C}(\Delta) \right)$  in a neighborhood of  $b$ . With similar choices we can find sections trivializing the other factors in (6.6.14) in a neighborhood of  $b$  and tensoring all such sections we get a section  $s$  such that  $s_{D,E}(\lambda, \mu) = s(b_{1,\lambda_1}, \dots, b_{2g,\mu_g})$ .



## 6.7 Extension of the Poincaré biextension over Néron models

Let  $C$  over  $\mathbb{Z}$  be a curve as in Section 2. Let  $q$  be a prime number that divides  $n$ . We also write  $C$  for  $C_{\mathbb{Z}_q}$ .

Let  $J$  be the Néron model over  $\mathbb{Z}_q$  of  $\text{Pic}_{C/\mathbb{Q}_q}^0$ , and  $J^0$  its fibre-wise connected component of 0. On  $(J \times_{\mathbb{Z}_q} J)_{\mathbb{Q}_q}$  we have  $\mathcal{M}$  as in Proposition 6.3.2, rigidified at  $0 \times J_{\mathbb{Q}_q}$  and  $J_{\mathbb{Q}_q} \times 0$ . We claim that  $\mathcal{M}$ , with these rigidifications, extends uniquely to  $J \times_{\mathbb{Z}_q} J^0$ . We prove this. First of all,  $J \times_{\mathbb{Z}_q} J^0$  is regular, hence Weil divisors and Cartier divisors are the same, and every invertible  $\mathcal{O}$ -module on  $(J \times_{\mathbb{Z}_q} J^0)_{\mathbb{Q}_q}$  has an extension to an invertible  $\mathcal{O}$ -module on  $J \times_{\mathbb{Z}_q} J^0$ . So let  $\mathcal{M}'$  be an extension of  $\mathcal{M}$ . Any extension  $\mathcal{M}''$  of  $\mathcal{M}$  is then of the form  $\mathcal{M}'(D)$ , with  $D$  a divisor on  $J \times_{\mathbb{Z}_q} J^0$  with support in  $(J \times_{\mathbb{Z}_q} J^0)_{\mathbb{F}_q}$ . Such  $D$  are  $\mathbb{Z}$ -linear combinations of the irreducible components of the  $D_i \times_{\mathbb{F}_q} J_{\mathbb{F}_q}^0$ , where the  $D_i$  are the irreducible components of  $J_{\mathbb{F}_q}$ . Now  $\mathcal{M}'|_{J \times 0}$  extends  $\mathcal{M}|_{J_{\mathbb{Q}_q} \times 0}$ , hence the rigidification of  $\mathcal{M}|_{J_{\mathbb{Q}_q} \times 0}$  is a rational section of  $\mathcal{M}'|_{J \times 0}$  whose divisor is a  $\mathbb{Z}$ -linear combination of the  $D_i$ . It follows that there is exactly one  $D$  as above such that the rigidification of  $\mathcal{M}$  extends to a rigidification of  $\mathcal{M}'(D)$  on  $J \times 0$ . That rigidification is compatible with a unique rigidification of  $\mathcal{M}'(D)$  on  $0 \times J^0$ . In what follows, we denote this extension  $\mathcal{M}'(D)$  of  $\mathcal{M}$  to  $J \times_{\mathbb{Z}_q} J^0$  by  $\mathcal{M}$ . Let us now prove that the  $\mathbb{G}_m$ -torsor  $\mathcal{M}^\times$  on  $J \times_{\mathbb{Z}_q} J^0$  has a unique biextension structure, extending that over  $\mathbb{Q}_q$ . Over  $J \times_{\mathbb{Z}_q} J \times_{\mathbb{Z}_q} J^0$  we have the invertible  $\mathcal{O}$ -modules whose fibres, at a point  $(x, y, z)$  (with values in some  $\mathbb{Z}_q$ -scheme) are  $\mathcal{M}(x + y, z)$  and  $\mathcal{M}(x, z) \otimes \mathcal{M}(y, z)$ . The biextension structure over  $\mathbb{Q}_q$  gives an isomorphism between these, that differs from an isomorphism over  $\mathbb{Z}_q$  by a divisor with support over  $\mathbb{F}_q$ . But the compatibility with the rigidification of  $\mathcal{M}$  over  $J \times_{\mathbb{Z}_q} 0$  proves that this divisor is zero. The other partial group law, and the required properties of them follow in the same way. We have now shown that  $\mathcal{M}^\times$  extends the Poincaré biextension.

## 6.8 Explicit description of the extended Poincaré bundle

Let  $C$  over  $\mathbb{Z}$  be a curve as in Section 2. Let  $q$  be a prime number that divides  $n$ . We also write  $C$  for  $C_{\mathbb{Z}_q}$ . By [7], Corollary 9.1.24,  $C$  is cohomologically flat over  $\mathbb{Z}_q$ , which means that for all  $\mathbb{Z}_q$ -algebras  $A$ ,  $\mathcal{O}(C_A) = A$ . Another reference for this is [10], (6.1.4), (6.1.6) and (7.2.1).

The relative Picard functor  $\text{Pic}_{C/\mathbb{Z}_q}$  sends a  $\mathbb{Z}_q$ -scheme  $T$  to the set of isomorphism classes of  $(\mathcal{L}, \text{rig})$  with  $\mathcal{L}$  an invertible  $\mathcal{O}$ -module on  $C_T$  and  $\text{rig}$  a rigidification at  $b$ . By cohomological flatness, such objects are rigid. But if the action of  $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$  on the set of irreducible components of  $C_{\overline{\mathbb{F}_q}}$  is non-trivial, then  $\text{Pic}_{C/\mathbb{Z}_q}$  is not representable by a  $\mathbb{Z}_q$ -scheme, only by an algebraic space over  $\mathbb{Z}_q$  (see [10], Proposition 5.5). Therefore, to not be annoyed by such inconveniences, we pass to  $S := \text{Spec}(\mathbb{Z}_q^{\text{unr}})$ , the maximal unramified extension of  $\mathbb{Z}_q$ . Then  $\text{Pic}_{C/S}$  is represented by a smooth  $S$ -scheme, and on  $C \times_S \text{Pic}_{C/S}$  there is a universal pair  $(\mathcal{L}^{\text{univ}}, \text{rig})$  ([10], Proposition 5.5, and Section 8.0). We note that  $\text{Pic}_{C/S} \rightarrow S$  is separated if and only if  $C_{\overline{\mathbb{F}_q}}$  is irreducible.

Let  $\text{Pic}_{C/S}^{[0]}$  be the open part of  $\text{Pic}_{C/S}$  where  $\mathcal{L}^{\text{univ}}$  is of total degree zero on the fibres of  $C \rightarrow S$ . It contains the open part  $\text{Pic}_{C/S}^0$  where  $\mathcal{L}^{\text{univ}}$  has degree zero on all irreducible components of  $C_{\overline{\mathbb{F}_q}}$ .

Let  $E$  be the closure of the 0-section of  $\text{Pic}_{C/S}$ , as in [10]. It is contained in  $\text{Pic}_{C/S}^{[0]}$ . By [10],

Proposition 5.2,  $E$  is represented by an  $S$ -group scheme, étale.

By [10], Theorem 8.1.4, or [4], Theorem 9.5.4, the tautological morphism  $\mathrm{Pic}_{C/S}^{[0]} \rightarrow J$  is surjective (for the étale topology) and its kernel is  $E$ , and so  $J = \mathrm{Pic}_{C/S}^{[0]}/E$ . Also, the composition  $\mathrm{Pic}_{C/S}^0 \rightarrow \mathrm{Pic}_{C/S}^{[0]} \rightarrow J$  induces an isomorphism  $\mathrm{Pic}_{C/S}^0 \rightarrow J^0$ .

Let  $C_i$ ,  $i \in I$ , be the irreducible components of  $C_{\mathbb{F}_q}$ . Then, as divisors on  $C$ , we have

$$(6.8.1) \quad C_{\mathbb{F}_q} = \sum_{i \in I} m_i C_i.$$

For  $\mathcal{L}$  an invertible  $\mathcal{O}$ -module on  $C_{\mathbb{F}_q}$ , its multidegree is defined as

$$(6.8.2) \quad \mathrm{mdeg}(\mathcal{L}): I \rightarrow \mathbb{Z}, \quad i \mapsto \deg_{C_i}(\mathcal{L}|_{C_i}),$$

and its total degree is then

$$(6.8.3) \quad \deg(\mathcal{L}) = \sum_{i \in I} m_i \deg_{C_i}(\mathcal{L}|_{C_i}).$$

The multidegree induces a surjective morphism of groups

$$(6.8.4) \quad \mathrm{mdeg}: \mathrm{Pic}_{C/S}(S) \rightarrow \mathbb{Z}^I.$$

Now let  $d \in \mathbb{Z}^I$  be a sufficiently large multidegree so that every invertible  $\mathcal{O}$ -module  $\mathcal{L}$  on  $C_{\mathbb{F}_q}$  with  $\mathrm{mdeg}(\mathcal{L}) = d$  satisfies  $H^1(C_{\mathbb{F}_q}, \mathcal{L}) = 0$  and has a global section whose divisor is finite. Let  $\mathcal{L}_0$  be an invertible  $\mathcal{O}$ -module on  $C$ , rigidified at  $b$ , with  $\mathrm{mdeg}(\mathcal{L}_0) = d$ . Then over  $C \times_S J^0$  we have the invertible  $\mathcal{O}$ -module  $\mathcal{L}^{\mathrm{univ}} \otimes \mathcal{L}_0$ , and its pushforward  $\mathcal{E}$  to  $J^0$ . Then  $\mathcal{E}$  is a locally free  $\mathcal{O}$ -module on  $J^0$ . Let  $E$  be the geometric vector bundle over  $J^0$  corresponding to  $\mathcal{E}$ . Then over  $E$ ,  $\mathcal{E}$  has its universal section. Let  $U \subset E$  be the open subscheme where the divisor of this universal section is finite over  $J^0$ . The  $J^0$ -group scheme  $\mathbb{G}_m$  acts freely on  $U$ . We define  $V := U/\mathbb{G}_m$ . As the  $\mathbb{G}_m$ -action preserves the invertible  $\mathcal{O}$ -module and its rigidification, the morphism  $U \rightarrow J^0$  factors through  $U \rightarrow V$  and gives a morphism  $\Sigma_{\mathcal{L}_0}: V \rightarrow J^0$ . Then on  $C \times_S V$  we have the universal effective relative Cartier divisor  $D^{\mathrm{univ}}$  on  $C \times_S V \rightarrow V$  of multidegree  $d$ , and  $\mathcal{L}^{\mathrm{univ}} \otimes \mathcal{L}_0$  together its rigidification at  $b$  is (uniquely) isomorphic to  $\mathcal{O}_{C \times_S V}(D^{\mathrm{univ}}) \otimes_{\mathcal{O}_V} b^* \mathcal{O}_{C \times_S V}(-D^{\mathrm{univ}})$  with its tautological rigidification at  $b$ , in a diagram:

$$(6.8.5) \quad \mathcal{L}^{\mathrm{univ}} \otimes \mathcal{L}_0 \longequal{\quad} \mathcal{O}_{C \times_S V}(D^{\mathrm{univ}}) \otimes_{\mathcal{O}_V} b^* \mathcal{O}_{C \times_S V}(-D^{\mathrm{univ}}).$$

Then  $\Sigma_{\mathcal{L}_0}$  sends, for  $T$  an  $S$ -scheme, a  $T$ -point  $D$  on  $C_T$  to  $\mathcal{O}_{C_T}(D) \otimes_{\mathcal{O}_T} b^* \mathcal{O}_{C_T}(-D) \otimes_{\mathcal{O}_C} \mathcal{L}_0^{-1}$  with its rigidification at  $b$ . Let  $s_0$  be in  $\mathcal{L}_0(C)$  such that its divisor  $D_0$  is finite over  $S$ , and let  $v_0 \in V(S)$  be the corresponding point.

On  $\mathrm{Pic}_{C/S}^{[0]} \times_S V \times_S C$  we have the universal  $\mathcal{L}^{\mathrm{univ}}$  from  $\mathrm{Pic}_{C/S}^{[0]}$  with rigidification at  $b$ , and the universal divisor  $D^{\mathrm{univ}}$ . Then on  $\mathrm{Pic}_{C/S}^{[0]} \times_S V$  we have the invertible  $\mathcal{O}$ -module  $\mathcal{N}_{q,d}$  whose fibre at a  $T$ -point  $(\mathcal{L}, \mathrm{rig}, D)$  is  $\mathrm{Norm}_{D/T}(\mathcal{L}) \otimes_{\mathcal{O}_T} \mathrm{Norm}_{D_0/T}(\mathcal{L})^{-1}$ , canonically trivial on  $\mathrm{Pic}_{C/S}^{[0]} \times_S v_0$ :

$$(6.8.6) \quad \mathcal{N}_{q,d}: \left( \mathrm{Pic}_{C/S}^{[0]} \times_S V \right) (T) \ni (\mathcal{L}, \mathrm{rig}, D) \longmapsto \mathrm{Norm}_{D/T}(\mathcal{L}) \otimes_{\mathcal{O}_T} \mathrm{Norm}_{D_0/T}(\mathcal{L})^{-1}.$$

Any global regular function on the integral scheme  $\text{Pic}_{C/S}^{[0]} \times_S V$  is constant on the generic fibre, hence in  $\mathbb{Q}_q^{\text{unr}}$ , and restricting it to  $(0, v_0)$  shows that it is in  $\mathbb{Z}_q^{\text{unr}}$ , and if it is 1 on  $\text{Pic}_{C/S}^{[0]} \times_S v_0$ , it is equal to 1. Therefore trivialisations on  $\text{Pic}_{C/S}^{[0]} \times_S v_0$  rigidify invertible  $\mathcal{O}$ -modules on  $\text{Pic}_{C/S}^{[0]} \times_S V$ .

The next proposition generalises [8], Corollary 2.8.6 and Lemma 2.7.11.2: there,  $C \rightarrow S$  is nodal (but not necessarily regular), and the restriction of  $\mathcal{M}$  to  $J^0 \times_S J^0$  is described.

**6.8.7 Proposition** *In the situation of Section 6.8, the pullback of the invertible  $\mathcal{O}$ -module  $\mathcal{M}$  on  $J \times_{\mathbb{Z}_q^{\text{unr}}} J^0$  to  $\text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^{[0]} \times_{\mathbb{Z}_q^{\text{unr}}} V$  by the product of the quotient map  $\text{quot}: \text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^{[0]} \rightarrow J$  and the map  $\Sigma_{\mathcal{L}_0}: V \rightarrow J^0$  is  $\mathcal{N}_{q,d}$ , compatible with their rigidifications at  $J \times 0$  and  $\text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^{[0]} \times v_0$ . In a diagram:*

$$(6.8.8) \quad \begin{array}{ccccc} P^\times & \longleftarrow & \mathcal{M}^\times & \longleftarrow & \mathcal{N}_{q,d}^\times \\ \downarrow & & \downarrow & & \downarrow \\ J \times_{\mathbb{Z}_q^{\text{unr}}} J^{\vee,0} & \xleftarrow{\text{id} \times j_b^{*, -1}} & J \times_{\mathbb{Z}_q^{\text{unr}}} J^0 & \xleftarrow{\text{quot} \times \Sigma_{\mathcal{L}_0}} & \text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^{[0]} \times_{\mathbb{Z}_q^{\text{unr}}} V. \end{array}$$

For  $T$  any  $\mathbb{Z}_q^{\text{unr}}$ -scheme, for  $x$  in  $J(T)$  given by an invertible  $\mathcal{O}$ -module  $\mathcal{L}$  on  $C_T$  rigidified at  $b$ , and  $y$  in  $J^0(T) = \text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^0(T)$  given by the difference  $D = D^+ - D^-$  of effective relative Cartier divisors on  $C_T$  of the same multidegree, we have

$$P(x, j_b^{*, -1}(y)) = \mathcal{M}(x, y) = \text{Norm}_{D^+/T}(\mathcal{L}) \otimes_{\mathcal{O}_T} \text{Norm}_{D^-/T}(\mathcal{L})^{-1}.$$

**Proof** The scheme  $\text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^{[0]} \times_{\mathbb{Z}_q^{\text{unr}}} V$  is smooth over  $\mathbb{Z}_q^{\text{unr}}$ , hence regular, it is connected, hence integral, and since  $V_{\overline{\mathbb{F}}_q}$  is irreducible, the irreducible components of  $(\text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^{[0]} \times_{\mathbb{Z}_q^{\text{unr}}} V)_{\overline{\mathbb{F}}_q}$  are the  $P^i \times_{\overline{\mathbb{F}}_q} V_{\overline{\mathbb{F}}_q}$ , with  $P^i$  the irreducible components of  $(\text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^{[0]})_{\overline{\mathbb{F}}_q}$ , with  $i$  in  $\pi_0((\text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^{[0]})_{\overline{\mathbb{F}}_q})$ , which, by the way, equals the kernel of  $\mathbb{Z}^I \rightarrow \mathbb{Z}$ ,  $x \mapsto \sum_{j \in I} m_j x_j$ .

We prove the first claim. Both  $\mathcal{N}_{q,d}$  and the pullback of  $\mathcal{M}$  are rigidified on  $\text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^{[0]} \times v_0$ . Below we will give, after inverting  $q$ , an isomorphism  $\alpha$  from  $\mathcal{N}_{q,d}$  to the pullback of  $\mathcal{M}$  that is compatible with the rigidifications. Then there is a unique divisor  $D_\alpha$  on  $\text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^{[0]} \times_{\mathbb{Z}_q^{\text{unr}}} V$ , supported on  $(\text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^{[0]} \times_{\mathbb{Z}_q^{\text{unr}}} V)_{\overline{\mathbb{F}}_q}$ , such that  $\alpha$  is an isomorphism from  $\mathcal{N}_{q,d}(D_\alpha)$  to the pullback of  $\mathcal{M}$ . Let  $i$  be in  $\pi_0((\text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^{[0]})_{\overline{\mathbb{F}}_q})$ , and let  $x$  be in  $\text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^{[0]}(\mathbb{Z}_q^{\text{unr}})$  specialising to an  $\overline{\mathbb{F}}_q$ -point of  $P^i$ , then restricting  $\alpha$  to  $(x_i, v_0)$  and using the compatibility of  $\alpha$  (over  $\mathbb{Q}_q^{\text{unr}}$ ) with the rigidifications, gives that the multiplicity of  $P^i \times V_{\overline{\mathbb{F}}_q}$  in  $D_\alpha$  is zero. Hence  $D_\alpha$  is zero.

Let us now give, over  $(\text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^{[0]} \times_{\mathbb{Z}_q^{\text{unr}}} V)_{\mathbb{Q}_q^{\text{unr}}}$ , an isomorphism  $\alpha$  from  $\mathcal{N}_{q,d}$  to the pullback of  $\mathcal{M}$ . Note that  $(\text{Pic}_{C/\mathbb{Z}_q^{\text{unr}}}^{[0]})_{\mathbb{Q}_q^{\text{unr}}} = J_{\mathbb{Q}_q^{\text{unr}}}$ , and that  $V_{\mathbb{Q}_q^{\text{unr}}} = C_{\mathbb{Q}_q^{\text{unr}}}^{(|d|)}$ , where  $|d| = \sum_i m_i d_i$  is the total degree given by the multidegree  $d$ . For  $T$  a  $\mathbb{Q}_q^{\text{unr}}$ -scheme,  $x \in J(T)$  given by  $\mathcal{L}$  an invertible  $\mathcal{O}_{C_T}$ -module rigidified at  $b$ , and  $v \in V(T)$  given by a relative Cartier divisor  $D$  of degree  $|d|$  on  $C_T$ , we have, using Proposition 6.3.2 and (6.8.6), the following isomorphisms (functorial in  $T$ ), respecting the rigidifications at  $v = v_0$ :

$$(6.8.9) \quad \begin{aligned} \mathcal{M}(x, \Sigma_{\mathcal{L}_0}(v)) &= \mathcal{M}(x, \Sigma(v) - \Sigma(v_0)) = \mathcal{M}(x, \Sigma(v)) \otimes \mathcal{M}(x, \Sigma(v_0))^{-1} \\ &= \text{Norm}_{D/T}(\mathcal{L}) \otimes_{\mathcal{O}_T} \text{Norm}_{D_0/T}(\mathcal{L})^{-1} = \mathcal{N}_{q,d}(x, v). \end{aligned}$$

This finishes the proof of the first claim of the Proposition. The second claim follows directly from the definition of  $\mathcal{N}_{q,d}$ , plus the compatibility at the end of Proposition 6.3.2.  $\square$

## 6.9 Integral points of the extended Poincaré torsor

Let  $C$  over  $\mathbb{Z}$  be a curve as in Section 2. Given a point  $(x, y) \in (J \times J^0)(\mathbb{Z})$  we want to describe explicitly the free  $\mathbb{Z}$ -module  $\mathcal{M}(x, y)$  when  $x$  is given by an invertible  $\mathcal{O}$ -module  $\mathcal{L}$  of total degree 0 on  $C$  rigidified at  $b$  and  $y$  is given as a relative Cartier divisor  $D$  on  $C$  of total degree 0 with the property that there exists a unique divisor  $V$  whose support is disjoint from  $b$  and contained in the bad fibres of  $C \rightarrow \text{Spec}(\mathbb{Z})$  such that  $\mathcal{O}(D+V)$  has degree zero when restricted to every irreducible component of any fibre of  $C \rightarrow \text{Spec}(\mathbb{Z})$ . Since  $\mathcal{M}(x, y)$  is a free  $\mathbb{Z}$ -module of rank 1 then it is a submodule of  $\mathcal{M}(x, y)[1/n]$  and writing  $D = D^+ - D^-$  as a difference of relative effective Cartier divisors, Proposition 6.3.2, with  $S = \text{Spec}(\mathbb{Z}[1/n])$ , gives

$$(6.9.1) \quad \mathcal{M}(x, y)[1/n] = (\text{Norm}_{D^+/\mathbb{Z}}(\mathcal{L}) \otimes_{\mathbb{Z}} \text{Norm}_{D^-/\mathbb{Z}}(\mathcal{L})^{-1}) [1/n]$$

and consequently there exist unique integers  $e_q$ , for  $q$  varying among the primes dividing  $n$ , such that, as submodules of  $(\text{Norm}_{D^+/\mathbb{Z}}(\mathcal{L}) \otimes_{\mathbb{Z}} \text{Norm}_{D^-/\mathbb{Z}}(\mathcal{L})^{-1}) [1/n]$ ,

$$(6.9.2) \quad \mathcal{M}(x, y) = \left( \prod_{q|n} q^{e_q} \right) \cdot (\text{Norm}_{D^+/\mathbb{Z}}(\mathcal{L}) \otimes_{\mathbb{Z}} \text{Norm}_{D^-/\mathbb{Z}}(\mathcal{L})^{-1}) .$$

We write  $V = \sum_{q|n} V_q$  where  $V_q$  is a divisor supported on  $C_{\mathbb{F}_q}$ . For every prime  $q$  dividing  $n$  let  $C_{i,q}$ ,  $i \in I_q$  the irreducible components of  $C_{\mathbb{F}_q}$  with multiplicity  $m_{i,q}$  and let  $V_{i,q}$  be the integers so that  $V_q = \sum_{i \in I_q} V_{i,q} C_{i,q}$ .

For every  $q$  dividing  $n$  let  $H_q$  be an effective relative Cartier divisor on  $C_{\mathbb{Z}_q}$  whose complement  $U_q$  is affine (recall that  $C$  is projective over  $\mathbb{Z}$ , take a high degree embedding and a hyperplane section that avoids chosen closed points  $c_{i,q}$  on the  $C_{i,q}$ ). The Chinese remainder theorem, applied to the  $\mathcal{O}_C(U_q)$ -module  $(\mathcal{O}_C(D+V))(U_q)$  and the (distinct) closed points  $c_{i,q}$ , provides an element  $f_q$  of  $(\mathcal{O}_C(D+V))(U_q)$  that generates  $\mathcal{O}_C(D+V)$  at all  $c_{i,q}$ . Let  $D_q = D_q^+ - D_q^-$  be the divisor of  $f_q$  as rational section of  $\mathcal{O}_C(D+V)$ . Then  $D_q^+$  and  $D_q^-$  are finite over  $\mathbb{Z}_q$ , and  $f_q$  is a rational function on  $C_{\mathbb{Z}_q}$  with

$$(6.9.3) \quad \text{div}(f_q) = (D_q^+ - D_q^-) - (D + V) = (D_q^+ + D^-) - (D^+ + D_q^-) - V .$$

This linear equivalence, restricted to  $\mathbb{Q}_q$ , gives the isomorphism (6.4.7)

$$(6.9.4) \quad \varphi: \text{Norm}_{(D^+ + D_q^-)/\mathbb{Q}_q}(\mathcal{L}) \longrightarrow \text{Norm}_{(D_q^+ + D^-)/\mathbb{Q}_q}(\mathcal{L}) .$$

Tensoring with  $\text{Norm}_{(D^- + D_q^-)/\mathbb{Q}_q}(\mathcal{L})^{-1}$  we obtain the isomorphism

$$(6.9.5) \quad \varphi \otimes \text{id}: \text{Norm}_{D^+/\mathbb{Q}_q}(\mathcal{L}) \otimes \text{Norm}_{D^-/\mathbb{Q}_q}(\mathcal{L})^{-1} \longrightarrow \text{Norm}_{D_q^+/\mathbb{Q}_q}(\mathcal{L}) \otimes \text{Norm}_{D_q^-/\mathbb{Q}_q}(\mathcal{L})^{-1}$$

using the identifications

$$(6.9.6) \quad \begin{aligned} \text{Norm}_{D^+/\mathbb{Q}_q}(\mathcal{L}) \otimes \text{Norm}_{D^-/\mathbb{Q}_q}(\mathcal{L})^{-1} &= \text{Norm}_{(D^+ + D_q^-)/\mathbb{Q}_q}(\mathcal{L}) \otimes \text{Norm}_{(D^- + D_q^-)/\mathbb{Q}_q}(\mathcal{L})^{-1} \\ \text{Norm}_{D_q^+/\mathbb{Q}_q}(\mathcal{L}) \otimes \text{Norm}_{D_q^-/\mathbb{Q}_q}(\mathcal{L})^{-1} &= \text{Norm}_{(D_q^+ + D^-)/\mathbb{Q}_q}(\mathcal{L}) \otimes \text{Norm}_{(D^- + D_q^-)/\mathbb{Q}_q}(\mathcal{L})^{-1} . \end{aligned}$$

Using the same method as for getting the rational section  $f_q$  of  $\mathcal{O}_C(D + V)$ , we get a rational section  $l$  of  $\mathcal{L}$  with the support of  $\text{div}(l)$  finite over  $\mathbb{Z}_q$  and disjoint from the supports of  $D$  and  $D_q$ , and from the intersections of different  $C_{i,q}$  and  $C_{j,q}$ . By Proposition 6.8.7, and the choice of  $l$ ,

$$(6.9.7) \quad \mathcal{M}(x, y)_{\mathbb{Z}_q} = \text{Norm}_{D_q^+/\mathbb{Z}_q}(\mathcal{L}) \otimes \text{Norm}_{D_q^-/\mathbb{Z}_q}(\mathcal{L})^{-1} = \mathbb{Z}_q \cdot \text{Norm}_{D_q^+/\mathbb{Z}_q}(l) \otimes \text{Norm}_{D_q^-/\mathbb{Z}_q}(l)^{-1},$$

and

$$(6.9.8) \quad \text{Norm}_{D^+/\mathbb{Z}_q}(\mathcal{L}) \otimes \text{Norm}_{D^-/\mathbb{Z}_q}(\mathcal{L})^{-1} = \mathbb{Z}_q \cdot \text{Norm}_{D^+/\mathbb{Z}_q}(l) \otimes \text{Norm}_{D^-/\mathbb{Z}_q}(l)^{-1}.$$

By (6.4.4), we have

$$(6.9.9) \quad \varphi \otimes \text{id}: \text{Norm}_{D^+/\mathbb{Q}_q}(l) \otimes \text{Norm}_{D^-/\mathbb{Q}_q}(l)^{-1} \mapsto f_q(\text{div}(l))^{-1} \cdot \text{Norm}_{D_q^+/\mathbb{Q}_q}(l) \otimes \text{Norm}_{D_q^-/\mathbb{Q}_q}(l)^{-1}.$$

Comparing with (6.9.2), we conclude that

$$(6.9.10) \quad e_q = v_q(f_q(\text{div}(l))).$$

We write  $\text{div}(l) = \sum_j n_j D_j$  as a sum of prime divisors. These  $D_j$  are finite over  $\mathbb{Z}_q$ , disjoint from the support of the horizontal part of  $\text{div}(f_q)$ , that is of  $D_q - D$ , and each of them meets only one of the  $C_{i,q}$ , say  $C_{s(j),q}$ . Then, for each  $j$ ,  $f_q^{m_{s(j),q}}$  and  $q^{-V_{s(j),q}}$  have the same multiplicity along  $C_{s(j),q}$ , and consequently they differ multiplicatively by a unit on a neighborhood of  $D_j$ . Then we have

$$(6.9.11) \quad \begin{aligned} v_q(f_q(D_j)) &= \frac{v_q(f_q^{m_{s(j),q}}(D_j))}{m_{s(j),q}} = \frac{v_q(q^{-V_{s(j),q}}(D_j))}{m_{s(j),q}} = \frac{v_q(\text{Norm}_{D_j/\mathbb{Z}_q}(q^{-V_{s(j),q}}))}{m_{s(j),q}} \\ &= \frac{-V_{s(j),q} \deg_{\mathbb{Z}_q}(D_j)}{m_{s(j),q}} = \frac{-V_{s(j),q} \cdot (D_j \cdot C_{\mathbb{F}_q})}{m_{s(j),q}} = \frac{-V_{s(j),q} \cdot (D_j \cdot m_{s(j),q} C_{s(j),q})}{m_{s(j),q}} \\ &= -V_{s(j),q}(D_j \cdot C_{s(j)}) = -V_q \cdot D_j. \end{aligned}$$

We get

$$(6.9.12) \quad e_q = v_q(f_q(\text{div}(l))) = -V_q \cdot \text{div}(l) = - \sum_{i \in I_q} V_{i,q}(C_i \cdot \text{div}(l)) = - \sum_{i \in I_q} V_{i,q} \deg_{\mathbb{F}_q}(\mathcal{L}|_{C_{i,q}}).$$

## 7 Description of the map from the curve to the torsor

The situation is as in Section 2, see (2.12). We describe the morphism  $\tilde{j}_b: U \rightarrow T$  in terms of invertible  $\mathcal{O}$ -modules on  $C \times C^{\text{sm}}$ . Since  $T$  is the product, over  $J$ , of the  $\mathbb{G}_m$ -torsors  $T_i := (\text{id}, m \cdot \circ \text{tr}_{c_i} \circ f_i)^* P^\times$  this amounts to describing, for each  $i$ , the morphism  $(\tilde{j}_b)_i: U \rightarrow T_i$ . Note that  $\text{tr}_{c_i} \circ f_i: J_{\mathbb{Q}} \rightarrow J_{\mathbb{Q}}$  is a morphism of groupschemes composed with a translation, and that all morphisms of schemes  $\alpha: J_{\mathbb{Q}} \rightarrow J_{\mathbb{Q}}$  are of this form. From now on we fix one such  $i$  and omit it from our notation.

Let  $\alpha: J_{\mathbb{Q}} \rightarrow J_{\mathbb{Q}}$  be a morphism of schemes, let  $\mathcal{L}_{\alpha}$  be the pullback of  $\mathcal{M}$  (see (6.3.3)) to  $C_{\mathbb{Q}} \times C_{\mathbb{Q}}$  via  $j_b \times (\alpha \circ j_b)$ , and let  $T_{\alpha} := (\text{id}, \alpha)^* \mathcal{M}^{\times}$  on  $J_{\mathbb{Q}}$ :

$$(7.1) \quad \begin{array}{ccccc} & & T_{\alpha} & \xrightarrow{\quad} & \mathcal{M}^{\times} \\ & & \downarrow & & \swarrow \\ C_{\mathbb{Q}} & \xrightarrow{j_b} & J_{\mathbb{Q}} & \xrightarrow{(\text{id}, \alpha)} & (J \times J)_{\mathbb{Q}} \\ \downarrow \text{diag} & & & & \uparrow j_b \times \text{id} \\ (C \times C)_{\mathbb{Q}} & \xrightarrow{\text{id} \times j_b} & (C \times J)_{\mathbb{Q}} & \xrightarrow{\text{id} \times \alpha} & (C \times J)_{\mathbb{Q}} \\ \uparrow & & & & \nwarrow \\ \mathcal{L}_{\alpha}^{\times} & \xrightarrow{\quad} & & & \mathcal{L}^{\text{univ}, \times} \end{array}$$

Then  $(b, \text{id})^* \mathcal{L}_{\alpha} = \mathcal{O}_{C_{\mathbb{Q}}}$ ,  $\mathcal{L}_{\alpha}$  is of degree zero on the fibres of  $\text{pr}_2: (C \times C)_{\mathbb{Q}} \rightarrow C_{\mathbb{Q}}$ , and:  $j_b^* T_{\alpha}$  is trivial if and only if  $\text{diag}^* \mathcal{L}_{\alpha}$  is trivial. Note that diagram (7.1) without the  $\mathbb{G}_m$ -torsors is commutative.

Conversely, let  $\mathcal{L}$  be an invertible  $\mathcal{O}$ -module on  $(C \times C)_{\mathbb{Q}}$ , rigidified on  $\{b\} \times C_{\mathbb{Q}}$ , and of degree 0 on the fibres of  $\text{pr}_2: (C \times C)_{\mathbb{Q}} \rightarrow C_{\mathbb{Q}}$ . The universal property of  $\mathcal{L}^{\text{univ}}$  gives a unique  $\beta_{\mathcal{L}}: C_{\mathbb{Q}} \rightarrow J_{\mathbb{Q}}$  such that  $(\text{id} \times \beta_{\mathcal{L}})^* \mathcal{L}^{\text{univ}} = \mathcal{L}$  (compatible with rigidification at  $b$ ). The Albanese property of  $j_b: C_{\mathbb{Q}} \rightarrow J_{\mathbb{Q}}$  then gives that  $\beta_{\mathcal{L}}$  extends to a unique  $\alpha_{\mathcal{L}}: J_{\mathbb{Q}} \rightarrow J_{\mathbb{Q}}$  such that  $\alpha_{\mathcal{L}} \circ j_b = \beta_{\mathcal{L}}$ . Then  $j_b^* T_{\alpha_{\mathcal{L}}}$  is trivial if and only if  $\text{diag}^* \mathcal{L}$  is trivial. Now assume that  $\text{diag}^* \mathcal{L} = \mathcal{O}_{C_{\mathbb{Q}}}$  (compatible with rigidifications at  $b$ ), and let

$$(7.2) \quad \ell \in (\text{diag}^* \mathcal{L}^{\times})(C_{\mathbb{Q}})$$

correspond to 1. Then  $m \cdot \circ \alpha_{\mathcal{L}}$  extends over  $\mathbb{Z}$  to  $m \cdot \circ \alpha_{\mathcal{L}}: J \rightarrow J^0$ , and the restriction of  $j_b^*(m \cdot \circ \alpha_{\mathcal{L}})^* \mathcal{M}$  on  $C^{\text{sm}}$  to  $U$  is trivial, giving a lift  $\tilde{j}_b$ , unique up to sign:

$$(7.3) \quad \begin{array}{ccccccc} & & T_{m \cdot \circ \alpha_{\mathcal{L}}} & \xrightarrow{\quad} & \mathcal{M}^{\times} & & \\ & \nearrow \tilde{j}_b & \downarrow & & \downarrow & & \\ U & \xrightarrow{\quad} & C^{\text{sm}} & \xrightarrow{j_b} & J & \xrightarrow{(\text{id}, m \cdot \circ \alpha_{\mathcal{L}})} & J \times J^0 \end{array}$$

The invertible  $\mathcal{O}$ -module  $\mathcal{L}$  on  $(C \times C)_{\mathbb{Q}}$  with its rigidification of  $(b, \text{id})^* \mathcal{L}$ , extends uniquely to an invertible  $\mathcal{O}$ -module on  $(C \times C)_{\mathbb{Z}[1/n]}$ , still denoted  $\mathcal{L}$ . For  $S$  a  $\mathbb{Z}[1/n]$ -scheme,  $d$  and  $e$  in  $\mathbb{Z}_{\geq 0}$ ,  $D \in C^{(d)}(S)$  and  $E \in C^{(e)}(S)$ , we claim that

$$(7.4) \quad \mathcal{M}(\Sigma(D), \alpha_{\mathcal{L}}(\Sigma(E))) = (\text{Norm}_{D/S}(\text{id}, b)^* \mathcal{L})^{\otimes (1-e)} \otimes \text{Norm}_{(D \times E)/S}(\mathcal{L}).$$

To prove this, we may and do assume (finite locally free base change on  $S$ ) that we have  $x_i$  and  $y_j$  in  $C(S)$ , such that  $D = \sum_i x_i$  and  $E = \sum_j y_j$ . Recall that, for  $c \in C(S)$ ,  $\beta_{\mathcal{L}}(c)$  in  $J(S)$  is  $(\text{id}, c)^* \mathcal{L}$  on  $C_S$ , with its rigidification at  $b$ . Then we have:

$$(7.4.1) \quad \begin{aligned} \mathcal{M}(\Sigma(D), \alpha_{\mathcal{L}}(\Sigma(E))) &= \mathcal{M}(\alpha_{\mathcal{L}}(\Sigma(E)), \Sigma(D)) \\ &= \mathcal{M}\left(\beta_{\mathcal{L}}(b) + \sum_j (\beta_{\mathcal{L}}(y_j) - \beta_{\mathcal{L}}(b)), \sum_i j_b(x_i)\right) \\ &= \left(\bigotimes_i \mathcal{L}(x_i, b)^{\otimes (1-e)}\right) \otimes \bigotimes_{i,j} \mathcal{L}(x_i, y_j). \end{aligned}$$

from which the desired equality follows.

For  $S$  a  $\mathbb{Z}[1/n]$ -scheme and  $x \in C(S)$ , applying (7.4) with  $D = E = x$  gives:

$$(7.5) \quad T_{m \circ \alpha_{\mathcal{L}}}(j_b(x)) = \mathcal{M}^\times(j_b(x), m \cdot \alpha_{\mathcal{L}}(j_b(x))) = \mathcal{L}^{\otimes m}(x, x)^\times = (\mathbb{G}_m)_S,$$

with the last equality coming from the rigidification at  $b$ .

Now let  $\mathcal{L}$  be any extension of  $\mathcal{L}$  with its rigidification of  $(b, \text{id})^*\mathcal{L}$  from  $(C \times C)_{\mathbb{Z}[1/n]}$  to  $C \times U$ . For  $q$  dividing  $n$ , let  $W_q$  be the valuation along  $U_{\mathbb{F}_q}$  of the rational section  $\ell$  of  $\text{diag}^*\mathcal{L}$  on  $U$ . Then  $\ell$ , multiplied by the product, over the primes  $q$  dividing  $n$ , of  $q^{-W_q}$ , generates  $\text{diag}^*\mathcal{L}$  on  $U$ :

$$(7.6) \quad \left( \prod_{q|n} q^{-W_q} \right) \cdot \ell \in (\text{diag}^*\mathcal{L}^\times)(U).$$

There is a unique divisor  $V$  on  $C \times U$  with support disjoint from  $(b, \text{id})U$  and contained in the  $(C \times U)_{\mathbb{F}_q}$  with  $q$  dividing  $n$ , such that

$$(7.7) \quad \mathcal{L}^m := \mathcal{L}^{\otimes m}(V) \quad \text{on } C \times U$$

has multidegree 0 on the fibres of  $\text{pr}_2: C \times U \rightarrow U$ . Then  $\mathcal{L}^m$  is the pullback of  $\mathcal{L}^{\text{univ}}$  via  $\text{id} \times (m \cdot \circ \alpha_{\mathcal{L}} \circ j_b): C \times U \rightarrow C \times J^0$ . Its restriction  $\mathcal{L}^m|_{C^{\text{sm}} \times U}$  is then the pullback of  $\mathcal{M}$  via  $j_b \times (m \cdot \circ \alpha_{\mathcal{L}} \circ j_b): C^{\text{sm}} \times U \rightarrow J \times J^0$ , because on  $C^{\text{sm}} \times J^0$  the restriction of  $\mathcal{L}^{\text{univ}}$  and  $(j_b \times \text{id})^*\mathcal{M}$  are equal (both are rigidified after  $(b, \text{id})^*$  and equal over  $\mathbb{Z}[1/n]$ ; here we use that, for all  $q|n$ ,  $J_{\mathbb{F}_q}^0$  is geometrically connected). Hence, on  $U$  we have  $j_b^*T_{m \circ \alpha_{\mathcal{L}}} = \text{diag}^*(\mathcal{L}^{\otimes m}(V)^\times)$ , compatible with rigidifications at  $b \in U(\mathbb{Z}[1/n])$ . Our trivialisation  $\tilde{j}_b$  on  $U$  of  $T_{m \circ \alpha_{\mathcal{L}}}$  is therefore a generating section of  $\mathcal{L}^{\otimes m}$ , multiplied by the product over the  $q$  dividing  $n$ , of the factors  $q^{-V_q}$ , where  $V_q$  is the multiplicity in  $V$  of the prime divisor  $(U \times U)_{\mathbb{F}_q}$ . Then, for  $x$  and  $S$  as in (7.5), we have the following description of  $\tilde{j}_b$ :

$$(7.8) \quad \tilde{j}_b(x) = \left( \prod_{q|n} q^{-mW_q - V_q} \right) \cdot \ell^{\otimes m} \quad \text{in } (T_{m \circ \alpha_{\mathcal{L}}}(j_b(x)))(S) = \mathcal{L}^{\otimes m}(x, x)^\times(S).$$

## 8 An example with genus 2, rank 2, and 14 points

Let  $C_0$  be the curve over  $\mathbb{Z}$  obtained from the following closed subschemes of  $\mathbb{A}_{\mathbb{Z}}^2$

$$\begin{aligned} V_1: \quad & y^2 + y = x^6 - 3x^5 + x^4 + 3x^3 - x^2 - x, \\ V_2: \quad & w^2 + z^3w = 1 - 3z + z^2 + 3z^3 - z^4 - z^5 \end{aligned}$$

by glueing the open subset of  $V_1$  where  $x$  is invertible with the open subset of  $V_2$  where  $z$  is invertible using the identifications  $z = 1/x$ ,  $w = y/x^3$ . The scheme  $C_0$  can be also described as a subscheme of the line bundle  $\mathcal{L}_3$  associated to the invertible  $\mathcal{O}$ -module  $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^1}(3)$  on  $\mathbb{P}_{\mathbb{Z}}^1$  with homogeneous coordinates  $X, Z$ : the map  $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^1}(3) \rightarrow \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^1}(6)$  sending a section  $Y$  to  $Y \otimes Y + Z^3 \otimes Y$  induces a map  $\varphi$  from  $\mathcal{L}_3$  to the line bundle  $\mathcal{L}_6$  associated to  $\mathcal{O}(6)$ ; then  $C_0$  is isomorphic to the inverse image by  $\varphi$  of the section  $s := X^6 - 3X^5Z + X^4Z^2 + 3X^3Z^3 - X^2Z^4 - XZ^5$  of  $\mathcal{L}_6$  and

since the map  $\varphi$  is finite of degree 2 then  $C_0$  is finite of degree 2 over  $\mathbb{P}_{\mathbb{Z}}^1$ . Hence  $C_0$  is proper over  $\mathbb{Z}$  and it is moreover smooth over  $\mathbb{Z}[1/n]$  with  $n = 3 \cdot 43$ . The generic fiber of  $C_0$  is a curve of genus  $g = 2$ , labeled 5547.b.16641.1 on [www.lmfdb.org](http://www.lmfdb.org). The only point where  $C_0$  is not regular is the point  $P_0 = (3, x - 2, y - 1)$  contained in  $V_1$  and the blow up  $C$  of  $C_0$  in  $P_0$  is regular.

In the rest of the article we apply our geometric method to the curve  $C$  and we prove that  $C(\mathbb{Z})$  contains exactly 14 elements. We use the same notation as in Sections 2 and 4.

The fiber  $C_{\mathbb{F}_{43}}$  is absolutely irreducible while  $C_{\mathbb{F}_3}$  is the union of two geometrically irreducible curves, a curve of genus 0 that lies above the point  $P_0$  and that we call  $K_0$ , and a curve of genus 1 that we call  $K_1$ . We define  $U_0 := C \setminus K_1$  and  $U_1 := C \setminus K_0$  so that  $C(\mathbb{Z}) = C^{\text{sm}}(\mathbb{Z}) = U_0(\mathbb{Z}) \cup U_1(\mathbb{Z})$  and both  $U_0$  and  $U_1$  satisfy the hypothesis of  $U$  in Section 2. We have  $K_0 \cdot K_1 = 2$  and consequently the self-intersections of  $K_0$  and  $K_1$  are both equal to  $-2$ . We deduce that all the fibers of  $J$  over  $\mathbb{Z}$  are connected except for  $J_{\mathbb{F}_3}$  which has group of connected components equal to  $\mathbb{Z}/2\mathbb{Z}$ . Hence  $m = 2$ .

The automorphism group of  $C$  is isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^2$ , generated by the automorphisms  $\iota$  and  $\eta$  lifting the extension to  $C_0$  of

$$\iota, \eta: V_1 \longrightarrow V_1, \quad \iota: (x, y) \longmapsto (x, -1 - y), \quad \eta: (x, y) \longmapsto (1 - x, -1 - y).$$

The quotients  $E_1 := C_{\mathbb{Q}}/\eta$  and  $E_2 := C_{\mathbb{Q}}/(\iota \circ \eta)$  are curves of genus 1 and the two projections  $C \rightarrow E_i$  induce an isogeny  $J \rightarrow \text{Pic}^0(E_1) \times \text{Pic}^0(E_2)$ . The elliptic curves  $\text{Pic}^0(E_i)$  are not isogenous and  $\rho = 2$ .

## 8.1 The torsor on the jacobian

Let  $\infty, \infty_- \in C(\mathbb{Z})$  be the lifts of  $(0, 1), (0, -1) \in V_2(\mathbb{Z}) \subset C_0(\mathbb{Z})$  and let us fix the base point  $b = \infty$  in  $C(\mathbb{Z})$ . Following Section 7 we describe a  $\mathbb{G}_m$ -torsor  $T \rightarrow J$  and maps  $\widetilde{j_{b,i}}: U_i \rightarrow T$  using invertible  $\mathcal{O}$ -modules on  $C \times C^{\text{sm}}$ . Let  $\Gamma_{\eta}$  denote the graph of the map  $\eta: C \rightarrow C$ . Over  $C_{\mathbb{Q}} \times C_{\mathbb{Q}}$  we consider

$$\mathcal{L} := \mathcal{O}_{C_{\mathbb{Q}} \times C_{\mathbb{Q}}}(\Gamma_{\eta, \mathbb{Q}} - \infty_- \times C_{\mathbb{Q}} - C_{\mathbb{Q}} \times \infty)$$

trivialised on  $b \times C_{\mathbb{Q}}$  through the section

$$l_b := 2 \quad \text{in } ((b, \text{id})^* \mathcal{L})(C_{\mathbb{Q}}) = \mathcal{O}_{C_{\mathbb{Q}}}(\eta(b) - b)(C_{\mathbb{Q}}) = \mathcal{O}_{C_{\mathbb{Q}}}(C_{\mathbb{Q}}).$$

Notice that  $(\text{id}, b)^* \mathcal{L}$  has degree 0 hence there exists a morphism of schemes  $\alpha_{\mathcal{L}}: J_{\mathbb{Q}} \rightarrow J_{\mathbb{Q}}$  such that  $\mathcal{L}$ , with its rigidification at  $b \times C_{\mathbb{Q}}$ , is the pull back of  $\mathcal{M}$  via  $j_b \times (\alpha_{\mathcal{L}} \circ j_b)$ . For every  $\overline{\mathbb{Q}}$ -point  $Q$  on  $C_{\mathbb{Q}}$  the invertible  $\mathcal{O}_{C_{\overline{\mathbb{Q}}}}$ -module  $(\text{id}, Q)^* \mathcal{L}$  is isomorphic to  $\mathcal{O}_{C_{\overline{\mathbb{Q}}}}(\eta(Q) - \infty_-)$  hence

$$\alpha = \text{tr}_c \circ f, \quad \text{with } f = \eta_* \text{ and } c = [D_0], D_0 := \infty - \infty_-.$$

When restricted to the diagonal  $\mathcal{L}$  is trivial since, compatibly with the trivialisation at  $(b, b)$ ,

$$\text{diag}^* \mathcal{L} = \mathcal{O}_{C_{\mathbb{Q}}}(\infty_- + \infty - \infty_- - \infty) = \mathcal{O}_{C_{\mathbb{Q}}} = l \cdot \mathcal{O}_{C_{\mathbb{Q}}} \quad \text{with } l := 1.$$

Following Section 7 we choose the extension of  $\mathcal{L}$  over  $C \times C^{\text{sm}}$

$$\mathcal{L} := \mathcal{O}(\Gamma_{\eta}|_{C \times C^{\text{sm}}} - \infty_- \times C^{\text{sm}} - C \times \infty)$$



trivialised along  $b \times C^{\text{sm}}$  through the section  $l_b = 2$  (the points  $\infty_-$  and  $b$  have a simple intersection over the prime 2). Following (7.5) the torsor  $T := T_{m \cdot \alpha_{\mathcal{L}}}$  on  $J$  satisfies, for  $S$  a  $\mathbb{Z}[1/n]$ -scheme and  $x$  in  $C(S)$ , using the trivialisation given by  $l$  and  $l_b$

$$\begin{aligned}
(8.1.1) \quad T(j_b(x)) &= \mathcal{M}^\times(j_b(x), m \cdot \alpha_{\mathcal{L}}(j_b(x))) = \mathcal{M}^\times(j_b(x), (\text{id}, x)^* \mathcal{L}^{\otimes m}) \\
&= x^*(\text{id}, x)^* \mathcal{L}^{\otimes m, \times} \otimes b^*(\text{id}, x)^* \mathcal{L}^{\otimes -m, \times} \\
&= \mathcal{L}^{\otimes m, \times}(x, x) \otimes \mathcal{L}^{\otimes m, \times}(b, x)^{-1} = \mathcal{L}^{\otimes m, \times}(x, x) = \mathcal{O}_S^\times.
\end{aligned}$$

Since  $l$  generates  $\text{diag}^*(\mathcal{L})$  on the whole  $C^{\text{sm}}$  then, we have  $W_3 = W_{43} = 0$  in (7.6) when computing both  $\widetilde{j_{b,0}}$  and  $\widetilde{j_{b,1}}$ .

The invertible  $\mathcal{O}$ -module  $\mathcal{L}^{\otimes m}$  has multidegree 0 over all the fibers  $C \times U_1 \rightarrow U_1$ , hence in order to compute  $\widetilde{j_{b,1}}$  we must take  $V = 0$  in (7.7), giving  $V_3 = V_{43} = 0$ . Hence for  $S$  and  $x$  as in (8.1.1), assuming moreover that 2 is invertible on  $S$ ,

$$(8.1.2) \quad \widetilde{j_{b,1}}(x) = l^2 \otimes l_b^{-2} = \frac{1}{4}(x^*1) \otimes (b^*1)^{-1} \quad \text{in}$$

$$T(j_b(x)) = x^*(\text{id}, x)^* \mathcal{L}^{\otimes m, \times} \otimes b^*(\text{id}, x)^* \mathcal{L}^{\otimes -m, \times} = x^* \mathcal{O}_{C_S}(\eta(x) - \infty_-)^\times \otimes b^* \mathcal{O}_{C_S}(\eta(x) - \infty_-)^\times,$$

where the last equality in (8.1.2) makes sense if the image of  $x$  is disjoint from  $\infty, \infty_-$  in  $C_S$ .

The restriction  $\mathcal{L}^{\otimes m}$  to  $C \times U_0$  has multidegree 0 over all the fibers  $C \times U_0 \rightarrow U_0$  of characteristic not 3, while if we consider a fiber of characteristic 3 it has degree 2 over  $K_0$  and degree  $-2$  over  $K_1$ . Hence for computing  $\widetilde{j_{b,0}}$  we take  $V = K_0 \times (K_0 \cap U_0)$  in (7.7) giving  $V_{43} = 0, V_3 = 1$ . Hence for  $S$  and  $x$  as in (8.1.1), assuming moreover that 2 is invertible on  $S$ ,

$$(8.1.3) \quad \widetilde{j_{b,0}}(x) = \frac{1}{3}l^2 \otimes l_b^{-2} = \frac{1}{12}(x^*1) \otimes (b^*1)^{-1} \quad \text{in}$$

$$T(j_b(x)) = x^*(\text{id}, x)^* \mathcal{L}^{\otimes m, \times} \otimes b^*(\text{id}, x)^* \mathcal{L}^{\otimes -m, \times} = x^* \mathcal{O}_{C_S}(\eta(x) - \infty_-)^\times \otimes b^* \mathcal{O}_{C_S}(\eta(x) - \infty_-)^\times,$$

where the last equality in (8.1.3) makes sense if the image of  $x$  is disjoint from  $\infty, \infty_-$  in  $C_S$ .

## 8.2 Some integral points on the biextension

On  $C_0$  we have the following integral points that lift uniquely to elements of  $C(\mathbb{Z})$

$$\begin{aligned}
\infty &= (0, 1), \quad \infty_- := (0, -1) \quad \text{in } V_2(\mathbb{Z}), \\
\alpha &:= (1, 0), \quad \beta := \eta(\alpha) = (0, -1), \quad \gamma := (2, 1), \quad \delta := \eta(\gamma) = (-1, -2) \quad \text{in } V_1(\mathbb{Z}).
\end{aligned}$$

Computations in Magma confirm that  $J(\mathbb{Z})$  is a free  $\mathbb{Z}$ -module of rank  $r = 2$  generated by

$$G_1 := \gamma - \alpha, \quad G_2 := \alpha + \infty_- - 2\infty.$$

The points in  $T(\mathbb{Z})$  are a subset of points of  $\mathcal{M}^\times(\mathbb{Z})$  that can be constructed, using the two group laws, from the points in  $\mathcal{M}^\times(G_i, m \cdot f(G_j))(\mathbb{Z})$  and  $\mathcal{M}^\times(G_i, m \cdot D_0)(\mathbb{Z})$  for  $i, j \in \{1, 2\}$ . Let us compute in detail  $\mathcal{M}^\times(G_1, m \cdot f(G_1))(\mathbb{Z})$ . As explained in Section 6.9

$$\begin{aligned}
\mathcal{M}(G_1, m \cdot f(G_1))^\times &= \mathcal{M}^\times(\gamma - \alpha, 2\delta - 2\beta) \\
&= 3^{e_3} 43^{e_{43}} \cdot \text{Norm}_{(2\delta)/\mathbb{Z}}(\mathcal{O}_C(\gamma - \alpha)) \otimes \text{Norm}_{(2\beta)/\mathbb{Z}}(\mathcal{O}(\gamma - \alpha))^{-1} \\
&= 3^{e_3} 43^{e_{43}} \cdot (2\delta - 2\beta)^* \mathcal{O}_C(\gamma - \alpha)
\end{aligned}$$

where, given a scheme  $S$ , an invertible  $\mathcal{O}$ -module  $\mathcal{L}$  on  $C_S$  and a divisor  $D_+ - D_- = \sum_i n_i P_i$  on  $C_S$  that is sum of  $S$ -points, we define the invertible  $\mathcal{O}_S$ -module

$$\left( \sum_i n_i P_i \right)^* \mathcal{L} := \bigotimes_i P_i^* \mathcal{L}^{n_i} = \text{Norm}_{D_+/S}(\mathcal{L}) \otimes \text{Norm}_{D_-/S}(\mathcal{L})^{-1}.$$

Since  $C_{\mathbb{F}_{43}}$  is irreducible then  $2f(G_1)$  has already multidegree 0 over 43, hence  $e_{43} = 0$ . If we look at  $C_{\mathbb{F}_3}$  then  $2f(G_1)$  does not have multidegree 0, while  $2f(G_1) + K_0$  has multidegree 0; hence by (6.9.12)

$$e_3 = -\deg_{\mathbb{F}_3} \mathcal{O}_C(\gamma - \alpha)|_{K_0} = -1.$$

Notice that over  $\mathbb{Z}[\frac{1}{2}]$  the divisor  $G_1$  is disjoint from  $\beta$  and  $\delta$  (to see that it is disjoint from  $\delta = (-1, -2, 1)$  over the prime 3 one needs to look at local equations of the blow up) thus  $\beta^* \mathcal{O}_C(\gamma - \alpha)$  and  $\delta^* \mathcal{O}_C(\gamma - \alpha)$  are generated by  $\beta^* 1$  and  $\delta^* 1$  over  $\mathbb{Z}[\frac{1}{2}]$ . Thus there are integers  $e_\beta, e_\delta$  such that  $\beta^* \mathcal{O}_C(\gamma - \alpha)$  and  $\delta^* \mathcal{O}_C(\gamma - \alpha)$  are generated by  $\beta^* 2^{e_\beta}$  and  $\delta^* 2^{e_\delta}$  over  $\mathbb{Z}$ . Looking at the intersections between  $\beta, \gamma, \alpha$  and  $\delta$  we compute that  $e_\beta = -1$   $e_\delta = 1$  hence

$$\begin{aligned} \mathcal{M}(G_1, m \cdot f(G_1)) &= 3^{-1} \cdot (\delta^* 2)^2 \otimes (\beta^* 2^{-1})^{-2} \cdot \mathbb{Z} = 2^4 \cdot 3^{-1} \cdot (\delta^* 1)^2 \otimes (\beta^* 1) \cdot \mathbb{Z} \quad \text{and} \\ Q_{1,1} &:= \pm 2^4 \cdot 3^{-1} \cdot (\delta^* 1)^2 \otimes (\beta^* 1)^{-2} \in \mathcal{M}_{G_1, m \cdot f(G_1)}^\times(\mathbb{Z}). \end{aligned}$$

With analogous computations we see that

$$\begin{aligned} Q_{2,1} &:= 2^{-2} \cdot (\delta^* 1)^2 \otimes (\beta^* 1)^{-2} && \text{generates } \mathcal{M}_{G_2, m \cdot f(G_1)} \\ Q_{1,2} &:= 2^{-2} \cdot (\beta^* 1)^2 \otimes (\infty^* 1)^2 \otimes (\infty^* 1)^{-4} && \text{generates } \mathcal{M}_{G_1, m \cdot f(G_2)} \\ Q_{2,2} &:= 2^{18} \cdot (\beta^* 1)^2 \otimes (\infty^* x)^2 \otimes (\infty^* z^2)^{-4} && \text{generates } \mathcal{M}_{G_2, m \cdot f(G_2)} \\ Q_{1,2} &:= (\infty^* 1)^2 \otimes (\infty^* 1)^{-2} && \text{generates } \mathcal{M}_{G_1, m \cdot D_0} \\ Q_{2,0} &:= 2^{-12} \cdot (\infty^* z^2)^2 \otimes (\infty^* x)^{-2} && \text{generates } \mathcal{M}_{G_2, m \cdot D_0}. \end{aligned}$$

### 8.3 Some residue disks of the biextension

Let  $p$  be a prime of good reduction for  $C$ . Given the divisors

$$D := \alpha - \infty, \quad E := 2\beta - 2\infty_- = (m \cdot \circ \text{tr}_c \circ \eta_*)(D) \quad \text{in } \text{Div}(C_{\mathbb{Z}/p^2})$$

we apply Section 6.6 and we give parameters on the residue disks in  $\mathcal{M}^\times(\mathbb{Z}/p^2)_{\overline{D}, \overline{E}}$  and  $T(\mathbb{Z}/p^2)_{\overline{D}}$ , with  $\overline{D}, \overline{E}$  the images of  $D, E$  in  $\text{Div}(C_{\mathbb{F}_p})$ .

We choose the “base points”  $b_1 = \alpha, b_2 = \infty, b_3 = \beta, b_4 = \infty$ , so that  $b_1 \neq b_2, b_3 \neq b_4$  and  $h^0(C_{\mathbb{F}_p}, b_1 + b_2) = h^0(C_{\mathbb{F}_p}, b_3 + b_4) = 1$ . As in Section 6.6 we define  $x_\alpha = x - 1, x_\infty = z, x_\beta = x$  and  $x_{D, \beta} = x_{D, \infty_-} = 1, x_{D, \infty} = z^{-1}$ . For  $Q$  in  $\{\infty, \beta, \alpha\}$  and  $a \in \mathbb{F}_p$  let  $Q_a$  be the unique  $\mathbb{Z}/p^2$ -point of  $C$  that is congruent to  $Q$  modulo  $p$  and such that  $x_Q(Q_a) = ap \in \mathbb{Z}/p^2$ . We have the bijections

$$\begin{aligned} \mathbb{F}_p^2 &\longrightarrow J(\mathbb{Z}/p^2)_{\overline{D}}, \quad \lambda \longmapsto D_\lambda := D + \alpha_{\lambda_1} - \alpha + \infty_{\lambda_2} - \infty = \alpha_{\lambda_1} + \infty_{\lambda_2} - 2\infty \\ \mathbb{F}_p^2 &\longrightarrow J(\mathbb{Z}/p^2)_{\overline{E}}, \quad \mu \longmapsto E_\mu := E + \beta_{\mu_1} - \beta + \infty_{\mu_2} - \infty = \beta + \beta_{\mu_1} + \infty_{\mu_2} - \infty - 2\infty_-. \end{aligned}$$

Following (6.6.6) for  $\lambda, \mu \in \mathbb{F}_p^2$  we define

$$s_{D,E}(\lambda, \mu) := (\beta^* 1) \otimes (\beta_{\mu_1}^* 1) \otimes (\infty_{\mu_2}^* \frac{z^2}{z - \lambda_2 p}) \otimes (\infty^* \frac{z^2}{z - \lambda_2 p})^{-1} \otimes (\infty_-^* 1)^{-2}$$

that, by Proposition 6.3.2 and Remark 6.3.12, generates  $E_\mu^* \mathcal{O}_{C_{\mathbb{Z}/p^2}}(D_\lambda) = \mathcal{M}_{D_\lambda, E_\mu}$ . The points in  $\mathcal{M}^\times(\mathbb{F}_p)$  projecting to  $(\overline{D}, \overline{E})$  are in bijection with the elements  $\xi$  in  $\mathbb{F}_p^\times$  and are exactly the points  $\xi \cdot s_{D,E}(0, 0)$ . Using  $(\mathbb{Z}/p^2)^\times = \mathbb{F}_p^\times \times (1+p\mathbb{F}_p)$ , for each  $\xi \in \mathbb{F}_p^\times$  we parametrise the residue disk of  $\xi \cdot s_{D,E}(0, 0)$  using bijection (6.6.7)

$$\mathbb{F}_p^5 \longrightarrow \mathcal{M}^\times(\mathbb{Z}/p^2)_{\xi \cdot s_{D,E}(0,0)}, \quad (\lambda_1, \lambda_2, \mu_1, \mu_2, \tau) \longmapsto (1 + p\tau)\xi \cdot s_{D,E}((\lambda_1, \lambda_2), (\mu_1, \mu_2)).$$

Since  $(m \cdot \circ \text{tr}_c \circ f)(D_\lambda) = E_{-2\lambda}$  then we have

$$T(\mathbb{Z}/p^2)_{\overline{D}} = \bigcup_{\lambda \in \mathbb{F}_p^2} T_{D_\lambda}(\mathbb{Z}/p^2) = \bigcup_{\lambda \in \mathbb{F}_p^2} \mathcal{M}_{D_\lambda, E_{-2\lambda}}^\times(\mathbb{Z}/p^2).$$

As  $\xi$  varies in  $\mathbb{F}_p^\times$  the point  $\xi \cdot s_{D,E}(0, 0)$  varies in all the points in  $\mathcal{M}^\times(\mathbb{F}_p)$  projecting to  $(\overline{D}, \overline{E})$  and we have the following bijection induced by parameters in  $\xi \cdot s_{D,E}(0, 0)$

$$(8.3.1) \quad \mathbb{F}_p^3 \longrightarrow T(\mathbb{Z}_p)_{\xi s_{D,E}(0,0)}, \quad (\lambda_1, \lambda_2, \tau) \longmapsto (1 + \tau p) \cdot \xi \cdot s_{D,E}((\lambda_1, \lambda_2), (-2\lambda_1, -2\lambda_2)).$$

If we apply (8.1.2) and (8.1.3) to  $Q = \alpha_\lambda$  and we use the symmetry of the Poincaré torsor explained in Proposition 6.3.2 and made explicit in Section 6.5 we obtain the following description of  $\widetilde{j_{b,i}}$  on  $C(\mathbb{Z}/p^2)_{\alpha_{\mathbb{F}_p}}$  when  $p \neq 2$

$$\widetilde{j_{b,1}}(\alpha_\lambda) = (1/4) \cdot s_{D,E}((\lambda, 0), (-2\lambda, 0)), \quad \widetilde{j_{b,0}}(Q) = (1/12) \cdot s_{D,E}((\lambda, 0), (-2\lambda, 0)).$$

If  $p = 5$  then 18 and  $-1$  are  $(p-1)$ -th roots of unity in  $(\mathbb{Z}/p^2)^\times$ , thus  $1/4 = (-1)(1+p)$  and  $1/12 = 3(1+2p)$  in  $(\mathbb{Z}/p^2)^\times = \mathbb{F}_p^\times \times (1+p\mathbb{F}_p)$ , hence

$$(8.3.2) \quad \widetilde{j_{b,1}}(\alpha_\lambda) = -(1+p) \cdot s_{D,E}((\lambda, 0), (-2\lambda, 0)), \quad \widetilde{j_{b,0}}(Q) = 3 \cdot (1+2p) \cdot s_{D,E}((\lambda, 0), (-2\lambda, 0)).$$

Since it is useful for computing the map  $\kappa_{\mathbb{Z}}$  in the residue disks of  $T(\mathbb{Z}/p^2)$  projecting to  $\overline{D}$ , let us apply the recipe in Section 6.6 also to the residue disks of  $\mathcal{M}^\times(\mathbb{Z}/p^2)$  lying over  $(\overline{D}, 0)$ ,  $(0, \overline{E})$  and  $(0, 0)$ . Hence for  $\lambda, \mu \in \mathbb{F}_p^2$  we define the divisors on  $C_{\mathbb{Z}/p^2}$

$$D_\lambda^0 := \alpha_{\lambda_1} - \alpha + \infty_{\lambda_2} - \infty, \quad E_\mu^0 := \beta_{\mu_1} - \beta + \infty_{\mu_2} - \infty$$

and the sections

$$\begin{aligned} s_{D,0}(\lambda, \mu) &:= (\beta_{\mu_1}^* 1) \otimes (\infty_{\mu_2}^* \frac{z^2}{z - \lambda_2 p}) \otimes (\beta^* 1)^{-1} \otimes (\infty^* \frac{z^2}{z - \lambda_2 p})^{-1} && \text{in } \mathcal{M}^\times(D_\lambda, E_\mu^0)(\mathbb{Z}/p^2) \\ s_{0,E}(\lambda, \mu) &:= (\beta^* 1) \otimes (\beta_{\mu_1}^* 1) \otimes (\infty_{\mu_2}^* \frac{z}{z - \lambda_2 p}) \otimes (\infty^* \frac{z}{z - \lambda_2 p})^{-1} \otimes (\infty_-^* 1)^{-2} && \text{in } \mathcal{M}^\times(D_\lambda^0, E_\mu)(\mathbb{Z}/p^2) \\ s_{0,0}(\lambda, \mu) &:= (\beta_{\mu_1}^* 1) \otimes (\infty_{\mu_2}^* \frac{z}{z - \lambda_2 p}) \otimes (\beta^* 1)^{-1} \otimes (\infty^* \frac{z}{z - \lambda_2 p})^{-1} && \text{in } \mathcal{M}^\times(D_\lambda^0, E_\mu^0)(\mathbb{Z}/p^2). \end{aligned}$$

## 8.4 Geometry mod $p$ of integral points

From now on  $p = 5$ . Let  $\overline{\alpha} \in C(\mathbb{Z}/p^2)$  be the image of  $\alpha \in C(\mathbb{Z})$ . In this subsection we compute the composition  $\overline{\kappa}: \mathbb{Z}^2 \rightarrow T(\mathbb{Z}/p^2)_{\widetilde{j_{b,1}}(\overline{\alpha})}$  of the map  $\kappa_{\mathbb{Z}}: \mathbb{Z}^2 \rightarrow T(\mathbb{Z}_p)_{\widetilde{j_{b,1}}(\overline{\alpha})}$  in (4.9) and the reduction map  $T(\mathbb{Z}_p)_{\widetilde{j_{b,1}}(\overline{\alpha})} \rightarrow T(\mathbb{Z}/p^2)_{\widetilde{j_{b,1}}(\overline{\alpha})}$ . With a suitable choice of parameters in  $\mathcal{O}_{T, \widetilde{j_{b,1}}(\overline{\alpha})}$ , the map  $\kappa_{\mathbb{Z}}$  is described by integral convergent power series  $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{Z}_p\langle z_1, z_2 \rangle$

and  $\overline{\kappa}$ , composed with the inverse of the parametrization (8.3.1), is given the images  $\overline{\kappa_1}, \overline{\kappa_2}, \overline{\kappa_3}$  of  $\kappa_1, \kappa_2, \kappa_3$  in  $\mathbb{F}_p[z_1, z_2]$ .

The divisor  $j_b(\overline{\alpha})$  is equal to the image of

$$\widetilde{G}_t := e_{0,1}G_1 + e_{0,2}G_2 \text{ with } e_{0,1} := 6, e_{0,2} := 3$$

in  $J(\mathbb{F}_p)$  and

$$\tilde{t} := Q_{1,0}^6 \otimes Q_{2,0}^3 \otimes Q_{1,1}^{6 \cdot 6} \otimes Q_{1,2}^{6 \cdot 3} \otimes Q_{2,1}^{3 \cdot 6} \otimes Q_{2,2}^{3 \cdot 3} \text{ in } \mathcal{M}^\times(\widetilde{D}_1, m \cdot (D_0 + \eta_* \widetilde{G}_t))(\mathbb{Z})$$

is a lift of  $\widetilde{j_{b,1}}(\overline{\alpha})$ . The kernel of  $J(\mathbb{Z}) \rightarrow J(\mathbb{F}_p)$  is a free  $\mathbb{Z}$ -module generated by

$$\widetilde{G}_1 := e_{1,1}G_1 + e_{1,2}G_2, \quad \widetilde{G}_2 := e_{2,1}G_1 + e_{2,2}G_2, \text{ with } e_{1,1} := 16, e_{1,2} := 2, e_{2,1} := 0, e_{2,2} := 5.$$

Let  $\widetilde{G_{t,2}}$  be the divisor  $m(D_0 + \eta_*(\widetilde{G}_t))$  representing  $(m \cdot \text{otr}_c \circ f)(\widetilde{G}_t) \in J^0(\mathbb{Z})$ . Following (4.1) for  $i, j \in \{1, 2\}$  we define

$$\begin{array}{ccc} P_{i,j} := \bigotimes_{m,l=1}^2 Q_{l,m}^{e_{i,l} \cdot e_{j,m}} & R_{i,\tilde{t}} := \bigotimes_{l=1}^2 Q_{l,0}^{e_{i,l}} \otimes \bigotimes_{m,l=1}^2 Q_{l,m}^{e_{i,l} \cdot e_{0,m}} & S_{\tilde{t},j} := \bigotimes_{m,l=1}^2 Q_{l,m}^{e_{0,l} \cdot e_{j,m}} \\ \downarrow & \downarrow & \downarrow \\ (\widetilde{G}_i, f(m\widetilde{G}_j)) & (\widetilde{G}_i, \widetilde{G_{t,2}}) & (\widetilde{G}_t, f(m\widetilde{G}_j)). \end{array}$$

Computations in  $C_{\mathbb{Z}/p^2}$  show the following linear equivalences of divisors

$$\widetilde{G}_t \sim D_{0,3}, \quad \widetilde{G}_1 \sim D_{4,0}^0, \quad \widetilde{G}_2 \sim D_{0,3}^0$$

and applying Section 6.4 and the functoriality of the norm we compute

(8.4.1)

$$\begin{array}{ll} P_{1,1} = (1+4p) \cdot s_{0,0}((4,0), (2,0)) & \text{in } \mathcal{M}^\times(\widetilde{G}_1, \widetilde{G}_1)(\mathbb{Z}/p^2) = \mathcal{M}^\times(D_{4,0}^0, E_{2,0}^0)(\mathbb{Z}/p^2), \\ P_{1,2} = (1+4p) \cdot s_{0,0}((4,0), (0,4)) & \text{in } \mathcal{M}^\times(\widetilde{G}_1, \widetilde{G}_2)(\mathbb{Z}/p^2) = \mathcal{M}^\times(D_{4,0}^0, E_{0,4}^0)(\mathbb{Z}/p^2), \\ P_{2,1} = (1+4p) \cdot s_{0,0}((0,3), (2,0)) & \text{in } \mathcal{M}^\times(\widetilde{G}_2, \widetilde{G}_1)(\mathbb{Z}/p^2) = \mathcal{M}^\times(D_{0,3}^0, E_{2,0}^0)(\mathbb{Z}/p^2), \\ P_{2,2} = (-1) \cdot (1+2p) \cdot s_{0,0}((0,3), (0,4)) & \text{in } \mathcal{M}^\times(\widetilde{G}_2, \widetilde{G}_2)(\mathbb{Z}/p^2) = \mathcal{M}^\times(D_{0,3}^0, E_{0,4}^0)(\mathbb{Z}/p^2), \\ R_{1,\tilde{t}} = s_{0,E}((4,0), (0,4)) & \text{in } \mathcal{M}^\times(\widetilde{G}_1, \widetilde{G_{t,2}})(\mathbb{Z}/p^2) = \mathcal{M}^\times(D_{4,0}^0, E_{0,4})(\mathbb{Z}/p^2), \\ R_{2,\tilde{t}} = (1+4p) \cdot s_{0,E}((0,3), (0,4)) & \text{in } \mathcal{M}^\times(\widetilde{G}_2, \widetilde{G_{t,2}})(\mathbb{Z}/p^2) = \mathcal{M}^\times(D_{0,3}^0, E_{0,4})(\mathbb{Z}/p^2), \\ S_{\tilde{t},1} = s_{D,0}((0,3), (2,0)) & \text{in } \mathcal{M}^\times(\widetilde{G}_t, \widetilde{G}_1)(\mathbb{Z}/p^2) = \mathcal{M}^\times(D_{0,3}, E_{2,0}^0)(\mathbb{Z}/p^2), \\ S_{\tilde{t},2} = (-1)(1+4p) \cdot s_{D,0}((0,3), (0,4)) & \text{in } \mathcal{M}^\times(\widetilde{G}_t, \widetilde{G}_2)(\mathbb{Z}/p^2) = \mathcal{M}^\times(D_{0,3}, E_{0,4}^0)(\mathbb{Z}/p^2), \\ \tilde{t} = (-1) \cdot (1+2p) \cdot s_{D,E}((0,3), (0,4)) & \text{in } \mathcal{M}^\times(\widetilde{G}_t, \widetilde{G_{t,2}})(\mathbb{Z}/p^2) = \mathcal{M}^\times(D_{0,3}, E_{0,4})(\mathbb{Z}/p^2). \end{array}$$

We now show these computations in the cases of  $\widetilde{G}_t$  and  $\tilde{t}$ . The Riemann-Roch space relative to the divisor  $\widetilde{G}_t + \infty + \alpha - D$  on  $C_{\mathbb{Z}/p^2}$  is generated by the inverse of the rational function

$$h_1 := \frac{x^9 - 5x^8 - 2x^7 + 7x^6 - 9x^5 - 5x^4 + 14x^3 + 7x^2 + 13x + 1 + (x^6 + 9x^5 - 5x^4 + 15x^3 - 5x^2 + 4x + 14)y}{15x^5 - x^4 + 4x^3 + 19x^2 + 4x + 9}$$

and indeed

$$\text{div}(h_1) = \widetilde{G}_t - D_{0,3} = (6\gamma + 3\infty_- - 3\alpha - 6\infty) - (\alpha + \infty_3 - 2\infty) \text{ in } \text{Div}(C_{\mathbb{Z}/p^2}).$$

Hence multiplication by  $h_1$  gives an isomorphism  $\mathcal{O}_{C_{\mathbb{Z}/p^2}}(\widetilde{G}_t) \rightarrow \mathcal{O}_{C_{\mathbb{Z}/p^2}}(D_{0,3})$  and by functoriality of the norm we get

$$\begin{aligned} \delta^* \mathcal{O}_C(\widetilde{G}_t) &\rightarrow \delta^* \mathcal{O}_{C_{\mathbb{Z}/p^2}}(D_{0,3}), & \delta^* 1 &\mapsto \delta^*(h_1) = h_1(\delta) \cdot \delta^* 1 = 12 \cdot \delta^* 1, \\ \beta^* \mathcal{O}_C(\widetilde{G}_t) &\rightarrow \beta^* \mathcal{O}_{C_{\mathbb{Z}/p^2}}(D_{0,3}), & \beta^* 1 &\mapsto \beta^*(h_1) = h_1(\beta) \cdot \beta^* 1 = 18 \cdot \beta^* 1, \\ \infty^* \mathcal{O}_C(\widetilde{G}_t) &\rightarrow \infty^* \mathcal{O}_{C_{\mathbb{Z}/p^2}}(D_{0,3}), & \infty^* z^6 &\mapsto \infty^*(z^6 h_1) = 13 \cdot \infty^* \frac{z^2}{z-3p}, \\ \infty_-^* \mathcal{O}_C(\widetilde{G}_t) &\rightarrow \infty_-^* \mathcal{O}_{C_{\mathbb{Z}/p^2}}(D_{0,3}), & \infty_-^* z^{-3} &\mapsto \infty_-^*(z^{-3} h_1) = (z^{-3} h_1)(\infty_-) \cdot \infty_-^* 1 = 6 \cdot \infty_-^* 1. \end{aligned}$$

Since  $\widetilde{G}_{t,2} = 12\delta + 4\infty_- - 6\beta - 10\infty$ , the above isomorphisms, tensored with the exponents, give the canonical isomorphism

$$(8.4.2) \quad \mathcal{M}(\widetilde{G}_t, \widetilde{G}_{t,2}) = \widetilde{G}_{t,2}^* \mathcal{O}_{C_{\mathbb{Z}/p^2}}(\widetilde{G}_t) \rightarrow \widetilde{G}_{t,2}^* \mathcal{O}_{C_{\mathbb{Z}/p^2}}(D_{0,3}) = \mathcal{M}(D_{0,3}, \widetilde{G}_{t,2})$$

$$\tilde{t} = 14 \cdot (\delta^* 1)^{12} \otimes (\beta^* 1)^{-6} \otimes (\infty^* z^6)^{-10} \otimes (\infty_-^* z^{-3})^4 \mapsto 14 \cdot (\delta^* 1)^{12} \otimes (\beta^* 1)^{-6} \otimes (\infty^* \frac{z^2}{z-3p})^{-10} \otimes (\infty_-^* 1)^4.$$

The Riemann-Roch space relative to the divisor  $\widetilde{G}_{t,2} + \infty + \alpha - E$  on  $C_{\mathbb{Z}/p^2}$  is generated by the inverse of the rational function

$$h_2 := \frac{x^{17} - 8x^{16} + x^{15} - 4x^{14} + 7x^{13} + 4x^{12} + 12x^{11} + x^{10} + 2x^9 - 5x^8 + x^7 + 3x^6 + 12x^5 - 6x^4 - 6x^3 + 4x^2 - 6}{20x^8 - 6x^7} + \frac{10x^2 + (x^{15} + 6x^{14} - 5x^{13} - x^{12} - 2x^{11} + 14x^{10} - 4x^9 + 14x^8 + 3x^7 + 8x^6 - 6x^5 - 3x^4 + 4x^3 + 13x^2 - x - 7)y}{20x^9 - 6x^8}$$

and indeed

$$\operatorname{div}(h_2) = \widetilde{G}_{t,2} - E_{0,4} = (12\delta + 4\infty_- - 6\beta - 10\infty) - (2\beta + \infty_4 - \infty - \infty_-) \quad \text{in } \operatorname{Div}(C_{\mathbb{Z}/p^2}).$$

Following the recipe in Section 6.4 we consider the following rational section of  $\mathcal{O}_{C_{\mathbb{Z}/p^2}}(D_{0,3})$

$$l := \frac{10x^4 + x^3 + 17x + 14 + (15x + 9)y}{10x^4 + 16x^3 + 7x^2 + 7x + 10}.$$

since it generates  $\mathcal{O}_{C_{\mathbb{Z}/p^2}}(D_{0,3})$  in a neighborhood of the supports of  $\widetilde{G}_{t,2}$  and  $E_{0,4}$ . Then

$$\operatorname{div}(l) = 3 \cdot (-1, 1) + (17, 23) + (15, 10) - 2 \cdot (12, 23) - 2 \cdot (5, 20) - (0, 1) \quad \text{in } \operatorname{Div}(V_{1, \mathbb{Z}/p^2}) \subset \operatorname{Div}(C_{\mathbb{Z}/p^2}).$$

Hence by (6.4.4) the canonical isomorphism

$$\mathcal{M}(D_{0,3}, \widetilde{G}_{t,2}) = \widetilde{G}_{t,2}^* \mathcal{O}_{C_{\mathbb{Z}/p^2}}(D_{0,3}) \longrightarrow E_{0,4}^* \mathcal{O}_{C_{\mathbb{Z}/p^2}}(D_{0,3}) = \mathcal{M}(D_{0,3}, E_{0,4})$$

described in Section 6.4 is characterised by

$$(8.4.3) \quad \widetilde{G}_{t,2}^* l \longmapsto h_2(\operatorname{div}(l)) \cdot E_{0,4}^* l = 14 \cdot E_{0,4}^* l.$$

where

$$\begin{aligned} \widetilde{G}_{t,2}^* l &:= (\delta^* l)^{12} \otimes (\beta^* l)^{-6} \otimes (\infty^* l)^{-10} \otimes (\infty_-^* l)^4 = -(\delta^* 1)^{12} \otimes (\beta^* 1)^{-6} \otimes (\infty^* \frac{z^2}{z-3p})^{-10} \otimes (\infty_-^* 1)^4, \\ E_{0,4}^* l &:= (\beta^* l)^2 \otimes (\infty_4^* l) \otimes (\infty^* l)^{-1} \otimes (\infty_-^* l)^{-2} = 16 \cdot (\beta^* 1)^2 \otimes (\infty_4^* \frac{z^2}{z-3p}) \otimes (\infty^* \frac{z^2}{z-3p})^{-1} \otimes (\infty_-^* 1)^{-2}. \end{aligned}$$

Equations (8.4.2) and (8.4.3) imply that  $\tilde{t} = -(1 + 2p) \cdot s_{D,E}((0, 3), (0, 4))$ .

Let  $\overline{A}_{\tilde{t}}, \overline{B}_{\tilde{t}}, \overline{C}$  and  $\overline{D}_{\tilde{t}}$  be the compositions of the reduction map  $\mathcal{M}^\times(\mathbb{Z}_p) \rightarrow \mathcal{M}(\mathbb{Z}/p^2)$  and respectively  $A_{\tilde{t}}, B_{\tilde{t}}, C$  and  $D_{\tilde{t}}$ , defined in (4.2), (4.3) and (4.4). Using (6.6.13) and (8.4.1) we get, for  $n$  in  $\mathbb{Z}^2$ ,

$$(8.4.4) \quad \begin{aligned} \overline{A}_{\tilde{t}}(n) &= (-1)^{n_2} (1 + (4n_2)t) \cdot s_{D,0}((0, 3), (2n_1, 4n_2)), \\ \overline{B}_{\tilde{t}}(n) &= (1 + (4n_2)p) s_{0,E}((4n_1, 3n_2), (0, 4)), \\ \overline{C}(n) &= (-1)^{n_2^2} (1 + (4n_1^2 + (4 + 4)n_1n_2 + 2n_2^2)p) \cdot s_{0,0}((4n_1, 3n_2), (2n_1, 4n_2)), \\ \overline{D}_{\tilde{t}}(n) &= -(1 + (4n_1^2 + 3n_1n_2 + 2n_2^2 + 3n_2 + 2)p) \cdot s_{D,E}((4n_1, 3 + 3n_2), (2n_1, 4 + 4n_2)), \\ \overline{\kappa}(n) &= -(1 + (4n_1^2 + 3n_1n_2 + 2n_2^2 + 2n_2 + 2)p) \cdot s_{D,E}((n_1, 3 + 2n_2), (3n_1, 4 + n_2)), \end{aligned}$$

hence, using the bijection (8.3.1),

$$(8.4.5) \quad \overline{\kappa}_1 = z_1, \quad \overline{\kappa}_2 = 3 + 2z_2, \quad \overline{\kappa}_3 = 4z_1^2 + 3z_1z_2 + 2z_2^2 + 2z_2 + 2.$$

## 8.5 The rational points with a specific image mod 5.

By 8.4.4 the image in  $T(\mathbb{F}_p)$  of a point  $\pm \overline{D}_{\tilde{t}}(n)$  for  $n \in \mathbb{Z}^2$  is always of the form  $\pm s_{D,E}(0, 0)$ , hence, looking at (8.1.3) we see that there is no point  $T(\mathbb{Z})$  with reduction  $\widetilde{j_{b,0}(\overline{\alpha})} \in T(\mathbb{F}_p)$ . Hence  $C(\mathbb{Z})_{\overline{\alpha}} = U_1(\mathbb{Z})_{\overline{\alpha}}$ .

Let  $f_1, f_2 \in \mathcal{O}(\widetilde{T}_t^p)^{\wedge p}$  be generators of the kernel of  $\widetilde{j_{b-1}^*} : \mathcal{O}(\widetilde{T}_t^p)^{\wedge p} \rightarrow \mathcal{O}(\widetilde{U}_u^p)^{\wedge p}$  as in Section 4. The bijection (8.3.1) gives an isomorphism  $\mathbb{F}_p \otimes \mathcal{O}(\widetilde{T}_t^p) = \mathbb{F}_p[\lambda_1, \lambda_2, \tau]$  and since the images  $\overline{f}_1, \overline{f}_2$  of  $f_1, f_2$  in  $\mathbb{F}_p \otimes \mathcal{O}(\widetilde{T}_t^p)$  are generators of the kernel of  $\widetilde{j_{b,1}^*} : \mathbb{F}_p \otimes \mathcal{O}(\widetilde{T}_t^p)^{\wedge p} \rightarrow \mathbb{F}_p \otimes \mathcal{O}(\widetilde{U}_u^p)^{\wedge p}$  we can suppose that

$$\overline{f}_1 = \lambda_2, \quad \overline{f}_2 = \tau - 1.$$

By (8.4.5) we have

$$\kappa^* \overline{f}_1 = \overline{\kappa}_2 = 3 + 2z_2, \quad \kappa^* \overline{f}_2 = \overline{\kappa}_3 - 1 = 4z_1^2 + 3z_1z_2 + 2z_2^2 + 2z_2 + 1.$$

Let  $A$  be  $\mathbb{Z}_p \langle z_1, z_2 \rangle / (\kappa^* f_1, \kappa^* f_2)$ . Then the ring

$$(8.5.1) \quad \overline{A} := A/pA = \mathbb{F}_p[z_1, z_2] / (\kappa^* \overline{f}_1, \kappa^* \overline{f}_2) = \mathbb{F}_p[z_1, z_2] / (z_2 - 1, 4z_1^2 + 3z_1)$$

has dimension 2 over  $\mathbb{F}_p$ , hence by Theorem 4.12  $U(\mathbb{Z})_{\overline{\alpha}}$  contains at most 2 points. Since both

$$\alpha \quad \text{and} \quad (12/7, 20/7) \in V_1(\mathbb{Z}[1/7])$$

reduce to  $\overline{\alpha}$  we deduce that  $C(\mathbb{Z})_{\overline{\alpha}} = U_1(\mathbb{Z})_{\overline{\alpha}}$  is made of the these two points.

## 8.6 Determination of all rational points

Denoting  $(3, -1) \in V_1(\mathbb{F}_p) \subset C(\mathbb{F}_p)$  as  $\varepsilon$  we have

$$C(\mathbb{F}_p) = \{\overline{\infty}, \overline{\infty}^-, \overline{\alpha}, \iota(\overline{\alpha}), \eta(\overline{\alpha}), (\iota \circ \eta)(\overline{\alpha}), \overline{\gamma}, \iota(\overline{\gamma}), \eta(\overline{\gamma}), (\iota \circ \eta)(\overline{\gamma}), \varepsilon, \iota(\varepsilon)\}.$$

Using that for any point  $Q$  in  $C(\mathbb{F}_p)$  the condition  $T(\mathbb{Z})_{\widetilde{j_{b,i}(Q)}} = \emptyset$  implies  $U_i(\mathbb{Z})_Q = \emptyset$  we get

$$U_0(\mathbb{Z})_{\overline{\infty}} = U_0(\mathbb{Z})_{\overline{\infty}^-} = U_1(\mathbb{Z})_{\varepsilon} = U_1(\mathbb{Z})_{\iota(\varepsilon)} = U_1(\mathbb{Z})_{\overline{\gamma}} = U_1(\mathbb{Z})_{\eta(\overline{\gamma})} = U_1(\mathbb{Z})_{\iota(\eta(\overline{\gamma}))} = U_1(\mathbb{Z})_{\iota(\eta(\overline{\gamma}))} = \emptyset.$$

Applying our method to  $\overline{\infty}$  we discover that  $U_1(\mathbb{Z})_{\overline{\infty}}$  contains at most 2 points and the same holds for  $U_1(\mathbb{Z})_{\overline{\infty}-}$ . Moreover the action of  $\langle \eta, \iota \rangle$  on  $C(\mathbb{Z})$  tells that  $U_1(\mathbb{Z})_{\iota(\overline{\alpha})}$ ,  $U_1(\mathbb{Z})_{\eta(\overline{\alpha})}$  and  $U_1(\mathbb{Z})_{\eta\iota(\overline{\alpha})}$  are sets containing exactly 2 elements. Hence

$$U_1(\mathbb{Z}) = U_1(\mathbb{Z})_{\overline{\alpha}} \cup U_1(\mathbb{Z})_{\iota(\overline{\alpha})} \cup U_1(\mathbb{Z})_{\eta(\overline{\alpha})} \cup U_1(\mathbb{Z})_{\eta\iota(\overline{\alpha})} \cup U_1(\mathbb{Z})_{\overline{\infty}-} \cup U_1(\mathbb{Z})_{\overline{\infty}}$$

contains at most 12 elements. Looking at the orbits of the action of  $\langle \eta, \iota \rangle$  on  $U_1(\mathbb{Z})$  we see that  $\#U_1(\mathbb{Z}) \equiv 2 \pmod{4}$ , hence  $\#U_1(\mathbb{Z}) \leq 10$ . Since  $U_1(\mathbb{Z})$  contains  $\infty, \infty_-$  and all the images by  $\langle \eta, \iota \rangle$  of  $U_1(\mathbb{Z})_{\overline{\alpha}}$  we conclude that  $\#U_1(\mathbb{Z}) = 10$ .

Applying our method to the point  $\overline{\gamma}$  we see that  $U_0(\mathbb{Z})_{\overline{\gamma}}$  contains at most two points, one of them being  $\gamma$ . Moreover solving the equations  $\kappa^* \overline{f}_i = 0$  we see that if there is another point  $\gamma'$  in  $U_0(\mathbb{Z})_{\overline{\gamma}}$  then there exist  $n_1, n_2 \in \mathbb{Z}$  such that

$$j_b(\gamma') = 39G_1 + 17G_2 + 5n_1\widetilde{G}_1 + 5n_2\widetilde{G}_2.$$

Confronting this information with the image of the map  $j_b: C(\mathbb{F}_7) \rightarrow J(\mathbb{F}_7)$  we derive a contradiction, hence  $U_0(\mathbb{Z})_{\overline{\gamma}} = \{\gamma\}$ . Applying our method to  $\varepsilon$  we see that  $U_0(\mathbb{Z})_{\varepsilon}$  contains at most 2 points corresponding to two different solutions to the equations  $\kappa^* \overline{f}_i = 0$ . We can see that one of the two solutions does not lift to a point in  $U_0(\mathbb{Z})_{\varepsilon}$  in the same way we excluded the existence of  $\gamma' \in U_0(\mathbb{Z})_{\overline{\gamma}}$ . Hence  $U_0(\mathbb{Z})_{\varepsilon}$  has cardinality at most 1. Using that for every  $Q \in C(\mathbb{F}_p)$  and every automorphism  $\omega$  of  $C$  we have  $\#U_0(\mathbb{Z})_Q = \#U_0(\mathbb{Z})_{\omega(Q)}$ , we deduce that

$$U_0(\mathbb{Z}) = U_0(\mathbb{Z})_{\overline{\gamma}} \cup U_0(\mathbb{Z})_{\iota(\overline{\gamma})} \cup U_0(\mathbb{Z})_{\eta(\overline{\gamma})} \cup U_0(\mathbb{Z})_{\eta\iota(\overline{\gamma})} \cup U_0(\mathbb{Z})_{\varepsilon} \cup U_0(\mathbb{Z})_{\iota(\varepsilon)}$$

contains at most 6 points. Looking at the orbits of the action of  $\langle \eta, \iota \rangle$  on  $U_0(\mathbb{Z})$  we see that  $\#U_0(\mathbb{Z}) \equiv 0 \pmod{4}$ , hence  $\#U_0(\mathbb{Z}) \leq 4$ , and since  $U_0(\mathbb{Z})$  contains the orbit of  $\gamma$  we conclude that  $\#U_0(\mathbb{Z}) = 4$ . Finally

$$\#C(\mathbb{Z}) = \#U_0(\mathbb{Z}) + \#U_1(\mathbb{Z}) = 4 + 10 = 14.$$

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