Southwest Center
for Arithmetic Geometry

ARIZONA WINTER SCHOOL 2017

Department of Mathematics
The University of Arizona®

**Deadline to apply for funding:
November 11, 2016**

http://swc.math.arizona.edu

PERFECTOID SPACES

Bhargav Bhatt
p-adic Hodge theory

Ana Caraiani
Shimura varieties

Kiran Kedlaya
Sheaves, stacks, and shtukas

Jared Weinstein
Adic spaces

with Peter Scholze

TUCSON, MARCH 11-15, 2017

Funded by the **National Science Foundation**
and organized in partnership
with the **Clay Mathematics Institute**

University of Arizona

Arizona Winter School 2017
Perfectoid Spaces

Notes By: Caleb McWhorter

March 2017

Contents

Part I
Talk Notes

1 Name: Lecture Title

1.1 Lecture 1

1.1.1 Lecture Name

Historic Remarks about the genesis of the paper "Perfectoid Spaces"

or why perfectoid spaces are a failed theory.

In 2007, Scholze went to Bonn as an undergrad and studied under M. Rapport.

Let X be a smooth projective scheme over $\mathbb{Q}_p$. Fix $i \geq 0$ and let $l \neq p$ be a prime. Consider the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representation $V = H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_l)$. There is a weight decomposition given by the following: if $\Phi \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is a geometric Frobenius, then

$$V = \bigoplus_{j=0}^{2i} V_j,$$

where Φ acts through the Weil numbers of weight j on V_j.

Rapport-Zink (1980) if X has semistable reduction

de Joung (1995) in general (reduction to semistable case).

Rapport gave Scholze the following problem to think about:

There is a monodromy operator $N : V \to V(\pm 1)$ (Tate twist) coming from the action of the inertia subgroup. In particular, $N : V_j \to V_{j-2}$. Then

$$\forall j = 0, \dots, i : N^j : V_{i+j} \xrightarrow{\sim} V_{i-j}.$$

Conjecture 1.1 (Weight-Monodromy Conjecture)**.** *Let X be a smooth projective scheme over $\mathbb{Q}_p$. Fix $n \geq 0$ and $l \neq p$ a prime. Consider the* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*-representation $V = H^i_{\acute{e}t}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_l)$*

Example 1.1.

(i) If X has good reduction, i.e. there exists a smooth projective $\mathfrak{X}/\mathbb{Z}_p$ with generic fiver X, then

$$\begin{array}{ccc} V & \cong & H^i_{\text{ét}}(\mathfrak{X}_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_l) \\ \circlearrowleft & & \circlearrowleft \\ \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) & \twoheadrightarrow & \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \end{array}$$

so that the inertia group acts trivially.

But then $N = 0$

$\forall j = 0, N^j = 0 : V'_{i+j} \xrightarrow{\sim} V_{i-j}$

so equiv, $V_j = 0\ \forall j \neq i$.

i.e. $V = V_i$.

But this follows from the Weil conjectures for $\mathfrak{X}_{\mathbb{F}_p}$.

(ii) If $X = E$ is an elliptic curve with multiplicative reduction

'$E = \mathbb{G}_m/q^{\mathbb{Z}}$', $0 \neq q \in \mathbb{Q}_p$, $|q| < 1$ as rigid-analytic/adic spaces

Then

$$H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_l) = H^1_{\text{ét}}(\mathbb{G}_{m,\overline{\mathbb{Q}}_p/q^{\mathbb{Z}}}, \overline{\mathbb{Q}}_l).$$

Then by Hochschild-Serre spectral sequence

$$H^i(\mathbb{Z}, \underbrace{H^j_{\text{ét}}(\mathbb{G}_{m,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_l)}_{=\begin{cases}\overline{\mathbb{Q}}_l, & j=0\\ \overline{\mathbb{Q}}_l(-1), & j=1\\ 0, & \text{otherwise}\end{cases}}) \Longrightarrow H^{i+j}_{\text{ét}}(\mathbb{G}_{m,\overline{\mathbb{Q}}_p/q^{\mathbb{Z}}}, \overline{\mathbb{Q}}_l).$$

with trivial $\mathbb{Z}$-action.

So

$$0 \longrightarrow \overline{\mathbb{Q}}_l \longrightarrow H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_l) \longrightarrow \overline{\mathbb{Q}}_l(-1) \longrightarrow 0.$$

So $V_2 = \overline{\mathbb{Q}}_l(-1)$, $V_0 = \overline{\mathbb{Q}}_l$

Splitting $V = V_0 \oplus V_2$ depends on choice of Φ.

Weight-monodromy predicts $N : V_2 \cong V_0$ can be checked by hand. Use that inertia action is trivial on l-power roots of q for $i = 1, 2$.

Remark.

(i) Conjecture is known for $i = 1, 2$. dim 1: reduce to abelian varieties or curves and use Néron models/semistable models. dim 2: Rapport-Zink + de Jong.

(ii) Known in equal characteristic p, i.e. over $\mathbb{F}_p((f))$.

Proved in Deligne's Weil 2 paper, uses that L-functions over function fields have good properties.

(iii) Conversely, weight-monodromy conjectures critical to understanding local factors of Hasse-Weil zeta functions at places of bad reduction ($\Leftrightarrow$ the Hasse-Weil zeta function "has no poles in region of absolute convergence.")

Rapport's suggestion: Try to reduce to case of equal characteristic after base change to some very ramified $K/\mathbb{Q}_p$.

Idea: If $\mathfrak{X}/\mathcal{O}_K$ integral (semistable, say) model of $X \times_{\mathbb{Q}_p} K$, then $\mathfrak{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_K/p$ lives over $\mathcal{O}_K/p \cong \mathbb{F}_q[t]/t^e$, where e is the ramification index of $K/\mathbb{Q}_p$.

If $e \gg 0$, this is almost $\mathbb{F}_p [\![t]\!]^0$.

Of course, this does not really work, as even if e is large, still not deform $\mathfrak{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_K/p$ from $\mathcal{O}_K/p = \mathbb{F}_p[t]/t^e$ to $\mathbb{F}_p [\![t]\!]$.

Usually, there are (a lot of) obstructions.

Also, in the end need to relate $V = H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_l)$ acting on $\operatorname{Gal}(\mathbb{F}_p((t))^{\text{sep}}/\mathbb{F}_p((t)))$, where $X^1/\mathbb{F}_p((f))$ is the generic fiber of deformation.

In semistable case, can use log-geometry to do this (related to isomorphism of tame quotients of $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\operatorname{Gal}(\mathbb{F}_p((t))^{\text{sep}}/\mathbb{F}_p((t)))$).

Turning these ideas in my head, lead

Theorem 1.1 (Fontaine-Wintenberger)**.** $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(p^{1/p^\infty})) \cong \operatorname{Gal}(\mathbb{F}_p((f))^{sep}/\mathbb{F}_p((t)))$, *and even canonically.*

proof involves Fontain's construction like

$$\varprojlim_{\text{Frob}} \mathcal{O}_{\overline{\mathbb{Q}}_p}/p.$$

Hard to understand what it means. Later, I learned from Faltings that

Theorem 1.2.
$$\pi^{\acute{e}t}(\operatorname{Spec} \mathbb{Q}_p(p^{1/p^\infty})\langle T^{\pm 1/p^\infty}\rangle) \cong \pi^{\acute{e}t}(\operatorname{Spec} \mathbb{F}_p((t))\langle T^{\pm 1}\rangle$$

Things started to resolve after I realized the following proof of Fontaine-Winterberger's Theorem

{finite étale $\mathbb{Q}_p(p^{1/p^\infty})$-alg} {almost finite étale $\mathbb{Z}_p[p^{1/p^\infty}]$-alg} {almost finite étale $\mathbb{Z}_p[p^{1/p^\infty}]/p$-alg} {almost finite étale $\mathbb{F}_p[t^{1/p^\infty}]/t$-alg} {finite étale $\mathbb{F}_p((t))(t^{1/p^\infty})$-alg} {finite étale $\mathbb{F}_p((t))$-alg}

This suggested what to do in the relative case. Find some notion of 'perfectpod'.

{perfectoid $\mathbb{Q}_p(p^{1/p^\infty})$-alg} {perfectoid almost $\mathbb{Z}_p[p^{1/p^\infty}]$-alg} (needs unique lifting property)

{perfectoid almost $\mathbb{Z}_p[p^{1/p^\infty}]/p$-alg} {perfectoid almost $\mathbb{F}_p[t^{1/p^\infty}]/t$-alg} $\vdots$ {perfectoid $\mathbb{F}_p((t))(t^{1/p^\infty})$-alg}

If R perfectoid (almost) $\mathbb{Z}_p[p^{1/p^\infty}]/p$-algebra, then cotangent complex of perfectoid almost $L_{R/(\mathbb{Z}[p^{1/p^\infty}]/p)} = 0$.

Lemma 1.1 (Gabber-Romero). *If $S \to R$ is a map of $\mathbb{F}_p$-algebras that is "relatively perfect", i.e. relative Frob $\Phi_{R/S} : R \otimes_S ??? \xrightarrow{\sim} R$ is an isomorphism, then $L_{R/S} \cong 0$.*

Proof (Sketch). $\Phi_{R/S}$ isomorphism of $L_{R/S}$ but also equal as $d(x^p) = px^{p-1}\,dx = 0$.

Definition. A perfectoid $\mathbb{Q}_p(p^{1/p^\infty})$-algebra is a uniform Banach $\mathbb{Q}_p(p^{1/p^\infty})$-algebra R such that $(R^0/p)/(\mathbb{Z}_p[p^{1/p^\infty}]/p)$ is relatively perfect, where R^0 is the set of powerbounded elements in R, equivalently, $\Phi : R^0/p \to R^0/p$, given by $x \mapsto x^p$, is surjective.

Corollary 1.1. *Set of R perfectoid $\mathbb{Q}_p(p^{1/p^\infty})$-algebra mapping to $R^\flat$ set of perfectoid $\mathbb{F}_p((t))(t^{1/p^\infty})$-algebras.*

This can be made explicit in terms of Fontaine's functor:

$$R^\flat = \varprojlim_{\text{Frob}} (R^0/p) \otimes_{\mathbb{F}_p[\![t]\!][t^{1/p^\infty}]} \mathbb{F}_p((t))(t^{1/p^\infty}).$$

pass to geometry.

Corollary 1.2. $(\mathbb{P}^{n,ad}_{\mathbb{F}_p((t))})_{\acute{e}t} = \varprojlim_{\varphi} (\mathbb{P}^{n,ad}_{\mathbb{Q}_p(p^{1/p^\infty})})_{\acute{e}t}$ $\varphi(x_0 : \cdots : x_n) = (x_0^p : \cdots : x_n^p)$

Now $X \subset \mathbb{P}^n_{\mathbb{Q}_p}$ is your smooth projective variety.

$$\begin{array}{ccc} \mathbb{P}^n_{\mathbb{F}_p((t))} & \xrightarrow{\quad\pi\quad} & \mathbb{P}^n_{\mathbb{Q}_p(p^{1/p^\infty})} \\ \circlearrowleft & & \circlearrowleft \\ \pi^{-1}(X_{\mathbb{Q}_p(p^{1/p^\infty})} & \longrightarrow & X_{\mathbb{Q}_p(p^{1/p^\infty})} \end{array}$$

Applying Deligne to bottom left

Problem: This is not algebraic. But how far can it be away from algebraic?

Easy case: If X is complete intersection, then any ϵ-neighborhood of $\pi^{-1}(X_{\mathbb{Q}_p} p^{1/p^\infty})$, there are algebraic varieties of same dimension enough to conclude.

2 Name: Lecture Title

2.1 Lecture 1

2.1.1 Lecture Name

Where do we go from here?

For example, Yves André has recently used perfectoid spaces to prove the following of Hochester (73):

Theorem 2.1 (Direct Summand Conjecture)**.** *Let R be a regular ring, $R \hookrightarrow S$. Then $R \hookrightarrow S$ has a splitting as R-modules.*

The forward direction is descent along $R \to S$.

Part of Hochster's "homological conjectures" refined by Bhatt, Ma, Schwede, ...

developed theory of test ideals in mixed characteristic

also: connections to algebraic toplogy via topological hochschild homology

but for the rest of talk, let's concentrate on "mixed-characteristic shtukas"

History of shtukas: function fields

Let $C/\mathbb{F}_q$ be a projective smooth geometrically coneccted. Let $G/\mathbb{F}_q$ be reductive groups, e.g. $G = \mathrm{GL}_2$ cufve. moduli space of shtukas over C with one leg.

$$f : \mathrm{Sht}_{\because\because} \longrightarrow C.$$

alongue of Shumura varieties

$$\mathrm{Sh} \longrightarrow \operatorname{Spec} \mathbb{Z}$$

$R^i f_* \overline{\mathbb{Q}}_l \circlearrowleft \pi_q(C) = \mathrm{Gal}(\overline{F}/F)^{curv}$

$$\begin{array}{ccccc} R^i f_* \overline{\mathbb{Q}}_l & \circlearrowright & \pi_q(C) & \cong & \mathrm{Gal}(\overline{F}/F)^{curv} \\ \circlearrowleft & & & & \\ G(\mathbb{A}) & & & & \end{array}$$

where F is the function field of C and $\mathbb{A} = \mathbb{A}_F$ are the adéles of F.

Theorem 2.2 (Drinfeld, L. Lafforge...)**.** $R^i f_* \overline{\mathbb{Q}}_l = \oplus_* \pi \otimes \sigma(\pi) \circlearrowleft \mathrm{Gal}(\overline{F}/F)$ ** certain automorphic cpr π of $G(A)$*

This association $\{$autom. rep. of $G(\mathbb{A})\}$ $\{$Gal rep$\}$ $\pi \mapsto \sigma(\pi)$. define the global Langlands correspondence (in some cases)

Unfortunately, not *all* autmorphic π.

Insight of Drinfeld: Cal get all π if one looks at spaces of shtukas with two legs.

2 legs:

$$f : \mathrm{Sht}_{\because\because} \to C \times C$$

$$\begin{array}{ccccc} R^i f_* \overline{\mathbb{Q}}_l & \circlearrowright & \pi_1(C \times C/\phi^{\mathbb{Z}}) & \cong & \pi_1(C) \\ \circlearrowleft & & & & \\ G(\mathbb{A}) & & & & \end{array}$$

where congruence Drinfeld's lemma

Theorem 2.3 (Same people)**.** *For good choices of data*

$$R^i f_* \overline{\mathbb{Q}}_l = \bigoplus_{*} \pi \otimes \sigma(\pi) \otimes o(\pi)^V$$

** all cuspidal automorphic reprep of $G(\mathbb{A})$ and $\pi_1(C) \circlearrowleft \sigma(\pi)$ and $\pi_!(C) \circlearrowleft o(\pi)^V$*

Get global langland's coorespondence for GL_2: Drinfeld GL_n: L. Lafforgue any G: V. Lafforgue

We would love to do the same over number fields.

Obvious problem: what is the analogue of $C \otimes_{\mathbb{F}_q} C$?

Magic of diamonds: Can we make sense of not $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$ but at last of $\operatorname{Spec}\mathbb{Q}_p \times_{\mathbb{F}_1} \operatorname{Spec}\mathbb{Q}_p$ (or even $\operatorname{Spec}\mathbb{Z}_p \times \operatorname{Spec}\mathbb{Z}_p =$ compeltion at $(p_1 p)$).

Namely, can take product $\operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{Q}_p$ in category of diamons get something 2-dimensional.

$$\operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{Q}_p = \operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{Q}_p^{tet}/\underline{\mathbb{Z}}_p^* = \operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{F}_p((t^{1/p^\infty}))/\underline{\mathbb{Z}}_p^* = (\tilde{\mathbb{D}}^*_{\mathbb{Q}_p})^\diamond/\underline{\mathbb{Z}}_p^*.$$

where last is perfectoid punctured open unit disk$/\mathbb{Q}_p$

analogue of Drinfeld's lemma:

Theorem 2.4. $\pi_1(\operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{Q}_p/\varphi^{\mathbb{Z}}) \cong \pi_1(\operatorname{Spd}\mathbb{Q}_p) \times \pi_1(\operatorname{Spd}\mathbb{Q}_p) = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \times \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. *Equivantely,*

$$\pi_1(\tilde{\mathbb{D}}^*_{\mathbb{Q}_p}/\mathbb{Q}_p^*) = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^2.$$

$\mathbb{Q}_p^* = \underline{\mathbb{Z}}_p^* \times \varphi^{\mathbb{Z}} = p^{\mathbb{Z}}$ *or* $\pi_1(\tilde{\mathbb{D}}^*_{\mathbb{C}_p}/\underline{\mathbb{Q}}_p^*) = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

moduli spaces of local, mixed class shtukas with one leg

$$\mathrm{Sht}^{\cdots}_{\cdots} \longrightarrow \operatorname{Spd}\mathbb{Q}_p.$$

There turn out to be (generalizations of) Rapport-ZInk spaces (local p-adic analogues of Shumura varities)

Example (lubin-tate spaces)

Let $H/\overline{\mathbb{F}}_p$ 1-dimensional formal group of height n (then is p-div. group)

deformation space of H:

$$\mathfrak{X}_H \cong \operatorname{Spf} W(\overline{F}_p)\,[\![u_1, \ldots, u_{k-1}]\!]$$

generic fibre $\mathcal{M}_H$ $(n-1)$-dimensional open unit disc

tower

$$\cdots \longrightarrow \mathcal{M}_{H,2} \longrightarrow \mathcal{M}_{H,0} = \mathcal{M}_H$$

$\mathcal{M}_{H,m}$ classifies isomorphisms

$$\mathcal{H}[p^m] \cong (\mathbb{Z}/p^m\mathbb{Z})^n,$$

where $\mathcal{H}$ universal deformation of H.

$$\mathcal{M}_{H,\infty} = \varprojlim_m \mathcal{M}_{H,m}$$

perfectoid space (S.-Weinstein)

Theorem 2.5 (S., Weinstein). *Let $C/\mathbb{Q}_p$ algebraically closed complete extension, let $\infty \in FF_{C^\flat}$ Fargue-Fontaine corresponding to $C^\flat$. Then*

$$\mathcal{M}_{H,\infty}(C) = \{\mathcal{O}^n \overset{f}{\hookrightarrow} \mathcal{O}(1/n) \text{ sthm coker } f \text{ is supported at } \infty\}$$

This can be also said in terms of shtukas with one leg at ∞

several legs: there is no obstruction to considering moduli spaces of shtukas with any number of legs.

Test objects: $S \in \mathrm{Pfd} = \{$perfectoid spaces of char $p\}$ legs at s-valued $x_1, \ldots, x_n : S \to \operatorname{Spd} \mathbb{Q}_p$ of $\operatorname{Spd} \mathbb{Q}_p$

These correspond to untilts $S_1^\#, \ldots, S_n^\#$ of S.

graph of $x_i S \to \operatorname{Spd} \mathbb{Q}_p Y_S = S \times \operatorname{Spd} \mathbb{Q}_p$

closed immersions of adic spaces $S_i^\# \xrightarrow{\sim} S$ can consider φ-modules over Y_S (or compactification of it) with poles zeroes at the divisors.

$f : \mathrm{Sht}_{\cdots}^{\cdots} \longrightarrow \operatorname{Spd} \mathbb{Q}_p \times \operatorname{Spd} \mathbb{Q}_p$

$$\begin{array}{ccc} R^i f_* \overline{\mathbb{Q}}_l & \circlearrowleft & \pi_1(\operatorname{Spd} \mathbb{Q}_p \times \operatorname{Spd} \mathbb{Q}_p / \phi^{\mathbb{Z}}) \\ \circlearrowleft & & \| \\ G(\mathbb{Q}_p) & & \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^2 \end{array}$$

Theorem 2.6. $\pi_0(THH(\mathcal{O}_C)_p^\wedge)^{hS^1} = \mathbb{A}_{inf}$

3 Bhargav Bhatt: p-adic Hodge Theory

3.1 Lecture 1

Hodge Decomposition
$X/\mathbb{C}$ smooth projective curve

Theorem 3.1. *There exists a natural isomorphism* $H^n(X^{an}, \mathbb{Q}) \otimes \mathbb{C} \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbb{C}})$

Example 3.1. $X = E$ elliptic curve over $\mathbb{C}$ then $E = \mathbb{C}/\Lambda$

Theorem 3.2.

$$\begin{array}{ccc} H^0(X, \Omega^1_X) & \hookrightarrow & H^1(X^{an}, \mathbb{Q}) \otimes \mathbb{C} \\ \| & & \| \\ \mathbb{C}\omega & & \mathrm{Hom}(\Lambda, \mathbb{C}) \end{array}$$

$$\omega \longmapsto r \in \Lambda \mapsto \int_\gamma \omega$$

Highly Transcendental

Corollary 3.1. *Say* $f : X \to Y$ *of smooth projective variety and* $f^* : H^n(Y, \mathbb{Q}) \xrightarrow{\sim} H^n(X, \mathbb{Q})$ *then* $H^i(X, \Omega^j_X) \xleftarrow{\sim} H^i(Y, \Omega^j_Y) : f^*$ *for all* $i + j = n$

ÉtaleCohomology
Say X is a scheme $A \in \{\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p\}$ Grothendieck then $H^*(X_{\text{ét}}, A)$ algebraically defined

Theorem 3.3 (Artin)**.** *Let* $X/\mathbb{C}$ *be a variety then* $H^*(X_{ét}, A) \xrightarrow{\sim} H^*(X^{an}, A)$.

Upshot: Say X is defined over $\mathbb{Q}$
Theorem + epsilon there exists a natural action $G_{\mathbb{Q}}$ on $H^*(X^{\text{an}}, A)$.

Example 3.2. (i) $X = E$ elliptic curve over $\mathbb{C}$ but defined over $\mathbb{C}$. Therefore, $E = \mathbb{C}/\Lambda$.

$$\begin{aligned} H^1(X^{\text{an}}, \mathbb{Z}/n\mathbb{Z}) &= \mathrm{Hom}(H_1(X^{\text{an}}, \mathbb{Z}/n\mathbb{Z}), \mathbb{Z}/n\mathbb{Z}) \\ &\cong \mathrm{Hom}(\Lambda, \mathbb{Z}/n\mathbb{Z}) \\ &\cong E[n]^\vee \end{aligned}$$

Theorem then $E[n]$ is defined over $\overline{\mathbb{Q}}$. Get action $G_{\mathbb{Q}}$ on $E[n]$.

Set $T_pE = \varprojlim E[p^n]$. Therefore, get a constant $G_{\mathbb{Q}}$-action on T_pE if and only if dual to the $G_{\mathbb{Q}}$-action on $H^1(X^{\text{an}}, \mathbb{Z}_p)$.

(ii) $X = \mathbb{G}_m$ some analysis shows

$$H^1(\mathbb{G}_n^{\text{an}}, \mathbb{Z}/n\mathbb{Z}) \cong \mu_n^\vee$$

Set $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}$. Therefore, get $G_{\mathbb{Q}}$-action on $\mathbb{Z}_p(1)$ if and only if $G_{\mathbb{Q}}$-action on $H^1(X^{\text{an}}, \mathbb{Z}_p)$

Notation: For any $\mathbb{Z}_p$-algebra R, set $R(i) := R \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^i$

Note: if $G_{\mathbb{Q}}$ acts on R, it also acts on $R(i)$

(iii) $X = \mathbb{P}^1$
$$H^2(\mathbb{P}^{\text{an}}, \mathbb{Q}_p) \cong H^1(\mathbb{G}_m^{\text{an}}, \mathbb{Q}_p) \cong \mathbb{Q}_p(-1)$$
as $G_{\mathbb{Q}}$-modules More generally, if X smooth projective of dimension d, then
$$H^{2d}(X^{\text{an}}, \mathbb{Q}_p) \cong \mathbb{Q}_p(-d)$$

Hodge-Tate Decomposition
Fix a prime p, $K/\mathbb{Q}_p$ finite extension
$$K \subset \overline{K} \subset \hat{\overline{K}} = \mathbb{C}_p$$
$G_K = \text{Gal}(\overline{K}/K)$ acts on $\overline{K}$ and G_K acts on $\mathbb{C}_p$.

Theorem 3.4 (Hodge-Tate Decomposition). *Say X/K is a smooth projective variety, then there is a natural G_K-equivariant isomorphism*
$$H^n(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K} \otimes_K \mathbb{C}_p(-1)$$
where G_K acts in the natural way on both sides.

To use this theorem use

Theorem 3.5 (Tate). *Fix $i \neq j \in \mathbb{Z}$*
$$\text{Hom}_{G_K}(\mathbb{C}_p(i), \mathbb{C}_p(j)) = 0$$
$$\text{Ext}^1_{G_K}(\mathbb{C}_p(i), \mathbb{C}_p(j)) = 0$$

Example 3.3. (i) $X = \mathbb{P}^1/K$, $n = 2$
$$\begin{array}{ccc} H^2(X_{\overline{K}}, \mathbb{Q}_p) \otimes \mathbb{C}_p & \cong & \left(H^2(X, \mathcal{O}_X) \otimes \mathbb{C}_p\right) \oplus \left(H^1(X, \Omega^1_X) \otimes \mathbb{C}_p(-1)\right) \oplus \left(H^0(X, \Omega^1_X) \otimes \mathbb{C}_p(-1)\right) \\ \| & & \| \\ \mathcal{O}_p(-1) \otimes \mathbb{C}_p & & 0 \oplus \mathbb{C}_p(-1) \oplus 0 \end{array}$$

$\mathbb{C}_p(-1)$

(ii) $X = E$ elliptic curve over K
$$\begin{array}{ccc} H^1(X_{\overline{K}}, \mathbb{Q}_p) \otimes \mathbb{C}_p & = & \left(H^1(X, \mathcal{O}_X) \otimes_K \mathbb{C}_p\right) \oplus \left(H^0(X, \Omega^1_X) \otimes_K \mathbb{C}_p(-1)\right) \\ \wr\| & & \| \\ T_p(E)^\vee \otimes \mathbb{C}_p & & \mathbb{C}_p \oplus \mathbb{C}_p(-1) \end{array}$$

$\text{Lie}(E^r) \otimes \mathbb{C}_p$ $\text{Lie}(E)^\vee \otimes \mathbb{C}_p(-1)$

Corollary 3.2. *X/K smooth projective then*
$$H^i(X, \Omega^j_{X/K}) \cong \left(H^{i+j}(X_{\overline{K}}, \mathbb{Q}_p) \otimes \mathbb{C}_p(j)\right)^{G_K}$$

Remark. Ito used this cor to reprove

Theorem 3.6. *X, Y Calabi-Yao varieties over $\mathbb{C}$, $X \overset{bir}{\sim} Y$, then* $\dim H^j(X, \Omega^i_{X/K}) =: h^{i,j}(X) = h^{i,j}(Y)$

Remark. There exists a good variant for general X

Hodge-Tate Spectral Sequence
Use perfectoid spaces to prove

Theorem 3.7 (HT,SS)**.** *$C/\mathbb{Q}_p$ complete and algebraically closed X/C proper smooth rigid-analytic space then there exists an E_2 spectral sequence*

$$E_2^{ij} : H^i(X, \Omega^j_{X/C})(-j) \longrightarrow H^{i+j}(X, \mathbb{Q}_p) \otimes \mathbb{C}$$

then get Hodge-Tate filtration on $H^n(X, \mathbb{Q}_p) \otimes \mathbb{C}$

Remark. (i) HT SS is functorial then if X is defined over K (with $K/\mathbb{Q}_p$ finite) then Tate's Theorem then get HT decomposition for X.

(ii) the HT SS always degenerates (Conrad-Gabbor) but not canonically so:

Example 3.4. Say $X = E$ elliptic curve. HT SS then low degree SES

$$0 \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathbb{Q}_p) \otimes \mathbb{C}_p \longrightarrow H^0(X, \Omega^1_X)(-1) \longrightarrow 0$$

maps go the wrong way cannot choose a splitting that varies well in family

4 Name: Lecture Title

4.1 Lecture 1

4.1.1 Lecture Name

5 Name: Lecture Title

5.1 Lecture 1

5.1.1 Lecture Name

6 Name: Lecture Title

6.1 Lecture 1

6.1.1 Lecture Name

7 Name: Lecture Title

7.1 Lecture 1

7.1.1 Lecture Name

8 Name: Lecture Title

8.1 Lecture 1

8.1.1 Lecture Name

9 Name: Lecture Title

9.1 Lecture 1

9.1.1 Lecture Name

10 Name: Lecture Title

10.1 Lecture 1

10.1.1 Lecture Name

11 Name: Lecture Title

11.1 Lecture 1

11.1.1 Lecture Name

12 Name: Lecture Title

12.1 Lecture 1

12.1.1 Lecture Name

13 Name: Lecture Title

13.1 Lecture 1

13.1.1 Lecture Name

14 Name: Lecture Title

14.1 Lecture 1

14.1.1 Lecture Name

15 Name: Lecture Title

15.1 Lecture 1

15.1.1 Lecture Name

16 Name: Lecture Title

16.1 Lecture 1

16.1.1 Lecture Name

17 Name: Lecture Title

17.1 Lecture 1

17.1.1 Lecture Name

18 Name: Lecture Title

18.1 Lecture 1

18.1.1 Lecture Name

Part II

Course/Project Outlines & Lecture Notes