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PERFECTOID SPACES

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p-adic Hodge theory

Ana Caraiani
Shimura varieties

Kiran Kedlaya
Sheaves, stacks, and shtukas

Jared Weinstein
Adic spaces

with Peter Scholze

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Perfectoid Spaces

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1 Opening Lecture

Historic Remarks about the genesis of the paper "Perfectoid Spaces"

or why perfectoid spaces are a failed theory.

In 2007, Scholze went to Bonn as an undergrad and studied under M. Rapport.

Let X be a smooth projective scheme over $\mathbb{Q}_p$. Fix $i \geq 0$ and let $l \neq p$ be a prime. Consider the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representation $V = H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_l)$. There is a weight decomposition given by the following: if $\Phi \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is a geometric Frobenius, then

$$V = \bigoplus_{j=0}^{2i} V_j,$$

where Φ acts through the Weil numbers of weight j on V_j.

Rapport-Zink (1980) if X has semistable reduction

de Joung (1995) in general (reduction to semistable case).

Rapport gave Scholze the following problem to think about:

There is a monodromy operator $N : V \to V(\pm 1)$ (Tate twist) coming from the action of the inertia subgroup. In particular, $N : V_j \to V_{j-2}$. Then

$$\forall j = 0, \ldots, i : N^j : V_{i+j} \xrightarrow{\sim} V_{i-j}.$$

Conjecture 1.1 (Weight-Monodromy Conjecture)**.** *Let X be a smooth projective scheme over $\mathbb{Q}_p$. Fix $n \geq 0$ and $l \neq p$ a prime. Consider the* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*-representation* $V = H^i_{ét}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_l)$

Example 1.1.

(i) If X has good reduction, i.e. there exists a smooth projective $\mathfrak{X}/\mathbb{Z}_p$ with generic fiver X, then

$$\begin{array}{ccc} V & \cong & H^i_{\text{ét}}(\mathfrak{X}_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_l) \\ \circlearrowleft & & \circlearrowleft \\ \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) & \twoheadrightarrow & \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \end{array}$$

so that the inertia group acts trivially.

But then $N = 0$

$\forall j = 0, N^j = 0 : V'_{i+j} \xrightarrow{\sim} V_{i-j}$

so equiv, $V_j = 0 \; \forall j \neq i$.

i.e. $V = V_i$.

But this follows from the Weil conjectures for $\mathfrak{X}_{\mathbb{F}_p}$.

(ii) If $X = E$ is an elliptic curve with multiplicative reduction

'$E = \mathbb{G}_m/q^{\mathbb{Z}}$', $0 \neq q \in \mathbb{Q}_p$, $|q| < 1$ as rigid-analytic/adic spaces

Then

$$H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_l) = H^1_{\text{ét}}(\mathbb{G}_{m,\overline{\mathbb{Q}}_p/q^{\mathbb{Z}}}, \overline{\mathbb{Q}}_l).$$

Then by Hochschild-Serre spectral sequence

$$H^i(\mathbb{Z}, \underbrace{H^j_{\text{ét}}(\mathbb{G}_{m,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_l)}_{=\begin{cases}\overline{\mathbb{Q}}_l, & j=0\\ \overline{\mathbb{Q}}_l(-1), & j=1\\ 0, & \text{otherwise}\end{cases}}) \Longrightarrow H^{i+j}_{\text{ét}}(\mathbb{G}_{m,\overline{\mathbb{Q}}_p/q^{\mathbb{Z}}}, \overline{\mathbb{Q}}_l).$$

with trivial $\mathbb{Z}$-action.

So

$$0 \longrightarrow \overline{\mathbb{Q}}_l \longrightarrow H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_l) \longrightarrow \overline{\mathbb{Q}}_l(-1) \longrightarrow 0.$$

So $V_2 = \overline{\mathbb{Q}}_l(-1)$, $V_0 = \overline{\mathbb{Q}}_l$

Splitting $V = V_0 \oplus V_2$ depends on choice of Φ.

Weight-monodromy predicts $N : V_2 \cong V_0$ can be checked by hand. Use that inertia action is trivial on l-power roots of q for $i = 1, 2$.

Remark.

(i) Conjecture is known for $i = 1, 2$. dim 1: reduce to abelian varieties or curves and use Néron models/semistable models. dim 2: Rapport-Zink + de Jong.

(ii) Known in equal characteristic p, i.e. over $\mathbb{F}_p((f))$.

Proved in Deligne's Weil 2 paper, uses that L-functions over function fields have good properties.

(iii) Conversely, weight-monodromy conjectures critical to understanding local factors of Hasse-Weil zeta functions at places of bad reduction ($\Leftrightarrow$ the Hasse-Weil zeta function "has no poles in region of absolute convergence.")

Rapport's suggestion: Try to reduce to case of equal characteristic after base change to some very ramified $K/\mathbb{Q}_p$.

Idea: If $\mathfrak{X}/\mathcal{O}_K$ integral (semistable, say) model of $X \times_{\mathbb{Q}_p} K$, then $\mathfrak{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_K/p$ lives over $\mathcal{O}_K/p \cong \mathbb{F}_q[t]/t^e$, where e is the ramification index of $K/\mathbb{Q}_p$.

If $e \gg 0$, this is almost $\mathbb{F}_p[\![t]\!]^0$.

Of course, this does not really work, as even if e is large, still not deform

$\mathfrak{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_K/p$ from $\mathcal{O}_K/p = \mathbb{F}_p[t]/t^e$ to $\mathbb{F}_p[\![t]\!]$.

Usually, there are (a lot of) obstructions.

Also, in the end need to relate $V = H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_l)$ acting on $\operatorname{Gal}(\mathbb{F}_p((t))^{\text{sep}}/\mathbb{F}_p((t)))$, where $X^1/\mathbb{F}_p((f))$ is the generic fiber of deformation.

In semistable case, can use log-geometry to do this (related to isomorphism of tame quotients of $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\operatorname{Gal}(\mathbb{F}_p((t))^{\text{sep}}/\mathbb{F}_p((t)))$).

Turning these ideas in my head, lead

Theorem 1.1 (Fontaine-Wintenberger)**.** $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(p^{1/p^\infty})) \cong \operatorname{Gal}(\mathbb{F}_p((f))^{sep}/\mathbb{F}_p((t)))$*, and even canonically.*

proof involves Fontain's construction like

$$\varprojlim_{\text{Frob}} \mathcal{O}_{\overline{\mathbb{Q}}_p}/p.$$

Hard to understand what it means. Later, I learned from Faltings that

Theorem 1.2.

$$\pi^{ét}(\operatorname{Spec} \mathbb{Q}_p(p^{1/p^\infty})\langle T^{\pm 1/p^\infty}\rangle) \cong \pi^{ét}(\operatorname{Spec} \mathbb{F}_p((t))\langle T^{\pm 1}\rangle$$

Things started to resolve after I realized the following proof of Fontaine-Winterberger's Theorem

{finite étale $\mathbb{Q}_p(p^{1/p^\infty})$-alg} {almost finite étale $\mathbb{Z}_p[p^{1/p^\infty}]$-alg} {almost finite étale $\mathbb{Z}_p[p^{1/p^\infty}]/p$-alg} {almost finite étale $\mathbb{F}_p[t^{1/p^\infty}]/t$-alg} {finite étale $\mathbb{F}_p((t))(t^{1/p^\infty})$-alg} {finite étale $\mathbb{F}_p((t))$-alg}

This suggested what to do in the relative case. Find some notion of 'perfectpod'.

{perfectoid $\mathbb{Q}_p(p^{1/p^\infty})$-alg} {perfectoid almost $\mathbb{Z}_p[p^{1/p^\infty}]$-alg} (needs unique lifting property)

{perfectoid almost $\mathbb{Z}_p[p^{1/p^\infty}]/p$-alg} {perfectoid almost $\mathbb{F}_p[t^{1/p^\infty}]/t$-alg} $\vdots$ {perfectoid $\mathbb{F}_p((t))(t^{1/p^\infty})$-alg}

If R perfectoid (almost) $\mathbb{Z}_p[p^{1/p^\infty}]/p$-algebra, then cotangent complex of perfectoid almost $L_{R/(\mathbb{Z}[p^{1/p^\infty}]/p)} = 0$.

Lemma 1.1 (Gabber-Romero). *If $S \to R$ is a map of $\mathbb{F}_p$-algebras that is "relatively perfect", i.e. relative Frob $\Phi_{R/S} : R \otimes_S ??? \xrightarrow{\sim} R$ is an isomorphism, then $L_{R/S} \cong 0$.*

Proof (Sketch). $\Phi_{R/S}$ isomorphism of $L_{R/S}$ but also equal as $d(x^p) = px^{p-1}\,dx = 0$.

Definition. A perfectoid $\mathbb{Q}_p(p^{1/p^\infty})$-algebra is a uniform Banach $\mathbb{Q}_p(p^{1/p^\infty})$-algebra R such that $(R^0/p)/(\mathbb{Z}_p[p^{1/p^\infty}]/p)$ is relatively perfect, where R^0 is the set of powerbounded elements in R, equivalently, $\Phi : R^0/p \to R^0/p$, given by $x \mapsto x^p$, is surjective.

Corollary 1.1. *Set of R perfectoid $\mathbb{Q}_p(p^{1/p^\infty})$-algebra mapping to $R^\flat$ set of perfectoid $\mathbb{F}_p((t))(t^{1/p^\infty})$-algebras.*

This can be made explicit in terms of Fontaine's functor:

$$R^\flat = \varprojlim_{\text{Frob}}(R^0/p) \otimes_{\mathbb{F}_p[\![t]\!][t^{1/p^\infty}]} \mathbb{F}_p((t))(t^{1/p^\infty}).$$

pass to geometry.

Corollary 1.2. $(\mathbb{P}^{n,ad}_{\mathbb{F}_p((t))})_{ét} = \varprojlim_{\varphi}(\mathbb{P}^{n,ad}_{\mathbb{Q}_p(p^{1/p^\infty})})_{ét}$ $\varphi(x_0 : \cdots : x_n) = (x_0^p : \cdots : x_n^p)$

Now $X \subset \mathbb{P}^n_{\mathbb{Q}_p}$ is your smooth projective variety.

$$\begin{array}{ccc}
\mathbb{P}^n_{\mathbb{F}_p((t))} & \xrightarrow{\pi} & \mathbb{P}^n_{\mathbb{Q}_p(p^{1/p^\infty})} \\
\circlearrowleft & & \circlearrowleft \\
\pi^{-1}(X_{\mathbb{Q}_p(p^{1/p^\infty})} & \longrightarrow & X_{\mathbb{Q}_p(p^{1/p^\infty})}
\end{array}$$

Applying Deligne to bottom left

Problem: This is not algebraic. But how far can it be away from algebraic?

Easy case: If X is complete intersection, then any ϵ-neighborhood of $\pi^{-1}(X_{\mathbb{Q}_p} p^{1/p^\infty})$, there are algebraic varieties of same dimension enough to conclude.

2 Closing Lecture

Where do we go from here?

For example, Yves André has recently used perfectoid spaces to prove the following of Hochester (73):

Theorem 2.1 (Direct Summand Conjecture). *Let R be a regular ring, $R \hookrightarrow S$. Then $R \hookrightarrow S$ has a splitting as R-modules.*

The forward direction is descent along $R \to S$.

Part of Hochster's "homological conjectures" refined by Bhatt, Ma, Schwede, ...

developed theory of test ideals in mixed characteristic

also: connections to algebraic toplogy via topological hochschild homology

but for the rest of talk, let's concentrate on "mixed-characteristic shtukas"

History of shtukas: function fields

Let $C/\mathbb{F}_q$ be a projective smooth geometrically coneccted. Let $G/\mathbb{F}_q$ be reductive groups, e.g. $G = \mathrm{GL}_2$ cufve. moduli space of shtukas over C with one leg.

$$f : \mathrm{Sht}_{\cdots}^{\cdots} \longrightarrow C.$$

alongue of Shumura varieties

$$\mathrm{Sh} \longrightarrow \operatorname{Spec} \mathbb{Z}$$

$R^i f_* \overline{\mathbb{Q}}_l \circlearrowleft \pi_q(C) = \mathrm{Gal}(\overline{F}/F)^{curv}$

$$\begin{array}{ccccc} R^i f_* \overline{\mathbb{Q}}_l & \circlearrowright & \pi_q(C) & \cong & \mathrm{Gal}(\overline{F}/F)^{curv} \\ \circlearrowleft & & & & \\ G(\mathbb{A}) & & & & \end{array}$$

where F is the function field of C and $\mathbb{A} = \mathbb{A}_F$ are the adéles of F.

Theorem 2.2 (Drinfeld, L. Lafforge...). $R^i f_* \overline{\mathbb{Q}}_l = \oplus_* \pi \otimes \sigma(\pi) \circlearrowleft \mathrm{Gal}(\overline{F}/F)$ * *certain automorphic cpr π of $G(A)$*

This association $\{$autom. rep. of $G(\mathbb{A})\}$ $\{$Gal rep$\}$ $\pi \mapsto \sigma(\pi)$. define the global Langlands correspondence (in some cases)

Unfortunately, not *all* autmorphic π.

Insight of Drinfeld: Cal get all π if one looks at spaces of shtukas with two legs.

2 legs:

$$f : \mathrm{Sht}_{\cdots}^{\cdots} \to C \times C$$

$$\begin{array}{ccccc} R^i f_* \overline{\mathbb{Q}}_l & \circlearrowright & \pi_1(C \times C/\phi^{\mathbb{Z}}) & \cong & \pi_1(C) \\ \circlearrowleft & & & & \\ G(\mathbb{A}) & & & & \end{array}$$

where congruence Drinfeld's lemma

Theorem 2.3 (Same people)**.** *For good choices of data*

$$R^i f_* \overline{\mathbb{Q}}_l = \bigoplus_{*} \pi \otimes \sigma(\pi) \otimes o(\pi)^V$$

* *all cuspidal automorphic reprep of* $G(\mathbb{A})$ *and* $\pi_1(C) \circlearrowleft \sigma(\pi)$ *and* $\pi_!(C) \circlearrowleft o(\pi)^V$

Get global langland's cooorespondence for GL_2: Drinfeld GL_n: L. Lafforgue any G: V. Lafforgue

We would love to do the same over number fields.

Obvious problem: what is the analogue of $C \otimes_{\mathbb{F}_q} C$?

Magic of diamonds: Can we make sense of not $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$ but at last of $\operatorname{Spec}\mathbb{Q}_p \times_{\mathbb{F}_1} \operatorname{Spec}\mathbb{Q}_p$ (or even $\operatorname{Spec}\mathbb{Z}_p \times \operatorname{Spec}\mathbb{Z}_p =$ compeltion at $(p_1 p)$).

Namely, can take product $\operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{Q}_p$ in category of diamons get something 2-dimensional.

$$\operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{Q}_p = \operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{Q}_p^{tet}/\underline{\mathbb{Z}}_p^* = \operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{F}_p((t^{1/p^\infty}))/\underline{\mathbb{Z}}_p^* = (\tilde{\mathbb{D}}^*_{\mathbb{Q}_p})^\diamond/\underline{\mathbb{Z}}_p^*.$$

where last is perfectoid punctured open unit disk$/\mathbb{Q}_p$

analogue of Drinfeld's lemma:

Theorem 2.4. $\pi_1(\operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{Q}_p/\varphi^{\mathbb{Z}}) \cong \pi_1(\operatorname{Spd}\mathbb{Q}_p) \times \pi_1(\operatorname{Spd}\mathbb{Q}_p) = \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \times \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. *Equivantely,*

$$\pi_1(\tilde{\mathbb{D}}^*_{\mathbb{Q}_p}/\mathbb{Q}_p^*) = \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^2.$$

$\mathbb{Q}_p^* = \underline{\mathbb{Z}}_p^* \times \varphi^{\mathbb{Z}} = p^{\mathbb{Z}}$ *or* $\pi_1(\tilde{\mathbb{D}}^*_{\mathbb{C}_p}/\underline{\mathbb{Q}}_p^*) = \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

moduli spaces of local, mixed class shtukas with one leg

$$\mathrm{Sht}^{\dots}_{\dots} \longrightarrow \operatorname{Spd}\mathbb{Q}_p.$$

There turn out to be (generalizations of) Rapport-ZInk spaces (local p-adic analogues of Shumura varities)

Example (lubin-tate spaces)

Let $H/\overline{\mathbb{F}}_p$ 1-dimensional formal group of height n (then is p-div. group)

deformation space of H:

$$\mathfrak{X}_H \cong \operatorname{Spf} W(\overline{F}_p)\,[\![u_1, \dots, u_{k-1}]\!]$$

generic fibre $\mathcal{M}_H$ $(n-1)$-dimensional open unit disc

tower

$$\cdots \longrightarrow \mathcal{M}_{H,2} \longrightarrow \mathcal{M}_{H,0} = \mathcal{M}_H$$

$\mathcal{M}_{H,m}$ classifies isomorphisms

$$\mathcal{H}[p^m] \cong (\mathbb{Z}/p^m\mathbb{Z})^n,$$

where $\mathcal{H}$ universal deformation of H.

$$\mathcal{M}_{H,\infty} = \varprojlim_m \mathcal{M}_{H,m}$$

perfectoid space (S.-Weinstein)

Theorem 2.5 (S., Weinstein)**.** *Let $C/\mathbb{Q}_p$ algebraically closed complete extension, let $\infty \in FF_{C^\flat}$ Fargue-Fontaine corresponding to $C^\flat$. Then*

$$\mathcal{M}_{H,\infty}(C) = \{\mathcal{O}^n \overset{f}{\hookrightarrow} \mathcal{O}(1/n) \textit{ sthm coker } f \textit{ is supported at } \infty\}$$

This can be also said in terms of shtukas with one leg at ∞

several legs: there is no obstruction to considering moduli spaces of shtukas with any number of legs.

Test objects: $S \in \mathrm{Pfd} = \{\text{perfectoid spaces of char } p\}$ legs at s-valued $x_1, \ldots, x_n : S \to \operatorname{Spd} \mathbb{Q}_p$ of $\operatorname{Spd} \mathbb{Q}_p$

These correspond to untilts $S_1^\#, \ldots, S_n^\#$ of S.

graph of $x_i S \to \operatorname{Spd} \mathbb{Q}_p Y_S = S \times \operatorname{Spd} \mathbb{Q}_p$

closed immersions of adic spaces $S_i^\# \xrightarrow{\sim} S$ can consider φ-modules over Y_S (or compactification of it) with poles zeroes at the divisors.

$f : \mathrm{Sht}^{\cdots}_{\cdots} \longrightarrow \operatorname{Spd} \mathbb{Q}_p \times \operatorname{Spd} \mathbb{Q}_p$

$$\begin{array}{ccc} R^i f_* \overline{\mathbb{Q}}_l & \circlearrowright & \pi_1(\operatorname{Spd} \mathbb{Q}_p \times \operatorname{Spd} \mathbb{Q}_p / \phi^{\mathbb{Z}}) \\ \circlearrowleft & & \| \\ G(\mathbb{Q}_p) & & \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^2 \end{array}$$

Theorem 2.6. $\pi_0(THH(\mathcal{O}_C)^\wedge_p)^{hS^1} = \mathbb{A}_{inf}$