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# TOPOLOGY AND ARITHMETIC

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*Lubin–Tate spaces: old and new questions*

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# Topology and Arithmetic

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# 1 Michael Hopkins: Lubin-Tate spaces: old and new questions

## 1.1 Lecture 1

Abelian Galois extensions of $\mathbb{Q}$ are $\mathbb{Q}(\zeta_n)$, where $\zeta_n$ is the $n$th root of 1.

Lubin-Tate: Get Galois extensions using formal groups. Formal group law over $R$. $F(x,y) = x +_F y = x + y + \cdots$

$x +_F 0 = 0 +_F x = x$ $x +_F y = y +_F x$ $(x +_F y) +_F z = x +_F (y +_F z)$

Lie variety over $R$ Objects $\mathbb{A}^n$, $n = 0, 1, \ldots$ maps $\mathbb{A}^n \to \mathbb{A}^1$, $f(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n]$ $\mathbb{A}^n = A^n \to \prod \mathbb{A}^1$

Question: How many formal group laws are there? How to construct formal group laws?

**Theorem 1.1** (Lazard). *$R \mapsto$ formal group laws over $R$, ring($L, R$), $L = \mathbb{Z}[x_1, x_2, \ldots]$.*

Isomorphism $F \xrightarrow{g} G$ $g(x)$ $g(x +_F y) = g(x) +_F g(y)$

Universal isomorphism over $L[s_1, s_2, \ldots]$

Algebraic Topology

Cohomology theories $E$ with Chern classes in complex line bunles, $V/X \to c_i(X) \in E^{2n}(X)$ $c_n(V\mathbb{C})_N = \sum_{i+j=n} c_i(Y)c_j(W)$

Not true in general $c_1(L_1 \otimes L_2) = c_1(L_1)$

**Theorem 1.2** (Quillen). *For general, $E$, there exists a formal group law $F$ $c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)) = c_1(L_1) +_F c_1(L_2)$*

$H^s(\mathfrak{m}_{FG}, \omega^*) \Rightarrow \pi_{2t-s}S^0 = \lim_{n\to\infty} \pi_{2t-s+n}S^n$ $\omega = \mathrm{Lie}\, F^*$

**Example 1.1.** $G_n$ $x + y$ $G_m$ $x + y - xy = 1 - (1-x)(1-y)$

Are these isomorphic?

$g(x) = 1 - e^{-x}$ $g(x+y)$ $\overset{=}{\text{maybe}}=$ $g(x)g(x) - g(x)g(y)$ over $\mathbb{Q}$-algebra So isomorphic over rationals.

Are they isomorphic over $\mathbb{F}_p$? Are there even homomorphisms between them? Suppose $g: G_n \to G_m$ is one such $g(x + \cdots + x) = 1 - (1 - g(x))^p$ $0 = g(0) = g(x)^4 = g^0(x^p)$ so that $g = 0$ so no homomorphisms from additive group to multiplicative group.

Height:

$R = k$ field of char $p > 0$ $f : G_1 \to G_2$ Then there exists unique $g(x)$, $g'(0) \neq 0$, $y = p^a$ $f(x) = g(x^q)$ $a$ is the height of $f$. Height of a formal group is by definition the height of mult by $p$ height $G_n = \infty$ height $G_m = 1$

**Theorem 1.3** (Dieudonne). *$k$ perfect algebraically closed, any two formal groups of the same height are isomorphic.*

Lubin-Tate deformation spaces $\Gamma$, $k$ field char $p > 0$, $B$ complete local $\mathfrak{m}$-maximal local

A deformation of $\Gamma$ to $B$ $B \xrightarrow{r} B/\mathfrak{m} \xleftarrow{i} k$

$(G, i, f)$, $G \xrightarrow{f} i^*\Gamma$ $\mathrm{Deform}_\Gamma(B) \leftarrow$ groupoid

**Theorem 1.4** (L-T). *$n =$ height of $\Gamma$ $\pi_0$ $\mathrm{Deform}_\Gamma(B) = \mathfrak{m}^{n-1}$*

We want to understand this set $\mathfrak{m}^{n-1}$ mod by automorphisms of $\Gamma$.

$G_{\text{univ}}$ universal deformation $W[[u_1, \ldots, u_{n-1}]]$ $W$ witt vectors of $k$ Universal deformation

$E_0 = W[[u_1 - u_{n-1}]]$ $E_* = W[[U_1 - u_{n-1}]][u, u^{-1}]$, $|u| = -2$

$\operatorname{Aut}\Gamma = S_n$, acts on $E_*$ $UE_0 = E_{-2}$ sections of Lie $G$ interested in $H^*(S_n; E_0)$, not the symmetric group $H^*(S_n; E_{2n})$

Question: Can one write down explicitly the action of $\operatorname{Aut}\Gamma$ on $W[[u_1 - u_{n-1}]]$.

Question: What is Pic(Lubin tate) $= H^1(\operatorname{Aut}\Gamma; E_0^*)$, conjectured answer enlists known $n = 2$, $p > 5$

Observation: $n = 2$, $H^*(S_n; W) \xrightarrow{\sim} H^*(S_n; E_0)$ $p > 3$ Shimomura $p \leq 3$ Beaudiy, Bobkova, Behrens, Wenn, ... True for $n > 2$?

## 1.2 Lecture 2

### 1.3 Lecture 3

## 1.4 Lecture 4

# 2 Jacob Lurie: Tamagawa numbers in the function field case

## 2.1 Lecture 1

**Definition.** $q$ and $q'$ are in the same genus if they are $\simeq$ mod $N$ for all $N > 0$.

If $q$ is a form over $\mathbb{Z}$ and $R$ a commutative ring.

$$\{A \in \mathrm{GL}_n(R)\colon q \circ A = q\} = O_q(R) \supseteq O_q(\mathbb{R}) \supseteq O_q(\mathbb{Z})$$

a compact Lie group of dimension $n(n-1)/2$.

$$\mathrm{Mass}(q) = \sum_{q' \text{of genus } q} \frac{1}{|O_{q'}(\mathbb{Z})|},$$

where the sum is taken over equivalence classes of quadratic forms.

**Definition** (Unimodular)**.** $q$ is unimodular if nondegenerate mod $p$ for all $p$

$x^2 + y^2 \equiv (x+y)^2 \mod 2$.
Mass Formula (Unimodular Case):
$8 \mid n$ $\mathrm{Mass}(q) =$ something else but

$$\mathrm{Mass}(q) = \sum_{q' \text{unimodular}} \frac{1}{|O_q(\mathbb{Z})|} = \frac{\zeta(n/2)\zeta(2)\zeta(4)\cdots\zeta(n-2)}{\mathrm{Vol}(S^1)\,\mathrm{Vol}(S^2)\cdots\mathrm{Vol}(S^{n-1})}$$

**Example 2.1.** $n = 8$

$$RHS = \frac{1}{2^{14}\cdot 3^5 \cdot 5^2 \cdot 7}$$

Then Mass-formula tells you there is a unique unimodular form in 3 variables.

**Example 2.2.** $n = 32$ RHS is approximately 40,000,000. Looking at left side, this implies there exists *a lot* of inequivalent unimodular forms in 32 variables.

Let $q, q'$ are in the same genus. $q = q' \circ A_N$ for some $A_n \in \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z})$. WLOG $\{A_N\} = A \in \mathrm{GL}_n(\hat{\mathbb{Z}})$ $\hat{\mathbb{Z}} = \operatorname{proj\,lim} \mathbb{Z}/N\mathbb{Z} = \prod_p \mathbb{Z}_p$ $q = q' \circ A \Rightarrow q, q'$ are equivalent over $\mathbb{Z}_p$ for all $p$ Then $q, q'$ are equivalent over $\mathbb{Q}_p = \mathbb{Z}[1/p]$.

Hasse-Minkowski: Then $q = q' \circ B$, where $B \in \mathrm{GL}_n(\mathbb{Q})$ $q = q' \circ A = q \circ B^{-1} \circ A$ $B^{-1} \circ A \in O_q(\mathbb{Q})/O_q(A^{\mathrm{fin}})/O_q(\hat{\mathbb{Z}}))$ Want to count size of this.

$B^{-1} \circ A \in O_q(\mathbb{Q})/O_q(A)/O_q(\hat{\mathbb{Z}} \times \mathbb{R}))$

$A$ has a natural topology that makes it into a locally compact ring containing $\mathbb{Q}$ as a discrete subring. This induces $O_q(A)$, which has the structure of a locally compact group with discrete subgroup $O_q(\mathbb{Q})$ and $O_q(\hat{Z} \times \mathbb{R})$, a compact open subgroup.

$O_q(\mathbb{Q})/O_q(\mathbb{A})$ acted on by $O_q(\hat{\mathbb{Z}} \times \mathbb{R})$

$$\#\text{ of orbits} = \frac{\mu(O_p(\mathbb{Q})\, O_q(\mathbb{A}))}{\mu(O_q(\hat{\mathbb{Z}} \times \mathbb{R}))}$$

Not quite correct.

$\mathrm{SO}_Q(A)$ has a canonical Haar measure called Tamagawa measure

$$2^k\,\mathrm{Mass}(q) = \frac{\mu(\mathrm{SO}_q(\mathbb{Q})/\,\mathrm{SO}_q(A))}{\mu(\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R}))}$$

$\mathrm{SO}_q(A) = \mathrm{SO}_q(\mathbb{R}) \times \prod_p^{\mathrm{res}} \mathrm{SO}_q(\mathbb{Q}_p)$ $V_{\mathbb{R}}$ is the space of translation invariant topological forms on $\mathrm{SO}_q(\mathbb{R})$. $V_{\mathbb{R}} \supseteq V_{\mathbb{Q}}$ the space of translation invariant topological forms on $\mathrm{SO}_q(\mathbb{Q})$

$V_{\mathbb{Q}_p}$ the space of translation invariant topological forms on $\mathrm{SO}(\mathbb{Q}_p)$

$\mathrm{SO}_q(\mathbb{Q}_p)$ is a $p$-adic analytic Lie group.

$0 \neq \omega \in V_{\mathbb{Q}} \mapsto \mu_{\omega,\mathbb{R}}$

$0 \neq \omega \in V_{\mathbb{Q}} \mapsto \mu_{\omega,\mathbb{Q}_p}$

Tamagawa Measure

$$\mu_{\mathrm{Tam}} = \prod_p \mu_{\omega,\mathbb{Q}_p} \times \mu_{\omega,\mathbb{R}}$$

independent of $\omega$

$$\mathrm{Mass}(q) = 2^{-k} \frac{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbb{Q}) / \mathrm{SO}_q(\mathbb{R}))}{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R}))}$$

$\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R}) = \mathrm{SO}_q(\mathbb{R}) \times \prod_p \mathrm{SO}_q(\mathbb{Z}_p)$ $\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R})) \stackrel{\mathrm{def}}{=} \mu_{\omega,\mathbb{R}}(\mathrm{SO}_q(\mathbb{R})) \times \prod_p \mu_{\omega,\mathbb{Q}_p}(\mathrm{SO}_q(\mathbb{Z}_p))$

Mass Formula (Tamagawa-Weil Version) $\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbb{Q}) / \mathrm{SO}_q(\mathbb{A})) = \mathbb{Z}$ $\mathrm{SO}_q$ has a two-sheeted double cover $\mathrm{Spin}_q$

Equivalent: $\mu_{\mathrm{Tam}}(\mathrm{Spin}_q(\mathbb{Q}) / \mathrm{Spin}_q(A)) = 1$

Conjecture (Weil)

Let $G$ be a simply connected semisimple algebraic group over $\mathbb{Q}$ $\mu_{\mathrm{Tam}}(G(\mathbb{Q})/G(\mathbb{A})) = 1$, where $G(\mathbb{Q})$ is $\tau_G$, the Tamagawa number of $G$.

Now a theorem, proved by Weil in many cases, Langlands when split group, ...,

## 2.2 Lecture 2

$X \to \mathrm{Spec}(\mathbb{F}_q)$, where $X$ smooth projective curve over $\mathbb{F}_q$, write $K_X$ for the fraction field of $X$. A field which arrives this way is called a function field.

Number Fields $\mathbb{Q}$ prime numbers $p$ and point at $\infty$ $\mathbb{Z}/p\mathbb{Z}$ $\mathbb{Z}_p$ $\mathbb{Q}_p$ or $\mathbb{R}$ $\mathbb{A}$ quadratic form $q_0$ over $\mathbb{Q}$ ($\mathrm{SO}_{q_0}$) $\mathrm{SO}_{q_0}(\mathbb{Q}) \subseteq \mathrm{SO}_{q_0}(\mathbb{A})$ $\mu_{\mathrm{Tam}}$ $\mu_{\mathrm{Tam}}(\mathrm{Spin}_{q_0}(\mathbb{Q})/\mathrm{Spin}_{q_0}(A))$ q quadratic form over $\mathbb{Z}$ $\mathrm{SO}_q(\mathbb{Z}/p\mathbb{Z})$ $\mathrm{Mass}(q) = \sum_{q'\text{quad at } q} \frac{1}{|O_q(\mathbb{Z})|}$ Mass Formula

Function Fields closed points $x \in X$ $k(x)$ field at $x$ $\mathcal{O}_x$ complete local ring of $X$ at $\mathcal{O}_x \cong k(x)[[z]]$ $K_a \sim k(x)((t))$ $\mathbb{A}_x = \prod^{\mathrm{res}}_{x\in X} K_x$ semisimple group $G_0$ over $K_x$ $G_0(K_x) \subseteq G_0(A_x)$ $\mu_{\mathrm{Tam}}$ $\mu_{\mathrm{Tam}}(G(K_x)/G_0(A_x)) = 1$ group scheme $G \to X$ (Ex: $G = X \times \mathrm{GL}_n$, $G = X \times \mathrm{SL}_n$) $G(X(x))$ $\sum_{\text{Prin G-bund P on X}} \frac{1}{|\mathrm{Aut}(P)|}$ Mass Formula $\sum_p \frac{1}{|\mathrm{Aut}(P)|} = q^D \prod_{x\in X} \frac{|??|}{|??|}$, where $d = \dim(G_0 Y K_x)$

$\mathrm{Bun}_G(X)$ the moduli stack of $G$-bundles

Maps: $\mathrm{Spec}\, R \to \mathrm{Bun}_G(X)$ similar $G$-bundle on $X \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}\, R$

Goal: Compute $\sum \frac{1}{|\mathrm{Aut}(S)|} =: |\mathrm{Bun}_G(X)(\mathbb{F}_q)|$

Digression $Y$ algebraic variety over $\mathbb{F}_q$ $|Y(\mathbb{F}_q)|$ Idea: $\overline{Y} := Y \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}(\overline{F}_q)$ Think of $Y(\mathbb{F}_q) \subseteq \overline{Y}$ $\overline{Y} \xrightarrow{u} \overline{Y}$, where $u$ is geometric frobenius

$$\overline{Y} \xrightarrow{u} \overline{Y}$$

$$\mathbb{P}^n \xrightarrow{u} \mathbb{P}^n$$

$[x_0 : \cdots : x_n]$, $[x_0^q : \cdots : x_n^q]$

$Y(\mathbb{F}_q)$ fixed points of $u$

Ideal (Weil) $|Y(\mathbb{F}_q)|$ should be $\sum(-1)^i \mathrm{Tr}(u \mid H^i(\overline{Y}))$

This is now a theorem of Grothendieck-Lefschetz Formula

Assume $Y$ smooth of dimension $d$

$H_i^i(\overline{Y}) \sim H^{2d-i}(\overline{Y})^\vee$ poincare duality Not $u$-equivariant

$\sum(-1)^i \mathrm{Tr}(u^{-1} \mid H^i(\overline{Y})) = \frac{|Y(\mathbb{F}_q)|}{q^?}$

Idea apply this to $Y = \mathrm{Bun}_G(X)$

**Definition.** $Y = \mathrm{Bun}_G(x)$ satisfies the trace formula if

$$\frac{\sum \frac{1}{|\mathrm{Aut}(P)|}}{q^{\dim \mathrm{Bun}_G(X)} = \sum(-1)^i \frac{\mathrm{Tr}(u^{-1})}{|H^i(\overline{\mathrm{Bun}_G(x)})|} =: \mathrm{Tr}\, |u^{-1}| H^*(\overline{\mathrm{Bun}_G(X)})} \; ?$$

Weil's conjecture follows from two assertions

1. $\mathrm{Bun}_G X$ satisfies GL $\frac{\sum 1/|\mathrm{Aut}(P)|}{/} q^D = \mathrm{Tr}(u^{-1} \mid H^*(\overline{\mathrm{Bun}_G(X)})) =$

2 $\prod_{x\in X} \left(\frac{|G(k(X))|}{|K(x)|^d}\right)^{-1}$

First equality in 1. shown by theorem of Behrend in case $G$ is a constant group, or everywhere semisimple.

Digression:

Let $x \in X$ be closed point. $\mathrm{Bun}_G(\{x\}) = BG_x$.

$\mathrm{Bun}_G(\{x\})(\mathbb{F}_q)$ is set of principle $G$-bundles on $\mathrm{Spec}(k(X))$ has one object, namely symmetry group is $G(K(x))$

$$\frac{|\,\mathrm{Bun}_G(\{x\})(\mathbb{F}_q)|}{q^{\dim \mathrm{Bun}_G(\{x\})}} = \frac{|k(x)|^d}{|G(k(x))|}$$

$\mathrm{Bun}_G(\{x\})$ satisfy GL trace formula

$$\frac{|k(x)|^d}{|G(k(x))|} = \mathrm{Tr}(u^{-1} \mid H^*(\overline{\mathrm{Bun}_G(\{x\})}))$$

$\mathrm{Tr}(u^{-1} \mid H^*(\mathrm{Bun}_G(X)) = \prod_{x \in X} \mathrm{Tr}(u^{-1} \mid H^*(\mathrm{Bun}_G(\{x\}))).$

$\mathrm{Bun}_G(X) = \prod_{x \in X}^{\mathrm{cont}} \mathrm{Bun}_G(\{x\})$

$H^?(\overline{\mathrm{Bun}_G(X)}) = \bigoplus_{x \in X}^{\mathrm{cont}} H^*(\mathrm{Bun}_G(\{x\}))$ Makes sense using theory of factorization homology

$\prod_{x \in X} \dfrac{1}{1 - 1/|k(x)|^2}$ $|\mathrm{SL}_2(\mathbb{F}_q)|/q^{\dim} = (q^3 - q)/q^3 = 1 - 1/q^2$

$\mathrm{Bun}_G(X) = \sqcup_{\,x \in \mathbb{Z}} \mathrm{Bun}_G^?(x)$

## 2.3 Lecture 3

## 2.4 Lecture 4

## 2.5 Lecture 5

# 3 Matthew Morrow: Topological Hochschild homology in arithmetic geometry

## 3.1 Lecture 1

Goal: Classical Hochschild/cyclic homology Topological versions relations to alg/arith geometry

Today: Classical Theory Fix commutative base ring $k$ For any $k$-algebra $A$ have hochschild complex

$$\mathrm{HH}(A/k) := A \leftarrow A \otimes_k A \leftarrow A \otimes_k A \otimes_k A$$

given by maps $a_0 \otimes a_1 = a_0 a_1 - a_1 a_0$ $a_0 \otimes a_1 \otimes a_2 \mapsto a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 + a_2 a_0 \otimes a_1$

hochschild homology $\mathrm{HH}_n(A/k)$, $n \geq 0$, are homology of $\mathrm{HH}(A/k)$.

1. $\mathrm{HH}_0(A/k) = A/\langle ab - ba \rangle = A/[A, A] = A$ (if $A$ is commutative)

2. If $A$ is commutative then $\mathrm{HH}_1(A/k) = A \otimes_k A/\langle ab \otimes c - a \otimes bc + ac \otimes b \colon a, b, c, d \rangle$ (Leibniz ??? $\leftrightarrow a \otimes b$)$= \Omega^1_{A/k}$

3. $\mathrm{HH}_*(A/k) = \bigoplus_{n \geq 0} \mathrm{HH}(A/k)$ as commutative $k$-algebra ($A$-algebra) $A$ commutative

4. 1.-s. then by universal property of $\Omega^*_{A/k} = \Lambda^*_A \Omega^1_{A/k}$

**Theorem 3.1** (Hochschild-Kashent-Rosenberg, 60s)**.** *If $A$ is smooth over $k$, then the maps $\epsilon_n : \Omega^n_{A/k} \to \mathrm{HH}_n(A/k)$ are isomorphisms.*

Philosophy (connes, feigir-Tsygs lodey-quillen) think of $\mathrm{HH}_*$ as generators of diff. forms (even if $A$ is noncommutative).

To prove HKR, adopt homological perspective on HH.

**Lemma 3.1.** *For any flat $k$-algebra $A$, $\mathrm{HH}(A/k) \cong A \otimes_{A \otimes A^{op}} A$.*

*Proof.* Explicit isom. of complexes

$$\mathrm{HH}(A/k) \cong A \otimes_{A \otimes A^{\mathrm{op}}} \underbrace{[A \otimes_k A \leftarrow A \otimes_k \otimes_k A \otimes_k A]}_{\text{Bar complex}}$$

Bar complex is resolution of $A$ by flat $A \otimes_k A^{\mathrm{op}}$ modules.

**Corollary 3.1.** $\mathrm{HH}_*(A/k) \cong \mathrm{Tor}_*^{A \otimes A^{op}}(A, A)$

*Proof.* (HKR thm) A smooth $k$-algebra must show that $\mathrm{HH}_*(A/k)$ is the exterior algebra on its degree 1 elements—this is well known for this graded algebra.

$$\mathrm{Tor}_*^B(C, C)$$

when $B \to C$ (surj?) has kernel is locally generated by a regular sequence $A \otimes_k A \to A$ (given $k \to A$ smooth).

Next: cyclic homology. $\mathrm{HH}(A/k) = A_{\mathbb{Z}/1} \leftarrow \underbrace{A \otimes_k A}_{\mathbb{Z}/2} \leftarrow \cdots$ where $\mathbb{Z}/n+1 C A^{\otimes n+1}$ generated $t_n : a_0 \otimes \cdots_n \mapsto a_n \otimes \cdots \otimes a_{n-1}$

Set 'norm' $N := \sum_{i=0}^n (-1)^n t_n : A^{\otimes n+1} \to A^{\otimes n+1}$

Extra dengenercy: $s : A^{\otimes n} \to A^{\otimes n+1}$ given by $a_0 \otimes a_n \mapsto 1 \otimes a_0 \otimes \cdots \otimes a_n$

Connes operator: $S : A^{\otimes n} \xrightarrow{N} A^{\otimes n} \xrightarrow{s} A^{\otimes n+1} \xrightarrow{1-(-1)^n t_n}$

Check: $B^2 = 0$, $Bb = -bB$, where $b$ is the boundary map in HH. 'mixed complex or an algebraic $S^1$-complex.

i.e. $B : \mathrm{HH}(A/k) \to \mathrm{HH}(A/k)[-1]$
Idea: This refers to de Rham diff. commutative diagram

$$\begin{tikzcd} \mathrm{HH}_n(A/k) \arrow[r, "B"] & \mathrm{HH}_{n+1}(A/k) \\ \Omega^n_{A/k} \arrow[u, "\epsilon_n"] \arrow[r, "d"] & \Omega^{n+1}_{A/k} \end{tikzcd}$$

Def: hochschild complex

$$\begin{tikzcd} A^{\otimes 4} \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "B"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\ A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] & \\ A^{\otimes 2} \arrow[d, "b"] & A \arrow[l, "B"'] & & \\ A & & & \end{tikzcd}$$

right section $x \geq 0$ while left is $\leq 0$
HP$(A/K)$ (periodic cyclic homology) product totalization of this complex
HC$(A/k)$ (cyclic homology) totalization of $x \geq 0$
HC$^-(A/k)$ (negative cyclic homology) totalization of $x \leq 0$

$$0 \longrightarrow \mathrm{HH} \longrightarrow \mathrm{HC} \xrightarrow{s} \mathrm{HC}[2] \longrightarrow 0$$

$$0 \longrightarrow \mathrm{HC}^-[-2] \xrightarrow{s} \mathrm{HC}^- \longrightarrow \mathrm{HH} \longrightarrow 0$$

Norm sequence: $0 \longrightarrow \mathrm{HC}^- \longrightarrow \mathrm{HP} \longrightarrow \mathrm{HC}[s] \longrightarrow 0$

$$\mathrm{HP} \cong \operatorname{proj\,lim}(\cdot\, \mathrm{HC}[-4] \xrightarrow{s} \mathrm{HC}[-1] \xrightarrow{3} \mathrm{HC}) \to S : \mathrm{HP} \xrightarrow{\cong} \mathrm{HP}[z]$$

$\mathrm{HP}_n(A/k) \cong \mathrm{HP}_{n+z}(A/k)$.
coarse info about hh gives coarse info about HP, HC$^-$, HC
Example: assume $\mathrm{HH}_{\mathrm{odd}}(A/k) = 0$, e.g. a perfectoidish. THen $\mathrm{HP}_0(A/k)$ is a complex filtered ring which ecnodes a lot of the above data. Here precisely $\mathrm{HP}_0(A/k)$ is a ring with filtered by ideals

$$\mathrm{Fil}^n \mathrm{HP}_0(A/k) = S^n(\mathrm{HC}^-_{2n}(A/k))$$

such that $\mathrm{HP}_0(A/k)/\mathrm{Fil}^n \cong \mathrm{HC}_{2n-2}(A/k)$ and $\mathrm{gr}^h \cong \mathrm{HH}_{2n}()$

## 3.2 Lecture 2

### 3.3 Lecture 3

### 3.4 Lecture 4

## 3.5 Lecture 5

# 4 Kirsten Wickelgren: $\mathbb{A}^1$-enumerative geometry

## 4.1 Lecture 1

Enumerative Geometry: counts algebraic-geometric objects satisfying conditions over $\mathbb{C}$

Goal: To record information about fields of definition

Arithmetic count of the lines on a smooth cubic surface

**Definition.** Cubic surface is $\{(x,y,z)\colon f(x,y,z)=0\}$, where $f$ is degree 3

Better: $X \subseteq \mathbb{P}^3 = \{[w,x,y,z]\}$, $[w,x,y,z] = [\lambda w, \lambda x, \lambda y, \lambda z]$, where $\lambda \in K^\times$ $X = \{[w,x,y,z]\colon f(w,x,y,z) = 0\}$

**Theorem 4.1** (Salmon, Cayley 1849). *Let $X$ be a smooth, cubic surface over $\mathbb{C}$. Then $X$ contains exactly 27 lines.*

**Example 4.1.** Fermat $f(w,x,y,z) = w^3+x^3+y^3+z^3$ $L = \{[S,-S,T,-T]\colon [S,T] \in \mathbb{P}^1\}$ $\lambda, \omega : \lambda^3 = \omega^3 = -1$

Lines $\{[S,\lambda S, T, \omega T]\colon S,T \in \mathbb{P}^1\}$ This produces $3\cdot 3\cdot 3 = 27$ llines

Modern proof: $\mathrm{Gr}(1,3)$, the Grassmannian parametrizing lines in $\mathbb{P}^3$, equivalently $W \subseteq \mathbb{C}^4$, $\dim W = 2$

Let $S \to \mathrm{Gr}(1,3)$ be the tautological bundle, $S_W = W$ $\mathrm{Sym}^3 S^* \to \mathrm{Gr}(1,3)$ $\mathrm{Sym}^3 S_W^*$ is the cubic polynomial on $W$, i.e. $\mathrm{Sym}^3 W^*$ $F$ determines element $\mathrm{Sym}^3(\mathbb{C}^4)^*$ then $f$ determines a section of $\mathrm{Sym}^3 S^*$ by $\sigma_f(W) = f|_W$

Note: the line $\mathbb{P}W$ corresponding to $W$ is in $X \Leftrightarrow \sigma_f(W) = 0$.

Want: to count zeros of $\sigma_f$ Euler class: $V \to M$ be a rank $r$ $\mathbb{R}$-vector bundle on a dimension $r$ $\mathbb{R}$-manifold $M$. Assume $V$ is oriented

Choose a section $\sigma$ with only isolated zeros. $\deg[S^{r-1}, S^{r-1}] \to \mathbb{Z}$, homotopy classes of maps

$P \in M$, $\sigma(p) = 0$ To define: $\deg_p \sigma \in \mathbb{Z}$

Here's how: choose local coordinates on $M$ around $p$. There's a small ball around $p$ with no other zeros. Choose local trivialization of $V$. Then $\sigma$ can be identified with a function $\sigma : \mathbb{R}^r \to \mathbb{R}^r$ given by $0 \mapsto 0$, $\sigma(\overline{B_0(1)} = 0) \subset \mathbb{R}^r = 0$ $S^{r-1} = OB_0(1) \xrightarrow{\overline{\sigma}} \partial B_0(1) = S^{r-1}$ given by $x \mapsto \sigma(x)/|\sigma(x)|$

Then $\deg_p \sigma = \deg(\overline{\sigma})$

Euler class $e(V) = \sum_{p:\sigma(p)=0} \deg_p \sigma$

Fact: $X$ smooth then $\deg_p \sigma = 1$, then number of lines on $X = e(\mathrm{Sym}^3 S^*)$. In particular, number of lines is independent of $X$ $e(\mathrm{Sym}^3 S^*) = 27$

Question: What about cubic surfaces over $\mathbb{R}$?

Segre in 20th century showed $X$ can have 3, 7, 15, or 27 real lines.

Segre 1942 distinguished between different hyperbolic and elliptic real lines on $X$

Recall: $L$ real line, $L \cong \mathbb{P}^1_{\mathbb{R}}$, $\mathrm{Aut}(L) \cong \mathbb{P}\,\mathrm{GL}_2(\mathbb{R})$ $I \leftrightarrow I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $z \mapsto \dfrac{az+b}{cz+d}$ $\mathrm{Fix}(I) = \{z\colon cz^2 + (d-a)z + b = 0\}$ either consists of 3 real points if and only if $I$ hyperbolic. a ?? conjugate pair of points elliptic.

We associate an involution $I$ to $L \subset X$ a real line on a real cubic surface.

$p \in L$ $T_pX \cap X = L \cup Q$ $Q \cap L =$ points $q$ such that $T_qX = T_pX = \{p, p'\}$ $I(p) = p'$

**Definition.** $L$ is elliptic/hyperbolic when $I$ is

Alternatively, spin structure.

**Example 4.2.** Fermat cubic surface $x^3 + y^3 + z^3 = -1$ hyperbolic

**Theorem 4.2** (Segre,Okonek,Teleman,...). *Number of hyperbolic lines $-$ number of elliptic lines $= 3$.*

$\mathbb{A}^1$-homotopy theory (due to Morel-Voemsky)

On smooth schemes over $k$, $k$ a field. Morel deg: $[\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \to GW(k)$ where $GW(k)$ is the Grothendieck-Witt=group completion of semiring $\oplus, \otimes$ isomorphism classes of (nondegenerate symmetric) bilinear forms $B : V \times V$, finite dimensional $k$ vector spaces

presentation: generators: $\langle a \rangle$, $a \in k^\times$ $\langle a \rangle : k \times k \to k$, $(x, y) \mapsto axy$ relations: $\langle ab^2 \rangle = \langle a \rangle$, $b \in k^\times$ $\langle a \rangle + \langle b \rangle = \langle a + b \rangle = \langle ab(a + b) \rangle$

**Example 4.3.** $GW(\mathbb{C}) \cong \mathbb{Z}$, rank, $B \mapsto \dim V$

**Example 4.4.** $GW(\mathbb{R}) \to \mathbb{Z} \times \mathbb{Z}$ by signatures $\times$ rank. Also iso to $\mathbb{Z} \times \mathbb{Z}$ $GW(\mathbb{F}_q) \to \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$ given by signatures $\times$ rank, isomorphism

There is an Euler class
$$e(V) = \sum_{p:\sigma(p)=0} \deg_p \sigma$$
$R$ field char not 2. $X$ a smooth cubic surface over $k$ line $L \subseteq X$ is a closed point of $\mathrm{Gr}(1,3)$ $L = \{[a, b, c, d]S + [a', b', c', d']T \colon [S, T] \in \mathbb{P}^1\}$ $k(L) = k(a, b, c, d, a', b', c', d')$ $\mathbb{P}^1_{k(L)} \cong L \subseteq X_{k(L)} \subseteq \mathbb{P}^3_{k(L)}$ Given a line $L$ on $X$, obtain involution $I \in \mathrm{Aut}(L) \cong \mathbb{P}\,\mathrm{GL}_2\, k(L)$ $\mathrm{Fix}(I)$ is either 2 $k(L)$ points or a conjugate pair of points in $k(L)[\sqrt{D}]$ for $D \in k(L)^*/(k(L)^*)^2$

**Definition.** $\mathrm{Type}(L) := \langle D \rangle \in GW(k(L))$

Equivalent to $D = ab - cd$, $I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\mathrm{Type}(L) = \langle -1 \rangle \deg I$

**Theorem 4.3** (Kass-W.). *char $R \neq 2$, $X$ smooth cubic surface*
$$\sum_{lines L of X} \mathrm{Tr}_{k(l)/k} \mathrm{Type}(L) = 15\langle 1 \rangle + 12\langle -1 \rangle$$
$\mathrm{Tr}_{k(L)/k} : GW(k(L)) \to GW(k)$ *given by* $(B : V \times V \to k(L)) \mapsto V \times V \xrightarrow{B} k(L) \to k$

$R = \mathbb{C}$, apply rank, number of lines is 27 $k = \mathbb{R}$ apply signature Number of hyperbolic lines $-$ number elliptic lines $= 3$

**Corollary 4.1.** *$k = \mathbb{F}_q$ Number of elliptic lines $L$ with $k(L) = \mathbb{F}_{q^{2n+1}}$ plus number of hyperbolic lines with $k(L) = \mathbb{F}_{q^{2n+1}}$ is equivalent to 0.*

### 4.2 Lecture 2

### 4.3 Lecture 3

## 4.4 Lecture 4