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TOPOLOGY AND ARITHMETIC

Michael Hopkins
Lubin–Tate spaces: old and new questions

Jacob Lurie
Tamagawa numbers in the function field case

Matthew Morrow
Topological Hochschild homology in arithmetic geometry

Kirsten Wickelgren
$\mathbb{A}^1$-enumerative geometry

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Part I
Talk Notes

1 Michael Hopkins: Lubin-Tate spaces: old and new questions

1.1 Lecture 1

We start with the famous theorem of Kronecker-Weber Theorem.

Theorem 1.1. *If $K/\mathbb{Q}$ is an abelian Galois extension, then $K = \mathbb{Q}(\zeta_n)$ for some n, where ζ_n is a (primitive) nth root of unity.*

Kronecker's famous JugendTraub. Want to construct? Using the roots of unity. to make abelian Galois extensions.

Lubin-Tate: Get Galois extensions using formal groups. Formal group law over R. $F(x,y) = x +_F y = x + y + \cdots$

$x +_F 0 = 0 +_F x = x$ $x +_F y = y +_F x$ $(x +_F y) +_F z = x +_F (y +_F z)$

Lie variety over R Objects $\mathbb{A}^n$, $n = 0, 1, \ldots$ maps $\mathbb{A}^n \to \mathbb{A}^1$, $f(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n]$ $\mathbb{A}^n = A^n \to \prod \mathbb{A}^1$

Question: How many formal group laws are there? How to construct formal group laws?

Theorem 1.2 (Lazard)**.** *$R \mapsto$ formal group laws over R, ring(L, R), $L = \mathbb{Z}[x_1, x_2, \ldots]$.*

Isomorphism $F \xrightarrow{g} G$ $g(x)$ $g(x +_F y) = g(x) +_F g(y)$

Universal isomorphism over $L[s_1, s_2, \ldots]$

Algebraic Topology

Cohomology theories E with Chern classes in complex line bunles, $V/X \to c_i(X) \in E^{2n}(X)$ $c_n(V\mathbb{C})_N = \sum_{i+j=n} c_i(Y)c_j(W)$

Not true in general $c_1(L_1 \otimes L_2) = c_1(L_1)$

Theorem 1.3 (Quillen)**.** *For general, E, there exists a formal group law F $c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)) = c_1(L_1) +_F c_1(L_2)$*

$H^s(\mathfrak{m}_{FG}, \omega^*) \Rightarrow \pi_{2t-s} S^0 = \lim_{n\to\infty} \pi_{2t-s+n} S^n$ $\omega = \mathrm{Lie}\, F^*$

Example 1.1. G_n $x + y$ G_m $x + y - xy = 1 - (1-x)(1-y)$

Are these isomorphic?

$g(x) = 1 - e^{-x}$ $g(x+y) \overset{=}{\text{maybe}} = g(x)g(x) - g(x)g(y)$ over $\mathbb{Q}$-algebra So isomorphic over rationals.

Are they isomorphic over $\mathbb{F}_p$? Are there even homomorphisms between them? Suppose $g : G_n \to G_m$ is one such $g(x + \cdots + x) = 1 - (1 - g(x))^p$ $0 = g(0) = g(x)^4 = g^0(x^p)$ so that $g = 0$ so no homomorphisms from additive group to multiplicative group.

Height:

$R = k$ field of char $p > 0$ $f : G_1 \to G_2$ Then there exists unique $g(x)$, $g'(0) \neq 0$, $y = p^a$ $f(x) = g(x^q)$ a is the height of f. Height of a formal group is by definition the height of mult by p height $G_n = \infty$ height $G_m = 1$

Theorem 1.4 (Dieudonne)**.** *k perfect algebraically closed, any two formal groups of the same height are isomorphic.*

Lubin-Tate deformation spaces Γ, k field char $p > 0$, B complete local $\mathfrak{m}$-maximal local

A deformation of Γ to B $B \xrightarrow{r} B/\mathfrak{m} \xleftarrow{i} k$

(G, i, f), $G \xrightarrow{f} i^*\Gamma$ $\mathrm{Deform}_\Gamma(B) \leftarrow$ groupoid

Theorem 1.5 (L-T). *$n =$ height of Γ π_0 Deform$_\Gamma(B) = \mathfrak{m}^{n-1}$*

We want to understand this set $\mathfrak{m}^{n-1}$ mod by automorphisms of Γ.

G_{univ} universal deformation $W[[u_1, \ldots, u_{n-1}]]$ W witt vectors of k Universal deformation

$E_0 = W[[u_1 - u_{n-1}]]$ $E_* = W[[U_1 - u_{n-1}]][u, u^{-1}]$, $|u| = -2$

$\operatorname{Aut}\Gamma = S_n$, acts on E_* $UE_0 = E_{-2}$ sections of Lie G interested in $H^*(S_n; E_0)$, not the symmetric group $H^*(S_n; E_{2n})$

Question: Can one write down explicitly the action of $\operatorname{Aut}\Gamma$ on $W[[u_1 - u_{n-1}]]$.

Question: What is Pic(Lubin tate) $= H^1(\operatorname{Aut}\Gamma; E_0^*)$, conjectured answer enlists known $n = 2$, $p > 5$

Observation: $n = 2$, $H^*(S_n; W) \xrightarrow{\sim} H^*(S_n; E_0)$ $p > 3$ Shimomura $p \leq 3$ Beaudiy, Bobkova, Behrens, Wenn, ... True for $n > 2$?

1.2 Lecture 2

Ex: G_m, $x + y - xy = 1 - (1-x)(1-y)$ over $\mathbb{Z}_p$ $\operatorname{Aut} G_m = \mathbb{Z}_p^\times$ $\lambda \in \mathbb{Z}_p^\times$ $x \mapsto 1 - (1-x)^\lambda$

Lubin-Tate ring $\mathbb{Z}_p$ $\operatorname{Aut} G_m$ acts trivially.

$E_0 = W[[u_1, \ldots, u_{n-1}]]$ $E_* = W[[u \cdots u_{n+1}[u^{\pm 1}]$, $|u| = -2$

$\mathbb{Z}_p[u^{1/2}]$ $\lambda \in \operatorname{Aut}(G_m) = \mathbb{Z}_p^\times$ How does λ act on u?

u^{-1} is an invariant differential on G_m u^{-1} is $dx + \cdots = (1-x)\, dx$

$1 - (1-x)^\lambda = \lambda x + \text{hot}$ $u^{-1} \mapsto \lambda u^{-1}$.

$H^*(\mathbb{Z}_p^\times : \mathbb{Z}_p[u^{\pm 1}])$, $p > 2$ $\lambda^{-1} \in \mathbb{Z}_p^\times$ λ^{-1} generates $\mathbb{Z}_p^\times$ map $(\lambda^{-1})^{p-1} \neq 1 \mod p^2$

$H^1(\mathbb{Z}_p^\times : u^n \mathbb{Z}_p) = \mathbb{Z}_p/(\lambda^n - 1)$ $n \neq 0$

Makes sense for $\lambda \mapsto \lambda^n$ Replace by $\operatorname{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$

$n = 2$ Γ is formal group over k of height 2 $H^*(\operatorname{Aut}\Gamma; W[[u_1]]) =?$

It turns out, $p > 3$ $\Lambda[x_1, x_3] = H^*(\operatorname{Aut}\Gamma; W) \xrightarrow{x} H^*(\operatorname{Aut}\Gamma; W[[u_1]])$

Dieudonne modules

k perfect field W ring of Witt vectors of k k acts on u, $u(x) = x^p$ W acts on u

Diuedonne-module: M free W-module of finite rank $F : u^*M \to M$ $F(am) = a^u F(m)$ for a$a \in W$,

$V : M \to u^*M$ $V(a^u m) = aUM$, $FU = UF = p$

Formal groups over $k \leftrightarrow$ diedonne modules

height $\leftrightarrow \dim_W M$ dim $\leftrightarrow \dim_k M/VM$

Ex: M basis $\gamma, U\gamma$ $F\gamma = V\gamma$

This $M \leftrightarrow$ height 2 formal group over k

$n \neq$

$\{\gamma, V\gamma, \ldots, V^{n-1}\gamma\}$ $F\gamma = V^{b-1}\gamma$

$\operatorname{Aut}\Gamma$, ht 2 $\gamma \to a\gamma + bV\gamma$ $V\gamma \to a^{\phi^{-1}} V\gamma + b^{\phi^{-1}} V^2\gamma = a^{\phi^{-1}} V\gamma + pb^{\phi^{-1}}\gamma$

$F\gamma \to a^\phi V\gamma + pb^\phi \gamma$ $a, b \in W\mathbb{F}_{p^2}$

$F\gamma = V\gamma$ $p\gamma = VF\gamma = V^2\gamma$

$\operatorname{Aut}(\Gamma) = \{ \begin{pmatrix} a & pb^{\phi^{-1}} \\ b & a^{\phi^{-1}} \end{pmatrix} : a, b \in W\mathbb{F}_{p^2} \}$.

Tapis de Cartier

$$\begin{tikzcd} M \arrow[r, dotted, "T"] \arrow[d] & W \arrow[d] \\ M/VM \arrow[r, "\sim"] & k \end{tikzcd}$$

lifts of Γ to W.

Dotted map W-linear specialization $\gamma \to 1$?

Bottom map $\gamma \mapsto 1$

$U_1(T) = T(V\gamma)$ $W[[u_1]]$

$\gamma \xrightarrow{y} a\gamma + bV\gamma$ $V\gamma \to a^{\phi^{-1}} V\gamma + pb^{\phi^{-1}}\gamma$ $y(u_1) = (a^{\phi^{-1}} u_1 + pb^{\phi^{-1}})/(bu_1 + a)$

Crystalline Approximation

$M \to E_{-2} = uW[[u_1 \cdots u_{n-1}]]$

$\gamma \to u$ $V\gamma \to uu_1 \vdots V^{n-1}\gamma \to uu_{n-1}$ $\operatorname{Aut}\Gamma$ equivalent.

G $k \leftarrow W \to u \otimes W$ $G \cong O_n$

Γ

$f(\gamma) = \log_G(x) = x + \cdots + u \otimes w[[x]]$ $f^{-1}(f(x) + f(y)) = x +_F y$

Ex: $\log_{G_m}(x) = \sum x^n/n$

$T : M \to W$ $f(x) = \sum T(F^n\gamma)x^{p^n}/p^n$ is the log of a formal ? over W
Ex: $G = G_m$ $M =?\{\gamma\}$ $F\gamma = \gamma$
$\log = \sum x^{p^n}/p^n$
Ex: ht$= 2$ $T(\gamma) = 1$ $T(v\gamma) = 0$ $f(y) = \sum x^{p^{2n}}/p^n = l(x)$
$W[[w_1]]$ $T(\gamma) = 1$ $T(V\gamma) = W_1$
$f(x) = l(x) + w_1/pl(x^p)$ $f^{-1}(f(y) + f(y))$ does not have coefficients in $?[[w_1]]$
It does have coefficient in the divided power completion $w << w_1 >>$.

1.3 Lecture 3

Last time: $l(x) = \sum x^{p^{2n}}/p^n$ is the log at a formal group over W $l^{-1}(l(x)+l(y)) \in W[[x,y]]$

$f(x) = l(x) + w_1/pl(x^p)$

not the log of a formal grouo

$W[[u_1 \cdots u_{n-1}]]$ acted on by ϕ, $\phi(u_i) = u_i^p$ ϕ Frob on W

$f(x) = x + u_1/p + f^{\phi}(x^p) + \cdots + u_{n-1}/pf^{\phi^{n-1}}(x^{p^n}) + 1/pf^{\phi^n}(x^{p^n})$

$n = 2$ $f(x) = \sum m_k x^{p^k}$ $m_0 = 1$ $m_1 = u_1/p$ $m_2 = 1/p + u_1^{1+p}/p^2$

$p \in I \trianglelefteq A \subset L$ $\phi : L \to L$ $\phi : A \to A$ $\phi(x) \epsilon x^p \mod I$

$s_1, \ldots, \in L$ $\forall \phi^i(s_j) \cdot I \subseteq A$.

Nazewinkel

$f(x) = x + s_1 f^{\phi}(x^p) + \cdots + s_n f^{\phi^n}(x^p)$

Then $f^{-1}(f(x)+f(y)) \in A[[x,y]]$

$l(x) = \sum x^{p^m}/p^n$ $l(x) = x + 1/pl(x^{p^2})$

$f(x) = l(x) + w_1/pl(x^p)$

If we took $\phi(w_1) = 0$ $f(x) = x + w_1/pf^{\phi}(x^p) + 1/pf^{\phi^2}(x^{p^2})$

$I = (p)$ $\phi(w_1) = 0$, need $w_1^p \equiv 0(p) = w_1^p/p = w^{cn}$

$\phi(w^{(1)}) = 0$ then $u^{(1)^p}/p$ if and only if $\forall n$, $w_1^n/n!$

Hazewinkel: $l(x) + w/pl(x^p)$ is the log of a formal group over $W << w_1 >>$, divided powers.

$W[[u_1]] \to W << w_1 >>$ $u_1 \to w_1 \mod p$

extends to an iso $W << u_1 >> \to W << u_2 >>$

Summary: $E_* = W[[u_1 \cdots u_{n-1}]][u_1 u^{-1}]]$, $|u| = -2$

CLaim over $W << u_1 - u_{n-1} >>$ $|u^{t-1}|$ there $w, w_1, \ldots, w_{n-1}$

$w \in u + \cdots$ $w_i \equiv u_i + \cdots$

such that $M \to W << u_1, \ldots, u_{n-1} >>$ $[u^{\pm 1}]$ $\gamma \to W$ $V^i\gamma \to ww_i$ is equivalent for $\operatorname{Aut}\Gamma$.

Q explicitly $w =?$ $ww_i =?$

$n = 2$ $l(x) = w_1/pl(x^p)$ $x + w_1/px^p + x^{p^2}/p^2 + w_1/p^2 x^{p^2}/p^2 \cdots$

Lubin-Tate log $x + m_1 x^p + m_2 x^{p^2}$ THen $w = \lim_{n\to\infty} p^n m_2 n$ $ww_1 = \lim_{n\to\infty} p^n m_{2n-1}$.

$A = A|u_1| = \begin{pmatrix} u_1/p & 1/p \\ 1 & 0 \end{pmatrix}$

Claim $\begin{pmatrix} w & 1/p(ww_1)^{\phi} \\ ww_1 & 1/pw^{\phi} \end{pmatrix} = \lim_{n\to\infty} A^{\phi^n} \cdots AA(i)^{-(n+1)}$

1.4 Lecture 4

2 Jacob Lurie: Tamagawa numbers in the function field case

2.1 Lecture 1

Definition. q and q' are in the same genus if they are $\simeq$ mod N for all $N > 0$.

If q is a form over $\mathbb{Z}$ and R a commutative ring.

$$\{A \in \mathrm{GL}_n(R) \colon q \circ A = q\} = O_q(R) \supseteq O_q(\mathbb{R}) \supseteq O_q(\mathbb{Z})$$

a compact Lie group of dimension $n(n-1)/2$.

$$\mathrm{Mass}(q) = \sum_{q' \text{of genus } q} \frac{1}{|O_{q'}(\mathbb{Z})|},$$

where the sum is taken over equivalence classes of quadratic forms.

Definition (Unimodular)**.** q is unimodular if nondegenerate mod p for all p

$x^2 + y^2 \equiv (x+y)^2 \mod 2$.
Mass Formula (Unimodular Case):
$8 \mid n \,\mathrm{Mass}(q) =$ something else but

$$\mathrm{Mass}(q) = \sum_{q' \text{unimodular}} \frac{1}{|O_q(\mathbb{Z})|} = \frac{\zeta(n/2)\zeta(2)\zeta(4)\cdots\zeta(n-2)}{\mathrm{Vol}(S^1)\,\mathrm{Vol}(S^2)\cdots\mathrm{Vol}(S^{n-1})}$$

Example 2.1. $n = 8$

$$RHS = \frac{1}{2^{14} \cdot 3^5 \cdot 5^2 \cdot 7}$$

Then Mass-formula tells you there is a unique unimodular form in 3 variables.

Example 2.2. $n = 32$ RHS is approximately 40,000,000. Looking at left side, this implies there exists *a lot* of inequivalent unimodular forms in 32 variables.

Let q, q' are in the same genus. $q = q' \circ A_N$ for some $A_n \in \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z})$. WLOG $\{A_N\} = A \in \mathrm{GL}_n(\hat{\mathbb{Z}})$ $\hat{\mathbb{Z}} = \operatorname{proj\,lim} \mathbb{Z}/N\mathbb{Z} = \prod_p \mathbb{Z}_p$ $q = q' \circ A \Rightarrow q, q'$ are equivalent over $\mathbb{Z}_p$ for all p Then q, q' are equivalent over $\mathbb{Q}_p = \mathbb{Z}[1/p]$.

Hasse-Minkowski: Then $q = q' \circ B$, where $B \in \mathrm{GL}_n(\mathbb{Q})$ $q = q' \circ A = q \circ B^{-1} \circ A$ $B^{-1} \circ A \in O_q(\mathbb{Q})/O_q(A^{\mathrm{fin}})/O_q(\hat{\mathbb{Z}}))$ Want to count size of this.

$B^{-1} \circ A \in O_q(\mathbb{Q})/O_q(A)/O_q(\hat{\mathbb{Z}} \times \mathbb{R}))$

A has a natural topology that makes it into a locally compact ring containing $\mathbb{Q}$ as a discrete subring. This induces $O_q(A)$, which has the structure of a locally compact group with discrete subgroup $O_q(\mathbb{Q})$ and $O_q(\hat{Z} \times \mathbb{R})$, a compact open subgroup.

$O_q(\mathbb{Q})/O_q(\mathbb{A})$ acted on by $O_q(\hat{\mathbb{Z}} \times \mathbb{R})$

$$\#\text{ of orbits} = \frac{\mu(O_p(\mathbb{Q})\, O_q(\mathbb{A}))}{\mu(O_q(\hat{\mathbb{Z}} \times \mathbb{R}))}$$

Not quite correct.

$\mathrm{SO}_Q(A)$ has a canonical Haar measure called Tamagawa measure

$$2^k \,\mathrm{Mass}(q) = \frac{\mu(\mathrm{SO}_q(\mathbb{Q})/\,\mathrm{SO}_q(A))}{\mu(\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R}))}$$

$\mathrm{SO}_q(A) = \mathrm{SO}_q(\mathbb{R}) \times \prod_p^{\mathrm{res}} \mathrm{SO}_q(\mathbb{Q}_p)$ $V_{\mathbb{R}}$ is the space of translation invariant topological forms on $\mathrm{SO}_q(\mathbb{R})$. $V_{\mathbb{R}} \supseteq V_{\mathbb{Q}}$ the space of translation invariant topological forms on $\mathrm{SO}_q(\mathbb{Q})$

$V_{\mathbb{Q}_p}$ the space of translation invariant topological forms on $\mathrm{SO}(\mathbb{Q}_p)$

$\mathrm{SO}_q(\mathbb{Q}_p)$ is a p-adic analytic Lie group.

$0 \neq \omega \in V_{\mathbb{Q}} \mapsto \mu_{\omega,\mathbb{R}}$

$0 \neq \omega \in V_{\mathbb{Q}} \mapsto \mu_{\omega,\mathbb{Q}_p}$

Tamagawa Measure

$$\mu_{\mathrm{Tam}} = \prod_p \mu_{\omega,\mathbb{Q}_p} \times \mu_{\omega,\mathbb{R}}$$

independent of ω

$$\mathrm{Mass}(q) = 2^{-k} \frac{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbb{Q}) / \mathrm{SO}_q(\mathbb{R}))}{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R}))}$$

$\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R}) = \mathrm{SO}_q(\mathbb{R}) \times \prod_p \mathrm{SO}_q(\mathbb{Z}_p)$ $\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R})) \stackrel{\mathrm{def}}{=} \mu_{\omega,\mathbb{R}}(\mathrm{SO}_q(\mathbb{R})) \times \prod_p \mu_{\omega,\mathbb{Q}_p}(\mathrm{SO}_q(\mathbb{Z}_p))$

Mass Formula (Tamagawa-Weil Version) $\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbb{Q}) / \mathrm{SO}_q(\mathbb{A})) = \mathbb{Z}$ SO_q has a two-sheeted double cover Spin_q

Equivalent: $\mu_{\mathrm{Tam}}(\mathrm{Spin}_q(\mathbb{Q}) / \mathrm{Spin}_q(A)) = 1$

Conjecture (Weil)

Let G be a simply connected semisimple algebraic group over $\mathbb{Q}$ $\mu_{\mathrm{Tam}}(G(\mathbb{Q})/G(\mathbb{A})) = 1$, where $G(\mathbb{Q})$ is τ_G, the Tamagawa number of G.

Now a theorem, proved by Weil in many cases, Langlands when split group, ...,

2.2 Lecture 2

$X \to \mathrm{Spec}(\mathbb{F}_q)$, where X smooth projective curve over $\mathbb{F}_q$, write K_X for the fraction field of X. A field which arrives this way is called a function field.

Number Fields $\mathbb{Q}$ prime numbers p and point at ∞ $\mathbb{Z}/p\mathbb{Z}$ $\mathbb{Z}_p$ $\mathbb{Q}_p$ or $\mathbb{R}$ $\mathbb{A}$ quadratic form q_0 over $\mathbb{Q}$ (SO_{q_0}) $\mathrm{SO}_{q_0}(\mathbb{Q}) \subseteq \mathrm{SO}_{q_0}(\mathbb{A})$ μ_{Tam} $\mu_{\mathrm{Tam}}(\mathrm{Spin}_{q_0}(\mathbb{Q})/\mathrm{Spin}_{q_0}(A))$ q quadratic form over $\mathbb{Z}$ $\mathrm{SO}_q(\mathbb{Z}/p\mathbb{Z})$ $\mathrm{Mass}(q) = \sum_{q' \text{quad at } q} \frac{1}{|O_q(\mathbb{Z})|}$ Mass Formula

Function Fields closed points $x \in X$ $k(x)$ field at x $\mathcal{O}_x$ complete local ring of X at $\mathcal{O}_x \cong k(x)[[z]]$ $K_a \sim k(x)((t))$ $\mathbb{A}_x = \prod_{x \in X}^{\mathrm{res}} K_x$ semisimple group G_0 over K_x $G_0(K_x) \subseteq G_0(A_x)$ μ_{Tam} $\mu_{\mathrm{Tam}}(G(K_x)/G_0(A_x)) = 1$ group scheme $G \to X$ (Ex: $G = X \times \mathrm{GL}_n$, $G = X \times \mathrm{SL}_n$) $G(X(x))$ $\sum_{\text{Prin G-bund P on X}} \frac{1}{|\mathrm{Aut}(P)|}$ Mass Formula $\sum_p \frac{1}{|\mathrm{Aut}(P)|} = q^D \prod_{x \in X} \frac{|??|}{|??|}$, where $d = \dim(G_0 Y K_x)$

$\mathrm{Bun}_G(X)$ the moduli stack of G-bundles

Maps: $\mathrm{Spec}\, R \to \mathrm{Bun}_G(X)$ similar G-bundle on $X \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}\, R$

Goal: Compute $\sum \frac{1}{|\mathrm{Aut}(S)|} =: |\mathrm{Bun}_G(X)(\mathbb{F}_q)|$

Digression Y algebraic variety over $\mathbb{F}_q$ $|Y(\mathbb{F}_q)|$ Idea: $\overline{Y} := Y \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}(\overline{F}_q)$ Think of $Y(\mathbb{F}_q) \subseteq \overline{Y}$ $\overline{Y} \xrightarrow{u} \overline{Y}$, where u is geometric frobenius

$$\overline{Y} \xrightarrow{u} \overline{Y}$$

$$\mathbb{P}^n \xrightarrow{u} \mathbb{P}^n$$

$[x_0 : \cdots : x_n]$, $[x_0^q : \cdots : x_n^q]$

$Y(\mathbb{F}_q)$ fixed points of u

Ideal (Weil) $|Y(\mathbb{F}_q)|$ should be $\sum (-1)^i \mathrm{Tr}(u \mid H^i(\overline{Y}))$

This is now a theorem of Grothendieck-Lefschetz Formula

Assume Y smooth of dimension d

$H_i^i(\overline{Y}) \sim H^{2d-i}(\overline{Y})^\vee$ poincare duality Not u-equivariant

$\sum (-1)^i \mathrm{Tr}(u^{-1} \mid H^i(\overline{Y})) = \frac{|Y(\mathbb{F}_q)|}{q^?}$

Idea apply this to $Y = \mathrm{Bun}_G(X)$

Definition. $Y = \mathrm{Bun}_G(x)$ satisfies the trace formula if

$$\frac{\sum \frac{1}{|\mathrm{Aut}(P)|}}{q^{\dim \mathrm{Bun}_G(X)} = \sum (-1)^i \frac{\mathrm{Tr}(u^{-1})}{|H^i(\overline{\mathrm{Bun}_G(x)})|} =: \mathrm{Tr}\, |u^{-1}| H^*(\overline{\mathrm{Bun}_G(X)})}^{?}$$

Weil's conjecture follows from two assertions

1. $\mathrm{Bun}_G\, X$ satisfies GL $\frac{\sum 1/|\mathrm{Aut}(P)|}{/} q^D = \mathrm{Tr}(u^{-1} \mid H^*(\overline{\mathrm{Bun}_G(X)})) =$

$2 \prod_{x \in X} \left(\frac{|G(k(X))|}{|K(x)|^d} \right)^{-1}$

First equality in 1. shown by theorem of Behrend in case G is a constant group, or everywhere semisimple.

Digression:

Let $x \in X$ be closed point. $\mathrm{Bun}_G(\{x\}) = BG_x$.

$\mathrm{Bun}_G(\{x\})(\mathbb{F}_q)$ is set of principle G-bundles on $\mathrm{Spec}(k(X))$ has one object, namely symmetry group is $G(K(x))$

$$\frac{|\,\mathrm{Bun}_G(\{x\})(\mathbb{F}_q)|}{q^{\dim \mathrm{Bun}_G(\{x\})}} = \frac{|k(x)|^d}{|G(k(x))|}$$

$\mathrm{Bun}_G(\{x\})$ satisfy GL trace formula

$$\frac{|k(x)|^d}{|G(k(x))|} = \mathrm{Tr}(u^{-1} \mid H^*(\overline{\mathrm{Bun}_G(\{x\})}))$$

$\mathrm{Tr}(u^{-1} \mid H^*(\mathrm{Bun}_G(X)) = \prod_{x \in X} \mathrm{Tr}(u^{-1} \mid H^*(\mathrm{Bun}_G(\{x\})))$.

$\mathrm{Bun}_G(X) = \prod^{\mathrm{cont}}_{x \in X} \mathrm{Bun}_G(\{x\})$

$H^?(\overline{\mathrm{Bun}_G(X)}) = \bigoplus^{\mathrm{cont}}_{x \in X} H^*(\mathrm{Bun}_G(\{x\}))$ Makes sense using theory of factorization homology

$\prod_{x \in X} \dfrac{1}{1 - 1/|k(x)|^2}\, |\mathrm{SL}_2(\mathbb{F}_q)|/q^{\dim} = (q^3 - q)/q^3 = 1 - 1/q^2$

$\mathrm{Bun}_G(X) = \sqcup_{\ x \in \mathbb{Z}} \mathrm{Bun}^?_G(x)$

2.3 Lecture 3

$Y \to \operatorname{Spec}(\mathbb{F}_q)$ $|Y(\mathbb{F}_q)| = \sum_{y \in Y(\mathbb{F}_q)\text{iso classes}} \dfrac{1}{|\operatorname{Aut}(y)|}$

Definition. Y satisfies the G-L trace formula if

$$\frac{|Y(\mathbb{F}_q)|}{q^D} = \operatorname{Tr}(u^{-1} \colon H^*(\overline{Y})) := sum(-1)^i \operatorname{Tr}(u^{-1} \colon H^1(\overline{Y}))$$

For example, true if Y is a variety

Example: G linear algebraic group over $\mathbb{F}_q$ $Y = BG$ $\operatorname{Spec} R \to Y = BG$ equivalent to principle G-bundle on $\operatorname{Spec} R$

Example: $G = \mathbb{G}_m$ $Y = B\mathbb{G}_m$ $Y(\mathbb{F}_q)$ is the category of 1-dimensional vector spaces over $\mathbb{F}_q$ $|Y(\mathbb{F}_q)| = 1/(q-1)$

$D = \dim B|G_m$ $B\mathbb{G}_m = */\mathbb{G}_m$ $\dim B\mathbb{G}_m = -1$

$$\frac{|Y(\mathbb{F}_q)|}{q^{\dim Y}} = \frac{q}{q-1}$$

RHS: $\operatorname{Tr}(u^{-1} \colon H^*(\overline{B\mathbb{G}_m}))$ $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ $B\mathbb{G}_m$, $B\mathbb{C}^*$, quotient of a contractible space by free action of $\mathbb{C}^*$

$\mathbb{C}^*$ acts freely on $V \setminus \{0\}$ V has dimension n $V \setminus \{0\} = S^{2n-1}$

If $\dim V = \infty$ $V \setminus \{0\}$ is contractible $B\mathbb{C}^* = (V \setminus \{0\})/\mathbb{C}^* =: \mathbb{CP}^\infty$

$H^*(\mathbb{CP}^\infty; A) \cong \Lambda[t]$, $\deg t = 2$

$H^*(\overline{B\mathbb{G}_m}) = \mathbb{Q}_\ell[t]$, where $\deg t = 2$

$u(t) = q^t$ $u(t^n) = q^n t^n$ $\operatorname{Tr}(u^{-1} \colon H^*(B\mathbb{G}_m)) = \sum_{n \geq 0} q^{-n} = q/(q-1)$

Conclusion: G-L is okay for $B\mathbb{G}_m$.

ℓ-adic homotopy:

$\overline{Y}$ is an algebraic-geometric object over an algebraically closed field $k = \overline{F}_q$ $y \in \overline{Y}(k)$

$\pi_1^{\text{et}}(\overline{Y}, y)$ profinite group. Assume $\overline{Y}$ is connected.

Then finite etale covers of $\overline{Y}$ are in correspondence with finite sets with continuous action $\pi_1^{\text{et}}(\overline{Y}, y)$.

$\pi_1^{\text{et}}(\overline{Y}, y)_\ell$ the maximal pro-ℓ quotient of $\pi_1^{\text{et}}(\overline{Y}, y)$, $\ell \neq 0$ in k

Artin-Mazur (ℓ-adic version)

Assume: $\pi_1(\overline{Y}, y)_\ell = 0$ if and only if $H^1_{\text{et}}(\overline{Y}, \mathbb{Z}/\ell) = 0$ Also: $H^n_{\text{et}}(\overline{Y}; \mathbb{Z}/\ell)$ finite.

To $\overline{Y}$, they associate a topological space Z (ℓ-adic homotopy type at y) with

1) Z is simply connected to $\pi_n Z$ is a finitely generated $\mathbb{Z}_\ell$-module for all n

2) $H^*_{\text{sing}}(Z; \mathbb{Z}/\ell) \simeq H^*_{\text{et}}(\overline{Y}; \mathbb{Z}/\ell)$

For each $n > 0$, $\pi_n(\overline{Y}) := \pi_n(Z)$ a finitely generated $\mathbb{Z}_\ell$-module

$\pi_n(\overline{Y})_{\mathbb{Q}_\ell} := \pi_n(\overline{Y})[1/\ell]$ finite dimensional vector space over $\mathbb{Q}_p$

Have a canonical pairing $b : \pi_n(\overline{Y})_{\mathbb{Q}_\ell} \times H^n(\overline{Y}) \to \mathbb{Q}_\ell$ $f : S^n \to Z$ $b(f, \eta) = f^*\eta \in H^n(S^n; \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$ $I = H^*_{\text{red}}(\overline{Y}) = \bigoplus_{n \geq 0} H^n(\overline{Y})$ $b : \pi_*(\overline{Y}) \times I \to \mathbb{Q}_\ell$ descends to a pairing $b : \pi_*(Y)_{\mathbb{Q}_\ell} \times I/I^2 \to \mathbb{Q}_\ell$, $\eta = \eta'\eta''$ $f^*(\eta) = f^*(\eta')f^*(\eta'')$

Assertion: If $H^*(\overline{Y})$ is polynomial ring (or even generators) then $\overline{b}$ is a perfect pairing.

$$\cdots \subseteq I^3 \subseteq I^2 \subseteq I \subseteq H^2(\overline{Y})$$

Ex: $\overline{Y} = \overline{B\mathbb{G}_m}$, this applies

Suppose $H(\overline{Y})$ is a polynomial ring $\overline{Y} = Y \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec}(\overline{F}_q)$

$\mathrm{Tr}(u^{-1}\colon H^*(\overline{Y})) := \sum(-1)^i \mathrm{Tr}(u^{-1}|H^i(\overline{Y}))$ $\pi_*\overline{Y}|_{\mathbb{Q}_\ell} \simeq (I/I^2)^\vee$ finite dimensional over $\mathbb{Q}_\ell$ u has complete eigenvalues $\lambda_1, \ldots, \lambda_n$ on $\pi_*(\overline{Y})_{\mathbb{Q}_\ell}$. u^{-1} has eigenvalues $\lambda_1, \ldots, \lambda_r$ $\mathrm{Tr}(u^{-1}|H^*(\overline{Y})) = \mathrm{Tr} * u^{-1}|\mathrm{gr}(H^*(\overline{Y})) = \mathrm{Tr}(u^{-1}|\operatorname{Sym}^*(I/I^2))$

$\mathrm{Tr}(u^{-1}|\operatorname{Sym}^*(I/I^2)) = \sum_{e_1,\ldots,e_n \geq 0} \lambda_1^{e_1}\lambda_2^{e_2}\cdots\lambda_n^{e_n} = \prod_{i=1}^n 1/(1-\lambda_i)$

$\mathrm{Tr}(u^{-1}|H^*(\overline{Y})) = (\det(1-u|(\pi_*\overline{Y})_{\mathbb{Q}_\ell}))^{-1}$

Ex $\overline{Y} = B\mathbb{G}_m$ $(\pi_*\overline{Y})_{\mathbb{Q}_\ell}$ 1-dimensional vector space acted on by $u = 1/q = \det(1-u) = 1-1/q$.

Ex: Let G be any connected linear algebraic group over $\mathbb{F}_q$ over $\mathbb{F}_q$, G-L trace formula for BG.

$$\frac{|BG(\mathbb{F}_q)|}{q^{\dim BG}} = \mathrm{Tr}(u^{-1}|H^*(\overline{BG}))$$

LHS is $q^{\dim G}/|G(\mathbb{F}_q)|$ RHS is $(\det(1-u|\pi_*(\overline{Y})_{\mathbb{Q}_\ell})^{-1}$

Steinberg's Formula $|G(\mathbb{F}_q)| = q^{\dim G}\det(1-u|\pi_*(\overline{BG})_{\mathbb{Q}_\ell})$

Ex: $G = \mathrm{GL}_n$ $H^*(\overline{BG}) = \mathbb{Q}_\ell[c_1, \ldots, c_n]$ $\pi_*(\overline{BG})_{\mathbb{Q}_\ell} = \mathbb{Q}_\ell\{e_1, \ldots, e_n\}$ $u(c_n) = q^i c_i$ $u(e_i) = q^{-i}e_i$

Steinberg $|\mathrm{GL}_n(\mathbb{F}_q)| = q^{n^2}(1-1/q)(1-1/q^2)\cdots(1-1/q^n)$

In general (not assuming $\overline{Y}$ is polynomial)

There is a spectral sequence $\operatorname{Sym}^*(\pi_*\overline{Y})^\vee_{\mathbb{Q}} \to H^*(\overline{Y})$

Gives some conclusion $\mathrm{Tr}(u^{-1}|H^*(\overline{Y})) = \det(1-u|\pi_*(\overline{Y}))^{-1} := \prod_i \det(1-u|\pi_1(\overline{Y}))^{(-1)^?}$ assuming everything converges. [For example, if $\pi_*(\overline{Y})_{\mathbb{Q}_\ell}$ is finite dimensional.]

This will apply when $Y = \mathrm{Bun}_G(X)$.

2.4 Lecture 4

2.5 Lecture 5

3 Matthew Morrow: Topological Hochschild homology in arithmetic geometry

3.1 Lecture 1

Goal: Classical Hochschild/cyclic homology Topological versions relations to alg/arith geometry

Today: Classical Theory Fix commutative base ring k For any k-algebra A have hochschild complex

$$\mathrm{HH}(A/k) := A \leftarrow A \otimes_k A \leftarrow A \otimes_k A \otimes_k A$$

given by maps $a_0 \otimes a_1 = a_0 a_1 - a_1 a_0$ $a_0 \otimes a_1 \otimes a_2 \mapsto a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 + a_2 a_0 \otimes a_1$

hochschild homology $\mathrm{HH}_n(A/k)$, $n \geq 0$, are homology of $\mathrm{HH}(A/k)$.

1. $\mathrm{HH}_0(A/k) = A/\langle ab - ba \rangle = A/[A, A] = A$ (if A is commutative)

2. If A is commutative then $\mathrm{HH}_1(A/k) = A \otimes_k A/\langle ab \otimes c - a \otimes bc + ac \otimes b \colon a, b, c, d \rangle$ (Leibniz ??? $\leftrightarrow a \otimes b$)$= \Omega^1_{A/k}$

3. $\mathrm{HH}_*(A/k) = \bigoplus_{n \geq 0} \mathrm{HH}(A/k)$ as commutative k-algebra (A-algebra) A commutative

4. 1.-s. then by universal property of $\Omega^*_{A/k} = \Lambda^*_A \Omega^1_{A/k}$

Theorem 3.1 (Hochschild-Kashent-Rosenberg, 60s)**.** *If A is smooth over k, then the maps $\epsilon_n : \Omega^n_{A/k} \to \mathrm{HH}_n(A/k)$ are isomorphisms.*

Philosophy (connes, feigir-Tsygs lodey-quillen) think of HH_* as generators of diff. forms (even if A is noncommutative).

To prove HKR, adopt homological perspective on HH.

Lemma 3.1. *For any flat k-algebra A, $\mathrm{HH}(A/k) \cong A \otimes_{A \otimes A^{op}} A$.*

Proof. Explicit isom. of complexes

$$\mathrm{HH}(A/k) \cong A \otimes_{A \otimes A^{\mathrm{op}}} \underbrace{[A \otimes_k A \leftarrow A \otimes_k \otimes_k A \otimes_k A]}_{\text{Bar complex}}$$

Bar complex is resolution of A by flat $A \otimes_k A^{\mathrm{op}}$ modules.

Corollary 3.1. $\mathrm{HH}_*(A/k) \cong \mathrm{Tor}_*^{A \otimes A^{op}}(A, A)$

Proof. (HKR thm) A smooth k-algebra must show that $\mathrm{HH}_*(A/k)$ is the exterior algebra on its degree 1 elements—this is well known for this graded algebra.

$$\mathrm{Tor}_*^B(C, C)$$

when $B \to C$ (surj?) has kernel is locally generated by a regular sequence $A \otimes_k A \to A$ (given $k \to A$ smooth).

Next: cyclic homology. $\mathrm{HH}(A/k) = A_{\mathbb{Z}/1} \leftarrow \underbrace{A \otimes_k A}_{\mathbb{Z}/2} \leftarrow \cdots$ where $\mathbb{Z}/n + 1 C A^{\otimes n+1}$ generated $t_n : a_0 \otimes \cdots_n \mapsto a_n \otimes \cdots \otimes a_{n-1}$

Set 'norm' $N := \sum_{i=0}^n (-1)^n t_n : A^{\otimes n+1} \to A^{\otimes n+1}$

Extra dengenercy: $s : A^{\otimes n} \to A^{\otimes n+1}$ given by $a_0 \otimes a_n \mapsto 1 \otimes a_0 \otimes \cdots \otimes a_n$

Connes operator: $S : A^{\otimes n} \xrightarrow{N} A^{\otimes n} \xrightarrow{s} A^{\otimes n+1} \xrightarrow{1-(-1)^n t_n}$

Check: $B^2 = 0$, $Bb = -bB$, where b is the boundary map in HH. 'mixed complex or an algebraic S^1-complex.

i.e. $B : \mathrm{HH}(A/k) \to \mathrm{HH}(A/k)[-1]$
Idea: This refers to de Rham diff. commutative diagram

$$\begin{tikzcd}
\mathrm{HH}_n(A/k) \arrow[r, "B"] & \mathrm{HH}_{n+1}(A/k) \\
\Omega^n_{A/k} \arrow[u, "\epsilon_n"] \arrow[r, "d"] & \Omega^{n+1}_{A/k}
\end{tikzcd}$$

Def: hochschild complex

$$\begin{tikzcd}
A^{\otimes 4} \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "B"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\
A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] & \\
A^{\otimes 2} \arrow[d, "b"] & A \arrow[l, "B"'] & & \\
A & & &
\end{tikzcd}$$

right section $x \geq 0$ while left is ≤ 0
$\mathrm{HP}(A/K)$ (periodic cyclic homology) product totalization of this complex
$\mathrm{HC}(A/k)$ (cyclic homology) totalization of $x \geq 0$
$\mathrm{HC}^-(A/k)$ (negative cyclic homology) totalization of $x \leq 0$

$$0 \longrightarrow \mathrm{HH} \longrightarrow \mathrm{HC} \xrightarrow{s} \mathrm{HC}[2] \longrightarrow 0$$

$$0 \longrightarrow \mathrm{HC}^-[-2] \xrightarrow{s} \mathrm{HC}^- \longrightarrow \mathrm{HH} \longrightarrow 0$$

Norm sequence: $0 \longrightarrow \mathrm{HC}^- \longrightarrow \mathrm{HP} \longrightarrow \mathrm{HC}[s] \longrightarrow 0$

$$\mathrm{HP} \cong \operatorname{proj\,lim}(\cdot\, \mathrm{HC}[-4] \xrightarrow{s} \mathrm{HC}[-1] \xrightarrow{3} \mathrm{HC}) \to S : \mathrm{HP} \xrightarrow{\cong} \mathrm{HP}[z]$$

$\mathrm{HP}_n(A/k) \cong \mathrm{HP}_{n+z}(A/k)$.
coarse info about hh gives coarse info about $\mathrm{HP}, \mathrm{HC}^-, \mathrm{HC}$
Example: assume $\mathrm{HH}_{\mathrm{odd}}(A/k) = 0$, e.g. a perfectoidish. THen $\mathrm{HP}_0(A/k)$ is a complex filtered ring which ecnodes a lot of the above data. Here precisely $\mathrm{HP}_0(A/k)$ is a ring with filtered by ideals

$$\mathrm{Fil}^n \mathrm{HP}_0(A/k) = S^n(\mathrm{HC}^-_{2n}(A/k))$$

such that $\mathrm{HP}_0(A/k)/\mathrm{Fil}^n \cong \mathrm{HC}_{2n-2}(A/k)$ and $\mathrm{gr}^h \cong \mathrm{HH}_{2n}()$

3.2 Lecture 2

A brief review

A a k-algebra $\to$ Hochschild complex

$\mathrm{HH}(A/k) := A \overset{k}{\leftarrow} A^{\otimes 2} \overset{k}{\leftarrow} A^{\otimes 3} \overset{k}{\leftarrow} \cdots$

$\mathrm{HH}_n(A/k) \leftrightarrow \Omega^n_{A/k}$

Norm sequence: $0 \longrightarrow \mathrm{HC}^-(A/k) \longrightarrow \mathrm{HP}(A/k) \longrightarrow \mathrm{HC}(A/k)[2] \longrightarrow 0$

If $\mathrm{HH}_{odd}(A/k) = 0$, then also for $\mathrm{HC}^-, \mathrm{HP}, \mathrm{HC}$ and get $0 \longrightarrow \mathrm{HC}^-_{2n}(A/k) \longrightarrow \mathrm{HP}_{2n}(A/k) \longrightarrow \mathrm{HC}_{2n-2}(A/k) \to 0$

$\mathrm{HP}_{2n}(A/k) \cong \mathrm{HP}_0(A/k)$

$\mathrm{HP}_0(A/k)$ is filtered ring with associated graded $= \mathrm{HH}_{2*}(A/k)$

Main theorem about smooth algebras (loday-quillen, Feign-Tsyon,Connes) If A is a smooth k-algebra and $k \supseteq \mathbb{Q}$, then the norm sequence looks like $\prod_{i\in\mathbb{Z}} ?[2i]$, where ? is

$$0 \longrightarrow \Omega^{\geq i}_{A/k} \longrightarrow \Omega^{\cdot}_{A/k} \longrightarrow \Omega^{<i}_{A/k} \longrightarrow 0$$

,i.e. HP $\leftrightarrow$ de Rham cohomology $\mathrm{HC}^- \leftrightarrow$ hodge filtration

Proof. Explicit map of chain complexes $\mathrm{HH}(A/k) \xrightarrow{\delta} [A \leftarrow \Omega^1_{A/k} \leftarrow \Omega^2_{A/k} \leftarrow \cdots]$ $a_0 \otimes \cdots \otimes a_n \mapsto \frac{1}{n!} a_0 da_1 \wedge \cdots \wedge da_n$ on $\mathrm{HH}_n(-)$, this splits $\epsilon : \Omega^n \to \mathrm{HH}_n$.

Because smooth algebra, maps are isomorphisms. Tells that δ is quism. So looking at double complex, replace up to quism columns by the 'dumb' complex here. Know b operator compatbile with de Rham so looks like sums of direct copies of de rham complexes. □

HC etc in characteristic p

Thm: R smooth over $\mathbb{F}_p$. Then the classical theorem is still true but the filtration is not naturally split, e.g. $\mathrm{HP}(R/\mathbb{F}_p)$ has a filtration whose graded pieces are $\Omega^i_{R/\mathbb{F}_p}[2i]$, $i \in \mathbb{Z}$.

'classical' proof: yoga if able ??

Today: Analysis if $\mathrm{HP}(R/\mathbb{F}_p)$ via perfectish map $-$ will generalize to topological case.

Idea: Don't study smooth algebra but instead quasiregular semiperfect (qrsp) $\mathbb{F}_p$-algebras

$-$ big (non-noetherian) $-$ but homologically simple

Def: An $\mathbb{F}_p$-algebra A is qrsp if there exist

a perfect $\mathbb{F}_p$-algebra B, $B \xrightarrow{\cong} B$, $b \mapsto b^p$ a regular ideal $I \subseteq B$ I/I^2 is a finite projective B/I-module such that $B/I = A$.

eg of regular ideal: generated by a regular sequence

Examples: 1) $\mathbb{F}_p[t^{1/p^\infty}]/(t)$ 2) if R smooth $\mathbb{F}_p$-algebra then its perfection $R_{\mathrm{perf}} := \operatorname{inj\,lim} R$, where limit over $x \mapsto x^p$ then $R_{\mathrm{perf}} \otimes_R \cdots \otimes_R R_{\mathrm{perf}}$ is qrsp

eg, $\mathbb{F}_p[t]_{\mathrm{perf}} \otimes_{\mathbb{F}_p[t]} \mathbb{F}_p[t]_{\mathrm{perf}} \cong \mathbb{F}_p[t^{1/p^\infty}] \otimes_{\mathbb{F}_p[t]} \mathbb{F}_p[t^{1/p^\infty}] = \mathbb{F}_p[t_1^{1/p^\infty}, t_2^{1/p^\infty}]/(t_1 - t_2)$

Technique: form qrsp to smooth all of our homology theories $F = \mathrm{HH}(-/\mathbb{F}_p), \mathrm{HC}(-/\mathbb{F}_p)$, etc, $\mathbb{F}_p$-algebra $\to D(\mathbb{F}_p)$

satisfy flat descent, meaning $S \to S'$ is a faithfully flat map of $\mathbb{F}_p$-algebras. then $F(S) \to \mathrm{Tot}(F(S') \rightrightarrows F(S' \otimes_S S') \cdots)$

eg, R smooth over $\mathbb{F}_p$, then $R \to R_{\mathrm{perf}}$ is faithfully flat.

So: $F(R) \to \mathrm{Tot}(F(R_{\mathrm{perf}})) \to\to F(R_{\mathrm{perf}} \otimes_R \to R)$

all are qrsp.

Most understand The other homologies $\mathrm{HC}^-, \mathrm{HP}, \mathrm{HC}$ of any group qrs $\mathbb{F}_p$;agebar$\mathbb{Q}_p$

Let A be qrsp

Step 1: $\mathrm{HH}_{odd}(A/\mathbb{F}_p) = 0$ and $\mathrm{HH}_0(A)$ $\mathrm{HH}_k = I/I^2 I$, where $A = B/I$ $\mathrm{HH}_{2n} = \Gamma^n_A(I/I^2)$ nth divided power of I/I^2

$\cong \mathrm{Sym}^n_A(I/I^2)$ but mult. is twised by ntm $m!, n!/(n!m!)$

$\to \mathrm{HH}_{2*}(A/\mathbb{F}_p) \cong \Gamma^{A^*}(I/I^2)$ Key words; cotangent complex.

Step 2: $\mathrm{HP}_0(A/\mathbb{F}_p)$ is a filtered ring with associated graded $\Gamma^*_A(I/I^2)$ What is it?

ANswer: hcc divided power enveolppe of completed B onto A, regular defin. able $f^n/n!$ to surject to A.

Computation: $\mathrm{HP}(R/\mathbb{F}_p)$ (R smooth) is buitl from copies of

$$\mathrm{Tot}(\mathrm{HP}_0(R_{\mathrm{perf}}/\mathbb{F}_p)) \to\to \mathrm{HP}_0(R_{\mathrm{perf}} \otimes \cdots$$

, where R smooth, contributed by divided power enevelopes

ALso show up in theory of derived de Rham cohomolology (Bhatt)

so tot is $\cong \Omega^{\cdot}_{R/\mathbb{F}_{[}}$

3.3 Lecture 3

From HH to Topological HH (HH over the sphere spectrum).

Categorify HH to adapt higher order point of view

$A \longleftrightarrow A^{\otimes 2}$

Def: A cyclic object in a category $\mathcal{C}$ is a simplicial object X in $\mathcal{C}$, i.e. $X : \Delta^{\mathrm{op}} \to \mathcal{C}$ such that each $X_n \in \mathcal{C}$ has action by $Z/n+1$ plus some axioms.

Eg, $\mathrm{HH}(A/k)$ is cyclic object m k-algebras (has universal property)

Conner's cyclic cateogry: there exists $\Lambda \supseteq \Delta$ such that

Objects $\Lambda =$ob $\Delta = \{[n] \colon n \geq 0\}$ $\mathrm{Aut}_\Lambda([n]) = \mathbb{Z}/n+1$ cyclic objects in $\mathcal{C} \leftrightarrow$ functors $\Lambda^{\mathrm{op}}\mathcal{C}$

The circle appears

Recall cat E then topological space $|N(E)|$

Fact: $|N(\Lambda)| = BS^1$ is the classifying space for the circle S^1.

Consequences: Any cyclic object X in k-modules give rise to a "object of $D(k)$ with S^1-action"

$\underline{X} : BS^1 \to D(k)$ simplicial set, alg/co-cat

Aside: cf, G, a finite group A functor $\underline{Y} : BG \to D(k)$ is a data of $* \mapsto$ some complex $Y \in D(k)$ $g \in G \mapsto gGY$ ie an action of G on Y, i.e. Y is a $k[G]$-module

Group cohom: $Y^{hG} := Rham_{k[G]}(k, Y) = \lim_{BG} \underline{Y}$

Group homol: $Y_{hG} := Y \otimes^{\mathcal{L}}_{k[G]} k = \mathrm{colim}_{BG} \underline{Y}$

Tate cohom $Y^{tG} := \underbrace{hofib}_{cone}(Y_{gH} \overset{Homo?,g}{\longrightarrow} Y^{hG})$

Back to S^1, $\underline{X} : BS^1 \to D(k)$ is the data of

Chain complex $\in D(k)$, Dold-Kan of X Module structure over $k[S^1]$. $k[S^1] := k[B\mathbb{Z}] \simeq k[\epsilon]/\epsilon^2$ where ϵ is in homological degree 1.

i.e. $\epsilon = B : X \to X[-1]$ Mimic earlier definitions $x^{hS^1} := Rham_{k[S^1]}(k, X) = \lim_{BS^1} \underline{X}$

$X_{hS^1} := k \otimes_{k[S^1]} X = \mathrm{colim}_{BS^1} \underline{X}$ $X^{tS^1} := hopfib(X_{hS^1}[1] \to normX^{hS^1} \in D(k)$

If $X = \mathrm{HH}(A/k)$ then $\mathrm{HH}(A/k)^{hS^1} \simeq \mathrm{HC}^-(A/k)$ $\mathrm{HH}(A/k)_{hS^1} \simeq \mathrm{HC}(A/k)$ $\mathrm{HH}(A/k)^{tS^1} \simeq \mathrm{HP}(A/k)$

May now replace $A \in k$-algebra $\subseteq D(k)$ by any rich enough symmetric monodial derived/∞-category eg spectral sp.

$D(k) \to$ restriction along $S \to k$ $Sp = D(S)$, symmetric monodial, stable ∞-category unit $= S$

Symmetric monodial stable ∞-categ unit$= k$

Aalg. $\to$ A Alg

then cyclic object $\mathrm{HH}(A/S) =: \mathrm{THH}(A) \in SP$ topological HH.

$\mathrm{THH}(A)^{hS^1} =: \mathrm{TC}^-(A)$ negative top cycl hom $\mathrm{THH}(A)_{hS^1} \neq$: top cyclic hom $\mathrm{THH}(A)^{tS^1} =: \mathrm{TP}(A)$ period top. cyclic hom

$\in Sp$

$\mathrm{THH}_n(A), \mathrm{TC}^-_n(A), \pi_n(\mathrm{THH}(A)_{hS^1})$

homotopy groups A-modules $\mathbb{Z}$-modules $\mathrm{TP}_n(A)$

Comparison with HH

$\mathrm{THH}_n(A) \to \mathrm{HH}_n(A)$

Kernel and cokernel killed by some $N = N(n)$

If $A \supseteq \mathbb{Q}$, then it is an isomorphism.

3.4 Lecture 4

A k-algebra $(D(k), \otimes_k) \in \mathrm{HH}(A/k) \subseteq (D(\mathbb{Z}), \otimes_{\mathbb{Z}}) \in \mathrm{HH}(A/\mathbb{Z})$, the last of which is 'nicer'.

$(D(k), \otimes_k) \in \mathrm{HH}(A/k) = \mathrm{HH}(A/\mathbb{Z}) \otimes_{\mathrm{HH}(k/\mathbb{Z})} k$

Let us check this. Left side built from $A \otimes_k \cdots \otimes_k A$ RHS built out of $(A \otimes_k \cdots \otimes_{\mathbb{Z}} A) \otimes_{k \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} k}$ k have map from bottom to top

$(?, \otimes_?) \supseteq (D(\mathbb{Z}), \otimes_{\mathbb{Z}}) \in \mathrm{HH}(A/\mathbb{Z})$ symmetric monodial $\otimes$ limits/cochains $\cdots$ dg category (or simplicial ∞-category.)

No! We cannot do better than $D(\mathbb{Z})$.

Yes but it is simplicial Spa or spectra $D(S)$.

$\mathbb{F}_p$, $\mathrm{HP}_0(\mathbb{F}_p/\mathbb{F}_p) = \mathbb{F}_p$ $\mathrm{HP}_0(\mathbb{F}_p/\mathbb{Z}) = \mathbb{Z}_p \otimes$ junk.

Today: THH$=$ $\mathrm{HH}(-/S)$ etc of $\mathbb{F}_p$-algebras

Thm: Bökshadt $\mathrm{THH}(\mathbb{F}_p) = 0$ and $\mathrm{THH}_{2\times}(\mathbb{F}_p) = \mathbb{F}_p[u]$, where $u \in \mathrm{THH}_2(\mathbb{F}_p)$

Consequence: For any $\mathbb{F}_p$-algebra A, get $\mathrm{THH}(A)[2] \xrightarrow{u} \mathrm{THH}(A) \to \mathrm{HH}(A/\mathbb{F}_p) = \mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathbb{F}_p)}$ $\mathbb{F}_p$, Bökstadt

Then take homotopy $\mathrm{THH}_0(A) \cong \mathrm{HH}_0(A/\mathbb{F}_p) = A$ $\mathrm{THH}_1(A) \cong \mathrm{HH}_1(A/\mathbb{F}_p) = \Omega_{A/\mathbb{F}_p}$

$$\longrightarrow \mathrm{THH}_1 = \Omega_1 \longrightarrow \mathrm{THH}_2 \longrightarrow \mathrm{HH}_2 \longrightarrow \mathrm{THH}(A) = A \longrightarrow \mathrm{THH}_2(A) \longrightarrow \mathrm{HH}_2(A/\mathbb{F}_p) \longrightarrow 0$$

thm: Hoschild HKR thm): If A is a smooth $\mathbb{F}_p$-algebra, then

$\mathrm{THH}_*(A) \cong \Omega^*_{A/\mathbb{F}_p}$

3.5 Lecture 5

4 Kirsten Wickelgren: $\mathbb{A}^1$-enumerative geometry

4.1 Lecture 1

Enumerative Geometry: counts algebraic-geometric objects satisfying conditions over $\mathbb{C}$

Goal: To record information about fields of definition

Arithmetic count of the lines on a smooth cubic surface

Definition. Cubic surface is $\{(x,y,z)\colon f(x,y,z)=0\}$, where f is degree 3

Better: $X \subseteq \mathbb{P}^3 = \{[w,x,y,z]\}$, $[w,x,y,z] = [\lambda w, \lambda x, \lambda y, \lambda z]$, where $\lambda \in K^\times$ $X = \{[w,x,y,z]\colon f(w,x,y,z) = 0\}$

Theorem 4.1 (Salmon, Cayley 1849)**.** *Let X be a smooth, cubic surface over $\mathbb{C}$. Then X contains exactly 27 lines.*

Example 4.1. Fermat $f(w,x,y,z) = w^3 + x^3 + y^3 + z^3$ $L = \{[S,-S,T,-T]\colon [S,T] \in \mathbb{P}^1\}$ λ, ω : $\lambda^3 = \omega^3 = -1$

Lines $\{[S, \lambda S, T, \omega T]\colon S, T \in \mathbb{P}^1\}$ This produces $3 \cdot 3 \cdot 3 = 27$ llines

Modern proof: $\mathrm{Gr}(1,3)$, the Grassmannian parametrizing lines in $\mathbb{P}^3$, equivalently $W \subseteq \mathbb{C}^4$, $\dim W = 2$

Let $S \to \mathrm{Gr}(1,3)$ be the tautological bundle, $S_W = W$ $\mathrm{Sym}^3 S^* \to \mathrm{Gr}(1,3)$ $\mathrm{Sym}^3 S_W^*$ is the cubic polynomial on W, i.e. $\mathrm{Sym}^3 W^*$ F determines element $\mathrm{Sym}^3(\mathbb{C}^4)^*$ then f determines a section of $\mathrm{Sym}^3 S^*$ by $\sigma_f(W) = f|_W$

Note: the line $\mathbb{P}W$ corresponding to W is in $X \Leftrightarrow \sigma_f(W) = 0$.

Want: to count zeros of σ_f Euler class: $V \to M$ be a rank r $\mathbb{R}$-vector bundle on a dimension r $\mathbb{R}$-manifold M. Assume V is oriented

Choose a section σ with only isolated zeros. $\deg[S^{r-1}, S^{r-1}] \to \mathbb{Z}$, homotopy classes of maps

$P \in M$, $\sigma(p) = 0$ To define: $\deg_p \sigma \in \mathbb{Z}$

Here's how: choose local coordinates on M around p. There's a small ball around p with no other zeros. Choose local trivialization of V. Then σ can be identified with a function $\sigma : \mathbb{R}^r \to \mathbb{R}^r$ given by $0 \mapsto 0$, $\sigma(\overline{B_0(1)} = 0) \subset \mathbb{R}^r = 0$ $S^{r-1} = OB_0(1) \xrightarrow{\overline{\sigma}} \partial B_0(1) = S^{r-1}$ given by $x \mapsto \sigma(x)/|\sigma(x)|$

Then $\deg_p \sigma = \deg(\overline{\sigma})$

Euler class $e(V) = \sum_{p:\sigma(p)=0} \deg_p \sigma$

Fact: X smooth then $\deg_p \sigma = 1$, then number of lines on $X = e(\mathrm{Sym}^3 S^*)$. In particular, number of lines is independent of X $e(\mathrm{Sym}^3 S^*) = 27$

Question: What about cubic surfaces over $\mathbb{R}$?

Segre in 20th century showed X can have 3, 7, 15, or 27 real lines.

Segre 1942 distinguished between different hyperbolic and elliptic real lines on X

Recall: L real line, $L \cong \mathbb{P}^1_{\mathbb{R}}$, $\mathrm{Aut}(L) \cong \mathbb{P}\,\mathrm{GL}_2(\mathbb{R})$ $I \leftrightarrow I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $z \mapsto \dfrac{az+b}{cz+d}$ $\mathrm{Fix}(I) = \{z\colon cz^2 + (d-a)z + b = 0\}$ either consists of 3 real points if and only if I hyperbolic. a ?? conjugate pair of points elliptic.

We associate an involution I to $L \subset X$ a real line on a real cubic surface.

$p \in L$ $T_pX \cap X = L \cup Q$ $Q \cap L =$ points q such that $T_qX = T_pX = \{p, p'\}$ $I(p) = p'$

Definition. L is elliptic/hyperbolic when I is

Alternatively, spin structure.

Example 4.2. Fermat cubic surface $x^3 + y^3 + z^3 = -1$ hyperbolic

Theorem 4.2 (Segre,Okonek,Teleman,...). *Number of hyperbolic lines − number of elliptic lines = 3.*

$\mathbb{A}^1$-homotopy theory (due to Morel-Voemsky)

On smooth schemes over k, k a field. Morel deg: $[\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \to GW(k)$ where $GW(k)$ is the Grothendieck-Witt=group completion of semiring $\oplus, \otimes$ isomorphism classes of (nondegenerate symmetric) bilinear forms $B : V \times V$, finite dimensional k vector spaces

presentation: generators: $\langle a \rangle$, $a \in k^\times$ $\langle a \rangle : k \times k \to k$, $(x, y) \mapsto axy$ relations: $\langle ab^2 \rangle = \langle a \rangle$, $b \in k^\times$ $\langle a \rangle + \langle b \rangle = \langle a + b \rangle = \langle ab(a + b) \rangle$

Example 4.3. $GW(\mathbb{C}) \cong \mathbb{Z}$, rank, $B \mapsto \dim V$

Example 4.4. $GW(\mathbb{R}) \to \mathbb{Z} \times \mathbb{Z}$ by signatures $\times$ rank. Also iso to $\mathbb{Z} \times \mathbb{Z}$ $GW(\mathbb{F}_q) \to \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2$ given by signatures $\times$ rank, isomorphism

There is an Euler class

$$e(V) = \sum_{p:\sigma(p)=0} \deg_p \sigma$$

R field char not 2. X a smooth cubic surface over k line $L \subseteq X$ is a closed point of $\mathrm{Gr}(1,3)$ $L = \{[a,b,c,d]S + [a',b',c',d']T \colon [S,T] \in \mathbb{P}^1\}$ $k(L) = k(a,b,c,d,a',b',c',d')$ $\mathbb{P}^1_{k(L)} \cong L \subseteq X_{k(L)} \subseteq \mathbb{P}^3_{k(L)}$ Given a line L on X, obtain involution $I \in \mathrm{Aut}(L) \cong \mathbb{P}\,\mathrm{GL}_2\, k(L)$ $\mathrm{Fix}(I)$ is either 2 $k(L)$ points or a conjugate pair of points in $k(L)[\sqrt{D}]$ for $D \in k(L)^*/(k(L)^*)^2$

Definition. $\mathrm{Type}(L) := \langle D \rangle \in GW(k(L))$

Equivalent to $D = ab - cd$, $I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\mathrm{Type}(L) = \langle -1 \rangle \deg I$

Theorem 4.3 (Kass-W.). *char $R \neq 2$, X smooth cubic surface*

$$\sum_{lines L of X} \mathrm{Tr}_{k(l)/k} \mathrm{Type}(L) = 15\langle 1 \rangle + 12\langle -1 \rangle$$

$\mathrm{Tr}_{k(L)/k} : GW(k(L)) \to GW(k)$ *given by* $(B : V \times V \to k(L)) \mapsto V \times V \xrightarrow{B} k(L) \to k$

$R = \mathbb{C}$, apply rank, number of lines is 27 $k = \mathbb{R}$ apply signature Number of hyperbolic lines − number elliptic lines = 3

Corollary 4.1. *$k = \mathbb{F}_q$ Number of elliptic lines L with $k(L) = \mathbb{F}_{q^{2n+1}}$ plus number of hyperbolic lines with $k(L) = \mathbb{F}_{q^{2n+1}}$ is equivalent to 0.*

4.2 Lecture 2

User's guide to $\mathbb{A}^1$-homotopy theory

Want: $\mathbb{P}^n/\mathbb{P}^{n-1}$, colimit

Ex:

$$\begin{tikzcd} \mathbb{P}^{n-1} \arrow[r] \arrow[d] & \mathbb{P}^n \arrow[d] \\ * \arrow[r] & \mathbb{P}^n/\mathbb{P}^{n-1} \end{tikzcd}$$

Ex: Open sets U, V

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \\ V \arrow[r] & U \cup V \end{tikzcd}$$

Want to glue, crash schemes like topological spaces

treat smooth schemes like manifolds construction of $\mathbb{A}^1$-homotopy theory (Morel-Voensky)

$\mathrm{Sm}_R =$ smooth schemes$/k$ $\mathrm{Sm}_k \to \mathrm{Func}(\mathrm{Sm}_k^{\mathrm{op}}, \mathrm{sset})$ $Y \mapsto \mathrm{Mor}(-, Y)$

Homotopy theory can mean: simplical model category or ∞-category $\mathrm{Pre}(\mathrm{Sm}_k) = \mathrm{Func}(\mathrm{Sm}_k^{\mathrm{op}}, \mathrm{sset})$ freely adding relations

Problem: had colimits from 2 in Sm_k

Fix: force certain classes of maps to be weak equivalences. Bousfield localization

For an open cover $B = \sqcup_k U_k \to X$ force $\mathrm{cosk}_x^{\cdot} \sqcup_x U_x \xrightarrow{\sim} X$ $\mathrm{Pre}(\mathrm{Sm}_k) \xrightarrow{L\tau} \mathrm{Sh}_R$, τ a Grothendieck topology.

(more open sets left to right) Choices: Zariski topology, Niesnevich, étale topology

Def: $f : X \to Y$ (not necessarily smooth, Sm_k) is etale at x if $T_x X \xrightarrow{\sim} T_{f(x)} Y$

Def: $U = \sqcup_x U_x \to X$ is an etale cover if it is etale and surjective

Def: $U = \sqcup_x U_x$ is a Nisnevich cover if it is an etale cover and for every $x \in X$ there exists $u \in U$ such that $u \mapsto x$, $k(x) \xrightarrow{\sim} k(u)$

Nice properties: $Z \hookrightarrow X$ in Sm_k can often be viewed as $\mathbb{A}^d \to \mathbb{A}^n$

$\mathrm{Sm}_k \longrightarrow \mathrm{PSh}_k = \mathrm{Func}(\mathrm{Sm}_l^{\mathrm{op}}, \mathrm{sset}) \xrightarrow{L_?} \mathrm{Sh}_k \xrightarrow{L_?} \mathrm{Spc}_k$ where last arrow force $X \times \mathbb{A}^1 \xrightarrow{\sim} X$ Spc_k is $\mathbb{A}^1$-homotopy theory.

Spheres:

Def: Given pointed spaces X, Y $X \wedge Y := (X \times Y)/(X * * U * * Y)$

Ex: $S^n \wedge S^m = S^{n+m}$

Spheres: S^1, $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ $S^{p+q\alpha} = S^{p+q,q} = (S^1)^{\wedge p} \wedge (\mathbb{G}_m)^?$

Ex:

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{A}^1 \simeq * \arrow[d] \\ \mathbb{A}^1 \simeq * \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

then $\mathbb{P}^1 \simeq \Sigma \mathbb{G}_m = S^1 \wedge \mathbb{G}_m$

Ex: $\mathbb{A}^n \setminus \{0\} \simeq (S^1)^{\wedge n-1} \wedge (\mathbb{G}_m)^{\wedge n}$ induction and

$$\begin{tikzcd} (\mathbb{A}^{n-1} \setminus \{0\}) \times (\mathbb{A}^{n-1} \setminus \{0\}) \arrow[r] \arrow[d] & (\mathbb{A}^{n-1} \setminus \{0\}) \times \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^n \times \mathbb{A}^1 \setminus \{0\} \arrow[r] & \mathbb{A}^1 \setminus \{0\} \end{tikzcd}$$

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & \Sigma X \wedge Y \end{tikzcd}$$

Ex: $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq (S^1)^{\wedge n}(\mathbb{G}_m)^{\wedge n}$ $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq \mathbb{P}^n/\mathbb{P}^h \setminus \{0\} \simeq \mathbb{A}^n/\mathbb{A}^n \setminus \{0\} \simeq */\mathbb{A}^n \setminus \{0\} \simeq \Sigma(\mathbb{A}^n \setminus \{0\})$.

Thom Space: Let $V \to X$ algebraic vector space $\mathrm{Th}(V) = V/V - X \simeq \mathbb{P}(V \oplus \mathcal{O})/\mathbb{P}(V)$

$S \in \mathrm{sset}$ $S \in \mathrm{Pre}(\mathrm{Sm}_k) = \mathrm{Func}(\mathrm{Sm}_k^{\mathrm{op}}\ \mathrm{sset})$

Purity Theorem: $Z \hookrightarrow X$ closed immersion in Sm_k $X/X - Z \simeq \mathrm{Th}(N_Z X)$

Ex: $\mathrm{Spec}\, k \hookrightarrow X$, where X is a smooth scheme U open neighborhood of z $U/U - z \simeq \mathbb{P}^n_{k(x)}/\mathbb{P}^{n-1}_{k(x)} \simeq \mathbb{P}^n/\mathbb{P}^{h-1} \wedge (\mathrm{Spec}\, k(z)+)$

Compare: z point on manifold, U small ball around z $\Sigma\partial U \simeq U/U - z$

$\mathrm{GW}(k), k_*^m(k)$:

$\mathrm{GW}(k)$ is group completion of isomorphism classes of symmetric, nondegenerate bilinear forms over k, $\otimes$ gives ring structure

Generators: $\langle a \rangle, a \in k^*$ $\langle a \rangle : k \times k \to k$ given by $(x, y) \mapsto axy$ relations: $\langle a, b^* \rangle = a \langle a \rangle \langle b \rangle = \langle ab \rangle$ $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$

then $h := \langle 1 \rangle + \langle -1 \rangle = \langle a \rangle + \langle -a \rangle$ for all a.

Acted on by hyperbolic forms

rank: $\mathrm{GW}(k) \to \mathbb{Z}$ given by $B : V \times V \to k \mapsto \dim V$

Fundamental ideal: $I := \ker \mathrm{rank}$

$\mathrm{GW}(k) \supseteq I \supseteq I^2 \supseteq \cdots$ $K_i^M = \bigoplus_{i=0}^{\infty} \bigotimes_{j=1}^{i} k^*/\langle a \otimes (1 - a) \rangle$, Milnor K-theory groups

Milnor conjecture, theorem of Voedosky $1 \to \mathbb{Z}/? \to k^* \to k^* \to 1$ $k^* \to H^1_{\text{ét}}(k, /\mathbb{Z}/?)$ $I^n/I^{n+1} \xleftarrow{\sim} K_n^M(k) \xrightarrow{\sim} H^n_{\text{ét}}(k, \mathbb{Z}/?)$ with left map $a_0 \otimes \cdots \otimes a_n \mapsto (\langle 1 \rangle - \langle a_1 \rangle) \cdots (\langle 1 \rangle - \langle a_n \rangle)$

view maps $I^n \to I^n/I^{n+1}$ as invariants on $\mathrm{GW}(k)$ $n = 0$ rank $n = 1$ discriminant $n = 2$ hasse-witt invariant

$B : V \times V \to k$ $\mathrm{disc}\, B = \det(B(v_i, v_j))$ $\{v_1, \ldots, v_n\}$ is a basis $n = 3$ Arason invariant

$K_*^{MW}(k)$ Milnor-Witt k-theory (hopkins-morel)

Generators: $[a]$ $a \in k^*$ $\deg 1, \deg -1$ η

relations: $\eta[a] = [a]\eta$ $[a][1 - a] = 0$ (Steinberg relation) $[ab] = [a] + [b] + \eta[a][b]$ $\eta h = 0$

$\mathrm{GW}(k) \cong K_0^{MW}(k)$ $\langle a \rangle \mapsto 1 + \eta[a]$ $h = \langle 1 \rangle + \langle -1 \rangle \mapsto h = 2 + \eta[-1]$

Degrees Theorem (Morel) $n \geq 2$

$[(S^1)^{\wedge n} \wedge \mathbb{G}_m^{\wedge j}, (S^1)^{\wedge n} \wedge \mathbb{G}_m^{\wedge ?}] \cong K_{r-j}^{MW}$ eg $[\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \cong \mathrm{GW}(k)$, $j = r = n$ $R = \mathbb{R}$

$$\begin{tikzcd} \left[S^{2n}, S^{2n}\right] \arrow[d, "\deg"] & \left[\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}\right] \arrow[l, "G-pts"'] \arrow[r, "R-pts"] \arrow[d, "\deg"] & \left[S^n, S^n\right] \arrow[d, "\deg"] \\ \mathbb{Z} & \mathrm{GW}(K) \arrow[l, "rank"] \arrow[r, "signature"] & \mathbb{Z} \end{tikzcd}$$

$\mathrm{GW}(?)K_*^{MW}(k)K_y^M(k)$ are global sections of sheaves.

Procedure for producing a sheaf K_*^{MW} from $K_*^{MW}(E)$, E finite type over k field.

plus data $V : E \to \mathbb{Z} \cup \{\infty\}$ valuation $\mathcal{O}_V = \{e \in E \colon v(e) \geq 0\}$ π uniformizer $v(\pi) = 1$ $k(V) := \mathcal{O}_V/\langle \pi \rangle$

$\partial_V^\pi : K_*^{MW}(E) \to K_{*-1}^{MW}(k(v)$ $\partial_V^\pi([\pi][a_1] \cdots [a_n]) = [\overline{a}_1] \cdots [\overline{a}_n]$ $q_i \in \mathcal{O}_V^*$ $\partial_V^\pi([a_1] \cdots [a_n]) = 0$

Correction: $\delta_x^\pi \eta = \eta \delta_V^\pi$ $\delta_x^\pi \eta = 0$

4.3 Lecture 3

Sheaves: K_*^{MW}, GW $GW(\operatorname{Spec} L \to \operatorname{Spec} L)$ = restruction $\mathcal{O}_K L$ or bilinear forms

Transfers: $K \subseteq L$ finite extension of finite schemes over R

$\mathrm{Tr}_{L/K} : GW(L) \to GW(K)$

geometric transfer, cohomological transfer, absolute transfer depends on generators, does not, twisted $\mathrm{Tr}_{L/K}(B : V \times V \to L) = V \times V \xrightarrow{B} \to L \xrightarrow{\mathrm{Tr}_{L/K}} K$ when $K \subseteq L$ is separable.

geometric: $L = k[z]/\langle f \rangle$

$\operatorname{Spec} L \overset{z}{\hookrightarrow} \mathbb{P}^1_k$ $\mathbb{P}^1_k \to \mathbb{P}^1_k/\mathbb{P}^1 \setminus \{z\}$ is a map $\mathrm{Tr}^{\mathrm{geo}}_{L/K}$ $GW(L) \to GW(K)$

CH: Chow groups $X \in \mathrm{Sm}_k$

$X^{(i)}$ is codimension i reduced, irreducible subschemes of X

$\mathrm{CH}^i(X) = \bigoplus_{X()} \mathbb{Z}/$rational equivalence

$V \subset X \times \mathbb{P}^1$ $V \wedge (X \times \{0\}) \sim V \wedge (X \times \{1\})$

useful in enumerative geometry: chern classes, pushforward pullbacks, ring structure Bloch $\mathrm{CH}^i * X (= H^i(X, K_i^M)$

Oriented Chow groups or Chow-Witt group

$\tilde{\mathrm{CH}}^i(X) = H^i(X, K_i^{MW})$ elements are formal rank $Z \in X^{(i)}$ and

Barge-Morel

Computed by Rost-Schmidt complex

$\oplus_{Z \in X^{(i+1)}} K_1^{MV}(k(x), \det) \to \oplus_{z \in X^i} GW(k(z), \det_{k(z)} T_z X) \to \oplus_{z \in X^{(i+1)}} K_{-1}^{MW}(k(z), \det)$

Fasel, M. Levine: pullbacks $f : X \to Y$ pushforward, non-commutative ring structure

E field, Λ 1-dimensional E vector space $K_i^{MW}(E, \Lambda) = K_i^{MW}(E) \otimes \mathbb{Z}$ $\mathbb{Z}[E]$ $\tilde{\mathrm{CH}}^i(X, L) = H^i(X, K_i^{MW}(i))$

$L \to X$ line bundle $f : X \to Y$ proper $\dim Y - \dim X = r$ $f_* \tilde{\mathrm{CH}}(X, \omega_{X/Z} \otimes f^* y)$ $\det TX \to \tilde{\mathrm{CH}}^{i-r}(Y, w_{Y/Z} BX)$

Degree via local degree

Algebraic topology

$f : S^n \to S^n$ $p \in S$ $\deg f = \sum \deg_q f$ $f^{-1}(p) = \{q_1, \ldots, q_n\}$

Differential topology formula for $\deg_{x_i} f$ choose coordinates $x_1, \ldots, x_n$ near q_i $y_1, \ldots, y_n$ near P $f : \mathbb{R}^n \to \mathbb{R}^n$

$\operatorname{Jac} f = \deg \frac{\partial f_i}{\partial x_i}$

$$\deg_{q_i} f = \begin{cases} 1, & \text{if } \operatorname{Jac} f > 0 \\ -1, & \text{if } \operatorname{Jac} f < 0 \end{cases}$$

$\mathbb{A}^1$-alg topology

Lanes/Morel: $f : \mathbb{P}^1 \to \mathbb{P}^1/k$ $p \in \mathbb{P}^1(k)$ $f^{-1}(p) = \{q_1, \ldots, q_n\}$ $\deg f = \sum(\mathrm{Jac}_{q_i} f) \in GW(k)$

this does not depend on P

Prop: (Global degree is a sum of local degrees)

$f : \mathbb{P}^n \to \mathbb{P}^n$ finite $f^{-1}(\mathbb{A}^n) = \mathbb{A}^n$ $\mathbb{P}^n/\mathbb{P}^{n-1} \xrightarrow{\overline{f}} \mathbb{P}^n/\mathbb{P}^{n-1}$

$\deg \overline{f} = \sum_{q \in f^{-1}(p)} \deg_q^{\mathbb{A}^1} f$ $p \in \mathbb{A}^n(k)$

where $\deg_q^{\mathbb{A}^1} f$ is degree of composite

$$\mathbb{P}^n/\mathbb{P}^{n-1} \cong U/U - q \to \mathbb{A}^n/\mathbb{A}^n - p \cong \mathbb{P}^n/\mathbb{P}^{n-1}$$

$k(q) = k$, $\mathrm{Th} N_p \mathbb{A}^*$

otherwise $\mathbb{P}^n/\mathbb{P}^{n-1} \to \mathbb{P}^n/\mathbb{P}^n - q$

If f is étale at q, then $\deg_q^{\mathbb{A}^1} f = \mathrm{Tr}_{k(q)/k}\langle \mathrm{Jac}\, f(q)\rangle$ and $k(q) \supseteq R$ separable.

A: Eisenbud-Lenine-Khinskdhflskhdfklashdgahsdgliadg lsdighasdgh Signature formula

$f : \mathbb{R}^n \to \mathbb{R}^n$ $0 \mapsto 0$ isolated zero

$\deg f =$ signature ω^{EKL}

ω^{EKL} is a bilinear form $Q = \mathbb{R}[x_1, \ldots, x_n]_0/\langle f_1, \ldots, f_n\rangle$

$\mathrm{Jac}\, f \in Q$ pick any $\eta : Q \to \mathbb{R}$ $\mathbb{R}$-linear so that $\eta(\mathrm{Jac}\, f) = \dim Q$ $\omega^{EKL} : Q \times Q \to \mathbb{R}$ $(a, b) \mapsto \eta(ab)$

Q (Eisenbud): ω^{EKL} could be a degree even replacing $\mathbb{R}$ with K. DOes this have an interpratation?

Thm: (Kass-W.) $\deg_0^{\mathbb{A}^1} f = \omega^{EKL}$

Project: remove $k(x) = k$hypothesis

Ex: ω^{EKL} for $f(x) = x^2$ $Q = k[x]/\langle x^2\rangle$ basis $\{1, x\}$ $\mathrm{Jac}\, f = \partial x$ $\eta : k[x]/\langle x^1\rangle \to k$ $\eta(2x) = 2$ $\eta(1) = 0$

$$\begin{array}{cccc} & 1 & x \\ 1 & 0 & 1 \\ x & 1 & 0 \end{array}$$

$\omega^{EKL} = \langle 1\rangle + \langle -1\rangle$

$\mathbb{A}^1$-milnor numbers

joint with Jesse Kass

Def: A point p on a scheme X is a node if after base change to $k^?$ $\hat{\mathcal{O}}_{X,p} \cong k^s[[x_1, \ldots, x_n]]/(x_1^2 + \cdots + x_n^2 + \text{hot})$

Let X be a hypersurface $X = \{f = 0\} \subseteq \mathbb{A}^n$ $p \in X$ be a singularity As X is perturbed in a family P bifercates into nodes for $(a_1, \ldots, a_n)$ have a family of hypersurfaces $f(x_1, \ldots, x_n) + a_1x_1 + \cdots + a_nx_n = f$ parametrized by $+$

Milnor $k = \mathbb{C}$ FOr any sufficiently small $(a_1, \ldots, a_n)$ the family contains $\mu(P)$ nodes

$\mu(P) =$ milnor # $= deg(gradf)(p)$

When k is not algebraically closed, nodes p contain arithmetic data

$R = \mathbb{R}$

nonsplit node, i.e. tangent directions not defiend over k

split node

Def: The type of a node $p \in \{f = 0\}$ $\deg_p^{\mathbb{A}^1} gradf$

Ex: Choose preimage of p after base change to $k(p)$

$\hat{\mathcal{O}}_{X,p} = k[[p]][[x_1, \ldots, x_n]]/(a_1x^2 + \cdots + a_nx^2 + \text{hot})$

$\mathrm{type}(p) = \mathrm{Tr}_{k(p)/k}\langle 2^n a_1a_2 \cdots a_n\rangle$, $k(p)/k$ always a seprable extension.

Ex: $\mathrm{type}(x^2 + y^2) = \langle a\rangle$

Def: p hypersurface singularity $p \in \{f = 0\}$ $M^{\mathbb{A}^1} = \deg_p gradf$

Thm: For generic $(a_1, \ldots, a_n)$ Crass-W.

$\sum_{Xnodesinfamily} type(X) = M^{\mathbb{A}^1}(p)$ in GW(k).

Ex: $f(x, y) = y^2 - x^2$ $gradf = (-3x^2, 2y)$ $M^{\mathbb{A}^1}(0) = \deg_0 gradf = \deg_0(x \mapsto -3x^1)\deg_0(y \mapsto 2y) = \langle -3\rangle(\langle 1\rangle + \langle -1\rangle)\langle 2\rangle = \langle -6\rangle + \langle 6\rangle = \langle 1\rangle + \langle -1\rangle = h$

Family parametrized by $+$ $y^2 = x^3 + ax + t$

$a = 0$

nodes occur when $x^3 + ax + t$ has a double root iff $= 27t^2 - 4a^3$

$\mathbb{F}_3 : \langle 1\rangle = \langle -1\rangle$ in a famiyl cant have one split and one nonsplit rational nodes

$\mathbb{F}_7 : \langle 1\rangle \neq \langle -1\rangle$ cant have 2 split or 2 nonsplit nodes.

4.4 Lecture 4

Part II

Course/Project Outlines & Lecture Notes

LUBIN-TATE SPACE: OLD AND NEW QUESTIONS
ARIZONA WINTER SCHOOL 2019

M. J. HOPKINS

Course outline

The Lubin-Tate formal groups and their corresponding deformation spaces have played many roles in mathematics, from their initial conception as an explicit means of realizing local class field theory, through their role in the local Langlands correspondence, and the many appearances they make in homotopy theory. For the most part the questions of importance in algebraic topology are very different from those of interest in number theory. One goal of this course is to lay out these diverse questions in the hopes of bridging these two audiences. There are also strong theoretical demands from both the algebraic topology side and number theoretic sides and wherever possible, the theory will be supplemented with explicit formulas.

Here is a tentative outline of the lecture series

(1) Origin story
  (a) Local class field theory
  (b) Deformations
  (c) The moduli stack of commutative 1-dimensional formal groups
(2) Period maps
  (a) Periods according to Abel and Jacobi
  (b) Periods and uniformization
  (c) Cartier's magic carpet
  (d) Crystalline Dieudonne theory
  (e) Finite orbifold quotients.
(3) Drinfeld covers
  (a) Level structures according to Drinfeld
  (b) Level structures in algebraic topology
  (c) Fricke involutions and the crystalline period map
(4) Picard groups
  (a) Picard groups and Iwasawa theory
  (b) Picard groups of Lubin-Tate spaces

Projects

Here are some possible projects the participants might work on in order to become better involved with this material. Some of these represent known results and some appear to be very difficult problems.

Tangent spaces to moduli stacks. Deformation theory is often controlled by the cotangent complex of a stack. The Lubin-Tate deformation theory can be interpreted in this language. It is illuminating to work out explicitly how this works.

Algebraic approximations to period maps. The period maps are p-adic analytic maps, given by solutions to p-adic Picard-Fuchs equations. In the case of Lubin-Tate space these solutions can be made very explicit. The crystalline period map comes from a map of formal schemes, though this map does not as yet have an interpretation in terms of moduli. Approximations to this map of formal schemes can be worked out explicitly. Doing so might give some hope of thinking of the period map in terms of formal moduli.

Uniformization of orbifold quotients. The quotients of the Lubin-Tate deformation space by finite groups of automorphisms of the Lubin-Tate group are extremely useful in algebraic topology. One good project here is to work out the classification of such finite groups. While one could find out the answer with a google search, doing so by hand gives some insights into the correspondence between Galois cohomology and division algebras. Building on this it is possible to show that the finite orbifold quotient stacks of the Lubin-Tate deformation space have particularly nice uniformizations.

Picard groups of Lubin-Tate spaces. Determining the Picard group of the Lubin-Tate stack is an important problem in algebraic topology. What is known is known by a lengthy computation. Trying to come to grips with this computation and perhaps generalize it would be an interesting project. Students could expect a good payoff in learning something, though the problem still appears to be quite challenging.

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Lectures on Lubin-Tate spaces
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Introduction

Lubin-Tate formal groups and their moduli play a surprising number of roles in mathematics, especially in number theory and algebraic topology. In number theory, the formal groups were originally used to construct the local class field theory isomorphism. Many years later a program emerged to use the moduli spaces to realize the local Langlands correspondence. In algebraic topology, thanks to Quillen, 1-dimensional commutative formal groups appear in the Adams-Novikov spectral sequence relating complex cobordism groups to stable homotopy groups, and through the work of Morava and many others, the Lubin-Tate groups lead to the "chromatic" picture of homotopy theory.

In both number theory and topology one is interested in the cohomology of Lubin-Tate spaces, but in number theory, one tends to be interested in something like the p-adic etale cohomology while in topology it is the coherent (stack) cohomology. Because of this the two fields have focused on a different set of fundamental questions. One goal of this lecture series is to expose aspects of Lubin-Tate spaces of interest in algebraic topology to a broader audience.

Prerequisites

I will assume the participants are familiar with the notion of a commutative formal group law and Lazard's theorems found in §1,3,5,7 of Part II of Adams [**1**] or the book of Fröhlich [**6**]. The Lubin-Tate formal groups are introduced in [**21**] and the deformation spaces in [**22**]. It will help to have read [**21**] (or Serre's article [**26**]) and to have some understanding of [**22**].

It will also help to know something about the classification of central simple division algebras over the p-adic rationals $\mathbb{Q}_p$. The notes of Serre [**25**] are very good.

Much of the material I will present is covered in the papers [**9, 10**] and [**5**]. Much of my presentation will follow [**5**].

There are many other good expositions of this material. The expository notes notes of Weinstein (`http://math.bu.edu/people/jsweinst/FRGLecture.pdf` are excellent.

Further references: [**18, 17, 19, 16, 15, 14**].

LECTURE 1

Lubin-Tate spaces

1.1. Overview

A *formal group law* (really, a commutative 1-dimensional formal group law) over a ring R is a power series

$$F(x,y) \in R[\![x,y]\!]$$

satisfying

$$\begin{aligned} F(x,y) &\equiv x+y \mod (x,y)^2 \\ F(x,0) &= F(0,x) = x \\ F(x,F(y,z)) &= F(F(x,y),z) \\ F(x,y) &= F(y,x). \end{aligned}$$

It's a lot easier to process these identities if one writes

$$x \underset{F}{+} y = F(x,y)$$

in which case they become the

$$\begin{aligned} x \underset{F}{+} y &\equiv x+y \mod (x,y)^2 \\ x \underset{F}{+} 0 &= 0 \underset{F}{+} x = x \\ (x \underset{F}{+} y) \underset{F}{+} z &= x \underset{F}{+} (y \underset{F}{+} z) \\ x \underset{F}{+} y &= y \underset{F}{+} x. \end{aligned}$$

Example 1.1.1. The *additive formal group law* $\mathbf{G}_a$ is given by

$$\mathbf{G}_a(x,y) = x+y.$$

Example 1.1.2. The *multiplicative formal group law* $\mathbf{G}_m$ is given by

$$\mathbf{G}_m(x,y) = x+y-xy = (1-(1-x)(1-y)).$$

A *homomorphism* $f : F \to G$ between formal group laws is a power series $f(x)$ satisfying

$$f(x \underset{F}{+} y) = f(x) \underset{G}{+} f(y).$$

There is also a notion of a (commutative 1-dimensional) *formal group* (without the word "law") which you should think of as a formal group law, without a choice of coordinate x. But that's not quite right. A formal group over R is what you get on R by (Zariski) descent datum consisting of formal groups and isomorphisms. The Zariski cotangent space to a formal group law at 0, over a ring R, is the free R-module $(x)/(x)^2$. The Zariski cotangent space at 0 to a formal group will be rank 1 projective R-module which might not be free. Mostly we will be working over

local rings so you won't lose anything by thinking of a formal group as a formal group law, only without commitment to a given coordinate.

1.2. Height

Here's a question. Is there a homomorphism from $\mathbf{G}_a$ to $\mathbf{G}_m$? Are they isomorphic? Over $\mathbb{Q}$-algebras the answer is yes (the map $1-\exp(-x)$ gives an isomorphism). It doesn't look so likely in characteristic $p > 0$. We will prove that there is no isomorphism. (We will do better in the next lecture).

Suppose that F and G are formal group laws over a field k of characteristic $p > 0$, and $f : F \to G$ is a homomorphism.

LEMMA 1.2.1. *If $f'(0) = 0$ there is a unique g with $f(x) = g(x^p)$.*

Proof: The derivative of f at 0 is zero. Since f is a group homomorphism, this means that the derivative of f is zero everywhere. □

REMARK 1.2.2. If you didn't like that proof, here it is in formulas. Take the identity

$$f(x \underset{F}{+} y) = f(x) \underset{G}{+} f(y)$$

and take the partial with respect to y at $y = 0$. One gets

$$f'(x)\partial_2 F(x, 0) = \partial_2 G(f(x), 0)f'(0) = 0,$$

where ∂_2 means "partial with respect to the second variable." Now

$$\partial_2 F(x, 0) = 1 + \dots$$

is a unit in $k[\![x]\!]$ so this means that $f'(x) = 0$. The expression $\partial_2 F(x, 0)\, dx$ is important and we will encounter it again later.

Iterating the above we get the following

PROPOSITION 1.2.3. *If $f : F \to G$ is a non-zero homomorphism of formal group laws over a field k of characteristic $p > 0$ there is a unique (g, n) with $g(x) \in k[\![x]\!]$ and $0 < n < \infty$ with*

$$f(x) = g(x^{p^n})$$
$$g'(0) \neq 0.$$

DEFINITION 1.2.4. The *height* of a non-zero homomorphism $f : F \to G$ is the integer n defined above.

One defines the height of the zero homomorphism to be ∞.

DEFINITION 1.2.5. Let Γ be a formal group law over a field k of characteristic $p > 0$. The *height* of Γ is the height of the homomorphism $p : \Gamma \to \Gamma$.

EXAMPLE 1.2.6. The height of $\mathbf{G}_a$ is ∞ while the height of $\mathbf{G}_m$ is 1. It follows that $\mathbf{G}_a$ is not isomorphic to $\mathbf{G}_m$.

EXERCISE 1.2.1. Show that any homomorphism between $\mathbf{G}_a$ and $\mathbf{G}_m$ must have infinite height. (HINT: In the situation of Lemma 1.2.1 show that the map g must also be a homomorphism.)

1.3. Deformations of formal groups

Let Γ be a formal group over a perfect field k, and B a local ring, with nilpotent maximal ideal $\mathfrak{m}$. It will be convenient to write $r : B \to B/\mathfrak{m}$ for the quotient map.

Definition 1.3.1. A *deformation* of Γ to B is a triple (G, i, f) with G a formal group over B, $i : k \to B/\mathfrak{m}$ a ring homomorphism, and $f : r^*B \to i^*\Gamma$ an isomorphism.

The collection of deformations of Γ to B forms a groupoid $\mathbf{Deform}_\Gamma(B)$ in which an isomorphism t from (G, i, f) to (G', i', f') exists only if $i = i'$, in which case it is isomorphism $t : G \to G'$ making the diagram

$$\begin{tikzcd} r^*G \arrow[r, "r^*"] \arrow[d, "f"'] & r^*G' \arrow[d, "f'"] \\ i^*\Gamma \arrow[r, "="'] & i^*\Gamma \end{tikzcd}$$

commute.

While it is sensible to regard $B \mapsto \mathbf{Deform}_\Gamma(B)$ as a sheaf of groupoids (ie a stack) in the B variable, it turns out that $\mathbf{Deform}_\Gamma(B)$ is *codiscrete* in the sense that there is at most one map between any two objects. This is a consequence of the following result, whose proof is left as an exercise.

Proposition 1.3.2. *Suppose that B is a local ring with nilpotent maximal ideal $\mathfrak{m}$, and $B/\mathfrak{m}$ has characteristic $p > 0$. Write $r : B \to B/\mathfrak{m}$ for the quotient map. Suppose that G_1 and G_2 are two formal group laws over B and that r^*G_1 and r^*G_2 have finite height. Show that if $f, g : G_1 \to G_2$ are two homomorphisms having the property that $r^*f = r^*g$ then $f = g$.*

Exercise 1.3.1. Prove Proposition 1.3.2. (I'll add some guidance later.)

This means we might as well replace the groupoid $\mathbf{Deform}_\Gamma(B)$ with the equivalent groupoid $\pi_0 \mathbf{Deform}_\Gamma(B)$ consisting of the set isomorphism classes of objects of $\mathbf{Deform}_\Gamma(B)$, with no non-identity maps.

Write $\mathbb{W}$ for the ring of Witt vectors of k.

Theorem 1.3.3 (Lubin-Tate). *Suppose that Γ is a formal group of height n. The functor $\pi_0 \mathbf{Deform}_\Gamma(-)$ is* representable. *More specifically, there is a deformation $(G_{univ}, i_{univ}, f_{univ})$ over the ring $\mathbb{W}[\![u_1, \ldots, u_{n-1}]\!]$ having the property that if $(G, i, f) \in \mathbf{Deform}_\Gamma(B)$ is any deformation, there is a unique ring homomorphism $\phi : \mathbb{W}[\![u_1, \ldots, u_{n-1}]\!] \to B$ with the property that $\phi^*(G_{univ}, i_{univ}, f_{univ})$ is isomorphic to (G, i, f).*

Remark 1.3.4. Strictly speaking the local ring $\mathbb{W}[\![u_1, \ldots, u_{n-1}]\!]$ is not an allowed B since the maximal ideal $\mathfrak{m} = (p, u_1, \ldots, u_{n-1})$ is not nilpotent. One can remedy this by either working with pro-nilpotent local rings, or just agreeing to extend the definition of "representable" to allow for this situation.

The ring $\mathbb{W}[\![u_1, \ldots, u_{n-1}]\!]$ is called the *Lubin-Tate* ring, and the associated formal scheme is called Lubin-Tate space. As described above, Lubin-Tate space represents the functor *isomorphisim classes of deformations of* Γ.

We now turn to the functorial dependence of $\mathbf{Deform}_\Gamma(B)$.

PROPOSITION 1.3.5. *Two formal groups Γ and Γ' over an algebraically closed field k of characteristic $p > 0$ are isomorphic if and only if they have the same height.*

Because of this result, we might as well suppose that k is the algebraic closure of $\mathbb{F}_p$ and that we've picked just one Γ. Then the only thing left to understand is the structure of the automorphism group $\mathrm{Aut}(\Gamma)$ and its action on $\mathbb{W}[\![u_1, \ldots, u_{n-1}]\!]$. In the algebraic topology the group $\mathrm{Aut}(\Gamma)$ is written

$$\mathbf{S}_n = \mathrm{Aut}(\Gamma)$$

and called the *Morava stabilizer group.* The structure on $\mathrm{Aut}(\Gamma)$ is well understood, and will be explained in the next lecture. Let me give a presentation now, just to make things specific.

Write $\mathbb{Z}_{p^n}$ for the ring of Witt vectors of the field $\mathbb{F}_{p^n}$ with p^n elements. There is a unique ring homomorphism (Frobenius)

$$\phi : \mathbb{Z}_{p^n} \to \mathbb{Z}_{p^n}$$

having the property that for all x, $\phi(x) \equiv x^p \mod p$. From time to time it will be a bit more convenient to write x^ϕ for $\phi(x)$. Let $F \in \mathrm{Gl}_n(\mathbb{Z}_{p^n})$ be the matrix

$$\Pi = \begin{pmatrix} 0 & 0 & \ldots & 0 & p \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

The proof of the following result will be given in the next lecture.

PROPOSITION 1.3.6. *The group $\mathbf{S}_n$ is isomorphic to the subgroup of $\mathrm{Gl}_n(\mathbb{Z}_{p^n})$ consisting of matrices A satisfying*

$$\Pi \cdot A = A^\phi \cdot \Pi.$$

Evidently the group $\mathbf{S}_n$ acts on the Lubin-Tate ring $E_0 = \mathbb{W}[\![u_1, \ldots, u_{n-1}]\!]$. The basic questions that will concern us in these lectures are the following.

QUESTION 1.3.7. Can one explicitly describe the action of $\mathbf{S}_n$ on E_0?

QUESTION 1.3.8. What are the cohomology groups $H^*(\mathbf{S}_n; E_0)$?

QUESTION 1.3.9. What is the group of $\mathbf{S}_n$ equivariant line bundle on $\mathrm{Spf}\, E_0$?

The reasons these are interesting and the approaches one can make on these problems will be spelled out over the course of these lectures and in the projects. For today I want to explain the basic construction of the Lubin-Tate group, and the proof of Theorem 1.3.3.

1.4. Formal complex multiplication in local fields

Lubin and Tate wrote two papers [**21, 22**] introducing what subsequently became known as the Lubin-Tate formal group laws and Lubin-Tate space. In [**21**] they gave a construction of the maximal ramified abelian extension of a local field, providing a new construction of part of the local class field theory isomorphism. Their work generalized the theorem of Kronecker-Weber that the abelian extensions of $\mathbb{Q}$ are contained in cyclotomic fields. I'm assuming you are familiar with

the Lubin-Tate paper, but as we will need some details of their construction it's worth reviewing them here.

Let's recall the setup. Suppose that K is a complete local field, with ring of integers $\mathcal{O} \subset K$. Let $\pi \in \mathcal{O}$ be a uniformizer, so that $(\pi) \subset A$ is the maximal ideal. We also assume that the residue field $k = A/(\pi)$ is finite, of order $q = p^n$ for a prime p. For the moment let

$$f(x) = x^q + \pi\, x,$$

and let $f^{(n)}(x)$ be the n-fold composition

$$f(f(\cdots(f(x)))).$$

Lubin and Tate showed that the field obtained from K by adjoining the roots of $f^n(x)$ is an abelian extension with Galois group $A/(\pi^n)^\times$.

Their idea was to show that there is a 1-parameter formal group law F over $\mathcal{O}$, with endomorphism ring A and in which $f^{(n)}$ is the power series representing "multiplication by p^n." Following Lubin and Tate, let $\mathcal{F}_\pi$ be the set of formal power series $f(x) \in \mathcal{O}[\![x]\!]$ satisfying

$$\begin{aligned} f(x) &= \pi x + O(x)^2 \\ f(x) &\equiv x^q \mod \pi. \end{aligned}$$

Given $f \in \mathcal{F}_\pi$ Lubin and Tate constructed a formal group law $F(x, y)$ with a map

$$[\,-\,] : A \to \mathrm{End}(F)$$

having the property that $[\pi](x) = f(x)$.

LEMMA 1.4.1. *Suppose that $L(x_1, \ldots, x_n) = a_1x_1 + \cdots + a_nx_n$ is any linear form in n-variables $x_1, \ldots, x_n$. There is a unique formal power series $F(x_1, \ldots, x_n)$ with*

$$F(x_1, \ldots, x_n) \equiv L(x_1, \ldots, x_n) \mod \deg 2 \tag{1.4.2}$$

and which commutes with f in the sense that

$$F(f(x_1), \ldots, f(x_n)) = f(F(x_1, \ldots, x_n)). \tag{1.4.3}$$

Proof: Suppose F is any power series satisfying (1.4.2) and satisfying (1.4.3) modulo degree $m > 1$. By assumption

$$f(F(x_1, \ldots, x_n)) - F(f(x_1), \ldots, f(x_n)) \equiv 0 \mod (\pi)$$

we have

$$f(F(\underline{x})) - F(f(\underline{x})) = \pi g(\underline{x}) + O(\underline{x})^{m+1}.$$

with for some homogeneous polynomial g of degree m. If $h(x_1, \ldots, x_n)$ is homogeneous of degree m then

$$\begin{aligned} f(F + h) &\equiv f(F) + hf'(0) \mod \deg(m+1) \\ &\equiv f(F) + h\pi \mod \deg(m+1) \end{aligned}$$

and

$$(F + h)(f) \equiv F(f) + \pi^m h(x_1, \ldots, x_n) \mod \deg(m+1). \tag{1.4.4}$$

This means that

$$(F+h)(f)-(f)(F+h)=\pi(\pi^{m-1}-1)h-\pi g \mod \deg(m+1).$$

Since $m>1$ there is a unique h for which

$$(F+h)(f)-(f)(F+h)\equiv 0 \mod \deg(m+1).$$

The claim follows easily from this. □

One gets a lot from this result. Taking $L(x,y)=x+y$ one constructs a unique powerseries

$$F(x,y)=xy_{+}$$

satisfying $F(f(x),f(y))=f(F(x,y)$, or

$$f(x\underset{F}{+}y)=f(x)\underset{F}{+}f(y).$$

Again, using the lemma one finds that this defines a commutative one dimensional formal group law F. For each $a\in A$ there is a unique power series

$$[a](x)=ax+\dots$$

satisfying

$$f([a](x))=[a](f(x)).$$

This series automatically satisfies

$$[a](x\underset{F}{+}y)=[a](x)\underset{F}{+}[a](y)$$

so that $a\mapsto[a](x)$ defines a ring homomorphism

$$A\to \mathrm{End}(F).$$

This makes F into a *formal A-module* over A in the sense of the definition below.

Definition 1.4.5. Suppose that $\phi: A\to R$ is a ring homomorphism from a not necessarily commutative ring A to a commutative ring R. A formal A-module is a formal group law F over R, equipped with a ring homomorphism

$$A\to \mathrm{End}(F)$$
$$a\mapsto [a](x)$$

having the property that

$$[a](x)\equiv \phi(a)\,x \mod \deg 2.$$

This situation is also described by saying that F has *complex multiplication* by A.

Using an obvious generalization of Lemma 1.4.1 one can also show that the formal group laws F_{f_1} and F_{f_2} are isomorphic (by a unique isomorphism with derivative 1 at 0) for any $f_1,f_2\in\mathcal{F}_\pi$, and furthermore that the group is independent of the choice of π. The resulting formal group is often called *the* Lubin-Tate group, however the term "Lubin-Tate" group is more commonly used to describe the induced formal group Γ over $k=A/(\pi)$.

Definition 1.4.6. The *Lubin-Tate* formal group is the formal group Γ obtained by reducing any choice of F_f modulo (π).

The formal group Γ is unique up to (non-unique) isomorphism.

1.5. Formal moduli for one-parameter formal Lie groups

In their second paper, Lubin and Tate turned to the question of describing deformations of the formal group Γ, and defined the famous Lubin-Tate spaces.

First note that the set of deformations is non-empty and in fact has a canonical element, namely the Lubin-Tate lift constructed in the previous section. To classify lifts it therefore suffices to find a means of comparing two different lifts. For this on proceeds by working modulo successive powers of $\mathfrak{m}$. Fix $k > 1$ and suppose that we have two formal group laws $F(x, y), G(x, y) \in B/\mathfrak{m}^{k+1}[\![x, y]\!]$ with

$$F(x, y) \equiv G(x, y) \mod \mathfrak{m}^k.$$

We study the difference by writing

$$G(x, y) = x \underset{F}{+} y \underset{F}{+} h(x, y)$$

for $h(x, y) \in \mathfrak{m}^k/\mathfrak{m}^{k+1}[\![x, y]\!]$. The commutativity of G gives the identity

$$h(x, y) = h(y, x).$$

Next we get an identity on h from the associativity law

$$G(x, y \underset{G}{+} z) = G(x \underset{G}{+} y, z).$$

Since

$$G(x, y \underset{G}{+} z) = x \underset{F}{+} (y \underset{G}{+} z) \underset{F}{+} h(x, y \underset{G}{+} z) = x \underset{F}{+} (y \underset{F}{+} z \underset{F}{+} h(y, z)) \underset{F}{+} h(x, y \underset{G}{+} z)$$

and

$$G(x \underset{G}{+} y, z) = (x \underset{G}{+} y) \underset{F}{+} z \underset{F}{+} h(x \underset{G}{+} y, z) = (x \underset{F}{+} y \underset{F}{+} h(x, y)) \underset{F}{+} z \underset{F}{+} h(x, y \underset{G}{+} z).$$

Using the associativity law, and formal "F" subtraction, one gets

$$h(y, z) \underset{F}{+} h(x \underset{G}{+} y, z) = h(x, y) \underset{F}{+} h(x, y \underset{G}{+} z).$$

This can be simplified a little but not a lot. Since the product of any two $h(-, -)$ terms is zero the $\underset{F}{+}$ can be replaced by $+$. Also the terms s and t in $h(s, t)$ depend only on their values modulo $\mathfrak{m}$. This means that $\underset{G}{+}$ may be replaced by $\underset{\Gamma}{+}$, So the condition in $h(x, y)$ may be written

$$h(y, z) - h(x \underset{\Gamma}{+} y, z) + h(x, y \underset{\Gamma}{+} z) - h(x, y) = 0.$$

Put differently, h is a symmetric 2-cocycle on Γ with values in $\mathfrak{m}^n$.

Suppose that F and G are isomorphic, by an isomorphism ϕ reducing to the identity modulo $\mathfrak{m}^m$. This means that $\phi(x)$ satisfies

$$\phi(x \underset{F}{+} y) = \phi(x) \underset{G}{+} \phi(y),$$

and so

$$\begin{aligned} \phi(x \underset{G}{+} y) &= \phi(x \underset{F}{+} y \underset{F}{+} h(x, y)) \\ &= \phi(x) \underset{G}{+} \phi(y) \underset{G}{+} \phi(h(x, y)). \end{aligned}$$

Since $\phi(x) \equiv x + \cdots$, using the above considerations we can rewrite this as

$$h(x, y) = \phi(x \underset{\Gamma}{+} y) - \phi(x) - \phi(y).$$

So if we declare F and G to be "isomorphic" if they are isomorphic by an isomorphism reducing to the identity modulo $\mathfrak{m}^m$ then we have found that the set of isomorphism classes of "lifts" of G from $B/\mathfrak{m}^m$to $B/\mathfrak{m}^{m+1}$ is given by

$$H^2_{\mathrm{sym}}(\Gamma; \mathfrak{m}^m/\mathfrak{m}^{m+1}) \approx H^2(\Gamma; k)_{\mathrm{sym}} \otimes \mathfrak{m}^m/\mathfrak{m}^{m+1}.$$

I will fill in the proof of the next result later.

Proposition 1.5.1. *If Γ has height n, the group $H^2_{sym}(\Gamma; k)$ has rank $n-1$.*

Using Proposition 1.5.1, Lubin and Tate then derive the following easy corollary.

Proposition 1.5.2. *Let G be a formal group law over $\mathbb{W}[\![u_1, \ldots, u_{n-1}]\!]$ deforming Γ. The group G is a universal deformation if*

$$x \underset{G}{+} y = x + y + u_1 C_p(x, y) + \cdots + u_{n-1} C_{p^{n-1}}(x, y) + C_{p^n}(x, y) \mod \mathfrak{m}^2.$$

What remains of the proof of Theorem 1.3.3 is the construction of a formal group law G satisfying the criterion of Proposition 1.5.2. There are many approaches to this and in one way or another this construction is a key to understanding Question 1.3.7.

1.6. Deformations a la Kodaira-Spencer

Working with a formula based approach is convenient for making computations and making new progress. However it opens the door to a lot of unintended choices, and it is useful to know how things look from a completely conceptual point of view. Think of it as the analogue of writing down a model in physics. You can get a good idea what a formula has to look like by thinking about what has to happen for the physical units to work out.

There are a lot of ways of getting to the general "deformation theory" picture. Here I'll describe the one I find easiest to remember. Suppose you have a family $p : E \to B$ of something. By family I might mean a locally trivial bundle, and everything needs to be appropriately smooth. In the classic example p is a family of Riemann surfaces over B. The derivative of p is a map of tangent bundles

$$dp : TE \to TB.$$

Now pick a point $b \in B$. For each tangent vector $v \in T_b B$ the object $dp^{-1}(b, v)$ is some kind of object related to $p^{-1}(b)$. The Kodaira-Spencer map is the map

$$T_b B \to \{\text{whatever classifies } dp^{-1}(b, v)\}.$$

That was kind of vague, so let me go through a couple of examples.

Example 1.6.1. Suppose that $E \to B$ is a family of Riemann surfaces (smooth projective algebraic curves.) Then $p^{-1}(b)$ is a specific curve Σ, and $dp^{-1}(b, 0)$ is the tangent $T\Sigma$ to Σ. So what is $dp^{-1}(b, v)$? Well, if you have a linear map $W \to V$, the inverse image of v is a torsor for the inverse image of 0. So $dp^{-1}(b, v)$ is a torsor for $T\Sigma$, and is classfied by an element of $H^1(\Sigma; T\Sigma)$. This gives a map

$$T_b B \to H^1(\Sigma; T\Sigma).$$

This is the Kodaira-Spencer map, and one expects it to be an isomorphism when B is the universal family.

Example 1.6.2. How does this work out in the case of Lubin-Tate space? In this case we imagine a family of formal groups $p : E \to B$ with $p^{-1}(b) = \Gamma$. In case $v = 0$, the group $p^{-1}(b, v)$ is the tangent bundle to Γ which sits in an exact sequence

$$0 \to \operatorname{Lie}\Gamma \to E \to \Gamma \to 0.$$

This sequence splits by the choice of "0" in each tangent space. When v is not zero there is no longer a canonical choice of splitting, and you get a commutative group extension of Γ by $\operatorname{Lie}\Gamma$. Such group extensions are classified by $H^2(\Gamma; \operatorname{Lie}\Gamma)$. This gives a map

$$T_b(B) \to H^2(\Gamma; \operatorname{Lie}\Gamma)$$

which, as Lubin-Tate showed, is an isomorphism in case B is Lubin-Tate space. This is important for getting explicit formulas, as we will see later. To summarize, the Kodaira-Spencer map exhibits an isomorphism

$$(\mathfrak{m}/\mathfrak{m}^2)^* \approx H^2(\Gamma; \operatorname{Lie}\Gamma)$$

where $\mathfrak{m} \subset E_0$ is the maximal ideal, and $(-)^*$ indicates vector space dual.

LECTURE 2

Toward explicit formulas

2.1. Differentials and the log

The Lubin-Tate construction gives a terrific way of constructing formal group laws. However it isn't always the best way of getting explicit formulas. Often it is easier to work with the "logarithm" of the formal group.

Exercise 2.1.1. Suppose that A is a $\mathbb{Q}$-algebra and F is a formal group law over A. There is a unique power series $\log_F(x)$ with the properties

$$\begin{aligned} d\log_F(0) &= 1 \\ \log_F(x \underset{F}{+} y) &= \log_F(x) + \log_F(y). \end{aligned}$$

To do this exercise you work modulo succesive powers of x and use the symmetric 2-cocycle lemma. The function $\log_F(x)$ is the unique isomorphism of F with the additive formal group law having derivative 1 at 0.

Write $\exp_F(y)$ for the inverse function of $\log_F(x)$. Then one has

$$x \underset{F}{+} y = \exp_F(\log_F(x) + \log_F(y)). \tag{2.1.1}$$

Most of the time it is easier to work with the log of a formal group law than it is to work with the coefficients of $F(x, y)$.

The expression "log of a formal group law" is used for lots of things. If F is a formal group law over a torsion free ring A, then F has a log over $A \otimes \mathbb{Q}$

$$x + m_1 x + \cdots + m_n x^{n+1} + \cdots \in A \otimes \mathbb{Q}[\![x]\!]. \tag{2.1.2}$$

This is also called the "log" of F. Given an arbitrary power series

$$\log_F(x) \in A \otimes \mathbb{Q}[\![x]\!]$$

the formal group law defined by (2.1.1) may or may not be defined over A. There are, however, some cool conditions that guarantee that it is. I will hopefully get to some of them in a later version of this lecture. At any rate the question isn't really as hard as it looks at first blush.

For any ring A one can find a torsion free ring $\tilde{A}$ and a surjective map $\tilde{A} \to A$. By Lazard's theorem, the formal group law F over A can be lifted to a formal group law $\tilde{F}$ over $\tilde{A}$. The formal group law $\tilde{F}$ has a log over $\tilde{A} \otimes \mathbb{Q}$ which is often called the log of F (or the "log of a lift.").

Let's return to the situation of a formal group law F over a torsion free ring A, and write

$$\log_F(x) = x + m_1 x^2 + \cdots + m_n x^{n+1} + \ldots.$$

Even though $\log_F(x)$ does not necessarily have coefficients in A, the form

$$d\log_F(x) = (1 + 2m_1 x + \cdots + (n+1)m_n x^n + \ldots)\, dx \tag{2.1.3}$$

actually does. Why is that? If we think if $\log_F(x)$ has a map

$$F \to \mathbf{G}_a$$

(given by $y = \log_F(x)$) then the 1-form (2.1.3) is the pullback along the log of the 1-form dy. Now dy can be characterized uniquely. It is the unique translation invariant 1-form whose value at 0 is dy. This has to pull back to the unique translation invariant 1-form on F whose value at 0 is dx. Since we've described it without mentioning $A \otimes \mathbb{Q}$ it must be defined over A.

There are two good ways to get a formula for the log of F. One is to start with the equation

$$\log_F(x \underset{F}{+} y) = \log_F(x) + \log_F(y)$$

and take the partial with respect to y at $y = 0$. One gets

$$\log_F'(x)\partial_2(F(x,0)) = 1$$

or

$$d\log_F(x) = \frac{dx}{\partial_2 F(x,0)}$$

where ∂_2 means "take the derivative with respect to the second variable." This is kind of a cool formula. The expression on the right is a formula for the invariant differential on F and is meant to remind you of the formula for the invariant differential on an elliptic curve.

You can also derive the formula for the invariant differential in the usual way. Suppose it is $g(x)\,dx$. The condition that $g(x)\,dx$ be invariant is that for every constant t

$$g(t \underset{F}{+} x)d(t \underset{F}{+} x) = g(t)\,dt$$

or

$$g(t \underset{F}{+} x)F_2(t,x) = g(t).$$

Setting $t = 0$ one gets

$$g(x)F_2(0,x) = g(0) = 1.$$

One can translate the Lubin-Tate criterion of Proposition 1.5.2 into a statement about the log.

Proposition 2.1.4. *Let G be a formal group law over $\mathbb{W}[\![u_1,\ldots,u_{n-1}]\!]$ deforming Γ and write*

$$\log_G(x) = \sum b_n x^{n+1},$$

with $b_n \in \mathbb{Q} \otimes \mathbb{W}[\![u_1,\ldots,u_{n-1}]\!]$. The group G is a universal deformation if for $i < n$,

$$pb_{p^i} \equiv u_i \mod (u_1,\ldots,u_{n-1})^2$$

and

$$pb_n \equiv u \mod (u_1,\ldots,u_{n-1})^2 \qquad u \in E_0^\times.$$

2.2. p-typical coordinates

Let's go back to the case of a torsion free ring A and a power series

$$f(x) = \sum a_n x^n \in (A \otimes \mathbb{Q})[\![x]\!]. \tag{2.2.1}$$

It turns out that if (2.2.1) is the log of a formal group law F over a p-local ring A, then so is

$$g(y) = \sum b_n y^{p^n} \tag{2.2.2}$$

where $b_n = a_{p^n}$. This seems somewhat surprising and hard to prove if you take it on directly. The trick is to find a substitution $y = x + \cdots \in A[\![x]\!]$ and show that the log of F, expressed in the coordinate y is given by (2.2.2).

Definition 2.2.3. Suppose that F is a formal group law over a ring A. A *curve on* F is a power series $\gamma(x) \in A[\![x]\!]$, with $\gamma(0) = 0$. The set of curves on F forms a group under the operation

$$\gamma_1(x) \underset{F}{+} \gamma_2(x).$$

Remark 2.2.4. In the language of formal geometry, a curve on F is a map of formal schemes

$$\mathbb{A}^1 \to F.$$

Exercise 2.2.1. Show that if A is a p-local ring, then the group of curves on A is a $\mathbb{Z}_{(p)}$-module.

There are operations that take curves to curves. The nth *Vershiebung operator* is the operator defined by

$$(V_n\gamma)(x) = \gamma(x^n)$$

and the nth *Frobenius operator* is given, formally as

$$F_n(\gamma(x)) = \sum{}^F \gamma(\zeta^i x^{1/n})$$

in which ζ is a primitive n^{th} root of unity and i is running from 0 to $(n-1)$.

Example 2.2.5. If F is the additive group and

$$\gamma(x) = \sum a_n x^n$$

then

$$(V_\ell\gamma)(x) = \sum a_n x^{\ell n}$$
$$(F_\ell\gamma)(x) = \sum \ell a_{n\ell} x^n.$$

Note that if A is p-local, and $(\ell, p) = 1$ then one can form the operator

$$\varepsilon_\ell = 1 - \frac{1}{\ell} V_\ell F_\ell.$$

In the case of the additive group one has

$$\varepsilon_\ell\big(\sum a_n x^n\big) = \sum_{(n,\ell)=1} a_n x^n.$$

From this one can check that ε_ℓ is a projection operator. Now let π be the composition of all of the projection operators ε_ℓ for primes $\ell \neq p$. Then in the additive group

$$\pi\big(\sum a_n x^n\big) = \sum a_{p^n} x^{p^n}.$$

It follows that if we take $\gamma(x) = x$ then $y = \pi\gamma$ is the desired new coordinate.

Definition 2.2.6. A curve γ is *p-typical* if for all $(\ell, p) = 1$, $F_\ell \gamma = 0$.

Remark 2.2.7. When A is p-local a curve γ is p-typical if and only if $\epsilon_\ell \gamma = \gamma$.

Definition 2.2.8. A curve $\gamma(x)$ on F is a *coordinate* if $\gamma'(0)$ is a unit in A.

We have shown that every formal group over a p-local ring has a p-typical coordinate. Unless otherwise stated, we will restrict our attention to p-typical formal group laws.

2.3. Hazewinkel's functional equation lemma

Building on work of Honda and others, on the functional equation satisfied by the logarithm of many formal group laws, Hazewinkel abstracted a general lemma saying that a power series satisfying a suitable functional equation was the log of a formal group. The proof is one of those "Frobenius contraction" arguments.

Before turning to the general result, let me give you an example. Consider the series

$$\ell(x) = x + \frac{x^p}{p} + \cdots \frac{x^{p^n}}{p^n} + \cdots .$$

It turns out that this is the log of a formal group law defined over $\mathbb{Z}_{(p)}$. How can we tell that? One way is to try and work out the height. Suppose that the multiplication by p map is

$$[p](x) = a_1 x + a_2 x^2 + \cdots .$$

Then from the identity

$$\ell([p](x)) = p\ell(x)$$

one works out that

$$\begin{aligned} a_1 &= p \\ a_i &= 0 \qquad 1 < i < p \\ a_p &= (1 - p^{p-1}) \equiv 1 \mod p. \end{aligned}$$

So if it is the log of a formal group its reduction has height 1 and by the Lubin-Tate theorem it isomorphic to the multiplicative formal group law. So to show that it defines a formal group law over $\mathbb{Z}_{(p)}$ is equivalent to showing that

$$\exp(\ell(x))$$

has coefficients in $\mathbb{Z}_p$. Now this is a classic result. This series is called the "Artin-Hasse" exponential. To prove it has integer coefficents one can observe it is the series gotten from the standard coordinate on the multiplicative group using the projection operator of §2.2. Alternatively one can use the following cool lemma of Dwork.

LEMMA 2.3.1. *A power series*

$$f(x) \in \mathbb{Q}[\![x]\!]$$

with $f(0) = 1$ *has coefficients in* $\mathbb{Z}_{(p)}[\![x]\!]$ *if and only if*

$$f(x^p)/f(x)^p$$

is in $1 + p\mathbb{Z}_{(p)}[\![x]\!]$.

Proof: The only if direction is obvious from Fermat's little theorem. The if direction is by induction on n. Write $f(x) = 1 + a_1 x + \dots$ and suppose by induction we have shown that $a_i \in \mathbb{Z}_{(p)}$ for $i < n$. Write

$$\tilde{f}(x) = \sum_{i<n} a_i x^i$$

so that $f(x) = \tilde{f}(x) + a_n x^n + \cdots$. Then

$$f(x)^p/f(x^p) = \tilde{f}(x)^p/\tilde{f}(x^p) + pa_n x^n + O[x]^{n+1}.$$

By the only if direction $\tilde{f}(x)^p/\tilde{f}(x^p)$ is in $1+p\mathbb{Z}_{(p)}[\![x]\!]$ and by assumption the entire expression is as well. This means that $pa_n \in p\mathbb{Z}_{(p)}$ and so $a_n \in \mathbb{Z}_p$. □

Using Dwork's Lemma it's easy to show that $\exp(\ell(x))$ has coefficients in $\mathbb{Z}_{(p)}$ and so $\ell(x)$ is the log of a formal group over $\mathbb{Z}_{(p)}$, and it is isomorphic to the multiplicative formal group. The key thing here was that

$$p\ell(x) - \ell(x^p) = px$$

or that $\ell(x)$ satisfied the functional equation

$$\ell(x) = x + \frac{1}{p}\ell(x^p).$$

Hazewinkel's general result implies that under suitable circumstances, a power series $f(x)$ satisfying a functional equation of the form

$$\ell(x) = g(x) + \frac{s_1}{p}\ell^{\phi}(x^p) + \frac{s_2}{p}\ell^{\phi^2}(x^{p^2}) + \cdots$$

will define a formal group law over a reasonable ring.

Here is the setup. We have a pair of rings $A \subset L$. For the moment picture that A is the ring of integers in a p-adic field L. We have an ideal $I \subset A$ which you can think of as the maximal ideal, but the only thing we really ask is that p is in I. We assume we have a "lift of (a power of) Frobenius" which will mean a choice of a fixed power q of p, and ring automorphism

$$\phi : L \to L$$

that restricts to an automorphism $\phi : A \to A$, having the property that

$$\phi(a) \equiv a^q \mod I.$$

Finally we have parameters $s_i \in L$ whose denominators are bounded in the sense that they satisfy

$$\phi^r(s_i)I \subset A$$

for all i and r. In this situation one has

THEOREM 2.3.2 (Hazewinkel). *Fix a power series $g(x) \in A[\![x]\!]$ with $g'(0) \in A^\times$, and let $f(x)$ be the power series defined recursively by the functional equation*

$$f(x) = g(x) + s_1 f^\phi(x^p) + \cdots s_i f^{\phi^i}(x^{p^i}) + \cdots$$

where f^ϕ is the power series obtained by applying the map ϕ to the coefficients of f. In this situation the power series

$$f^{-1}(f(x) + f(y))$$

has coefficients in A and so $f(x)$ is the log of a formal group law over A.

There are several other aspects to the functional equation lemma which are useful. The next result asserts that the formal group you get this way depends only on the parameters s_i and not on g.

THEOREM 2.3.3. *In the situation of Theorem 2.3.3, suppose f and $\tilde{f}$ are defined from the data (g, s_i) and $(\tilde{g}, s_i)$ respectively. Then the power series $f^{-1}(\tilde{f}(x))$ has coefficients in A and so gives an isomorphism of the formal group $F_{(g,s_i)}$ and $F_{(\tilde{g},s_i)}$.*

2.4. The Hazewinkel parameters

Consider the ring $A = \mathbb{Z}_p[v_1, v_2, \dots]$ with $I = (p)$ and $\phi(v_i) = v_i^p$. Then the series $f(x)$ defined by the functional equation

$$f(x) = x + \sum \frac{v_i}{p} f^{\phi^i}(x^{p^i})$$

is the log of a formal group over A. If one writes

$$f(x) = \sum m_n x^{p^n}$$

then from the functional equation one gets the recursive relation

$$m_n = \frac{v_1}{p} m_{n-1}^\phi + \cdots + \frac{v_{n-1}}{p} m_1^{\phi^{n-1}} + \frac{v_n}{p}.$$

It's not hard to expand this out and get a complete expression for m_n as a polynomial in the v_i with coefficients in $\frac{1}{p}\mathbb{Z}$. The first few terms are

$$\begin{aligned}
m_1 &= \frac{v_1}{p} \\
m_2 &= \frac{v_2}{p} + \frac{v_1^{1+p}}{p^2} \\
m_3 &= \frac{v_3}{p} + \frac{v_1 v_2^p}{p^2} + \frac{v_2 v_1^{p^2}}{p^2} + \frac{v_1^{1+p+p^2}}{p^3} \\
m_n &= \sum_{i_1+\cdots+i_k=n} \frac{v_{i_1} v_{i_2}^{p^{i_1}} \cdots v_{i_k}^{p^{1_1+i_2+\cdots+i_{k-1}}}}{p^k}
\end{aligned}$$

The above formula formula defines parameters v_n for any formal group law. The v_n are called the *Hazewinkel* parameters (or "generators") and one has the following result

PROPOSITION 2.4.1. *Suppose that A is a $\mathbb{Z}_{(p)}$-algebra. A p-typical series*

$$f(x) = \sum m_n x^{p^n}$$

is the log of a formal group law over A if and only if the Hazewinkel parameters v_n are in A for all n. □

This result can be derived from Lazard's theorem, and the result of §2.2.

Hazewinkel's result gives a formal group law over

$$\mathbb{Z}_{(p)}[v_1, \ldots,].$$

One way of constructing the Lubin-Tate universal deformation is to change base along the map

$$\mathbb{Z}_{(p)}[v_1, \ldots,] \to \mathbb{W}[\![u_1, \ldots, u_{n-1}]\!]$$

defined by

$$v_i \mapsto \begin{cases} u_i & i < n \\ 1 & i = n \\ 0 & i > n. \end{cases}$$

2.5. Dieudonné modules

There is a really great way of understanding formal groups over a perfect field k. Let's start with the mechanics of the Dieudonné theory and get to the "theory" part of the theory later. A word of warning. I'm going to talk about the *covariant* Dieudonné module. There are various reasons why it isn't as good as the *contraviariant* Dieudonné theory, but it's also got some advantages. I'll try and lay out the relative merits a bit later.

Let k be a perfect field of characteristic $p > 0$ and write $\mathbf{FGL}_k$ for the category of commutative formal groups of finite dimension over k and homomorphisms. I could have said formal group *laws* over k and gotten an equivalent category. The Dieudonné theory gives an equivalence of $\mathbf{FGL}_k$ with a category easily described in the language of linear algebra.

Let $\mathbb{W}$ be the ring of Witt vectors of k,

$$\begin{aligned} \phi : \mathbb{W} &\to \mathbb{W} \\ a &\mapsto a^\phi \end{aligned}$$

the Frobenius automorphism. Let $\mathbf{Dieud}_k^{\text{big}}$ be the category of left $\mathbb{W}$ modules M equipped with abelian group homomorphisms

$$F, V : M \to M$$

satisfying

$$\begin{aligned} F(am) &= a^\phi F(m) \qquad a \in \mathbb{W} \\ V(a^\phi m) &= aV(m) \qquad a \in \mathbb{W} \\ FV &= VF = p. \end{aligned}$$

(and $\mathbb{W}$-linear maps compatible with F and V). Let $\mathbf{Dieud}_k \subset \mathbf{Dieud}_k^{\text{big}}$ be the full subcategory of modules M satisfying the following conditions

i) M is free of finite rank over $\mathbb{W}$

ii) The map $M \to \varprojlim M/V^j M$ is an isomorphism.

Remark 2.5.1. The assumptions imply that for each j, the map $V^j : M/VM \to V^jM/V^{j+1}M$ is an isomorphism. The map is obviously surjective. To see that it is injective, suppose that $V^j\gamma = V^{j+1}\eta$. Multiply both sides by F^j to get

$$p^j(\gamma - V\eta) = 0.$$

Since we have assumed that M is p-torsion free this means that $\gamma = V\eta$.

The Dieudonné theory provides an equivalence of categories between $\mathbf{FGL}_k$ and $\mathbf{Dieud}_k$. Let's just assume this for the moment and work out a few things. Suppose that $M \in \mathbf{Dieud}_k$ is a Dieudonné module. The properties of M imply that M/VM naturally has the structure of a k-vector space.

Definition 2.5.2. The *height* of a Dieudonné module is the rank of M over $\mathbb{W}$. The *dimension* of M is the dimension $\dim_k M/VM$.

Under the equivalece of categories, the height and dimension of a Dieudonné module correspond to the height and dimension of the corresponding formal group.

When working with the covariant Dieudonné theory, the elements of M are called *p-typical curves* or just *curves* for short. A *coordinate system* on M is an ordered set $(\gamma_1, \ldots, \gamma_m) \subset M$ of curves whose image in M/VM form a k-basis. Coordinate systems on M correspond to (p-typical) coordinate systems on the corresponding formal group law. It follows that what we have been studying, (p-typical) formal group *laws* of dimension 1 over k correspond to Dieudonne modules M equipped with a choice of curve $\gamma \in M$ with the property that the image of γ in M/VM generates M/VM as a k-vector space.

Exercise 2.5.1. Show that if k is algebraically closed, two Dieudonné modules of the same height and dimension are isomorphic. If this seems hard just try it in the case of dimension 1. The case of dimension 1 gives a proof of Proposition 1.3.5.

Example 2.5.3. Let M_n be the Dieudonné module with basis $\{\gamma, V\gamma, \cdots, V^{n-1}\gamma\}$ and $F\gamma = V^{n-1}\gamma$. You can check that this defines a Dieudonné module of dimension 1 and height n. Under the equivalence of categories, this is the Dieudonné module corresponding to the formal group law Γ.

Exercise 2.5.2. Choose integers $a, b > 0$ with $(a, b) = 1$. Show that there is a free Dieudonné module $M_{a,b}$ generated by a curve γ subject to the relation $F^a\gamma = V^b\gamma$. What are the dimension and height of $M_{a,b}$?

Under the Dieudonné correspondence the Lubin-Tate formal group Γ corresonds to M_n. Using the equivalence we can easily determine the endomorphism ring of Γ, as well as the automorphism group of Γ. Let's do that.

We will suppose for convenience that k is algebraically closed. Obviously any endomorphism is determined by where the generator γ goes. So suppose

$$\gamma \mapsto a_1\gamma + \cdots + a_nV^{n-1}\gamma.$$

Using the fact that ϕ is an isomorphism (since k was assumed to be perfect) we can apply the relation $V^n = p$ to both sides and get

$$p\gamma \mapsto pa_1^{\phi^{-n}}\gamma + \cdots pa_n^{\phi^{-n}}V^{n-1}\gamma$$

from which we conclude that

$$a_i^{\phi^n} = a_i.$$

I claim that is the only condition. We're supposed to check that our map is compatible with the relation $F\gamma = V^{n-1}\gamma$, but this is now automatic:

$$\begin{aligned} F\gamma &\mapsto F(a_1\gamma + \cdots + a_n V^{n-1}\gamma) \\ &= a_1^{\phi} V^{n-1}\gamma + \cdots + a_i^{\phi} V^{i-1+(n-1)}\gamma + \cdots \\ V^{n-1}\gamma &\mapsto V^{n-1}(\sum a_i V^{i-1}\gamma) \\ &= \sum a_i^{\phi^{-(n-1)}} V^{i-1+n-1}\gamma \\ &= \sum a_i \phi V^{i-1+n-1}\gamma. \end{aligned}$$

Writing the above out in matrix form gives the computation described in Proposition 1.3.6.

Remark 2.5.4. The automorphism denoted Π in Proposition 1.3.6 is usually called F, but this get's kind of confusing because of the operator F. The operator F acts on the right of M and is ϕ-linear. The automorphism Π is W-linear and you can think of it as acting on the right. The matrix for Π looks a lot like the one for V but the operator V is ϕ^{-1} linear.

There is another useful way to describe this endomorphism ring. Let $T : M \to M$ be the map sending γ to $V\gamma$. (Usually T is called F). One can check that $\mathrm{End}(M)$ is isomorphic to the algebra of not quite commutative power series

$$\mathbb{W}\langle T\rangle/(Ta = a^{\phi}T, T^n = p).$$

This is the maximal order in the division algebra $D = D_n$ with Hasse invariant $\frac{1}{n}$. It's a little more professional to write it as $\mathcal{O}_D$ and state Proposition 1.3.6 as an isomorphism $\mathbf{S}_n = \mathcal{O}_D^{\times}$.

It's easy to check that every non-zero endomorphism of M_n divides a power of p, so that $\mathbb{Q} \otimes \mathrm{End}(M_n)$ is a central simple division algebra over $\mathbb{Q}_p$.

Exercise 2.5.3. Show that the Hasse invariant of this division algebra is $\frac{1}{n}$. More generally show that $\mathbb{Q} \otimes \mathrm{End}(M_{a,b})$ is a central simple division algebra with Hasse invariant $\frac{a}{a+b}$.

2.6. Dieudonné modules and logarithms

Here's a question. How can we get our hands on the formal group law associated to a Dieudonné module M equipped with a coordinate γ? Here is how Choose a $\mathbb{W}$-linear homomorphism $T : M \to \mathbb{W}$ having the property that

$$T(\gamma) = 1.$$

Proposition 2.6.1. *The power series*

$$\ell(x) = \sum T(F^n\gamma)\frac{x^{p^n}}{p^n} \in \mathbb{W}[\![x]\!]$$

is the log of a formal group law over $\mathbb{W}$.

The formal group law associated to M is the reduction of this formal group law to $k = \mathbb{W}/p$.

We will talk about ways to prove Proposition 2.6.1. One is to try and get it from the Lubin-Tate argument and the other is to use Hazewinkel's Functional

Equation Lemma (Theorem 2.3.2). Except in special cases these methods aren't totally straightforward. We will deduce it from another result below.

2.7. Tapis de Cartier

Cartier showed [**4**] that in fact the above construction gives bijection between the set of equivalence class of lifts of Γ and a suitable set of lifts

$$\begin{tikzcd} & \mathbb{W} \arrow[d] \\ M \arrow[r] \arrow[ur, dashed, "T"] & k \end{tikzcd} .$$

in which the bottom map is the reduction M/VM followed by a choice of isomorphism of k-vector spaces $M/VM \approx k$. Let me state this more formally.

Theorem 2.7.1 (Tapis de Cartier). *Suppose that Γ is a formal group law over k of dimension 1 and height $n < \infty$ and let M be the Diedonné module of Γ. Let $\gamma \in M - VM$ be a p-typical coordinate, and write $\epsilon : M \to k$ for the unique map satisfying*

$$\begin{aligned} \epsilon(VM) &= 0 \\ \epsilon(\gamma) &= 1. \end{aligned}$$

The formula of Proposition 2.6.1 gives a bijection between the set of isomorphism classes of lifts of Γ to $\mathbb{W}$ and the set of $\mathbb{W}$-linear homomorphism

$$T : M \to \mathbb{W}$$

with $T(\gamma) = 1$.

We will give a proof of both Proposition 2.6.1 and Theorem 2.7.1 below. But let's just go along a bit and believe in the two statements. The proof will kind of pop out of that.

In principle Theorem 2.7.1 should allow us to get a good description of the Lubin-Tate ring. It tells us that the lifts to $\mathbb{W}$ correspond to suitable ring homomorphisms $\mathrm{Sym}(M) \to \mathbb{W}$, and this suggests that there is some relationship between $\mathrm{Sym}(M)$ and the Lubin-Tate ring E_0. That would be a good thing, as it's easy to understand how $\mathbf{S}_n$ acts on $\mathrm{Sym}(M)$. There are a lot of little things to straighten out, but once we get that done we will see that there is indeed such a relation, and it can be described geometrically in terms of the crystalline period mapping.

Let's try and directly make this work. We will take $M = M_n$ and give it the basis $\{\gamma, V\gamma, \ldots, V^{n-1}\gamma\}$. Our map $\epsilon : M \to k$ is the one sending γ to 1 and $V^i\gamma$ to 0 for $i > 0$. Let's define variable w and ww_i on the space of lifts T by

$$\begin{aligned} w(T) &= T(\gamma) \\ w_i(T) &= T(V^i\gamma)/T(\gamma) \qquad i = 1, \ldots, n-1. \end{aligned}$$

By Proposition 2.6.1 the log of the lift corresponding to T is

$$\log(x) = w\ell(x) + \frac{ww_1}{p}\ell(x^p) + \cdots + \frac{ww_{n-1}}{p}\ell(x^{p^{n-1}})$$

where

$$\ell(x) = \sum_{k\geq 0} \frac{x^{p^{nk}}}{p^k}.$$

The element w will be invertible, so changing the coordinate on the additive group we could write the log as

$$f(x) = \ell(x) + \frac{w_1}{p}\ell(x^p) + \cdots + \frac{w_{n-1}}{p}\ell(x^{p^{n-1}}). \tag{2.7.2}$$

Now this looks promising. This series certainly satisfies the criterion of Proposition 2.1.4. If it worked out to be the log of a formal group law over $\mathbb{W}[\![w_1, \ldots, w_{n-1}]\!]$ we would have succeeded in writing down the universal deformation in a manner well related to the action of $\mathbf{S}_n$. Promising as it looks, it doesn't work. You can check this yourself by trying to work out the Hazewinkel parameters. You get

$$\begin{aligned} w_1 &= pm_1 = v_1 \\ w_2 &= pm_2 = v_2 + m_1 v_1^p = v_2 + \frac{v_1^{1+p}}{p}, \end{aligned}$$

and you see that v_1^{1+p} has to be divisible by p.

All is not lost though. The expression looks a lot like it would work if we could use Hazewinkel's functional equation lemma. In fact if we set $\phi(w_i) = 0$ for all i then $f(x)$ satisfies the functional equation

$$f(x) = x + \frac{w_1}{p} f^{\phi}(x^p) + \cdots + \frac{w_{n-1}}{p} f^{\phi^{p^{n-1}}}(x^{p^{n-1}}) + \frac{1}{p} f^{\phi^{p^n}}(x^{p^n}).$$

Now why aren't we allowed to do this? We have to specify and ideal $I \subset R$ containing p with the properties that

$$\phi(x) \equiv x^p \mod I$$

and among other things having the property that

$$\begin{aligned} \phi^j(w_i/p)I &\subset R \\ \phi^j(1/p)I &\subset R. \end{aligned}$$

This last condition means that I is contained in pR, so we must have $I = (p)$. So the only way this can work is if for all i, $w_i^p \in pR$. But this isn't quite enough. If $x = \phi(w_i^p)/p$ is in R then x^p must also be divisible by p. If you think this through a bit you'll come to the conclusion that this will work if the ideal $(w_1, \ldots, w_{n-1})$ has divided powers. So let

$$\mathbb{W}\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle$$

be the "divided power completion" of the complete local ring $\mathbb{W}[\![w_1, \cdots, w_n]\!]$. Over this ring the functional equation lemma implies that (2.7.2) is the log of a formal group law. This defines a deformation of Γ to $\mathbb{W}\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle$. It is classified by a map

$$E_0 \to \mathbb{W}\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle$$

and one can easily write down the Hazewinkel parameters u_i in terms of the w_i. We will do this in detail a little later. For now let's just note that, by construction, the leading terms are

$$\begin{aligned} u_i &= w_i + \cdots \qquad 1 < i < n \\ u_n &= 1. \end{aligned}$$

This means that the group we have constructed is the universal deformation of Γ to local rings $(B, \mathfrak{m})$ equipped with a divided power structure on $\mathfrak{m}$. We also have excellent control on the action of $\mathbf{S}_n$.

Here is one immediate consequence of this. Since the ideal $(p) \subset \mathbb{W}$ has a (unique) divided power structure, the deformations of Γ to $\mathbb{W}$ are classified by (continuous) ring homomorphisms

$$\mathbb{W}\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle \to \mathbb{W},$$

and so have a logarithm given by (2.7.2). This proves both Proposition 2.6.1 and Theorem 2.7.1.

There's another remarkable thing that comes out of this formula. It's pretty clear that the formula of Proposition 2.6.1 is gotten by applying T to something that depends only on the Dieudonné module M. Let's try and work it out. We want to make sense of the formula

$$f(x) = \sum F^n \gamma \frac{x^{p^n}}{p^n}$$

but a small amount of typechecking shows this doesn't quite make sense. It would if we chose a basis for M and wrote F as a matrix A, and if the symbol x stood for an h-tuple

$$x = (x_1, \ldots, x_h)$$

with $h = \dim_{\mathbb{W}} M$ the height of the Dieudonné module. In this case case $f(x)$ would be an h-tuple of power series in the variables x_i. One might hope that $f(x)$ is the log of a formal group law of dimension h over $\mathbb{W}$. This is indeed the case, and follows easily from the higher dimensional version of Hazewinkel's functional equation lemma. Indeed, this f satisfies the functional equation

$$f(x) = x + \frac{A}{p} f^{\phi}(x^p)$$

and works out to be

$$f(x) = x + \sum A A^{\phi} \cdots A^{\phi^n} \frac{x^{p^n}}{p^n}.$$

This construction is also called the "tapis de Cartier." Magically, it associates to a formal group law Γ of finite height h over a perfect field of characteristic p, with Dieudonné module M, an h-dimensional formal group law $\mathcal{G}$ over the ring of Witt vectors of k. The assertions of Proposition 2.6.1 and 2.7.1 imply that every lift G of Γ to $\mathbb{W}$ is a quotient of $\mathcal{G}$:

$$1 \to V \to \mathcal{G} \to G \to 1. \tag{2.7.3}$$

By construction one has $\operatorname{Lie} \mathcal{G} = M$, and the map on Lie algebras associated to (2.7.3) becomes, after choosing an isomorphism $\operatorname{Lie} G \approx \mathbb{W}$, the sequence

$$0 \to \operatorname{Lie} V \to M \xrightarrow{T} \operatorname{Lie} G \approx \mathbb{W} \to 0.$$

One can check that V is isomorphic to the additive group and so determined by its Lie algebra. The sequence (2.7.3) turns out to be the *universal extension of G by an additive group* in the sense that the map

$$\hom(V, \mathbf{G}_a) \to \operatorname{Ext}(G, \mathbf{G}_a)$$

gotten by cobase change of (2.7.3) along $V \to \mathbf{G}_a$, is an isomorphism. It is possible to construct $\mathcal{G}$ directly as the universal additive extension, and show that it is

independent of the choice of lift G. This gives a definition of the Dieudonné module as the "Lie algebra of the universal additive extension of a lift." At least this would be a definition if one make it a functor of Γ alone. This can be done. See [**23**]. Getting this straight was one of the motivations for the introduction of the crystalline topos. We will return to this a bit later.

2.8. Cleaning things up

Before turning to a more invariant version of the theory we clean a few things up. If M is a Diedonné module of a formal group Γ then M/VM is naturally isomorphic to the Lie algebra $\operatorname{Lie}\Gamma$. The choice of curve $\gamma \in M$ is used only through the isomorphism $M/VM \approx k$ or in other words only through the induced isomorphism $\operatorname{Lie}\Gamma \approx k$. The lift G_T associate to $T : M \to W$ comes equipped with an isomorphism $\operatorname{Lie} G_T \approx \mathbb{W}$. This makes it clear that we are really working with formal groups G which are *rigidified* in the sense that they come equipped with a basis $\operatorname{Lie} G$ of the Lie algebra. Let's step back a bit and systematically incorporate this choice.

Suppose that G is a 1-dimensional formal group law over a ring R. The Lie algebra $\operatorname{Lie} G$ is then a locally free R-module of rank 1, with dual the cotangent space $\omega = \omega_G$ to G at the identity section. Geometrically, $\operatorname{Lie} G$ is a line bundle over $\operatorname{Spec} R$ (or $\operatorname{Spf} R$). For a map $R \to S$, an isomorphism

$$S \underset{R}{\otimes} \operatorname{Lie} G$$

corresponds to a lift

$$\begin{tikzcd} & P \arrow[d] \\ \operatorname{Spec} S \arrow[ur, dashed] \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

to the principal Gl_1-bundle underlying $\operatorname{Lie} G$. This means we should probably be working over P to begin with. Now the ring of functions on P is graded by the eigenspaces of the Gl_1-action. So given Γ over k we should work with the graded ring k_* whose homogeneous components correspond to sections of $\operatorname{Lie}\Gamma^{\otimes j}$ or $\omega_\Gamma^{\otimes j}$. The convention in the algebraic topology literature is to make the ring k_* be *evenly graded* with

$$k_{2n} \approx H^0(\operatorname{Spec} k; \omega_\Gamma^n).$$

An isomorphism $k \to \operatorname{Lie}\Gamma$ corresponds to an invertible element $u \in k_{-2}$, and once one is chosen one gets an isomorphism

$$k_* \approx k[u, u^{-1}] \qquad |u| = -2.$$

To connect the Dieudonné theory to deformations, we should also work on the principal $\operatorname{Lie} G$-bundle of a deformation G. This motivates looking at the graded ring E_* associated to the ring of formal functions on the universal deformation of Γ. Again, following the conventions in topology the homogeneous component E_{2n} is the space of sections of $\omega_{G_{\mathrm{univ}}}$. A choice of isomorphism $u : E_0 \approx \operatorname{Lie} G_{\mathrm{univ}}$ specifies an element $u \in E_{-2}$ and enables us to write

$$E_* \approx \mathbb{W}[\![u_1, \ldots, u_{n-1}]\!][u, u^{-1}].$$

But it's important to remember that the parameters u_i and u are not canonically chosen. In fact a *good* choice would be one in which one could understand the action of $\operatorname{Aut}\Gamma$.

OK. How does the Tapis de Cartier match the Dieudonné theory with this graded ring? Let

$$\mathcal{G}_{\mathrm{univ}} \to G_{\mathrm{univ}}$$

be the universal additive extension of G_{univ} by an additive group. There is then a map

$$\operatorname{Lie}\mathcal{G}_{\mathrm{univ}} \to \operatorname{Lie}G_{\mathrm{univ}}.$$

If we imagine that the Dieudonné module M is somehow related to $\operatorname{Lie}\mathcal{G}_{\mathrm{univ}}$ then we would have a map, or relationship between M and the module of sections of $\operatorname{Lie}G_{\mathrm{univ}}$. In terms of our graded ring, this sets up the expectation that there is a map, or at least a relationship between M and E_{-2}. This will be the subject of the next two lectures.

Exercise 2.8.1. Can you work out how this squares with the Kodaira-Spencer picture of deformations from §1.6?

LECTURE 3

Crystals

3.1. A further formula

In the last section we introduced deformation parameters w and w_i which can be used to study deformations of formal group to local rings whose maximal ideal has a divided power structure. There are also Hazewinkel's deformation parameters u and u_i. Can we get a formula relating these?

With Hazewinkel's parameters, the log of the universal deformation satisfies the functional equation

$$f(x) = x + \frac{u_1}{p} f^{\phi}(x^p) + \cdots + \frac{u_{n-1}}{p} f^{\phi^{n-1}}(x^{p^{n-1}}) + \frac{1}{p} f^{\phi^n}(x^{p^n}).$$

Writing

$$f(x) = \sum m_n x^{p^n}$$

these expand into the recursive formuals

$$\begin{aligned} m_i &= 0 \qquad i < 0 \\ m_0 &= 1 \\ pm_\ell &= u_1 m_{\ell-1}^{\phi} + \cdots + u_{n-1} m_{\ell-(n-1)}^{\phi^{n-1}} + u_{n-\ell}. \end{aligned}$$

These are easily solved and one can write down a closed (albeit a bit complicated) formual for m_ℓ. Here is what you get if $n = 2$.

$$\begin{aligned}
m_0 &= 1 \\
m_1 &= \frac{u_1}{p} \\
m_2 &= \frac{1}{p} + \frac{u_1^{p+1}}{p^2} \\
m_3 &= \frac{u_1}{p^2} + \frac{u_1^{p^2}}{p^2} + \frac{u_1^{p^2+p+1}}{p^3} \\
m_4 &= \frac{1}{p^2} + \frac{u_1^{p+1}}{p^3} + \frac{u_1^{p^3+1}}{p^3} + \frac{u_1^{p^3+p^2}}{p^3} + \frac{u_1^{p^3+p^2+p+1}}{p^4} \\
m_5 &= \frac{u_1}{p^3} + \frac{u_1^{p^4}}{p^3} + \frac{u_1^{p^2}}{p^3} + \frac{u_1^{p^2+p+1}}{p^4} + \frac{u_1^{p^4+p+1}}{p^4} + \frac{u_1^{p^4+p^3+1}}{p^4} + \frac{u_1^{p^4+p^3+p^2}}{p^4} + \frac{u_1^{p^4+p^3+p^2+p+1}}{p^5} \\
m_6 &= \frac{1}{p^3} + \frac{u_1^{p+1}}{p^4} + \frac{u_1^{p^5+1}}{p^4} + \frac{u_1^{p^5+p^4}}{p^4} + \frac{u_1^{p^5+p^4+p+1}}{p^5} + \frac{u_1^{p^5+p^2+p+1}}{p^5} + \frac{u_1^{p^3+1}}{p^4} + \frac{u_1^{p^5+p^4+p^3+1}}{p^5} \\
&\quad + \frac{u_1^{p^5+p^2}}{p^4} + \frac{u_1^{p^3+p^2+p+1}}{p^5} + \frac{u_1^{p^3+p^2}}{p^4} + \frac{u_1^{p^5+p^4+p^3+p^2}}{p^5} + \frac{u_1^{p^5+p^4+p^3+p^2+p+1}}{p^6}
\end{aligned}$$

There is a visible pattern here. The terms $p^n m_{2n}$ seem to be converging on something, as do the terms $p^n m_{2n-1}$.

Now the log $g(x)$ for the w_i deformation is given by

$$g(x) = x + \frac{w_1}{p}\ell(x^p) + \cdots + \frac{w_{n-1}}{p}\ell(x^{p^{n-1}}) + \frac{1}{p}\ell(x^{p^n}). \tag{3.1.1}$$

Writing

$$g(x) = \sum \eta_k x^{p^k}$$

one sees that

$$\eta_\ell = \frac{w_r}{p^t}$$

in which we have written

$$\begin{aligned} \ell &= tn - r \qquad 0 \le r < n \\ w_n &= 1. \end{aligned}$$

It might be easier to process this when $n = 2$:

$$g(x) = x + \frac{w_1}{p}x^p + \frac{1}{p}x^{p^2} + \frac{w_1}{p^2}x^{p^3} + \frac{1}{p^2}x^{p^4} + \cdots$$

so

$$\begin{aligned} \eta_{2m} &= \frac{1}{p^m} \\ \eta_{2m-1} &= \frac{w_1}{p^m}. \end{aligned}$$

This suggests the formula

$$ww_r = \lim_{t\to\infty} p^n m_{tn-r} \qquad (w_0 = 1)$$

This is in fact correct and is proved in [**5**, Theorem 4.4] from this point of view. There is a nice way of writing this formula. Let's do this when $n = 2$. The recursion formula (3.1.1) defining m_n is

$$\begin{aligned} m_{-1} &= 0 \\ m_0 &= 1 \\ m_n &= \frac{u_1}{p}m_{n-1}^{\phi} + \frac{1}{p}m_{n-1}^{\phi^2} \end{aligned}$$

can be written as the matrix equation

$$\begin{bmatrix} m_n & \frac{1}{p}m_{n-1}^{\phi} \\ m_{n-1} & \frac{1}{p}m_{n-2}^{\phi} \end{bmatrix} = \begin{bmatrix} m_{n-1} & \frac{1}{p}m_{n-2}^{\phi} \\ m_{n-2} & \frac{1}{p}m_{n-3}^{\phi} \end{bmatrix}^{\phi} \begin{pmatrix} \frac{u_1}{p} & \frac{1}{p} \\ 1 & 0 \end{pmatrix}$$

from which one gets the formula

$$\begin{bmatrix} p^n m_{2n} \\ p^n m_{2n-1} \end{bmatrix} = p^n \begin{pmatrix} \frac{u_1^{p^{2n-1}}}{p} & \frac{1}{p} \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \frac{u_1}{p} & \frac{1}{p} \\ 1 & 0 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Writing

$$A = \begin{pmatrix} \frac{u_1}{p} & \frac{1}{p} \\ 1 & 0 \end{pmatrix} \qquad A(0) = \begin{pmatrix} 0 & \frac{1}{p} \\ 1 & 0 \end{pmatrix},$$

one can also write the term on the right as

$$A^{\phi^{2n-1}} \cdots A \cdot A(0)^{-2n}$$

and one gets the formula

$$\begin{pmatrix} w & (ww_1)^\phi \\ ww_1 & w^\phi \end{pmatrix} = \lim_{m\to\infty} A^{\phi^m} A^{\phi^{m-1}} \cdots A \cdot A(0)^{-(m+1)}.$$

This expression turns out to have a nice geometric interpretation.

In the height n case one writes

$$\begin{aligned} w &= \varinjlim p^k m_{nk} = 1 + \cdots \\ ww_i &= \varinjlim p^{k+1} m_{nk+i} = u_i + \cdots . \end{aligned}$$

Let ϕ be the ring homomorphism with $\phi(u_i) = u_i^p$. Then Hazewinkel's recursion relation

$$m_\ell = \frac{u_1}{p} m_{\ell-1}^\phi + \cdots + \frac{u_{n-1}}{p} m_{\ell-(n-1)}^{\phi^{n-1}} + \frac{1}{p} m_{\ell-n}^{\phi^n}$$

becomes

$$\begin{aligned} ww_1 &= u_1 w^\phi + u_2 (ww_{n-1})^{\phi^2} + \cdots + u_{n-1}(ww_2)^{\phi^{n-1}} + (ww_1)^{\phi^n} \\ ww_2 &= \frac{1}{p} u_1 (ww_1)^\phi + u_2 w^{\phi^2} + \cdots + u_{n-1}(ww_3)^{\phi^{n-1}} + (ww_2)^{\phi^n} \\ &\vdots \\ ww_{n-1} &= \frac{1}{p} u_1 (ww_{n-2})^\phi + \cdots + \frac{1}{p} u_{n-1}(ww_1)^{\phi^{n-1}} + w^{\phi^{n-1}} + (ww_{n-1})^{\phi^n} \\ w &= \frac{u_1}{p}(ww_1)^\phi + \cdots + \frac{u_{n-1}}{p}(ww_{n-1})^{\phi^{n-1}} + w^{\phi^n} \end{aligned} \tag{3.1.2}$$

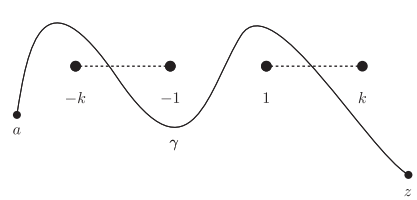
We will express this as a matrix equation in §3.4.

3.2. Periods

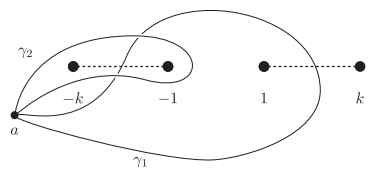
The classical story of Abelian integrals and their periods arises when one considers the complex analytic function

$$\int_a^z \frac{dx}{\sqrt{(x^2-1)(x^2-k^2)}}. \tag{3.2.1}$$

In order to even talk about the integral one needs to choose branch cuts between the branch points so one can be careful about which branch of the square root one is using. The branch points are at $x = \pm 1$ and $x = \pm k$. In the picture below I've chosen some branch cuts, and drawn a typical curve



Now the value of the integral is almost independent of the choice of curve. The only ambiguity comes from adding the values of the integrals around γ_1 and γ_2 below



Let's write

$$\eta_1 = \oint_{\gamma_1} \frac{dx}{\sqrt{(x^2-1)(x^2-k^2)}}$$
$$\eta_2 = \oint_{\gamma_2} \frac{dx}{\sqrt{(x^2-1)(x^2-k^2)}}.$$

It turns out that η_1 and η_2 are linearly independent over $\mathbb{R}$ and generate a lattice $\Lambda = \Lambda_k \subset \mathbb{C}$. The function (3.2.1) thus determines two pieces of data: the lattice $\Lambda_k \subset \mathbb{C}$ and what turns out to be an analytic isomorphism

$$X \to \mathbb{C}/\Lambda$$

from (the desingularizatin of) the plane curve X defined by the equation

$$y^2 = (x^2-1)(x^2-k^2) \tag{3.2.2}$$

to a complex torus.

The generalization of this to higher genus curves is as follow. For a smooth compact analytic curve X of genus g on finds a basis

$$\{\omega_1, \ldots, \omega_g\}$$

for the space of holomorphic differentials and a basis

$$\gamma_1, \ldots, \gamma_{2g} \subset H_1(X; \mathbb{Z})$$

for the first homology group with integer coefficients. The vectors

$$\int_{\gamma_i} \omega_j \in \mathbb{C}^g$$

generate a lattice $\Lambda \subset \mathbb{C}^g$ and the set of integrals

$$\int_a^z \omega_j \qquad j = 1, \cdots, g$$

an embedding $X \to \mathbb{C}^g/\Lambda$. This is the *Abel-Jacobi map*, the complex torus $\mathbb{C}^g/\Lambda$ is the *Jacobian variety* of X, and the integrals used to define it are called *Abelian integrals*. Suitably formulated the association $X \mapsto \Lambda$ gives an emedding of the moduli space of curves into and appropriate moduli space of lattices. This is the famous *Torelli Theorem*. The *Shottky problem* is the problem of characterizing the image.

The remarkable thing about this is that much of the story can be made to work for formal groups! We have to set things up a bit differently to make it work. Once we do, the display of identities at the end of §3.1 will have an interpretation in terms of periods.

The key is to look at the Hodge sequence

$$0 \to H^0(X;\Omega^1) \to H^1_{\mathrm{DR}}(X;\mathbb{C}) \to H^1(X;\Omega^0) \to 0.$$

The vector space $H^0(X/;\Omega^1)$ is the space of holomorphic differentials, while, by de Rham's Theorem (and the universal coefficient theorem) the de Rham cohomology group in the middle is isomorphic to

$$\hom(H_1(X/\mathbb{Z}),\mathbb{C}).$$

The left map is the map sending a differential ω to the complex "period"

$$\gamma \mapsto \int_\gamma \omega.$$

With point of view the Jacobian variety is

$$H^0(\Omega^1 X)^*/H_1(X;\mathbb{Z}).$$

At this point this just looks like a structureless embedding of a complex vector space of dimension g into on of dimension $2g$. But the two groups have very different characters. To see this look at the integral (3.2.1), and think k as a parameter specifying the "modulus" of the curve X. The form $\omega = \frac{dx}{\sqrt{(x^2-1)(x^2-k^2)}}$ varies with k, however the integer cohomology (and hence the de Rham cohomology) depends only on the topological space underlying X and not on its complex structure. So it is, in some sense, rigid.

Now it's kind of surprising but one can account for the rigidity of de Rham cohomology in purely "algebraic" terms. Let's go back to our motivating example of the curve defined by (3.2.2), but now think of k as a parameter. To emphasize that I'll write X_k for the specific curve. Then what we really have is a map

$$p : X \to K$$

where X is the totality of all the curves X_k and K is the parameter space of values of k (in our example $K = \mathbb{C} \setminus \{\pm 1\}$). The fiber of p over a point $k \in K$ is our curve X_k. Now X is something 2-dimensional and so it's de Rham complex in degree 1 will have things that look like

$$f(x,y,k)\,dx + g(x,y,k)\,dy + h(x,y,k)\,dk,$$

and from (3.2.2) one has

$$2y\,dy = 2x\big((x^2-k^2)+(x^2-1)\big)\,dx - 2k\,dk.$$

When we think about the de Rham cohomology of an *individual* X_k we are ignoring the terms $h(x,y,k)\,dk$. More specifically, we are modding them out. This situation was analyzed in a beautiful paper of Katz and Oda [**13**]. Here is how it goes.

Let's write $\Omega^* X$ for the de Rham complex of X and $\Omega^*(K)$ for the de Rham complex of K. Then the pullback of forms gives a map of differential graded algebras

$$\Omega^*(K) \to \Omega^*(X).$$

Let

$$I = \bigoplus_{n\geq 1} \Omega^n K \subset \Omega^*(K)$$

be the (differential graded) ideal of forms of positive degree. Then we have

$$\Omega^*(K)/I = \Omega^0(K)$$

in which the latter is regarded as a differential graded algebra with $d = 0$ and concentrated entirely in degree 0. The relative de Rham complex $\Omega^*(X/K)$is the (derived) quotient

$$\Omega^*(X/K) = \Omega^*(X)/I = \Omega^*(X) \underset{\Omega^*(K)}{\otimes} \Omega^0 K.$$

It is the cohomology of $\Omega^*(X/K)$ that is the aggregate of the individual de Rham cohomology groups of the X_k.

Now this setup is part of a spectral sequence. It is the Leray spectral sequence of the map p for de Rham cohomology, and you can think of it as constructed by filtering $\Omega^* K$ by powers of the ideal I. The E_1-term of the spectral sequence is

$$H^s_{\mathrm{DR}}((X/K)) \otimes \Omega^t(K)$$

and the above term contributes to $H^{s+t}_{\mathrm{DR}}(X)$. The first differential is rather interesting. It gives a map

$$H^s_{\mathrm{DR}}(X/K) \to H^s_{\mathrm{DR}}(X/K) \otimes \Omega^1(K).$$

This piece of structure is a connection on $H^*_{\mathrm{DR}}(X/K)$, and the fact that $d^2 = 0$ shows that the connection is flat. This is the *Gauss-Manin* connection and was analyzed in this form by Katz and Oda.

Remark 3.2.3. In fact Katz goes further ([**12**, p. 186]) and analyzes the next differential, showing that it is given by "cup product with the Kodaira-Spencer class." This is the piece of structure mentioned in§1.6. Let's at least check that everything lives in the right place. First of all where do the differentials go? Like the Serre spectral sequence for a fibration, the differential d_r goes from sub quotient of $H^s_{\mathrm{DR}}(X/K) \otimes \Omega^t(K)$ to a sub quotient of $H^{s-r+1}_{\mathrm{DR}}(X/K) \otimes \Omega^{t+r}(K)$, so d_2 maps a subgroup of $H^s_{\mathrm{DR}}(X/K) \otimes \Omega^t(K)$ to a quotient of $H^{s-1}_{\mathrm{DR}}(X) \otimes \Omega^{t+2}(K)$. As described in §1.6, the Kodaira-Spencer map is a map from $T_k K$ to $H^1(X_k; TX_k)$. Put differently it is an element of

$$\Omega^1(K; H^1(X/K; TX/K)). \tag{3.2.4}$$

Now an element of $H^s_{\mathrm{DR}}(K; H^t_{\mathrm{DR}}(X/K))$ is represented by a element of

$$\Omega^s(K) \otimes \Omega^t(X/K).$$

Multiplying these by (3.2.4) and contracting the tangent vectors against the forms give an element of $\Omega^{s+1}(K) \otimes \Omega^{t-1}(X/K)$ which is what we wanted.

3.3. Crystals

3.3.1. Formal Lie Varieties. The takeaway from the previous section is that we probably ought to be looking at de Rham cohomology. We now turn to doing so. Much of this section follows the presentation of Katz [**11**].

3.3.1.1. *De Rham cohomology.* Let S be a ring. Following Katz [**11**] we define the category $\mathbf{FormalLie}_S$ of *pointed formal Lie varieties over* S to be the category with objects $\{\mathbb{A}^n = \mathbb{A}^n_S \mid n = 0, 1, \dots\}$ in which a map $\mathbb{A}^n_S \to \mathbb{A}^1_S$ is a formal power series

$$f(x_1, \dots, x_n) \in S[\![x_1, \dots, x_n]\!]$$

satisfying $f(0, \dots, 0) = 0$. The maps x_i have the property that

$$\mathbb{A}^n_S \xrightarrow{(x_1,\dots,x_n)} \left(\mathbb{A}^1_S\right)^n$$

is an isomorphism, and the composition of maps is given by composition of power series. Thus a map of formal Lie varieties $\mathbb{A}^n_S \to \mathbb{A}^m_S$ is a column vector

$$\big[f_1, \dots, f_m\big]^T$$

of functions $f_i(x_1, \dots, x_n)$.

The category of formal Lie varieties over S is functorial in S. Given a map $S \to T$ there is a functor from $\mathbf{FormalLie}_S \to \mathbf{FormalLie}_T$. To ease the burden on the typist I will often write $\mathbb{A}^n$ instead of $\mathbb{A}^n_S$ when the ground ring S is understood.

Remark 3.3.1. In this language a formal group law of dimension n is a group structure on $\mathbb{A}^n$.

Remark 3.3.2. The category $\mathbf{FormalLie}_S$ is the opposite of the category with objects the augmented formal power series rings

$$\epsilon : S[\![x_1, \dots, x_n]\!] \to S$$
$$\epsilon(x_i) = 0$$

and ring homomorphisms of augmented rings.

3.3.1.2. *Differential forms and the De Rham complex.* Just as in differential topology one can talk about k-forms on $\mathbb{A}^n_S$. The space $\Omega^k(\mathbb{A}_S/S)$ of (relative) of k-forms on $\mathbb{A}^n_S$ is a free module over the ring of formal functions on $\mathbb{A}^n_S$ with basis the set of

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

in which $I = (i_1 < \cdots < i_k)$ is running through the k-element subsets of $\{1, \dots, n\}$. The usual formulas for exterior differentiation make sense and one can construct the de Rham complex

$$\Omega^*(\mathbb{A}^n_S/S) = \Omega^0(\mathbb{A}^n_S/S) \xrightarrow{d} \Omega^1(\mathbb{A}^n_S/S) \to \cdots \xrightarrow{d} \Omega^n(\mathbb{A}^n_S/S).$$

The ith cohomology of the de Rham complex will be denoted $H^i_{\mathrm{DR}}(\mathbb{A}^n_S/S)$.

Exercise 3.3.1. Show that $H^0_{\mathrm{DR}}(\mathbb{A}^1_S/S) = S$. Show that if S is the ring $\mathbb{Z}_p$ of p-adic numbers, then

$$H^1_{\mathrm{DR}}(\mathbb{A}^1_S/S) = \prod_{(n,p)=1} \prod_{k \geq 1} \mathbb{Z}/p^k \tag{3.3.3}$$

in which the factor $\mathbb{Z}/p^k$ with index n is the form

$$d\big(x^{np^k}/p^k\big) = x^{np^k - 1}\, dx.$$

When S is a $\mathbb{Q}$-algebra the proof of the usual Poincaré Lemma applies and one has

Proposition 3.3.4. *When S is a $\mathbb{Q}$-algebra then for all $i > 0$,*

$$H^i_{DR}(\mathbb{A}^n_S/S) = 0.$$

3.3.1.3. *Functorial properties of De Rham cohomology.* The de Rham cohomology has surprising functorial properties.

Definition 3.3.5. Suppose that S is a torsion free ring, which we will regard as a subring of $S \otimes \mathbb{Q}$. A *divided power ideal* is ideal $I \subset S$ with the property that if $x \in I$ then for all n, the element $x^n/n!$ is in I.

Example 3.3.6. The ideal (p) in $\mathbb{Z}_p$ has divided powers.

Remark 3.3.7. When $I \subset S$ is an ideal which is not torsion free (over $\mathbb{Z}$), one has to talk about a *divided power structure* on I. This means a sequence of functions $\gamma^n : I \to I$, often just written as $x \mapsto x^{(n)}$ satisfying the formal algebraic properties of $x^n/n!$. The advantage of sticking with the torsion free case is that the existence of divided powers is a condition and not data.

Exercise 3.3.2. Suppose that S is a torsion free ring and $I \subset S$ is an ideal with divided powers. Show that if $f, g : \mathbb{A}^n_S \to \mathbb{A}^m_S$ are two maps satisfying $f \equiv g \mod I$ then for all $i \geq 0$

$$f^* = g^* : H^i_{\mathrm{DR}}(\mathbb{A}^m_S) \to H^i_{\mathrm{DR}}(\mathbb{A}^m_S).$$

Here $f$and g are column vectors of formal power series over S in n variables and $f \equiv g \mod I$ means that all of the the coefficients of these power series are the same modulo I.

3.3.1.4. *De Rham cohomology with coefficients.* The results of the previous section apply to what one might think of as de Rham cohomology with coefficients. We're not going to use them in that form, but it's a convenient way to introduce modules with connection, which will play an important role.

Let's suppose that V is a formal Lie variety of dimension n over S. In our main example, V will not be a formal group, but will be Lubin-Tate space itself. Suppose we have a vector bundle M over V of rank k. Since the ring

$$\mathbf{FormalLie}_S(V, \mathbb{A}^1) = S[\![x_1, \ldots, x_n]\!]$$

of functions on V is a local ring the $\mathbf{FormalLie}_S(V, \mathbb{A}^1)$-module of sections of M is free of rank k. We want to add structure to M that allows us to form the de Rham complex

$$\Omega^0(V) \otimes M \xrightarrow{d} \Omega^1(V) \otimes M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(V) \otimes M \tag{3.3.8}$$

and be able to talk about $H^*_{\mathrm{DR}}(V; M)$. The first thing we need is a map

$$M \xrightarrow{d} \Omega^1(V) \otimes M$$

which, written terms of a section m of M is

$$m \mapsto \sum D_i(m)\, dx_i.$$

The operator D_i has the property that

$$D_i(fm) = \frac{\partial f}{\partial x_i} + f D_i(m) \qquad f : V \to \mathbb{A}^1.$$

The operators D_i endow M with a *connection*, and you can think of $D_i(m)$ as $\partial(m)/\partial x_i$.

What corresponds to the condition that $d^2 = 0$? In terms of the operators D_i, the 2-form d^2m is given by

$$\sum_{i,j} D_i D_j(m)\, dx_i\, dx_j = \sum_{i<j} (D_i D_j(m) - D_j D_i(m))\, dx_i\, dx_j.$$

The condition that $d^2 = 0$ is thus equivalent to the condition that the operators D_i and D_j commute. A connection with these properties is called *integrable*. A section m of M is called *horizontal* if $D_i m = 0$ for all i and so $H^0_{\mathrm{DR}}(M)$ is the space of *horizontal sections* of M. The results of 3.3.1.3 apply to the cohomology groups

$$H^*_{\mathrm{DR}}(V; M)$$

of a formal Lie variety with coefficients in a vector bundle with an integrable connections.

Suppose that (M, D_i^M) and (N, D_i^N) are modules equipped with with integrable connections. A module map $T : M \to N$ is *horizontal* if it is compatible with the connection:

$$T^*(D_i^M(m)) = D_i^N(T(m)).$$

A horizontal map gives a map of de Rham cohomology with coefficients

$$H^*_{\mathrm{DR}}(V; M) \to H^*_{\mathrm{DR}}(V; N).$$

3.3.1.5. *De Rham cohmology of groups.* Let G be a formal group law of dimension n over S, thought of as a group structure on $\mathbb{A}^n_S$. Now here's an annoying thing about the notation. In the usual theory of groups, when a group is a set endowed with a composition law, a group is denoted G, the underlying set is also denoted G, and no special notation is used for the composition law. I'd like to follow this notation when we are working with formal groups. Thus a formal group G of dimension n will consist of a formal Lie variety (which abstractly isomorphic to $\mathbb{A}^n$) equipped with a group structure. I'd like to use the symbol G for this variety and just write the composition law as a map

$$\mu : G \times G \to G$$

and try not to mention μ all that much. A coordinate system on G is a choice of isomorphism $G \to \mathbb{A}^n$ and in terms of the coordinate system the map μ is given by an n-tuple of power series in variables $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{y} = (y_1, \ldots, y_n)$. The symbol G is also used to denote this power series:

$$G(\underline{x}, \underline{y}) = \begin{bmatrix} G_1(\underline{x}, \underline{y}) \\ \vdots \\ G_n(\underline{x}, \underline{y}) \end{bmatrix}.$$

Hopefully overloading the symbol G like this won't cause confusion.

Let's go back to the situation in which S is a torsion free ring and G is a formal group over S. In that case the composition law gives the de Rham cohomology of G an additional piece of structure, namely a map

$$\mu^* : H^i_{\mathrm{DR}}(G) \to H^i_{\mathrm{DR}}(G \times G).$$

Definition 3.3.9. An element $\omega \in H^i_{\mathrm{DR}}(G)$ is *primitive* if

$$\mu^*(\omega) = \mathrm{pr}_1^*\, \omega + \mathrm{pr}_2^*\, \omega.$$

The group of primitive elements of $H^i_{\mathrm{DR}}(G)$ will be denoted $\mathbf{D}^i(G)$.

Let's do a few examples to get the swing of this. We will focus on $\mathbf{D}^1(G)$. We will also restrict our attention to the case in which S is $\mathbb{Z}$-torsion free and so may be regarded as a subring of $S \otimes \mathbb{Q}$. In that case, since every closed one form $\omega \in \Omega^1(G_S)$ is exact over $S \otimes \mathbb{Q}$, we may identify $H^1_{\mathrm{DR}}(G)$ with the quotient

$$\{f \in \Omega^0(G_{S\otimes\mathbb{Q}}) \mid df \in \Omega^1(G_S)\}/\Omega^0(G_S).$$

From this point of view the elements of $\mathbf{D}^1(G)$ are the elements f above satisfying the additional condition that

$$f(\underline{x} \underset{G}{+} \underline{y}) - f(x) - f(y) \in \Omega^0((G \underset{S}{\times} G)). \tag{3.3.10}$$

Now for some examples. We will begin with the case in which G is the one dimensional additive formal group over $\mathbb{Z}_p$. As we saw above $H^1_{\mathrm{DR}}(G)$ is the product of cyclic groups generated by

$$d\big(\frac{x^{np^k}}{p^k}\big).$$

When is such an element primitive? First note that

$$(x+y)^{np^k} - x^{np^k} - y^{np^k}$$

is not even divisible by p unless $n = 1$. In that case it is divisible by p but not by p^2. These observations easily imply that

$$\mathbf{D}^1(G) = \prod_{k\geq 0} \mathbb{Z}/p$$

in which the factor with index k is represented by the function x^{p^k}/p.

Before turning to other formal groups let's make a general observation. Suppose that

$$f(x) = \sum a_n x^n \in \mathbb{Q}[\![x]\!]$$

represents an element of $\mathbf{D}^1(G)$. If it happens that some coefficients $\{a_i \mid i \in I\}$ are in S then $f(x) - \sum_{i\in I} a_i x^i$ will also be quasi-primitive and will represent the same element of $\mathbf{D}^1(G)$ as f. We may therefore suppose that the first non-zero term of f is of the form

$$a_k x^k$$

with $0 \neq a_k \in S \otimes \mathbb{Q}/\mathbb{Z}$. Next note that the fact that

$$x \underset{G}{+} y = x + y + \cdots$$

implies that

$$a_k((x+y)^k - x^k - y^k) \in S.$$

As we saw above, this means that k must be a power of p and $pa_k \in S$.

Now let's turn to the case of the Lubin-Tate formal group of height n with logarithm

$$\ell(x) = \sum \frac{x^{p^{nk}}}{p^k} = x + \frac{x^{p^n}}{p} + \cdots .$$

Because

$$\ell(x \underset{G}{+} y) = \ell(x) + \ell(y)$$

this element is actually primitive. This can be used to cancel a term $\frac{x^{p^n}}{p}$. We can get a few more this way. We have

$$p^{m-1}\ell(x) = p^{m-1}x + p^{m-2}x^p + \cdots + x^{p^{(m-1)k}} + \frac{1}{p}x^{p^{(m)k}} + \cdots$$

so we can also cancel leading terms like $x^{p^{mk}}/p$ as well.

How about x^p/p? Well here there is something interesting. I claim that

$$\frac{1}{p}\ell(x^p)$$

is also primitive. Why? This is a little easier to explain in a more general setting.

Let's start with a torsion free p-local ring S and a formal group law Γ over S/pS given y If

$$x \underset{\Gamma}{+} y = \sum a_{ij}x^i y^j$$

We can construct a new formal group law $\phi^*\Gamma$ with

$$x \underset{\phi^*\Gamma}{+} y = \sum \phi(a_{ij})x^i y^j.$$

The fact that

$$\big(\sum a_{ij}x^i y^j\big)^p = \sum a_{ij}(x^p)^i(y^p)^j$$

means that the we have a map

$$\begin{aligned} \Gamma &\to \phi^*\Gamma \\ x &\mapsto x^p. \end{aligned}$$

This map is the *Frobenius isogeny.*

Now suppose that S has a map $\phi : S \to S$ which is a lift of Frobenius, in the sense that

$$\phi(x) \equiv x^p \mod pS.$$

Given a power series

$$f(x) = \sum b_i x^i \in (S \otimes \mathbb{Q})[\![x]\!]$$

let

$$f^\phi(x) = \sum \phi(b_i)x^i$$

I claim that if $f(x)$ is the log of a formal group G over S then for any k, the series

$$\frac{1}{p}f^{\phi^k}(x^{p^k})$$

satisfies (3.3.10).

Before turning to the proof I need another general fact about the log of a formal group I haven't had a chance to mention. Suppose $L(x)$ is the log of a formal group G over S. Then by Taylor's theorem

$$L(x+py) = L(x) + \sum_{n>0} L^{(n)}(x)\frac{(py)^n}{n!}.$$

However every term on the right is in pS. We already saw that the derivative of the log has coefficients in S, and you checked above that (p) has divided powers, so $\frac{p^n}{n!}$ is divisible by p.

Returning to G, note that by by naturality, f^{ϕ^k} is a log of G^{ϕ^k}. The fact that

$$(x \underset{G}{+} y)^{p^k} \equiv x^{p^k} \underset{G^{\phi^k}}{+} y^{p^k} \mod pS$$

means that

$$\begin{aligned} f^{\phi^k}(x^{p^k}) + f^{\phi^k}(y^{p^k}) &= f^{\phi^k}(x^{p^i} \underset{G^{\phi^k}}{+} y^{p^k}) \\ &= f^{\phi^k}(x^{p^k} + y^{p^k} + p\epsilon) \\ &= f^{\phi^k}(x^{p^k}) + f^{\phi^k}(y^{p^k}) \mod pS[\![x]\!]. \end{aligned}$$

This implies that $\frac{1}{p} f^{\phi^k}(x^{p^k})$ also satisfies (3.3.10).

Putting this all together, this shows that when G is the Lubin-Tate group with logarithm

$$\ell(x) = \sum x^{p^{nk}}/p^k$$

then $\mathbf{D}^1(G)$ is the free $\mathbb{Z}_p$-module of rank n with basis

$$\{\ell(x), \frac{1}{p}\ell^{\phi}(x^p), \ldots, \frac{1}{p}\ell^{\phi^{n-1}}(x^{p^{n-1}})\}. \tag{3.3.11}$$

3.3.1.6. *Change of base.* Note that the de Rham cohomology $H^*_{\mathrm{DR}}(\mathbb{A}^n_S/S)$ is functorial in S. Given a ring homomorphism $S \to T$ there is a map

$$T \underset{S}{\otimes} H^*_{\mathrm{DR}}(\mathbb{A}^n_S/S) \to H^*_{\mathrm{DR}}(\mathbb{A}^n_T/T).$$

This map is not an isomorphism, but it fails to be for a particular reason. Let's look at the case $n = 1$ Then describe above. we have

$$H^1_{\mathrm{DR}}(\mathbb{A}^1_S/S) = \prod_{(n,p)=1} \prod_{k \geq 1} S/(p^k S).$$

The trouble then is that the tensor product does not, in general, commute with finite products. It does however either T is a finitely presented S module, or if we can avail ourselves of some mechanism of picking out a natural fintely generated sub module of H^{ast}_{DR}. Now when G has finite height, the module $\mathbf{D}(G)$ is exactly such a submodule. This implies the following

Proposition 3.3.12. *Suppose that $f : S \to T$ is a ring homomorphism. If G is a formal group of finite height (and finite dimension) over S then the map*

$$T \underset{S}{\otimes} \mathbf{D}(G/S) \to \mathbf{D}(f^*G/T)$$

is an isomorphism. □

3.3.1.7. *Computing Frobenius.* Suppose that S is a torsion free ring and G_0 is a formal group law over S/p, and that G and G' are lifts of G_0 to S. write Γ for the reduction of G to S/pS. Since

$$x \underset{G}{+} y \equiv x \underset{G'}{+} y \mod (p)$$

and since (p) has divided powers the two group laws

$$(\mathbb{A}^1 \times \mathbb{A}^1)_S \to A^1_S$$

induce the *same* homomorphism on $H^1_{\mathrm{DR}}(\mathbb{A}^1)$. This means that $\mathbf{D}^1(G)$ is *equal* to $\mathbf{D}^1(G')$ when regarded as subsets of $H^1_{\mathrm{DR}}(\mathbb{A}^1_S)$.

In fact it's even better than that. The induced map in de Rham cohomolgy

$$H^1_{\mathrm{DR}}(G) \to H^1_{\mathrm{DR}}(G \times G)$$

can be computed using *any* power series

$$\sum \tilde{a}_{ij} x^i y^j$$

whose reduction mod p is $x \underset{\Gamma}{+} y$. It doesn't actually have to define a group structure on $\mathbb{A}^1$.

All of this means that the object $\mathbf{D}^1(G_S)$ is actually a functor of G_0 and in particular, maps of formal groups over $S/(p)$ give maps of the modules $\mathbf{D}^1(-)$.

Here is an important example. Consider the Frobenius isogeny

$$\begin{aligned} F : G_0 &\to \phi^* G_0 \\ x &\mapsto x^p. \end{aligned}$$

What is its effect on $\mathbf{D}^1(-)$? Let G be a lift of G_0 and consider the map $x \mapsto x^p$. This does not quite define a homomorphism $G \to \phi^* G$ but since it lifts a map which is a homomomorphism mod p, it can be used to compute the map

$$\phi^* \mathbf{D}^1(G) = \mathbf{D}^1(\phi^{*G}) \to \mathbf{D}^1(\Gamma).$$

This means that F induces the map

$$\ell^{\phi^k}(x) \mapsto \ell^{\phi^{k+1}}(x^p).$$

Consider the case in which G is the Lubin-Tate group with log

$$\ell(x) = \sum \frac{x^{p^{nk}}}{p^k}.$$

In terms of the basis (3.3.11), the map F has the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ p & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Now let's work out the matrix of F in case of the universal deformation. We take $S = \mathbb{W}[\![u_1, \dots u_{n-1}]\!]$, $\phi : S \to S$ given by Frobenius on $\mathbb{W}$ and $\phi(u_i) = u_i^p$, and G the formal group law whose log $f(x)$ is defined by the functional equation

$$f(x) = x + \frac{u_1}{p} f^{\phi}(x^p) + \cdots + \frac{u_{n-1}}{p} f^{\phi^{n-1}}(x^{p^{n-1}}) + \frac{1}{p} f^{\phi^n}(x^{p^n}).$$

Then, as above, a basis of $\mathbf{D}^1(\Gamma)$ is

$$\{f(x), \frac{1}{p} f^{\phi}(x^p), \cdots \frac{1}{p} f^{\phi^{n-1}}(x^{p^{n-1}})\},$$

and from the functional equation for f, the map F is given by

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ p & 0 & \cdots & 0 & -u_1 \\ 0 & 1 & \cdots & 0 & -u_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -u_{n-1} \end{pmatrix} \tag{3.3.13}$$

3.3.2. Crystals. To go further it is useful to describe in more abstract terms the nature of the object $\mathbf{D}^1(G)$ that we constructed.

Suppose that S is a ring. Let $\mathbf{Crys}(S)$ be the category of triples $T = (\pi, i, \delta)$ consisting of a diagram of ring homomorphisms

$$\begin{tikzcd} & T \arrow[d, "\pi"] \\ S \arrow[r, "i"'] & U \end{tikzcd} \tag{3.3.14}$$

in which $S \to U$ is a Zariski localization, $T \to U$ is surjective and δ is a divided power structure on the kernel I of π. When I is $\mathbb{Z}$-torsion free the structure δ, if it exists, is unique.

A map in $\mathbf{Crys}(S)$ is map of the above data

$$\begin{tikzcd} & T \arrow[r] \arrow[d, "\pi"] & T' \arrow[d, "\pi'"] \\ S \arrow[r] & U \arrow[r] & U' \end{tikzcd} \tag{3.3.15}$$

in which the map $T \to T'$ is compatible with the divided power structures.

A *crystal* on R consists of a T-module M_T for each diagram (3.3.14) and for each map (3.3.15) in $\mathbf{Crys}(S)$ an isomorphism

$$T' \underset{T}{\otimes} M_T \to M_{T'}$$

satsfying the cocycle condition that if

$$\begin{tikzcd} & T \arrow[r] \arrow[d, "\pi"] & T' \arrow[r] \arrow[d, "\pi'"] & T'' \arrow[d, "\pi''"] \\ S \arrow[r] & U \arrow[r] & U' \arrow[r] & U'' \end{tikzcd} \tag{3.3.16}$$

is a composition in $\mathbf{Crys}(S)$ then the diagram

$$\begin{tikzcd} T'' \underset{T'}{\otimes} T' \underset{T'}{\otimes} M_T \arrow[r] \arrow[d] & T'' \underset{T'}{\otimes} M_{T'} \arrow[d] \\ T'' \underset{T}{\otimes} M_T \arrow[r] & M_{T''} \end{tikzcd} \tag{3.3.17}$$

commutes.

Example 3.3.18. Suppose that S_0 is complete local $\mathbb{F}_p$-algebra and G_0 is deformation to S_0 of a formal group law of finite height. Given a diagram (3.3.14) choose any lift G of i^*G_0 to T and define $M_T = M_T^\Gamma = \mathbf{D}^1(G/T)$. The conditions on G_0 guarantee that $\mathbf{D}^1(G)$ is a finite free T-module and so by all of above discussion this defines a crystal. This is the (contravariant) *Dieudonné crystal* and will be denoted $\mathcal{M}(G_0)$. Thus

$$\mathcal{M}(G_0)_T = \mathbf{D}^1(G)$$

where G is any lift of i^*G to T.

For the crystalline period mapping we will only require the information summarized by Example 3.3.18. But to connect it with the classical period mapping requires another point of view on crystals. Since the language used in describing this situation is derived from the classical period mapping I'll describe it. You can safely skip this section if you wish.

In the situation we are interested in, S is the mod p reduction $k[\![u_1, \ldots, u_{n-1}]\!]$ of the Lubin-Tate ring. Since the ideal

$$(p) \subset \mathbb{W}[\![u_1, \ldots, u_{n-1}]\!] = E_0$$

has divided powers, a crystal M on S provides an E_0-module M. Given any solid arrow diagram

$$\begin{tikzcd} \mathbb{W}[\![u_1, \ldots, u_{n-1}]\!] \arrow[r, dotted, "f"] \arrow[d] & T \arrow[d] \\ S \arrow[r] & U \end{tikzcd}$$

a dotted arrow f exists which is compatible with the divided power structure on the kernel of $T \to U$. This means that M_T is determine by M as

$$M_T = T \underset{E_0}{\otimes} M.$$

So a crystal on S is just a module over E_0 equipped with some extra structure.

What is this extra structure? Look at the ring

$$T = E_0\langle\langle h_1, \ldots, h_{n-1}\rangle\rangle \tag{3.3.19}$$

of formal divided power series in variables h_i. There are two maps

$$g_1, g_2 : E_0 \to E_0\langle\langle h_1, \ldots, h_{n-1}\rangle\rangle$$

given by

$$\begin{aligned} g_1(u_i) &= u_i \\ g_2(u_i) &= u_i + h_i. \end{aligned}$$

By definition, if M is to be a crystal, then it must come equipped with an isomorphism

$$\tau : g_1^* M \to g_2^* M.$$

The map τ can be thought of as a map

$$\tau : M \underset{\mathbb{W}[\![\vec{u}]\!]}{\otimes} \mathbb{W}[\![\vec{u}]\!]\langle\langle \vec{h}\rangle\rangle$$

(in which $\vec{u} = (u_1, \ldots, u_{n-1})$, etc) with the property that

$$\tau(f(\vec{u})m) = f(\vec{u} + \vec{h})m. \tag{3.3.20}$$

Write

$$\tau(m) = m + \sum D_i(m)h_i + \cdots .$$

Then (3.3.20) gives

$$D_i(fm) = fD_i(m) + (\partial f/\partial u_i)\, m$$

so that M becomes equipped with a connection.

The operation $\tau = \tau_{\vec{u}}^{\vec{h}}$ satisfies a cocycle condition over

$$\mathbb{W}[\![u_1, \ldots, u_{n-1}]\!]\langle\langle h_1, \cdots, h_{n-1}, h_1', \cdots, h_{n-1}'\rangle\rangle$$

which, in what is hopefully evident notation, is the condition that the diagram

$$\begin{tikzcd} M_{\vec{u}} \arrow[r, "\tau^{\vec{u}}_{\vec{h}}"] \arrow[dr, "\tau^{\vec{u}}_{\vec{h}+\vec{h}'}"'] & M_{\vec{u}+\vec{h}} \arrow[d, "\tau^{\vec{u}+\vec{h}}_{\vec{h}'}"] \\ & M_{\vec{u}+\vec{h}+\vec{h}'} \end{tikzcd}$$

commutes. In terms of the expansion (3.3.20) this implies that the operators D_i commute and so define an integrable connection. In fact the operators D_i determine the operator τ (this is called *integrating the connection form*). In this way one gets an equivalence of categories between crystals on $k[\![u_1, \ldots, u_{n-1}]\!]$ and the category of modules over $\mathbb{W}[\![u_1, \ldots, u_{n-1}]\!]$ equipped with an integrable connection.

3.4. The crystalline period map

Now let's put this all together. Γ be a 1-dimensional formal group of height n over the algebraic closure k of $\mathbb{F}_p$, G the universal deformation of Γ over $E_0 = \mathbb{W}[\![u_1, \ldots, u_{n-1}]\!]$ and G_0 the pullback of G to $k[\![u_1, \ldots, u_{n-1}]\!]$. Write $\mathcal{M}_\Gamma$ and $\mathcal{M}_{G_0}$ for the Dieudonné crystals. We have seen that the de Rham cohomology $\mathcal{M}_{G_0}(E_0) = \mathbf{D}^1(G)$ depends only on G_0. The lift G gives us some extra structure. Namely, a particular 1-dimensional subspace generated by

$$d\log_G(x).$$

This is subspace of primitive elements in $H^0(G; \Omega^1)$, and gives a line $H^{0,1} \subset \mathbf{D}^1(G)$. This is called the *Hodge line* in $\mathbf{D}^1(G)$. In this language, the crystalline Dieudonne theory tells us that $\mathbf{D}^1(G)$ determines G_0 while the Tapis de Cartier tells us that the lift G is determined by $\mathbf{D}^1(G)$, equipped with its Hodge structure, which in this case is the Hodge line

$$H^{0,1}(G) \subset \mathbf{D}^1(G).$$

Now lets pull everything back to $\mathbb{W}\langle\langle u_1, \ldots, u_{n-1}\rangle\rangle$. Over this ring the two maps

$$\begin{aligned} u_i &\mapsto u_i \\ u_i &\mapsto 0 \end{aligned}$$

agree modulo the maximal ideal of $\mathbb{W}\langle\langle u_1, \ldots, u_{n-1}\rangle\rangle$, which has divided powers. This means that we have a $\mathbb{W}$-linear map

$$B : M \to \mathbf{D}^1(G)$$

which extends to an isomorphism of E_0-modules

$$M \otimes E_0 \to \mathbf{D}^1(G).$$

So here is the picture, over $\mathbb{W}\langle\langle u_1, \ldots, u_{n-1}\rangle\rangle$

$$\begin{tikzcd} & M \arrow[d, "B"] \\ H^{0,1}(G) \arrow[r] & \mathbf{D}^1(G), \end{tikzcd}$$

analogous to the diagram

$$\begin{array}{ccc} & & H^1(X;\mathbb{Z}) \\ & & \downarrow \\ H^0(X;\Omega^1) & \longrightarrow & H^1_{\mathrm{DR}}(X). \end{array}$$

Using this we have an abstract period map from the subspace of Lubin-Tate space defined by $\mathbb{W}\langle\langle u_1,\ldots,u_{n-1}\rangle\rangle$ to the projective space $P(M)$. Everything is functorial in Γ so this map is $\mathrm{Aut}(\Gamma)$ equivariant.

How to compute this map? Here there is an amazing thing: the map B is determined by the fact that it is compatible with Frobenius. Let's see how. Let A be the matrix of Frobenius

$$F : \phi^*\mathbf{D}^1(G) \to \mathbf{D}^1(G)$$

computed in §3.3.1.7 and A_0 the one for M. Then specifically

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ p & 0 & \cdots & 0 & -u_1 \\ 0 & 1 & \cdots & 0 & -u_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -u_{n-1} \end{pmatrix}$$

and

$$A_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ p & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Then the compatibilty of B with Frobenius is a commutative diagram

$$\begin{array}{ccc} \phi^*M & \xrightarrow{A_0} & M \\ {\scriptstyle \phi^*B}\downarrow & & \downarrow{\scriptstyle B} \\ \phi^*\mathbf{D}^1(G) & \xrightarrow[A]{} & \mathbf{D}^1(G). \end{array}$$

In terms of matrices this is the identity

$$BA(0) = AB^\phi$$

or,

$$B = AB^\phi A(0)^{-1}.$$

Now just keep substituting this into itself

$$B = AA^\phi \cdots A^{\phi^n} B^{\phi^{n+1}} A(0)^{-(n+1)}$$

and take the limit as $n \to \infty$, noting that

$$\lim_{n\to\infty} B^{\phi^{n+1}}$$

is the identity matrix. This gives

$$B = \lim_{n\to\infty} AA^\phi \cdots A^{\phi^n} A(0)^{-(n+1)}.$$

In our setup it's actually the first column of B^{-1} that we want. It's not too hard to expand this all out. But for our purposes we don't really need to. All we need to know is that B is uniquely determined by A from the properties

$$B(0) = I_n$$
$$BA(0) = AB^{\phi}.$$

Since it's really B^{-1} we want I'll write these as

$$B^{-1}(0) = I_n$$
$$\left(B^{-1}\right)^{\phi} A(0) = AB^{-1}.$$

Now the matrix A is given by (3.3.13) and if you check it you'll see that the identites (3.1.2) are equivalent to the assertion that the matrix

$$\begin{pmatrix} w & \frac{1}{p}(ww_{n-1})^{\phi} & \frac{1}{p}(ww_{n-2})^{\phi^2} & \cdots & \frac{1}{p}(ww_1)^{\phi^{n-1}} \\ ww_1 & w^{\phi} & (ww_{n-1})^{\phi^2} & \cdots & (ww_2)^{\phi^{n-1}} \\ ww_2 & \frac{1}{p}(ww_1)^{\phi} & w^{\phi^2} & \cdots & (ww_3)^{\phi^{n-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ww_{n-1} & \frac{1}{p}(ww_{n-1})^{\phi} & \frac{1}{p}(ww_{n-3})^{\phi^2} & \cdots & w^{\phi^{n-1}} \end{pmatrix}$$

Satisfies the identites for B^{-1} described above. (in this matrix the $\frac{1}{p}$ factors disappear in a row as soon as you get to the w term). This means that the period mapping is given in homogeneous coordinates by

$$[w, ww_1, \ldots, ww_{n-1}].$$

LECTURE 4

The crystalline approximation

LECTURE 5

Projects

5.1. Project: finite subgroups

Let $D = D_n$ be the division algebra over $\mathbb{Q}_p$ and $\mathcal{O}_D$ the maximal compact sub algebra, so that the group of units $\mathcal{O}_d^\times$ is the automorphism group of the Lubin-Tate group. The purpose of this project is to work out as many of the finite subgroups of $D^\times$. This is something that appears in print (see [**7**]), but in a given instance there is often something more straightforward one can do. There are three facts that you will use.

Fact 5.1.1. The maximal commutative subalgebras of D are the field extension of $\mathbb{Q}_p$ of degree n. If $L/\mathbb{Q}_p$ is a field extension of degree n then there is an algebra embedding $L \hookrightarrow D$ and any two embeddings of L in D are conjugate.

Fact 5.1.2. Let $L \subset D$ be a maximal commutative subalgebra, and $N \subset D^\times$ the normalizer of L. The natural map $N/L^\times \to \mathrm{Gal}(L/\mathbb{Q}_p)$ is an isomorphism.

Fact 5.1.3. Write $G = \mathrm{Gal}(L/\mathbb{Q}_p)$. The group $H^2(G; L^\times)$ is cyclic of order n with generator the element corresponding to the extension

$$1 \to L^\times \to N \to G \to 1 \tag{5.1.4}$$

ocurring in the central simple division alebra with Hasse invariant $1/n$

Remark 5.1.5. One goes from an extension (5.1.4) to an algebra D by

$$D = \mathbb{Z}[N] \underset{\mathbb{Z}[L^\times]}{\otimes} L.$$

When the extension (5.1.4) is a generator of the group this is a division algebra (see [**25**]).

Exercise 5.1.1. Show that any element of finite order in $D^\times$ is actually in $\mathcal{O}_D^\times$.

Exercise 5.1.2. Show that any finite abelian subgroup of $D^\times$ is cyclic. Suppose that $D^\times$ contains an element of order k. Show that if $(k, p) = 1$ then k divides $p^n - 1$. Show that if $k = p^m$ then $(p-1)p^{m-1}$ divides n. Suppose if $D^\times$ contains an element of order ap^b with $(a, p) = 1$. Show that the maximal finite cyclic subgroups of $D^\times$ have order $(p^f - 1)p^m$ where $f(p-1)p^{m-1}$ divides n.

Now let's work out an example. Suppose that $p = 3$ and $n = 2$. We wish to find all of the finite subgroups of $D^\times$. By Exercise 5.1.2, the elements of finite order must have order dividing 8 ($f = 2$ in the previous exercise), or 6 ($f = 1$, $m = 1$ in the previous exercise). Suppose that $\omega \in D^\times$ has order 3. Then the field extension L of $\mathbb{Q}_3$ generated by ω is a degree 2 extension, and so the Galois group has order 2. It follows that there is a group extension

$$1 \to L^\times \to N \to \mathbb{Z}/2 \to 1.$$

The first order of business is to work out if restricts to an extension

$$1 \to \mathcal{O}_L^\times \to \tilde{N} \to \mathbb{Z}/2 \to 1.$$

This is a question about the long exact cohomology sequence

$$\begin{aligned} H^1(\mathbb{Z}/2; \mathcal{O}_L^\times) \to H^1(\mathbb{Z}/2; L^\times) \to H^1(\mathbb{Z}/2; \mathbb{Z}) \\ \to H^2(\mathbb{Z}/2; \mathcal{O}_L^\times) \to H^2(\mathbb{Z}/2; L^\times) \to H^2(\mathbb{Z}/2; \mathbb{Z}) \end{aligned}$$

Since L is ramified over $\mathbb{Q}_3$ there is no invariant uniformizer. This means that the map $H^2(\mathbb{Z}/2; L^\times) \to H^2(\mathbb{Z}/2; \mathbb{Z})$ is the zero map so the extension does lift.

Next note that the group of units of $\mathcal{O}_L$ has the structure

$$\mathcal{O}_L^\times \approx \mu_6 \times (1 + p\mathcal{O}_L)^\times.$$

Since $H^2(\mathbb{Z}/2; M)$ is trivial if 2 is invertible in M this means that this extension

$$1 \to \mu_6 \to G \to \mathbb{Z}/2 \to 1.$$

splits over $\mu_3 \times (1 + p\mathcal{O}_L)^\times$, and does not over μ_2 (since the cohomology class must generate H^2). This shows that at $p = 3$ the group $\mathbf{S}_2$ contains a finite subgroup isomorphic to the semidirect product

$$Z/3 \rtimes \mathbb{Z}/4.$$

You can check that this is a maximal finite subgroup, and that any two are conjugate.

Exercise 5.1.3. Show that the only p-groups in ℓ-adic division algebras are cyclic if $p > 2$, and either $\mathbb{Q}_8$ if $p = 2$ or cyclic. For which n do the $\mathbb{Q}_8$'s occur?

Exercise 5.1.4. Show that if $\mathcal{O}_D^\times$ contains an element of order p^k then n is divisible by $p^{k-1}(p-1)$.

Exercise 5.1.5. Show that for $p > 2$ the group $\mathbf{S}_{p-1}$ contains a maximal finite subgroup and that it has order $p(p-1)^2$. Show that any two subgroups of this order are conjugate. What happens with $\mathbf{S}_{p(p-1)}$?

5.2. Project: Action of finite subgroups

As described in an earlier lecture, it is proved in [**8**] that if $G \subset \operatorname{Aut}\Gamma$ is finite subgroup whose p-Sylow subgroup is cyclic, then there are deformation parameters for which the crystalline approximation is an isomorphism. This can be checked by hand at heights $(p-1)$ and at height 2 for $p = 2$ (where there is a twist). In this project you will explore some explicit ways of checking this result in these cases.

5.2.1. Height $(p-1)$.

Exercise 5.2.1. Prove this result directly for $p = 2$.

Now assume $p > 2$ and write $n = (p-1)$. In this case there is a maximal finite subgroup which can be written as $\mathbb{Z}/p \rtimes \mathbb{Z}/n^2$ (this is worked out in another one of the projects for this course.) The goal is to find a G-equivariant map

$$M \to E_{-2}$$

inducing the Cartier isomorphism

$$M/pM \approx E_{-2}/(p, \mathfrak{m}^2).$$

Since one can always average over groups of order prime to p you can reduce this to the problem of finding a $\mathbb{Z}/p$-equivariant map. Let $g \in G$ be an element of order p.

EXERCISE 5.2.2. Show that it suffices to find an invertible element $u \in E_{-2}$ with

$$u + gu + \cdots + g^{(p-1)}u = 0.$$

and $u \notin \mathfrak{m}E_{-2}$.

You will do this by finding an element $v \in E_{2(p-1)}$ with the analogous properties, and setting

$$u = v \cdot N(\tilde{u})$$

where $\tilde{u} \in E_{-2}$ is any invertible element, and

$$N(\tilde{u}) = \prod_{i=0}^{p-1} g^i(\tilde{u}).$$

Consider the power series $[p](x)$ representing multiplication by p on the universal deformation. By Lemma 1.2.1 one has

$$[p](x) = v_1 x^p + \cdots .$$

EXERCISE 5.2.3. Show that this formula defines an element $v_1 \in E_{2(p-1)} \otimes \mathbb{Z}/p$ and that this element is in fact invariant under $\operatorname{Aut}\Gamma$.

Now lift v_1 to an element $\tilde{v}_1 \in E_{2(p-1)}$ and define $v \in E_{2(p-1)}$ be defined by

$$pv = g(v_1) - v_1.$$

EXERCISE 5.2.4. Why is the right side divisible by p? Show that the element v satisfies

$$v + gv + \cdots + g^{(p-1)}v = 0.$$

It remains to show that v is not zero modulo the maximal ideal. Let $\mathcal{O} = \mathbb{Z}_p[\zeta_p]$ and $A = \mathbb{W} \underset{\mathbb{Z}_p}{\otimes} \mathcal{O}$, where ζ_p is a primitive p^{th} root of unity. Let $\Gamma_{\mathcal{O}}$ be the $\mathcal{O}$-module over A gotten by change of base of the Lubin-Tate formal $\mathcal{O}$-module along the map $\mathcal{O} \to A$. By construction, the group $G_{\mathcal{O}}$ is a deformation of Γ, and so classified by a map

$$E_0 \to A.$$

The group $G_{\mathcal{O}}$ has complex multiplicaton by ζ_p so the map $E_0 \to A$ is equivariant for the action of G (which acts trivially on A). You can prove that v is not in $\mathfrak{m}E_{2(p-1)}$ by mapping over to $A_{2(p-1)}$ and doing the computation there.

5.3. Project: Line bundles in height 2

This project also concerns material that is well represented in the literature. But it is a very informative exercise to explore. Also the results on cohomology from this point of view are derived in Kohlhaase [**17**]. Behrens [**3**] describes the computation of the Picard group. More details can be found in Beaudry et. al. [**2**] and in the thesis of Lader [**20**]. In this project you will work out the computation of the Picard group of the Lubin-Tate stack at $p \geq 5$ and height $n = 2$. More concretely you will prove the following theorem

THEOREM 5.3.1. *For $p \geq 5$, the restriction map*

$$H^1(\mathbf{S}_2; E_0^\times)^{\mathrm{Gal}} \to \hom(\mathbb{W}^\times, \mathbb{W}^\times)^{\mathrm{Gal}}$$

is a monomorphism, with image the subgroup topologically generated by the identity map and the norm.

Here Gal is the Galois group of k over $\mathbb{F}_p$. I'll leave it to you to pass to Galois invariants at the end.

We will use the crystalline approximation to the action of the $\mathbf{S}_2$ on the ring

$$E_* = \mathbb{W}[u^{\pm 1}][\![u_1]\!]$$

via the map

$$\begin{aligned} M &\to E_{-2}/(p, u_1^p) \\ \gamma &\mapsto u \\ V\gamma &\mapsto uu_1. \end{aligned}$$

Not quite following the conventions in [**5**]) write we will write an automorphism $g \in \mathbf{S}_2$ as

$$\begin{aligned} g(\gamma) &\mapsto a\gamma + bV\gamma \\ g(V\gamma) &\mapsto pb^{\phi^{-1}}\gamma + a^{\phi^{-1}}V\gamma \end{aligned}$$

and so corresponding to a matrix

$$g = \begin{pmatrix} a & pb^{\phi^{-1}} \\ b & a^{\phi^{-1}}. \end{pmatrix}$$

Note that since $n = 2$ one has $\phi^2 = 1$ and so you won't go wrong if you write ϕ^{-1}

The determinant is a map

$$\det : \mathbf{S}_2 \to \mathbb{Z}_p^\times \subset \mathbb{W}^\times.$$

There is an inclusion

$$\begin{aligned} \mathbb{Z}_p^\times \times &\to \mathbf{S}_2 \\ a &\mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}. \end{aligned}$$

The composite is the squaring map and so, since p is odd, is an isomorphism on the 1-units

$$(1 + p\mathbb{Z}_p)^\times.$$

It follows that $\mathbf{S}_2$ can be written as a product

$$\mathbb{Z}_p \times \left(\mathbf{S}_2^s \rtimes \mu_{p^2-1}\right)$$

in which $\zeta \in \mu_{p^2-1}$ corresponds to the matrix

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^p \end{pmatrix} \tag{5.3.2}$$

and

$$\mathbf{S}_2^s \subset \mathbf{S}_2$$

is the subgroup of matrices of determinant 1. The factor of $\mathbb{Z}_p$ is acting trivially on everything we will consider and can just be put in at the end. We can make

our computation with $\mathbf{S}_2^s$ as long as we project to the invariant part of the action of μ_{p^2-1}.

Write $q = p^2$. Note that reducing the matrices modulo p gives a map

$$\alpha\langle p-1\rangle : \mathbf{S}_2^s \to \mathbb{F}_q = \mu_{q-1}$$
$$\begin{pmatrix} a & pb^{\phi^{-1}} \\ b & a^{\phi^{-1}}. \end{pmatrix} \mapsto b.$$

For ζ as in (5.3.2), we have

$$\alpha\langle p-1\rangle(\zeta g \zeta^{-1}) = \zeta^{p-1}\alpha\langle p-1\rangle(g),$$

so that $\alpha\langle p-1\rangle$ is an element of $H^1(\mathbf{S}_2^s;\mathbb{F}_q)$ transforming in the ζ^{p-1} eigenspace. The element

$$\alpha\langle 1-p\rangle \in H^1(\mathbf{S}_2^s;\mathbb{F}_q)$$

given by

$$\alpha\langle 1-p\rangle(g) = b^{\phi}$$

is in the $\zeta^{(1-p)}$ eigenspace. One easily checks that these form a basis for $H^1(\mathbf{S}_2^s;\mathbb{F}_q)$. Taking the Bockstein of these gives two more elements

$$\beta\langle p-1\rangle = \beta(\alpha\langle p-1\rangle) \in H^2(\mathbf{S}_2^s;\mathbb{F}_q)$$
$$\beta\langle p-1\rangle = \beta(\alpha\langle p-1\rangle) \in H^2(\mathbf{S}_2^s;\mathbb{F}_q).$$

Notation 5.3.3. I've adopted some non-standard notation here in order to make the eigenspace decomposition more transparent. In the topology literature one finds

$$\alpha\langle p-1\rangle \leftrightarrow h_0 u^{p-1}$$
$$\alpha\langle 1-p\rangle \leftrightarrow h_1 u^{p(p^2-1)}$$
$$\beta\langle p-1\rangle \leftrightarrow g_0 u^{p^2+p-2} \leftrightarrow b_1 u^{p^2(p-1)}$$
$$\beta\langle 1-p\rangle \leftrightarrow g_1 u^{(p)(p^2+p-2)} \leftrightarrow b_0 u^{p(p-1)}$$

Exercise 5.3.1. Show that for $p > 3$, the cohomolgy ring $H^*(\mathbf{S}_2^s;\mathbb{F}_q)$ is the graded commutative ring generated by the classes $\alpha\langle 1-p\rangle$, $\alpha\langle p-1\rangle$, $\beta\langle 1-p\rangle$, $\beta\langle p-1\rangle$ subject to the relations

$$\alpha\langle 1-p\rangle\beta\langle p-1\rangle = \alpha\langle p-1\rangle\beta\langle 1-p\rangle$$
$$\beta\langle p-1\rangle^2 = \beta\langle 1-p\rangle^2 = 0.$$

(This is not straightforward, but it is doable and it is informative. I'll add some guidance an a later revision of these notes.)

The computation will be gased on the following striking theorem of Shimomura and Tamura [**27**].

Theorem 5.3.4. *The map*

$$H^*(\mathbf{S}_2^s;k) \to H^*(\mathbf{S}_2^s;E_0/p)$$

is an isomorphism on μ_{p^2-1} invariants.

Corollary 5.3.5. *The map $H^*(\mathbf{S}_2;\mathbb{W}) \to H^*(\mathbf{S}_2;E_0)$ is an isomorphism.*

Exercise 5.3.2. Deduce the Corollary from the Theorem. (The main point of this is to come to grips with what kind of cohomology one is talking about here).

The proof of Theorem 5.3.4 requires some details of the proof of 5.3.5 to which we not turn. It is done by filtering the ring

$$k[\![u_1]\!]$$

by powers of the maximal ideal and running the spectral sequence. The E_1-term is the sum of the cohomology groups

$$H^*(\mathbf{S}_2^s; (u_1)^s/(u_1)^{s+1}).$$

From the crystalline approximation, for $g \in \mathbf{S}_2^s$ (and working modulo p) we have

$$\begin{aligned} g \cdot u &= au + buu_1 + O[u_1]^p \\ &= u + buu_1 + O[u_1]^p \\ g \cdot uu_1 &= pb^{\phi^{-1}} u + a^{\phi^{-1}} uu_1 + O[u_1]^p \\ &= uu_1 + O[u_1]^p \\ g \cdot u_1 &= \frac{a^{\phi^{-1}} u_1 + pb^{\phi^{-1}}}{bu_1 + a} + O[u_1]^p \\ &= \frac{u_1}{1 + bu_1}. \end{aligned}$$

From the above formulas one sees that modulo $(u_1)^2$ the element u_1 transforms in the representation $\zeta^{(p-1)}$, so that the classes

$$\alpha\langle 1-p\rangle u_1 \text{ and } \alpha\langle p-1\rangle u_1^p$$

are invariant. This leads to the following basis for the μ_{p^2-1}-invariant part of the cohomology of the associated graded ring, in which $m = 0, 1, 2, \ldots$

$$\begin{aligned} H^0 &: \quad \{u_1^{m(p+1)}\} \\ H^1 &: \quad \{\alpha\langle 1-p\rangle u_1^{m(p+1)+1}\} \cup \{\alpha\langle p-1\rangle u_1^{m(p+1)+p}\} \\ H^2 &: \quad \{\beta\langle 1-p\rangle u_1^{m(p+1)+1}\} \cup \{\beta\langle p-1\rangle u_1^{m(p+1)+p}\} \\ H^3 &: \quad \{\alpha\langle 1-p\rangle \beta\langle p-1\rangle u_1^{m(p+1)}\}. \end{aligned}$$

Now for the differentials. The first ones are in Miller-Ravenel-Wilson [**24**]. From the transformation formula

$$g \cdot u_1 = \frac{u_1}{1 + b^{\phi} u_1} + O[u_1]^p \mod p$$

we get, for $g \in \mathbf{S}_2^s$

$$u_1 \mapsto u_1 - b^{\phi^{-1}} u_1^2 + O[u_1]^3$$

leading to the differential

$$u_1^{s(p+1)} \mapsto -s u_1^{(s-1)(p+1)+1} \alpha\langle 1-p\rangle.$$

From the formula

$$u_1^p \mapsto \frac{u_1^p}{1 + bu_1^p} + O[u_1]^{p^2}$$

we get for $g \in \mathbf{S}_2^s$

$$u_1^p \mapsto u_1^p - bu_1^{2p} + O[u_1]^{3p}$$

and so the differential

$$du_1^{sp(p+1)} = -su_1^{sp(p+1)+p}\alpha\langle p-1\rangle.$$

This looks pretty good, but at the next step there is a bit of trouble. For $g \in \mathbf{S}_2^s$ we have

$$\begin{aligned} u_1^{p^2} &\mapsto \frac{u_1^{p^2}}{1+b^\phi u_1^{p^2}} + O[u_1]^{p^3} \\ &\equiv u_1^{p^2} - b^\phi u_2^{2p^2} + \cdots + O[u_1]^{p^3}. \end{aligned}$$

However since

$$u_1^{2p^2-1} \mapsto u_1^{2p^2-1} + b^\phi u_1^{2p^2} + \cdots$$

we can add this term to the above and get a longer differential. If you do this you find that

$$u_1^{(2)} := u_1^{p^2} + u_1^{2p^2-1} = u_1^{p^2}(1+u_1^{p^2-1})$$

satisfies

$$g(u_1^{(2)}) = u_1^{(2)} + O[u_1^{2p^2+p-1}]$$

and so to go further one requires a more accurate expression for the action. It's not so easy to get the final answer from this point of view. If you want to know how to do it, you should consult the references [**18, 20, 2**].

I'll now just report on all of the differentials

5.3.1. Differentials from H^0 to H^1.

$$\begin{aligned} u_1^{s(p+1)} &\mapsto -su_1^{s(p+1)}u_1\alpha\langle 1-p\rangle \\ u_1^{sp(p+1)} &\mapsto -s\big(u_1^{sp(p+1)}\big)u_1^p\alpha\langle p-1\rangle \\ u_1^{sp^n(p+1)} &\mapsto -2s\big(u_1^{sp^n(p+1)}\big)u_1^{p^n+p^{n-1}-1}\alpha\langle p-1\rangle \qquad n>1 \end{aligned}$$

5.3.2. Differentials from H^1 to H^2. We start with the differentials on the classes $u_1^{s(p+1)}u_1\alpha\langle 1-p\rangle$. The classes with $s \not\equiv 0 \mod p$ are in the image of the differentials from H^0 so are in the kernel of the differential from H^1 to H^2. I won't remark on this kind of thing further, and will leave it to you to check that all of the necessary differentials have been reported.

$$u_1^{sp(1+p)}u_1\alpha\langle 1-p\rangle \mapsto u_1^{sp(1+p)}u_1u_1^{p-1}\beta\langle p-1\rangle. \tag{5.3.6}$$

Exercise 5.3.3. These differentials are within the range of the cryatalline approximation. Can you account for them? (I haven't tried this).

The differentials on the classes

$$u_1^{s(p+1)}u_1^p\alpha\langle p-1\rangle$$

break into cases. For $s \not\equiv 1 \mod p^2$ one has

$$d(u_1^{(s-1)(1+p)}\,u_1^p\alpha\langle p-1\rangle) \mapsto -\binom{s}{2}u_1^{s(1+p)}\,u_1\beta\langle 1-p\rangle \tag{5.3.7}$$

(5.3.8)

$$d(u_1^{(sp^n-1)(1+p)}\,u_1^p\alpha\langle p-1\rangle) \mapsto -\frac{(-1)^n}{2}\binom{s}{2}u_1^{sp^n(1+p)}\,u_1^{(p+1)\frac{p^n-1}{p-1}}\,u_1\beta\langle 1-p\rangle$$

For $s \equiv 1 \mod p^2$ one has

(5.3.9)

$$d(u_1^{(s-1)(1+p)}\, u_1^p \alpha\langle p-1\rangle) \mapsto -u_1^{sp^n(1+p)}\, u_1^{(1+p)(p-1)}\, u_1\beta\langle 1-p\rangle$$

(5.3.10)

$$d(u_1^{(sp^n-1)(1+p)}\, u_1^p \alpha\langle p-1\rangle) \mapsto -\frac{(-1)^n}{4} u_1^{sp^n(1+p)}\, u_1^{(1+p)(\frac{p^n-1}{p-1}+p^n(p-1))}\, u_1\beta\langle 1-p\rangle$$

(5.3.11)

5.3.3. Differentials from H^2 to H^3. The differentials from H^2 to H^3 are actually determined by Poincaré duality. I'll record them here and if there is interest set up a project/guided exercise for deducing them.

First the differentials on the elements

$$u_1^{(p+1)m} u_1\beta\langle 1-p\rangle.$$

Every integer m can be written uniquely in the form

$$m = 1 + p + \cdots + tp^{n-1}$$

with $t \not\equiv 1 \mod p$. The differential on the classes $u_1^{(p+1)\ell} u_1^p \alpha\langle p-1\rangle$ hit the classes with

$$t \not\equiv 0 \mod p.$$

This leaves the elements of the form

$$m = 1 + \cdots + p^{n-2} + sp^n.$$

In this way we associate n and s to each eligible m. The differential is

$$u_1^{(p+1)m} u_1\beta\langle 1-p\rangle \mapsto -2(s-1) u_1^{(p+1)m}\, u_1^{(p+1)p^{n-1}} \alpha\langle p-1\rangle\beta\langle 1-p\rangle.$$

The differentials on the classes

$$u_1^{(p+1)m} u_1^p \beta\langle p-1\rangle$$

follows from the differentials on $u_1^{(p+1)s}$ with $s \not\equiv 0 \mod p$. The formula is

$$u_1^{(p+1)s} u_1^p \beta\langle p-1\rangle \mapsto -s u_1^{(p+1)s} u_1^{(p+1)} \alpha\langle 1-p\rangle\beta\langle p-1\rangle.$$

Exercise 5.3.4. How accurately would one need the formula for $g(u_1)$ to account for all of these differentials?

Exercise 5.3.5. Assuming these differentials deduce Theorem 5.3.4. (This is just a matter of bookkeeping, but it's worthwhile straightening it out.)

We can now turn to the proof of Theorem 5.3.1.

Exercise 5.3.6. Show that there is a short exact sequence of $\mathbf{S}_2^s$-modules

$$0 \to E_0 \xrightarrow{\exp(px)} E_0^\times \to (E_0/p)^\times \to 1.$$

This short exact sequence gives a long exact sequence in cohomology. You can work out one third of the terms from Theorem 5.3.5, so the computation reduces to knowing

$$H^1(\mathbf{S}_2^s; (E_0/p)^\times) = H^1(\mathbf{S}_2^s; k[\![u_1]\!]^\times).$$

You do this by filtering by kernels of the map

$$k[\![u_1]\!]^\times \to k[\![u_1]\!]/(u_1^j)^\times$$

and studying the spectral sequence

$$\bigoplus H^*(\mathbf{S}_2^s; (1+u_1^\ell k[\![u_1]\!])^\times/(1+u_1^{\ell+1}k[\![u_1]\!])^\times) \Rightarrow H^*(\mathbf{S}_2^s; (k[\![u_1]\!]^\times). \tag{5.3.12}$$

You can finish the project by doing this. There are two observations you will need.

Observation 5.3.13. For each n, the module E_{-n} is invertible. It maps the element of $H^1(\mathbf{S}_2^s; k[\![u_1]\!]^\times)$ given by the crossed homomorphism

$$g \mapsto g(u^n)/u^n.$$

You can work out what you need of this from the chromatic approximation. This gives a bunch of elements that you know survive this spectral sequence.

Observation 5.3.14. The group

$$(1+u_1^\ell k[\![u_1]\!])^\times/(1+u_1^{2\ell}k[\![u_1]\!])^\times$$

is isomorphic to $(u_1^\ell)/(u_1^{2\ell})$ so in a certain range the "multiplicative" spectral sequence (5.3.12) is isomorphic to the "additive" spectral sequence analyzed by Shimomura, and you can use those differentials.

Armed with these two observations it is possible to prove Theorem 5.3.1.

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Tamagawa Numbers in the Function Field Case

September 25, 2018

Let G be a semisimple algebraic group defined over the field $\mathbf{Q}$ of rational numbers and let $G(\mathbf{Q})$ denote the group of rational points of G. Then $G(\mathbf{Q})$ can be regarded as a discrete subgroup of the locally compact group $G(\mathbf{A})$ of *adelic points* of G. Moreover, the group $G(\mathbf{A})$ carries a canonical (bi-invariant) measure, called *Tamagawa measure*. A celebrated conjecture of Weil asserts that, if the group G is simply connected, then the Tamagawa measure of the quotient $G(\mathbf{Q})\backslash G(\mathbf{A})$ is equal to 1. Weil's conjecture is now a theorem of Kottwitz, building on earlier work of Langlands and Lai. More recently, Gaitsgory and Lurie proved a version of Weil's conjecture in the setting of function fields, using techniques inspired by algebraic topology (specifically, the theory of factorization homology). The goal of this lecture series is to explain some of the ideas surrounding the proof.

- In the first lecture, we will review the origin of Weil's conjecture as a generalization of the *mass formula* of Smith-Minkowski-Siegel. We will then discuss how to interpret the function field analogue of Weil's conjecture as a mass formula for counting principal G-bundles on algebraic curves (over finite fields).
- When X is an algebraic curve over a field k, principal G-bundles on X are parametrized by an algebraic stack $\mathrm{Bun}_G(X)$, called the *moduli stack of G-bundles*. In the second lecture, we will review the Grothendieck-Lefschetz trace formula, which reduces the problem of counting principal G-bundles (over a finite field) to the problem of understanding the cohomology $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$ (over an algebraically closed field). We then give a heuristic formula for this cohomology as a "continuous tensor product" $\bigotimes_{x \in X} \mathrm{H}^*(\mathrm{Bun}_G(\{x\}); \mathbf{Q}_\ell)$ which explains the mass formula predicted by Weil's conjecture.
- In the third lecture, we introduce an algebro-geometric version of *factorization homology* and use it to give a precise formulation of the heuristic from Lecture 2. This takes the form of a "product formula" characterizing the cochain algebra $C^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$ (rather than the cohomology algebra $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$) by a universal property.
- In the fourth lecture, we specialize to the case where G and X are defined over a finite field $\mathbf{F}_q$, and explain how to use the product formula of Lecture 3 (together with the Grothendieck-Lefschetz trace formula) to count principal G-bundles on X, thereby obtaining a proof of Weil's conjecture.

1 Project

In order to reduce the proof of Weil's conjecture to a statement about the cohomology of $\mathrm{Bun}_G(X)$, one needs to know that $\mathrm{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula. This does not follow formally from the classical version of the Grothendieck-Lefschetz trace formula, because the moduli stack $\mathrm{Bun}_G(X)$ is not quasi-compact. Nevertheless, the trace formula is still valid. This was proved by Behrend in the case where G is everywhere semisimple, using the Harder-Narasimhan stratification of $\mathrm{Bun}_G(X)$. In [2], this argument is extended to the case of groups of bad reduction by a somewhat ad-hoc method. However, it seems likely that there is a better proof.

In this project, students will study a potential generalization of the Harder-Narasimhan stratification to the case where G is a group scheme over X which is only *generically* reductive, but can have bad (but

parahoric) reduction at finitely many points of X. In the case where the generic fiber of G is GL_n, this generalization can be described very explicitly in terms of vector bundles on algebraic curves. Further extensions could be considered, depending on the expertise of the group.

This project is somewhat orthogonal to the material covered in the lectures, and does not require any knowledge of algebraic topology. However, it will require some algebraic geometry: students should be familiar with the classical Harder-Narasimhan theory for vector bundles on algebraic curves.

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Weil's Conjecture for Function Fields

February 12, 2019

1 First Lecture: The Mass Formula and Weil's Conjecture

Let R be a commutative ring and let V be an R-module. A *quadratic form* on V is a map $q : V \to R$ satisfying the following conditions:

(a) The construction $(v, w) \mapsto q(v + w) - q(v) - q(w)$ determines an R-bilinear map $V \times V \to R$.

(b) For every element $\lambda \in R$ and every $v \in V$, we have $q(\lambda v) = \lambda^2 q(v)$.

A *quadratic space* over R is a pair (V, q), where V is a finitely generated projective R-module and q is a quadratic form on V.

One of the basic problems in the theory of quadratic forms can be formulated as follows:

Question 1.1. Let R be a commutative ring. Can one classify quadratic spaces over R (up to isomorphism)?

Let us begin by describing some cases where Question 1.1 admits a complete answer.

Example 1.2 (Quadratic Forms over $\mathbf{C}$)**.** Let $R = \mathbf{C}$ be the field of complex numbers (or, more generally, any algebraically closed field of characteristic different from 2). Then every quadratic space over R is isomorphic to $(\mathbf{C}^n, q)$, where q is given by the formula

$$q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_r^2$$

for some $0 \leqslant r \leqslant n$. Moreover, the integer r is uniquely determined: it is an isomorphism-invariant called the *rank* of the quadratic form q.

Example 1.3 (Quadratic Forms over the Real Numbers)**.** Let $R = \mathbf{R}$ be the field of real numbers. Then every quadratic space over R is isomorphic to $(\mathbf{R}^n, q)$, where the quadratic form q is given by

$$q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_a^2 - x_{a+1}^2 - x_{a+2}^2 - \cdots - x_{a+b}^2$$

for some pair of nonnegative integers a and b satisfying $a + b \leqslant n$. Moreover, Sylvester's *invariance of signature theorem* asserts that the integers a and b are uniquely determined. The difference $a - b$ is an isomorphism-invariant of q, called the *signature of* q. We say that a quadratic space (V, q) is *positive-definite* if it has signature n: that is, if it is isomorphic to the standard Euclidean space $(\mathbf{R}^n, q_0)$ with $q_0(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$. Equivalently, a quadratic form q is positive-definite if it satisfies $q(v) > 0$ for every nonzero vector $v \in V$.

Example 1.4 (Quadratic Forms over p-adic Fields)**.** Let $R = \mathbf{Q}_p$ be the field of p-adic rational numbers, for some prime number p. One can show that a *nondegenerate* quadratic space (V, q) over $\mathbf{Q}_p$ is determined up to isomorphism by its *discriminant* (an element of the finite group $\mathbf{Q}_p^\times / \mathbf{Q}_p^{\times 2}$) and its *Hasse invariant* (an element of the group $\{\pm 1\}$). In particular, if p is odd and $n \gg 0$, then there are exactly eight isomorphism classes of nondegenerate quadratic spaces of dimension n over $\mathbf{Q}_p$. When $p = 2$, there are sixteen isomorphism classes.

If (V, q) is a quadratic space over a commutative ring R and $f : R \to S$ is a ring homomorphism, then we can extend scalars along f to obtain a quadratic space (V_S, q_S) over S; here $V_S = S \otimes_R V$ and q_S is the unique extension of q to a quadratic form on V_S. Note that if two quadratic spaces (V, q) and (V', q') over R are isomorphic, then they remain isomorphic after extending scalars to S. In the case $R = \mathbf{Q}$, this assertion has a converse:

Theorem 1.5 (The Hasse Principle)**.** *Let (V, q) and (V', q') be quadratic spaces over the field $\mathbf{Q}$ of rational numbers. Then (V, q) and (V', q') are isomorphic if and only if the following conditions are satisfied:*

- (a) *The quadratic spaces $(V_{\mathbf{R}}, q_{\mathbf{R}})$ and $(V'_{\mathbf{R}}, q'_{\mathbf{R}})$ are isomorphic over $\mathbf{R}$ (this can be tested by comparing signatures).*
- (b) *For every prime number p, the quadratic spaces $(V_{\mathbf{Q}_p}, q_{\mathbf{Q}_p})$ and $(V'_{\mathbf{Q}_p}, q'_{\mathbf{Q}_p})$ are isomorphic.*

Remark 1.6. Theorem 1.5 is known as the *Hasse-Minkowski* theorem: it is originally due to Minkowski, and was later generalized to arbitrary number fields by Hasse.

Remark 1.7. Theorem 1.5 asserts that the canonical map

$$\{\text{Quadratic spaces over } \mathbf{Q}\}/\sim \to \prod_K \{\text{Quadratic spaces over } K\}/\sim$$

is injective, where K ranges over the collection of all completions of $\mathbf{Q}$. It is possible to explicitly describe the image of this map (using the fact that the theory of quadratic forms over real and p-adic fields are well-understood; see Examples 1.3 and 1.4 above). We refer the reader to [17] for details.

Let us now specialize to the case $R = \mathbf{Z}$. Note that if (V, q) is a quadratic space over $\mathbf{Z}$, then q determines (and is determined by) a bilinear form

$$b : V \times V \to \mathbf{Z} \qquad b(x, y) = q(x + y) - q(x) - q(y).$$

The bilinear form b has the property that, for each $x \in V$, $b(x, x) = q(2x) - q(x) - q(x) = 2q(x)$ is *even*; conversely, if $b : V \times V \to \mathbf{Z}$ is any symmetric bilinear form with the property that each $b(x, x)$ is even, then we can equip V with a quadratic form given by $q(x) = \frac{b(x,x)}{2}$. For this reason, quadratic spaces over $\mathbf{Z}$ are often called *even lattices*.

We can now ask if the analogue of Theorem 1.5 holds over the integers.

Exercise 1.8. Let (V, q) and (V', q') be quadratic spaces over $\mathbf{Z}$. Show that the following conditions are equivalent:

(a) The quadratic spaces (V, q) and (V', q') become isomorphic after extending scalars along the projection map $\mathbf{Z} \to \mathbf{Z}/N\mathbf{Z}$, for every positive integer N.

(b) The quadratic spaces (V, q) and (V', q') become isomorphic after extending scalars along the projection map $\mathbf{Z} \to \mathbf{Z}/p^k\mathbf{Z}$, for each prime number p and each $k \geqslant 0$.

(c) The quadratic spaces (V, q) and (V', q') become isomorphic after extending scalars along the inclusion $\mathbf{Z} \hookrightarrow \mathbf{Z}_p$ for each prime number p. Here $\mathbf{Z}_p$ denotes the ring of p-adic integers.

(d) The quadratic spaces (V, q) and (V', q') become isomorphic after extending scalars along the inclusion $\mathbf{Z} \hookrightarrow \widehat{\mathbf{Z}}$. Here $\widehat{\mathbf{Z}} = \varprojlim_{N>0}(\mathbf{Z}/N\mathbf{Z}) \simeq \prod_p \mathbf{Z}_p$ is the *profinite completion of* $\mathbf{Z}$.

Definition 1.9. Let (V, q) be a quadratic form over $\mathbf{Z}$. We say that q is *positive-definite* if the quadratic form $q_{\mathbf{R}}$ is positive-definite (Example 1.3). We say that two positive-definite quadratic spaces (V, q) and (V', q') *have the same genus* if they satisfy the equivalent conditions of Exercise 1.8.

Let (V, q) and (V', q') be positive-definite quadratic spaces over $\mathbf{Z}$. If (V, q) and (V', q') are isomorphic, then they are of the same genus. The converse is generally false. However, it is *almost* true in the following sense: for a fixed positive definite quadratic space (V, q) over $\mathbf{Z}$, there are only finitely many quadratic spaces of the same genus (up to isomorphism). In fact, there is even a *formula* for the number of such quadratic spaces, counted with multiplicity. To state it, we first need to introduce some notation.

Notation 1.10. Let (V, q) be a quadratic space over a commutative ring R. We let $\mathrm{O}_q(R)$ denote the automorphism group of (V, q): that is, the group of R-module isomorphisms $\alpha : V \to V$ such that $q = q \circ \alpha$. We will refer to $\mathrm{O}_q(R)$ as the *orthogonal*

group of the quadratic space (V, q). More generally, if $\phi : R \to S$ is a map of commutative rings, we let $\mathrm{O}_q(S)$ denote the automorphism group of the quadratic space (V_S, q_S) over S obtained from (V, q) by extension of scalars to S.

Example 1.11. Suppose q is a positive-definite quadratic form on an real vector space V of dimension n. Then $\mathrm{O}_q(\mathbf{R})$ can be identified with the usual orthogonal group $\mathrm{O}(n)$, generated by rotations and reflections in Euclidean space. In particular, $\mathrm{O}_q(\mathbf{R})$ is a compact Lie group of dimension $\frac{n^2-n}{2}$.

Example 1.12. Let (V, q) be a positive-definite quadratic space over $\mathbf{Z}$. Then we can identify $\mathrm{O}_q(\mathbf{Z})$ with a subgroup of the orthogonal group $\mathrm{O}_q(\mathbf{R})$: it consists of rotations and reflections in Euclidean space $V_{\mathbf{R}}$ which preserve the lattice $V \subseteq V_{\mathbf{R}}$. In particular, it is a finite group.

Theorem 1.13 (Smith-Minkowski-Siegel Mass Formula)**.** *Let (V, q) be a positive-definite quadratic space over $\mathbf{Z}$ of rank $\geqslant 2$, and let D be the discriminant of q. Then we have*

$$\sum_{q'} \frac{1}{|\,\mathrm{O}_{q'}(\mathbf{Z})|} = \frac{2|D|^{(n+1)/2}}{\prod_{m=1}^{n} \mathrm{Vol}(S^{m-1})} \prod_p c_p,$$

where the sum on the left hand side is taken over all isomorphism classes of quadratic spaces (V', q') in the genus of (V, q), $\mathrm{Vol}(S^{m-1}) = \frac{2\pi^{m/2}}{\Gamma(m/2)}$ denotes the volume of the standard $(m-1)$-sphere, and the product on the right ranges over all prime numbers p, with individual factors c_p satisfying $c_p = \frac{2p^{kn(n-1)/2}}{|\,\mathrm{O}_q(\mathbf{Z}/p^k\mathbf{Z})|}$ for $k \gg 0$.

Example 1.14. Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, and let $q : \mathcal{O}_K \to \mathbf{Z}$ be the norm map. Then $(\mathcal{O}_K, q)$ is a positive-definite quadratic space of rank 2 over $\mathbf{Z}$. In this case, Theorem 1.13 reduces to the class number formula for K.

We can greatly simplify the statement of Theorem 1.13 by restricting our attention to a special case.

Definition 1.15. Let (V, q) be a quadratic space over $\mathbf{Z}$. We say that q is *unimodular* if, for every prime number p, the quadratic form $q_{\mathbf{F}_p}$ is nondegenerate. That is, the associated symmetric bilinear form

$$b(x, y) = q_{\mathbf{F}_p}(x + y) - q_{\mathbf{F}_p}(x) - q_{\mathbf{F}_p}(y)$$

is nondegenerate on the $\mathbf{F}_p$-vector space $\mathbf{F}_p^n$.

Warning 1.16. Unimodularity is a very strong condition. Note that the "standard" quadratic form $q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$ is not unimodular: the associated bilinear form is identically zero modulo 2. One can show that if a quadratic (V, q) is both unimodular and positive-definite, then the rank of V must be divisible by 8.

Let (V, q) and (V', q') be positive-definite quadratic spaces over $\mathbf{Z}$ (of the same rank). It follows immediately from the definition that if (V, q) and (V', q') are of the same genus, then (V, q) is unimodular if and only if (V', q') is unimodular. This assertion has a converse: if (V, q) and (V', q') are both unimodular, then they are of the same genus. In other words, if we fix the number of variables (assumed to be a multiple of 8), then the unimodular quadratic spaces *comprise* a genus. This is in some sense the simplest genus, and for this genus the statement of Theorem 1.13 simplifies: the discriminant D has absolute value 1 (this is equivalent to unimodularity), and the Euler factors c_p can be explicitly evaluated.

Theorem 1.17 (Mass Formula: Unimodular Case)**.** *Let n be an integer which is a positive multiple of* 8*. Then*

$$\begin{aligned} \sum_q \frac{1}{|\operatorname{O}_q(\mathbf{Z})|} &= \frac{2\zeta(2)\zeta(4)\cdots\zeta(n-4)\zeta(n-2)\zeta(n/2)}{\operatorname{Vol}(S^0)\operatorname{Vol}(S^1)\cdots\operatorname{Vol}(S^{n-1})} \\ &= \frac{B_{n/4}}{n}\prod_{1\leqslant j<n/2}\frac{B_j}{4j}. \end{aligned}$$

Here ζ denotes the Riemann zeta function, B_j denotes the jth Bernoulli number, and the sum is taken over all isomorphism classes of positive-definite unimodular quadratic spaces (V, q) of rank n.

Example 1.18. Let $n = 8$. The right hand side of the mass formula evaluates to $\frac{1}{696729600}$. The integer $696729600 = 2^{14}3^5 5^2 7$ is the order of the Weyl group of the exceptional Lie group E_8, which is also the automorphism group of the root lattice of E_8 (which is an even unimodular lattice). Consequently, the fraction $\frac{1}{696729600}$ also appears as one of the summands on the left hand side of the mass formula. It follows from Theorem 1.17 that no other terms appear on the left hand side: that is, the root lattice of E_8 is the *unique* positive-definite even unimodular lattice of rank 8, up to isomorphism.

Remark 1.19. Theorem 1.17 allows us to count the number of positive-definite even unimodular lattices of a given rank with multiplicity, where a lattice (V, q) is counted with multiplicity $\frac{1}{|\operatorname{O}_q(\mathbf{Z})|}$. If the rank of V is positive, then $\operatorname{O}_q(\mathbf{Z})$ has order at least 2 (since $\operatorname{O}_q(\mathbf{Z})$ contains the group $\langle \pm 1 \rangle$), so that the left hand side of Theorem 1.17 is at most $\frac{C}{2}$, where C is the number of isomorphism classes of positive-definite even unimodular lattices. In particular, Theorem 1.17 gives an inequality

$$C \geqslant \frac{\Gamma(\frac{1}{2})\Gamma(\frac{2}{2})\cdots\Gamma(\frac{n}{2})\zeta(2)\zeta(4)\cdots\zeta(n-4)\zeta(n-2)\zeta(\frac{n}{2})}{2^{n-2}\pi^{n(n+1)/4}}.$$

The right hand side of this inequality grows very quickly with n. For example, when $n = 32$, we can deduce the existence of more than eighty million pairwise nonisomorphic (positive-definite) even unimodular lattices of rank n.

We now describe a reformulation of Theorem 1.13, following ideas of Tamagawa and Weil. Suppose we are given a positive-definite quadratic space (V, q) over $\mathbf{Z}$, and that we wish to classify other positive-definite quadratic spaces (V', q') of the same genus. Note that (V, q) and (V', q') must then become isomorphic after extending scalars to $\widehat{\mathbf{Z}}$ (Exercise 1.8). We can therefore choose an isomorphism

$$\alpha : \widehat{\mathbf{Z}} \otimes V \simeq \widehat{\mathbf{Z}} \otimes V'$$

satisfying $q_{\widehat{\mathbf{Z}}} = q'_{\widehat{\mathbf{Z}}} \circ \alpha$. It follows that (V, q) and (V', q') are isomorphic after extending scalars to $\mathbf{Z}_p$, for every prime number p, hence also after extending scalars to $\mathbf{Q}_p = \mathbf{Z}_p[1/p]$. Since (V, q) and (V', q') are also isomorphic over $\mathbf{R}$ (by virtue of our assumption that both are positive-definite), the Hasse principle (Theorem 1.5) guarantees that they are isomorphic over $\mathbf{Q}$. That is, we can choose an isomorphism of rational vector spaces

$$\beta : \mathbf{Q} \otimes V \simeq \mathbf{Q} \otimes V'$$

satisfying $q_{\mathbf{Q}} = q'_{\mathbf{Q}} \circ \beta$.

Let $\mathbf{A}_f$ denote the ring of *finite adeles*: that is, the tensor product $\widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q}$. The isomorphism $\widehat{\mathbf{Z}} \simeq \prod_p \mathbf{Z}_p$ induces an injective map

$$\mathbf{A}_f \simeq \widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q} \hookrightarrow \prod_p (\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathbf{Q}) \simeq \prod_p \mathbf{Q}_p,$$

whose image is the *restricted product* $\prod_p^{\mathrm{res}} \mathbf{Q}_p \subseteq \prod_p \mathbf{Q}_p$: that is, the subset consisting of those elements $\{x_p\}$ of the product $\prod_p \mathbf{Q}_p$ such that $x_p \in \mathbf{Z}_p$ for all but finitely many prime numbers p. The quadratic spaces (V, q) and (V', q') become isomorphic after extension of scalars to $\mathbf{A}_f$ in two different ways: via the isomorphism α which is defined over $\widehat{\mathbf{Z}}$, and via the isomorphism β which is defined over $\mathbf{Q}$. Consequently, the composition $\beta^{-1} \circ \alpha$ can be regarded as an element of the orthogonal group $\mathrm{O}_q(\mathbf{A}_f)$. This element depends not only the quadratic space (V', q'), but also on our chosen isomorphisms α and β. However, any other isomorphism between $(V_{\widehat{\mathbf{Z}}}, q_{\widehat{\mathbf{Z}}})$ and $(V'_{\widehat{\mathbf{Z}}}, q'_{\widehat{\mathbf{Z}}})$ can be written in the form $\alpha \circ \gamma$, where $\gamma \in \mathrm{O}_q(\widehat{\mathbf{Z}})$. Similarly, the isomorphism β is well-defined up to right multiplication by elements of $\mathrm{O}_q(\mathbf{Q})$. Consequently, the composition $\beta^{-1} \circ \alpha$ is really well-defined as an element of the set of double cosets

$$\mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A}_f) / \mathrm{O}_q(\widehat{\mathbf{Z}}).$$

Let us denote this double coset by $[V', q']$. Note that $[V', q']$ is the identity double coset if and only if it possible to arrange that $\alpha = \beta$: in this case, the map $\alpha = \beta$ is defined

over the subring $\mathbf{Z} = \widehat{\mathbf{Z}} \cap \mathbf{Q}$ (where the intersection is formed in the ring $\mathbf{A}_f$), and therefore defines an isomorphism of quadratic space $(V, q) \simeq (V', q')$ over $\mathbf{Z}$. Using a more elaborate version of the same argument, one obtains the following:

Proposition 1.20. *Fix a positive-definite quadratic space (V, q) over $\mathbf{Z}$. Then the construction $(V', q') \mapsto [V', q']$ induces a bijection*

$$\begin{array}{c} \{\text{Quadratic spaces } (V', q') \text{ in the genus of } (V, q)\}/\text{isomorphism} \\ \Big\downarrow \sim \\ \mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A}_f) / \mathrm{O}_q(\widehat{\mathbf{Z}}). \end{array}$$

Exercise 1.21. Give an explicit description of the inverse bijection (this is actually somewhat easier, because it does not depend on the Hasse principle of Theorem 1.5).

At this point, it will be convenient to modify the preceding discussion. Let $\mathbf{A}$ denote the ring of *adeles*, given by the product $\mathbf{A}_f \times \mathbf{R}$. Then the collection of double cosets appearing in Proposition 1.20 can also be written as

$$\mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A}) / \mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}).$$

Note that $\mathbf{A}$ has the structure of a locally compact commutative ring, containing $\mathbf{Q}$ as a closed subring (with the discrete topology). It follows that $\mathrm{O}_q(\mathbf{A})$ inherits the structure of a locally compact topological group, which contains $\mathrm{O}_q(\mathbf{Q})$ as a discrete subgroup and $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$ as a compact open group.

Let μ be a Haar measure on the group $\mathrm{O}_q(\mathbf{A})$. One can show that the group $\mathrm{O}_q(\mathbf{A})$ is unimodular: that is, the measure μ is invariant under both right and left translations. In particular, μ determines a measure on the quotient space $\mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A})$, which is invariant under the right action of $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$. We will abuse notation by denoting this measure also by μ. Write $\mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A})$ as a union of orbits $\bigcup_{x \in X} O_x$ for the action of the group $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$. If $x \in X$ is a double coset represented by an element $\gamma \in \mathrm{O}_q(\mathbf{A})$, then we can identify the orbit O_x with the quotient of $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$ by the finite subgroup $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}) \cap \gamma^{-1} \mathrm{O}_q(\mathbf{Q}) \gamma$. We therefore have

$$\begin{aligned} \sum_{\gamma} \frac{1}{|\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}) \cap \gamma^{-1} \mathrm{O}_q(\mathbf{Q}) \gamma|} &= \sum_{x \in X} \frac{\mu(O_x)}{\mu(\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))} && (1) \\ &= \frac{\mu(\mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A}))}{\mu(\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))}. && (2) \end{aligned}$$

Exercise 1.22. Show that, if $\gamma \in \mathrm{O}_q(\mathbf{A})$ is an element representing a double coset $x \in X$, then the intersection $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}) \cap \gamma^{-1} \mathrm{O}_q(\mathbf{Q}) \gamma$ is the finite group $\mathrm{O}_{q'}(\mathbf{Z})$, where (V', q') is the quadratic space corresponding to x under the bijection of Proposition 1.20.

Combining Proposition 1.20 with Exercise 1.22, we obtain an identity

$$\sum_{q'} \frac{1}{|\operatorname{O}_{q'}(\mathbf{Z})|} = \frac{\mu(\operatorname{O}_q(\mathbf{Q})\backslash \operatorname{O}_q(\mathbf{A}))}{\mu(\operatorname{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))}.$$

Here the sum on the left is taken over isomorphism classes of positive-definite quadratic spaces (V', q') in the genus of (V, q), as in the Smith-Minkowski-Siegel mass formula (Theorem 1.13). The right hand side is *a priori* dependent on a choice of Haar measure μ on the locally compact group $\operatorname{O}_q(\mathbf{A})$. Such a measure is not uniquely determined: it is only well-defined up to multiplication by a positive scalar. However, the measure μ appears both in the numerator and denominator on the right hand side, and therefore does not depend on μ.

To proceed further, it will be convenient to make another modification. For each commutative ring R, let $\operatorname{SO}_q(R) \subseteq \operatorname{O}_q(R)$ denote the *special orthogonal group* of q over R: that is, the subgroup of $\operatorname{O}_q(R)$ consisting of automorphisms of V_R which preserves the quadratic form q_R and have determinant 1. Then $\operatorname{SO}_q(\mathbf{A})$ is also a locally compact group, on which we can choose a Haar measure μ'. It is not difficult to show that there is an equality

$$\frac{\mu(\operatorname{O}_q(\mathbf{Q})\backslash \operatorname{O}_q(\mathbf{A}))}{\mu(\operatorname{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))} = 2^{k-1} \frac{\mu'(\operatorname{SO}_q(\mathbf{Q})\backslash \operatorname{SO}_q(\mathbf{A}))}{\mu'(\operatorname{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))},$$

where k is the number of primes p for which $\operatorname{SO}_q(\mathbf{Z}_p) = \operatorname{O}_q(\mathbf{Z}_p)$. Consequently, to understand Theorem 1.13, it will suffice to compute the ratio

$$\frac{\mu'(\operatorname{SO}_q(\mathbf{Q})\backslash \operatorname{SO}_q(\mathbf{A}))}{\mu'(\operatorname{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))}$$

appearing on the right hand side. The virtue of this maneuver is that on the group $\operatorname{SO}_q(\mathbf{A})$, there is actually a *canonical* choice for the Haar measure μ' (called *Tamagawa measure*), which will allow us to evaluate the numerator and denominator individually.

Note that the group $\operatorname{SO}_q(\mathbf{A})$ can be identified with the *restricted* product

$$\operatorname{SO}_q(\mathbf{R}) \times \prod_p{}^{\mathrm{res}} \operatorname{SO}_q(\mathbf{Q}_p) \subseteq \operatorname{SO}_q(\mathbf{R}) \times \prod_p \operatorname{SO}_q(\mathbf{Q}_p),$$

where the superscript indicates that we allow only those tuples $\{\gamma_p \in \operatorname{SO}_q(\mathbf{Q}_p)\}$ where γ_p belongs to $\operatorname{SO}_q(\mathbf{Z}_p) \subseteq \operatorname{SO}_q(\mathbf{Q}_p)$ for all but finitely many prime numbers p. Let us first focus our attention on the group $\operatorname{SO}_q(\mathbf{R})$. Since q is positive-definite, this is a familiar group: it is the connected Lie group of rotations in Euclidean space. In particular, it is a smooth manifold of dimension $d = \frac{n^2-n}{2}$. Every differential form ω of degree d on $\operatorname{SO}_q(\mathbf{R})$ determines a measure on $\operatorname{SO}_q(\mathbf{R})$ (given by integrating the absolute value $|\omega|$). Moreover, this measure is a Haar measure if and only if ω is invariant under

translation (and nonzero). The collection of translation-invariant differential forms forms a 1-dimensional vector space $\Omega_{\mathbf{R}}$ over the real numbers, and every nonzero element $\omega \in \Omega_{\mathbf{R}}$ determines a Haar measure on $\mathrm{SO}_q(\mathbf{R})$, which we will denote by $\mu_{\omega,\mathbf{R}}$.

Note that the Lie group $\mathrm{SO}_q(\mathbf{R})$ is rather special: it is an *linear algebraic group* over $\mathbf{R}$. In other words, it can be described explicitly as the group

$$\mathrm{SO}_q(\mathbf{R}) = \{A \in \mathrm{GL}_n(\mathbf{R}) : \det(A) = 1, q \circ A = q\} \subseteq \mathrm{GL}_n(\mathbf{R})$$

of invertible matrices over $\mathbf{R}$ whose matrix coefficients satisfy some polynomial equations. Moreover, these polynomial equations have coefficients in $\mathbf{Q}$ (or even in $\mathbf{Z}$), and therefore define a linear algebraic group over $\mathbf{Q}$ that we will denote by SO_q. (In the language of schemes, we identify SO_q with the $\mathbf{Q}$-scheme representing the functor $R \mapsto \mathrm{SO}_q(R)$ defined above). It follows that the vector space $\Omega_{\mathbf{R}}$ (which is one-dimensional over the field of real numbers) contains a subspace $\Omega_{\mathbf{Q}}$ (which is one-dimensional over the rational numbers) consisting of translation-invariant *algebraic* differential forms on the algebraic group SO_q. Any nonzero element $\omega \in \Omega_{\mathbf{Q}}$ is also a nonzero element of $\Omega_{\mathbf{R}}$, and therefore determines a Haar measure $\mu_{\omega,\mathbf{R}}$ on the compact Lie group $\mathrm{SO}_q(\mathbf{R})$.

Let us now consider the p-adic factors $\mathrm{SO}_q(\mathbf{Q}_p)$. These are not Lie groups in the usual sense. However, it is an example of a *p-adic analytic Lie group*. In the setting of p-adic analytic groups, one can tell a parallel story: there is a 1-dimensional vector space $\Omega_{\mathbf{Q}_p}$ of differential forms of top degree on $\mathrm{SO}_q(\mathbf{Q}_p)$, and any nonzero element $\omega \in \Omega_{\mathbf{Q}_p}$ determines a Haar measure on $\mathrm{SO}_q(\mathbf{Q}_p)$ which we will denote by $\mu_{\omega,\mathbf{Q}_p}$. Moreover, since the p-adic analytic Lie group $\mathrm{SO}_q(\mathbf{Q}_p)$ comes from the same algebraic group SO_q over the rational numbers, we have $\Omega_{\mathbf{Q}_p} \simeq \mathbf{Q}_p \otimes_{\mathbf{Q}} \Omega_{\mathbf{Q}}$, where $\Omega_{\mathbf{Q}}$ is the 1-dimensional $\mathbf{Q}$-vector space described above (consisting of algebraic differential forms of top degree on SO_q). It follows that every nonzero element $\omega \in \Omega_{\mathbf{Q}}$ can also be regarded as a nonzero element of $\Omega_{\mathbf{Q}_p}$, and therefore determines a Haar measure $\mu_{\omega,\mathbf{Q}_p}$ on the group $\mathrm{SO}_q(\mathbf{Q}_p)$.

Construction 1.23 (Tamagawa)**.** Fix a nonzero element $\omega \in \Omega_{\mathbf{Q}}$. We let μ_{Tam} denote the invariant measure on the group $\mathrm{SO}_q(\mathbf{A})$ given informally by the infinite product

$$\mu_{\omega,\mathbf{R}} \times \prod_p \mu_{\omega,\mathbf{Q}_p}.$$

We refer to μ_{Tam} as *Tamagawa measure* on the group $\mathrm{SO}_q(\mathbf{A})$.

Remark 1.24. To define Tamagawa measure more precisely, recall that the group $\mathrm{SO}_q(\mathbf{A})$ contains the group

$$\mathrm{SO}_q(\hat{\mathbf{Z}} \times \mathbf{R}) \simeq \mathrm{SO}_q(\mathbf{R}) \times \prod_p \mathrm{SO}_q(\mathbf{Z}_p)$$

as a compact open subgroup. Consequently, for every positive real number λ, there is a unique Haar measure μ on $\mathrm{SO}_q(\mathbf{A})$ satisfying

$$\mu(\mathrm{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R})) = \lambda.$$

To obtain Tamagawa measure $\mu = \mu_{\mathrm{Tam}}$, we take λ to be the real number

$$\mu_{\omega,\mathbf{R}}(\mathrm{SO}_q(\mathbf{R})) \times \prod_p \mu_{\omega,\mathbf{Q}_p}(\mathrm{SO}_q(\mathbf{Z}_p)).$$

To guarantee that this is well-defined, we need to know that the infinite product $\prod_p \mu_{\omega,\mathbf{Q}_p}(\mathrm{SO}_q(\mathbf{Z}_p))$ converges; this is our motivation for working with the group SO_q in place of the larger group O_q (for which the analogous product would not converge).

Remark 1.25. The Tamagawa measure of Construction 1.23 is *a priori* dependent on a choice of nonzero element $\omega \in \Omega_{\mathbf{Q}}$. However, it turns out to be independent of this choice. If λ is any nonzero rational number, we have equalities

$$\mu_{\lambda\omega,\mathbf{R}} = |\lambda| \mu_{\omega,\mathbf{R}} \qquad \mu_{\lambda\omega,\mathbf{Q}_p} = |\lambda|_p \mu_{\omega,\mathbf{Q}_p},$$

where $|\lambda|$ denotes the usual absolute value of λ and $|\lambda|_p$ denotes the p-adic absolute value of λ. Consequently, replacing ω by $\lambda\omega$ will later the choice of Tamagawa measure by an overall factor of

$$|\lambda| \cdot \prod_p |\lambda|_p,$$

which is equal to 1 by the product formula.

Returning to our previous discussion, we obtain a formula

$$\sum \frac{1}{|\mathrm{O}_{q'}(\mathbf{Z})|} = 2^{k-1} \frac{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbf{Q}) \backslash \mathrm{SO}_q(\mathbf{A}))}{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R})};$$

here k (as above) is the number of primes p for which $\mathrm{O}_q(\mathbf{Z}_p) = \mathrm{SO}_q(\mathbf{Q}_p)$. Moreover, the denominator on the right hand side can be evaluated explicitly (Remark 1.24). This allows us to restate the Smith-Minkowski-Siegel mass formula in a much simpler form (for simplicity, let us assume that the number of variables is at least 3):

Theorem 1.26 (Mass Formula, Tamagawa-Weil Version)**.** *Let (V, q) be a nondegenerate quadratic space over $\mathbf{Q}$ of rank $\geqslant 3$. Then the Tamagawa measure*

$$\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbf{Q}) \backslash \mathrm{SO}_q(\mathbf{A}))$$

is equal to 2.

Remark 1.27. In the statement of Theorem 1.26, it is not necessary to assume that (V, q) is defined over $\mathbf{Z}$ or that the quadratic form q is positive-definite.

Theorem 1.26 can be refined a bit further. Let (V, q) be a nondegenerate quadratic space over $\mathbf{Q}$ of rank n. Then the functor $R \mapsto \mathrm{SO}_q(R)$ is (representable by) an algebraic group over $\mathbf{Q}$, which we denote by SO_q. If $n \geqslant 3$, then this algebraic group is semisimple. However, it is not simply connected: instead, it has a simply-connected double cover $\mathrm{Spin}_q \to \mathrm{SO}_q$. We can regard the group of adelic points $\mathrm{Spin}_q(\mathbf{A})$ as a locally compact group, and a variant of Construction 1.23 produces a canonical measure on $\mathrm{Spin}_q(\mathbf{A})$ which we will also denote by μ_{Tam} and refer to as *Tamagawa measure*. We then have the following result (from which one can recover the mass formula of Smith-Minkowski-Siegel).

Theorem 1.28 (Weil)**.** *Let (V, q) be a nondegenerate quadratic space over $\mathbf{Q}$ of rank $\geqslant 3$. Then the Tamagawa measure $\mu_{\mathrm{Tam}}(\mathrm{Spin}_q(\mathbf{Q})\backslash \mathrm{Spin}_q(\mathbf{A}))$ is equal to 1.*

More generally, if G is any semisimple algebraic group over $\mathbf{Q}$, then $G(\mathbf{A})$ is a locally compact group which is equipped with a canonical measure μ_{Tam} (given by an analogue of Construction 1.23). One can show that the measure

$$\tau_G := \mu_{\mathrm{Tam}}(G(\mathbf{Q})\backslash G(\mathbf{A}))$$

is finite; we refer to τ_G as the *Tamagawa number* of the group G. Using this terminology, Theorem 1.26 asserts that $G = \mathrm{SO}_q$ has Tamagawa number 2, and Theorem 1.28 asserts that Spin_q has Tamagawa number 1. Motivated by this (another other examples), Weil conjectured the following:

Conjecture 1.29 (Weil's Conjecture on Tamagawa Numbers)**.** Let G be a simply-connected semisimple algebraic group over $\mathbf{Q}$. Then the Tamagawa number $\tau_G := \mu_{\mathrm{Tam}}(G(\mathbf{Q})\backslash G(\mathbf{A}))$ is equal to 1.

Conjecture 1.29 was proven by Weil in a number of special cases. The general case was proven by Langlands in the case of a split group ([12]), by Lai in the case of a quasi-split group ([10]), and by Kottwitz for arbitrary simply connected algebraic groups satisfying the Hasse principle ([9]; this is now known to be all simply-connected semisimple algebraic groups over $\mathbf{Q}$, by work of Chernousov).

2 Second Lecture: The Function Field Case

We now discuss the analogue of Conjecture 1.29 in the function field setting. First, we need to establish some terminology.

Notation 2.1. Let $\mathbf{F}_q$ denote a finite field with q elements, and let X be an algebraic curve over $\mathbf{F}_q$ (which we assume to be smooth, projective, and geometrically connected). We let K_X denote the function field of the curve X (that is, the residue field of the generic point of X). Fields which arise in this manner are known as *function fields*.

We will write $x \in X$ to mean that x is a *closed* point of the curve X. For each point $x \in X$, we let $\kappa(x)$ denote the residue field of X at the point x. Then $\kappa(x)$ is a finite extension of the finite field $\mathbf{F}_q$. We will denote the degree of this extension by $\deg(x)$ and refer to it as the *degree* of x. We let $\mathcal{O}_x$ denote the completion of the local ring of X at the point x: this is a complete discrete valuation ring with residue field $\kappa(x)$, noncanonically isomorphic to a power series ring $\kappa(x)[[t]]$. We let K_x denote the fraction field of $\mathcal{O}_x$.

We let $\mathbf{A}_X$ denote the *adele ring* of X, given by the subset $\mathbf{A}_X = \prod_{x \in X}^{\text{res}} K_x \subseteq \prod_{x \in X} K_x$ consisting of those tuples $\{f_x \in K_x\}_{x \in X}$ such that f_x belongs to $\mathcal{O}_x$ for all but finitely many values of x. Then $\mathbf{A}_X$ has a canonical topology with respect to which it is locally compact. It contains the product $\prod_{x \in X} \mathcal{O}_x$ as a compact open subring and the function field K_X as a discrete subring.

Let G_0 be a semisimple algebraic group over K_X. For every K_X-algebra R, we let $G_0(R)$ denote the group of R-valued points of G_0. The topology on $\mathbf{A}_X$ determines a topology on the group $G_0(\mathbf{A}_X)$. With respect to this topology, $G_0(\mathbf{A}_X)$ is a locally compact topological group, containing $G_0(K_X)$ as a discrete subgroup. Using an analogue of Construction 1.23, one can specify a canonical Haar measure μ_{Tam} on the group $G_0(\mathbf{A}_X)$, allowing us to define the *Tamagawa number*

$$\tau_{G_0} = \mu_{\text{Tam}}(G_0(K_X) \backslash G_0(\mathbf{A}_X)).$$

We can then formulate the function field analogue of Weil's conjecture as follows:

Conjecture 2.2 (Weil's Conjecture, Function Field Case)**.** Let X be an algebraic curve over a finite field $\mathbf{F}_q$ and let G_0 be a linear algebraic group over the function field K_X which is semisimple and simply connected. Then the Tamagawa number $\tau_{G_0} := \mu_{\text{Tam}}(G_0(K_X) \backslash G_0(\mathbf{A}_X))$ is equal to 1.

In the first lecture, we arrived at the problem of computing the Tamagawa number τ_G (in the special case $G = \mathrm{SO}_q$ for some quadratic space (V, q) over $\mathbf{Z}$) as a reformulation of counting the number of quadratic spaces in the genus of (V, q). Conjecture 2.2 admits a similar interpretation. To formulate it, we need to make an additional choice: a *group scheme* G over the entire curve X, having the algebraic group G_0 as its generic fiber. We will assume that the map $G \to X$ is smooth and affine, with connected fibers (this can always be arranged).

Warning 2.3. Let G be a smooth affine group scheme over X with generic fiber G_0. For each closed point $x \in X$, let G_x denote the fiber of G over the point x, which can

then be regarded as a linear algebraic group over the finite field $\kappa(x)$. Our assumption that G_0 is semisimple and simply connected guarantees that the algebraic groups G_x are also semisimple and simply connected with finitely many exceptions. However, it is not always possible to arrange that the fibers G_x are semisimple for *all* x: in general, there will be finitely many points of X at which the group scheme G has "bad reduction."

Note that since G_0 is an algebraic group over K_X, it can be evaluated on any K_X-algebra R to obtain a group $G_0(R)$ of R-valued points of G_0. Taking R to be the local field K_x (for any choice of closed point $x \in X$), we obtain a locally compact group $G_0(K_x)$. However, the choice of a group scheme G provides some additional structure: to any map of schemes $f : \mathrm{Spec}(R) \to X$, we can associate a group $G(R)$ of R-valued points of G. In the special case where f factors through the generic point $\mathrm{Spec}(K_X) \hookrightarrow X$, we have $G(R) = G_0(R)$; in particular, $G(K_x) = G_0(K_x)$ is the locally compact group defined above. However, we can also evaluate G on the local ring $\mathcal{O}_x$ to obtain a compact open subgroup $G(\mathcal{O}_x) \subseteq G(K_x)$, and on the residue field $\kappa(x)$ to obtain a finite group $G(\kappa(x))$ (which is a quotient of $G(\mathcal{O}_x)$). Note that the product

$$\prod_{x \in X} G(\mathcal{O}_x) \simeq G(\prod_{x \in X} \mathcal{O}_x)$$

can be regarded as a compact open subgroup of the adelic group $G(\mathbf{A}_X)$. We can then consider the collection of double cosets

$$G(K_X) \backslash G(\mathbf{A}_X) / G(\prod_{x \in X} \mathcal{O}_x).$$

This collection of double cosets can be described by the following analogue of Proposition 1.20:

Proposition 2.4.

$$\begin{array}{c} \{\text{Principal } G\text{-bundles on } X\}/\text{isomorphism} \\ \downarrow \sim \\ G(K_X) \backslash G(\mathbf{A}_X) / G(\prod_{x \in X} \mathcal{O}_x). \end{array}$$

Moreover, if $\mathcal{P}$ is a principal G-bundle on X which corresponds under this bijection to a $G(\prod_{x \in X} \mathcal{O}_x)$-orbit O on $G(K_X) \backslash G(\mathbf{A}_X)$, then the stabilizer group of O can be identified with the automorphism group $\mathrm{Aut}(\mathcal{P})$ of the principal G-bundle $\mathcal{P}$.

Proposition 2.4 is not entirely formal. It relies on two non-obvious facts about principal G-bundles on X:

- If $\mathcal{P}$ is a principal G-bundle on X, then $\mathcal{P}$ is trivial when restricted to every closed point $x \in X$ This follows from a theorem of Lang, which depends on our assumption that the fibers of G are connected; see [11].
- If $\mathcal{P}$ is a principal G-bundle on X, then $\mathcal{P}$ is trivial at the generic point of X. This is a theorem of Harder, which uses the assumption that G_0 is simply connected; it can be regarded a version of the Hasse principle for G-bundles (analogous to Theorem 1.5); see [8].

Using Proposition 2.4, we obtain the following interpretation of the Tamagawa number τ_{G_0}: it is given by

$$\tau_{G_0} = \mu_{\mathrm{Tam}}(G(K_X)\backslash G(\mathbf{A}_X)) = \mu_{\mathrm{Tam}}(\prod_{x \in X} G(\mathcal{O}_x)) \cdot \sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|};$$

here the sum is taken over all isomorphism classes of principal G-bundles on X (note that each of the automorphism groups $\operatorname{Aut}(\mathcal{P})$ is finite, by virtue of the fact that X is a proper curve over a finite field). Here the first factor $\mu_{\mathrm{Tam}}(\prod_{x \in X} G(\mathcal{O}_x))$ is easy to evaluate, because Tamagawa measure is essentially *defined* as a product of measures on the individual groups $G(K_x)$. (Of course, to actually evaluate it, we would need to spell out the definition of Tamagawa measure in more detail: see [7] for a discussion). This allows us to reformulate Conjecture 2.2 as follows:

Conjecture 2.5 (Weil's Conjecture, Mass Formula Version)**.** Let d be the dimension of the algebraic group G_0. Then we have an equality

$$\sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|} = q^D \prod_{x \in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|}.$$

Here the sum on the left hand side is taken over all isomorphism classes of principal G-bundles on X, and we have $D = d(g-1) - \deg(\mathfrak{g})$, where g is the genus of the curve X and $\mathfrak{g}$ is the vector bundle on X given by the Lie algebra of the group scheme G (we will give a better interpretation of the integer D below).

Warning 2.6. Neither the sum $\sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|}$ nor the product $\prod_{x \in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|}$ appearing in the statement of Conjecture 2.5 is finite. Hence part of the content of the conjecture is that both of these expressions are convergent.

The assertion of Conjecture 2.5 can be regarded as a function field version of the Smith-Minkowski-Siegel mass formula. More precisely, we have the following table of analogies:

Classical Mass Formula	Conjecture 2.5
Number field $\mathbf{Q}$	Function field K_X
Quadratic space $(V_{\mathbf{Q}}, q_{\mathbf{Q}})$ over $\mathbf{Q}$	Algebraic Group G_0
Quadratic space (V, q) over $\mathbf{Z}$	Group Scheme G
Quadratic space (V', q') of the same genus	Principal G-bundle $\mathcal{P}$
$\sum_{q'} \frac{1}{\lvert \mathrm{O}_{q'}(\mathbf{Z}) \rvert}$	$\sum_{\mathcal{P}} \frac{1}{\lvert \mathrm{Aut}(\mathcal{P}) \rvert}$.

Note however that there is reason to expect that Conjecture 2.5 should be easier to prove than Theorem 1.13, because function fields tend to be "easier" than number fields. More precisely, there are tools available for attacking Conjecture 2.5 that have no analogue in the number field case. If (V, q) is a quadratic space over $\mathbf{Z}$, then isomorphism classes of quadratic spaces in the genus of (V, q) form a set without any obvious additional structure. However, if G is a group scheme over X, then the collection of principal G-bundles admits an algebro-geometric parametrization.

Definition 2.7. If Y is a scheme equipped with a map $Y \to X$, we define a *G-bundle* on Y to be a principal homogeneous space for the group scheme $G_Y = Y \times_X G$ over Y. The collection of G-bundles on Y forms a category (in which all morphisms are isomorphisms).

For every $\mathbf{F}_q$-algebra R, we let $\mathrm{Bun}_G(X)$ denote the category of G-bundles on the relative curve $X_R = \mathrm{Spec}(R) \times_{\mathrm{Spec}(\mathbf{F}_q)} X$. The construction $R \mapsto \mathrm{Bun}_G(X)(R)$ is an example of an *algebraic stack*, which we will denote by $\mathrm{Bun}_G(X)$. We will refer to $\mathrm{Bun}_G(X)$ as the *moduli stack of G-bundles on X*.

Remark 2.8. The algebraic stack $\mathrm{Bun}_G(X)$ is smooth over $\mathbf{F}_q$, and its dimension is the integer $D = d(g-1) - \deg(\mathfrak{g})$ appearing in the statement of Conjecture 2.5. By definition, the category $\mathrm{Bun}_G(X)(\mathbf{F}_q)$ is the category of G-bundles on X. We will denote the sum $\sum_{\mathcal{P}} \frac{1}{\lvert \mathrm{Aut}(\mathcal{P}) \rvert}$ by $\lvert \mathrm{Bun}_G(X)(\mathbf{F}_q) \rvert$: we can think of it as a (weighted) count of the objects of $\mathrm{Bun}_G(X)(\mathbf{F}_q)$, which properly takes into account the fact that $\mathrm{Bun}_G(X)(\mathbf{F}_q)$ is a category rather than a set. With this notation, we can rephrase Conjecture 2.5 as an equality

$$\frac{\lvert \mathrm{Bun}_G(X)(\mathbf{F}_q) \rvert}{q^{\dim(\mathrm{Bun}_G)}} = \prod_{x \in X} \frac{\lvert \kappa(x) \rvert^d}{\lvert G(\kappa(x)) \rvert}.$$

Variant 2.9. For each closed point $x \in X$, let G_x denote the fiber $G \times_X \operatorname{Spec}(\kappa(x))$, which we regard as a linear algebraic group over $\kappa(x)$, and let BG_x denote the moduli stack of principal G_x-bundles. Since the fibers of G are connected, every G-bundle on $\operatorname{Spec}(\kappa(x))$ is trivial (by virtue of Lang's theorem), and the trivial G-bundle on $\operatorname{Spec}(\kappa(x))$ has automorphism group $G(\kappa(x))$. We may therefore write

$$\frac{|\kappa(x)|^d}{|G(\kappa(x))|} = \frac{|\operatorname{BG}_x(\kappa(x))|}{|\kappa(x)|^{\dim \mathrm{BG}_x}}.$$

Conjecture 2.5 can therefore be written in the more suggestive form

$$\frac{|\operatorname{Bun}_G(X)(\mathbf{F}_q)|}{q^{\dim \operatorname{Bun}_G(X)}} = \prod_{x \in X} \frac{|\operatorname{BG}_x(\kappa(x))|}{|\kappa(x)|^{\dim \mathrm{BG}_x}}.$$

The problem of counting the number of points on algebraic varieties over $\mathbf{F}_q$ is the subject of another very famous idea of Weil. Let Y be a quasi-projective variety over $\mathbf{F}_q$, so that there exists an embedding $j : Y \hookrightarrow \mathbf{P}^n_{\mathbf{F}_q}$ for some $n \geqslant 0$. Set $\overline{Y} = \operatorname{Spec}(\overline{\mathbf{F}}_q) \times_{\operatorname{Spec}(\mathbf{F}_q)} Y$, so that j determines an embedding $\overline{j} : \overline{Y} \to \mathbf{P}^n_{\overline{\mathbf{F}}_q}$. There is a canonical map from $\mathbf{P}^n_{\overline{\mathbf{F}}_q}$ to itself, given in homogeneous coordinates by

$$[x_0 : \cdots : x_n] \mapsto [x_0^q : \cdots x_n^q].$$

Since $\overline{Y}$ is characterized by the vanishing and nonvanishing of homogeneous polynomials with coefficients in $\mathbf{F}_q$, this construction determines a map $\varphi : \overline{Y} \to \overline{Y}$, which we will refer to as the *geometric Frobenius map*. Then the finite set $Y(\mathbf{F}_q)$ of $\mathbf{F}_q$-points of Y can be identified with the *fixed locus* of the map $\varphi : \overline{Y} \to \overline{Y}$. Weil made a series of conjectures about the behavior of the integers $|Y(\mathbf{F}_q)|$, which are informed by the following heuristic:

Idea 2.10 (Weil)**.** There exists a good cohomology theory for algebraic varieties having the property that the fixed point set $Y(\mathbf{F}_q) = \{y \in \overline{Y}(\overline{\mathbf{F}}_q) : \varphi(y) = y\}$ is given by the Lefschetz fixed point formula

$$Y(\mathbf{F}_q) = \sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(\varphi | \operatorname{H}^i_c(\overline{Y})).$$

Weil's conjectures were eventually proven by Grothendieck and Deligne by introducing the theory of *ℓ-adic cohomology*, which we will denote by $\overline{Y} \mapsto \operatorname{H}^*(\overline{Y}; \mathbf{Q}_\ell)$ (there is also an analogue of ℓ-adic cohomology with compact supports which appears in the fixed point formula above, which we will denote by $\operatorname{H}^*_c(\overline{Y}; \mathbf{Q}_\ell)$). Here ℓ denotes some fixed prime number which does not divide q.

For our purposes, it will be convenient to write the Grothendieck-Lefschetz trace formula in a slightly different form. Suppose that Y is a smooth variety of dimension n

over $\mathbf{F}_q$. Then, from the perspective of ℓ-adic cohomology, Y behaves as if it were a smooth manifold of dimension $2n$. In particular, it satisfies Poincare duality: that is, there is a perfect pairing

$$\mu : \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell) \to \mathbf{Q}_\ell \,.$$

This pairing is not quite φ-equivariant: instead, it fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell) \arrow[r, "\mu"] \arrow[d, "\varphi\otimes\varphi"] & \mathbf{Q}_\ell \arrow[d, "q^n"] \\
\mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell) \arrow[r, "\mu"] & \mathbf{Q}_\ell \,.
\end{tikzcd}$$

It follows formally that

$$q^{-n} \operatorname{Tr}(\varphi | \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell)) \simeq \operatorname{Tr}(\varphi^{-1} | \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell)).$$

We may therefore rewrite the Grothendieck-Lefschetz trace formula as an equality

$$\sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(\varphi^{-1} | \mathrm{H}^i(\overline{Y}; \mathbf{Q}_\ell)) = \frac{|Y(\mathbf{F}_q)|}{q^{\dim(Y)}}.$$

This suggests the possibility of breaking Conjecture 2.5 into two parts.

Notation 2.11. In what follows, we fix an embedding $\mathbf{Q}_\ell \hookrightarrow \mathbf{C}$, so that we can regard the trace of any endomorphism of a $\mathbf{Q}_\ell$-vector space as a complex number. If Y is an algebraic stack over a finite field $\mathbf{F}_q$, we write $\overline{Y}$ for the fiber product $\operatorname{Spec}(\overline{\mathbf{F}}_q) \times_{\operatorname{Spec}(\mathbf{F}_q)} Y$, and we write $\operatorname{Tr}(\varphi^{-1} | \mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell))$ for the alternating sum

$$\sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(\varphi^{-1} | \mathrm{H}^i(\overline{Y}; \mathbf{Q}_\ell).$$

Conjecture 2.12. The algebraic stacks $\mathrm{Bun}_G(X)$ and BG_x satisfy the Grothendieck-Lefschetz trace formulae

$$\frac{|\mathrm{Bun}_G(X)(\mathbf{F}_q)|}{q^{\dim(\mathrm{Bun}_G(X))}} = \operatorname{Tr}(\varphi^{-1} | \mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Q}_\ell)).$$

$$\frac{|\mathrm{BG}_x(\kappa(x))|}{|\kappa(x)|^{\dim \mathrm{BG}_x}} = \operatorname{Tr}(\varphi_x^{-1} | \mathrm{H}^*(\overline{\mathrm{BG}}_x; \mathbf{Q}_\ell)).$$

Conjecture 2.13. There is an equality

$$\operatorname{Tr}(\varphi^{-1} | \mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Q}_\ell)) = \prod_{x \in X} \operatorname{Tr}(\varphi_x^{-1} | \mathrm{H}^*(\overline{\mathrm{BG}}_x; \mathbf{Q}_\ell)).$$

Warning 2.14. The ℓ-adic cohomology rings $\mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Q}_\ell)$ and $\mathrm{H}^*(\mathrm{BG}_x; \mathbf{Q}_\ell)$ are typically nonzero in infinitely many degrees, so the sum

$$\sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(\varphi^{-1} | \mathrm{H}^i(\overline{\mathrm{Bun}}_G(X); \mathbf{Q}_\ell))$$

appearing in Conjectures 2.12 and 2.13 is generally infinite. Consequently, the convergence of this sum is not automatic, but part of the substance of the conjecture.

The proofs of Conjecture 2.12 and 2.13 involve completely different ideas. In the case where G has good reduction (that is, every fiber G_x is a semisimple algebraic group over $\kappa(x)$), the Grothendieck-Lefschetz trace formula for $\mathrm{Bun}_G(X)$ was proved by Behrend in [4]. This is extended to the case of bad reduction in [7] (by a somewhat ad-hoc argument). The Grothendieck-Lefschetz trace formula for BG_x is essentially a formula for the order of the finite group $G(\kappa(x))$ which we will discuss in the next lecture. Along the way, we will develop some ideas which can be applied to Conjecture 2.13 as well.

3 Third Lecture: Steinberg's Formula

Let Y be an algebraic variety over the finite field $\mathbf{F}_q$, and set $\overline{Y} = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} Y$. In the previous lecture, we discussed the *Grothendieck-Lefschetz fixed point formula*, which asserts that the number of $\mathbf{F}_q$-valued points of Y is given by the trace

$$\operatorname{Tr}(\varphi | \mathrm{H}^*_c(\overline{Y}; \mathbf{Q}_\ell)) := \sum_{n \geqslant 0} (-1)^n \operatorname{Tr}(\varphi | \mathrm{H}^n_c(\overline{Y}; \mathbf{Q}_\ell)).$$

In the case where Y is smooth, this formula can be rewritten as

$$\frac{|Y(\mathbf{F}_q)|}{q^{\dim(Y)}} = \operatorname{Tr}(\varphi^{-1} | \mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell)).$$

In this lecture, we will discuss a variation on this theme, which relates the number of $\mathbf{F}_q$-valued points of Y to the (ℓ-adic) *homotopy groups* of $\overline{Y}$, rather than to its cohomology.

We begin with some general remarks. Let $\overline{Y}$ be any scheme, and let y denote a *geometric point* of $\overline{Y}$: that is, a map $y : \mathrm{Spec}(k) \to \overline{Y}$, where k is a separably closed field. To the pair $(\overline{Y}, y)$, Grothendieck associated a profinite group $\pi_1(\overline{Y}, y)$, called the *étale fundamental group of* $\overline{Y}$ (with base point y). Assume (for simplicity) that $\overline{Y}$ is connected. Then the fundamental group $\pi_1(\overline{Y}, y)$ is essentially characterized by the

requirement that there is an equivalence of categories

$$\begin{array}{c} \{\text{Finite sets with continuous action of } \pi_1(\overline{Y}, y)\ \} \\ \downarrow \sim \\ \{\text{Finite étale covers of } \overline{Y}\}. \end{array}$$

In what follows, it will be convenient to focus our attention on a particular prime number ℓ; we let $\pi_1(\overline{Y}, y)_\ell$ denote the maximal pro-ℓ quotient of $\pi_1(\overline{Y}, y)$ (that is, the quotient $\pi_1(\overline{Y}, y)/N$, where N is the smallest closed normal subgroup of $\pi_1(\overline{Y}, y)$ for which $\pi_1(\overline{Y}, y)/N$ is a pro-ℓ group).

In [2], Artin and Mazur introduced a refinement of the étale fundamental group. To any scheme $\overline{Y}$, they associate an invariant called the *étale homotopy type* of $\overline{Y}$. To simplify the discussion, let us fix a prime number ℓ and assume that $\overline{Y}$ satisfies the following conditions:

($*$) The scheme $\overline{Y}$ is connected, the étale cohomology groups $\mathrm{H}^n(\overline{Y}; \mathbf{Z}/\ell\mathbf{Z})$ are finite for every integer n, and $\mathrm{H}^1(\overline{Y}; \mathbf{Z}/\ell\mathbf{Z})$ vanishes (this last condition is equivalent to the vanishing of the group $\pi_1(\overline{Y}, y)_\ell$, for any choice of geometric point y; that is, it asserts that $\overline{Y}$ is "ℓ-adically simply connected").

In this case, the ℓ-adic version of the Artin-Mazur construction produces a topological space T (or, if you prefer, a simplicial set) with the following properties:

(a) The space T is simply connected, and the homotopy groups of T are finitely generated $\mathbf{Z}_\ell$-modules.

(b) There is a canonical isomorphism of cohomology rings

$$\mathrm{H}^*(T; \mathbf{Z}/\ell\mathbf{Z}) \simeq \mathrm{H}^*(\overline{Y}; \mathbf{Z}/\ell\mathbf{Z}),$$

where the left hand side is denotes the singular cohomology of the topological space T, and the right hand side indicates the étale cohomology of the scheme $\overline{Y}$ (and similarly for coefficients in $\mathbf{Z}/\ell^m\mathbf{Z}$, $\mathbf{Z}_\ell$, or $\mathbf{Q}_\ell$).

Moreover, the space T is essentially characterized (up to weak homotopy equivalence) by (a) together with a stronger form of condition (b). We refer to T as the *ℓ-adic homotopy type* of $\overline{Y}$. For every integer n, we let $\pi_n(\overline{Y})_\ell$ denote the homotopy group $\pi_n(T)$ (note that since T is simply connected, the group $\pi_n(T)$ is canonically independent of the choice of base point). By virtue of (a), this is a free $\mathbf{Z}_\ell$-module of finite rank. We let $\pi_n(\overline{Y})_{\mathbf{Q}_\ell}$ denote the $\mathbf{Q}_\ell$-vector space given by $\pi_n(\overline{Y})_\ell[\ell^{-1}]$.

We will refer to the vector spaces $\pi_n(\overline{Y})_{\mathbf{Q}_\ell}$ as the *rational ℓ-adic homotopy groups of* $\overline{Y}$. They are related to the ℓ-adic cohomology groups of $\overline{Y}$. Recall that, for any topological space T (and any choice of base point $t \in T$), we have a canonical homomorphism

$$\pi_n(T, t) \to \mathrm{H}_n(T; \mathbf{Z}_\ell)$$

called the *Hurewicz map*. Combining this with the canonical pairing $\mathrm{H}^n(T; \mathbf{Z}_\ell) \times \mathrm{H}_n(T; \mathbf{Z}_\ell) \to \mathbf{Z}_\ell$ we obtain a bilinear map

$$\mathrm{H}^n(T; \mathbf{Z}_\ell) \times \pi_n(T, t) \to \mathbf{Z}_\ell.$$

Specializing to the case where T is the ℓ-adic homotopy type of $\overline{Y}$ and inverting the prime number ℓ, we get a bilinear map of $\mathbf{Q}_\ell$-vector spaces

$$b_n : \mathrm{H}^n(\overline{Y}; \mathbf{Q}_\ell) \times \pi_n(\overline{Y})_{\mathbf{Q}_\ell} \to \mathbf{Q}_\ell\,.$$

Summing over all $n > 0$, we obtain a pairing of graded $\mathbf{Q}_\ell$-vector spaces

$$b : \mathrm{H}^*_{\mathrm{red}}(\overline{Y}; \mathbf{Q}_\ell) \times \pi_*(\overline{Y})_{\mathbf{Q}_\ell} \to \mathbf{Q}_\ell\,.$$

Here $\mathrm{H}^*_{\mathrm{red}}(\overline{Y}; \mathbf{Q}_\ell) = \bigoplus_{n>0} \mathrm{H}^n(\overline{Y}; \mathbf{Q}_\ell)$ denotes the *reduced* ℓ-adic cohomology of $\overline{Y}$. Let us denote $\mathrm{H}^*_{\mathrm{red}}(\overline{Y}; \mathbf{Q}_\ell)$ by I, and view it as an ideal in the ring $\mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell)$. Then the square I^2 is contained in the kernel of the bilinear pairing b. In other words, for $m, n > 0$, every triple of elements $x \in \mathrm{H}^m(\overline{Y}; \mathbf{Q}_\ell)$, $y \in \mathrm{H}^n(\overline{Y}; \mathbf{Q}_\ell)$, $z \in \pi_{m+n}(\overline{Y})_{\mathbf{Q}_\ell}$, we have

$$b(x \cup y, z) = 0 \in \mathbf{Q}_\ell\,.$$

Passing back to topology, it is easy to see why this should be true: we can assume without loss of generality that z comes from a homotopy class of continuous maps $f : S^{m+n} \to T$. In this case, $b(x \cup y, z)$ is obtained by viewing $x \cup y$ a cohomology class on T, pulling it back to the sphere S^{m+n}, and "integrating" (by evaluating on a generator of the group $\mathrm{H}_{m+n}(S^{m+n}; \mathbf{Z}) \simeq \mathbf{Z}$). However, the pullback $f^*(x \cup y) = f^*(x) \cup f^*(y)$ is zero, since the sphere S^{m+n} has vanishing cohomology in degrees m and n. It follows that b restricts to a bilinear map

$$I/I^2 \times \pi_*(\overline{Y})_{\mathbf{Q}_\ell} \to \mathbf{Q}_\ell\,.$$

This pairing is a bit more useful, by virtue of the following:

Proposition 3.1. *Let $\overline{Y}$ satisfy condition $(*)$, suppose that the cohomology ring $\mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell)$ is a polynomial ring (on generators of even degree), and let $I = \mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell)$ denote the augmentation ideal. Then the preceding construction gives a* perfect *pairing*

$$I/I^2 \times \pi_*(\overline{Y})_{\mathbf{Q}_\ell} \to \mathbf{Q}_\ell\,.$$

In other words, it identifies $\pi_*(\overline{Y})_{\mathbf{Q}_\ell}$ with the dual of I/I^2 (as a graded $\mathbf{Q}_\ell$-vector space).

At a first glance, Proposition 3.1 would appear to be useless: its hypotheses are essentially never satisfied. For example, if $\overline{Y}$ is an algebraic variety over an algebraically closed field k, then the ℓ-adic cohomology groups groups $\mathrm{H}^n(\overline{Y}; \mathbf{Q}_\ell)$ vanish for $n > 2\dim(\overline{Y})$; in particular, $\mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell)$ can never be a polynomial ring (except in the trivial case $\mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell) \simeq \mathbf{Q}_\ell$). However, the entirety of the preceding discussion can be extended to algebro-geometric objects other than schemes: for example, we can apply it when $\overline{Y}$ is an algebraic stack. In this case, we can find many interesting examples Proposition 3.1 applies.

Example 3.2. Let G be a connected linear algebraic group over a field k, let BG be the classifying stack of G, and set $\overline{\mathrm{BG}} = \mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} \mathrm{BG}$ where $\overline{k}$ is an algebraic closure of k. If ℓ is invertible in k, then the polynomial ring $\mathrm{H}^*(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)$ is a polynomial ring in r generators, where r is the reductive rank of the group G. It follows that Proposition 3.1 supplies a duality of $\pi_*(\overline{\mathrm{BG}})_{\mathbf{Q}_\ell}$ with I/I^2.

To illustrate this more concretely, let us specialize to the case where $G = \mathrm{SL}_2$. Then G can be viewed as an algebraic group over *any* field k (or even as a *group scheme* over the integers). Let us write G_k for the group SL_2, viewed as an algebraic group over k, and let BG_k denote its classifying stack. One can show that if k is algebraically closed, then the ℓ-adic cohomology $\mathrm{H}^*(\mathrm{BG}_k; \mathbf{Q}_\ell)$ does not depend on k, as long as ℓ is invertible in k (in fact, this is even true for the ℓ-adic homotopy type of BG_k). Consequently, there is no loss of generality in assuming that $k = \mathbf{C}$ is the field of complex numbers. In this case, one can show that the cohomology ring of the *classifying stack* BG_k (in the sense of algebraic geometry) is the same as the cohomology of the *classifying space* of the topological group $G(\mathbf{C}) = \mathrm{SL}_2(\mathbf{C})$ of complex points of $\mathbf{C}$ (in the sense of topology). Here, there is no harm in replacing $\mathrm{SL}_2(\mathbf{C})$ with its maximal compact subgroup $\mathrm{SU}(2)$, which we will identify with the group of *unit quaternions* $\{x \in \mathbf{H} : |x| = 1\}$. The classifying space $\mathrm{BSU}(2)$ can then be defined as the quotient $E/\mathrm{SU}(2)$, where E is a contractible space equipped with a free action of $\mathrm{SU}(2)$. For each $n \geqslant 0$, let $\mathbf{HP}^n$ denote projective space of dimension n over $\mathbf{H}$, given concretely by the quotient

$$(\mathbf{H}^{n+1} - \{0\})/\mathbf{H}^\times \simeq \{(x_0, \ldots, x_n) \in \mathbf{H}^{n+1} : |x_0|^2 + \cdots + |x_n|^2 = 1\}/\mathrm{SU}(2).$$

The description on the right exhibits the sphere S^{2n+3} as a principal $\mathrm{SU}(2)$-bundle over $\mathbf{HP}^n$.

Identifying each $\mathbf{H}^n$ with the subspace $\{(x_0, \ldots, x_n) \in \mathbf{H}^{n+1} : x_n = 0\} \subseteq \mathbf{H}^{n+1}$, we obtain a compatible system of principal $\mathrm{SU}(2)$-bundles

$$\begin{tikzcd}
S^3 \arrow[r] \arrow[d] & S^7 \arrow[r] \arrow[d] & S^{11} \arrow[r] \arrow[d] & \cdots \\
\mathbf{HP}^0 \arrow[r] & \mathbf{HP}^1 \arrow[r] & \mathbf{HP}^2 \arrow[r] & \cdots
\end{tikzcd}$$

Setting $\mathbf{HP}^\infty = \varinjlim \mathbf{HP}^n$, we obtain a principal SU(2)-bundle $E \to \mathbf{HP}^\infty$, where $E \simeq \varinjlim S^{4n+3}$ is contractible. It follows that $\mathbf{HP}^\infty$ is a classifying space for the compact Lie group SU(2), so we have isomorphisms

$$\mathrm{H}^*(\mathrm{BG}_k; \mathbf{Q}_\ell) \simeq \mathrm{H}^*(\mathbf{HP}^\infty; \mathbf{Q}_\ell)$$

The group on the right is easy to compute: it is a polynomial ring $\mathbf{Q}_\ell[t]$, where t is a class in degree 4. One can think of t as playing the role of the *second Chern class*. If X is any k-scheme equipped with a principal SL_2-bundle $\mathscr{E}$ (which we can view as a vector bundle of rank 2 with a trivialization of its determinant), then $\mathscr{E}$ determines a map of algebraic stacks $X \to \mathrm{BG}_k$, hence a map of cohomology rings

$$\mathbf{Q}_\ell[t] \simeq \mathrm{H}^*(\mathrm{BG}_k; \mathbf{Q}_\ell) \to \mathrm{H}^*(X; \mathbf{Q}_\ell),$$

which (for appropriate normalizations) carries t to the Chern class $c_2(\mathscr{E}) \in \mathrm{H}^4(X; \mathbf{Q}_\ell)$.

Remark 3.3. In Example 3.2, we are being a bit sloppy: the second Chern class of the vector bundle $\mathscr{E}$ should really be an element of the cohomology group $\mathrm{H}^4(X; \mathbf{Q}_\ell(2))$, where $\mathbf{Q}_\ell(2)$ denotes the second Tate twist of $\mathbf{Q}_\ell$. In particular, if we regard $G = \mathrm{SL}_2$ as an algebraic group over the finite field $\mathbf{F}_q$, then the geometric Frobenius φ acts on the ℓ-adic cohomology ring $\mathrm{H}^*(\overline{\mathrm{BG}}; \mathbf{Q}_\ell) \simeq \mathbf{Q}_\ell[t]$ by the formula $\varphi(t) = q^2 t$. We therefore have

$$\begin{aligned} \mathrm{Tr}(\varphi^{-1} | \, \mathrm{H}^*(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)) &= \sum_{n \geqslant 0} \mathrm{Tr}(\varphi^{-1} | \, \mathrm{H}^{4n}(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)) \\ &= \sum_{n \geqslant 0} q^{-2n} \\ &= \frac{1}{1 - q^{-2n}}. \end{aligned}$$

Consequently, the Grothendieck-Lefschetz trace formula for BG asserts that we have an identity

$$\frac{1}{1 - q^{-2n}} = \frac{q^3}{|\, \mathrm{SL}_2(\mathbf{F}_q)|},$$

or equivalently $|\, \mathrm{SL}_2(\mathbf{F}_q)| = q^3 - q$ (which is easy to verify directly).

Let us now generalize the preceding discussion. Let G be a connected linear algebraic group over a finite field $\mathbf{F}_q$, set $\overline{\mathrm{BG}} = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{BG}$, and let $I = \mathrm{H}^*_{\mathrm{red}}(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)$ so that Example 3.2 supplies a duality of graded $\mathbf{Q}_\ell$-vector spaces between I/I^2 and $\pi_*(\overline{\mathrm{BG}})_{\mathbf{Q}_\ell}$. The *Grothendieck-Lefschetz trace formula for* BG asserts that we have an equality

$$\mathrm{Tr}(\varphi^{-1} | \, \mathrm{H}^*(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)) := \sum_{i \geqslant 0} (-1)^i \, \mathrm{Tr}(\varphi^{-1} | \, \mathrm{H}^i(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)) = \frac{|\, \mathrm{BG}(\mathbf{F}_q)|}{q^{\dim(\mathrm{BG})}},$$

where $|\operatorname{BG}(\mathbf{F}_q)|$ denotes the sum $\sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|}$, taken over all isomorphism classes of principal G-bundles $\mathcal{P}$ on $\operatorname{Spec}(\mathbf{F}_q)$. If G is connected, then every principal G-bundle on $\operatorname{Spec}(\mathbf{F}_q)$ is trivial (by a theorem of Lang [11]), so this sum has only one term: the fraction $\frac{1}{|G(\mathbf{F}_q)|}$. We can therefore rewrite this equality as

$$\operatorname{Tr}(\varphi^{-1}|\operatorname{H}^*(\overline{\operatorname{BG}};\mathbf{Q}_\ell)) = \frac{q^{\dim(G)}}{|G(\mathbf{F}_q)|}.$$

Here we have fixed an embedding $\mathbf{Q}_\ell \hookrightarrow \mathbf{C}$. Let $\lambda_1, \ldots, \lambda_r$ be the complex numbers given by the generalized eigenvalues of φ on $\pi_*(\operatorname{BG}_{\overline{\mathbf{F}}_q})_{\mathbf{Q}_\ell}$ (listed with multiplicity). These are then the same as the generalized eigenvalues of φ^{-1}, acting on the quotient I/I^2. Since $\operatorname{H}^*(\overline{\operatorname{BG}};\mathbf{Q}_\ell)$ is a polynomial ring, the generalized eigenvalues of φ^{-1} on $\operatorname{H}^*(\overline{\operatorname{BG}};\mathbf{Q}_\ell)$ are the same as the generalized eigenvalues of φ on the symmetric algebra $\operatorname{Sym}^*(I/I^2)$: that is, they are given by $\{\lambda_1^{e_1}\cdots\lambda_r^{e_r}\}_{e_1,\cdots,e_r\geqslant 0}$. We can therefore compute

$$\begin{aligned}\operatorname{Tr}(\varphi^{-1}|\operatorname{H}^*(\overline{\operatorname{BG}};\mathbf{Q}_\ell) &= \sum_{e_1,\ldots,e_r\geqslant 0} \lambda_1^{e_1}\cdots\lambda_r^{e_r} \\ &= \prod_{1\leqslant i\leqslant r} \frac{1}{1-\lambda_i} \\ &= \det(1-\varphi|\pi_*(\overline{\operatorname{BG}})_{\mathbf{Q}_\ell})^{-1};\end{aligned}$$

here the calculation depends on the fact that each of the eigenvalues λ_i has complex absolute value < 1 (otherwise, the trace on the left hand side would not be well-defined). We can therefore restate the Grothendieck-Lefschetz trace formula for BG as follows:

Proposition 3.4 (Steinberg's Formula)**.** *Let G be a connected linear algebraic group over $\mathbf{F}_q$. Then the order of the finite group $G(\mathbf{F}_q)$ is given by*

$$q^{\dim(G)}\det(1-\varphi|\pi_*(\overline{\operatorname{BG}})_{\mathbf{Q}_\ell}).$$

Remark 3.5. In the statement of Proposition 3.4, we do not assume that G is reductive. However, there is no real loss in restricting to the reductive case: a general connected linear algebraic group G fits into a short exact sequence

$$0 \to U \to G \to G/U \to 0$$

where U is unipotent and G/U is reductive. We then have a short exact sequence of finite groups

$$0 \to U(\mathbf{F}_q) \to G(\mathbf{F}_q) \to (G/U)(\mathbf{F}_q) \to 0$$

(this follows from Lang's theorem), hence an equality

$$|G(\mathbf{F}_q)| = |(G/U)(\mathbf{F}_q)|\cdot|U(\mathbf{F}_q)| = |(G/U)(\mathbf{F}_q)|\cdot q^{\dim(U)}.$$

Moreover, replacing G by G/U does not change the ℓ-adic homotopy groups $\pi_*(\mathrm{BG}_{\overline{\mathbf{F}}_q})_{\mathbf{Q}_\ell}$, so that the validity of Proposition 3.4 for the group G follows from its validity for the reductive quotient G/U.

Let us now suppose that $\overline{Y}$ is an algebraic stack satisfying condition $(*)$ above, but whose ℓ-adic cohomology $\mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell)$ is not necessarily a polynomial ring. In this case, the relationship between the homotopy and cohomology of $\overline{Y}$ is more subtle. In place of the isomorphism $\mathrm{Sym}^*(\pi_*(\overline{Y})^\vee_{\mathbf{Q}_\ell}) \simeq \mathrm{gr}(\mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell))$, one has a *spectral sequence*

$$\mathrm{Sym}^*(\pi_*(\overline{Y})^\vee_{\mathbf{Q}_\ell}) \Rightarrow \mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell),$$

whose second page is the (graded) symmetric algebra on the dual of $\pi_*(\overline{Y})_{\mathbf{Q}_\ell}$ and whose last page is the associated graded for a suitable filtration on $\mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell)$ (not necessarily the I-adic filtration for $I = \mathrm{H}^*_{\mathrm{red}}(Y;\mathbf{Q}_\ell)$). The existence of this spectral sequence is good enough to draw many of the conclusions mentioned above. Suppose that $\overline{Y} = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} Y$ for some algebraic stack Y over a finite field $\mathbf{F}_q$, so that the cohomology ring $\mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell)$ and the $\mathbf{Q}_\ell$-homotopy groups $\pi_*(\overline{Y})_{\mathbf{Q}_\ell}$ are equipped with a geometric Frobenius automorphism φ. Assume the following:

$(*')$ The homotopy groups $\pi_n(\overline{Y})_{\mathbf{Q}_\ell}$ vanish for $n \gg 0$, and every eigenvalue of φ on $\pi_*(\overline{Y})_{\mathbf{Q}_\ell}$ has complex absolute value < 1.

One can then calculate

$$\begin{aligned}
\mathrm{Tr}(\varphi^{-1}|\,\mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell)) &= \mathrm{Tr}(\varphi^{-1}|\,\mathrm{gr}(\mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell))) \\
&= \mathrm{Tr}(\varphi^{-1}|E^{*,*}_\infty) \\
&= \mathrm{Tr}(\varphi^{-1}|E^{*,*}_2) \\
&= \mathrm{Tr}(\varphi^{-1}|\,\mathrm{Sym}^*(\pi_*(\overline{Y})^\vee_{\mathbf{Q}_\ell}) \\
&= \det(1-\varphi|\pi_*(\overline{Y})_{\mathbf{Q}_\ell})^{-1}
\end{aligned} \tag{3}$$

where the last term is shorthand for the alternating product

$$\prod_{n\geqslant 0} \det(1-\varphi|\pi_n(\overline{Y})_{\mathbf{Q}_\ell})^{(-1)^{n+1}}$$

Consequently, if $\overline{Y}$ is a smooth algebraic stack, then the Grothendieck-Lefschetz trace formula for $\overline{Y}$ (in the sense of the previous lecture) can be written as an equality

$$\frac{|Y(\mathbf{F}_q)|}{q^{\dim(Y)}} = \det(1-\varphi|\pi_*(\overline{Y})_{\mathbf{Q}_\ell})^{-1}.$$

Warning 3.6. If $\overline{Y}$ is an ordinary algebraic variety over $\overline{\mathbf{F}}_q$, the assumption that $\pi_n(\overline{Y})_{\mathbf{Q}_\ell}$ vanishes for $n \gg 0$ is usually not satisfied: this is an algebro-geometric analogue of requiring that $\overline{Y}$ is "elliptic" in the sense of rational homotopy theory.

Example 3.7. Let $\mathbf{CP}^n$ be complex projective space of dimension n, regarded as a topological space. Then there are two interesting elements of the homotopy of $\mathbf{CP}^n$:

- A class $x \in \pi_2(\mathbf{CP}^n)$, represented by the inclusion $S^2 \simeq \mathbf{CP}^1 \hookrightarrow \mathbf{CP}^n$.
- A class $y \in \pi_{2n+1}(\mathbf{CP}^n)$, represented by the Hopf map $S^{2n+1} \to \mathbf{CP}^n$ (whose cone is the projective space $\mathbf{CP}^{n+1}$).

One can show that x and y form a basis for the $\mathbf{Q}$-vector space $\pi_*(\mathbf{CP}^n)_{\mathbf{Q}}$.

The situation in algebraic geometry is completely analogous. If k is any algebraically closed field (and ℓ is a prime number which does not vanish in k), then the $\mathbf{Q}_\ell$-homotopy of the algebraic variety $\mathbf{P}^n_k$ is a vector space of dimension 2, generated by classes

$$x \in \pi_2(\mathbf{P}^n_k)_{\mathbf{Q}_\ell} \qquad y \in \pi_{2n+1}(\mathbf{P}^n_k)_{\mathbf{Q}_\ell}.$$

Moreover, when $k = \overline{\mathbf{F}_q}$ is the algebraic closure of a finite field $\mathbf{F}_q$, then the geometric Frobenius acts on $\pi_*(\mathbf{P}^n_k)_{\mathbf{Q}_\ell}$ by the formula

$$\varphi(x) = \frac{x}{q} \qquad \varphi(y) = \frac{y}{q^{n+1}}.$$

Consequently, the Grothendieck-Lefschetz trace formula can be written as

$$\begin{aligned} \frac{|\mathbf{P}^n(\mathbf{F}_q)|}{q^n} &= \det(1 - \varphi | \pi_*(\mathbf{P}^n_k)_{\mathbf{Q}_\ell})^{-1} \\ &= \frac{\det(1 - \varphi | \pi_{2n+1}(\mathbf{P}^n_k)_{\mathbf{Q}_\ell})}{\det(1 - \varphi | \pi_2(\mathbf{P}^n_k))} \\ &= \frac{1 - q^{-n-1}}{1 - q^{-1}} \\ &= \frac{1}{q^n} \cdot \frac{q^{n+1} - 1}{q - 1}. \end{aligned}$$

Multiplying both sides by q^n, we obtain the equality $|\mathbf{P}^n(\mathbf{F}_q)| = \frac{q^{n+1}-1}{q-1} = 1 + q + \cdots + q^n$.

4 Fourth Lecture: The Product Formula

Let X be an algebraic variety defined over a finite field $\mathbf{F}_q$. Recall that the *zeta function of* X is a meromorphic function $\zeta_X(s)$ on the complex plane, which for $\Re(s) \gg 0$ is given by the formula

$$\zeta_X(s) = \prod_{x \in X} \frac{1}{1 - |\kappa(x)|^{-s}},$$

where the product is taken over all closed points $x \in X$ (so that the residue field $\kappa(x)$ is a finite extension of $\mathbf{F}_q$). Using the Grothendieck-Lefschetz fixed point formula, one can give an alternate formula for $\zeta_X(s)$ in terms of the action of the geometric Frobenius automorphism on the ℓ-adic cohomology of $\overline{X} = X \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q)$: it is given by

$$\zeta_X(s) = \det(1 - q^{-s}\varphi | \operatorname{H}^*_c(\overline{X}; \mathbf{Q}_\ell))^{-1} := \prod_{n \geqslant 0} \det(1 - q^{-s}\varphi | \operatorname{H}^n_c(\overline{X}; \mathbf{Q}_\ell))^{(-1)^{n+1}}.$$

The zeta function $\zeta_X(s)$ has a generalization. Let $\mathscr{F}$ be an ℓ-adic sheaf on X. For each closed point $x \in X$, we let $\mathscr{F}_{\overline{x}}$ denote the stalk of $\mathscr{F}$ at the geometric point $\overline{x} : \mathrm{Spec}(\overline{\kappa(x)}) \to X$ determined by a choice of algebraic closure $\overline{\kappa(x)}$ of the residue field $\kappa(x)$. Then $\mathscr{F}_{\overline{x}}$ is a finite-dimensional $\mathbf{Q}_\ell$-vector space equipped with a local Frobenius operator φ_x. The L-function $L_{X,\mathscr{F}}(s)$ is then a meromorphic function of a complex variable s, whose value for $\Re(s) \gg 0$ is given by the formula

$$L_{X,\mathscr{F}}(s) = \prod_{x \in X} \det(1 - |\kappa(x)|^{-s}\varphi_x | \mathscr{F}_{\overline{x}})^{-1}.$$

By virtue of the Grothendieck-Lefschetz trace formula, this can also be written as

$$L_{X,\mathscr{F}}(s) = \prod_{n \geqslant 0} \det(1 - q^{-s}\varphi | \operatorname{H}^n_c(\overline{X}; \mathscr{F}))^{(-1)^{n+1}}.$$

We will need a slightly more general version of this construction. Let $D^b(X)$ denote the *bounded derived category* of ℓ-adic sheaves on X. Then every object $\mathscr{F} \in D^b(X)$ has *cohomology sheaves* $\mathcal{H}^i(\mathscr{F})$, which are ℓ-adic sheaves on X in the usual sense. Moreover, these sheaves vanish for almost every integer i (since we are working with the *bounded* derived category). We can then define a meromorphic function $L_{X,\mathscr{F}}(s)$ by the formula $L_{X,\mathscr{F}}(s) = \prod_{i \in \mathbf{Z}} L_{X,\mathcal{H}^i(\mathscr{F})}(s)^{(-1)^i}$. This function has an Euler product expansion (convergent for $\Re(s) \gg 0$)

$$L_{X,\mathscr{F}}(s) = \prod_{x \in X} \det(1 - |\kappa(x)|^{-s}\varphi_x | \mathscr{F}_{\overline{x}})^{-1},$$

where $\det(A | \mathscr{F}_{\overline{x}})$ is shorthand for the product $\prod_{i \in \mathbf{Z}} \det(A | \mathcal{H}^i(\mathscr{F})_{\overline{x}})^{(-1)^i}$. We also have a cohomological formula

$$L_{X,\mathscr{F}}(s) = \prod_{n \in \mathbf{Z}} \det(1 - q^{-s}\varphi | \operatorname{H}^n_c(\overline{X}; \mathscr{F}))^{(-1)^{n+1}},$$

where $\operatorname{H}^n_c(\overline{X}; \mathscr{F})$ now denotes the (compactly supported) *hypercohomology* of $\overline{X} = X \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q)$ with coefficients in $\mathscr{F}$.

We now describe the context in which we would like to apply these ideas.

Claim 4.1. *Let X be a scheme and let Y be an algebraic stack equipped with a smooth morphism $Y \to X$ and a choice of section $s : X \to Y$. Under certain hypotheses, one can construct an object $\mathscr{F}_{Y/X} \in D^b(X)$ with the property that, for each geometric point $\overline{x} : \mathrm{Spec}(k) \to X$, we have a canonical isomorphism*

$$\mathcal{H}^{-*}(\mathscr{F}_{Y/X})_x \simeq \pi_*(Y_{\overline{x}})_{\mathbf{Q}_\ell};$$

here $Y_{\overline{x}} = Y \times_X \mathrm{Spec}(k)$ denotes the fiber of $Y \to X$ over the geometric point x, and $\pi_*(Y_{\overline{x}})_{\mathbf{Q}_\ell}$ denotes the $\mathbf{Q}_\ell$-homotopy groups of the stack $Y_{\overline{x}}$ in the previous lecture (note that the section s provides a base point $s(\overline{x})$ on $Y_{\overline{x}}$). Moreover, this construction has the following properties

(*i*) *If we are given a pullback square*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X, \end{array}$$

then $\mathscr{F}_{Y'/X'} \in D^b(X')$ can be identified with the pullback $f^*\mathscr{F}_{Y/X}$.

(*ii*) *For fixed X, the object $\mathscr{F}_{Y/X} \in D^b(X)$ depends functorially on Y: any map $Y' \to Y$ of smooth algebraic stacks over X (satisfying the relevant hypotheses) determines a map $\mathscr{F}_{Y'/X} \to \mathscr{F}_{Y/X}$ in the derived category $D^b(X)$.*

Remark 4.2. The construction of $\mathscr{F}_{Y/X} \in D^b(X)$ appears implicitly in [7]: it can be defined as the Verdier dual of the cotangent fiber of the relative cohomology sheaf $[Y]_X$ which appears in [7].

The hypotheses needed for Claim 4.1 are relative versions of conditions $(*)$ and $(*')$ from the previous lecture, which will guarantee in particular that the ℓ-adic homotopy groups $\pi_*(Y_{\overline{x}})_{\mathbf{Q}_\ell}$ are well-behaved (and vanish for $* \gg 0$). Under these assumptions, we can consider the L-function $L_{Y/X}(s) = L_{X,\mathscr{F}_{Y/X}}(s)$, given for $\Re(s) \gg 0$ by the formula

$$\begin{aligned} L_{Y/X}(s) &= \prod_{x \in X} \det(1 - |\kappa(x)|^{-s}\varphi_x | \mathscr{F}_{Y/X,x})^{-1} \\ &= \prod_{x \in X} \prod_{n>0} \det(1 - |\kappa(x)|^{-s}\varphi_x | \pi_n(Y_{\overline{x}})_{\mathbf{Q}_\ell})^{(-1)^{n+1}}. \end{aligned}$$

If the eigenvalues of Frobenius on the homotopy groups $\pi_n(Y_x)_{\mathbf{Q}_\ell}$ are sufficiently small, then this product will converge when $s = 0$ and (by virtue of calculation (3) from Lecture 3) yield the identity

$$L_{Y/X}(0) = \prod_{x \in X} \mathrm{Tr}(\varphi_x^{-1} | \mathrm{H}^*(Y_{\overline{x}}; \mathbf{Q}_\ell)).$$

If, in addition, we assume that each fiber $Y_{\overline{x}}$ satisfies the Grothendieck-Lefschetz trace formula, we can rewrite this formula as

$$L_{Y/X}(0) = \prod_{x \in X} \frac{|Y_x(\kappa(x))|}{|\kappa(x)|^{\dim(Y_x)}}.$$

Let us now specialize to the situation which is relevant to Weil's conjecture. Let X be an algebraic curve over $\mathbf{F}_q$ (assumed to be smooth, projective, and geometrically connected), and let G be a smooth affine group scheme over X. Let BG_X denote the classifying stack of G. More precisely, BG_X is an algebraic stack whose R-valued points can be identified with pairs $(\eta, \mathcal{P})$, where $\eta : \mathrm{Spec}(R) \to X$ is a morphism of schemes and $\mathcal{P}$ is a principal G-bundle on $\mathrm{Spec}(R)$. Then the construction $(\eta, \mathcal{P}) \mapsto \eta$ determines a projection map $\mathrm{BG}_X \to X$. If the fibers of G are connected, then each fiber of the map $\mathrm{BG}_X \to X$ can be identified with the classifying stack of a connected linear algebraic group. It therefore satisfies conditions $(*)$ and $(*')$ of Lecture 3, as well as the Grothendieck-Lefschetz trace formula (Proposition 3.4). We can then apply the construction of Claim 4.1 to obtain an object $\mathscr{F}_{\mathrm{BG}_X/X} \in D^b(X)$. If we further assume that the generic fiber of G is semisimple, then Euler product for $L_{\mathrm{BG}_X/X}(s)$ converges at $s = 0$, and therefore yields an identity

$$L_{\mathrm{BG}_X/X}(0) = \prod_{x \in X} \mathrm{Tr}(\varphi_x^{-1} | \, \mathrm{H}^*(\overline{\mathrm{BG}}_x; \mathbf{Q}_\ell))$$

where the right hand side is the infinite product appearing in the statement of Conjecture 2.13. Since each of the classifying stacks BG_x satisfies the Grothendieck-Lefschetz trace formula (Proposition 3.4), we can also write this as

$$L_{\mathrm{BG}_X/X}(0) = \frac{|\,\mathrm{BG}_x(\kappa(x))|}{|\kappa(x)|^{\dim(\mathrm{BG}_x)}} = \prod_{x \in X} \frac{|\kappa(x)|^{\dim(G_x)}}{|G_x(\kappa(x))|}.$$

To relate the product expansion to the left hand side of Conjecture 2.13, we note that the classifying stack BG_X is closely related to the moduli stack of G-bundles $\mathrm{Bun}_G(X)$. Note that there is a canonical *evaluation map*

$$e : X \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Bun}_G(X) \to \mathrm{BG}_X \, .$$

An R-valued point of the left hand side is given by a pair $(\eta, \mathcal{P})$ where $\eta : \mathrm{Spec}(R) \to X$ is an R-valued point of X and $\mathcal{P}$ is a G-bundle on the *entire* product $X_R = \mathrm{Spec}(R) \times_{\mathrm{Spec}(\mathbf{F}_q)} X$, and in these terms the map e is given by $e(\eta, \mathcal{P}) = (\eta, \eta^* \mathcal{P})$. Letting $f : X \to \mathrm{Spec}(\mathbf{F}_q)$ denote the projection map, we see that properties (i) and (ii) above supply a comparison map

$$\rho : f^* \mathscr{F}_{\mathrm{Bun}_G(X)/\mathrm{Spec}(\mathbf{F}_q)} \simeq \mathscr{F}_{X \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Bun}_G(X)/X} \to \mathscr{F}_{\mathrm{BG}_X/X}$$

in the derived category $D^b(X)$. We then have the following:

Theorem 4.3 (Gaitsgory-L)**.** *Let X be a smooth projective geometrically connected curve over $\mathbf{F}_q$ and let G be a smooth affine group scheme over X. Assume that the fibers of G are connected and that the generic fiber of G is semisimple and simply connected. Then the map ρ above induces an isomorphism*

$$\mathscr{F}_{\mathrm{Bun}_G(X)/\mathrm{Spec}(\mathbf{F}_q)} \to Rf_* \mathscr{F}_{\mathrm{BG}_X/X}$$

in the derived category $D^b(\mathbf{F}_q)$. In other words, it induces an isomorphism of graded $\mathbf{Q}_\ell$-vector spaces

$$\pi_*(\overline{\mathrm{Bun}}_G(X))_{\mathbf{Q}_\ell} \to \mathrm{H}^{-*}(\overline{X}; \mathscr{F}_{\mathrm{BG}_X/X}),$$

where $\overline{\mathrm{Bun}}_G(X)_{\mathbf{Q}_\ell}$ denotes the product $\mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Bun}_G(X)$.

Warning 4.4. In the statement of Theorem 4.3, we cannot omit the hypothesis that the generic fiber of G is simply connected. Note that the statement of Theorem 4.3 implicitly assumes that the sheaf $\mathscr{F}_{\mathrm{Bun}_G(X)/\mathrm{Spec}(\mathbf{F}_q)}$ is well-defined: this requires that the moduli stack $\overline{\mathrm{Bun}}_G(X)$ satisfies conditions $(*)$ and $(*')$ of the previous lecture. If the generic fiber of G is not simply connected, then these conditions can fail: for example, $\overline{\mathrm{Bun}}_G(X)$ can fail to be connected.

Assuming Theorem 4.3, we can complete the proof of Conjecture 2.13. Applying the Grothendieck-Lefschetz trace formula to the object $\mathscr{F} \in D^b(X)$, we obtain the identity

$$\begin{aligned} L_{\mathrm{BG}_X/X}(s) &= \prod_{n \in \mathbf{Z}} \det(1 - q^{-s}\varphi | \, \mathrm{H}^n_c(\overline{X}; \mathscr{F}_{\mathrm{BG}_X/X}))^{(-1)^{n+1}} \\ &= \prod_{n \in \mathbf{Z}} \det(1 - q^{-s}\varphi | \pi_{-n}(\overline{\mathrm{Bun}_G(X)})_{\mathbf{Q}_\ell})^{(-1)^{n+1}} \\ &= \det(1 - q^{-s}\varphi | \pi_*(\overline{\mathrm{Bun}}_G(X))_{\mathbf{Q}_\ell}). \end{aligned}$$

Evaluating at $s = 0$ and using calculation (3) of Lecture 3, we obtain the identity

$$L_{\mathrm{BG}_X/X}(0) = \mathrm{Tr}(\varphi^{-1} | \, \mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Q}_\ell)).$$

Remark 4.5. The L-functions appearing the above discussion are a bit of a red herring, since we do not use them for any purpose other than evaluation at $s = 0$.

Let us close this lecture with a few remarks concerning Theorem 4.3. Theorem 4.3 is essentially a *geometric* statement, rather than an arithmetic one: it is valid more generally for an algebraic curve X defined over any field k (in which ℓ is invertible), and the proof can be immediately reduced to the case where k is algebraically closed. It can be formulated in terms of cohomology, rather than homotopy: roughly speaking, it

corresponds to the assertion that the ℓ-adic cochain complex $C^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$ admits a "continuous Künneth decomposition"

$$C^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell) \simeq \bigotimes_{x \in X}^{\mathrm{cont}} C^*(\mathrm{BG}_x; \mathbf{Q}_\ell).$$

We refer the reader to [7] for a precise statement, where the right hand side is defined using an algebro-geometric version of *factorization homology*. When $k = \mathbf{C}$ is the field of complex numbers, the existence of this decomposition is essentially equivalent to the fact that principal G-bundles on X are governed by an *h-principle*: more precisely, the moduli stack $\mathrm{Bun}_G(X)$ of algebraic G-bundles on X is homotopy equivalent (in a suitable sense) to the classifying space for *topological* G-bundles on X. When k has positive characteristic, the proof requires a different set of ideas.

5 Project: Parahoric Harder-Narasimhan Theory

For motivation, let us begin with a few remarks about Conjecture 2.12. Let Y be a smooth algebraic stack over a finite field $\mathbf{F}_q$, and set $\overline{Y} = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} Y$. We let $|Y(\mathbf{F}_q)|$ denote the sum $\sum_\eta \frac{1}{|\mathrm{Aut}(\eta)|}$, where η ranges over the collection of isomorphism classes of $\mathbf{F}_q$-valued points of Y (each of which has a finite automorphism group $\mathrm{Aut}(\eta)$). Fix a prime number ℓ which is invertible in $\mathbf{F}_q$ and an embedding $\mathbf{Q}_\ell \hookrightarrow \mathbf{C}$. We say that Y *satisfies the Grothendieck-Lefschetz trace formula* if there is an equality

$$\frac{|Y(\mathbf{F}_q)|}{q^{\dim(Y)}} = \mathrm{Tr}(\varphi^{-1} | \mathrm{H}^*(Y; \mathbf{Q}_\ell)) := \sum_{n \geqslant 0} (-1)^n \, \mathrm{Tr}(\varphi^{-1} | \mathrm{H}^n(Y; \mathbf{Q}_\ell))$$

(which requires, in particular, that the sums on both sides are convergent).

If Y is a smooth algebraic variety over $\mathbf{F}_q$, then it satisfies the Grothendieck-Lefschetz trace formula. More generally, if Y is given as a *global quotient stack* $Z//H$, where Z is a smooth variety over $\mathbf{F}_q$ and H is a linear algebraic group acting on Z, then one can deduce that the Grothendieck-Lefschetz trace formula holds for Y by knowing that it holds for Z and H (for a more general statement, see [5]). Conjecture 2.12 asserts that the Grothendieck-Lefschetz trace formula holds for $\mathrm{Bun}_G(X)$, where X is an algebraic curve over $\mathbf{F}_q$ and G is a smooth affine group scheme over X with semisimple generic fiber. One can show that $\mathrm{Bun}_G(X)$ is *locally* a global quotient stack: that is, every quasi-compact open substack $U \subseteq \mathrm{Bun}_G(X)$ can be written as a quotient $Z//H$ as above. However, there is still work to be done, because the moduli stack $\mathrm{Bun}_G(X)$ is almost never quasi-compact.

In [4], Behrend shows that $\mathrm{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula when the group scheme G is *constant*: that is, when it comes from a semisimple

algebraic group over $\mathbf{F}_q$. The proof makes use of the *Harder-Narasimhan* stratification of $\mathrm{Bun}_G(X)$: this is a decomposition of $\mathrm{Bun}_G(X)$ into (infinitely many) locally closed substacks which are quasi-compact, and therefore global quotient stacks for which the Grothendieck-Lefschetz trace formula is known. In [7], this approach is generalized to prove Conjecture 2.12 in general, including in cases where G fails to be semisimple at finitely many points. The general strategy of argument is the same: one constructs a suitable stratification of $\mathrm{Bun}_G(X)$, and deduces the trace formula for $\mathrm{Bun}_G(X)$ from the fact that the trace formula is known for each stratum. However, the stratification used in [7] is a bit *ad hoc*. It is probably possible to give a better proof, using an appropriate generalization of the Harder-Narasimhan stratification to the setting of *parahoric* group schemes over curves. In a series of guided exercises, we consider how this might work in the special case where G is a group scheme with generic fiber GL_n (beware that in this case one does not expect the Grothendieck-Lefschetz trace formula to hold for the entire stack $\mathrm{Bun}_G(X)$, because $\mathrm{Bun}_G(X)$ will have infinitely many connected components). We will also work over an algebraically closed ground field k, rather than a finite field $\mathbf{F}_q$.

Remark 5.1. There is a bit of recent literature on this subject, at least over fields of characteristic zero: see [3] and the final section of [1].

5.1 Normed Vector Spaces

We begin with some standard facts about normed vector spaces over nonarchimedean fields. Let $\widehat{K}$ be a field which is complete with respect to a nonarchimedean absolute value $|\bullet|_{\widehat{K}} : \widehat{K} \to \mathbf{R}_{\geqslant 0}$. We will assume that the absolute value $|\bullet|_{\widehat{K}}$ is nontrivial and discretely valued, so that

$$\widehat{\mathcal{O}} = \{x \in \widehat{K} : |x|_K \leqslant 1\}$$

is a discrete valuation ring with residue field $\widehat{K}$.

Definition 5.2. Let V be a vector space over $\widehat{K}$. A *norm* on E is a function

$$|\bullet|_V : V \to \mathbf{R}_{\geqslant 0}$$

satisfying the following conditions:

- For $v \in V$, we have $|v|_V = 0$ if and only if $v = 0$.
- For $v \in V$ and $f \in \widehat{K}$, we have $|fv|_V = |f|_{\widehat{K}} \cdot |v|_V$.
- For $u, v \in V$, we have $|u + v|_V \leqslant \max(|u|_V, |v|_V)$.

Exercise 5.3. Let V be a finite-dimensional vector space over $\widehat{K}$ equipped with a norm $|\bullet|_V$. Prove the following:

(a) Every subspace $V' \subseteq V$ is closed (with respect to the topology defined by the metric $d(u,v) = |u - v|_V$).

(b) For every subspace $V' \subseteq V$, the quotient V/V' inherits a norm, given by the formula

$$|\overline{v}|_{V/V'} = \inf_{v \to \overline{v}} |v|_V;$$

here the infimum is taken over all elements $v \in V$ which map to $\overline{v}$ in the quotient V/V'.

(c) For every element $\overline{v} \in V/V'$, the infimum of (b) is actually achieved: that is, one can choose an element $v \in V$ representing $\overline{v}$ such that $|\overline{v}|_{V/V'} = |v|_V$.

(d) The vector space V has a basis $v_1, v_2, \ldots, v_n$ with respect to which the norm $|\bullet|_V$ is given by

$$|f_1 v_1 + \cdots + f_n v_n|_V = \max(c_1 |f_1|_{\widehat{K}}, \cdots, c_n |f_n|_{\widehat{K}}),$$

for some positive real numbers $c_1, c_2, \ldots, c_n \in \mathbf{R}$ (which are then given by given by $c_i = |v_i|_V$).

(e) The norm $|\bullet|_V$ makes V into a Banach space: that is, V is complete with respect to the metric $d(u,v) = |u - v|_V$.

Hint: prove all of these assertions simultaneously by induction on the dimension of V.

Exercise 5.4. Let V be a finite-dimensional vector space over $\widehat{K}$ equipped with a norm $|\bullet|_V$. For each positive real number λ, set $V^{\leqslant \lambda} = \{v \in V : |v|_V \leqslant \lambda\}$. Show that $V^{\leqslant \lambda}$ is a $\widehat{\mathcal{O}}$-lattice in V. That is, $V^{\leqslant \lambda}$ is a free $\widehat{\mathcal{O}}$-module of rank $\dim_{\widehat{\mathcal{K}}}(V)$ satisfying $V \simeq \widehat{K} \times_{\widehat{\mathcal{O}}} V^{\leqslant \lambda}$.

Exercise 5.5. Let V be a finite-dimensional vector space over $\widehat{K}$ and let $V_0 \subseteq V$ be a $\widehat{\mathcal{O}}$-lattice in V. For each $v \in V$, set

$$|v|_V = \inf\{|f|_{\widehat{K}} : v \in f V_0\}.$$

Show that $|\bullet|_V$ is a metric on V, and that $V_0 = V^{\leqslant 1}$.

Exercise 5.6. Let Γ denote the *value group* of $\widehat{K}$: that is, the set of all positive real numbers which have the form $|f|_{\widehat{K}}$. Let V be a finite-dimensional vector space over $\widehat{K}$. Show that the construction of Exercise 5.5 induces a bijection from the set of $\widehat{\mathcal{O}}$-lattices in V to the set of all norms

$$|\bullet|_V : V \to \mathbf{R}_{\geqslant 0}$$

which take values in $\Gamma \cup \{0\}$.

Exercise 5.7. Let V be a finite-dimensional vector space over $\widehat{K}$, let

$$| \bullet |_V : V \to \mathbf{R}_{\geqslant 0}$$

be a norm on V, and let $V_0 \subseteq V$ be a $\widehat{\mathcal{O}}$-lattice. Show that there exists a $\widehat{\mathcal{O}}$-basis $v_1, \ldots, v_n$ for V_0 and positive real numbers $c_1, \ldots, c_n \in \mathbf{R}$ such that

$$|f_1 v_1 + \cdots + f_n v_n|_V = \max(c_1 |f_1|_{\widehat{K}}, \cdots, c_n |f_n|_{\widehat{K}}).$$

Exercise 5.8. In the situation of Exercise 5.7, show that the real numbers $c_1, \ldots, c_n \in \mathbf{R}$ are uniquely determined (by the choice of norm $| \bullet |_V$ the choice of lattice $V_0 \subseteq V$). In particular, the product $\prod_{i=1}^n c_n$ is a well-defined positive real number, which we will denote by $\mathrm{Vol}(V_0)$ and refer to as the *volume* of V_0.

5.2 Generalized Vector Bundles

Now suppose that X is an algebraic curve defined over an *algebraically closed* field k. Let K_X denote the fraction field of X. We write $x \in X$ to indicate that x is a *closed* point of X (or, equivalently, a k-valued point of x). We let $\widehat{\mathcal{O}}_{X,x}$ denote the completed local ring of X at the point x, and $\widehat{K}_{X,x}$ its fraction field. Then $\widehat{K}_{X,x}$ is complete with respect to the absolute value given by

$$|f|_{\widehat{K}_{X,x}} = \exp(-\mathrm{ord}_x(f)),$$

where $\mathrm{ord}_x(f)$ denotes the order of vanishing of f at the point x.

Notation 5.9. Let V be a finite-dimensional vector space over the fraction field K_X. For each closed point $x \in X$, we let V_x denote the tensor product $\widehat{K}_{X,x} \otimes_{K_X} V$.

Definition 5.10. A *generalized vector bundle on* X is a pair $(V, \{| \bullet |_{V_x}\}_{x \in X})$, where V is a finite-dimensional vector space over the fraction field K_X and each $| \bullet |_{V_x}$ is a norm on the vector space $V_x = \widehat{K}_{X,x} \otimes_{K_X} V$, satisfying the following condition:

($*$) Let $v_1, \ldots, v_n$ be a basis for the vector space V. Then, for all but finitely many closed points $x \in X$, the norm $| \bullet |_{V_x}$ is given by the formula

$$|f_1 v_1 + \cdots + f_n v_n|_{V_x} = \max(|f_i|_{\widehat{K}_{X,x}})$$

(see Exercise 5.5).

Exercise 5.11. Show that condition ($*$) of Definition 5.10 does not depend on the choice of basis $v_1, \ldots, v_n \in V$.

Definition 5.12. Let $(V, \{|\bullet|_{V_x}\}_{x \in X})$ and $(V', \{|\bullet|_{V'_x}\}_{x \in X})$ be generalized vector bundles on X. A *morphism of generalized vector bundles* from $(V, \{|\bullet|_{V_x}\}_{x \in X})$ to $(V', \{|\bullet|_{V'_x}\}_{x \in X})$ is a K_X-linear map $F : V \to V'$ such that, for each point $x \in X$, the induced map $F_x : V_x \to V'_x$ is a contraction: that is, we have $|F_x(v)|_{V'_x} \leqslant |v|_{V_x}$ for each $v \in V_x$.

Exercise 5.13. Let $\mathscr{E}$ be a vector bundle on X. Then $\mathscr{E}$ determines a generalized vector bundle $(V, \{| \bullet |_{V_x}\}_{x \in X})$, by taking V to be the stalk of $\mathscr{E}$ at the generic point of X and each $| \bullet |_{V_x}$ to be the norm determined by the lattice given by the completed stalk $\widehat{\mathscr{E}}_x$ (see Exercise 5.5). Show that this construction induces a fully faithful embedding

$$\{\text{Vector bundles on } X\} \to \{\text{Generalized vector bundles on } X\}.$$

Moreover, a generalized vector bundle $(V, \{| \bullet |_{V_x}\}_{x \in X})$ belongs to the essential image of this embedding if and only if each of the norms $| \bullet |_{V_x}$ takes values in the set $\exp(\mathbf{Z}) \cup \{0\}$ (Exercise 5.6).

5.3 Degrees of Generalized Vector Bundles

We maintain the notations of §5.2.

Construction 5.14. Let $\mathscr{V} = (V, \{| \bullet |_{V_x}\}_{x \in X})$ be a generalized vector bundle on X, and choose a basis $v_1, \ldots, v_n$ for the vector space V. For each closed point $x \in X$, let $V_{x0} \subseteq V_x$ be the $\widehat{\mathcal{O}}_{X,x}$-lattice generated by the elements $v_1, \ldots, v_n$. We set

$$\deg(\mathscr{V}) = -\sum_{x \in X} \log(\mathrm{Vol}(V_{x0})),$$

where $\mathrm{Vol}(V_{x0})$ denotes the volume of the lattice V_{x0} with respect to the norm $| \bullet |_{V_x}$ (see Exercise 5.8). We will refer to $\deg(\mathscr{V})$ as the *degree* of the generalized vector bundle $\mathscr{V}$.

Exercise 5.15. Show that the degree $\deg(\mathscr{V})$ of Construction 5.14 is well-defined. That is, the sum $\sum_{x \in X} \log(\mathrm{Vol}(V_{x0}))$ is finite, and does not depend on the choice of basis $v_1, \ldots, v_n \in V$.

Exercise 5.16. Let $\mathscr{E}$ be a vector bundle on X, and let $\mathscr{V}$ be the generalized vector bundle associated to $\mathscr{E}$ in Exercise 5.13. Show that the degree of $\mathscr{V}$ (in the sense of Construction 5.14) is equal to the degree of $\mathscr{E}$ (in the usual sense).

Warning 5.17. If $\mathscr{E}$ is a vector bundle on X, then the degree $\deg(\mathscr{E})$ is an integer. However, the degree of a generalized vector bundle on X can be any real number.

Notation 5.18. Let $\mathscr{V} = (V, \{| \bullet |_{V_x}\}_{x \in X})$ be a generalized vector bundle on X. A *subbundle* of $\mathscr{V}$ is a pair $\mathscr{V}' = (V', \{| \bullet |_{V'_x})$, where $V' \subseteq V$ is a subspace and each of the norms $| \bullet |_{V'_x}$ is given by the restriction of $| \bullet |_{V_x}$ to the subspace $V'_x \subseteq V_x$. If this condition

is satisfied, we let $\mathscr{V}/\mathscr{V}'$ denote the generalized vector bundle $(V/V', \{|\bullet|_{V_x/V'_x}\}_{x\in X})$, where each $|\bullet|_{V_x/V'_x}$ is defined as in Exercise 5.3. In this case, we will refer to the diagram

$$0 \to \mathscr{V}' \to \mathscr{V} \to \mathscr{V}/\mathscr{V}' \to 0$$

as a *short exact sequence* of generalized vector bundles.

Exercise 5.19. Show that for every short exact sequence of generalized vector bundles

$$0 \to \mathscr{V}' \to \mathscr{V} \to \mathscr{V}'' \to 0,$$

we have $\deg(\mathscr{V}) = \deg(\mathscr{V}') + \deg(\mathscr{V}'')$.

Exercise 5.20. Let $F : \mathscr{V} \to \mathscr{W}$ be a morphism of generalized vector bundles, and suppose that the underlying map of K_X-vector spaces $V \to W$ is an isomorphism. Show that $\deg(\mathscr{V}) \leqslant \deg(\mathscr{W})$, and that equality holds if and only if F is an isomorphism of generalized vector bundles.

5.4 Slopes and Semistability

We maintain the notations of §5.2.

Definition 5.21. Let $\mathscr{V} = (V, \{|\bullet|_{V_x}\}_{x\in X})$ be a generalized vector bundle on X. The *rank* of $\mathscr{V}$ is the dimension of V as a vector space over K_X. We denote the rank by $\mathrm{rk}(\mathscr{V})$. If $V \neq 0$, we set $\mathrm{slope}(\mathscr{V}) = \frac{\deg(\mathscr{V})}{\mathrm{rk}(\mathscr{V})}$. We refer to $\mathrm{slope}(\mathscr{V})$ as the *slope* of $\mathscr{V}$.

Exercise 5.22. Suppose we have a short exact sequence of (nonzero) generalized vector bundles

$$0 \to \mathscr{V}' \to \mathscr{V} \to \mathscr{V}'' \to 0.$$

Show that:

- If $\mathrm{slope}(\mathscr{V}') = \mathrm{slope}(\mathscr{V}'')$, then $\mathrm{slope}(\mathscr{V}') = \mathrm{slope}(\mathscr{V}) = \mathrm{slope}(\mathscr{V}'')$.
- If $\mathrm{slope}(\mathscr{V}') < \mathrm{slope}(\mathscr{V}'')$, then $\mathrm{slope}(\mathscr{V}') < \mathrm{slope}(\mathscr{V}) < \mathrm{slope}(\mathscr{V}'')$.
- If $\mathrm{slope}(\mathscr{V}') > \mathrm{slope}(\mathscr{V})$, then $\mathrm{slope}(\mathscr{V}') > \mathrm{slope}(\mathscr{V}) > \mathrm{slope}(\mathscr{V}'')$.

Definition 5.23. Let $\mathscr{V} = (V, \{|\bullet|_{V_x}\}_{x\in X})$ be a generalized vector bundle on X. We say that $\mathscr{V}$ is *semistable* if, for every subbundle $\mathscr{V}' \subseteq \mathscr{V}$, we have $\mathrm{slope}(\mathscr{V}') \leqslant \mathrm{slope}(\mathscr{V})$.

Exercise 5.24. Suppose we have a short exact sequence of generalized vector bundles

$$0 \to \mathscr{V}' \to \mathscr{V} \to \mathscr{V}'' \to 0.$$

Show that, if $\mathscr{V}$ is semistable of slope λ, then $\mathrm{slope}(\mathscr{V}') \leqslant \lambda \leqslant \mathrm{slope}(\mathscr{V}'')$

Exercise 5.25. Let $\mathscr{V}$ and $\mathscr{W}$ be generalized vector bundles on X which are semistable of slopes λ and μ, respectively. Show that if $\lambda > \mu$, then every map of generalized vector bundles $\mathscr{V} \to \mathscr{W}$ is zero.

Exercise 5.26. Fix a real number λ, and let $\mathcal{C}$ be the category whose objects are generalized vector bundles of slope λ (where, by convention, we include the zero object). Show that $\mathcal{C}$ is an abelian category (beware that the analogous statement is *not* true for the entire category of generalized vector bundles, or for the usual category of vector bundles on X).

Exercise 5.27. Suppose we have a short exact sequence of generalized vector bundles

$$0 \to \mathscr{V}' \to \mathscr{V} \to \mathscr{V}'' \to 0.$$

Show that, if $\mathscr{V}'$ and $\mathscr{V}''$ are semistable of the same slope λ, then $\mathscr{V}$ is also semistable of slope λ.

5.5 The Harder-Narasimhan Filtration

We maintain the notations of §5.2.

Definition 5.28. Let $\mathscr{V}$ be a generalized vector bundle on X. We say that a sequence of generalized subbundles

$$0 = \mathscr{V}_0 \subsetneq \mathscr{V}_1 \subseteq \mathscr{V}_2 \subsetneq \cdots \subseteq \mathscr{V}_m = \mathscr{V}$$

is a *Harder-Narasimhan filtration* if the following conditions are satisfied:

- Each of the quotients $\mathscr{V}_i/\mathscr{V}_{i-1}$ is semistable of some slope λ_i.
- The slopes λ_i are strictly decreasing: that is, we have $\lambda_1 > \lambda_2 > \cdots > \lambda_m$.

Exercise 5.29. Let $\mathscr{V}$ be a generalized vector bundle on X. Show that a Harder-Narasimhan filtration of $\mathscr{V}$ is unique if it exists.

Exercise 5.30. Let $\mathscr{V}$ be a generalized vector bundle on X, and let S be the collection of all real numbers of the form slope($\mathscr{V}'$), where $\mathscr{V}'$ is a nonzero subbundle of $\mathscr{V}$. Show that:

(a) The set S is discrete (with respect to the usual topology on $\mathbf{R}$).

(b) The set S is bounded above.

(c) The set S has a largest element.

Hint: to prove (b), choose a filtration

$$0 = \mathscr{V}_0 \subseteq \mathscr{V}_1 \subseteq \cdots \subseteq \mathscr{V}_n = \mathscr{V}$$

where each successive quotient $\mathscr{V}_m/\mathscr{V}_{m-1}$ has rank 1, and show that $\max(\deg(\mathscr{V}_m/\mathscr{V}_{m-1}))$ is an upper bound for S.

Exercise 5.31. Show that every generalized vector bundle $\mathscr{V}$ on X admits a Harder-Narasimhan filtration. Hint: choose a nonzero subbundle $\mathscr{V}' \subseteq \mathscr{V}$ of the largest possible slope (and largest possible rank for that slope), and proceed by induction.

5.6 Parahoric Group Schemes: Pointwise Case

We return to the notations of §5.1. Let V be a finite-dimensional vector space over a discretely valued field $\widehat{K}$. We let $\mathrm{GL}(V)$ denote the algebraic group over $\widehat{K}$ whose R-valued points are given by

$$\mathrm{GL}(V)(R) = \mathrm{Aut}_R(R \otimes_{\widehat{K}} V).$$

Note that any choice of basis $v_1, \ldots, v_n \in V$ determines an isomorphism of algebraic groups $\mathrm{GL}(V) \simeq \mathrm{GL}_n$.

If $V_0 \subseteq V$ is a $\widehat{\mathcal{O}}$-lattice, we let $\mathrm{GL}(V_0)$ denote the group scheme over $\widehat{\mathcal{O}}$ whose R-valued points are given by

$$\mathrm{GL}(V_0)(R) = \mathrm{Aut}_R(R \otimes_{\widehat{\mathcal{O}}} V_0).$$

Then $\mathrm{GL}(V_0)$ is again isomorphic to GL_n (this time as group schemes over $\widehat{\mathcal{O}}$), with an isomorphism determined by any choice of basis for V_0 as a $\widehat{\mathcal{O}}$-module. Moreover, we can identify the algebraic group $\mathrm{GL}(V)$ with the generic fiber of $\mathrm{GL}(V_0)$; in particular, we have a canonical map of $\widehat{\mathcal{O}}$-schemes $\mathrm{GL}(V) \to \mathrm{GL}(V_0)$.

Definition 5.32. Let V be a finite-dimensional vector space over $\widehat{K}$ and let $|\bullet|_V$ be a norm on V. Let $S \subseteq \mathbf{R}_{>0}$ be a finite set of real numbers with the property that for each nonzero element $v \in V$, we have $|fv|_V \in S$ for some $f \in \widehat{K}$ (exercise: show that S exists). We let $\mathrm{GL}(V, |\bullet|_V)$ denote the scheme-theoretic image of the map

$$\mathrm{GL}(V) \to \prod_{\lambda \in S} \mathrm{GL}(V^{\leqslant \lambda});$$

here the product on the right hand side is formed in the category of $\mathcal{O}$-schemes.

Exercise 5.33. In the situation of Definition 5.32, show that $\mathrm{GL}(V, |\bullet|_V)$ is a group scheme over $\widehat{\mathcal{O}}$, with generic fiber $\mathrm{GL}(V)$. (More precisely, it is a closed subgroup of the product $\mathrm{GL}(V^{\leqslant \lambda})$). In other words, it is an *integral model* for the group scheme $\mathrm{GL}(V)$. Integral models which arise in this way (that is, which come from norms on V) are called *parahoric*.

Exercise 5.34. In the situation of Definition 5.32, show that the map $\mathrm{GL}(V, |\bullet|_V) \to \mathrm{Spec}(\widehat{\mathcal{O}})$ is smooth and has connected fibers.

Exercise 5.35. Let V be a finite-dimensional vector space over $\widehat{K}$ equipped with a norm $|\bullet|_V$. Show that the group $\mathrm{GL}(V, |\bullet|_V)(\widehat{\mathcal{O}})$ of $\widehat{\mathcal{O}}$-valued points of $\mathrm{GL}(V, |\bullet|_V)$ can be identified with the subgroup of $\mathrm{GL}(V)(\widehat{K}) = \mathrm{Aut}_{\widehat{K}}(V)$ consisting of those automorphisms of V which preserve the norm $|\bullet|_V$ (that is, $\widehat{K}$-linear maps $F : V \to V$ satisfying $|F(v)|_V = |v|_V$ for each $v \in V$).

5.7 Parahoric Group Schemes over a Curve

We now return to the notations of §5.2.

Exercise 5.36. Let $\mathscr{V} = (V, \{|\bullet|_{V_x}\}_{x \in X})$ be a generalized vector bundle on X. Show that there is an essentially unique group scheme G over X equipped with isomorphisms

$$\mathrm{Spec}(K_X) \times_X G \simeq \mathrm{GL}(V)$$

$$\mathrm{Spec}(\widehat{\mathcal{O}}_{X,x}) \times_X G \simeq \mathrm{GL}(V_x, |\bullet|_{V_x})$$

which are compatible with restricted to $\widehat{K}_{X,x}$, for each closed point $x \in X$. We will denote the group scheme G by $\mathrm{GL}(\mathscr{V})$.

Exercise 5.37. Let $\mathscr{V} = (V, \{|\bullet|_{V_x}\}_{x \in X})$ be a generalized vector bundle on X, and let $\mathrm{GL}(\mathscr{V})$ be the group scheme of Exercise 5.36. Show that the groupoid of principal $\mathrm{GL}(\mathscr{V})$-bundles on X can be identified with the groupoid of generalized vector bundles $\mathscr{W} = (W, \{|\bullet|_{W_x}\}_{x \in X})$ having the property that, for each closed point $x \in X$, there exists an isomorphism of normed vector spaces $(V_x, |\bullet|_{V_x}) \simeq (W_x, |\bullet|_{W_x})$.

From this point forward, we fix a generalized vector bundle $\mathscr{V}$ on X of rank n. Let $\mathrm{Bun}_{\mathrm{GL}(\mathscr{V})}(X)$ denote the moduli stack of principal $\mathrm{GL}(\mathscr{V})$-bundles, where $\mathrm{GL}(\mathscr{V})$ is the group scheme of Exercise 5.36. By virtue of Exercise 5.37, we can identify k-valued points of $\mathrm{Bun}_{\mathrm{GL}(\mathscr{V})}(X)$ with generalized vector bundles $\mathscr{W}$ which are locally isomorphic to $\mathscr{V}$.

Question 5.38. What can one say about the locus of principal $\mathrm{GL}(\mathscr{V})$-bundles on X which correspond to *semistable* generalized vector bundles of some fixed slope λ? Are they parametrized by an open substack $\mathrm{Bun}^{\mathrm{ss},\lambda}_{\mathrm{GL}(\mathscr{V})}(X) \subseteq \mathrm{Bun}_{\mathrm{GL}(\mathscr{V})}(X)$? If so, is it quasi-compact?

Question 5.39. Suppose we are given real numbers $d_1, \ldots, d_m$ and positive integers $n_1, n_2, \ldots, n_m$ satisfying

$$\frac{d_1}{n_1} > \frac{d_2}{n_2} > \cdots > \frac{d_m}{n_m} \qquad n = n_1 + \cdots + n_m.$$

What can one say about the locus of principal $\mathrm{GL}(\mathscr{V})$-bundles on X which correspond to generalized vector bundles $\mathscr{W}$ which admit a Harder-Narasimhan filtration

$$0 = \mathscr{W}_0 \subsetneq \mathscr{W}_1 \subsetneq \cdots \subsetneq \mathscr{W}_m = \mathscr{W}$$

where each quotient $\mathscr{W}_i/\mathscr{W}_{i-1}$ is a generalized vector bundle of rank n_i and degree d_i. Are they parametrized by a locally closed substack of $\mathrm{Bun}_{\mathrm{GL}(\mathscr{V})}(X)$? If so, is it quasi-compact? Can it be described in terms of moduli stacks of the form $\mathrm{Bun}_{\mathrm{GL}(\mathscr{U}_1)}(X) \times \cdots \times \mathrm{Bun}_{\mathrm{GL}(\mathscr{U}_m)}(X)$, for suitable generalized vector bundles $\mathscr{U}_i$ of rank n_i?

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AWS 2019: Topological cyclic homology in arithmetic geometry

Matthew Morrow

The goal of this course will be to introduce the audience to certain aspects of the theory of topological Hochschild and cyclic homology, particularly those of interest in algebraic and arithmetic geometry.

The classical Hochschild homology of an algebra A over a field k is defined via the explicit Hochschild complex

$$HH(A/k) := \qquad A \leftarrow A \otimes_k A \leftarrow A \otimes_k A \otimes_k A \leftarrow \cdots$$

The extra data of cyclic permutations on each term $A^{\otimes_k n}$ gives $HH(A/k)$ the structure of a so-called *cyclic complex* and was used by Connes and Feĭgin–Tsygan in the 1980s to construct the cyclic homology, negative cyclic homology, and period cyclic homology of A; these fit together into a fibre sequence in the derived category of k-modules

$$HC(A/k)[1] \longrightarrow HC^-(A/k) \longrightarrow HP(A/k).$$

In the case in which k is of characteristic zero and A is smooth over k, they established natural decompositions

$$HP(A/k) = \prod_{n\in\mathbb{Z}} \Omega^\bullet_{A/k}[2n], \qquad HC^-(A/k) = \prod_{n\in\mathbb{Z}} \Omega^{\geq n}_{A/k}[2n]$$

In short, periodic and negative cyclic homology encode the de Rham cohomology of A and its Hodge filtration; similarly statements apply in the case of smooth varieties over k.

A profound idea of Goodwillie was to transport the above constructions to the setting of structured ring spectra, replacing the base k by the sphere spectrum $\mathbb{S}$ and A (now any commutative ring) by its associated Eilenberg–Maclane spectrum HA:

$$THH(A) := |HA \leftarrow HA \otimes_{\mathbb{S}} HA \leftarrow HA \otimes_{\mathbb{S}} HA \otimes_{\mathbb{S}} HA \leftarrow \cdots|$$

This again inherits extra structure, now that of a so-called *cyclotomic spectrum*. The theory of cyclotomic spectra has been substantially clarified by recent work of Nikolaus–Scholze [4], whose point of view we will adopt in the course. Essentially, $THH(A)$ is not merely equipped with an action by the circle S^1, but moreover with a certain Frobenius map for each prime number p

$$\varphi_p : THH(A) \to THH(A)^{tC_p},$$

where the right side denotes the Tate construction associated to the action of the cyclic group $C_p \subseteq S^1$. From this data one builds the topological cyclic homology, negative topological cyclic homology, and periodic cyclic homology of A, fitting into a fibre sequence of spectra

$$TC(A) \longrightarrow TC^-(A) \xrightarrow{\varphi-1} TP(A)$$

(we warn the reader that this does not quite correspond to the fibre sequence in the classical theory).

The style of result which will interest us in the course is the following analogue of the de Rham calculation from the classical theory:

Theorem 0.1 ([2])**.** *Let A be a smooth algebra over a perfect field k of characteristic p. Then $TP(A)$ and $TC^-(A)$ admit $\mathbb{Z}$-indexed complete filtrations whose graded pieces are respectively shifts of*

$$W\Omega^\bullet_{A/k}, \qquad \mathcal{N}^{\geq i} W\Omega^\bullet_{A/k}.$$

Here $W\Omega^{\bullet}_{A/k}$ denotes the de Rham–Witt complex of Bloch–Deligne–Illusie while $\mathcal{N}^{\geq i}W\Omega^{\bullet}_{A/k}$ is its Nygaard filtration, defined as the largest subcomplex on which the absolute Frobenius is divisible by p^i. In other words, the classical de Rham calculation in characteristic zero has a natural analogue in characteristic p in terms of crystalline cohomology, except that the direct sum decomposition is no longer split, i.e., there is merely a complete filtration. This is analogous to the motivic filtration on algebraic K-theory.

A similar theorem holds in mixed characteristic, e.g., A smooth over the ring of integers of a perfectoid field, where the graded steps of the filtrations are now described in terms of $A\Omega_A$, the $\mathbb{E}_\infty$-ring constructed in [1] (and re-constructed as prismatic cohomology in [3]), which simultaneously interpolates certain de Rham, crystalline, and p-adic étale cohomologies.

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AWS 2019
Topological Hochschild homology in arithmetic geometry

Matthew Morrow

v.1 (1 Feb.) – updates to follow during February.
v.2 (8 Feb.) – Sections 2 and 3 rewritten (except §2.4 and §3.4); several projects expanded.
v.3 (9 Feb.) – Fixed mistakes in Project C.
v.4 (19 Feb.) – Fixed typos; added details to some projects; minor changes to §4.
v.5 (27 Feb.) – Added a fun exercise to 2.18, some remarks on transitivity properties of HH, also a section on the mixed characteristic story.
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1 Introduction

This course is an introduction to Hochschild and cyclic homology and their topological variants, aimed mainly at readers whose background is in number theory and arithmetic geometry rather than algebraic topology and homotopy theory. Useful references to have at hand while reading these notes are the following:

- [29] The standard textbook presenting cyclic homology in a classical fashion; available free on SpringerLink if your institution subscribes.
- [28] Notes based on lectures given by Nikolaus on topological cyclic homology.
- [21] Notes from the Arbeitsgemeinschaft on topological cyclic homology at Oberwolfach, April 2018.
- [32] Nikolaus and Scholze's new approach to topological cyclic homology.
- [8] The article constructing "motivic" filtrations on all variants of (topological) cyclic homology and relating these theories to crystalline cohomology and p-adic Hodge theory.

Given a commutative base ring k and a k-algebra A, the Hochschild homology groups $HH_n(A/k)$, arising as the homology groups of the explicit Hochschild complex $HH(A/k)$, were introduced already in the 1950s and are closely related to algebraic differential forms: to be precise, if A is commutative then there exist maps $\Omega^n_{A/k} \to HH_n(A/k)$, which are even isomorphisms when A is smooth over k. In the 1980s it was realised by Connes and Feigin–Tsygan that the usual de Rham differential $d : \Omega^n_{A/k} \to \Omega^{n+1}_{A/k}$ was simply a shadow of the fact that the complex $HH(A/k)$ could be equipped with an action by the circle S^1, in a sense which we will take time to explain in the course. One may then study the homology, cohomology, and Tate cohomology of this S^1-action on $HH(A/k)$, giving rise respectively to cyclic homology, negative cyclic homology, and periodic cyclic homology; these fit together into a fibre sequence

$$HC(A/k)[1] \longrightarrow HC^-(A/k) \longrightarrow HP(A/k).$$

In §2 we will discuss these constructions from the classical point of view of explicit double complexes and the theory of mixed complexes, finally presenting a proof of the following classical description in the case of smooth algebras over a characteristic zero base ring:

Theorem 1.1 (Connes, Feigin–Tsygan)**.** *If A is a smooth k-algebra and $k \supseteq \mathbb{Q}$, then there are natural decompositions*

$$HP(A/k) = \prod_{n\in\mathbb{Z}} \Omega^{\bullet}_{A/k}[2n], \qquad HC^{-}(A/k) = \prod_{n\in\mathbb{Z}} \Omega^{\geq n}_{A/k}[2n]$$

In short, periodic and negative cyclic homology encode the de Rham cohomology of A and its Hodge filtration. The main interest of Connes was therefore to view these cohomology theories as replacements for de Rham cohomology, even if A is non-commutative. In this course we will rather be interested in the case in which A is commutative, but not necessarily smooth nor of characteristic zero. In particular our first goal will be to explain the following generalisation of the previous theorem to finite characteristic:

Theorem 1.2. *If A is a smooth k-algebra, where k is a field of characteristic $p > 0$, then $HP(A/k)$ and $HC^{-}(A/k)$ admit $\mathbb{Z}$-indexed, complete filtrations whose n^{th} graded pieces are given respectively by*

$$\Omega^{\bullet}_{A/k}[2n], \qquad \Omega^{\geq n}_{A/k}[2n].$$

In other words, the classical theorem of Connes and Feigin–Tsygan extends beyond characteristic zero, at the expense of replacing the product decomposition by a possibly non-split filtration.

The techniques which we will use to prove Theorem 1.2 in §3 are radically different to its characteristic zero predecessor. We will see that the cohomology theories involved can be determined by their behaviour on large $\mathbb{F}_p$-algebras, namely perfect rings and certain well-behaved quotients thereof (called quasiregular semiperfect rings). This leads us to the study of these cohomology theories on quasiregular semiperfect rings, and to the formalism of the quasisyntomic site.

The problem with Theorem 1.2 is that de Rham cohomology is not the most interesting cohomology theory in characteristic p; we would prefer to replace it by cristalline cohomology. However, it is manifestly impossible to build cristalline cohomology from $HH(A/k)$ via any $\mathbb{Z}$-linear construction: since $HH(A/k)$ is a complex of $\mathbb{F}_p$-vector spaces, so too will be anything constructed from it $\mathbb{Z}$-linearly. The trick is therefore to leave the $\mathbb{Z}$-linear world, i.e., $D(\mathbb{Z})$, by passing to the world of spectra Sp in the sense of homotopy theory; constructions in Sp are no longer $\mathbb{Z}$-linear, merely $\mathbb{S}$-linear where $\mathbb{S}$ is the sphere spectrum.

The idea of developing Hochschild and cyclic homology in this "brave new world" of ring spectra is due to Goodwillie and Wahldhausen. The subject has seen enormous progress in recent years, particularly thanks to pioneering work of Hesselholt and then the refoundation of the subject by Nikolaus–Scholze, whose framework we will adopt. To any ring A one may associated its topological Hochschild homology

$$THH(A)\text{“} := HH(A/\mathbb{S})\text{”} \in \mathrm{Sp},$$

which is once again equipped with an action by the circle S^1; passage to homology, cohomology, and Tate cohomology (which are no longer $\mathbb{Z}$-linear constructions!) then yields an analogous fibre sequence as classically

$$THH(A)_{hS^1}[1] \longrightarrow TC^{-}(A) \longrightarrow TP(A)$$

In §4 we attempt to gradually familiarise the reader with the concept of what it means for S^1 to act on an algebraic object like a chain complex, then categorify the construction of Hochschild homology in order to construct topological Hochschild homology.

Section 5 presents the main results concerning topological Hochschild homology of $\mathbb{F}_p$-algebras, after which there will be a brief discussion of the mixed characteristic situation.

Notation

Both chain and cochain complex appear in this theory, which is an easy source of confusion. We follow the standard convention of denoting the former by lower indexes (and often by leftwards arrows)

$$\cdots \leftarrow C_n \leftarrow C_{n+1} \leftarrow \cdots$$

and the latter by upper indexes

$$\cdots \rightarrow C^n \rightarrow C^{n+1} \rightarrow \cdots .$$

For shifting complexes we follow the convention $C[1]^n := C^{n+1}$, whence for chain complexes we have $C[1]_n = C_{n-1}$, i.e., given an abelian group A, the shift $A[n]$ is supported in cohomological degree $-n$ and in homological degree n; the differentials on $C[1]$ are given by the negatives of the differentials on C. Under this convention, the Eilenberg–Maclane functor takes the shift [1] to the suspension Σ. There are bound to be some mistakes involving signs of shifts...

2 Classical Hochschild and cyclic homology

In this section we give a classical presentation of Hochschild and cyclic homology and their variants, namely negative cyclic homology and periodic cyclic homology. We will define these in terms of the explicit double complexes which were introduced by Connes and Feigin–Tsygan and establish the periodicity and norm sequences relating the different theories. We then abstract the constructions by introducing the classical concepts of cyclic objects and mixed complexes, which will be helpful when discussing two important examples. In the final subsection we explain how the definitions should be modified when treating non-flat algebras, which is really just an excuse to recall the cotangent complex.

A standard reference for this material is chapters 1–5 of the book of Loday [29], also [39, §9] and [30]; we will give quite a streamlined presentation of the results we need, and consultations of these sources will probably be helpful for readers seeing the material for the first time.

2.1 Hochschild homology and differential forms

Let k be a commutative ring and A a flat k-algebra (the flatness hypothesis will be discussed and removed in §2.4). Then the associated *Hochschild complex*, denoted by $HH(A/k)$ or more classically by $HH^k(A)$, is the chain complex of k-modules

$$A \xleftarrow{b} A \otimes_k A \xleftarrow{b} A \otimes_k A \otimes_k A \xleftarrow{b} \cdots$$

(where A lies in degree 0) with boundary maps traditionally denoted by b given by

$$\begin{aligned} b : A^{\otimes_k n+1} \to A^{\otimes_k n}, \qquad a_0 \otimes \cdots \otimes a_n \mapsto\ & a_0 a_1 \otimes \cdots \otimes a_n \\ & - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_n \\ & + \cdots \\ & + (-1)^n a_0 \otimes \cdots \otimes a_{n-1} a_n \\ & + (-1)^{n+1} a_n a_0 \otimes \cdots \otimes a_{n-1} \end{aligned}$$

For example, $b(a \otimes b) = ab - ba$ and $b(a \otimes b \otimes c) = ab \otimes c - a \otimes bc + ca \otimes b$. The homology groups of the Hochschild complex are denoted by $HH_n(A/k)$ are called the *Hochschild homology* of the k-algebra A.

Example 2.1 (Low degrees)**.** (i) $n = 0$. The image of boundary map $A \otimes_k A \to A$ is precisely the k-submodule generated by elements of the form $ab - ba$, i.e., the commutator $[A, A]$ of A; so $HH_0(A/k) = A/[A, A]$. In particular, if A is commutative then $HH_0(A/k) = A$.

(ii) $n = 1$. Assume A is commutative. Then we have just seen that the first boundary map n the Hochschild complex is zero, so that

$$HH_1(A/k) = A \otimes_k A / \langle ab \otimes c - a \otimes bc + ca \otimes b : a, b, c \in A \rangle,$$

where $\langle\ \rangle$ denotes here the abelian group (or equivalently A-module) generated by the indicated elements. The relation by which we are quotienting is precisely the Leibniz rule, and so we see that there is an isomorphism of A-modules

$$HH_1(A/k) \xrightarrow{\simeq} \Omega^1_{A/k}, \qquad a \otimes b \mapsto a\, db.$$

Example 2.2 (The case $A = k$)**.** If $A = k$ then the boundary maps in the Hochschild complex are alternately zero and the identity, whence

$$HH_n(k/k) = \begin{cases} k & n = 0 \\ 0 & n > 0. \end{cases}$$

The higher Hochschild homology groups vanish more generally whenever A is a commutative étale k-algebra, as will follow from the Tor description of Remark 2.4.

Remark 2.3 (Simplicial perspective)**.** Recall that a simplicial object in a category consists of objects A_n, for $n \geq 0$, and morphisms

$$\begin{aligned} \text{“face maps”} \quad & d_i : A_n \to A_{n-1}, \quad i = 0, \dots, n, \\ \text{“degeneracy maps”} \quad & s_i : A_n \to A_{n+1}, \quad i = 0, \dots, n, \end{aligned}$$

satisfying the rules $d_i d_j = d_{j-1} d_i$ for $i < j$, and $s_i s_j = s_{j+1} s_i$ for $i \leq j$, and

$$d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ \mathrm{id} & i = j \text{ or } j+1 \\ s_j d_{i-1} & i > j+1. \end{cases}$$

Diagramatically, we tend to forget about the degeneracy maps and draw such an object as

$$A_0 \Lleftarrow A_1 \Lleftarrow\!\!\!\!\Lleftarrow A_2 \Lleftarrow\!\!\!\!\Lleftarrow\!\!\!\!\Lleftarrow \cdots$$

A simplicial object is equivalently a functor to the category from Δ^{op}, where Δ is the simplicial category.

A simplicial object $A_\bullet$ in any additive category gives rise to a chain complex $A_0 \xleftarrow{d} A_1 \xleftarrow{d} A_2 \xleftarrow{d} \cdots$ where the boundary maps are given by the alternating sum of the face maps, namely $d := \sum_{i=0}^{n} (-1)^i d_i : A_n \to A_{n-1}$; this is known as the *un-normalised chain complex* associated to $A_\bullet$ (and, by a common abuse of notation, usually still denoted by $A_\bullet$).

In particular, to our flat k-algebra A we may associate a simplicial k-module defined by $A_n := A^{\otimes_k n+1}$ with face and degeneracy maps

$$\begin{aligned} d_i(a_0 \otimes \cdots \otimes a_n) &:= \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & i < n, \\ a_n a_0 \otimes \cdots \otimes a_{n-1} & i = n, \end{cases} \\ s_i(a_0 \otimes \cdots \otimes a_n) &:= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n, \end{aligned}$$

whose associated un-normalised chain complex is precisely the Hochschild complex $HH(A/k)$ as defined above. We will interchangeably use $HH(A/k)$ to denote both this simplicial k-module and the associated complex.

Remark 2.4 (Bar complex and Tor description)**.** Don't confuse $HH_n(A/k)$ with the *bar complex*

$$B_\bullet(A/k) := \qquad A \otimes_k A \xleftarrow{b'} A \otimes_k A \otimes_k A \xleftarrow{b'} \cdots$$

$$b' : A^{\otimes_k n} \to A^{\otimes_k n-1}, \qquad a_0 \otimes \cdots \otimes a_{n-1} \mapsto \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} a_i \otimes \cdots \otimes a_{n-1}$$

(i.e., same as the Hochschild complex except that the product of the first and last tensor factors does not appear), which can alternatively be described simplicially as

$$\begin{aligned} d_i(a_0 \otimes \cdots \otimes a_{n-1}) &:= a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ s_i(a_0 \otimes \cdots \otimes a_{n-1}) &:= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

In fact, the bar complex $B_\bullet(A/k)$ is acyclic except possibly in degree 0, with multiplication $b' = \mu : A \otimes_k A \to A$ defining a quasi-isomorphism $B_\bullet(A/k) \xrightarrow{\sim} A$; this follows formally from the existence of

the "extra degeneracy" $s : A^{\otimes_k n} \to A^{\otimes_k n+1}$, $a_0 \otimes \cdots \otimes a_{n-1} \mapsto 1 \otimes a_0 \otimes \cdots \otimes a_{n-1}$, which serves as a contracting homotopy for $B_\bullet(A/k)$ as a simplicial object [29, 1.1.12].

The bar complex nevertheless plays a role in the theory in several ways. In particular, observe that it consists of left $A \otimes_k A^{\mathrm{op}}$-modules, under the rule $(f \otimes g) \cdot (a_0 \otimes \cdots \otimes a_{n-1}) := f a_0 \otimes a_1 \otimes \cdots \otimes a_{n-2} \otimes a_{n-1} g$, which are moreover flat since A is flat over k. Thus $B_\bullet(A/k) \xrightarrow{\sim} A$ serves as a resolution of A by flat $A \otimes_k A^{\mathrm{op}}$-modules (where we view A as a left $A \otimes_k A^{\mathrm{op}}$-module). But one checks directly that there is an isomorphism of complexes (even of simplicial objects)

$$A \otimes_{A \otimes_k A^{\mathrm{op}}} B_\bullet(A/k) \xrightarrow{\simeq} HH(A/k), \qquad a_0 \otimes (f \otimes a_1 \otimes \cdots \otimes a_n \otimes g) \mapsto g a_0 f \otimes a_1 \otimes \cdots \otimes a_n$$

(the fact that f and g change order is an inevitable consequence of the fact that the tensor product $A \otimes_{A \otimes_k A^{\mathrm{op}}}$ involves viewing A as a right $A \otimes_k A^{\mathrm{op}}$-module) whence we see that $HH(A/k)$ is a model for the derived tensor product $A \otimes^{\mathbb{L}}_{A \otimes_k A^{\mathrm{op}}} A$; in particular, $HH_*(A/k) \cong \mathrm{Tor}_*^{A \otimes_k A^{\mathrm{op}}}(A, A)$.

Remark 2.5 (The case A commutative)**.** In this remark we assume that A is a commutative k-algebra, which is our main case of interest. Then the face and degeneracy maps in $HH(A/k)$ are ring homomorphisms, which are even A-linear if we declare A to act on each $A^{\otimes_k n}$ by multiplication on the left-most tensor factor. In this way $HH(A/k)$ becomes a simplicial A-algebra and the boundary maps b are A-linear. It follows formally that the homology groups $HH_n(A/k)$ are A-modules, and that $HH_*(A/k) = \bigoplus_{n \geq 0} HH_n(A/k)$ has the structure of a graded-commutative A-algebra.

Similarly, $B_\bullet(A/k)$ is a simplicial $A \otimes_k A$-algebra and the identification $A \otimes_{A \otimes_k A^{\mathrm{op}}} B_\bullet(A/k) \cong HH(A/k)$ is compatible with multiplicative structure, whence $HH(A/k)$ models the simplicial commutative ring $A \otimes^{\mathbb{L}}_{A \otimes_k A} A$.

Remark 2.6 (Étale extensions)**.** Continue to suppose that A is a commutative, flat k-algebra, and let A' be an étale A-algebra. Then the canonical map $HH(A/k) \otimes_A A' \to HH_n(A'/k)$ is an isomorphism for any $n \geq 0$. This does not follow tautologically from the definition of Hochschild homology (even if A' is a localisation of A) but may be proved in several ways; here we present one approach, for another see [40].

Since $A \to A'$ is flat, it suffices to check that $HH(A/k) \otimes^{\mathbb{L}}_A A' \xrightarrow{\sim} HH(A'/k)$, which reduces via the Hochschild–Kostant–Rosenberg filtration of Proposition 2.28 to showing that $\mathbb{L}^i_{A/k} \otimes_A A' \xrightarrow{\sim} \mathbb{L}^i_{A'/k}$ for all $i \geq 0$ (where $\mathbb{L}$ is the cotangent complex, which will be introduced in §2.4). But this follows easily from the fundamental properties of the cotangent complex which will be presented in Proposition 2.27.

The cautious reader should be concerned about circular reasoning, given that the proof of Proposition 2.28 will Theorem 2.8, which in turn uses this remark, but in fact Proposition 2.28 only requires Theorem 2.8 in the case of polynomial algebras, whereas this remark will be used to extend from polynomial algebras to general smooth algebras.

Remark 2.7 (Transitivity)**.** Suppose that we are given a flat morphism of flat k-algebras $A \to B$, where A is assumed to be commutative. Then we claim that the canonical map of simplicial A-algebras is an isomorphism

$$HH(B/k) \otimes_{HH(A/k)} A \xrightarrow{\simeq} HH(B/A).$$

Here we use that A is commutative so that the adjunction $HH(A/k) \to HH_0(A/k)$ is a map to A. Since tensor products of simplicial rings are by definition computed degree-wise, the claimed isomorphism is simply the assertion that

$$(B \otimes_k \cdots \otimes_k B) \otimes_{A \otimes_k \cdots \otimes_k A, \mu} A \xrightarrow{\simeq} B \otimes_A \cdots \otimes_A B$$

for any tensor power, where we base change along the multiplication map μ. But this is obvious: the universal property of the right side yields the inverse map.

We finish this foundational subsection by discussing the case of smooth algebras. Assuming for a moment that A is any commutative, flat k-algebra, then the identification $\Omega^1_{A/k} \xrightarrow{\simeq} HH_1(A/k)$ of Example 2.1(i) induces a map of graded algebras $\Omega^*_{A/k} \to HH_*(A/k)$, simply because $HH_*(A/k)$ is a graded-commutative A-algebra by Remark 2.5. By explicitly examining the definition of the product

structure on the homotopy groups of a simplicial commutative rings (known as the *shuffle product*), one can check that this map of graded algebras is given in each degree by the maps

$$\varepsilon_n : \Omega^n_{A/k} \longrightarrow HH_n(A/k), \qquad a\, db_1 \wedge \cdots \wedge db_n \mapsto \sum_{\sigma \in \mathrm{Sym}_n} (-1)^{\mathrm{sign}(\sigma)} a \otimes b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(n)}$$

for each $n \geq 0$ (however, we will try to avoid using this explicit formula). These are often known as the *anti-symmetrisation maps*.

We may now present the fundamental description of Hochschild homology of smooth algebras:

Theorem 2.8 (Hochschild–Kostant–Rosenberg Theorem 1962)**.** *If A is a smooth k-algebra, then the antisymmetrisation maps $\Omega^n_{A/k} \to HH_n(A/k)$ are isomorphisms of A-modules for all $n \geq 0$.*

Proof. The goal is to establish that the morphism of graded-commutative rings $\Omega^*_{A/k} \to HH_*(A/k)$, which is already known to be an isomorphism in degrees ≤ 1, is in fact an isomorphism in all degrees. It is therefore equivalent to prove that the canonical map $\bigwedge^*_A HH_1(A/k) \to HH_*(A/k)$ is an isomorphism; by now appealing to the Tor description of Hochschild homology from Remarks 2.4 and 2.5, we see that it is equivalent to check that the canonical map $\bigwedge^*_A \mathrm{Tor}_1^{A\otimes_k A}(A, A) \to \mathrm{Tor}_*^{A\otimes_k A}(A, A)$ is an isomorphism, i.e., that the Tor algebra $\mathrm{Tor}_*^{A\otimes_k A}(A, A)$ is the free graded-commutative A-algebra generated in degree 1 by $\mathrm{Tor}_1^{A\otimes_k A}(A, A) = I/I^2$, where $I = \mathrm{Ker}(A \otimes_k A \xrightarrow{\mu} A)$. This is well-known to be true if I is generated by a regular sequence (just compute the graded Tor ring using a Koszul complex).

We have completed the proof whenever $\mathrm{Ker}\,\mu$ is generated by a regular sequence, e.g., if A is a polynomial k-algebra.

But A is smooth over k, so by arguing Zariski locally on $\mathrm{Spec}\, A$ (here we use that $HH_n(A/k)\otimes_A A[\frac{1}{f}] = HH_n(A[\frac{1}{f}]/k)$ for all $f \in A$, thanks to Remark 2.6) we may reduce to the case that A is étale over a polynomial algebra; the result for A then follows by the polynomial case and another use of Remark 2.6. □

Remark 2.9 (Detecting smoothness with *HH*)**.** The HKR Theorem admits various converses; for example, if A is a non-smooth, but finitely generated, algebra over a field k, then $HH_n(A/k)$ is non-zero for infinitely many $n \geq 0$ [3].

Question 2.10. Which $\mathbb{E}_\infty$-rings A have the property that $THH_*(A) = \bigwedge^*_A THH_1(A)$? This might be known, I have not yet asked any experts.

2.2 Cyclic homology and its variants; cyclic objects and mixed complexes

The goal of this subsection is first to define classical cyclic homology and its variants, namely negative and periodic cyclic homology; then we will axiomatise the formalism by summarising Connes' theory of cyclic objects and Kassel–Burghelea's mixed complexes.

For the moment we continue to let A be a flat algebra over a commutative base ring k. The starting point for cyclic homology is the observation that each tensor power $A^{\otimes_k n+1}$ in the Hochschild complex carries a k-linear action by the cyclic group $\mathbb{Z}/n+1$ via permutation, i.e., the generator $t_n = 1 \in \mathbb{Z}/n+1$ acts[1] as

$$t_n \cdot a_0 \otimes \cdots \otimes a_n := a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

Following classical notation, we also introduce the k-linear maps

$$\text{“norm”} \quad N := \sum_{i=0}^{n} ((-1)^n t_n)^i : A^{\otimes_k n+1} \to A^{\otimes_k n+1},$$

$$\text{“extra degeneracy”} \quad s : A^{\otimes_k n} \to A^{\otimes_k n+1}, \qquad a_0 \otimes \cdots \otimes a_{n-1} \mapsto 1 \otimes a_0 \otimes \cdots \otimes a_{n-1},$$

and the composition

$$\text{“Connes' (boundary) operator”} \quad B := (1 - (-1)^n t_n) s N : A^{\otimes_k n} \longrightarrow A^{\otimes_k n+1}.$$

[1] Compared to the first version of these notes, the sign convention has changed.

When n is clear from the context one tends to write simply t instead of t_n to simplify notation; with this in mind we will adopt the (non-standard) convention of writing $\pm t := (-1)^n t_n$ so that, for example, $N = \sum_{i=0}^{n} (\pm t)^i$ and $B = (1 - \pm t)sN$.

Lemma 2.11. *The above operators satisfy the following identities:*

(i) $(1 - \pm t)b' = n(1 - \pm t)$ *and* $b'N = Nb$.

(ii) $sb' + b's = 1$, $B^2 = 0$, *and* $Bb = -bB$.

Proof. We leave these to the reader as an exercise; see [29, Lem. 2.1.1]. □

Remark 2.12 (Connes' operator vs. the de Rham differential)**.** Connes' boundary operator clearly induces k-linear maps $B : HH_n(A/k) \to HH_{n+1}(A/k)$ for $n \geq 0$. When A is commutative, as we assume in the rest of the remark, then this turns the graded algebra $HH_*(A/k)$ into a commutative differential graded k-algebra [To do: insert proof of this.] Since $HH_0(A/k) = A$, the universal property of $\Omega^*_{A/k}$ (namely, it is the initial cdg k-algebra which is A in degree 0) induces a map of differential graded k-algebras $\Omega^*_{A/k} \to HH_*(A/k)$. However this is nothing other than the anti-symmetrisation map constructed before Theorem 2.8, since the map

$$B : A = HH_0(A/k) \longrightarrow \Omega^1_{A/k} = HH_1(A/k)$$

is easily seen to be precisely the de Rham differential (the reader should check).

In particular, it follows that the following diagrams commute

$$\begin{tikzcd}
HH_n(A/k) \arrow[r, "B"] & HH_{n+1}(A/k) \\
\Omega^n_{A/k} \arrow[u, "\varepsilon_n"] \arrow[r, "d"'] & \Omega^{n+1}_{A/k} \arrow[u, "\varepsilon_{n+1}"']
\end{tikzcd}$$

i.e., B is a refinement of the de Rham differential. Part of Connes' motivation when developing cyclic homology was to consider B as a generalisation of the de Rham differential even when A was non-commutative.

The classical double complex approach to defining cyclic homology and its variants are based on the following (anticommuting) bicomplex of k-modules, which is horizontally 2-periodic

$$\begin{tikzcd}
\cdots & A^{\otimes 3} \arrow[d, "-b'"] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "1-\pm t"'] \arrow[d, "-b'"] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "1-\pm t"'] \arrow[d, "-b'"] & \cdots \\
\cdots & A^{\otimes 2} \arrow[d, "-b'"] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-\pm t"'] \arrow[d, "-b'"] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-\pm t"'] \arrow[d, "-b'"] & \cdots \\
\cdots & A & \underline{A} \arrow[l, "N"'] & A \arrow[l, "1-\pm t"'] & A \arrow[l, "N"'] & A \arrow[l, "1-\pm t"'] & \cdots
\end{tikzcd}$$

(the term in degree $(0, 0)$ is underlined, a convention we will follow throughout this subsection), i.e., the even columns are the Hochschild complex $HH(A/k)$ and the odd columns are the augmented bar complex $B_\bullet(A/k) \to A$ from Remark 2.4. Restricting this bicomplex to the right half plane $x \geq 0$ gives a first quadrant bicomplex whose totalisation we define to be the *cyclic homology* $HC(A/k)$. Similarly, restricting to the left half plane $x \leq 0$ gives a second quadrant bicomplex whose product totalisation is defined to be *negative cyclic homology* $HC^-(A/k)$ (also denoted by $HN(A/k)$ in the literature). Finally, the product totalisation of the entire 2-periodic bicomplex is *periodic cyclic homology* $HP(A/k)$.

However, these double complexes can be simplified by recalling from Remark 2.4 that the odd columns in the above bicomplex are all acyclic, with contracting homotopy given by the extra degeneracy s; this allows us formally to remove all odd columns from the bicomplex by mapping between adjacent even columns via $(1 - \pm t)sN = B$. More precisely, it follows that $HP(A/k)$ is quasi-isomorphic to its subcomplex given by totalising the *BP-bicomplex*

$$\begin{tikzcd}
\cdots & A^{\otimes 4} \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "B"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\
\cdots & A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] & \\
\cdots & A^{\otimes 2} \arrow[d, "b"] & \underline{A} \arrow[l, "B"'] & & \\
\cdots & A & & &
\end{tikzcd}$$

Similarly, $HC^-(A/k)$ and $HC(A/k)$ are quasi-isomorphic to their subcomplexes given respectively by totalising the restriction of the BP-bicomplex to $x \leq 0$ or $x \geq 0$, which we call the *BC^--bicomplex* and *BC-bicomplex*

$$\begin{tikzcd}
\cdots & A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & \underline{A} \arrow[l, "B"'] \\
\cdots & A^{\otimes 2} \arrow[d, "b"] & A \arrow[l, "B"'] & \\
\cdots & A & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\
A^{\otimes 2} \arrow[d, "b"] & A \arrow[l, "B"'] & \\
\underline{A} & &
\end{tikzcd}$$

For some further details on this process of discarding all all columns of the original bicomplexes, we refer the reader to [29, §2.1.7] or [30, Prop. 1.5].

Over the next sequence of remarks we explain various formal properties of cyclic homology and its variants, all of which follow by elementary examination of the double complexes defining them.

Remark 2.13 (Periodicity sequence for HC)**.** Here we explain the periodicity sequence through which cyclic homology is built from successive shifts of Hochschild homology.

Viewing $HC(A/k)$ as the totalisation of the latter BC-bicomplex, the inclusion of the zero-th column defines an inclusion $I : HH(A/k) \hookrightarrow HC(A/k)$; the cokernel of I is the totalisation of what remains after removing the first column from the BC-bicomplex, which is simply another copy of the BC-bicomplex, but shifted by bidegree $(1, 1)$. This gives a short exact sequence of complexes

$$0 \longrightarrow HH(A/k) \xrightarrow{I} HC(A/k) \xrightarrow{S} HC(A/k)[2] \longrightarrow 0$$

with associated boundary map $B : HC(A/k)[1] \to HH(A/k)$. The projection map $S : HC(A/k) \to HC(A/k)[2]$ from removing the zero-th column is known as *Connes' periodicity operator*.

In other words, $\mathrm{Fil}^i := \mathrm{Ker}(S^i : HC(A/k) \to HC(A/k)[2i])$ defines an exhaustive, increasing, $\mathbb{N}$-indexed filtration on $HC(A/k)$ whose i^{th} graded piece is $HH(A/k)[2i]$ for each $i \geq 0$. For explanation of our adjectives concerning filtrations, see Remark 2.29

Remark 2.14 (Periodicity sequences for HC^-)**.** Similarly to Remark 2.14, projecting the BC^--bicomplex to its zero-th column defines a short exact sequence of complexes

$$0 \longrightarrow HC^-(A/k)[-2] \xrightarrow{S} HC^-(A/k) \xrightarrow{h} HH(A/k) \longrightarrow 0,$$

and the images of all powers of S define a complete, descending, $\mathbb{N}$-indexed filtration on $HC^-(A/k)$ with graded pieces $HH(A/k)$[-2i], $i \geq 0$.

Remark 2.15 (Norm sequence)**.** Continuing the theme of Remarks 2.13 and 2.14, the inclusion I of the BC^--bicomplex into the BP-bicomplex has complement given by a shift of the BC-bicomplex, and there are projections h from the BC^--bicomplex to the BC-bicomplex, and from the BP-bicomplex to the BC-bicomplex. Using the sequences of the previous remarks, the summary is a commutative diagram of short exact sequences as follows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & HC^-(A/k)[-2] \arrow[r, equal] \arrow[d, "S"'] & HC^-(A/k)[-2] \arrow[d, "IS"'] & & \\
0 \arrow[r] & HC^-(A/k) \arrow[r, "I"] \arrow[d, "h"'] & HP(A/k) \arrow[r, "Sh"] \arrow[d, "h"'] & HC(A/k)[2] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & HH(A/k) \arrow[r, "I"'] \arrow[d] & HC(A/k) \arrow[r, "S"'] \arrow[d] & HC(A/k)[2] \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The connecting map $HC(A/k)[1] \to HC^-(A/k)$ associated to the short exact sequence in the middle is represented by the norm map N (this is easier to make precise using the original 2-periodic (b, b')-bicomplexes).

Remark 2.16 (Periodicity of HP)**.** The restriction of the BP-bicomplex to any right half plane $x \geq -n$ is of course a copy of the BC-bicomplex shifted by bidegree $(-n, -n)$. Taking the inverse limit as $n \to \infty$ shows that

$$HP(A/k) \simeq \mathrm{Rlim}_{n\to\infty} HC(A/k)[2n],$$

where the transition maps are shifts of the periodicity operator $S : HC(A/k) \to HC(A/k)[2]$. In other words, periodic cyclic homology is constructed by forcing the periodicty operator to be a quasi-isomorphism, and in particular it is 2-periodic: $S : HP(A/k) \xrightarrow{\sim} HP(A/k)[2]$.

Next we axiomatise the essential properties of the Hochschild complex which were used in the above constructions. However, we suggest that the reader first jumps ahead to the examples of §2.3 to see why this axiomatisation is useful.

Definition 2.17 (Connes [14])**.** A *cyclic object* in a category $\mathcal{C}$ is a simplicial object $X_\bullet$ in $\mathcal{C}$ such that each X_n is equipped with an automorphism t_n of order $n+1$, and these are required to satisfy:

$$t_n d_i = \begin{cases} d_{i-1} t_{n-1} & 1 \leq i \leq n \\ d_n & i = 0 \end{cases} \qquad t_n s_i = \begin{cases} d_{i-1} s_{n-1} & 1 \leq i \leq n \\ s_n t_{n+1}^2 & i = 0. \end{cases}$$

Equivalently, a cyclic object is a functor $\Lambda^{\mathrm{op}} \to \mathcal{C}$, where Λ is Connes' *cyclic category*, characterised as follows:

- Λ contains Δ as a full subcategory having the same objects;
- $\mathrm{Aut}_\Lambda([n]) = \mathbb{Z}/n+1$ for each $n \geq 0$ (whereas $\mathrm{Aut}_\Delta([n]) = 1$);
- The above relations between $t_n \in \mathbb{Z}/n+1$ and the face and degeneracy maps hold.

Checking that there really exists such a category Λ characterised by these properties requires some mild combinatorics which we omit; see [29, §6.1].

Example 2.18. The reader will easily observe that the simplicial k-module $HH(A/k)$ is indeed a cyclic object in k-modules, with the automorphisms t_n as defined at the start of the subsection.

Assume now that A is commutative, whence $HH(A/k)$ is even a cyclic object in k-algebras. In fact, as an exercise prove the following universal characterisation of the $HH(A/k)$: it is initial among the collection of pairs $(X_\bullet, A \to X_\bullet)$, where $X_\bullet$ is a cyclic object in k-algebras and $A \to X_\bullet$ is a morphism of simplicial k-algebras (i.e., the underlying simplicial object of $X_\bullet$ is given the structure of a simplicial A-module).

When viewed as a complex, the essential data on $HH(A/k)$ was instead as follows:

Definition 2.19 (Burghelea [13], Kassel [25])**.** A *mixed complex*, or *algebraic S^1-complex*, $(A_\bullet, b, B)$ over k is the data of a chain complex of k-modules

$$A_\bullet = \qquad \cdots \xleftarrow{b} A_n \xleftarrow{b} A_{n+1} \xleftarrow{b} \cdots$$

supported in homological degrees ≥ 0, together with maps of k-modules $B : A_n \to A_{n+1}$ satisfying both $Bb = -bB$ (i.e., B is a morphism of complexes $A \to A[1]$) and $B^2 = 0$.

Given a mixed complex $(A_\bullet, b, B)$, its *periodic cyclic homology* $HP(A_\bullet, b, B)$, *negative cyclic homology* $HC^-(A_\bullet, b, B)$, and *cyclic homology* $HC(A_\bullet, b, B)$, are the product totalisations of the following bicomplex:

$$\begin{tikzcd}
\cdots & A_3 \arrow[d, "b"] & A_2 \arrow[l, "B"'] \arrow[d, "b"] & A_1 \arrow[l, "B"'] \arrow[d, "b"] & A_0 \arrow[l, "B"'] \\
\cdots & A_0 \arrow[d, "b"] & A_1 \arrow[l, "B"'] \arrow[d, "b"] & A_0 \arrow[l, "B"'] & \\
\cdots & A_1 \arrow[d, "b"] & \underline{A_0} \arrow[l, "B"'] & & \\
\cdots & A_0 & & &
\end{tikzcd}$$

resp. its restriction to the left half plane $x \leq 0$, resp. its restriction to the right half plane $x \geq 0$.

Example 2.20. (i) The Hochschild complex $HH(A/k)$ is a mixed complex with respect to Connes' operator B, by Lemma 2.11; in this case Definition 2.19 of course reduces to the earlier definitions of cyclic, negative cyclic, and periodic cyclic homology.

(ii) Given any cochain complex $\cdots \xrightarrow{B} C^n \xrightarrow{B} C^{n+1} \xrightarrow{B} \cdots$ supported in cohomological degrees ≥ 0, we can form a mixed complex $(C_\bullet, 0, B)$ by setting $C_n := C^n$ and declaring the b-differentials to be zero. Then the double complex of Definition 2.19 is a product of infinitely many copies of shifts of

the cochain complex $(C^\bullet, B)$; truncating to the left or right half plane involves naively truncating this cochain complex. Taking care of the indexing, one sees that

$$HC^-(C_\bullet, 0, B) = \prod_{i\in\mathbb{Z}} C^{\geq i}[2i], \qquad HP(C_\bullet, 0, B) = \prod_{i\in\mathbb{Z}} C^\bullet[2i], \qquad HC(C_\bullet, 0, B) = \prod_{i\geq 0} C^{\leq i}[2i].$$

Of particular interest will be the mixed complex $(\Omega^\bullet_{A/k}, 0, d)$, where $\Omega^\bullet_{A/k}$ is the usual de Rham complex of a commutative k-algebra A.

(Warning: there is an unfortunate conflict of notation here: usually if we view a cochain complex $C^\bullet$ as a chain complex then it implicitly means that $C_n := C^{-n}$ with the same differential as in $C^\bullet$; but mixed complexes will appear sufficiently rarely that this conflict should not cause any problems.)

(iii) At the other extreme to example (ii), one can also consider mixed complexes $(A_\bullet, b, 0)$ in which the B-differential is zero. Then the double complex of Definition 2.19 is a product of infinitely many copies of shifts of the chain complex $A_\bullet$, and one sees that

$$HC^-(A_\bullet, b, 0) = \prod_{i\leq 0} A_\bullet[2i], \qquad HP(A_\bullet, b, 0) = \prod_{i\in\mathbb{Z}} A_\bullet[2i], \qquad HC(A_\bullet, b, B) = \prod_{i\geq 0} A_\bullet[2i].$$

Remark 2.21. Remarks 2.13–2.16 extend verbatim to arbitrary mixed complexes, from which one obtains the following sort of consequence (the reader should check if uncertain): given a morphism of mixed complexes $(A_\bullet, b, B) \to (A'_\bullet, b', B')$ (the notion should be clear) such that $A_\bullet \to A'_\bullet$ is a quasi-isomorphism of chain complexes, then $HC(A_\bullet, b, B) \to HC(A'_\bullet, b', B')$ is also a quasi-isomorphism, and similarly for HC^- and HP.

Now let $A_\bullet$ be a cyclic object in k-modules. Then we claim that its un-normalised chain complex

$$A_\bullet := \quad A_0 \xleftarrow{b} A_1 \xleftarrow{b} \cdots, \qquad b = \sum_{i=0}^{n} d_i$$

admits the structure of a mixed complex. Indeed, defining the "extra degeneracy" $s : A_{n-1} \to A_n$ to be $s := t_n s_n$, we may then define the norm N and boundary operator B exactly as at the start of the subsection and repeat the arguments of Lemma 2.11 verbatim to see that $(A_\bullet, b, B)$ is a mixed complex.

There is moreover an analogue of the bar complex in this generality, namely the chain complex

$$A_0 \xleftarrow{b'} A_1 \xleftarrow{b'} \cdots \qquad b' = \sum_{i=0}^{n-1} d_i,$$

which is again acyclic since s serves as a contracting homotopy. Just as for Hochschild homology we may therefore form the bicomplex

$$\begin{tikzcd}
\cdots & A_2 \arrow[d, "-b'"] & A_2 \arrow[l, "N"'] \arrow[d, "b"] & A_2 \arrow[l, "1-\pm t"'] \arrow[d, "-b'"] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"] & A_2 \arrow[l, "1-\pm t"'] \arrow[d, "-b'"] & \cdots \\
\cdots & A_1 \arrow[d, "-b'"] & A_1 \arrow[l, "N"'] \arrow[d, "b"] & A_1 \arrow[l, "1-\pm t"'] \arrow[d, "-b'"] & A_1 \arrow[l, "N"'] \arrow[d, "b"] & A_2 \arrow[l, "1-\pm t"'] \arrow[d, "-b'"] & \cdots \\
\cdots & A_0 & A_0 \arrow[l, "N"'] & \underline{A_0} \arrow[l, "1-\pm t"'] & A_0 \arrow[l, "N"'] & A_0 \arrow[l, "1-\pm t"'] & \cdots
\end{tikzcd}$$

and argue that $HP(A_\bullet, b, B)$, $HC^-(A_\bullet, b, B)$, and $HC(A_\bullet, b, B)$ could instead have been defined (up to quasi-isomorphism) as the totalisation of this bicomplex, resp. its truncation to $x \leq 0$, resp. to $x \geq 0$.

We will return to cyclic objects and mixed complexes, and their relation, from a more highbrow point of view in §4.

2.3 Two examples: smooth algebras and group algebras

2.3.1 Smooth algebras in characteristic zero

The main classical example in cyclic homology which will be important to us (albeit only as motivation) is the case of smooth algebras in characteristic zero, when it can be described in terms of de Rham cohomology. For any commutative k-algebra A, we denote as usual by $\Omega^\bullet_{A/k}$ its de Rham complex and by

$$\Omega^{\leq i}_{A/k} := \qquad A \xrightarrow{d} \cdots \xrightarrow{d} \Omega^i_{A/k} \longrightarrow 0 \longrightarrow \cdots$$

its naive truncations; similarly define $\Omega^{\geq i}_{A/k}$, so that $0 \to \Omega^{\geq i}_{A/k} \to \Omega^\bullet_{A/k} \to \Omega^{\leq i-1}_{A/k} \to 0$ is a short exact sequence of complexes.

Theorem 2.22 (Connes, Loday–Quillen, Feigin–Tsygan 1980s)**.** *Let k be a commutative base ring containing $\mathbb{Q}$ and A a smooth k-algebra. Then there are natural equivalences*

$$HC^-(A/k) \simeq \prod_{i\in\mathbb{Z}} \Omega^{\geq i}_{A/k}[2i], \qquad HP(A/k) \simeq \prod_{i\in\mathbb{Z}} \Omega^\bullet_{A/k}[2i], \qquad HC(A/k) \simeq \prod_{i\geq 0} \Omega^{\leq i}_{A/k}[2i].$$

Proof. We define maps

$$\pi : A^{\otimes_k n+1} \longrightarrow \Omega^n_{A/k}, \qquad a \otimes b_1 \otimes \cdots \otimes b_n \mapsto a\, db_1 \wedge \cdots \wedge db_n$$

and leave it to the reader to check directly that the following diagrams commute:

$$\begin{tikzcd} A^{\otimes_k n} \arrow[d, "\pi"'] & A^{\otimes_k n+1} \arrow[l, "b"'] \arrow[d, "\pi"] \\ \Omega^{n-1}_{A/k} & \Omega^n_{A/k} \arrow[l, "0"] \end{tikzcd} \qquad \begin{tikzcd} A^{\otimes_k n} \arrow[r, "B"] \arrow[d, "\pi"'] & A^{\otimes_k n+1} \arrow[d, "\pi"] \\ \Omega^{n-1}_{A/k} \arrow[r, "(n+1)d"'] & \Omega^n_{A/k} \end{tikzcd}$$

In other words, the maps π define a morphism of mixed complexes from $HH(A/k)$ to $(\Omega^\bullet_{A/k}, 0, (\bullet+1)d) :=$ the mixed complex with zero b-differential associated to the cochain complex

$$A \xrightarrow{2d} \Omega^1_{A/k} \xrightarrow{3d} \Omega^2_{A/k} \xrightarrow{4d} \cdots .$$

We also leave it to the reader to check that the composition $\Omega^n_{A/k} \xrightarrow{\varepsilon_n} HH_n(A/k) \xrightarrow{H_n(\pi)} \Omega^n_{A/k}$ is simply multiplication by $n!$, where ε_n is the anti-symmetrisation map which appeared in Theorem 2.8.

Now we use the hypotheses on A. Since it is smooth over k, Theorem 2.8 implies that ε_n is an isomorphism; since $k \supseteq \mathbb{Q}$, it follows that $H_n(\pi)$ is also an isomorphism. Replacing π by $\frac{1}{n!}\pi_n$ constructs a morphism of mixed complexes from $HH(A/k)$ to $(\Omega^\bullet_{A/k}, 0, d)$, which is a quasi-isomorphism of underlying complexes. Therefore it induces quasi-isomorphisms on HC, HC^-, and HP by Remark 2.21, whence Example 2.20(ii) completes the proof. □

2.3.2 Group algebras

Let k be a commutative base ring and let G be an abelian group, written multiplicatively. Here we describe some relations between the homology of G and the cyclic homology of the group algebra $k[G]$.

The *cyclic bar construction*

$$B^{\mathrm{cycl}}G = \quad G \Leftarrow\!\!\!

To analyse $B^{\mathrm{cycl}}G$ we decompose it as follows. Firstly, the simplicial set

$$BG = \quad 1 \Leftarrow G \Lleftarrow G^2 \Llleftarrow \cdots$$

denotes the usual bar construction; it may be equipped with the structure of a cyclic set by declaring the cyclic operator $t_n \in \mathbb{Z}/(n+1)\mathbb{Z}$ to act on $B_nG = G^n$ as

$$(g_1, \ldots, g_n) \mapsto ((g_1 \cdots g_n)^{-1}, g_1, \ldots, g_{n-1}).$$

More generally, for any $z \in G$ there is a "twisted" bar construction $B(G, z)$ which is again a cyclic set: as a simplicial set it is the same as BG, but the cyclic operator t_n is now defined by $(g_1, \ldots, g_n) \mapsto (z(g_1 \cdots g_n)^{-1}, g_1, \ldots, g_{n-1})$; obviously $B(G, 1) = BG$.

For any $z \in G$ there is an inclusion of cyclic sets

$$B(G, z) \hookrightarrow B^{\mathrm{cylc}}G, \qquad (g_1, \ldots, g_n) \mapsto (z(g_1 \cdots g_n)^{-1}, g_1, \ldots, g_n)$$

with image $\{(g_0, \ldots, g_n) \in B_n^{\mathrm{cylc}}G : g_0 \cdots g_n = z)\}$; these assemble to define a decomposition of the cyclic bar construction $B^{\mathrm{cylc}}G = \bigsqcup_{z \in G} B(G, z)$. Forgetting the cyclic structure, this decomposition may be written as one of simplicial sets $B^{\mathrm{cylc}}G = \bigsqcup_{z \in G} zBG$, where zBG is the left translation of BG.

By then k-linearising we see that we have defined a direct sum decomposition of cyclic k-modules

$$HH(k[G]/k) = \bigoplus_{z \in G} k[B(G, z)].$$

Forgetting the cyclic structure, this may be written as a direct sum decomposition of simplicial modules $HH(k[G]/k) = \bigoplus_{z \in G} zk[BG] = k[G] \otimes_k k[BG]$, which we summarise in the next lemma:

Lemma 2.23. *The natural map of simplicial $k[G]$-modules*

$$k[G] \otimes_k k[BG] \longrightarrow HH(k[G]/k), \qquad f \otimes (g_1, \ldots, g_n) \mapsto f(g_1 \cdots g_n)^{-1} \otimes g_1 \otimes \cdots \otimes g_n,$$

where $g_1, \ldots, g_n \in G$ and $f \in k[G]$, is an isomorphism. Passing to homology yields a natural isomorphism of graded $k[G]$-algebras

$$k[G] \otimes_k H_*(G, k) \xrightarrow{\sim} HH_*(k[G]/k).$$

Proof. The only statement still requiring justification is the homology calculation, which follows from the fact that the simplicial k-module $k[BG]$ models the group homology $R\Gamma(G, k)$, where G acts trivially on k. □

The cyclic structure on the twisted bar constructions is somewhat subtle, so we focus now only on the cyclic k-module $k[BG]$, whose associated cyclic homology etc. we denote by $HC(G/k)$ etc. It is known that the S^1-action on the geometric realisation of BG is trivial whence the cyclic homology and its variants behave as in Example 2.20 (since we do not need the following proposition, we refer to [29, §7.3.9 & Prop. 7.4.8] for further explanation):

Proposition 2.24. *There are natural equivalences*

$$HC^-(G/k) \simeq \prod_{i \leq 0} R\Gamma(G, k)[2i], \qquad HP(G/k) \simeq \prod_{i \in \mathbb{Z}} R\Gamma(G, k)[2i], \qquad HC(A/k) \simeq \prod_{i \geq 0} R\Gamma(G, k)[2i].$$

2.4 The cotangent complex and Hochschild homology for non-flat algebras

We briefly summarise the theory of the cotangent complex; the classical standard references are Quillen [35, 34], Illusie [23, 24], and M. André [1]. Let k be a commutative base ring and A a commutative k-algebra. Then there exists a simplicial resolution $P_\bullet \xrightarrow{\sim} A$ of A by polynomial k-algebras (possibly in infinitely many variables); this resolution is unique up to homotopy, whence the same is true of the simplicial A-module

$$\mathbb{L}_{A/k} := \Omega^1_{P_\bullet/k} \otimes_{P_\bullet} A$$

which is known as the *cotangent complex* of $k \to A$. By a standard abuse, $\mathbb{L}_{A/k}$ is often identified with its associated complex via Dold–Kan in $D(A)$ (whence the terminology). Similarly, the wedge powers of the cotangent complex are $\mathbb{L}^i_{A/k} := \bigwedge^i_A \mathbb{L}_{A/k} = \Omega^i_{P_\bullet/k} \otimes_{P_\bullet} A$.

Remark 2.25 (Left Kan extension)**.** Particularly in §3.1 it will be helpful to adopt a more modern perspective on the above construction, namely that of left Kan extensions. Let $\mathcal{D}$ be any ∞-category which admits sifted colimits (e.g., $D(k)$ or Sp), and write $k\text{-algs}_\Sigma$ for the category of polynomial k-algebras in finitely many variables. One says that a functor $\mathcal{F} : k\text{-algs} \to \mathcal{D}$ is *left Kan extended* (from $k\text{-algs}_\Sigma$) if the canonical map $\mathcal{F}(A) \to \varinjlim_{P\to A} \mathcal{F}(P)$ is an equivalence for each k-algebra A, where the colimit in $\mathcal{D}$ is taken over all $P \in k\text{-algs}_\Sigma$ mapping to A.

Conversely, given a functor $\mathcal{G} : k\text{-algs}_\Sigma \to \mathcal{D}$, we may left Kan extend it to $\mathbb{L}\mathcal{G} : k\text{-algs} \to \mathcal{D}$ by using the above formula: $\mathcal{G}(A) := \varinjlim_{P\to A} \mathcal{F}(P)$. We thus obtain an equivalence between functors $k\text{-algs}_\Sigma \to \mathcal{D}$ and left Kan extended functors $k\text{-algs} \to \mathcal{D}$.

Returning to the more concrete point of view, the left Kan extension is given as follows: first extend $\mathcal{G}$ to all polynomial k-algebras (possibly in infinitely many variables) by taking filtered colimits, then extend it to all k-algebras by the rule

$$\mathbb{L}\mathcal{G}(A) := |\mathcal{G}(P_\bullet)| = \text{geometric realisation of the simplicial } \mathcal{D}\text{-object } \mathcal{G}(P_\bullet)$$

where $P_\bullet \to A$ is any simplicial resolution of A by polynomial k-algebras. For example, $\mathbb{L}^i_{-/k}$ is precisely the left Kan extension of $\Omega^i_{-/k} : k\text{-algs}_\Sigma \to D(k)$ and could more correctly be denoted by $\mathbb{L}\Omega^i_{-/k}$.

Example 2.26 (Low degrees)**.** (i) For any k-algebra A and $n \geq 0$, the augmentation $\mathbb{L}^n_{A/k} \to \Omega^n_{A/k}$ induces $\pi_0(\mathbb{L}^n_{A/k}) \xrightarrow{\simeq} \Omega^n_{A/k}$. Indeed, it is clearly surjective since $P_0 \twoheadrightarrow A$ induces $\Omega^1_{P_0/k} \twoheadrightarrow \Omega^1_{A/k}$; but the differential $d : P_\bullet \to \Omega^1_{P_\bullet/k}$ induces a derivative $d : A \to \pi_0(\mathbb{L}_{A/k})$ which in turn induces a map $\Omega^1_{A/k} \to \pi_0(\mathbb{L}_{A/k})$ splitting the surjection.

(ii) If $A = k/I$ is a quotient of k, then $\pi_0(\mathbb{L}_{A/k}) = 0$ and $\pi_1(\mathbb{L}_{A/k}) = I/I^2$. The vanishing of π_0 follows from (i), so it remains to justify the description of π_1. Unfortunately there doesn't follow straight from the definitions: one either relates the cotangent complex to algebra extensions [23, Corol. II.1.2.8.1], or else reads it off Quillen's "fundamental spectral sequence" [34, Thm. 6.3].

Next we summarise the fundamental abstract properties of the cotangent complex:

Proposition 2.27. *Let k be a commutative ring and A a commutative k-algebra.*

(i) (Base change) For any base change $k \to k'$ (where k' is commutative) such that $\mathrm{Tor}^k_i(A, k') = 0$ for $i > 0$, then $\mathbb{L}_{A/k} \otimes^{\mathbb{L}}_k k' \xrightarrow{\sim} \mathbb{L}_{A'/k}$ where $A' := A \otimes_k k'$.

(ii) (Künneth) For any other commutative k-algebra B such that $\mathrm{Tor}^k_i(A, B) = 0$ for $i > 0$, then $\mathbb{L}_{A\otimes_k B/k} \simeq (\mathbb{L}_{A/k} \otimes^{\mathbb{L}}_k B) \oplus (\mathbb{L}_{B/k} \otimes^{\mathbb{L}}_k A)$.

(iii) (Transitivity sequence) Given a commutative A-algebra B, the resulting sequence

$$\mathbb{L}_{A/k} \otimes^{\mathbb{L}}_A B \longrightarrow \mathbb{L}_{B/k} \longrightarrow \mathbb{L}_{B/A}$$

is a fibre sequence.

(iv) (Localisation) If $S \subseteq k$ is a multiplicative system, then $\mathbb{L}_{S^{-1}k/k} \simeq 0$

(v) (Étale) If A is étale over k, then $\mathbb{L}_{A/k} \simeq 0$.

(vi) (Smooth) If A is smooth over k then $\mathbb{L}_{A/k}$ is supported in degree zero, whence $\mathbb{L}_{A/k} \xrightarrow{\sim} \Omega^1_{A/k}[0]$.

Proof. For (i), just pick a simplicial resolution $P_\bullet \xrightarrow{\sim} A$ by polynomial k-algebras and observe that $P_\bullet \otimes_k k' \xrightarrow{\sim} A'$ is a resolution of A' by polynomial k'-algebras (indeed, the failure of it to be a resolution is precisely the higher Tors which we have assumed vanish); then use that $\Omega^1_{P_n/k} \otimes_k k' = \Omega^1_{P_n\otimes_k k'/k'}$ for each $n \geq 0$.

(ii): Let $P_\bullet \xrightarrow{\sim} A$ be as in the previous paragraph, and $Q_\bullet \xrightarrow{\sim} B$ similarly for B. Then $P_\bullet \otimes_k Q_\bullet$ is a simplicial resolution of $A \otimes_k B$ by polynomial k-algebras, whence $\mathbb{L}_{A\otimes_k B/k} = \Omega^1_{P_\bullet\otimes_k Q_\bullet} \otimes_{P_\bullet\otimes_k Q_\bullet} (A \otimes_k B)$. Rewriting this using $\Omega^1_{P_n\otimes_k Q_n/k} = \Omega^1_{P_n/k} \oplus \Omega^1_{Q_n/k}$ for each n easily gives the desired result.

To prove the transitivity sequence one first uses some standard properties of simplicial commutative rings (or, more precisely, of their model structure; we refer to the proof of [34, Thm. 5.1] for details) to build a commutative diagram

$$\begin{tikzcd}
P_\bullet \arrow[r] \arrow[dr] & Q_\bullet \arrow[r] \arrow[dr] & Q_\bullet \otimes_{P_\bullet} A \arrow[d] \\
k \arrow[u] \arrow[r] & A \arrow[r] \arrow[ur] & B
\end{tikzcd}$$

where $P_\bullet$ is a simplicial resolution of A by free k-algebras, $Q_\bullet$ is a simplicial resolution of B by a cofibrant $P_\bullet$-algebra (this means that $Q_\bullet$ is a retract of a free $P_\bullet$-algebra), and $Q_\bullet \otimes_{P_\bullet} A$ is a simplicial resolution of B by free A-algebras (or perhaps only retracts of free A-algebras). Since the maps $k \to P_n \to Q_n$ are all smooth, there is an exact Jacobi–Zariski sequence $0 \to \Omega^1_{P_n/k} \otimes_{P_n} Q_n \to \Omega^1_{Q_n/k} \to \Omega^1_{Q_n/P_n} \to 0$ for each n. Tensoring by $- \otimes_{Q_n} B$ and rearranging some of the tensor products gives a fibre sequence $\Omega^1_{P_\bullet/k} \otimes_{P_\bullet} A \to \Omega^1_{Q_\bullet/k} \otimes_{Q_\bullet} B \to \Omega^1_{Q_\bullet \otimes_{P_\bullet} A/A} \otimes_{Q_\bullet \otimes_{P_\bullet} A} B$, as desired.

To prove (iv) and (v) we follow a clever argument apparently due to M. André. Given a multiplicative system $S \subseteq k$, we have $S^{-1}k \otimes_k S^{-1}k = S^{-1}k$ and so $\mathbb{L}_{S^{-1}k/k} \otimes_k S^{-1}k = \mathbb{L}_{S^{-1}k/k} \otimes_{S^{-1}k} (S^{-1}k \otimes_k S^{-1}k) = \mathbb{L}_{S^{-1}k/k}$; but part (i) shows that $\mathbb{L}_{S^{-1}k/k} \otimes_k S^{-1}k = \mathbb{L}_{S^{-1}k \otimes_k S^{-1}k/S^{-1}k} = \mathbb{L}_{S^{-1}k/S^{-1}k} = 0$.

For (v), we recall that étale means that $k \to A$ is flat and that $\mu : A \otimes_k A \to A$ induces an open immersion on Spec. Therefore, for any prime ideal $\mathfrak{p} \subseteq A$ we have $A_\mathfrak{p} = (A \otimes_k A)_\mathfrak{q}$ with $\mathfrak{q} := \mu^{-1}(\mathfrak{p})$, and so

$$\begin{aligned}
\mathbb{L}_{A/k} \otimes_A A_\mathfrak{p} &= \mathbb{L}_{A/k} \otimes_A (A \otimes_k A) \otimes_{A \otimes_k A} (A \otimes_k A)_\mathfrak{q} \\
&= \mathbb{L}_{A \otimes_k A/A} \otimes_{A \otimes_k A} (A \otimes_k A)_\mathfrak{q} && \text{(by part (i))} \\
&= \mathbb{L}_{(A \otimes_k A)_\mathfrak{q}/A} && \text{(by transitivity and } \mathbb{L}_{(A \otimes_k A)_\mathfrak{q}/A \otimes_k A} = 0 \text{ by (iv))} \\
&= \mathbb{L}_{A_\mathfrak{p}/A} \\
&= 0 && \text{(by (iv))}
\end{aligned}$$

Since this holds for all prime ideals $\mathfrak{p} \subseteq A$, we have $\mathbb{L}_{A/k} = 0$.

To prove (vi) we then argue as follows: it is enough to prove the result Zariski locally on Spec A (using (i)), so we may suppose that A is étale over $k[t_1, \ldots, t_d]$ for some $d \geq 0$. But then the result follows at once from the following: $\mathbb{L}_{k[\underline{t}]/k} = \Omega^1_{k[\underline{t}]/k}$ (since $k[\underline{t}]$ serves as a polynomial resolution of itself); $\mathbb{L}_{A/k[\underline{t}]} = 0$ by (v); the transitivity sequence for $k \to k[\underline{t}] \to A$; the well-known fact that $\Omega_{k[\underline{t}]/k} \otimes_{k[\underline{t}]} A \xrightarrow{\simeq} \Omega^1_{A/k}$ since $k[\underline{t}] \to A$ is étale. □

Our interest in the the cotangent complex stems from the following relation to Hochschild homology

Proposition 2.28 (Hochschild–Kostant–Rosenberg filtration)**.** *Let A be a commutative k-algebra.*

(i) Then the Hochschild complex $HH(A/k)$ (viewed as an object of $D(A)$) admits a natural, complete, descending $\mathbb{N}$-indexed filtration whose i^{th} graded piece is equivalent to $\mathbb{L}^i_{A/k}[i]$, for $i \geq 0$.

(ii) Similarly, $HC(A/k)$ admits a natural, complete, descending $\mathbb{N}$-indexed filtration whose i^{th}-graded piece is $\bigoplus_{n \geq 0} \mathbb{L}^i_{A/k}[i + 2n]$, for $i \geq 0$.

Remark 2.29 (Complete filtrations)**.** Although we have already used terminology surrounding filtrations in §2.2, we did not give definitions; we do so now. Let us begin with the case of an honest complex C over a ring; by a *descending, $\mathbb{N}$-indexed filtration* we mean a descending chain of subcomplexes $C = \operatorname{Fil}^0 C \supseteq \operatorname{Fil}^1 C \supseteq \cdots$. To say that it is *complete* means that the canonical map $C_n \to \varprojlim_i C_n / \operatorname{Fil}^i C_n$ is an isomorphism for each degree n.

However, even when C is an explicit complex like $HH(A/k)$, we always prefer to view C merely as an object of the derived category (where it does not make sense to discuss subcomplexes or the individual terms C_n); in that case a descending, $\mathbb{N}$-indexed filtration means simply that we are given complexes and maps between then $C = \operatorname{Fil}^0 C \leftarrow \operatorname{Fil}^1 C \leftarrow \operatorname{Fil}^2 C \leftarrow \cdots$, and complete means that the canonical map $C \to \operatorname{Rlim}_i C / \operatorname{Fil}^i C$ is an equivalence (informally, we are asking that C can be completely recovered from all the $C / \operatorname{Fil}^i C$). Here we write $C / \operatorname{Fil}^i C$ for the cofiber of the map $\operatorname{Fil}^i C \to C$, so the completeness condition is equivalent to asking that $\operatorname{Rlim}_i \operatorname{Fil}^i C \simeq 0$.

Occasionally we will also encounter ascending filtrations, in which case *exhaustive* means that the filtered colimit up the filtration coincides (up to equivalence) with the complex itself.

We have deliberately stated the proposition without assuming that A is necessarily flat over k; therefore we must first define Hochschild and cyclic homology in this greater degree of generality. Although the definitions of §2.1–2.2 do work without assuming such flatness, the resulting theory would less closely related to topological Hochschild homology (in which all tensor products will be automatically derived, thereby automatically overcoming the flatness issues) and Proposition 2.28 would not hold.

To define the Hochschild and cyclic homologies of a k-algebra which is not necessarily flat, we essentially replace $\otimes_k$ by $\otimes_k^{\mathbb{L}}$. More precisely, we define the Hochschild homology $HH(A/k)$ to be the diagonal of the bisimplicial k-module $HH(P_\bullet/k)$, where $P_\bullet \to A$ is a simplicial resolution of A by polynomial k-algebras. In other words (recalling that the diagonal of a bisimplicial object models the geometric realisation), we are defining $HH(-/k) : k\text{-algs} \to D(k)$ to be the left Kan extension of $HH(-/k) : k\text{-algs}_\Sigma \to D(k)$. Note that if A is flat over k, then the augmentations $P_\bullet^{\otimes_k n} \to A^{\otimes_k n}$ are equivalences for all $n \geq 0$, whence the diagonal of the bisimplicial k-module $HH(P_\bullet/k)$ is equivalent to $HH(A/k)$ as it was defined in the flat case, i.e., the two definitions agree in the flat case.

Similarly, we define $HC(A/k)$ as the totalisation of the simplicial cochain complex $HC(P_\bullet/k)$, or in other words as the left Kan extension of $HC(-/k) : k\text{-algs}_\Sigma \to D(k)$. This agrees with the old definition if A is flat over k, and the periodicity sequence and increasing filtration of Remark 2.13 formally remain valid for general A.

One can alternatively define $HH(A/k)$ and $HC(A/k)$ using a resolution of A by a flat differential graded k-algebra; this approach was adopted in [37, 38], for which reason $HH(A/k)$ of non-flat algebras is sometimes called Shukla homology.

Proof of Proposition 2.28. (i): Since $HH(A/k)$ is obtained by totalising the bisimplicial object $HH(P_\bullet/k)$, where $P_\bullet \to A$ is a simplicial resolution by free k-algebras, it follows formally that the Postnikov filtration on each $HH(P_n/k)$ naturally induces a complete, descending $\mathbb{N}$-indexed filtration on $HH(A/k)$ whose i^{th} graded piece is given by $HH_i(P_\bullet/k)[i]$, $i \geq 0$. But each P_n is a polynomial k-algebra, so Theorem 2.8 implies that $HH_i(P_n/k) \cong \Omega^i_{P_n/k}$; therefore $HH_i(P_\bullet/k) \cong \Omega^i_{P_\bullet/k}$, which we may rewrite up to equivalence as $\Omega^i_{P_\bullet/k} \otimes^{\mathbb{L}}_{P_\bullet} P_\bullet = \Omega^i_{P_\bullet/k} \otimes_{P_\bullet} A$ since each $\Omega^i_{P_n/k}$ is a free P_n-module; but the latter simplicial module is precisely $\mathbb{L}^i_{A/k}$.

In short, the desired filtration on $HH(-/k) : k\text{-algs} \to D(k)$ is the left Kan extension of the Postnikov filtration on $HH(-/k) : k\text{-algs}_\Sigma \to D(k)$.

(ii): We begin by supposing that $A \in k\text{-algs}_\Sigma$, or more generally that A is smooth over k. In terms of $HC(A/k)$ as the totalisation of the BC-bicomplex, we define $\mathrm{Fil}^i_{\mathrm{HKR}} HC(A/k)$ to be its subcomplex obtained by totalising

$$\begin{tikzcd}
A^{\otimes_k i+3} \arrow[d, "b"] & A^{\otimes_k i+2} \arrow[l, "B"'] \arrow[d, "b"] & \operatorname{Ker} b \arrow[l, "B"'] \arrow[d, "b"] & 0 \arrow[l, "B"'] \\
A^{\otimes_k i+2} \arrow[d, "b"] & \operatorname{Ker} b \arrow[l, "B"'] \arrow[d, "b"] & 0 \arrow[l, "B"'] & \\
\operatorname{Ker} b \arrow[d, "b"] & 0 \arrow[l, "B"'] & & \\
0 & & &
\end{tikzcd}$$

(i.e., canonical truncation on the columns). The ith graded step of this filtration is clearly the totalisation

of

$$\begin{tikzcd}
0 \arrow[d, "b"] & A^{\otimes_k i+2}/\operatorname{Ker} b \arrow[l, "B"'] \arrow[d, "b"] & \operatorname{Ker} b \arrow[l, "B"'] \arrow[d, "b"] & 0 \arrow[l, "B"'] \\
A^{\otimes_k i+2}/\operatorname{Ker} b \arrow[d, "b"] & \operatorname{Ker} b \arrow[l, "B"'] \arrow[d, "b"] & 0 \arrow[l, "B"'] & \\
\operatorname{Ker} b \arrow[d, "b"] & 0 \arrow[l, "B"'] & & \\
0 & & &
\end{tikzcd}$$

i.e., $\bigoplus_{n\geq 0}[\cdots \leftarrow 0 \leftarrow \underline{\operatorname{Ker} b} \leftarrow A^{\otimes_k i+2}/\operatorname{Ker} b \leftarrow 0 \leftarrow \cdots][i+2n]$ where we underline the term in degree 0. But each of the complexes in the direct sum is equivalent to $HH_i(A/k) \cong \Omega^i_{A/k}$, whence the graded piece is equivalent to $\bigoplus_{n\geq 0} \Omega^i_{A/k}[i+2n]$ as desired.

To construct the analogous filtration on $HH(A/k)$, with graded pieces $\bigoplus_{n\geq 0} \mathbb{L}^i_{A/k}[i+2n]$, for arbitrary A there are two (essentially equivalent) ways to argue: either rewrite the above argument replacing all the columns by $HH(P_\bullet/k)$, or else just formally declare the desired filtration on $HC(-/k) : k\text{-algs} \to D(k)$ to be the left Kan extension of the just-construction filtration on $HC(-/k) : k\text{-algs}_\Sigma \to D(k)$.

It remains to check that the filtration on $HC(A/k)$ is complete, i.e., that $\operatorname{Rlim}_i \operatorname{Fil}^i_{\mathrm{HKR}} HC(A/k) \simeq 0$. But this is a formal consequence of the fact that $\operatorname{Fil}^i_{\mathrm{HKR}} HC(A/k)$ is supported in homological degrees $\geq i$ by construction. □

Remark 2.30. The antisymmetrisation maps $\Omega^n_{A/k} \to HH_n(A/k)$ from Theorem 2.8 continue to exist even if A is not flat over k; indeed, they are precisely the edge maps in the spectral sequence which arises from the filtration of Proposition 2.28.

Remark 2.31 (Transitivity in general)**.** Let $A \to B$ be a morphism of k-algebras, where A is commutative. Then there is a natural equivalence of simplicial A-algebras

$$HH(B/k) \otimes^{\mathbb{L}}_{HH(A/k)} A \xrightarrow{\sim} HH(B/A),$$

generalising the isomorphism of Remark 2.7 in the flat case. This can be proved either by left Kan extending the aforementioned isomorphism, or else picking compatible resolutions and arguing as in the proof of Proposition 2.27(iii).

Example 2.32. The Hochschild homology groups of the $\mathbb{Z}$-algebra $\mathbb{F}_p$ are given by

$$HH_n(\mathbb{F}_p/\mathbb{Z}) \cong \begin{cases} \mathbb{F}_p & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Regarding multiplicative structure, $HH_*(\mathbb{F}_p/\mathbb{Z}) = \mathbb{F}_p\langle u\rangle$ (divided power algebra on a single variable u) with $u \in HH_2(\mathbb{F}_p/\mathbb{Z})$ any basis element. In particular, the powers $u^2, \ldots, u^{p-1}$ serve as basis elements for $HH_4(\mathbb{F}_p/\mathbb{Z}), \ldots, HH_{2(p-1)}(\mathbb{F}_p/\mathbb{Z})$ respectively, but $u^p = 0$ in $HH_{2p}(\mathbb{F}_p/\mathbb{Z})$. This is usually viewed as pathological behaviour, which will be fixed by topological Hochschild homology.

There are two ways to prove this. Either use the HKR filtration of Theorem 2.28 and a standard calculation of the wedge powers of the cotangent complex for a ring modulo a regular element (see Remark 3.9, which also shows that $HH_2(\mathbb{F}_p/\mathbb{Z})$ naturally identifies with $\pi_1(\mathbb{L}_{\mathbb{F}_p/\mathbb{Z}}) = p\mathbb{Z}/p^2\mathbb{Z}$; this gives a preferred choice of u, namely the class of p mod p^2), or use the standard resolution of $\mathbb{F}_p$ by the Koszul complex $K(p;\mathbb{Z}) = [\mathbb{Z} \xleftarrow{p} \mathbb{Z}]$ (which is a flat cdg $\mathbb{Z}$-algebra) and compute $HH(\mathbb{F}_p/\mathbb{Z})$ from the point of view of Shukla homology. We refer to [28, Prop. 2.6].

3 Hochschild and cyclic homology of $\mathbb{F}_p$-algebras

In Theorem 2.22 we saw that the cyclic homology theories of a smooth algebra in characteristic zero were given by de Rham cohomology and its Hodge filtration. The goal of this section is twofold; firstly, we will present a proof of the following analogous result in characteristic p:

Theorem 3.1. *Let k be a perfect field of characteristic p and R a smooth k-algebra. Then $HC^-(R/k)$, $HP(R/k)$, and $HC(R/k)$ admit natural complete, descending, $\mathbb{Z}$-indexed filtrations whose i^{th} graded pieces are respectively given by*

$$\Omega^{\geq i}_{R/k}[2i], \qquad \Omega^{\bullet}_{R/k}[2i], \qquad \Omega^{\leq i}_{R/k}[2i].$$

In short, Theorem 2.22 remains true in characteristic p except that the filtration need not be split. Theorem 3.1 is far from the best possible generalisation of Theorem 2.22 beyond characteristic zero; in fact, Antieau has generalised it to smooth algebras over arbitrary base rings [2] (and has even removed the smoothness hypothesis at the expense of replacing de Rham cohomology by its derived version). However, the second goal of this section is to introduce the reader to the methods of [8], in which "motivic" filtrations are constructed on the cyclic homology and its topological variants of $\mathbb{F}_p$- and p-adic algebras. Therefore we will prove Theorem 3.1 following the main technique of [8], namely descent to the case of quasiregular semiperfect rings.

In §3.1 we present this technique by explaining how suitable cohomology theories can be determined by their behaviour on certain semiperfect rings. This technique is then applied in §3.2 in order to prove Theorem 2.22. Finally in §3.4 the methods are abstracted and extended by introduction of the quasisyntomic site.

3.1 Flat descent via quasiregular semiperfect algebras

Given a commutative base ring k, suppose that we have a functor

$$\mathcal{F} : k\text{-algs} \to D(\mathbb{Z}) \text{ or } D(k) \text{ or } \mathrm{Sp} \text{ or in general any } \infty\text{-category } \mathcal{D} \text{ with sifted colimits.}$$

In Remark 2.25 we recalled the theory of left Kan extension, which allows us to make precise the idea that $\mathcal{F}$ might be determined by its value on polynomial algebras, and so in particular by its value on smooth algebras. Now we consider a converse problem: can we recover the behaviour of $\mathcal{F}$ on smooth algebras from its values on some class of "large algebras" (where, perhaps perversely, we hope that $\mathcal{F}$ is easier to understand)?

Definition 3.2. $\mathcal{F}$ is said to satisfy *flat descent* (or to be an *fpqc ∞-sheaf,* where *fpqc* means fidèlement plat et quasi-compact) if, for every faithfully flat map $A \to B$ of k-algebras, the induced morphism

$$\mathcal{F}(A) \to \lim \big(\mathcal{F}(B) \rightrightarrows \mathcal{F}(B \otimes_A B) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{F}(B \otimes_A B \otimes_A B) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots\big)$$

in $\mathcal{D}$ is an equivalence. Inside the bracket we are applying $\mathcal{F}$ termwise to the usual Cech nerve

$$\mathrm{Cech}(B/A) := \qquad B \rightrightarrows B \otimes_A B \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} B \otimes_A B \otimes_A B \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots$$

(which is a cosimplicial object in $\mathcal{D}$), and the recall that the limit may be interpreted as totalisation in case $\mathcal{D} = D(\mathbb{Z})$ or $D(k)$.

Now we suppose that k is a perfect field of characteristic p and explain how flat descent allows us to answer the converse problem we have posed. Let R be a smooth k-algebra, and recall first that the Frobenius morphism $\varphi : R \to R$ is therefore flat. Indeed, to prove this we may work locally on $\operatorname{Spec} R$ and so assume that there exists an étale morphism $k[t_1, \ldots, t_d] \to R$ for some $d \geq 0$; the étaleness implies that the diagram

$$\begin{tikzcd} k[t_1, \ldots, t_d] \arrow[r] & R \\ k[t_1, \ldots, t_d] \arrow[r] \arrow[u, "\varphi"] & R \arrow[u, "\varphi"'] \end{tikzcd}$$

is a pushout, and so the desired flatness follows from the obvious flatness of φ on $k[t_1, \dots, t_d]$. By taking the colimit over iterations of φ, we see that the colimit perfection $R_{\mathrm{perf}} := \varinjlim_{\varphi} R$ is a flat R-algebra. Moreover φ always induces a homeomorphism on Spec, so $\operatorname{Spec} R_{\mathrm{perf}} \to \operatorname{Spec} R$ is a homeomorphism, in particular surjective; so we have shown that $R \to R_{\mathrm{perf}}$ is faithfully flat.

Assuming that $\mathcal{F}$ satisfies flat descent, we deduce that

$$\mathcal{F}(R) \xrightarrow{\sim} \lim \big(\mathcal{F}(R_{\mathrm{perf}}) \rightrightarrows \mathcal{F}(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{F}(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots\big),$$

which describes $\mathcal{F}(R)$ in terms of $\mathcal{F}(R_{\mathrm{perf}})$ and all the $\mathcal{F}(R_{\mathrm{perf}} \otimes_R \cdots \otimes_R R_{\mathrm{perf}})$. The tensor powers $R_{\mathrm{perf}} \otimes_R \cdots \otimes_R R_{\mathrm{perf}}$ are no longer perfect, but they are locally quotients of perfect k-algebras by regular sequences. Indeed, assuming as in the previous paragraph that there is an étale morphism $k[t_1, \dots, t_d] \to R$, one obtains

$$R_{\mathrm{perf}} \otimes_R \cdots \otimes_R R_{\mathrm{perf}} = \big(k[\underline{t}]_{\mathrm{perf}} \otimes_{k[\underline{t}]} \cdots \otimes_{k[\underline{t}]} k[\underline{t}]_{\mathrm{perf}}\big) \otimes_{k[\underline{t}]} R,$$

where $k[\underline{t}]_{\mathrm{perf}} = k[t_1^{1/p^\infty}, \dots, t_d^{1/p^\infty}]$. But the large bracketed term is indeed a perfect k-algebra modulo a regular sequence, for example

$$k[t^{1/p^\infty}] \otimes_{k[t]} k[t^{1/p^\infty}] \cong k[x^{1/p^\infty}, y^{1/p^\infty}]/(x-y), \qquad t^{1/p^j} \otimes 1 \mapsto x^{1/p^j}, \quad 1 \otimes t^{1/p^j} \mapsto y^{1/p^j}.$$

For practical reasons, being locally a quotient of a perfect algebra modulo a regular sequence is not a good class of rings, since regular sequences are not the right notion for non-Noetherian rings; it is better to work instead with the following wider class, which avoids any finiteness hypotheses:

Definition 3.3. Quillen calls an ideal I of a ring A *quasiregular* if and only if I/I^2 is a flat A/I-module and $\pi_n(\mathbb{L}_{A/I/A}) = 0$ for $n > 1$ (whence $\mathbb{L}_{A/I/A} \simeq I/I^2[1]$ by Example 2.26) [34, Thm. 6.13]. For example, if I is locally generated by a regular sequence then it is quasiregular.

We say that an $\mathbb{F}_p$-algebra A is *quasiregular semiperfect* (qrsp) if and only if it is semiperfect (i.e., the Frobenius $\varphi : A \to A$ is surjective) and $\mathbb{L}_{A/\mathbb{F}_p}$ is a flat A-module supported in homological degree 1.

Lemma 3.4. *An $\mathbb{F}_p$-algebra A is quasiregular semiperfect if and only if there exists a perfect $\mathbb{F}_p$-algebra S and a quasiregular ideal $I \subseteq S$ such that $S/I = A$.*

Proof. We first recall the following general result: if $S' \to S$ is any morphism between perfect $\mathbb{F}_p$-algebras, then $\mathbb{L}_{S/S'} \simeq 0$. Indeed, let $P_\bullet \to S$ be a simplicial resolution by polynomial S'-algebras and observe that the absolute Frobenius φ induces the zero map on $\Omega^1_{P_\bullet/S'}$, since $\varphi(df) = df^p = pf^{p-1}dp = 0$; but S and S' are perfect, so this zero map $\varphi : \mathbb{L}_{S/S'} \to \mathbb{L}_{S/S'}$ is also an equivalence.

Both hypotheses in the statement of the lemma include that A is semiperfect, so let S be any perfect $\mathbb{F}_p$-algebra surjecting onto A, and let I denote the kernel; the standard choice is $A^\flat := \varprojlim_{\varphi} A$, the inverse limit perfection (aka. tilt) of A. Then $\mathbb{L}_{S/\mathbb{F}_p} \simeq 0$ by the previous paragraph, so the transitivity sequence shows that $\mathbb{L}_{A/\mathbb{F}_p} = \mathbb{L}_{A/S}$; by definition this is a flat A-module supported in homological degree 1 if and only if I is quasiregular. $\square$

In light of the previous lemma and definition, the prior discussion may therefore be summarised by the following proposition:

Proposition 3.5. *Let k be a perfect field of characteristic p and $\mathcal{F} : k\text{-algs} \to \mathcal{D}$ (where $\mathcal{D} = D(\mathbb{Z})$ or $D(k)$ or Sp) a functor satisfying flat descent. Then, for any smooth k-algebra R, we have*

$$\mathcal{F}(R) \xrightarrow{\sim} \lim \big(\mathcal{F}(R_{\mathrm{perf}}) \rightrightarrows \mathcal{F}(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{F}(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots\big),$$

and each term $R_{\mathrm{perf}} \otimes_R \cdots \otimes_R R_{\mathrm{perf}}$ appearing in the Cech nerve is quasiregular semiperfect.

And then here is the resulting filtration on $\mathcal{F}(R)$ which we have promised to construct in Theorem 3.1 (we remark that in all the cases of interest, $\mathcal{F}_i$ will vanish on any qrsp algebra for i odd and so we will discard the odd terms and reindex the filtration):

Corollary 3.6. *Under the same hypotheses as Proposition 3.5, suppose that $\mathcal{D} = D(\mathbb{Z})$, $D(k)$, or* Sp. *Then $\mathcal{F}(R)$ has a natural, complete, descending $\mathbb{Z}$-indexed filtration with i^{th}-graded piece given by the $[i]$-shift of the cosimplicial abelian group*

$$\mathcal{F}_i(R_{\text{perf}}) \rightrightarrows \mathcal{F}_i(R_{\text{perf}} \otimes_R R_{\text{perf}}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{F}_i(R_{\text{perf}} \otimes_R R_{\text{perf}} \otimes_R R_{\text{perf}}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots$$

(or more precisely by its totalisation in $\mathcal{D}$, i.e., associated cochain complex via Dold–Kan), where $\mathcal{F}_i(-) := \pi_i\mathcal{F}(-)$.

Proof. For each $A = R_{\text{perf}} \otimes_R \cdots \otimes_R R_{\text{perf}}$, we equip $\mathcal{F}(A)$ with its natural complete, descending, $\mathbb{Z}$-indexed Postnikov/Whitehead filtration $\tau_{>i}\mathcal{F}(A)$. This formally induces a filtration on $\mathcal{F}(R)$ by Proposition 3.5, which is still complete since the limits $\mathcal{F}(A) \xrightarrow{\sim} \lim_i \tau_{\leq i}\mathcal{F}(A)$ may be commuted through the totalisation. The graded pieces of this filtration on $\mathcal{F}(R)$ are precisely

$$\lim\big(\mathcal{F}_i(R_{\text{perf}}) \rightrightarrows \mathcal{F}_i(R_{\text{perf}} \otimes_R R_{\text{perf}}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{F}_i(R_{\text{perf}} \otimes_R R_{\text{perf}} \otimes_R R_{\text{perf}}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots\big)[i],$$

as desired. □

3.2 Cyclic homology of quasiregular semiperfect algebras

We have seen in §3.1, especially Proposition 3.5 and Corollary 3.6, how flat descent provides a general method for constructing filtrations, by descending the Postnikov filtration from quasiregular semiperfect $\mathbb{F}_p$-algebras. Now we wish to put this technique into practice to prove Theorem 3.1. The first step is to check that cyclic homology and its variants satisfy the necessary axiom for the technique, namely:

Lemma 3.7. *For any commutative ring k, the $D(k)$-valued functors*

$$\mathbb{L}^i_{-/k}, \quad HH(-/k), \quad HC(-/k), \quad HC^-(-/k), \quad HP(-/k)$$

*on k-*algs *satisfy flat descent.*

Proof. The fact that $\mathbb{L}^i_{-/k}$ satisfies flat descent is due to Bhatt; his first proof is given in [4], and his shorter proof may be found in [8, Thm. 3.1].

Since the property of satisfying flat descent is itself a derived limit and hence closed under derived limits (including fibre sequences), Proposition 2.28 and the previous paragraph imply that $HH(-/k)$ and $HC(-/k)$ satisfy flat descent. The complete filtration on $HC^-(-/k)$ from Remark 2.14 then similarly implies that $HC^-(-/k)$ satisfies flat descent, then deduce it for $HP(-/k)$ thanks to the norm fibre sequence of Remark 2.15. □

The second step is ideally to compute the cyclic homology and its variants of quasiregular semiperfect algebras; we begin with the following coarse information:

Lemma 3.8. *Let A be a qrsp $\mathbb{F}_p$-algebra. Then $HH_*(A/\mathbb{F}_p)$, $HC^-_*(A/\mathbb{F}_p)$, $HP_*(A/\mathbb{F}_p)$, and $HC_*(A/\mathbb{F}_p)$ are all supported in even degrees. Moreover, the $\mathbb{F}_p$-algebra $HP_0(A/\mathbb{F}_p)$ admits a complete, descending, $\mathbb{N}$-indexed filtration by ideals such that*

$$\mathrm{Fil}^i \cong HC^-_{2i}(A/\mathbb{F}_p), \qquad HP_0(A/\mathbb{F}_p)/\mathrm{Fil}^i \cong HC_{2i-2}(A/\mathbb{F}_p), \qquad \mathrm{gr}^i \cong HH_{2i}(A/\mathbb{F}_p) \cong \pi_i(\mathbb{L}^i_{A/\mathbb{F}_p})$$

for all $i \geq 0$.

Remark 3.9. In the setting of the lemma, the hypothesis is that $\mathbb{L}_{A/\mathbb{F}_p} = N[1]$ is given by a flat A-module supported in homological degree 1. Standard results on divided wedge powers then imply that $\mathbb{L}^i_{A/\mathbb{F}_p}$ is given by a flat A-module supported in homological degree i, namely $\Gamma^i_A(N)$, compatibly with multiplication as i varies [23, Prop. I.4.3.2.1].

Here $\Gamma^*_A(N) = \bigoplus_{i\geq 0}\Gamma^i_A(N)$ is the divided power algebra associated to the module N. Since N is flat, each A-module $\Gamma^i_A(N)$ is isomorphic to $\mathrm{Sym}^i_A(N)$, but the algebra structure on $\Gamma^*_A(N)$ is given by

$$x \cdot y := \tfrac{(i+j)!}{i!j!}xy \qquad x \in \Gamma^i_A(N),\ y \in \Gamma^j_A(N),$$

where $x \cdot y$ represents the product in $\Gamma^*_A(N)$ and xy represents the product in $\mathrm{Sym}^*_A(N)$.

Proof of 3.8. As mentioned in the remark, the fact that $\mathbb{L}^1_{A/\mathbb{F}_p} = N[1]$ implies that $\mathbb{L}^i_{A/\mathbb{F}_p} = \Gamma^i_A(N)[i]$. Therefore the spectral sequence associated to the filtration in Proposition 2.28 degenerates, implying that $HH_*(A/\mathbb{F}_p)$ is supported in even degrees and given as an A-algebra by $HH_{2*}(A/\mathbb{F}_p) = \Gamma^*_A(N)$.

The rest of the proof is quite general, we use only that $k := \mathbb{F}_p \to A$ is a morphism whose HH is supposed in even degrees. It follows formally that the various periodicity sequences in Remark 2.15 all break into short exact sequence, and that $HC_*(A/k)$, $HC^-_*(A/k)$, and $HP_*(A/k)$ are also supported in even degree:

$$0 \longrightarrow HH_{2i}(A/k) \xrightarrow{I} HC_{2i}(A/k) \xrightarrow{S} HC_{2i-2}(A/k) \longrightarrow 0$$

$$0 \longrightarrow HC^-_{2i+2}(A/k) \xrightarrow{S} HC^-_{2i}(A/k) \xrightarrow{h} HH_{2i}(A/k) \longrightarrow 0$$

$$0 \longrightarrow HC^-_{2i}(A/k) \xrightarrow{I} HP_{2i}(A/k) \xrightarrow{Sh} HC_{2i-2}(A/k) \longrightarrow 0$$

$$HP_{2i}(A/k) = \varprojlim_{s \text{ wrt } S} HC_{2i+2s}(A/k), \qquad S : HP_{2i}(A/k) \xrightarrow{\sim} HP_{2i-2}(A/k)$$

There is therefore an identification of graded rings $HP_*(A/k) = HP_0(A/k)[u^{\pm 1}]$, with $u := S(1) \in HP_2(k/k)$ corresponding to the periodicity operator S. Furthermore, $I : HC^-_*(A/k) \to HP_*(A/k)$ is an inclusion of graded rings which is an isomorphism in degrees ≤ 0; therefore

$$\mathrm{Fil}^i\, HP_0(A/k) := u^{-i} I(HC^-_{2i}(A/k)) = S^n I((HC^-_{2i}(A/k)),$$

for $i \geq 0$, defines a complete decreasing filtration of $HP_0(A/k)$ by ideals with quotients and graded pieces given respectively by

$$HP_0(A/k)/\,\mathrm{Fil}^i \xrightarrow{h}_{\sim} HC_{2i-2}(A/k), \qquad \mathrm{gr}^i \xrightarrow{h}_{\sim} HH_{2i}(A/k).$$

□

Remark 3.10. The moral of the previous lemma and its proof is the following: since $HH_*(A/\mathbb{F}_p)$ is supported in even degree, the data of HC^-, HP, and HC are captured (up to extension problems) by the $\mathbb{F}_p$-algebra $HP_0(A/\mathbb{F}_p)$ and its filtration coming from HC^-.

The coarse information afforded by Lemma 3.8 will actually be sufficient to prove Theorem 3.1, but conceptually it is better to first try to explicitly identify the $\mathbb{F}_p$-algebra $HP_0(A/\mathbb{F}_p)$, for any qrsp algebra A. We now construct two possible candidates; they will both be equipped with a complete, descending, $\mathbb{N}$-indexed filtration by ideals.

Definition 3.11 (de Rham construction)**.** The first construction comes from the theory of derived de Rham cohomology. For each $i \geq 0$, we let $\mathbb{L}\Omega^{\leq i}_{-/\mathbb{F}_p} : \mathbb{F}_p\text{-algs} \to D(\mathbb{F}_p)$ be the left Kan extension of the naively truncated de Rham complex $\Omega^{\leq i}_{-/\mathbb{F}_p}$; recall from Remark 2.25 that this concretely means the following: given an $\mathbb{F}_p$-algebra A, we pick a simplicial resolution $P_\bullet \xrightarrow{\sim} A$ by polynomial algebras over $\mathbb{F}_p$ (possibly in infinitely many variables), and define $\mathbb{L}\Omega^{\leq i}_{A/\mathbb{F}_p}$ to be the geometric realisation of the simplicial cochain complex $\Omega^{\leq i}_{P_\bullet/\mathbb{F}_p}$. The *Hodge completed derived de Rham complex* of the $\mathbb{F}_p$-algebra A is then defined to be

$$\widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p} := \mathrm{Rlim}_i\, \mathbb{L}\Omega^{\leq i}_{A/\mathbb{F}_p},$$

which is an algebra object in $D(\mathbb{F}_p)$ (possibly unbounded in both directions).

From the fibre sequences $\mathbb{L}^i_{A/\mathbb{F}_p} \to \mathbb{L}\Omega^{\leq i}_{A/\mathbb{F}_p} \to \mathbb{L}\Omega^{\leq i-1}_{A/\mathbb{F}_p}$ we see using Remark 3.9 that, assuming A is quasiregular semiperfect, then each $\mathbb{L}\Omega^{\leq i}_{A/\mathbb{F}_p}$ is supported in degree 0 and the transition maps are surjective as i increases; therefore $\widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p}$ is still supported in degree 0, given by an $\mathbb{F}_p$ algebra which is complete with respect to its induced filtration by the ideals $\mathrm{Ker}(\widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p} \twoheadrightarrow \mathbb{L}\Omega^{\leq i}_{A/\mathbb{F}_p})$ for $i \geq 0$.

Definition 3.12 (Divided power construction)**.** The second construction comes from the theory of divided powers. Assuming that A is quasiregular semiperfect (though in fact the definition only really requires that A be semiperfect), let $A^\flat := \lim_\varphi A$ be its inverse limit perfection and $I := \mathrm{Ker}(A^\flat \twoheadrightarrow A)$, so that $\mathbb{L}_{A/\mathbb{F}_p} = I/I^2[1]$ (see the proof of Lemma 3.4). Let $D_{A^\flat}(I)$ denote the divided power envelope of $A^\flat$ along I, and $\widehat{D}_S(I)$ its completion with respect to the divided power filtration (equipped with the induced filtration).

The following is the promised improvement of Lemma 3.8 which completely describes the cyclic, negative cyclic, and periodic cyclic homology groups of quasiregular semiperfect $\mathbb{F}_p$-algebras:

Proposition 3.13. *Let A be a quasiregular semiperfect $\mathbb{F}_p$-algebra. Then there are natural isomorphisms of filtered rings*

$$HP_0(A/\mathbb{F}_p) \cong \widehat{D}_{A^\flat}(I) \cong \widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p},$$

where $HP_0(A/\mathbb{F}_p)$ is equipped with the filtration coming from HC^- defined in Lemma 3.8.

Comments on the proof. We begin with a reality check that the isomorphisms are plausible, or more precisely valid for the associated graded rings. By construction of the filtered ring $\widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p}$, its graded pieces are precisely $\pi_i(\mathbb{L}^i_{A/\mathbb{F}_p})$ for $i \geq 0$. The same is true of the filtered ring $HP_0(A/\mathbb{F}_p)$, by Lemma 3.8. Meanwhile, it is a classical general result in the theory of divided powers that if I is generated by a regular sequence of $A^\flat$, then the filtered ring $D_{A^\flat}(I)$ has associated graded ring $\Gamma^*_{A^\flat}(N)$, where $N = I/I^2$. Since we know from Remark 3.9 that $\Gamma^*_{A^\flat}(N)$ and $\pi_*(\mathbb{L}^*_{A/\mathbb{F}_p})$ are naturally isomorphic, we have indeed shown that the three rings of the proposition have isomorphic graded rings (at least if I is generated by a regular sequence).

The obvious way to proceed is therefore to construct natural morphisms between the three rings of the proposition respecting the filtrations and to check that they induce the aforementioned isomorphisms on each graded piece; since the three rings are complete with respect to their filtrations, these morphisms will therefore be the desired isomorphisms. Fortunately the ring $D_{A^\flat}(I)$ has a universal property since it is a divided power envelope, so a priori we can construct maps out of it by showing that the target ring has suitable divided powers. Unfortunately it seems hard to directly write down natural divided power structures on $HP_0(A/\mathbb{F}_p)$ and $\widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p}$. See project F if you want to try.

Instead, Bhatt has constructed a comparison map $\widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p} \to \widehat{D}_{A^\flat}(I)$, by left Kan extending Berthelot's comparison from de Rham to crystalline cohomology, and shown that it is an isomorphism [5, Prop. 3.25 & Corol. 3.40] (assuming I is generated by a regular sequence). Some tricks in the style of the proofs of [8, Props. 8.12 & 8.15] should extend this to arbitrary quasiregular I.

To then compare to $HP_0(A/\mathbb{F}_p)$ there are two options. The first of these is based on Theorem 3.1, which we will prove in a moment without using the current proposition. Upon left Kan extending that theorem to all $\mathbb{F}_p$-algebras, it yields in particular natural isomorphisms $HC_{2i}(A/\mathbb{F}_p) \cong \mathbb{L}\Omega^{\leq i}_{A/\mathbb{F}_p}$; letting $i \to \infty$ gives $HP_0(A/\mathbb{F}_p) \cong \widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p}$, as desired.

The second approach for comparing to $HP_0(A/\mathbb{F}_p)$ is to use topological cyclic homology; indeed, we will see in Corollary 5.5 that the topological periodic cyclic homology $TP_0(A)$ is a p-adically complete, p-torsion-free ring such that $TP_0(A)/p = HP_0(A/\mathbb{F}_p)$. The p-adic completeness and p-torsion-freeness mean that existence of divided powers on $TP_0(A)$ (which would then induce divided powers on the quotient $HP_0(A/\mathbb{F}_p)$) becomes a condition on certain elements of the ring, rather than extra data to be specified. This allows the approach proposed in the second paragraph to be carried out; we refer to [8, Thm. 8.15] for the details. □

3.3 Cyclic homology of smooth algebras; proof of Theorem 3.1

This subsection is devoted to indicating the main ideas of proof of Theorem 3.1, so we let R denote a smooth algebra over a perfect field k of characteristic p. We will restrict to the central case of periodic cyclic homology; the analogous results for HC^- and HC follow by working with a suitable step of the filtrations instead, or respectively the quotient by a step of the filtration.

First proof of Theorem 3.1. If we allow ourselves to use Proposition 3.13, then Theorem 3.1 follows formally from the descent machinery as follows. Thanks to the general technique of Corollary 3.6, the verification in Lemma 3.7 that $HP(-/k)$ satisfies flat descent, and the fact from Lemma 3.8 that $HP_*(-/k)$ vanishes in odd degrees on any qrsp ring, we see that $HP(R/k)$ has a complete, descending, $\mathbb{Z}$-indexed filtration with graded pieces given by the $[2i]$-shifts of the cosimplicial k-module $HP_{2i}(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k)$

$$HP_{2i}(R_{\mathrm{perf}}/k) \rightrightarrows HP_{2i}(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}/k) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} HP_{2i}(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}/k) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots$$

We wish to show that this (or rather its totalisation) is equivalent to the de Rham complex $\Omega^\bullet_{R/k}$; we may suppose that $i = 0$, since HP_* is 2-periodic.

Proposition 3.13 shows that $HP_0(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k)$ is the same as

$$\widehat{\mathbb{L}\Omega}_{R_{\mathrm{perf}}/k} \rightrightarrows \widehat{\mathbb{L}\Omega}_{R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}/k} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \widehat{\mathbb{L}\Omega}_{R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}/k} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots.$$

But the totalisation of this is equivalent to $\widehat{\mathbb{L}\Omega}_{R/k} = \mathrm{Rlim}_i\, \mathbb{L}\Omega^{\leq i}_{R/k}$ since the functor $\widehat{\mathbb{L}\Omega}_{-/k}$ satisfies flat descent (indeed, it is complete with respect to a filtration whose graded pieces are $\mathbb{L}^i_{-/k}$, and these satisfy flat descent by Lemma 3.7. But R is smooth over k, so the adjunction $\mathbb{L}\Omega^{\leq i}_{R/k} \to \Omega^{\leq i}_{R/k}$ is an equivalence and therefore $\widehat{\mathbb{L}\Omega}_{R/k} \simeq \mathrm{Rlim}\, \Omega^{\leq i}_{R/k} = \Omega^\bullet_{R/k}$, as desired. □

Second proof of Theorem 3.1. We now sketch a proof of Theorem 3.1 which uses only the coarse information of Lemma 3.8 and does not rely on Proposition 3.13; as we explained in the proof of the latter, it can then be deduced via left Kan extension from the proof of the current theorem.

As in the first proof, the goal is to show that the totalisation of $HP_0(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k)$ is equivalent to $\Omega^\bullet_{R/k}$. The filtration on HP_0 of any qrsp ring from Lemma 3.8 induces a complete, descending, $\mathbb{N}$-indexed filtration on $HP_0(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k)$ with i^{th}-graded piece

$$\pi_i \mathbb{L}^i_{R_{\mathrm{perf}}/k} \rightrightarrows \pi_i \mathbb{L}^i_{R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}/k} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \pi_i \mathbb{L}^i_{R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}/k} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots$$

Each of the cotangent complexes appearing here is supported in degree i, so we may remove the π_i and instead shift by $[i]$; its totalisation is then precisely $\Omega^i_{R/k}[i]$, thanks to flat descent for $\mathbb{L}^i_{-/k}$.

To summarise, the coarse information of Lemma 3.8 is enough to show that $HP_0(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k)$ has a complete descending filtration with graded pieces $R[0]$, $\Omega^1_{R/k}[1]$, $\Omega^2_{R/k}[2]$,.... It follows quite formally[2] that $HP_0(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k)$ is necessarily equivalent to an actual complex of the form $R \to \Omega^1_{R/k} \to \Omega^2_{R/k} \to \cdots$, where each differential is the boundary map $H^i \to H^{i+1}$ induced by a fibre sequence coming from the filtration (see the footnote for details). But the complex is even a differential

[2] Here is a precise statement; for more details, including compatibility with symmetric monoidal structures, see [8, §5.1]:

Lemma 3.14. *Let R be a commutative ring and $C \in D(R)$ a complex equipped with a descending, $\mathbb{N}$-indexed filtration $C = \mathrm{Fil}^0 C \leftarrow \mathrm{Fil}^1 C \leftarrow \cdots$ such that, for each $i \geq 0$, the graded piece $\mathrm{gr}^i C$ is supported in degree i. Then C is naturally equivalent to the actual complex*

$$0 \longrightarrow H^0(\mathrm{gr}^0 C) \longrightarrow H^1(\mathrm{gr}^1 C) \longrightarrow H^2(\mathrm{gr}^2 C) \longrightarrow \cdots$$

where the boundary maps are the Bocksteins of the fibre sequences $\mathrm{gr}^{i+1} C \to \mathrm{Fil}^i C/\mathrm{Fil}^{i+2} C \to \mathrm{gr}^i C$; moreover, the filtration on C corresponds to the naive upwards truncations of this actual complex, and this process is compatible with symmetric monoidal structures.

Proof. By proving the result for each $C/\mathrm{Fil}^{n+1} C$ and then letting $n \to \infty$ we may suppose that the filtration on C is finite of length $n+1$, and by induction we may suppose that the result has been proved for any complex with a filtration of length n. Then we have a fibre sequence and equivalence

$$H^n(\mathrm{gr}^n C)[-n] = \mathrm{Fil}^n C \longrightarrow C \longrightarrow C/\mathrm{Fil}^n C \simeq [H^0(\mathrm{gr}^0 C) \to H^1(\mathrm{gr}^1 C) \to \cdots \to H^{n-1}(\mathrm{gr}^{n-1} C) \to 0],$$

whence C is the homotopy fibre of

$$[H^0(\mathrm{gr}^0 C) \to H^1(\mathrm{gr}^1 C) \to \cdots \to H^{n-1}(\mathrm{gr}^{n-1} C) \to 0] \xrightarrow{\beta} H^n(\mathrm{gr}^n C)[-n+1]$$

But β is precisely given by the desired Bockstein $H^{n-1}(\mathrm{gr}^{n-1} C) \to H^n(\mathrm{gr}^n C)$ and the homotopy fibre is given in the desired way. □

graded k-algebra since the above construction is compatible with product structures, so it is enough so check that $R \to \Omega^1_{R/k}$ is the usual de Rham differential. By picking an element of R and looking at the corresponding map $k[t] \to R$, functoriality reduces this to the case $R = k[t]$, in which case one computes explicitly to check it. $\square$

3.4 The quasisyntomic site

This subsection can be omitted when reading the notes for the first time. In fact, in the current version of the notes it serves no purpose.

The arguments of §3.1–3.2 depended crucially not only on the cohomologiy theories of interest satisfying flat descent, but also on existence of the flat cover $R \to R_{\mathrm{perf}}$, for the smooth k-algebra R, having the following properties:

(a) All terms of the Cech resolution $R_{\mathrm{perf}} \otimes_R \cdots \otimes_R R_{\mathrm{perf}}$ are qrsp.

(b) The flat cover R_{perf} is functorial in R (to ensure naturality of the filtrations we constructed).

Although we argued directly to prove (a), it is also easy to check (we will do it in Lemma 3.16) that it is a consequence of $\mathbb{L}_{R_{\mathrm{perf}}/R}$ being supported in homological degree 1, where it is given by a flat R_{perf}-module. There do exist other rings admitting such a nicely behaved flat cover:

Definition 3.15. Given a morphism $A \to B$ of $\mathbb{F}_p$-algebras, we say that it is *quasisyntomic* if it is flat and $\mathbb{L}_{B/A}$ has cohomological Tor amplitude in $[-1, 0]$, i.e., for any B-module M, the complex $\mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B M$ is supported in cohomological degrees $[-1, 0]$. We say that it is a *quasisyntomic cover* if, in addition, it is faithfully flat.

We say that an $\mathbb{F}_p$-algebra A is *quasisyntomic* if $\mathbb{F}_p \to A$ is quasisyntomic in the above sense. Let QSyn denote the category of quasisyntomic $\mathbb{F}_p$-algebras (and all morphisms, not just quasisyntomic ones).

Quasisyntomic rings are precisely those which have a cover behaving similarly to $R \to R_{\mathrm{perf}}$:

Lemma 3.16. *An $\mathbb{F}_p$-algebra A is quasisyntomic if and only if there exist a qrsp $\mathbb{F}_p$-algebra S and a quasisyntomic cover $A \to S$. When this holds, all terms $S \otimes_A \cdots \otimes_A S$ of the Cech nerve are qrsp.*

To prove Lemma 3.16 it is helpful to first note the following general properties, which follow from the basic properties of the cotangent complex, especially transitivity (Proposition 2.27), so we leave the verifications to the reader:

Lemma 3.17. *(i) Let $A \to B$ be a quasisyntomic cover of $\mathbb{F}_p$-algebras; then A is quasisyntomic if and only if B is quasisyntomic.*

(ii) A composition of quasisyntomic morphism (resp. quasisyntomic covers) again a quasisyntomic morphism (resp. quasisyntomic cover).

(iii) A pushout of a quasisyntomic morphism (resp. quasisyntomic cover) along an arbitrary morphism is again a quasisyntomic morphism (resp. quasisyntomic cover).

Proof of Lemma 3.16. The implication $\Leftarrow$ is a consequence of Lemma 3.17(i). For the converse, suppose that A is quasisyntomic, let $\mathbb{F}_p[x_i : i \in I] \twoheadrightarrow A$ be a surjection from a polynomial algebra and let $\mathbb{F}_p[x_i^{1/p^\infty}] = \mathbb{F}_p[x_i : i \in I]_{\mathrm{perf}}$ be its colimit-perfection, which is easily seen to be a quasisyntomic cover of $\mathbb{F}_p[x_i]$; finally let $S := A \otimes_{\mathbb{F}_p[x_i]} \mathbb{F}_p[x_i^{1/p^\infty}]$ be the resulting pushout. Then $A \to S$ is the pushout of a quasisyntomic cover, hence is a quasisyntomic cover (by Lemma 3.17); finally note that S is semiperfect, since it is quotient of $\mathbb{F}_p[x_i^{1/p^\infty}]$.

To prove the Cech statement, let $A \to S$ be any quasisyntomic cover where S is qrsp. By taking repeated pushouts and compositions, Lemma 3.17(ii)&(iii) shows that $A \to S^{\otimes_A n}$ is quasisyntomic whence $S^{\otimes_A n}$ is quasisyntomic; but it is also clearly semiperfect since S is semiperfect. $\square$

Remark 3.18. A stronger condition than quasisyntomic is *quasismooth*, which means that $A \to B$ is flat and $\mathbb{L}_{B/A}$ has Tor amplitude in $[0,0]$, i.e., $\mathbb{L}_{B/A}$ is supported in degree 0 and $\Omega^1_{B/A}$ is a flat B-module. Just as in Lemma 3.17(ii)&(iii), compositions and pushouts of quasismooth maps are again quasismooth.

Example 3.19. If k is a perfect field (in fact, any perfect ring) of characteristic p and A is a smooth k-algebra (resp. perfect k-algebra), then A is quasismooth; if $I \subseteq A$ is a quasiregular ideal (e.g., generated by a regular sequence, then A/I is quasisyntomic). These are our main examples of interest, but the advantage of adopting an abstract definition in terms of the cotangent complex is that it avoids any finiteness hypothesis (which allows us to use, for example, polynomial rings in infinitely many variables, possibly over non-Noetherian perfect rings).

It follows from Lemma 3.17 that the opposite of QSyn is a site, with covers declared to be quasisyntomic covers. In other words, we have cut down the flat site on all $\mathbb{F}_p$-algebras by restricting to quasisyntomic algebras and by insisting that flat covers also be quasisyntomic. Lemma 3.16 gives us interesting covers with the same properties as (a), but without any naturality properties as in (b); next we explain the formalism which overcomes this lack of naturality.

Note that an $\mathbb{F}_p$-algebra A is quasiregular semiperfect if and only if it is quasisyntomic and semiperfect; so we write QRSP $\subseteq$ QSyn for the subcategory of qrsp algebras. The first assertion of the Lemma 3.16 implies in particular that $\mathrm{QRSP}^{\mathrm{op}}$ forms a basis for $\mathrm{QSyn}^{\mathrm{op}}$, as we showed that each object in the former site can be covered by an object in the latter. It follows that the categories of abelian sheaves on $\mathrm{QSyn}^{\mathrm{op}}$ and on $\mathrm{QRSP}^{\mathrm{op}}$ are equivalent via the obvious restriction functor. Using the second assertion of Lemma 3.16, this equivalence extends quite formally from abelian sheaves to ∞-sheaves taking values in any presentable ∞-category $\mathcal{D}$:

$$\mathrm{Sh}(\mathrm{QSyn}^{\mathrm{op}}, \mathcal{D}) \to \mathrm{Sh}(\mathrm{QRSP}^{\mathrm{op}}, \mathcal{D})$$

The key point is that, given an ∞-sheaf $\mathcal{F} : \mathrm{QRSP}^{\mathrm{op}} \to \mathcal{C}$, we may extend it to an ∞-sheaf $\mathcal{F}^{\beth} : \mathrm{QSyn}^{\mathrm{op}} \to \mathcal{C}$ by setting

$$\mathcal{F}^{\beth}(A) := \lim \big(\mathcal{F}(S) \rightrightarrows \mathcal{F}(S \otimes_A S) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{F}(S \otimes_A S \otimes_A S) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots\big),$$

where $A \in \mathrm{QSyn}$ and $A \to S$ is a chosen quasisyntomic cover with $S \in \mathrm{QRSP}$. The equivalence of ∞-categories assures us that this definition of $\mathcal{F}^{\beth}$ is independent (up to equivalence) of the chosen cover $A \to S$, which is how we overcome the lack of the functorial cover $R \to R_{\mathrm{perf}}$ which we had in the smooth case. Following [8], the extension $\mathcal{F}^{\beth}$ will be called the *unfolding* of $\mathcal{F}$.

In particular, suppose that $\mathcal{F}$ is a presheaf of abelian groups on $\mathrm{QSyn}^{\mathrm{op}}$ satisfying the following Cech condition:

> For any quasisyntomic cover $A \to B$ between qrsp algebras, the Cech complex $0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(B \otimes_A B) \to \cdots$ is exact, i.e., the higher Cech cohomology of $\mathcal{F}$ on A with respect to the cover B vanishes, and the 0th Cech cohomology agrees with $\mathcal{F}(A)$. Or, in other words $\mathcal{F}|_{\mathrm{QRSP}}$, viewed as functor to $D(\mathbb{Z})$ is an ∞-sheaf on $\mathrm{QRSP}^{\mathrm{op}}$.

Then a general result about sites [Stacks Project, 03AN & 03F9] implies that $\mathcal{F}|_{\mathrm{QRSP}}$ is an abelian sheaf on $\mathrm{QRSP}^{\mathrm{op}}$ and that it has no higher (site-theoretic) cohomology on any object of $\mathrm{QRSP}^{\mathrm{op}}$. It follows also that the above Cech complex defining $(\mathcal{F}|_{\mathrm{QRSP}})^{\beth}(A)$ is nothing other than the cohomology $R\Gamma_{\mathrm{QSyn}}(A, \mathcal{F})$ (we should more correctly write the sheafification $\mathcal{F}^{\mathrm{sh}}$ of $\mathcal{F}$ here instead, but we will usually lighten the notation by leaving the sheafification implicit when taking cohomology).

The previous paragraph applies in particular to our sheaves of interest:

Lemma 3.20. *The following abelian presheaves $\mathcal{F}$ on $\mathrm{QSyn}^{\mathrm{op}}$ all satisfy the previous Cech condition:*

(i) $\pi_n(\mathbb{L}^i_{-/\mathbb{F}_p})$ *for any* $i, n \geq 0$.

(ii) $HH_n(-/\mathbb{F}_p)$, $HC_n(-/\mathbb{F}_p)$, $HC^-_n(-/\mathbb{F}_p)$, $HP_n(-/\mathbb{F}_p)$ *for any* $n \in \mathbb{Z}$.

Proof. (i): If $i \neq n$ then $\pi_n(\mathbb{L}^i_{-/\mathbb{F}_p})$ is identically zero on QRSP by Remark 3.9, so it remains to deal with the case $i = n$. But then we have $\mathbb{L}_{-/\mathbb{F}_p} = \pi_1(\mathbb{L}_{-/\mathbb{F}_p})[1]$, so we must check that for any quasisyntomic cover $A \to B$ between qrsp algebras we have

$$\mathbb{L}^i_{A/\mathbb{F}_p} \xrightarrow{\sim} \lim \big(\mathbb{L}^i_{B/\mathbb{F}_p} \rightrightarrows \mathbb{L}^i_{B\otimes_A B/\mathbb{F}_p} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathbb{L}^i_{B\otimes_A B\otimes_A B/\mathbb{F}_p} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots\big)$$

But this is exactly flat descent for the cotangent complex (Lemma 3.7).

(ii): As we saw at the start of Lemma 3.8, each $HH_n(-/\mathbb{F}_p)$ is either zero or given by $\pi_{n/2}(\mathbb{L}^{n/2}_{-/\mathbb{F}_p})$, to which we apply part (i). The Cech condition for the others then follows from the filtrations of Lemma 3.8. □

As a concrete example, the lemma and prior discussion mean that the Cech complex $HP_0(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k)$ which we studied in §3.2 can alternatively be described as

$$HP_0(\mathrm{Cech}(R_{\mathrm{perf}}/R)/k) = R\Gamma_{\mathrm{QSyn}}(R, HP_{2i}(-/\mathbb{F}_p)),$$

the quasi-syntomic cohomology of (the sheafification of) $HP_0(-/k)$; and the same holds for $HC^-_n(-/\mathbb{F}_p)$, $HP_n(-/\mathbb{F}_p)$, and $HC_n(-/\mathbb{F}_p)$.

4 S^1-actions (three points of view) and *THH*

So far we have studied Hochschild and cyclic homology in terms of relatively explicit, but unmotivated, double complexes. In this section we will explain that there has been an action of the circle S^1 lurking in the background; although it may initially appear absurd to discuss S^1-actions on algebraic objects such as chain complexes, we will see several ways of making this precise. This S^1-action will play an essential role once we pass to topological cyclic homology, where the algebraic manifestations of this action will no longer be available.

First, in §4.1, we will explain that mixed complexes (Definition 2.19) may be viewed as complexes equipped with an S^1-action. Then in §4.2 we we drop any linearity assumptions and return to cyclic objects (Definition 2.17), which provide a different way of algebraically encoding an S^1-action. In §4.3 we will discover an even better (and our final) point of view, namely that of functors out of the classifying space of S^1. Once that formalism is available, we explain in §4.4 that the classical definition of Hochschild homology can be transported to any suitable symmetric monoidal (∞-)category, such as that of spectra; the result is topological Hochschild homology.

4.1 As $k[\varepsilon]/\varepsilon^2$-modules, i.e., mixed complexes

Given a fixed commutative base ring k, here we explain how mixed complexes over k provide a way of modelling what it means for a complex to be equipped with an S^1-action.

Note first that in the case of a discrete group G, a module M or complex with G-action is simply a module or complex over the group algebra $k[G]$; moreover, the group homology and cohomology are then given respectively by the derived coinvariants $k \otimes^{\mathbb{L}}_{k[G]} M$ and derived invariants $\mathrm{RHom}_{k[G]}(k, M)$.

Remark 4.1. The case in which G is a finite group is particularly interesting. Then the *norm map* $k \to k[G]$, $1 \mapsto \sum_{g\in G} g$ is well-defined and $k[G]$-linear (viewing k as a $k[G]$-algebra via the augmentation map), thereby inducing a natural transformation $k\otimes^{\mathbb{L}}_{k[G]} - \to \mathrm{RHom}_{k[G]}(k, -)$. Given a k-module M with G-action, the cofiber of this norm map $k \otimes^{\mathbb{L}}_{k[G]} M \to \mathrm{RHom}_{k[G]}(k, M)$ is known as the *Tate cohomology* (the same Tate cohomology which often appears, when G is a cyclic group, in class field theory).

Now we turn to the case of a topological group G. Possibly replacing G by a weakly equivalent topological group we may assume that G is the geometric realisation of a simplicial group $G_\bullet$ (for example, the Kan complex of simplices $\mathrm{Sing}(G)$ is a simplicial group whose geometric realisation is weakly equivalent to G, though in practice we will be interested in choosing some $G_\bullet$ which is much smaller than $\mathrm{Sing}(G)$). Then, in any category $\mathcal{C}$, we could define "an object with G-action" to be a simplicial object $X_\bullet$ of $\mathcal{C}$ which is equipped with an action by $G_\bullet$; i.e., each X_n is equipped with an action by G_n in a manner compatible with the face and degeneracy maps. In particular, if $X_\bullet$ were a simplicial set, or simplicial k-module, equipped with a $G_\bullet$-action in this sense, then its geometric realisation $|X_\bullet|$ would be a topological space equipped with a continuous action by $G = |G_\bullet|$ (simply because geometric realisation is functorial and compatible with products). Conversely, if X were any topological space equipped with a continuous action by $G = |G_\bullet|$, then $\mathrm{Sing}(X)$ would be equipped with an action by $\mathrm{Sing}(|G_\bullet|)$, which is equivalent to $G_\bullet$. In conclusion, up to issues of homotopy equivalence,

actions by the simplicial group $G_\bullet$ offer a combinatorial model for actions by G. We will return to this point of view in §4.2, but now we wish to linearise by restricting to complexes of k-modules: then k-linear actions are modelled by modules over the simplicial k-algebra $k[G_\bullet]$, or equivalently connected differential graded modules over the dg k-algebra $Nk[G_\bullet]$ associated to the simplicial k-algebra $k[G_\bullet]$.

Remark 4.2. We have just used the following fact: given a simplicial ring $A_\bullet$, then its normalised cochain complex $NA_\bullet$ naturally admits the structure of a differential graded ring, with product given by the so-called "shuffle product". Note also that if $A_\bullet$ is commutative, then $NA_\bullet$ is strictly graded commutative.

We note also that passing from simplicial modules over $k[G_\bullet]$ to connective dg modules over $Nk[G_\bullet]$ does not lose any information: the two model categories are Quillen equivalent via the normalised complex construction [36, Thm. 1.1(2)].

We specialise now to the circle S^1, which is equivalent to the geometric realisation of $B\mathbb{Z}$, the classifying space construction on the infinite cyclic group:

$$B\mathbb{Z} = \quad 0 \Leftarrow \mathbb{Z} \Lleftarrow \mathbb{Z}\times\mathbb{Z} \Llleftarrow \cdots$$

$$d_i(a_1,\dots,a_n) := \begin{cases} (a_2,\dots,a_n) & i=0 \\ (a_1,\dots,a_i+a_{i+1},\dots,a_n) & 0<i<n \\ (a_1,\dots,a_{n-1}) & i=n \end{cases} \qquad s_i(a_1,\dots,a_n) := (a_1,\dots,a_i,0,a_{i+1},\dots,a_n)$$

Lemma 4.3. *The dg k-algebra $Nk[B\mathbb{Z}]$ associated to the simplicial k-algebra $k[B\mathbb{Z}]$ is quasi-isomorphic to the dg k-algebra $k[\varepsilon]/\varepsilon^2$, where ε lies in homological degree 1 and the differential $k\varepsilon \to k$ is zero.*

Proof. Slick proof: since $B\mathbb{Z}$ is equivalent to $\mathrm{Sing}(S^1)$, the associated simplicial group algebra $k[B\mathbb{Z}]$ is equivalent to $\mathrm{Sing}(S^1;k)$, which in degree n is by definition the k-module generated by all n-simplices $\Delta^n \to S^1$. Passing to dgas, we must compute the usual dga computing $H_*(S^1,k)$, which is indeed $k[\varepsilon]/\varepsilon^2$.

Hands-on proof: the dga $Nk[B\mathbb{Z}]$ is the normalised complex $N_0 \overset{d_0}{\leftarrow} N_1 \overset{d_0}{\leftarrow} N_2 \overset{d_0}{\leftarrow} \dots$, where $N_n = \bigcap_{i=1}^n \operatorname{Ker} d_n \subseteq k[\mathbb{Z}^n] = k[t_1^{\pm 1},\dots,t_n^{\pm 1}]$. It is a good exercise in understanding definitions to check that this is indeed quasi-isomorphic to the desired $k[\varepsilon]/\varepsilon^2$. □

Therefore, once again up to issues of homotopy, we may view a connective complex C of k-modules as being equipped with an action by S^1 as soon as it is given the structure of a differential graded $k[\varepsilon]/\varepsilon^2$-module. Concretely, this means specifying a k-linear map $\varepsilon : C \to C[1]$ such that $\varepsilon^2 = 0$; but this is precisely the notion of a mixed complex from Definition 2.19! In fact, when mixed complexes were first introduced by Burghelea, they were called *algebraic S^1-complexes*.

Next we discuss cyclic and negative cyclic homology from this point of view. Observe that k, as a $k_\varepsilon := k[\varepsilon]/\varepsilon^2$-module, has a periodic resolution by free k_ε-modules

$$k \overset{\sim}{\leftarrow} k_{\mathrm{per}} := [k_\varepsilon \overset{\varepsilon}{\leftarrow} k_\varepsilon[1] \overset{\varepsilon}{\leftarrow} k_\varepsilon[2] \overset{\varepsilon}{\leftarrow} \cdots].$$

Given a mixed complex C over k, i.e., a connective dg $k[\varepsilon]/\varepsilon^2$-module, one easily checks the following:

- $HC(C/k) = [k_\varepsilon \overset{\varepsilon}{\leftarrow} k_\varepsilon[1] \overset{\varepsilon}{\leftarrow} \cdots] \otimes_{k[\varepsilon]/\varepsilon^2} C$, which represents $k \otimes^{\mathbb{L}}_{k[\varepsilon]/\varepsilon^2} C$, i.e., the "group homology of the action $S^1 \circlearrowright C$"
- $HC^-(C/k) = \mathrm{Hom}_{k[\varepsilon]/\varepsilon^2}(k_{\mathrm{per}}, C)$, which represents $\mathrm{RHom}_{k[\varepsilon]/\varepsilon^2}(k,C)$, i.e., the "group cohomology of the action $S^1 \circlearrowright C$"
- Moreover, the inclusion $k[1] = k\varepsilon \hookrightarrow k[\varepsilon]/\varepsilon^2$ is $k[\varepsilon]/\varepsilon^2$-linear and plays the role of the norm map from Remark 4.1; it induces $k \otimes^{\mathbb{L}}_{k[\varepsilon]/\varepsilon^2} C[1] \to \mathrm{Rhom}_{k_\varepsilon}(k,C)$, i.e., $HC(C/k)[1] \to HC^-(C/k)$. The reader should check that this map is precisely the boundary map in the "norm sequence" of Remark 2.15, whence its cofiber "Tate cohomology of the action $S^1 \circlearrowright C$" is $HP(C/k)$.

In conclusion, S^1-actions on connective chain complexes may be modelled as mixed complexes, in which case HC, HC^- and HP represent respectively group homology, cohomology, and Tate cohomology of the action.

4.2 As cyclic objects

Before restricting to the k-linear situation in §4.1, we observed that "an object with G-action" could be modelled as a simplicial object in the category $\mathcal{C}$ equipped with an action by the simplicial group $G_\bullet$ (where $G \simeq |G_\bullet|$). Specialising to the case of S^1, realised by the simplicial group $B\mathbb{Z}$, we could therefore study simplicial objects $X_\bullet$ equipped with an action by $B\mathbb{Z}$, i.e., each X_n should be equipped with n commuting automorphisms satisfying certain compatibilities with respect to the face and degeneracies. Unfortunately it seems that few concrete examples of such $B\mathbb{Z}$-actions exist; in particular, the Hochschild complex does not admit such an action on the nose (as far as the author is aware).

The problem is that although $B\mathbb{Z}$ is a simplicial group realising S^1, it is enormous when compared to the standard simplicial set S^1_{simp} realising S^1, whose only non-denerate simplices are one vertex and one 1-simplex:

$$S^1_{\text{simp}} = \qquad \{v\} \Leftarrow \{v, e\} \Lleftarrow \{v, s_1 e, s_0 e\} \Llleftarrow \cdots$$

Unfortunately, S^1_{simp} is not a simplicial group (it is not even a Kan complex) and therefore it does not make sense to discuss its actions in the above sense. Remarkably, it does however admit the structure of a so-called *crossed simplicial group*, a notion introduced in the late 1980s by Krasauskas and Fiedorowicz–Loday [18] in order to provide much smaller simplicial models of certain topological groups by slightly weakening the concept of the simplicial group.

Definition 4.4. A *crossed simplicial group* is a simplicial set $\mathfrak{G}$ such that each $\mathfrak{G}_n$ is a group, together with group homomorphisms $\mathfrak{G}_n \to \mathrm{Aut}(\{0, \ldots, n\})$, such that for all $0 \le i \le n$

(i) $d_i(gh) = d_i(g) d_{g^{-1}(i)}(h)$, $s_i(gh) = s_i(g) s_{g^{-1}(i)}(h)$, and

(ii) the following diagrams of sets commute

$$\begin{tikzcd} {[n-1]} \arrow[r, "d_{g^{-1}(i)}"] \arrow[d, "d_i(g)"'] & {[n]} \arrow[d, "g"] \\ {[n-1]} \arrow[r, "d_i"'] & {[n]} \end{tikzcd} \qquad \begin{tikzcd} {[n+1]} \arrow[r, "s_{g^{-1}(i)}"] \arrow[d, "s_i(g)"'] & {[n]} \arrow[d, "g"] \\ {[n-1]} \arrow[r, "s_i"'] & {[n]} \end{tikzcd}$$

An *action* of a crossed simplicial group $\mathfrak{G}_\bullet$ on a simplicial set $X_\bullet$ (or many generally on a simplicial object in an arbitrary category) is the data of actions of $\mathfrak{G}_n$ on X_n, for each $n \ge 0$, such that $d_i(gx) = d_i(g)(d_{g^{-1}(i)}(x))$ and $s_i(gx) = s_i(g)(s_{g^{-1}(i)}(x))$ for all $0 \le i \le n$.

Example 4.5. (i) Any simplicial group $G_\bullet$ may be viewed a crossed simplicial group, by declaring the representations $G_n \to \mathrm{Aut}(\{0, \ldots, n\})$ to be trivial.

(ii) Connes' *cyclic* crossed simplicial group $\mathfrak{C}$ is defined by $\mathfrak{C}_n = \mathbb{Z}/n+1$, with the representation $\mathbb{Z}/n+1 \to \mathrm{Aut}(\{0, \ldots, n\})$ being cyclic permutation and with simplicial structure isomorphic to S^1_{simp} given by the bijections

$$\mathbb{Z}/n+1 = \{0, \ldots, n\} \cong S^1_{\text{simp},n}, \quad 0 \mapsto v, \quad 0 < i \mapsto s_{n-1} \cdots s_i s_{i-2} \cdots s_0(e).$$

It can be shown that the discussion after Remark 4.1 extends to crossed simplicial groups:

Proposition 4.6 ([18, §5])**.** *Let $\mathfrak{G}$ be a crossed simplicial group. Then:*

(i) Its geometric realisation $|\mathfrak{G}|$ naturally admits the structure of a topological group.

(ii) Given an action of $\mathfrak{G}$ on a simplicial set $X_\bullet$, there is an induced continuous action of $|\mathfrak{G}|$ on $|X_\bullet|$.

(iii) Adjoint: if X is a topological space equipped with a continuous action by $|\mathfrak{G}|$, then there is an induced action of $\mathfrak{G}$ on the simplicial set $\mathrm{Sing}(X)$.

Proof. The key observation is that, given any simpicial set $X_\bullet$, there exists a simplicial set $F_{\mathfrak{G}}(X_\bullet)$ with $\mathfrak{G}$-action which models the product $\mathfrak{G} \times X_\bullet$. More precisely, define $F_{\mathfrak{G}}(X_\bullet)$ as follows:

$$F_{G_\bullet}(X_\bullet)_n := \mathfrak{G}_n \times X_n, \qquad d_i(g,x) = (d_i(g), d_{g^{-1}(i)}(x)), \qquad s_i(g,x) = (s_i(g), s_{g^{-1}(i)}(x))$$

It is easy to check that $F_{\mathfrak{G}}(X_\bullet)$ admits a $\mathfrak{G}$ action (by the obvious left action of each $\mathfrak{G}_n$ on $\mathfrak{G}_n \times X_n$), and that there is even a monadic structure $F_{\mathfrak{G}}(F_{\mathfrak{G}}(X_\bullet)) \to F_{\mathfrak{G}}(X_\bullet)$, $(g,h,x) \mapsto (gh,x)$. Importantly, it can be shown that there is a natural homeomorphism of geometric realisations

$$|F_{\mathfrak{G}}(X_\bullet)| \xrightarrow{\simeq} |\mathfrak{G}| \times |X_\bullet|.$$

So, assuming now that $X_\bullet$ is equipped with a $\mathfrak{G}$-action, whence there is an action map $F_{\mathfrak{G}}(X_\bullet) \to X_\bullet$ (equivalently, $X_\bullet$ is an algebra over the monad $F_{\mathfrak{G}}$), passing to geometric realisations provides a continuous map $|\mathfrak{G}| \times |X_\bullet| \to |X_\bullet|$. Consideration of the special case $X_\bullet = \mathfrak{G}$ and the axioms for a monad easily proves (i) and (ii).

Part (iii) is slightly more involved so we refer the reader to [18, §5]. □

Remark 4.7. Part (i) of the proposition can be made more precise. For each $n \geq 0$, there is an inclusion of $\mathfrak{G}_n$ into $|\mathfrak{G}|$ as a discrete subgroup, given by $g \mapsto [g, (\frac{1}{n+1}, \ldots, \frac{1}{n+1})]$. When $n+1$ divides $n'+1$ these maps are moreover compatible with the degeneracy $[n'] \to [n]$ sending the first $\frac{n'+1}{n+1}$ elements to 0, the next $\frac{n'+1}{n+1}$ elements to 1, etc., so that $\mathfrak{G}_n \subseteq \mathfrak{G}_{n'} \subseteq |\mathfrak{G}|$. Passage to the limit identifies $\bigcup_{n \geq 0} \mathfrak{G}_n$ (a filtered union under the ordering $n+1 | n'+1$) as a dense subgroup of $|\mathfrak{G}|$.

Thanks to Proposition 4.6 we see that, for any crossed simplicial group $\mathfrak{G}$, a combinatorial way to model $|\mathfrak{G}|$-actions is via $\mathfrak{G}$-actions. In particular, we may model $S^1 \simeq |\mathfrak{C}|$-actions by studying simplicial objects equipped with an action by Connes' cyclic group $\mathfrak{C}$. But, applying Definition 2.19 in the case of crossed simplicial group $\mathfrak{C}$, we see that an object with $\mathfrak{C}$-action is precisely a cyclic object in the sense of 2.17. In conclusion, cyclic objects in a category model S^1-actions.

Having heuristically claimed that both mixed complexes and cyclic objects model S^1-actions, we should compare the two approaches. As we saw in Section 2.2, the un-normalised chain complex construction defines a functor

$$\{\text{cyclic } k\text{-modules}\} \longrightarrow \{\text{mixed complexes over } k\}$$

This functor is indeed an equivalence up to issues of homotopy (more precisely, it can be upgraded to a Quillen equivalence of model categories, or an equivalence of ∞-categories). We refer to [20, §II.2] [16, 17] [22, Corol. 2.4] for further details.

4.3 As functors $BS^1 \to \mathcal{C}$

Given a discrete group G, the data of an object of a category $\mathcal{C}$ equipped with a G-action is the same as giving a functor $G \to \mathcal{C}$, where we follow a standard abuse of notation of denoting by G the category with a single object $*$ having endomorphisms $\mathrm{End}_G(*) = G$. Given that the process of replacing a category by its nerve is fully faithful, this is then the same data as a morphism of simplicial sets (i.e., functor of ∞-categories) $BG = N(G) \to N(\mathcal{D})$.

However, let us now replace $N(\mathcal{D})$ by a homotopically richer category $\mathcal{D}$, such as a dg, simplicial, or ∞-category (in practice it will be $D(k)$ or Sp); we will begin to use some elementary language of ∞-categories (we refer the reader to §4.5 for a short introduction). Then a functor $BG \to \mathcal{D}$ is now slightly more subtle: it represents an object $X \in \mathcal{D}$, an auto-equivalence $g : X \xrightarrow{\sim} X$ for each $g \in G$, a 2-cell

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend right, "hg"'] & X \arrow[r, "h"] & X \end{tikzcd},$$

for each $g, h \in G$, 3-cells connecting the 2-cells, and so on. That is, the action of G on X is no longer associative on the nose, but only up to higher homopoties in a coherent fashion.

This hopefully motives the following definition:

Definition 4.8. Given a topological group G, an *object of $\mathcal{D}$ with G-action* is a functor $BG \to \mathcal{D}$, where BG is a fixed Kan complex whose geometric realisation is a classifying space for G (as a topological group). The ∞-category of objects of $\mathcal{D}$ with G-action is thus $\mathcal{D}^{BG} := \mathrm{Fun}(BG, \mathcal{D})$.

Remark 4.9. (i) Any two choices of simplicial classifying space BG are homotopy equivalent, hence categorically equivalent as ∞-categories since they are Kan complexes, and therefore $\mathcal{D}^{BG}$ is well-defined up to categorical equivalence. If G is given to us as the geometric realisation of a simplicial group $G_\bullet$, then there are two common choices for BG, both built from the bisimplicial set $BG_\bullet$: either its diagonal or its $\overline{W}$-construction (which can be checked directly to be equivalent [**?**, **?**]).

(ii) In the special case that $\mathcal{D} = N(\mathcal{C})$ is the nerve of a usual category $\mathcal{C}$, then the adjunction between nerves and homotopy categories shows that objects of $N(\mathcal{C})$ with G-action are simply objects of $\mathcal{C}$ with an action by the discrete group $\pi_0(G)$ in the usual sense. To get interesting examples it is essential that $\mathcal{D}$ be homotopically richer.

To compare Definition 4.8 to mixed complexes and cyclic objects we will consider two possible choices of simplicial classifying space BS^1.

Firstly, it is well-known that the classifying space of S^1 is $\mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2) \simeq |BB\mathbb{Z}|$ [Mixed complex point of view to be added]

Our second model for the classifying space for S^1 comes from Connes' cyclic category:

Lemma 4.10. *The geometric realisation $|N(\Lambda)|$ of the cyclic category Λ is a classifying space for S^1.*

Proof. We will sketch one method of proof; for further details see [18, 5.9–5.12], for other approaches, see [32, App. T] [28].

One applies Quillen's Theorem B to the inclusion of categories $i : \Delta \hookrightarrow \Lambda$; recall that this is faithful and bijective on objects. For any object $[m] \in \Lambda$, the comma category $[m]\backslash i$ has objects given by all morphisms f in Λ with codomain $[m]$, and has morphisms given by commutative diagrams

$$\begin{tikzcd} {[n]} \arrow[dr, "f"] \arrow[d, "\lambda"'] & \\ {[n']} \arrow[r, "f'"'] & {[m]} \end{tikzcd}$$

where $\lambda \in \mathrm{Hom}_\Delta([n], [m]) \subseteq \mathrm{Hom}_\Lambda([n], [m])$. Thus its nerve $N([m]\backslash i)$ is the simplicial set $\mathrm{Hom}_\Lambda(-, [m])$. But any morphism $[n] \to [m]$ in Λ may be unique decomposed as an automorphism of $[n]$, i.e., an element of $\mathbb{Z}/n + 1$, followed by a morphism $[n] \to [m]$ in Δ; this is part of the combinatorics which we omitted when defining the cyclic category. In other words, $\mathrm{Hom}_\Lambda([n], [m]) = \mathbb{Z}/n + 1 \times \mathrm{Hom}_\Delta([n], [m])$; one then checks that the simplicial structure on the right side as n varies is precisely that which appeared in the construction $F_{\mathfrak{C}}$ in the proof of Proposition 4.6, i.e., $\mathrm{Hom}_\Lambda(-, [m]) = F_{\mathfrak{C}}(\Delta^m)$, where $\Delta^m := \mathrm{Hom}_\Delta(-, [m])$ is the m-simplex. [To finish; in the meantime, also for the details of the next paragraph, see [18]]

Quillen's Theorem B therefore implies that there is a homotopy fibre sequence

$$S^1 \simeq |\mathfrak{C}| \longrightarrow |N(\Delta)| \longrightarrow |N(\Lambda)|$$

But the geometric realisation of Δ is contractible (since, e.g., Δ has an initial object), and so $S^1 \simeq \Omega|N(\Lambda)|$. Taking care that this sufficiently preserves group structures (this is where we are omitting some details), one completes the proof by recalling that any space X serves as a classifying space for ΩX. $\square$

The canonical unit map of simplicial sets $N(\Lambda) \to \mathrm{Sing}(|N(\Lambda)|)$ may therefore be written as $N(\Lambda) \to BS^1$, which by restriction induces a morphism

$$\mathcal{D}^{BS^1} \longrightarrow \mathrm{Fun}(N(\Lambda), \mathcal{D}),$$

i.e., each object X of $\mathcal{D}$ with S^1-action determines a cyclic object $X_\bullet$ of $\mathcal{D}$ (this is a simplicial object $X_\bullet$ of $\mathcal{D}$ equipped with $t_n : X_n \xrightarrow{\sim} X_n$ for each $n \geq 0$, such that the axioms of Definition 2.17 are satisfied up

to higher homotopies in a coherent fashion). Informally, $X_\bullet$ is defined by taking the constant simplicial object $X_n := X$ for all n, with the cyclic operator t_n on X_n corresponding to the action of the cyclic group $\mathbb{Z}/n+1 \subseteq S^1$ on X.

We can also go the other direction (assuming that $\mathcal{D}$ admits geometric realisations of simplicial objects, which is true in cases like $D(k)$ or Sp): there is also a functor

$$\mathrm{Fun}(N(\Lambda), \mathcal{D}) \longrightarrow \mathcal{D}^{BS^1}$$

which takes a cyclic object $X_\bullet$ in $\mathcal{D}$ and gives $|X_\bullet| \in \mathcal{D}$ (the geometric realisation of the underlying simplicial object of $X_\bullet$) equipped with a certain natural S^1-action in the above sense. This is the essentially surjective right adjoint to the functor of the previous paragraph.

Remark 4.11. The previous two paragraphs are probably better known to the reader in the simpler case of Δ instead of Λ. Then $|N(\Delta)|$ is contractible and so the canonical unit map $N(\Delta) \to \mathrm{Sing}(|N(\Delta)|)$ may be written as $N(\Delta) \to *$. This induces the adjoint pair

$$\begin{tikzcd} \mathcal{D} = \mathrm{Fun}(*, \mathcal{D}) \arrow[r, bend left, "\text{constant simplicial object}"] & \mathrm{Fun}(N(\Delta), \mathcal{D}) \arrow[l, bend left, "\text{geometric realisation}"] \end{tikzcd}$$

Given an object of $\mathcal{D}$, in general we do not care whether/how it arose as the geometric realisation of some simplicial diagram, so we work in the left category rather than the right.

Analogously to the remark, we do not care whether/how a given object of $\mathcal{D}^{BS^1}$ arose as the geometric realisation of some cyclic object. In conclusion, although we may pass through various constructions to get there, the final home for our objects with S^1-actions will be $\mathcal{D}^{BS^1}$.

4.4 The cyclic bar construction

Having hopefully justified in the section thus far that the correct framework in which to study S^1-actions is $\mathcal{D}^{BS^1}$, we should now offer an example. We will abstract the construction of Hochschild homology, rapidly leading to its topological analogue.

Recall that an *(associative) algebra object* in a symmetric monoidal category $(\mathcal{C}, \otimes, 1_\mathcal{C})$ is the data of an object $A \in \mathcal{C}$ together with morphisms $\mu : A \otimes A \to A$ (=multiplication) and $1_\mathcal{C} \to A$ (= multiplicative identity) which satisfy the usual commutative diagrams specifying the axioms of an algebra. Given such an algebra object A, one may write down a simplicial object

$$\begin{tikzcd} A & A \otimes A \arrow[l, shift right] \arrow[l, shift left] & A \otimes A \otimes A \arrow[l, shift right] \arrow[l] \arrow[l, shift left] & \cdots \arrow[l, shift right] \arrow[l] \arrow[l, shift left] \end{tikzcd}$$

in $\mathcal{C}$ by copying the rules for the face and degeneracy maps used in the definition of Hochschild homology from Remark 2.3. For example, the three face maps $d_0, d_1, d_2 : A \otimes A \otimes A \to A \otimes A$ are given by $\mu \otimes \mathrm{id}$, $\mathrm{id} \otimes \mu$, $\mu \otimes \mathrm{id} \circ t$, where t is the cyclic endomorphism of $A \otimes A \otimes A$ given by $a \otimes b \otimes c \mapsto c \otimes a \otimes b$. Moreover, by declaring t_n to be the analogous cyclic endomorphism of $A^{\otimes n+1}$ one upgrades this simplicial object to have the structure of a cyclic object in $\mathcal{C}$, known as the *cyclic bar construction* of A; it is sometimes denoted by $B^{\mathrm{cyc}}_\bullet(A)$.

We now wish to categorise this construction so that it may be applied to associative ring spectra, i.e., algebra objects for which the axioms of an algebra are only satisfied up to higher homotopties. The first step is to observe that algebra objects in $\mathcal{C}$ are classified by functors $\mathrm{Ass}^\otimes_{\mathrm{act}} \to \mathcal{C}$ (Lemma 4.13 below), where the domain category is defined as follows:

Definition 4.12 (Associative algebras)**.** The category $\mathrm{Ass}^\otimes_{\mathrm{act}}$ is defined as follows: objects are finite sets, and a morphism from T to S is the data of a map of sets $f : T \to S$ together with linear orderings $\leq_s$ on the preimages $f^{-1}(s)$ for all $s \in S$. The composition of f with $g : S \to R$ is given by the set-theoretic composition $gf : T \to R$ together with the lexicographic orderings on the preimages; i.e., given $r \in R$ and $t, t' \in (gf)^{-1}(r)$, then $t <_r t'$ iff "$f(t) <_r f(t')$" or "$f(t) = f(t') =: s$ and $t <_s t'$".

The category $\mathrm{Ass}^{\otimes}_{\mathrm{act}}$ is symmetric monoidal under disjoint union, with unit element $\emptyset$. Note that the element $\{1\} \in \mathrm{Ass}^{\otimes}_{\mathrm{act}}$ admits the structure of an algebra object, with multiplication $\mu : \{1,2\} = \{1\} \sqcup \{1\} \to \{1\}$ corresponding to the ordering $1 < 2$. In fact, as promised, $\{1\} \in \mathrm{Ass}^{\otimes}_{\mathrm{act}}$ represents algebra objects in general:

Lemma 4.13. *The category of algebra objects in $\mathcal{C}$ (the notion of a morphism of algebra objects should be clear) identifies with the category of monoidal functors $F : \mathrm{Ass}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$, via $F \mapsto F(\{1\})$.*

Proof. A monoidal functor obviously sends algebra objects to algebra objects, so the functor $F \mapsto F(\{1\})$ with induced multiplication is indeed an algebra object in $\mathcal{C}$.

Conversely, from any algebra object $(A, \mu : A \otimes A \to A, 1_{\mathcal{C}} \to A)$, we may define a monoidal functor $A^{\otimes} : \mathrm{Ass}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$ by sending a finite set S to $A^{\otimes}(S) := \otimes_{s \in S} A$, and by sending a morphism $f : T \to S$ to

$$\otimes_{t \in T} A \longrightarrow \otimes_{s \in S} A, \qquad \otimes_{t \in T} a_t \mapsto \otimes_{s \in S} \prod_{t \in f^{-1}(s)} a_t.$$

Here we abusively write the product μ as though A were a set equipped with a law of multiplication, leaving the reader the write a categorical formula; the important point is that the product $\prod_{t \in f^{-1}(s)} a_t := a_{t_1} \cdots a_{t_m}$ is well-defined, since $f^{-1}(s) = \{t_1 <_s \cdots <_s t_m\}$ is equipped with a total order. $\square$

The algebra object $\{1\} \in \mathrm{Ass}^{\otimes}_{\mathrm{act}}$ has an associated cyclic bar construction $B^{\mathrm{cryc}}_{\bullet}(\{1\})$

$$\{1\} \Leftarrow \{1,2\} \Lleftarrow \{1,2,3\} \Llleftarrow \cdots.$$

This is a cyclic object in $\mathrm{Ass}^{\otimes}_{\mathrm{act}}$, so corresponds to a certain functor $\Lambda^{\mathrm{op}} \to \mathrm{Ass}^{\otimes}_{\mathrm{act}}$, which is denoted by Cut (and which can be described explicitly should one wish [32, App. T] [28]). Purely formally, it follows that the cyclic bar construction $B^{\mathrm{cryc}}_{\bullet}(A)$ of a general algebra object $A \in \mathcal{C}$ is now given by the functor

$$\Lambda^{\mathrm{op}} \xrightarrow{\mathrm{Cut}} \mathrm{Ass}^{\otimes}_{\mathrm{act}} \xrightarrow{A^{\otimes}} \mathcal{C},$$

where $A^{\otimes}$ is the functor corresponding to A in the sense of Lemma 4.13.

We are now prepared to extend the constructions to the ∞-world:

Definition 4.14. Let $\mathcal{D}$ be a symmetric monoidal infinity category. An *(associative) algebra object* in $\mathcal{D}$ is a functor $A : N(\mathrm{Ass}^{\otimes}_{\mathrm{act}}) \to \mathcal{D}$; note that this is the data of objects $A_1, A_2, \cdots \in \mathcal{D}$, various equivalences $A_n \otimes A_m \simeq A_{n+m}$, a multiplication map $A_2 \to A$, an identity map $1_{\mathcal{D}} \to A_1$, homotopies between them to express "associativity up to homotopy", etc. Also note that this is equivalent to the operadic definition of $\mathbb{E}_1$-algebras in $\mathcal{D}$ (see [28, Def./Prop. 3.3] and [31, Prop. 2.2.4.9]).

The *cyclic bar construction* $B^{\mathrm{cyc}}_{\bullet}(A)$ of an algebra object is now defined to be the functor

$$B^{\mathrm{cyc}}_{\bullet}(A) : N(\Lambda^{\mathrm{op}}) \xrightarrow{N(\mathrm{Cut})} N(\mathrm{Ass}^{\otimes}_{\mathrm{act}}) \xrightarrow{A} \mathcal{D},$$

which is simply making precise the statement there is a cyclic object in $\mathcal{D}$ which looks like

$$A \Leftarrow A \otimes A \Lleftarrow A \otimes A \otimes A \Llleftarrow \cdots.$$

Finally, assuming that $\mathcal{D}$ admits geometric realisations of simplical objects, we may define the Hochschild homology of the algebra object A to be

$$HH(A/\mathcal{D}) := |B^{\mathrm{cyc}}_{\bullet}(A)| \in \mathcal{D}^{BS^1}$$

(recall from the end of §4.3 the geometric realisation of a cyclic object gives an object with S^1-action).

Thus $HH(-/\mathcal{D})$ is a (well-behaved) functor from the ∞-category of algebra objects in $\mathcal{D}$ to $\mathcal{D}^{BS^1}$.

Example 4.15. (i) Let k be a base ring and A a flat k-algebra. We leave it to the reader to write down a precise statement that Definition 4.14 recovers $HH(A/k)$.

(ii) More generally, if A is not necessarily a flat k-algebra, then it induces an algebra object A in the symmetric monoidal infinity category $D(k)$ such that $HH(A/D(k))$ is the derived form of $HH(A/k)$ introduced in §2.4 (since the monoidal structure in $D(k)$ is given by the derived tensor product $\otimes_k^{\mathrm{L}}$, which we modelled earlier by replacing A by a simplicial resolution $P_\bullet$).

(iii) Let A be an associate ring spectrum, i.e., an algebra object in Sp. Then its topological Hocshchild homology is

$$THH(A) := HH(A/\operatorname{Sp}) \in \operatorname{Sp}^{BS^1}.$$

We will discuss this further in the next subsection.

4.5 Topological Hochschild homology and its variants

A profound idea of Goodwillie and Waldhausen in the 1980s was to consider the formalism of Hochschild and cyclic homology not only for rings, but more generally for ring spectra. A well behaved theory of ring spectra was not available at the time (in particular, there was no strictly commutative and associative smash product, nor was the language of ∞-categories yet developed in order to overcome the higher coherence issues), but nevertheless Bökstedt succeeded in giving a rigorous definition of topological Hochschild homology (for "functors with smash product") in 1985 [10, 11]. Fortunately, the ∞-category of spectra is now available to us, and Nikolaus–Scholze have redeveloped the subject from this point of view. (Here we have omitted the entire history of the subject between 1985 and 2017, with apologies to all those concerned.)

We begin with an informal discussion of ∞-categories and spectra, for readers who have not previously encountered them. As is well-known, the triangulated category $D(k)$ of complexes of modules over a ring k up to quasi-isomorphism suffers from many problems, which can be resolved by keeping track of which quasi-isomorphisms were used to identify complexes. Of course, as soon as one keeps track of quasi-isomorphisms, one also needs to keep track of which homopoties between quasi-isomorphisms were used, and so on. The theory of ∞-categories (modelled as Joyal and Lurie's "quasi-categories") does this: one constructs a simplicial set $D(k)$ whose 0-simplices are chain complexes over k and whose higher simplices precisely encode the aforementioned data. To understand these notes, the reader should not run into too much trouble by treating ∞-categories as derived categories in which the usual problems of functoriality/uniqueness (of cones, homotopy limits, etc.) disappear, or rather become contractible in a precise sense. A good introduction to the foundation of the theory are the notes by Groth.

Next we turn to spectra. In the course of the historical development of mathematics, each new discovery of a rings of arithmetic interest was larger than the previous, i.e., $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$, but let us nevertheless consider the hypothetical situation of only knowing the complex numbers. This suffices to define complex manifolds, but in order to do Hodge theory we must first construct the integers, real numbers, and the functors

$$\begin{tikzcd} \mathbb{Z}\text{-mod} \arrow[r, bend left, "-\otimes_{\mathbb{Z}}\mathbb{R}"] & \mathbb{R}\text{-mod} \arrow[l, bend left, "\text{restriction}"] \arrow[r, bend left, "-\otimes_{\mathbb{R}}\mathbb{C}"] & \mathbb{C}\text{-mod} \arrow[l, bend left, "\text{restriction}"] \end{tikzcd}$$

The faithful, non-full restriction functor $\mathbb{C}$-mod $\to$ $\mathbb{R}$-mod is particular important, as without it we would never see complex conjugation; in other words, it is essential to have the freedom to view our complex vector spaces merely as real vector spaces. Once we have done so, we can say that $\mathbb{R} \to \mathbb{C}$ is a morphism of algebra objects in $\mathbb{R}$-mod and thereby reconstruct $\mathbb{C}$-mod.

In a similar way, the theory of spectra provides a ring (in a generalised sense) $\mathbb{S}$ called the *sphere spectrum* and a restriction functor from $\mathbb{Z}$-modules to $\mathbb{S}$-modules which corresponds to a morphism of algebra objects $\mathbb{S} \to \mathbb{Z}$. Analogously to the naive Hodge theory example, it is now widely accepted that it is important to allow ourselves the freedom of viewing objects over $\mathbb{Z}$ as merely over $\mathbb{S}$: this allows for maps between them which are no longer $\mathbb{Z}$-linear, but merely $\mathbb{S}$-linear. Of course none of this is true in a naive set-theoretic fashion: only the derived category of $\mathbb{S}$-modules exists. So to be more precise there exist a symmetric monoidal ∞-category Sp of *spectra*, whose unit object is $\mathbb{S}$ and whose monidal

structure is written $\otimes = \otimes_{\mathbb{S}}$, and functors

$$\begin{tikzcd} \mathrm{Sp} \arrow[r, bend left, "-\otimes_{\mathbb{S}}\mathbb{Z}"] & D(\mathbb{Z}) \arrow[l, bend left, "\text{restriction}"] \end{tikzcd}$$

corresponding to a morphism of algebra objects $\mathbb{S} \to \mathbb{Z}$ in Sp. The restriction functor is known as the *Eilenberg–Maclane* construction, but we prefer not to introduce any notion for it and instead allow ourselves to view any complex as a spectrum. Any spectrum C has abelian "homotopy groups" $\pi_n(C)$, for $n \in \mathbb{Z}$, which are the usual homology groups $H_n(C)$ in case C comes via restriction from $D(\mathbb{Z})$ (or from $D(k)$ for some other ring k).

By restriction, any algebra object A in $D(\mathbb{Z})$ (e.g., a usual ring, or something more exotic like a differential graded ring) gives rise to an algebra object in Sp, i.e., a *ring spectrum*; we may then form its topological Hochschild homology as in Example 4.15(iii):

$$THH(A) := HH(A/\mathrm{Sp}) = |A \leftleftarrows A \otimes_{\mathbb{S}} A \leftleftarrows\!\!\leftarrow A \otimes_{\mathbb{S}} A \otimes_{\mathbb{S}} A \leftleftarrows\!\!\leftarrow\!\!\leftarrow \cdots| \in \mathrm{Sp}^{BS^1}$$

(Of course, as in Example 4.15(iii), we could also apply this construction to an arbitrary ring spectrum which does not necessarily come from $D(\mathbb{Z})$, but we are interested mainly in *THH* of usual rings.) The homotopy groups of this spectrum are denoted by $THH_n(A) := \pi_n THH(A)$ and are known as the *topological Hochschild homology groups* of A.

Remark 4.16 (Low degrees, rationalising, and stable homotopy groups of spheres)**.** The homotopy groups of the sphere spectrum $\mathbb{S}$ are given by

$$\pi_n(\mathbb{S}) = \begin{cases} 0 & n < 0 \\ \mathbb{Z} & n = 0 \\ n^{\text{th}} \text{ stable homotopy group of spheres} & n > 0 \end{cases}.$$

Two concrete consequences for the canonical map $THH(A) \to HH(A/\mathbb{Z})$ as are follows.

(i) It is an isomorphism in degrees ≤ 2, i.e., $THH_n(A) \xrightarrow{\sim} HH_n(A/\mathbb{Z})$ for $n \geq 2$ (also both sides vanish for $n < 0$)

(ii) It is an isomorphism rationally (since the stable homotopy groups of spheres are known to be finite), or more precisely the map $THH_n(A) \to HH_n(A/\mathbb{Z})$ has kernel and cokernel killed by some integer depending only on n. In particular, if $A \supseteq \mathbb{Q}$ (whence the two sides are $\mathbb{Q}$-vector spaces), it is an isomorphism: topological Hochschild homology offers us nothing new in characteristic zero!

Let $\mathcal{D} = D(k)$ or Sp. Given any object with S^1-action $X \in \mathcal{D}^{BS^1}$, we may form its group cohomology, i.e., derived/homotopy S^1-invariants

$$X^{hS^1} = \lim_{BS^1} X \in \mathcal{D}$$

(where, on the right side, we recall that X is a functor $BS^1 \to \mathcal{D}$). Similarly we may form its group homology, i.e., derived/homotopy S^1-coinvariants

$$X_{hS^1} = \mathrm{colim}_{BS^1} X \in \mathcal{D}.$$

These are respectively the right and left adjoints to the functor $\mathcal{D} \to \mathcal{D}^{BS^1}$ which equips an object of $\mathcal{D}$ with the trivial action. There exists moreover a certain "norm" morphism $N : X_{hS^1}[1] \to X^{hS^1}$, whose cofiber is known as the *Tate cohomology* of the action and denoted by X^{tS^1}, i.e.,

$$X_{hS^1}[1] \xrightarrow{N} X^{hS^1} \longrightarrow X^{tS^1}.$$

Example 4.17. Suppose that the object with S^1-action in the preceding discussion is the Hochschild homology $HH(A/k) \in D(k)^{BS^1}$ of some k-algebra A. In §4.1 we informally explained that $HC(A/k)$,

$HC^-(A/k)$, and $HP(A/k)$ may be considered the homology, cohomology, and Tate cohomology of the S^1-action. To be precise, one does indeed have natural equivalences in $D(k)$

$$HC(A/k) \simeq HH(A/k)_{hS_1}, \qquad HC^-(A/k) \simeq HH(A/k)^{hS_1}, \qquad HP(A/k) \simeq HH(A/k)^{tS_1}.$$

These folklore identifications are obtained by picking particular models to compute the homotopy invariants and coinvariants, then comparing to the explicit double complexes of §2.2 (similarly to what we did at the end of §4.1); see [22].

Furthermore, if we rearrange the above norm sequence into a fibre sequence $X^{hS^1} \to X^{tS^1} \to X_{hS^1}[2]$, then the case $X = HH(A/k)$ recovers the sequence of Remark 2.15.

Definition 4.18. Transporting the previous definition to the world of spectra, we define the *topological negative cyclic homology* and *topological periodic cyclic homology* of a ring A to be

$$TC^-(A) := THH(A)^{hS^1} \in \mathrm{Sp}, \qquad TP(A) := THH(A)^{tS^1} \in \mathrm{Sp}\,.$$

We will also be interested in $THH(A)_{hS^1}$, which is an analogue of cyclic homology, but we stress that it is not what is known as topological cyclic homology. The homotopy groups of these spectra are denoted by $TC_n^-(A)$, $TP_n(A)$, $\pi_n THH(A)_{hS^1}$.

In the setting of the previous definition, the general norm sequence looks like

$$THH(A)_{hS^1}[1] \xrightarrow{N} TC^-(A) \longrightarrow TP(A).$$

Since $THH(A)_{hS^1}$ is supported in homotopical degree ≥ 0 (as this is true of A and preserved by colimits), we see that $TC_n^-(A) \xrightarrow{\simeq} TP_n(A)$ for $n \leq 0$.

Remark 4.19 (Multiplicative structure)**.** Assuming that A is a commutative ring, which is once again our only case of interest, then $THH(A)$ is itself a ring spectrum. Moreover, taking the Tate construction or homotopy fixed points are known to be lax monoidal functors, so that $TP(A)$ and $TC^-(A)$ are ring spectra. Their homotopy groups

$$\bigoplus_{n\geq 0} THH_n(A), \qquad \bigoplus_{n\in\mathbb{Z}} TP_n(A), \qquad \bigoplus_{n\in\mathbb{Z}} TC_n^-(A)$$

therefore naturally have the structure of graded commutative rings.

5 Topological Hochschild homology of $\mathbb{F}_p$-algebras

In this section we restrict our study of topological Hochschild homology to the case of $\mathbb{F}_p$-algebras. We begin in §5.1 by studying $\mathbb{F}_p$ itself and deriving some analogues for general $\mathbb{F}_p$-algebras A; in particular, we will see that $TP(A)$ is a 2-periodic lifting of the classical periodic cyclic homology $HP(A/\mathbb{F}_p)$ from characteristic p to mixed characteristic, and establish an analogue of the HKR filtration of Proposition 2.28. We remark that these results do not have analogues for the topological Hochschild homology of arbitrary rings (though they do if we replace our base ring $\mathbb{F}_p$ by a perfectoid ring).

Then in §5.2 we analyse topological periodic and negative cyclic homologies of smooth algebras by reduction to qrsp algebras, analogously to §3.

5.1 The case of $\mathbb{F}_p$ itself and consequences

The following fundamental highly non-trivial result is sadly beyond the techniques of this course:

Theorem 5.1 (Bökstedt)**.** *The homotopy groups of $THH(\mathbb{F}_p)$ are given by*

$$THH_n(\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p & n \text{ even} \geq 0 \\ 0 & n \text{ otherwise} \end{cases}$$

Regarding multiplicative structure $THH_*(\mathbb{F}_p) = \mathbb{F}_p[u]$ (polynomial algebra on a single variable u) with $u \in THH_2(\mathbb{F}_p/\mathbb{Z}_p) = HH_2(\mathbb{F}_p/\mathbb{Z})$ (same element as in Example 2.32).

Although we have omitted the proof of Bökstedt's theorem, we will be able to use it as a blackbox to obtain interesting consequences, starting with a calculation of the homotopy groups of the topological periodic and negative cyclic homology of $\mathbb{F}_p$:

Proposition 5.2. *The homotopy groups of $TP(\mathbb{F}_p)$ are given by*

$$TP_n(\mathbb{F}_p) \cong \begin{cases} \mathbb{Z}_p & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Regarding multiplicative structure $TP_*(\mathbb{F}_p) = \mathbb{Z}_p[\sigma^{\pm 1}]$, where σ is any generator of the invertible $\mathbb{Z}_p$-module $TP_2(\mathbb{F}_p)$. The map of graded rings $TC^-_*(\mathbb{F}_p) \to TP_*(\mathbb{F}_p)$ is injective, with image

$$\mathrm{Im}(TC^-_{2n}(\mathbb{F}_p) \to TP_{2n}(\mathbb{F}_p)) = \begin{cases} p^n TP_{2n}(\mathbb{F}_p) & n \geq 0 \\ TP_{2n}(\mathbb{F}_p) & n \leq 0 \end{cases},$$

i.e., $TC^-_*(\mathbb{F}_p) = \mathbb{Z}_p[p\sigma, \sigma^{-1}]$, and the canonical morphism $TC^-(\mathbb{F}_p) \to THH(\mathbb{F}_p)$ sends $p\sigma$ to u (equivalently, without making any choices, $TC^-_*(\mathbb{F}_p)/p \xrightarrow{\simeq} THH_*(\mathbb{F}_p)$ for $* \geq 0$).

Remark 5.3. It may be helpful to compare to Lemma 3.8 and Remark 3.10, where HP_0 inherited an interesting filtration coming from HC^-_{2n}, $n \geq 0$. Here we witness the same phenomenon: the homotopy groups of $TC^-(\mathbb{F}_p)$ is inducing the p-adic filtration on the homotopy groups of $TP(\mathbb{F}_p)$.

Proof of Proposition 5.2. Recall that there is a class $u \in THH_2(\mathbb{F}_p)$ which corresponds to $p \in p\mathbb{Z}/p^2\mathbb{Z}$ under the identifications $THH_2(\mathbb{F}_p) \xrightarrow{\simeq} HH_2(\mathbb{F}_p/\mathbb{Z}) = p\mathbb{Z}/p^2\mathbb{Z}$. We will also use the cohomology class $t \in H^2(S^1, \mathbb{F}_p)$ which corresponds to the extension $0 \to \mathbb{F}_p \to S^1 \xrightarrow{p} S^1 \to 0$; a standard fact about group cohomology states that $H^*(S^1, \mathbb{F}_p) \cong \mathbb{F}_p[t]$.

Using Bökstedt's theorem and the previous paragraph, the homotopy-fixed points spectral sequence for the action of S^1 on $THH(\mathbb{F}_p)$ looks as follows and converges to $TC^-_{i+j}(\mathbb{F}_p)$:

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 \\
H^4(S^1, THH_2(\mathbb{F}_p)) = \mathbb{F}_p t^2 u & 0 & H^2(S^1, THH_2(\mathbb{F}_p)) = \mathbb{F}_p tu & 0 & H^0(S^1, THH_2(\mathbb{F}_p)) = \mathbb{F}_p u \\
0 & 0 & 0 & 0 & 0 \\
H^4(S^1, THH_0(\mathbb{F}_p)) = \mathbb{F}_p t^2 & 0 & H^2(S^1, THH_0(\mathbb{F}_p)) = \mathbb{F}_p t & 0 & H^0(S^1, THH_0(\mathbb{F}_p)) = \mathbb{F}_p
\end{array}$$

(the bottom right corner is in bidegree $(0,0)$). Since all non-zero terms are in even bidegree, all differentials on all pages are necessarily zero, i.e., the spectral sequence is degenerate.

The key (in fact, only) non-trivial calculation is to check the following: the usual spectral sequence map

$$\mathrm{Fil}^2_{\text{abutement}} TC^-_0(\mathbb{F}_0) = \mathrm{Ker}\left(TC^-_0(\mathbb{F}_p) \xrightarrow{\text{edge}} H^0(S^1, THH_0(\mathbb{F}_p))\right) \longrightarrow H^2(S^1, THH_2(\mathbb{F}_p))$$

sends p to tu. This is proved as follows: the part of the spectral sequence which we have displayed (i.e., involving $THH_n(\mathbb{F}_p)$ for $n \leq 2$) is the same as the analogous spectral sequence for the Hochschild homology $HH(\mathbb{F}_p/\mathbb{Z})$, and therefore the calculation is a claim purely about usual Hochschild and negative cyclic homology. One resolves $\mathbb{F}_p$ by a flat dg $\mathbb{Z}$-algebra and proceeds by a direct calculation, for which we refer to [28, Prop. 2.12].

By multiplicativity of the spectral sequence it follows that $p^i \in \mathrm{Fil}^{2i}_{\mathrm{abut}} TC^-_0(\mathbb{F}_p)$ for all $i \geq 0$, and that the canonical map $\mathrm{Fil}^{2i}_{\mathrm{abut}} TC^-_0(\mathbb{F}_p) \to H^{2i}(S^1, THH_{2i}(\mathbb{F}_p))$ sends p^i to $t^i u^i$. That is, the unique ring homomorphism $\mathbb{Z} \to TC^-_0(\mathbb{F}_p)$ induces isomorphism $p^i\mathbb{Z}/p^{i+1}\mathbb{Z} \xrightarrow{\simeq} \mathrm{gr}^{2i}_{\mathrm{abut}} TC^-_0(\mathbb{F}_p)$ for all $i \geq 0$. It follows that $\mathbb{Z}_p \xrightarrow{\simeq} TC^-_0(\mathbb{F}_p)$.

Pick lifts $u \in TC^-_2(\mathbb{F}_p)$ and $t \in TC^-_{-2}(\mathbb{F}_p)$, along the edge maps, of u and t respectively (it would be more correct to write $\widetilde{u}$ and $\widetilde{t}$ for the lifts, but we follow a standard abuse of notation). The multiplicativity of the spectral sequence forces $TC^-_{2i}(\mathbb{F}_p) = \mathbb{Z}_p u^i$ for $i \geq 0$, and $TC^-_{2i}(\mathbb{F}_p) = \mathbb{Z}_p t^i$ for $i \leq 0$, with the relation $ut = p \cdot \text{unit}$ in $TC^-_0(\mathbb{F}_p) = \mathbb{Z}_p$. □

Remark 5.4. Theorem 5.1 and Proposition 5.2 remain true if $\mathbb{F}_p$ is replaced by an arbitrary perfect $\mathbb{F}_p$-algebra k and $\mathbb{Z}_p$ is replaced by the ring of p-typical Witt vectors $W(k)$.

As we promised earlier, from the proposition one obtains some interesting general consequences which will allow us to control the topological theory in characteristic p. The first is that topological periodic cyclic homology of $\mathbb{F}_p$-algebras is always 2-periodic; despite its name, this is not always the case, notably $TP(\mathbb{Z})$. The second is that it lifts classical periodic cyclic homology; since we already know that the latter is related to de Rham cohomology, this indicates that the former might be related to the standard lift of de Rham cohomology, namely crystalline cohomology (as we shall see is indeed true).

Corollary 5.5. *Let A be a commutative $\mathbb{F}_p$-algebra. Then:*

(i) There is a natural, S^1-equivariant fibre sequence $THH(A)[2] \xrightarrow{u} THH(A) \to HH(A/\mathbb{F}_p)$. (This will let us control THH in terms of HH and so serve as a replacement for the classical periodicity sequence.)

(ii) $TP(A)$ is 2-periodic, i.e., the $TP_0(A)$-module $TP_2(A)$ is invertible and $TP_n(A) \otimes_{TP_0(A)} TP_2(A) \xrightarrow{\simeq} TP_{n+2}(A)$ for all $n \in \mathbb{Z}$.

(iii) There is a natural equivalence $TP(A)/p \to HP(A/\mathbb{F}_p)$.

Proof. We need the following general result: given a map of commutative rings $k \to A$, then in the world of $\mathbb{E}_\infty$-ring spectra one has $THH(A) \otimes_{THH(k)} k \simeq HH(A/k)$. This is proved either by checking that both sides satisfy an identical universal property similar to that of Example 2.18, or by transferring the proof of Remark 2.7 to the $\mathbb{E}_\infty$ context.

To prove (i), it is therefore enough to construct an S^1-equivariant fibre sequence

$$THH(\mathbb{F}_p)[2] \to THH(\mathbb{F}_p) \to \mathbb{F}_p,$$

since we can then base change by $THH(A) \otimes_{THH(\mathbb{F}_p)} -$. The class $u := p\sigma \in TC^-_2(\mathbb{F}_p) = \mathrm{Hom}_{h(\mathrm{Sp})}(\mathbb{S}[2], TC^-(\mathbb{F}_p))$ corresponds to a morphism $u : \mathbb{S}[2] \to TC^-(\mathbb{F}_p)$ uniquely up to homotopy; composing with $TC^-(\mathbb{F}_p) \to THH(\mathbb{F}_p)$ gives us a S^1-equivariant map $\mathbb{S}[2] \to THH(\mathbb{F}_p)$, which then linearises to a morphism $u : THH^-(\mathbb{F}_p)[2] \to THH^-(\mathbb{F}_p)$ whose effect on homotopy groups is multiplication by x thanks to the final sentence of Proposition 5.2. We can also read off that proposition that the cofiber of $\times u$ is $\mathbb{F}_p[0]$, as desired.

(ii) follows formally from the fact that $TP_*(A)$ is a graded module over $TP_*(\mathbb{F}_p) \cong \mathbb{Z}_p[\sigma^{\pm 1}]$.

(iii): Taking Tate constructions in the fibre sequence of (i) gives

$$TP(A)[2] \xrightarrow{u} TP(A) \to HP(A/\mathbb{F}_p)$$

But by construction u is given by

$$\begin{tikzcd} TP(A)[2] \arrow[rr, "u"] \arrow[dr, "\sigma", "\cong"'] & & TP(A) \\ & TP(A) \arrow[ur, "p"'] & \end{tikzcd}$$

□

Suppose now that R is a smooth algebra over a perfect field k of characteristic p (in fact, we could suppose that k is any perfect $\mathbb{F}_p$-algebra without any change to the argument). Then we have the anti-symmstrisation map $\Omega^1_{R/\mathbb{F}_p} = \Omega^1_{R/\mathbb{Z}} \to HH_1(R/\mathbb{Z}) = THH_1(R)$ which we know induces by multiplication $\Omega^2_{R/\mathbb{F}_p} \to HH_2(R/\mathbb{Z}) = THH_2(R)$; since $THH_*(R)$ is a commutative graded ring, multiplicativity formally implies that the antisymmstrisation map extends to all degrees $\Omega^*_{R/k} \to THH_*(R)$. Combining this with the map of graded rings $THH_*(\mathbb{F}_p) = \mathbb{F}_p[u] \to THH_*(R)$, we arrive at $\Omega^*_{R/k} \otimes_{\mathbb{F}_p} THH_*(\mathbb{F}_p) \to THH_*(R)$, where $\otimes$ denotes tensor product of graded rings. One has the following analogue of the classical HKR theorem (it may initially appear strange that *THH* is built from multiple copies of de Rham groups, but when passing to *TP* these will perfectly stack up on top of each other to form cristalline cohomology):

Proposition 5.6 (Hesselholt's HKR theorem)**.** *Let R be a smooth k-algebra as in the previous paragraph. Then the map of graded rings $\Omega^*_{R/k} \otimes_{\mathbb{F}_p} THH_*(\mathbb{F}_p) \to THH_*(R)$ is an isomorphism. In particular, $THH_n(R)$ is isomorphic as an R-module to $\bigoplus_{i\geq 0} \Omega^{n-2i}_{R/k}$.*

Proof. By construction the composition

$$\Omega^*_{R/k} \subseteq \Omega^*_{R/k} \otimes_{\mathbb{F}_p} THH_*(\mathbb{F}_p) \to THH_*(R) \to HH_*(R)$$

is the usual antisymmetrisation map, which is an isomorphism by the usual HKR theorem. So $THH_*(A) \to HH_*(A/\mathbb{F}_p)$ is surjective on homotopy groups and the fibre sequence of Corollary 5.5(i) breaks into short exact sequence, whence we have an isomorphism of graded rings $THH_*(A)/u \xrightarrow{\sim} HH_*(A/\mathbb{F}_p)$.

In other words, the map of graded $\mathbb{F}_p[x]$-modules

$$\Omega^*_{R/k} \otimes_{\mathbb{F}_p} THH_*(\mathbb{F}_p) = \Omega^*_{R/k} \otimes_{\mathbb{F}_p} \mathbb{F}_p[x] \longrightarrow THH_*(R)$$

is an isomorphism modulo x. It follows formally (by induction up the degree) that it is an isomorphism. □

Hesselholt's HKR theorem leads to an analogue of the HKR filtration from Proposition 2.28, which gives us a technique to control *THH* via the cotangent complex.

Corollary 5.7 (HKR filtration on *THH*)**.** *Let A be an $\mathbb{F}_p$-algebra. Then $THH(A)$ admits a descending, $\mathbb{N}$-indexed, complete filtration whose n^{th}-graded piece is $\bigoplus_{i=0}^{\lfloor n/2 \rfloor} \mathbb{L}^{n-2i}_{A/\mathbb{F}_p}[n]$.*

Proof. Just left Kan extend the previous proposition; see [8, Corol. 6.10] for some details. □

In order to extend the techniques of §3.1–3.2 to topological Hochschild homology and its variants, we of course need to know that they satisfy flat descent; although there is a technique to deduce this in general from the case of *HH* (see [8, Corol. 3.3], the following special case is enough for us:

Corollary 5.8. *The* Sp*-valued functors*

$$THH(-), \qquad THH(-)_{hS^1}, \qquad TC^-(-), \qquad TP(-)$$

on $\mathbb{F}_p$-algs *satisfy flat descent.*

Proof. The argument is similar to Lemma 3.7. The previous corollary and flat descent for all the $\mathbb{L}^i_{-/\mathbb{F}_p}$ yield flat descent for $THH(-)$. The other cases then follow by suitably taking limits and colimits, taking care of the connectivity issues which arise for $THH(-)_{hS^1}$. □

5.2 Topological periodic & negative cyclic homology of smooth and qrsp algebras

The main theorem we wish to discuss is the following analogue of Theorem 3.1 in which *HP* is replaced by *TP* and de Rham cohomology is replaced by crystalline cohomology:

Theorem 5.9. *If R is a smooth k-algebra, where k is a perfect field of characteristic $p > 0$, then $TP(R)$ and $TC^-(R)$ admit natural, complete, descending $\mathbb{Z}$-indexed filtrations whose n^{th} graded pieces are given respectively by*

$$R\Gamma_{\text{crys}}(R/W(k))[2n], \qquad \mathcal{N}^{\geq i} R\Gamma_{\text{crys}}(R/W(k))[2n],$$

where $\mathcal{N}^{\geq i}$ refers to a certain filtration on crystalline cohomology known as the Nygaard filtration.

The filtrations of the theorem are precisely those arising from the general technique presented in Corollary 3.6, which is valid since we verified flat descent for TP and TC^- in Corollary 5.8. Therefore, following the same method of proof of §3.1–3.2, we must attempt to identify TP and TC^- of the qrsp rings $R_{\text{perf}} \otimes_R \cdots \otimes_R R_{\text{perf}}$. Let we begin with some coarse information analogous to Lemma 3.8

Lemma 5.10. *Let A be a qrsp algebra and set $N := \pi_1(\mathbb{L}_{A/\mathbb{F}_p})$. Then $THH_*(A)$, $TC^-_*(A)$, $TP_*(A)$, and $\pi_* THH(A)_{hS^1}$ are all supported in even degrees. Moreover, $THH_{2i}(A)$ admits a finite increasing filtration with graded pieces $A, N, \Gamma^2_A(N), \ldots, \Gamma^i_A(N)$, and $TP_0(A)$ admits a complete, descending, $\mathbb{N}$-indexed filtration by ideals such that*

$$\mathrm{Fil}^i \cong TC^-_{2i}(A), \qquad TP_0(A)/\mathrm{Fil}^i \cong \pi_{2i-2} THH(A)_{hS^1}, \qquad \mathrm{gr}^i \cong THH_{2i}(A) \cong \pi_i(\mathbb{L}^i_{A/\mathbb{F}_p})$$

for all $i \geq 0$.

Proof. The proof is entirely analogous to that of Lemma 3.8; firstly, as already argued there, each wedge powers $\mathbb{L}^i_{A/\mathbb{F}_p} = \Gamma^i_A(N)[i]$ is supported in degree i. The HKR filtration of Corollary 5.8 therefore proves that $THH_*(A)$ is supported in even degree and gives the desired finite increasing filtration (we leave it to the reader to carefully check that the degrees are ok).

The rest of the structure can be formally read off Corollary 5.5 and the norm sequence $THH(A)_{hS^1} \to TC^-(A) \to TP(A)$, in a very similar way to the arguments of Lemma 3.8. □

Continuing to following the same line of argument we present in §3.2, we would now like to identify explicitly the filtered ring $TP_0(A)$. In the case of classical periodic cyclic homology we could do this in two ways, either in terms of a divided power envelope or in terms of derived de Rham cohomology. Now we lift these two constructions from characteristic p to mixed characteristic.

5.2.1 First construction: derived de Rham(–Witt) and crystalline cohomology

Let R be a smooth algebra over a perfect field k of characteristic p, and let $\widetilde{R}$ be a p-adically complete, formally smooth $W(k)$-algebra lifting R (i.e., $\widetilde{R}/p = R$). Let $\Omega^\bullet_{\widetilde{R}/W(k)}$ be the p-adically complete de Rham complex (we should really add a hat to indicate the p-adic completion, but we prefer to keep notation light and will never consider the non-complete version).

Assuming that $\widetilde{R}$ is equipped with a lift of the absolute Frobenius (i.e., there exists a ring homomorphism $\widetilde{\varphi} : \widetilde{R} \to \widetilde{R}$ compatible with the usual Frobenius on $W(k)$ and such that $\widetilde{\varphi}(f) \equiv f^p \bmod p\widetilde{R}$ for all $f \in \widetilde{R}$), then we define a filtration on $\Omega^\bullet_{\widetilde{R}/W(k)}$ by setting, for $i \geq 0$,

$$p^{\max(i-\bullet,0)}\Omega^\bullet_{\widetilde{R}/W(k)} := \qquad p^i\widetilde{R} \xrightarrow{d} p^{i-1}\Omega^1_{\widetilde{R}/W(k)} \xrightarrow{d} \cdots \xrightarrow{d} p\Omega^{i-1}_{\widetilde{R}/W(k)} \xrightarrow{d} \Omega^i_{\widetilde{R}/W(k)} \xrightarrow{d} \Omega^{i+1}_{\widetilde{R}/W(k)} \xrightarrow{d} \cdots$$

(although the definition of this filtration does not depend on $\widetilde{\varphi}$, it is only reasonable to define it when $\widetilde{\varphi}$ exists; this is partly because of the next lemma).

Lemma 5.11. *The graded pieces of the above filtration on $\Omega^\bullet_{\widetilde{R}/W(k)}$ are given (up to quasi-isomorphism) by*

$$\widetilde{\varphi}/p^i : \mathrm{gr}^i_{\mathcal{N}} R\Gamma_{\text{crys}}(R/W(k)) \xrightarrow{\sim} \tau^{\leq i}\Omega_{R/k}$$

(where $\tau^{\leq i}$ denotes canonical – not naive – truncation).

Proof. Note that $\widetilde{\varphi}$ induces an endomorphism of $\Omega^n_{\widetilde{R}/W(k)}$ which is divisible by p^n, so there is indeed an induced map as indicated. The graded pieces of the filtration are given by $R \xrightarrow{0} \Omega^1_{R/k} \xrightarrow{0} \cdots \xrightarrow{0} \Omega^i_{R/k}$ and the induced map on cohomology looks like $\Omega^n_{R/k} \to H^n_{\text{dR}}(R/k)$ (for $n \leq i$); one checks that it is precisely the Cartier isomorphism. □

As an object of the derived category $D(W(k))$, the complex $\Omega^\bullet_{\widetilde{R}/W(k)}$ depends only on R. Indeed, the theory of crystalline cohomology implies that $\Omega^\bullet_{\widetilde{R}/W(k)}$ is equivalent to the crystalline cohomology $R\Gamma_{\mathrm{crys}}(R/W(k))$, or equivalently to the de Rham–Witt complex $W\Omega^\bullet_{R/k}$. This independence remains true for the filtration we have just defined:

Lemma 5.12. *The above filtration on $\Omega^\bullet_{\widetilde{R}/W(k)}$ also depends only on R. (More precisely $\Omega^\bullet_{\widetilde{R}/W(k)}$, as an object of the filtered derived category over $W(k)$, depends naturally on R.)*

Proof. There are two ways to check this, either via crystalline cohomology or via de Rham–Witt theory. For the crystalline approach one uses a result of Berthelot–Ogus, stating that the Frobenius φ induces an equivalence $\varphi : R\Gamma_{\mathrm{crys}}(R/W(k)) \xrightarrow{\sim} L\eta_p R\Gamma_{\mathrm{crys}}(R/W(k))$, where $L\eta_p$ refers to the décalage functor of [8]. There is a natural filtration on the right side arising from the décalage functor (see ...), which therefore induces a filtration on the left side. Under the equivalence $\Omega^\bullet_{\widetilde{R}/W(k)} \simeq \Gamma_{\mathrm{crys}}(R/W(k))$, this is precisely the above filtration on $\Omega^\bullet_{\widetilde{R}/W(k)}$.

For the de Rham–Witt approach, recall that $\Omega^\bullet_{\widetilde{R}/W(k)}$ identifies up to equivalence with $W\Omega^\bullet_{R/k}$; then the above filtration can be shown to identify with the so-called *Nygaard filtration* on $W\Omega^\bullet_{R/k}$ given by

$$\mathcal{N}^{\geq i}W\Omega^\bullet_{R/k} := \quad p^{i-1}VW(R) \xrightarrow{d} p^{i-2}VW\Omega^1_R \xrightarrow{d} \cdots \xrightarrow{d} pVW\Omega^{i-2}_R \xrightarrow{d} VW\Omega^{i-1}_R \xrightarrow{d} W\Omega^i_R \xrightarrow{d} W\Omega^{i+1}_R \xrightarrow{d} \cdots .$$

□

In conclusion, we have naturally associated to each smooth k-algebra R a filtered complex $R\Gamma_{\mathrm{crys}}(R/W(k))$ (even an $\mathbb{E}_\infty$-$W(k)$-algebra). If we left Kan extend and complete with respect to the resulting filtration then the result, in the case of a qrsp algebra A, is a complete filtered ring $\widehat{\mathbb{L}W\Omega}_{A/\mathbb{F}_p}$ such that $\widehat{\mathbb{L}W\Omega}_{A/\mathbb{F}_p}/p = \widehat{\mathbb{L}\Omega}_{A/\mathbb{F}_p}$. More details to be added.

5.2.2 Second construction: a divided power envelope

Let A be qrsp. We denote by $\mathbb{A}^\circ_{\mathrm{crys}}(A)$ the divided power envelope of the composition $W(A^\flat) \twoheadrightarrow A^\flat \twoheadrightarrow A$, and by $\mathbb{A}_{\mathrm{crys}}(A)$ its p-adic completion. For example, $\mathbb{A}_{\mathrm{crys}}(\mathcal{O}_{\overline{\mathbb{Q}}_p}/p)$ is the period ring $\mathbb{A}_{\mathrm{crys}}$ of p-adic Hodge theory.

The usual Witt vector Frobenius φ induces a Frobenius endomorphism φ of $\mathbb{A}_{\mathrm{crys}}(A)$, using which we define its *Nygaard filtration* as follows:

$$\mathcal{N}^{\geq i}\mathbb{A}_{\mathrm{crys}}(A) := \{f \in \mathbb{A}_{\mathrm{crys}}(A) : \varphi(f) \in p^i\mathbb{A}_{\mathrm{crys}}(A)\}.$$

Finally, write $\widehat{\mathbb{A}}_{\mathrm{crys}}(A) := \varprojlim_{i\to\infty} \mathbb{A}_{\mathrm{crys}}(A)/\mathcal{N}^{\geq i}\mathbb{A}_{\mathrm{crys}}(A)$ for the completion with respect to the Nygaard filtration, and $\widehat{\mathbb{A}}_{\mathrm{crys}}(A) := \mathrm{Ker}(\widehat{\mathbb{A}}_{\mathrm{crys}}(A) \twoheadrightarrow \mathbb{A}_{\mathrm{crys}}(A)/\mathcal{N}^{\geq i}\mathbb{A}_{\mathrm{crys}}(A))$ for the induced filtration.

The advantage of $\widehat{\mathbb{A}}_{\mathrm{crys}}(A)$ is that it is reasonably explicit.

5.2.3 Back to *TP*

The following is the analogue for the topological theories of Proposition 3.13:

Proposition 5.13. *Let A be a qrsp algebra. Then there are natural isomorphisms of filtered rings*

$$TP_0(A) \cong \widehat{\mathbb{A}}_{\mathrm{crys}}(A) \cong \widehat{\mathbb{L}W\Omega}_{A/\mathbb{F}_p}$$

which modulo p recover the filtered isomorphisms of Corollary 3.13.

Proof. The proof of this is relatively self-contained so we refer to [8, Thm. 8.15]. □

The proposition implies Theorem 5.9, similarly to the second proof of Theorem 3.1.

6 (Topological) Hochschild and cyclic homologies of $\mathbb{Z}_p$-algebras – a glimpse

It is beyond the scope of this course to discuss in detail the topological Hochschild and cyclic homologies of p-adic algebras, but we nevertheless indicate some of the main results without proof. The goal is essentially to overview the analogous results of §3 and §5 in the case of p-adic algebras.

Remark 6.1 (p-completeness convention)**.** When the input ring A is p-adically complete (but not killed by a power of p), then $HH(A/\mathbb{Z}_p)$, $THH(A)$, etc., as we have already defined them, will contain large amounts of undesirable, junk data. (For example, if S is any perfect $\mathbb{F}_p$-algebra, then we would like $HH(W(S)/\mathbb{Z}_p)$ to be supported in degree 0, since $\mathbb{L}_{W(S)/\mathbb{F}_p}$ should vanish; but these statements are only true after p-completing the complexes.) In this section we therefore adopt the following convention: if A is a p-adically complete ring, then we write $HH(A/\mathbb{Z}_p)$, $THH(A)$, etc. to mean $HH(A/\mathbb{Z}_p)\widehat{}_p$, $THH(A)\widehat{}_p$, where the hat denotes derived p-adic completion.

6.1 The case of perfectoid rings and consequences

We begin with the analogue of Bökstedt's calculation (Theorem 5.1) and the consequences for topological periodic and topological negative cyclic homologies (analogue of Proposition 5.2). Although we stated these only for $\mathbb{F}_p$ we remarked that they held for arbitrary perfect $\mathbb{F}_p$-algebras; we therefore begin by recalling the p-adic analogue of a perfect $\mathbb{F}_p$-algebra:

Definition 6.2. A ring A is *perfectoid* if it satisfies the following conditions:

(i) it is p-adically complete and separated;

(ii) A/pA is semiperfect;

(iii) there exist $\pi \in A$ and $u \in A^\times$ such that $\pi^p = pu$;

(iv) the kernel of Fontaine's map $\theta_A : A_{\mathrm{inf}}(A) \to A$ is principal.

It is not our intention here to review the theory of perfectoid rings, in particular do not discuss Fontaine's ring $A_{\mathrm{inf}}(A) := W(A^\flat)$ or the associated map $\theta_A : A_{\mathrm{inf}}(A) \to A$. We refer to [7] or to [Lecture IV of Bhatt's notes on prisms, available on his webpage].

Example 6.3. (i) An $\mathbb{F}_p$-algebra is perfect if and only if it is perfectoid.

(ii) Let C be a perfectoid field (e.g., an algebraically closed field which is complete under a rank one valuation) of mixed characteristic containing all p-power roots of unity, and let $\mathcal{O}_C \subseteq C$ denote its ring of integers. Then $\mathcal{O}_C$ is a perfectoid ring, which often serves as the base ring in the theory.

Theorem 6.4. *Let A be a perfectoid ring. Then the homotopy groups of $THH(A)$ are given by*

$$THH_n(A) \cong \begin{cases} A & n \text{ even } \geq 0 \\ 0 & n \text{ otherwise} \end{cases}$$

Regarding multiplicative structure, $THH_*(A) = A[u]$ (polynomial on a single variable u), with u any generator of the invertible A-module $THH_n(A) = HH_2(A/\mathbb{Z}_p) = \operatorname{Ker}\theta_A/(\operatorname{Ker}\theta_A)^2$.

Sketch of proof. This is the analogue of Bökstedt's theorem. It is proved by performing various base changes to capture all the data either from characteristic p (i.e., Bökstedt) or rationally (i.e., usual Hochschild homology). See [8, Thm. 6.1]. □

Let $\xi \in A_{\mathrm{inf}}(A)$ be a generator of $\operatorname{Ker}\theta_A$.

Theorem 6.5. *The homotopy groups of $TP(A)$ are given by*

$$TP_n(A) \cong \begin{cases} A_{\mathrm{inf}}(A) & n \text{ even } \geq 0 \\ 0 & n \text{ otherwise} \end{cases}$$

Regarding multiplicative structure, $TP_*(A) = A_{\mathrm{inf}}[\sigma^{\pm 1}]$, where σ is any generator of the invertible $A_{\mathrm{inf}}(A)$-module $TP_2(A)$. The map of graded rings $TC^-_*(A) \to TP_*(A)$ is injective, with image

$$\mathrm{Im}(TC^-_{2n}(A) \to TP_{2n}(A) = \begin{cases} \xi^n TP_{2n}(A) & n \geq 0 \\ TP_{2n}(A) & n \leq 0 \end{cases},$$

i.e., $TC^-_*(A) = A_{\mathrm{inf}}(A)[\xi\sigma, \sigma^{-1}]$, and the canonical morphism $TC^-(A) \to THH(A)$ sends $\xi\sigma$ to u (equivalently, without making any choices, $TC^-_*(A)/\xi \xrightarrow{\simeq} THH_*(A)$ for $* \geq 0$).

Proof. This is proved similarly to Proposition 5.2, noting that the universal property of Fontaine's ring $A_{\mathrm{inf}}(A)$ provides a map $A_{\mathrm{inf}}(A) \to TP_0(A)$ which serves as a replacement for the map $\mathbb{Z}_p \to TP_0(\mathbb{F}_p)$ which we used in the earlier proof. □

The previous theorems lead to analogues of results 5.5–5.8; see [8, §6.3].

6.2 Quasiregular semiperfectoids

The mixed characterisation of a quasiregular semiperfect $\mathbb{F}_p$-algebra is as follows:

Definition 6.6. A ring A is called *quasiregular semiperfectoid* if it satisfies the following conditions:

(i) it is p-adically complete and A/pA is semiperfect;

(ii) it has bounded p^∞-torsion;

(iii) there exists a perfectoid ring S and a map $S \to A$ such that the p-completion of $\mathbb{L}_{A/S}$ is supported in homological degree 1 where it is given by a module $N := H_1(\widehat{\mathbb{L}}_{A/S})$ such that N/p^r is a flat A/p^r-module for all $r \geq 0$.

When A is qrsp in this sense, we remark that the condition on the cotangent complex is then satisfied for *any* map from a perfectoid ring to A (which may in fact be chosen to be surjective).

Example 6.7. (i) Any quasiregular semiperfect $\mathbb{F}_p$-algebra is of course quasiregular semiperfectoid. Nevertheless, we obtain some new presentations: for example, the quasiregular semiperfect $\mathbb{F}_p$-algebra $\mathcal{O}_C^\flat/p^\flat$ is the same as $\mathcal{O}_C/p$.

(ii) The prototypical example of a quasiregular semiperfectoid not of characteristic p is $\mathcal{O}_C\langle T^{\pm 1/p^\infty}\rangle/(T-1)$.

Arguing as in Lemmas 3.8 and 5.10, one again sees that TP_0 of a quasiregular semiperfectoid ring admits a filtration coming from negative cyclic homology:

Lemma 6.8. *Let A be a quasiregular semiperfectoid ring, fix an integral perfectoid ring S and map $S \to A$, and set $N := \pi_1(\widehat{\mathbb{L}}_{A/S})$. Then $THH_*(A)$, $TC^-_*(A)$, $TP_*(A)$, and $\pi_* THH(A)_{hS^1}$ are all supported in even degrees. Moreover, $THH_{2i}(A)$ admits a finite increasing filtration with graded pieces given by the p-completions of $A, N, \Gamma^2_A(N), \ldots, \Gamma^i_A(N)$, and $TP_0(A/\mathbb{F}_p)$ admits a complete, descending, $\mathbb{N}$-indexed filtration by ideals such that*

$$\mathrm{Fil}^i \cong TC^-_{2i}(A), \qquad TP_0(A)/\mathrm{Fil}^i \cong \pi_{2i-2} THH(A)_{hS^1}, \qquad \mathrm{gr}^i \cong THH_{2i}(A) \cong \pi_i(\widehat{\mathbb{L}}^i_{A/S})$$

for all $i \geq 0$.

In the case of classical periodic cyclic homology we saw two descriptions of HP_0 of qusasiregular semiperfect rings: either in terms of derived de Rham cohomology or an explicit divided power envelope. We had two similar descriptions of TP_0, either in terms of derived crystalline cohomology or again via a divided power envelope construction. We would like a similar description of TP_0 of any quasiregular semiperfectoid:

Theorem 6.9. *Let A be a qrsp $\mathcal{O}_C$-algebra. Then there are natural isomorphisms of filtered rings*

$$TP_0(A) \cong \widehat{\mathbf{\Delta}}_{A/A_{\mathrm{inf}}} \cong \widehat{A\Omega}_A$$

(The middle and right filtered rings are defined below.)

Proof. This is proved by a rather intricate set of arguments bouncing back and forth between qrsp and formally smooth algebras, and also requires Bhatt–Scholze's forthcoming work on prismatic cohomology. We aren't going to say anything about it. □

6.2.1 Derived $A\Omega$ cohomology

Let R be the p-adic completion of a smooth $\mathcal{O}_C$-algebra; assume that R is *small* in the sense that there exists a formally étale morphism $\mathcal{O}\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle \to R$ for some $d \geq 0$ (but do not fix any such morphism). In [7] the authors introduced an $\mathbb{E}_\infty$-A_{inf}-algebra $A\Omega_R \in D(A_{\mathrm{inf}})$ which may be described in any of the following ways:

(i) (Faltings-style Galois cohomology) Assuming that $\operatorname{Spf} R$ is connected, let $\overline{R[\frac{1}{p}]}$ be a "universal cover" of R, i.e., the filtered union of all finite étale extensions of $R[\frac{1}{p}]$ inside a fixed algebraic closure of $\operatorname{Frac} R$. In particular, the geometric Galois group $\Delta := \pi_1^{\text{ét}}(\operatorname{Spec} R)$ acts naturally on $\overline{R[\frac{1}{p}]}$. Next let $\overline{R}$ be the p-adic completion of the integral closure of R inside $\overline{R[\frac{1}{p}]}$ and observe that $\overline{R}$ inherits a continuous action by Δ. It turns out (here the smallness assumption is used) that $\overline{R}$ is integral perfectoid, so we may form its associated Fontaine ring $A_{\mathrm{inf}}(\overline{R})$, which is again equipped with a continuous action by Δ, and the resulting Galois cohomology $R\Gamma_{\mathrm{cont}}(\Delta, A_{\mathrm{inf}}(\overline{R}))$. The first definition of $A\Omega_R$ is

$$A\Omega_R := L\eta_\mu R\Gamma_{\mathrm{cont}}(\Delta, A_{\mathrm{inf}}(\overline{R})),$$

where $L\eta_\mu$ is the décalge functor from [7].

(ii) (q-de Rham complex) The previous definition of $A\Omega_R$ is a priori uncomputable since the geometric Galois group Δ is inaccessible. However, the almost purity theorem and a curious property of the décalage functor overcome this difficulty as follows. Fix a formally étale morphism $\mathcal{O}\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle \to R$ and define R_∞ to be the p-adic completion of $R \otimes_{\mathcal{O}\langle \underline{T}^{\pm 1}\rangle} \mathcal{O}\langle \underline{T}^{\pm 1/p^\infty}\rangle$. Then $R_\infty \subseteq \overline{R}$, and the Δ-action on $\overline{R}$ restrictions to R_∞: in fact, on R_∞ the action factors through $\Delta \to \Gamma := \mathbb{Z}_p(1)^d$, where $\mathbb{Z}_p(1)^d$ acts on R_∞ by rescaling the p-power roots of the variables by suitable p-power roots of unity. A consequence of the almost purity theorem for perfectoid rings is that the induced morphism $R\Gamma_{\mathrm{cont}}(\Gamma, A_{\mathrm{inf}}(R_\infty)) \to R\Gamma_{\mathrm{cont}}(\Delta, A_{\mathrm{inf}}(\overline{R}))$ is in fact an almost equivalence; moreover, the almost difference remarkably disappears after applying the décalage functor, i.e.,

$$L\eta_\mu R\Gamma_{\mathrm{cont}}(\Gamma, A_{\mathrm{inf}}(R_\infty)) \xrightarrow{\sim} L\eta_\mu R\Gamma_{\mathrm{cont}}(\Delta, A_{\mathrm{inf}}(\overline{R})) = A\Omega_R$$

But the left side is defined in terms of an explicit action of a pro-cyclic group and can be computed in terms of Koszul complexes; it turns out to be a "q-de Rham complex", for which we refer to [7] for further details.

(iii) (pro-étale cohomology) The problem with both of the previous definitions is that they do not enjoy strong functoriality properties: the q-de Rham description is completely non-functorial as it depends on the chosen coordinates, while the Galois cohomology descriptions depends at least on the choice of an algebraic closure of $\operatorname{Frac} R$. The solution to this problem adopted in [7] is to replace the Galois cohomology $R\Gamma_{\mathrm{cont}}(\Delta, A_{\mathrm{inf}}(\overline{R}))$ by the site-theoretic pro-étale cohomology $R\Gamma_{\text{proét}}(\operatorname{Spa}(R[\frac{1}{p}], R), \mathbb{A}_{\mathrm{inf}})$ of a period sheaf $\mathbb{A}_{\mathrm{inf}}$ on the adic spectrum $\operatorname{Spa}(R[\frac{1}{p}], R)$. Similarly to (ii), these two cohomologies are almost equivalent and become equivalent after décalge, i.e., $A\Omega_R \simeq R\Gamma_{\text{proét}}(\operatorname{Spa}(R[\frac{1}{p}], R), \mathbb{A}_{\mathrm{inf}})$. This yields a definition of $A\Omega_R$ which is functorial and can be easily extended to arbitrary smooth, formal $\mathcal{O}$-schemes.

Definition 6.10. For an arbitrary p-complete $\mathcal{O}_C$-algebra A, let $A\Omega_A$ denote the (p, ξ)-adic completion of the left Kan extension of $A\Omega_-$ from p-adic completions of smooth $\mathcal{O}$-algebras.

When R is a formally smooth $\mathcal{O}_C$-algebra, then $A\Omega_R$ is moreover equipped with a complete filtration whose graded pieces are given by truncations of the de Rham complex of R. By left Kan extending this filtration and mimicking the arguments of §5.2.1, one checks that if A is a qrsp $\mathcal{O}_C$-algebra then $A\Omega_A$ is a supported in degree 0, where it is given by a complete filtered ring whose graded pieces are related to derived de Rham cohomology; in that case we write $\widehat{A\Omega}_A$ for the completion of A with respect to this filtration.

6.2.2 Prisms and δ-rings (following Bhatt–Scholze)

Let A be a qrsp $\mathcal{O}_C$-algebra. The new theory of prismatic cohomology, in which (very roughly) divided powers are replaced by p-derivations, associates to A its prismatic cohomology $\mathbf{\Delta}_{A/A_{\mathrm{inf}}}$. Since A is qrsp this cohomology is supported in degree zero, where it is given by a sort of period ring constructed in terms of p-derivations. We now explain this construction.

Fix an integral perfectoid $\mathcal{O}_C$-algebra S equipped with a surjection $S \to A$. Let $A_{\mathrm{inf}}(S)[\frac{1}{\xi}]^{\widehat{}}$ be the p-adic completion of $A_{\mathrm{inf}}(S)[\frac{1}{\xi}]$ and observe that the usual Witt vector Frobenius φ on $A_{\mathrm{inf}}(S) = W(S^\flat)$ extends uniquely to a Frobenius lift on this completion (indeed, one just needs to check that $\varphi(\xi)$ is a unit mod p in $A_{\mathrm{inf}}(S)[\frac{1}{\xi}]$, but this is clear since $\varphi(\xi) \equiv \xi^p \bmod p$). We denote by $\delta : A_{\mathrm{inf}}(S)[\frac{1}{\xi}]^{\widehat{}} \to A_{\mathrm{inf}}(S)[\frac{1}{\xi}]^{\widehat{}}$ the associated p-derivation, i.e., $\delta(f) := \frac{1}{p}(\varphi(f) - f^p)$.

Let $\mathbf{\Delta}_{A/A_{\mathrm{inf}}}$ be the (p,ξ)-adic completion of the $A_{\mathrm{inf}}(S)$-subalgebra of $A_{\mathrm{inf}}(S)[\frac{1}{\xi}]^{\widehat{}}$ generated by terms of the form $\delta^n(\frac{f}{\xi})$ for all $f \in \mathrm{Ker}(A_{\mathrm{inf}}(S) \xrightarrow{\theta} S \to A)$ and $n \geq 0$. We equip $\mathbf{\Delta}_{A/A_{\mathrm{inf}}}$ with a Nygaard-style filtration $\mathcal{N}^{\geq i}\mathbf{\Delta}_{A/A_{\mathrm{inf}}} := \{g \in \mathbf{\Delta}_{A/A_{\mathrm{inf}}} : \varphi(g) \in \xi^i \mathbf{\Delta}_{A/S}\}$ for $i \geq 0$, and let $\widehat{\mathbf{\Delta}}_{A/A_{\mathrm{inf}}}$ be its completion for this filtration.

6.3 Smooth algebras

It remains only to say something about the topological cyclic homologies of a smooth $\mathcal{O}_C$-algebra; for these we have the following analogue of Theorems 3.1 and 5.9:

Theorem 6.11. *Let R be a p-adic formally smooth $\mathcal{O}_C$-algebra. Then $TP(R)$ and $TC^-(R)$ admit natural, complete, descending $\mathbb{Z}$-indexed filtrations whose n^{th} graded pieces are given respectively by*

$$\mathcal{N}^{\geq n} A\Omega_R, [2n] \qquad A\Omega_R[2n],$$

where we use the $A\Omega_R$ complex and its filtration from §6.2.1.

Sketch of proof. Similar to the proof of the aforementioned theorems: in a p-adic version of the quasisyntomic topology, R may be covered by quasiregular semiperfectoid $\mathcal{O}_C$-algebras. On the TP and TC^- of these latter algebras we impose the two-speed Postnikov filtration, whose graded pieces are calculated by Theorem 6.9. Descending back down to $TP(R)$ and $TC^-(R)$ then implies the theorem. □

Projects

A Motivic cohomology with modulus in characteristic p

(Themes: derived de Rham–Witt complexes)

A.1 Motivation: classical motivic cohomology

To any finite type scheme X over a field k, Bloch associated his *cycle complex*

$$z^n(X) = [z^n(X,0) \leftarrow z^n(X,1) \leftarrow \cdots]$$

This is constructed more naturally as a simplicial abelian group, in which $z^n(X,i)$ consists of formal sums of irreducible, codimension-n subvarieties of $X \times_k \Delta^i_k$ (where Δ^i_k is the spectrum of $k[t_1,\ldots,t_i]/(t_1 + \cdots + t_i - 1)$) which intersect all faces correctly. The homology of this chain complex is Bloch's higher Chow groups $CH^n(X,i) := H_i(z^n(X))$; up to reindexing these are the same as Voevodsky's motivic cohomology $H^i_{\mathcal{M}}(X,\mathbb{Z}(q)) := CH^q(X,2q-p) = H^p(z^q(X)[-2q])$.

We recall also that each presheaf $X \supseteq U \mapsto z^n(U,i)$ turns out to be a Zariski sheaf, and that this complex of Zariski sheaves $U \mapsto z^n(U)$ even satisfies Zariski descent; therefore the above definitions can be rephrased in terms of the hypercohomology of this complex of sheaves. In particular, introducing the complex of Zariski sheaves $\mathbb{Z}(i)_X : U \mapsto z^n(U)[-2n]$, we have $H^i_{\mathcal{M}}(X,\mathbb{Z}(q)) = \mathbb{H}^i_{\mathrm{Zar}}(Z,\mathbb{Z}(q)_X)$. (In an attempt to avoid mistakes when shifting and passing between (c)ohomological indexing, it might be helpful to note the following: $\mathbb{Z}(q)_X$ cohomologically vanishes in degrees $> 2q$, since each $z^q(U)$ is homologically supported in degrees ≥ 0; also, $\mathbb{Z}(q)_X$ should be cohomologically supported in degrees ≥ 0, but this is precisely the Beilinson–Soulé vanishing conjecture.)

Here is the statement we really care about:

Theorem A.1 (Bloch–Lichtenbaum, Levine, Voevodsky)**.** *Assuming X is smooth over k (otherwise one should replace $K(X)$ by $K'(X)$ in what follows), then the K-theory spectrum $K(X)$ admits a complete, descending $\mathbb{N}$-indexed filtration whose i^{th} graded pieces is*

$$R\Gamma_{\mathcal{M}}(X,\mathbb{Z}(i))[2i] := \mathbb{H}_{\mathrm{Zar}}(Z,\mathbb{Z}(q)_X)[2q] = z^q(X)$$

Therefore there is an associated "Atiyah–Hirzebruch" spectral sequence converging to the K-groups $K_*(X)$.

A.2 Motivic cohomology beyond the smooth case

An open problem is to find an analogous motivic filtration on the K-theory of arbitrary schemes (and to relate it to existing objects, such as algebraic cycles). The particular case which has attracted most attention in recent years had aimed at constructing such a motivic filtration on the relative K-group $K(X,D)$, where X is smooth and $D \hookrightarrow X$ is a simple normal crossing divisor (which is not necessarily reduced), such that the graded pieces are related to so-called "Chow groups with modulus".

Let X be a smooth variety over a perfect field k of characteristic p, and let $D = \bigcup_{i=1}^c D_i \hookrightarrow X$ be a reduced normal crossing divisor. For a tuple of positive integers $\underline{r} = (r_1,\ldots,r_c)$, we write $\underline{r}D := \cup_{i=1}^c r_i D_i$, where $r_i D_i$ means the r_i-infinitesimal thickening of D. In other words, if D is defined locally by the equation $t_1 \cdots t_c = 0$, then $\underline{r}D$ is defined by $t_1^{r_1} \cdots t_c^{r_c} = 0$.

As explained above, the relative K-theory $K(X,\underline{r}D)$ should be equipped with a motivic filtration whose graded pieces are related to higher Chow groups with modulus, which themselves are known to be close to de Rham–Witt groups and similar objects.

The K-theory $K(X,\underline{r}D)$ is built from $K(X,D)$ and $K(\underline{r}D,D)$, which one may refer to respectively as the *tame* and *wild* parts. Since $D \hookrightarrow \underline{r}D$ is an infinitesimal thickening in characteristic p, the trace map induces an equivalence $K(\underline{r}D,D) \simeq TC(\underline{r}D,D)$; the latter is built from $TC(\underline{r}D)$ and $TC(D)$, which carry filtrations from [8] with i^{th} graded piece given by the Zariski hypercohomology of

$$\mathrm{hofib}\left(\mathcal{N}^{\geq i}\mathbb{L}W\Omega_{\underline{r}D/k} \xrightarrow{\frac{\varphi}{p^i}-1} \mathbb{L}W\Omega_{\underline{r}D/k}\right)$$

(resp. replacing $\underline{r}D$ by D), where $\mathbb{L}W\Omega_{-/k}$ is the derived de Rham–Witt complex equipped with its Nygaard filtration.

Question A.2. Can this be used to build the desired motivic filtration on $K(X, \underline{r}D)$? In other words:

(i) Firstly, can the above Frobenius fixed points be related to other objects appearing in the theory of motivic cohomology with modulus? How does the derived de Rham-Witt complex of a simple normal crossing divisor look? Is Bhatt–Lurie–Mathew's saturated de Rham–Witt complex better adapted?

(ii) Secondly, once we have the right filtration on the wild part, can it be glued to a filtration on the tame part to get the desired filtration on $K(X, \underline{r}D)$?

Question A.3. A closely related but more immediately accessible problem is simply to compute the THH, etc. (including K-theory) of rings B such as $k[t]/t^r$ and $k[t_1, \ldots, t_c]/(t_1^{r_1}, \ldots, t_c^{r_c})$ (warning: the latter ring is not the one defining the ncd $\underline{r}D$ above). Many such results are already available (see particularly Hesselholt–Madsen and Angeltveit–Gerhardt–Hill–Lindenstrauss respectively), but it would be nice to have a more natural presentation in terms of (derived) de Rham–Witt theory. The point is that the relative K-groups can be computed in terms of the topological cyclic homology $TC(B)$, which fits into a fibre sequence

$$TC(B) \longrightarrow TC^-(B) \xrightarrow{\varphi - 1} TP(B),$$

all of which carry filtrations similar to the discussion above.

More ambitiously, what about rings like $\mathbb{Z}/p^r$ or $\mathcal{O}/p^r$, where $\mathcal{O}$ is the ring of integers of a perfectoid field of characteristic zero? (This seems hard without the forthcoming theory of prismatic cohomology of Bhatt–Scholze, which will provide an interpretation of the objects to be calculated in terms of p-derivations.)

B Quasisyntomic coefficients

(Themes: quasisyntomic site, crystals)

Let R be a smooth algebra over a perfect field of characteristic p (more generally one could allow R to be any quasisyntomic $\mathbb{F}_p$-algebra, or replace $\operatorname{Spec} R$ by a non-affine variety).

On the quasisyntomic site we have a sheaf TP_0^{sh} which associates to each quasiregular semiperfect A the (filtered) ring

$$TP_0(A) \cong \widehat{\mathbb{A}}_{\mathrm{crys}}(A) \cong \widehat{\mathbb{L}W\Omega}_{A/\mathbb{F}_p}.$$

Question B.1. How do locally free sheaves of TP_0^{sh}-modules on the quasisyntomic site over R look? Are they related to crystals on the crystalline site of R? It might be helpful to first consider the mod p-situation $HP_0^{\mathrm{sh}} = TP_0^{\mathrm{sh}}/p = \widehat{\mathbb{L}\Omega}_{-/\mathbb{F}_p}$. It might also be better to study sheaves of filtered modules, or perhaps modules with a Frobenius.

How are such objects related to modules of cyclotomic spectra over $THH(R)$?

In theory the analogue of the previous question should already have been answered if R is a smooth algebra over a field k of characteristic 0: then one expects modules with S^1-action over $HH(R/k)$ to be related to R-modules equipped with a filtration and connection satisfying Griffiths transversality. But I do not know of a reference.

C Smoothness conditions and Cartier isomorphisms
(Theme: cotangent complex)

C.1 Smoothness conditions

There are three possible nilpotence condition which one can impose on an ideal I:

- *nilpotent* means that there exists $n \geq 1$ such that $I^n = 0$;
- *nil* means that there exists $n \geq 1$ such that $x^n = 0$ for all $x \in I$.
- *locally nilpotent* means that for each $x \in I$ there exists $n \geq 1$ such that $x^n = 0$.

Obviously nilpotent $\Rightarrow$ nil $\Rightarrow$ locally nilpotent.

A ring homomorphism $A \to B$ is called *formally smooth* if for each A-algebra C and nilpotent ideal $I \subseteq C$, any A-algebra homomorphism $B \to C/I$ lifts to an A-algebra homomorphism $B \to C$; when the lifting is moreover always unique, one says that $A \to B$ is *formally étale*. By replacing "nilpotent" in this definition by "nil" or "locally nilpotent", we may similarly define *n-formally smooth/étale* and *ln-formally smooth/étale*. (This terminology is totally non-standard; the concept of n-formally smooth appears elsewhere in the literature, notably in the theory of crystalline cohomology and especially in the thesis of Berthelot, where it is called "quasi-smooth".)

One can also ask whether B is a filtered colimit of smooth étale A-algebras, in which case we say it is *ind-smooth*; one defines *ind-étale similarly*. In summary we arrive at the following implications:

$$\text{ind-smooth} \Rightarrow \text{ln-formally smooth} \Rightarrow \text{n-formally smooth} \Rightarrow \text{formally smooth}$$

and similarly for étale.

There is in fact also a class which is between ind-étale and and ln-formally étale (and strictly larger than ind-étale), namely weakly étale. Recall here Bhatt–Scholze's notion of *weakly étale*, which means that both $A \to B$ and $\mu : B \otimes_A B \to B$ are flat. According to [9, Thm. 1.3], if $A \to B$ is weakly étale then there exists a faithfully flat map $B \to C$ such that $A \to B$ is ind-étale (i.e., weakly étale maps are locally ind-étale in the flat topology), whence flat descent implies that $A \to B$ is ln-formally étale. It seems difficult to construct an ln-formally étale map which is not weakly eétale, and so Nikolaus has raised the following question:

Question C.1. Is ln-formally étale actually equivalent to weakly étale? Consider first the case of an extension of fields $k \to k'$ (in which case weakly étale is equivalent to ind-étale, i.e., that k' is an algebraic separable extension of k).

We can also consider conditions coming from the cotangent complex: say that $A \to B$ is *L-smooth* if $\mathbb{L}_{B/A}$ is supported in degree 0 and $\Omega^1_{B/A}$ is a projective B-module; say it is *L-étale* if $\mathbb{L}_{B/A} \simeq 0$. It is known that

$$\text{L-smooth} \Rightarrow \text{formally śmooth}$$

whence (since formally étale is equivalent to formally smooth and vanishing of Ω^1) we get

$$\text{L-étale} \Rightarrow \text{formally étale}$$

but the converse is false by the next example:

Example C.2. See [Stacks project, Lem. 102.37.1] for an example of a formally étale morphism with non-vanishing cotangent complex.

Gabber has given an example of a non-reduced (hence not ind-smooth) $\mathbb{F}_p$-algebra B such that $\mathbb{L}_{B/\mathbb{F}_p} \simeq 0$, i.e., L-étale does not imply ind-smooth. You can find the construction of B on Bhatt's webpage. According to the remark between 1.9 and 1.10 of [12], Gabber claims that the construction can be modified (by replacing the perfections of the B_i which occur in Bhatt's write-up by the extensions obtained by adjoining all squares of elements) to produce a $\mathbb{Q}$-algebra with similar properties (see the next question).

Question C.3. (i) Carry out Gabber's suggestion to carefully construct a $\mathbb{Q}$-algebra B which is quasismooth (see below) but not ind-smooth.

(ii) Bhatt asks the following: Does there exist an L-étale $\mathbb{Q}$-algebra which is not ind-étale?

(iii) Remaining on the subject of special base rings, is every formally étale $\mathbb{F}_p$-algebra automatically L-étale?

(iv) Does L-smooth imply ln-formally smooth? Similarly for étale?

Recall from Remark 3.18 that we say $A \to B$ is *quasismooth* if $\mathbb{L}_{B/A}$ is supported in degree 0 and $\Omega^1_{B/A}$ is a flat B-module. So

$$\text{L-smooth} \Rightarrow \text{quasismooth}$$

It turns out that if A and B are Noetherian then $A \to B$ is quasismooth if and only if it is ind-smooth, and so all our smoothness conditions are then equivalent. This is a consequence of Néron–Popescu desingularisation and a hard theorem of M. André [1, Thm. 30, pg. 331]. André's theorem is actually more general (for example, he can say something even if A is not Noetherian, as long as $A \to B$ is flat), and it might be useful.

C.2 Incorporating the Cartier isomorphism

Let A be an $\mathbb{F}_p$-algebra. Then the *inverse Cartier map*

$$C^{-1} : \Omega^n_{A/\mathbb{F}_p} \to H^n(\Omega^\bullet_{A/\mathbb{F}_p})$$

is the linear map characterised by

$$C^{-1}(f dg_1 \wedge \cdots \wedge dg_n) = f^p g_1^{p-1} \cdots g_n^{p-1} dg_1 \wedge \cdots \wedge dg_n$$

for all $f, g_1, \ldots, g_n \in A$. In [26], we say that A is *Cartier smooth* if the following conditions are satisfied:

(Sm1) $\mathbb{F}_p \to A$ is quasismooth;

(Sm2) the inverse Cartier maps are isomorphisms for all $n \geq 0$.

The above criteria are of course satisfied if A is a smooth $\mathbb{F}_p$-algebra, or more generally if A is a regular Noetherian $\mathbb{F}_p$-algebra by Néron–Popescu desingularisation (an alternative proof avoiding Néron–Popescu may be found in the recent work of Bhatt–Lurie–Mathew [6, Thm. 9.5.1]), or simply an ind-smooth $\mathbb{F}_p$-algebra, or even a smooth algebra over a perfect $\mathbb{F}_p$-algebra. But They are also satisfied if A is a valuation ring of characteristic p: criterion (Sm1) is due to Gabber–Ramero [19, Thm. 6.5.8(ii) & Corol. 6.5.21], while the Cartier isomorphism (Sm2) is a result of Gabber obtained by refining his earlier work with Ramero, the proof of which may be found in the appendix of [27].

Note that condition (Sm2) is not a consequence of condition (Sm1), as explained in [6, Warning 9.6.3]: Bhatt–Gabber's Example C.2 is a semiperfect, non-perfect $\mathbb{F}_p$-algebra B such that $\mathbb{L}_{B/\mathbb{F}_p} \simeq 0$; then the inverse Cartier map in degree $n = 0$ identifies with $\varphi : A \to A$, which has non-zero kernel. Condition (Sm2) is designed to overcome this sort of pathological behaviour: for example, Cartier smooth + L-étale is equivalent to perfect, which implies formally étale (even n-formally étale).

Question C.4. (i) Assuming that A is quasismooth over $\mathbb{F}_p$, observe that condition (Sm2) is equivalent to the adjunction $\mathbb{L}\Omega^\bullet_{A/\mathbb{F}_p} \to \Omega^\bullet_{A/\mathbb{F}_p}$ being an equivalence (c.f., [6, 9.6.2]). Take care of the difference between $\mathbb{L}\Omega^\bullet$ and its Hodge completion $\widehat{\mathbb{L}\Omega^\bullet}$.

(ii) Is there a good notion (say, closed under composition) of what it means for a morphism of $\mathbb{F}_p$-algebras to be Cartier smooth? If so, can you prove that $\mathcal{O}_{C_p}/p \to \mathcal{O}/p$ is Cartier smooth? (For the notation see Project C.3. Maybe one can replace $\mathcal{O}$ by $\overline{\mathcal{O}}$ and then note that $\mathcal{O}_{C_p}/p \to \overline{\mathcal{O}}/p$ can be rewritten as $\mathcal{O}^\flat_{C_p}/p^\flat \to \overline{\mathcal{O}}^\flat/p^\flat$, to which one might be able to apply existing results in characteristic p?)

(iii) How does Cartier smooth compare to Bhatt–Lurie–Mathew's "universal Cartier isomorphism"?

C.3 Extending to mixed characteristic

The relation of Cartier smoothness to the material of §5 is as follows: for any $\mathbb{F}_p$-algebra A, we write $\mathbb{L}W\Omega_{A/\mathbb{F}_p} := \mathrm{Rlim}_r\, \mathbb{L}W_r\Omega_{A/\mathbb{F}_p}$, or equivalently the p-adic completion of the left Kan extension of $W\Omega_{-/\mathbb{F}_p}$ on A, and we let $\widehat{\mathbb{L}W\Omega}_{A/\mathbb{F}_p}$ be its Nygaard completion; these are connected by natural morphisms

$$\mathbb{L}W\Omega_{A/\mathbb{F}_p} \longrightarrow \widehat{\mathbb{L}W\Omega}_{A/\mathbb{F}_p} \longrightarrow W\Omega_{A/\mathbb{F}_p}$$

The left object is perhaps the most classical, the middle object is the right one to describe $TP(A)$, and the right one is the most computable. The key point is the following: if A is Cartier smooth over $\mathbb{F}_p$ (e.g., a valuation ring of characteristic p) then the morphisms are equivalences (this can be extracted from the arguments of [26]).

The goal of the rest of this project is to establish a similar result in mixed characteristic; some familiarity with [7] is required.

Let C_p be an algebraically closed, non-archimedean field of mixed characteristic, e.g., $\mathbb{C}_p$, and let $\mathcal{O}_{C_p}$ denote its ring of integers. Let C be a complete valued field extension of C_p (possibly highly transcendental and not necessarily algebraically closed) with corresponding ring of integers $\mathcal{O} \supseteq \mathcal{O}_{C_p}$ (if C has rank > 1, then we could even let $\mathcal{O}$ be a non-maximal valuation subring). The project concerns various cohomology theories attached to $\mathcal{O}$. Note that Gabber–Ramero [19, Thm. 6.5.8(ii) & Corol. 6.5.21] again tells us that $\mathcal{O}_C \to \mathcal{O}$ is quasismooth. (In contrast, although $\mathbb{Z}_p \to \mathcal{O}_C$ also has cotangent complex supported in degree 0, the $\mathcal{O}_C$-module $\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}$ is far from being flat: its p-torsion $\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}[p]$ is a rank one $\mathcal{O}_C/p$-module.)

Let $\overline{C}$ be the completed algebraic closure of C, and $\overline{\mathcal{O}} \subseteq \overline{C}$ its ring of integers; let Δ be the absolute Galois group of C. Let $\overline{\mathcal{O}}^\flat$ be the tilt of $\overline{\mathcal{O}}$. The first cohomology we are interested in is the Galois cohomology $R\Gamma(\Delta, \overline{\mathcal{O}})$ and its décalage $\widetilde{\Omega}_{\mathcal{O}} := L\eta_{\zeta_p - 1} R\Gamma(\Delta, \overline{\mathcal{O}})$ (Note: similarly to the smooth case studied in [7], it should be equivalent to define $\widetilde{\Omega}_{\mathcal{O}}$ as $L\eta_{\zeta_p-1} R\Gamma_{\text{proét}}(X, \widehat{\mathcal{O}}^+_X)$ where $X := \mathrm{Spa}(\mathcal{O}, C)$). The following Hodge–Tate comparison plays the role of the Cartier isomorphism in mixed characteristic:

Question C.5. Do there exist natural (but ignoring Breuil–Kisin–Fargues twists) isomorphisms

$$\Omega^i_{\mathcal{O}/\mathcal{O}_{C_p}} \cong H^i(\widetilde{\Omega}_{\mathcal{O}})$$

for $i \geq 0$? Compare with [7, §8] for the case of a smooth $\mathcal{O}_{C_p}$-algebra in place of $\mathcal{O}$. By mimicking the argument in the smooth case (and taking advantage of the aforementioned result of Gabber–Ramero), it should be possible to construct a comparison map in the direction $\to$. To check that it is an isomorphism, maybe reduce modulo p and relate it to the Cartier map for $\mathcal{O}/p$; I suspect one needs to know that $\mathcal{O}_{C_p}/p \to \mathcal{O}/p$ is Cartier smooth, which is why I asked it in Question C.4.

Update: After some discussions with Bhatt, we have decided that the case of an arbitrary complete valued field extension C is probably too ambitious; one should perhaps assume that the $\mathbb{F}_p$-algebra $\mathcal{O}/p$ admits a finite set of generators over its subring of pth-powers. Lifting these to generators to units $t_1, \ldots, t_d \in \mathcal{O}$ should provide a basis $dt_1, \ldots, dt_d$ of the module $\Omega^1_{\mathcal{O}/\mathcal{O}_C}$; meanwhile, the resulting map $\mathcal{O}\langle T_1^{\pm 1}, \ldots, T_d^{\pm 1}\rangle \to \mathcal{O}$ provides the analogue of the framing which usually appears in p-adic Hodge theory. Without this map it seems hard to attack the question. Note that such valuation rings $\mathcal{O}$ are reasonably abundant: take any smooth $\mathcal{O}_C$-algebra R and define $\mathcal{O}$ to be the p-adic completion of $R_{\mathfrak{m}R}$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_C$.

Next let $A_{\mathrm{inf}}(\overline{\mathcal{O}}) := W(\overline{\mathcal{O}}^\flat)$ be Fontaine's infinitesimal period ring construction of $\mathcal{O}$, and

$$A\Omega_{\mathcal{O}} := L\eta_\mu R\Gamma(\Delta, A_{\mathrm{inf}}(\overline{\mathcal{O}})).$$

Assuming that the previous question can be answered in the affirmative, then I am confident that it can be shown that $A\Omega_{\mathcal{O}}$ is exactly the prismatic cohomology of $\mathcal{O}$, that it is already Nygaard complete, and that the motivic filtration on $TP(\mathcal{O})$ has graded pieces given by $A\Omega_{\mathcal{O}}[2i]$, for $i \in \mathbb{Z}$.

Question C.6. Is there any remotely explicit description of the complexes $\widetilde{\Omega}_{\mathcal{O}}$ or $A\Omega_{\mathcal{O}}$, comparable to the q-de Rham complex in the case of a smooth $\mathcal{O}_{C_p}$-algebra in place of $\mathcal{O}$?

D Computing $THH(\mathbb{Z})$

Theme: spectral sequence calculation

Here are two useful results for analysing the topological Hochschild homology of a ring A.

(i) M. Bökstedt's [15, Thm. 4.1.0.1] calculation of the groups $THH_n(\mathbb{Z})$; they are $= \mathbb{Z}/m\mathbb{Z}$ if $n = 2m \geq 0$, and otherwise they vanish.

(ii) Pirashvili–Waldhausen's [33, Thm. 4.1] first quadrant spectral sequence

$$E^2_{ij} = HH_i(A, THH_j(\mathbb{Z}, A)) \implies THH_{i+j}(A).$$

Here $THH(\mathbb{Z}, A)$ refers to THH of the integers with coefficients in A, whose homotopy groups will be given by

$$THH_n(\mathbb{Z}, A) \cong \begin{cases} A/mA & n = 2m-1 \\ A[m] & n = 2m. \end{cases}$$

Similarly $HH(A, M)$ denotes HH of A with coefficients in the A-module M.

For example, if one combines these two results with Bökstedt's calculation of $THH(\mathbb{F}_p)$, then carefully chasing through the spectral sequence can be used to compute THH_* of the ring of integers $\mathcal{O}$ of a mixed characteristic perfectoid field. However, it is also possible to calculate $THH_*(\mathcal{O})$ without using (i), as was done in [8, Thm. 6.1].

Question D.1. By using the description of $THH_*(\mathcal{O})$, as we vary the residue characteristic of $\mathcal{O}$, can the P-W spectral sequence be run backwards to reprove Bökstedt's description of $THH_*(\mathbb{Z})$?

E A filtration on HH etc. depending on coordinates

(Theme: classical cyclic homology)

Let k be a base ring and $S := k[x_1, \ldots, x_n]$ a polynomial ring over k; let V be the free k-module on $x_1, \ldots, x_n$, so that $S = \mathrm{Sym}_k V$; more generally the following works for any free V-module k, or for S being étale over $\mathrm{Sym}_k V$.

A different proof of the HKR isomorphism of Theorem 2.8 uses an argument in the case of polynomial algebras which is a quasi-isomorphism of mixed complexes

$$\text{mixed complex with zero } B\text{-differential corresponding to } \Omega^\bullet_{S/k} \xrightarrow{\sim} (HH(S/k), B)$$

See [29, 3.2.2&3.2.3]. The maps defining this depend on writing $\Omega^n_{S/k} = S \otimes_k \bigwedge_k^n V$, which of course depends on the choice of co-ordinates (more precisely, only on choice of the free module V they span). But from this quasi-isomorphism, we see that the statement of Theorem 2.22 remains true for S, even though we did not assume that $k \supseteq \mathbb{Q}$; the problem is that the decompositions appears a priori to depend on the coordinates.

Question E.1. When k is a perfect field of characteristic p, are the resulting filtrations on $HC^-(S/k)$, $HP(S/k)$, $HC(S/k)$ the same as the filtrations we constructed in Theorem 3.1? Note that if this is true, then there will be two interesting consequences: (1) The filtration arising from the above construction does not depend on choice of coordinates; (2) The filtrations of Theorem 3.1 are split (though the splitting depends on choice of coordinates).

F An elementary description of $HP_0(A/\mathbb{F}_p)$

Let A be a quasiregular semiperfect $\mathbb{F}_p$-algebra. Logically the following question is redundant, but it is nevertheless tantalising:

Question F.1. Is it possible to give a direct proof of the isomorphisms of Proposition 3.13 without using topological periodic cyclic homology or Kan extending from the smooth case?

More precisely, as we explained in the second paragraph of the proof of Proposition 3.13, it would be nice to directly construct a comparison map $D_{A^\flat}(I) \to HP_0(A/\mathbb{F}_p)$. Since the target $HP_0(A/\mathbb{F}_p)$ is an algebra over $HP_0(A^\flat/\mathbb{F}_p) = A^\flat$ (to prove this equality, use the vanishing of $\mathbb{L}_{A^\flat/\mathbb{F}_p}$ to see that $HP(A^\flat/\mathbb{F}_p) \xrightarrow{\simeq} HP(A^\flat/A^\flat)$), it remains to show that $HP_0(A/\mathbb{F}_p)$ admits divided powers along its ideal $\mathrm{Fil}^1 = \mathrm{Ker}(HP_0(A/\mathbb{F}_p) \to A)$ (where we use the filtration of Lemma 3.8).

By using some formal tricks (as in [8, Props. 8.12 & 8.15]) one can try to reduce the existence of divided powers to the universal case $A = \mathbb{F}_p[t^{1/p^\infty}]/t-1$; this ring is the group algebra of $G := \mathbb{Q}_p/\mathbb{Z}_p$ over $\mathbb{F}_p$. Since G is p-divisible, a standard group homology calculation (e.g., Brown, Homology of groups §V.6) identifies the graded $\mathbb{F}_p$-algebra $H_*(G, \mathbb{F}_p)$ with the divided power algebra $\Gamma^*_{\mathbb{F}_p}(V)$ on the p-torsion $V := G[p] = H_2(G, \mathbb{F}_p)$. Proposition 2.24 therefore identifies $HP_0(G/\mathbb{F}_p)$ with $\widehat{\Gamma}^*_{\mathbb{F}_p}(V)$, the completion of $\Gamma^*_{\mathbb{F}_p}(V)$ with respect to the divided power filtration. In conclusion, the natural map $HP_0(G/\mathbb{F}_p) \to HP_0(\mathbb{F}_p[G]/\mathbb{F}_p) = HP_0(A/\mathbb{F}_p)$ shows that the target does contain some divided powers.

One of the problems is showing that this production of divided powers in $HP_0(A/\mathbb{F}_p)$ is independent of the chosen identification $A = \mathbb{F}_p[t^{1/p^\infty}]/t-1$. To do this it might be helpful to instead work with the surjection $\mathbb{F}_p[A^\times] \to A$, having reduced to the case that A is local and its units are p-divisible. Or try to lift to the p-torsion-free case by using Kaledin's Hochschild–Witt construction?

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$\mathbb{A}^1$-ENUMERATIVE GEOMETRY

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We will discuss how to use $\mathbb{A}^1$-homotopy theory to include arithmetic-geometric information into results from classical enumerative geometry, such as the result that over the complex numbers, there are 27 lines on a smooth cubic surface. Over non-algebraically closed fields, we replace arguments from classical algebraic topology, for example computations of Euler classes, with arguments from $\mathbb{A}^1$-homotopy theory. There has been a lot of recent progress in this direction due to mathematicians including Marc Hoyois, Jesse Kass, Marc Levine, Padmavathi Srinivasan, Matthias Wendt, and the lecturer. These lectures will introduce this recent direction of study. The project will first identify classical enumerative geometry results over $\mathbb{C}$ amenable to such enrichment. We will then choose one to attempt to enrich to a corresponding result giving information over non-algebraically closed fields using these methods.

ARIZONA WINTER SCHOOL: $\mathbb{A}^1$-ENUMERATIVE GEOMETRY

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ABSTRACT. We use $\mathbb{A}^1$-homotopy theory to include arithmetic-geometric information into results from classical enumerative geometry.

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1. Lecture 1: Introduction and an arithmetic count of the lines on a smooth cubic surface

Enumerative geometry counts algebro-geometric objects satisfying certain conditions. Classically, such counts are performed over an algebraically closed field to obtain "invariance of number" results. For example, the number of solutions in $\mathbf{C}$ of a degree n polynomial is always n, but this is no longer true over $\mathbf{Q}$. We use $\mathbb{A}^1$-homotopy theory to give enumerative results over general fields, including number fields, local fields, finite fields etc. As the number of objects is not itself invariant, we do not obtain formulas for these numbers; rather, information about the fields of definition will be recorded with a bilinear form, and the sum will satisfy an "invariance of bilinear form" principle.

We start with an example from joint work with Jesse Leo Kass [KW17]. A cubic surface X over $\mathbf{C}$ is the space of solutions to a cubic polynomial in three variables. Namely,

$$X = \{(x, y, z) \in \mathbf{C}^3 : f(x, y, z) = 0\}$$

where $f \in \mathbf{C}[x, y, z]$ is a degree 3 polynomial. X is *smooth* if the partials of X do not simultaneously vanish on X, which is equivalent to X being a manifold. We can substitute an arbitrary field k for $\mathbf{C}$, and view X as a variety or scheme over k. When $k = \mathbf{R}$, the real points of a cubic surface can be naturally embedded as a 2-dimensional surface in our 3-dimensional room, and there are pretty models and pictures, for example, the model photographed in Figure 1. For the invariance of number principle invoked above, it is better to compactify, viewing X as the subscheme of $\mathbf{P}^3$ determined by a homogeneous degree 3 polynomial f in $k[w, x, y, z]$.

It is a lovely theorem of Salmon and Cayley proven in 1849 [Cay49] that there are exactly 27 lines on a smooth cubic surface over $\mathbf{C}$.

Theorem 1. *[Salmon, Cayley] Let X be a smooth cubic surface over $\mathbf{C}$. Then X contains exactly 27 lines.*

See [Dol05] for interesting historical remarks.

Example 2. *The polynomial*

$$f(w, x, y, z) = w^3 + x^3 + y^3 + z^3$$

determines the Fermat Cubic Surface, whose lines can be described as follows. Let $[S, T]$ denote homogeneous coordinates on $\mathbf{P}^1 = \operatorname{Proj} \mathbf{C}[S, T]$. The line

$$\{[S, -S, T, -T] : [S, T] \in \mathbf{P}^1\}$$

is on the Fermat Cubic Surface by inspection. Moreover, for any λ, ω such that $\lambda^3 = \omega^3 = -1$, the line

$$\{[S, \lambda S, T, \omega T] : [S, T] \in \mathbf{P}^1\}$$

is also on the Fermat Cubic Surface by the same principle. Permuting the coordinates produces more lines, giving a total of $\frac{\binom{4}{2}}{2} \cdot 3 \cdot 3 = 27$ lines. It can be checked in an elementary manner that these are the only lines; see for instance `https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2014/alggeom-2014-c11.pdf`.

Here is a modern proof of Salmon and Cayley's theorem [EH16, Chapter 6, especially 6.2 and 6.4].

Proof. Let $\mathrm{Gr}(1,3)$ denote the Grassmannian parametrizing lines in $\mathbf{P}^3$ or equivalently 2-dimensional subspaces W of a 4-dimensional vector space. The *tautological bundle* (not to be confused with the canonical bundle) is the vector bundle $\mathcal{S} \to \mathrm{Gr}(1,3)$ whose fiber over a W is W itself. The third symmetric power of the dual of $\mathcal{S}$ is the vector bundle $\mathrm{Sym}^3 \mathcal{S}^* \to \mathrm{Gr}(1,3)$, whose fiber over the point corresponding to W of the Grassmannian is the space of cubic polynomials on W, i.e., $\mathrm{Sym}^3 W^*$. The cubic surface X is defined by a cubic polynomial f on the entire 4 dimensional vector space. f therefore determines an element of $\mathrm{Sym}^3 W^*$ for every subspace W by restriction. In other words, f determines a section σ_f of $\mathrm{Sym}^3 \mathcal{S}^* \to \mathrm{Gr}(1,3)$ by

$$\sigma_f(W) = f|_W.$$

A line $\mathbf{P}W$ is contained in X exactly when the polynomial f vanishes on W. In other words, the zeros of the section σ_f are in bijection with the lines in X.

Aside on the Euler Class 3. *There is a tool for counting the zeros of a section of a vector bundle. This tool is the* Euler Class *and we use it in the case when generically there are a finite number of zeros, or equivalently, we use it on a (relatively oriented) vector bundle V of rank r on a dimension r (real, respectively) complex manifold M. Choose such a $V \to M$, and then choose a section σ so that all the zeros of σ are* isolated *in the sense that there for every p in M with $\sigma(p) = 0$, there is an open neighborhood U of p such that the only zero of σ in U is p.*

The Euler Class can be defined using the degree *of a map $S^d \to S^d$ between oriented topological spheres of the same dimension. The degree only depends on the homotopy class of the map, namely we have a map*

$$\deg : [S^d, S^d] \to \mathbf{Z}.$$

Now for the Euler class of V, computed using σ. Let p be a point of M such that $\sigma(p) = 0$. By assumption, p is an isolated zero *of σ, so we may choose local coordinates around p and a local trivialization of V (compatible with the relative orientation) and identify σ with a function, which by an abuse of notation we denote $\sigma : \mathbf{C}^n \to \mathbf{C}^n$ (respectively $\sigma : \mathbf{R}^n \to \mathbf{R}^n$) in such a way that the point p corresponds to the origin in the domain. Then we may choose a small ball B around the origin such that p is the only zero of σ in B. We thus obtain a function between oriented spheres*

$$\overline{\sigma} : \partial B \to \partial\{x \in \mathbf{C}^n : \|x\| = 1\}$$

(respectively $\overline{\sigma} : \partial B \to \partial\{x \in \mathbf{R}^n : \|x\| = 1\}$) given by

$$\overline{\sigma}(x) = \frac{\sigma(x)}{\|\sigma(x)\|}.$$

Then set the local degree *or* index *of σ at p to be*

$$\deg_p \sigma = \deg(\overline{\sigma}).$$

Then define the Euler class $e(V)$ by

$$e(V) = \sum_{p:\sigma(p)=0} \deg_p \sigma, \tag{1}$$

and one can show that this is independent of the choice of σ.

Now return to the lines on a smooth cubic surface X. We saw above that the lines $L = \mathbf{P}W$ on X are in bijection with the zeros L of σ_f. Now, some multivariate calculus and the smoothness of X implies that every zero L of σ_f is isolated and the local index $\deg_L \sigma_f = 1$ is one [EH16, Section 2.4.2, Corollary 6.17] [KW17, Corollary 52]. We therefore conclude that the number of lines on X is $e(\mathrm{Sym}^3 \mathcal{S}^*)$, and in particular, this number is independent of surface! The example of the Fermat Cubic Surface then completes the proof, or alternatively one can use tools from algebraic topology such as the Splitting Principle, cohomology of Grassmannians, etc.

□

We now consider cubic surfaces over $\mathbf{R}$. This is also a classical topic with many beautiful results, among the first of which are those of Schläfli [Sch58] in 1858. For our purposes, it is useful to say that he showed that the number of real lines depends on the chosen cubic surface: there can be 3,7,15, or 27 real lines.

Segre distinguished between two types of real lines: *hyperbolic* and *elliptic* lines [Seg42]. Given a physical model of a real cubic surface, such as the one pictured if Figure 1, one can feel if the line is hyperbolic or elliptic using (s)pin structures as follows. Place your index finger so it lies on the line and let your palm rest on the tangent plane to the surface. Then move your hand along the line. If your hand becomes "twisted" the line is elliptic and if it remains "untwisted" the line is hyperbolic. More precisely, the line is a copy of $\mathbf{RP}^1$ which is topologically a circle. At every point on the line, your index finger, thumb and a perpendicular through your palm give a frame of the tangent space to $\mathbf{P}^3$. This data describes a loop in SO_3 which is either trivial or non-trivial. Alternatively, this data describes a loop in the frame bundle which can either lift or not lift to its double cover corresponding to the nontrivial loop in SO_3.

Example 4. *The three real lines on the Fermat Cubic Surface are hyperbolic. You can check this with the above procedure and Diagram 2, which consists of two views of the real points of Fermat Cubic Surface. The three real lines lie in a plane and the regions of the surface surrounded by the lines are either entirely above the plane (these are marked with a +) or entirely below the plane (these are marked with a −).*

A more algebraic approach to define elliptic and hyperbolic lines uses two Segre points of the line over the algebraic closure where the tangent space "pauses." More explicitly, suppose L is a real line on X. We define an involution $I : L \to L$ as follows. Suppose x is a point of L. The tangent space T_xX is a 2-dimensional plane, and the intersection $T_xX \cap X$ of the tangent space and the cubic surface is a degree 3 plane curve by Bézout's theorem. This intersection contains L, and therefore must also contain a degree 2 curve C, i.e.,

$$T_xX \cap X = L \cup C.$$

By Bézout's theorem, the intersection $L \cap C$ consists of two points, counted with multiplicity. Note that the tangent space to X at any point of $L \cap C$ contains T_xX, and therefore equals T_xX because X is smooth. The same reasoning works backwards, showing that $L \cap C$ consists of those points p of L such that $T_pX = T_xX$. Thus $L \cap C = \{x, y\}$ for some well-defined point y of L. Define I by $I(x) = y$. Choosing a coordinate on L gives an

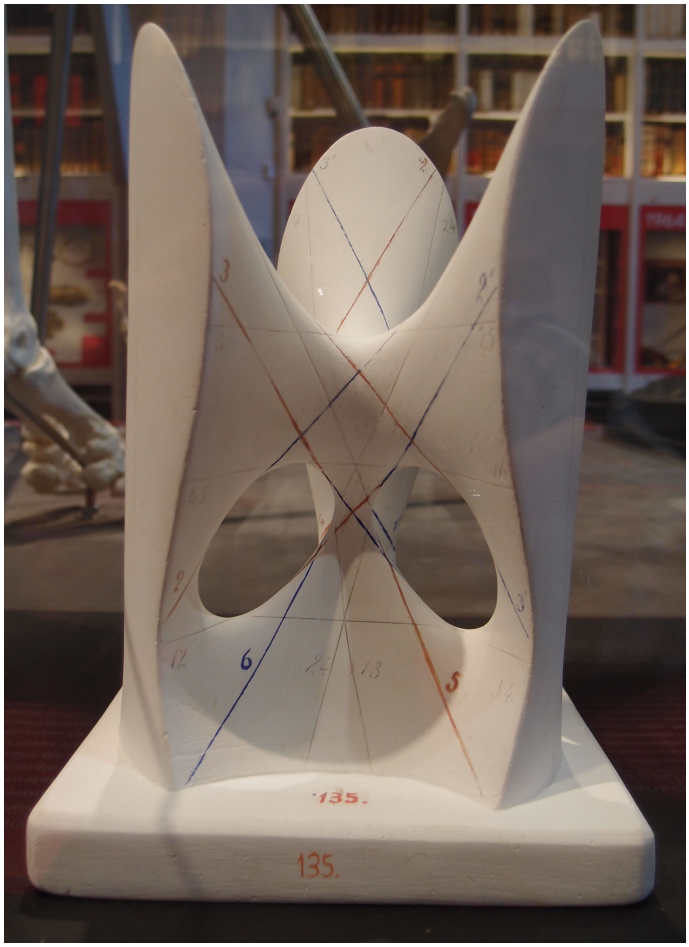

FIGURE 1. **R**-points of Clebsch Cubic Surface. Model in the collection of mathematical models and instruments, Georg-August-Universität Göttingen. By Oliver Zauzig. Published under CC BY-SA 3.0 on universitaetssammlungen.de

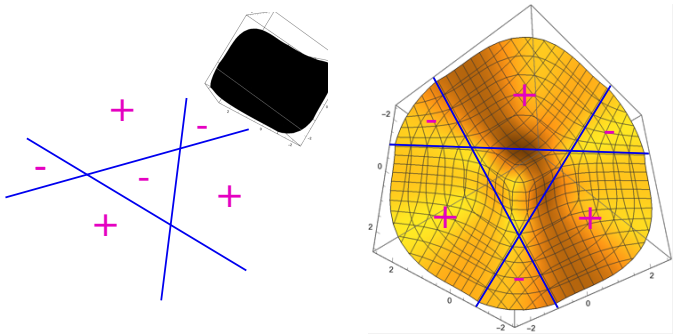

FIGURE 2. **R**-points of Fermat Cubic Surface

isomorphism $\mathrm{Aut}\, L = \mathbf{P}\,\mathrm{GL}_2\,\mathbf{R}$, so I can be represented as a matrix

$$I = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2}$$

or as the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$. I has two fixed points, namely the roots of the equation $cz^2 + (d-a)z - b = 0$ with real coefficients. These roots are thus either both real or a complex conjugate pair of points. In the first case, the line L is hyperbolic and in the second case the line is elliptic.

Remark 5. *For future reference, note that this procedure generalizes to fields other than* $\mathbf{R}$*. Let* X *be a smooth cubic surface over a field* k*, of characteristic not 2. Let* L *be a line on* X *defined over a field extension* $k(L)$ *of* k*. Then there is an associated involution* I *in* $\operatorname{Aut} L \cong \mathbf{P}\,\mathrm{GL}_2\, k(L)$*.*

Although the number of lines on a real cubic surface depends on the surface, a certain signed count does not:

Theorem 6. *[Segre, Benedetti–Silhol, Okonek–Teleman, Finashin–Kharlamov, Horev–Solomon] Let* X *be a smooth cubic surface over* $\mathbf{R}$*. Let* h *denote the number of hyperbolic lines. Let* e *denote the number of elliptic lines. Then*

$$h - e = 3.$$

Interestingly, although Theorem 6 follows from the work of Segre, it does not seem to have been noticed by him [OT14]. The proofs in [OT14] and [FK13] are along similar lines to the proof given above for the complex case (but harder). They replace the complex Grassmannian with the real one, and again compute the Euler class of $\mathrm{Sym}^3 \mathcal{S}^* \to \mathrm{Gr}(1,3)(\mathbf{R})$. The local index $\deg_L \sigma_f$ is shown to be $+1$ (respectively -1) when L is hyperbolic (respectively elliptic), and then the Euler class is computed to be 3. They obtain an analogous result for hypersurfaces of degree $2n-3$ in $\mathbf{P}^n$. Note that the Euler class computation of 3 for $\mathrm{Sym}^3 \mathcal{S}^* \to \mathrm{Gr}(1,3)(\mathbf{R})$ follows from Example 4. The proof in [BS95] uses (s)pin structures, and that of [HS12] uses Gromov–Witten theory.

There is a general principle that a result which is true over $\mathbf{C}$ and $\mathbf{R}$ may be a result in $\mathbf{A}^1$-homotopy theory. $\mathbf{A}^1$-homotopy theory was developed by Morel and Voevodsky in the late 1990's [MV99], and it allows us to treat smooth schemes like manifolds in certain respects. In some sense, there are small spheres equivalent to $\mathbf{P}^n/\mathbf{P}^{n-1}$ around any point of a smooth scheme. Morel defined a degree homomorphism

$$\deg : [\mathbf{P}^n/\mathbf{P}^{n-1}, \mathbf{P}^n/\mathbf{P}^{n-1}] \to \mathrm{GW}(k), \tag{3}$$

whose target is the Grothendieck–Witt group of bilinear forms over a field. Elements of $\mathrm{GW}(k)$ are formal differences of symmetric, non-degenerate, k-valued bilinear forms on finite dimensional vector spaces. We will discuss $\mathrm{GW}(k)$ in detail in Section 2.3, but we claim that this group is both interesting and computable. Since such forms can be (stably) diagonalized, $\mathrm{GW}(k)$ is generated by elements $\langle a \rangle$ for a in $k^*/(k^*)^2$, which correspond to the isomorphism classes of the rank one bilinear forms $k \times k \to k$ defined $(x, y) \mapsto axy$.

Example 7. $\mathrm{GW}(\mathbf{C}) \cong \mathbf{Z}$*. An isomorphism is given by the* rank *homomorphism, taking a bilinear form on a vector space* V *to the dimension of* V*.*

Example 8. *A bilinear form over* $\mathbf{R}$ *can be diagonalized so only* 1*'s and* -1*'s appear on the diagonal. The* signature *is the number of* 1*'s minus the number of* -1*'s.*

$$\mathrm{GW}(\mathbf{R}) \cong \mathbf{Z} \times \mathbf{Z}.$$

An isomorphism is induced by the rank and signature*.*

Example 9.

$$\mathrm{GW}(\mathbf{F}_q) \cong \mathbf{Z} \times \mathbf{F}_q^*/(\mathbf{F}_q^*)^2.$$

An isomorphism is given by the rank and discriminant, *where the discriminant takes a bilinear form* $\beta : V \times V \to k$ *to the determinant of a Gram matrix* $\big(\beta(v_i, v_j)\big)_{i,j}$ *representing* β, *where* $\{v_1, \ldots, v_r\}$ *is a basis of* V.

GW also admits transfer maps

$$\mathrm{Tr}_{E/k} : \mathrm{GW}(E) \to \mathrm{GW}(k)$$

for finite degree field extensions $k \subseteq E$. When $k \subset E$ is a separable extension, these can be described using the trace map $\mathrm{Tr}_{E/k} : E \to k$ from Galois theory which takes an element of E to the sum of its Galois conjugates in k. Namely, for a bilinear form $\beta : V \times V \to E$ on an E-vector space V, the image $\mathrm{Tr}_{E/k}(\beta)$ of the isomorphism class of β under $\mathrm{Tr}_{E/k}$ is represented by the bilinear form

$$V \times V \xrightarrow{\beta} E \xrightarrow{\mathrm{Tr}_{E/k}} k$$

where V is now viewed as a k-vector space.

Morel's degree and Equation (1) can be used to define an Euler class in this context, which we will discuss in detail in Section 4. Then one can repeat the previous proof to count lines on a smooth cubic surface over a field k. A line L in $\mathbf{P}^3$ is determined by two linear equations $aw + bx + cy + dz = 0$ and $a'w + b'x + c'y + d'z = 0$. We will allow $a, b, c, d, a', b', c', d'$ to be elements of an algebraic extension of k, so a line determines a closed subscheme of $\mathbf{P}^3_{k(L)}$ isomorphic to $\mathbf{P}^1_{k(L)}$, where $k(L) = k[a, b, c, d, a', b', c', d']$. Equivalently, a line L is a closed point of the Grassmannian. A line L of X determines an involution I n $\mathrm{Aut}\, L \cong \mathbf{P}\,\mathrm{GL}_2\, k(L)$ as above. The two fixed points of I are either two $k(L)$-points or a conjugate pair of points defined over some quadratic extension $k(L)[\sqrt{D}]$ of $k(L)$ for a unique D in $k(L)^*/(k(L)^*)^2$.

Definition 10. *The* type *of* L *is* $\mathrm{Type}(L) = \langle D \rangle$ *in* $\mathrm{GW}(k(L))$

It can be shown that

$$\mathrm{Type}(L) = \langle -1 \rangle \deg I$$

and

$$\mathrm{Type}(L) = \langle ad - bd \rangle$$

when I is represented as the matrix in Equation (2).

The main theorem of [KW17] is:

Theorem 11. *[Kass – W.] Let* k *be a field of characteristic not* 2. *Let* X *be a smooth cubic surface over* k. *Then*

$$\sum_{\textit{lines } L \textit{ on } X} \mathrm{Tr}_{k(L)/k}\, \mathrm{Type}(L) = 15\langle 1 \rangle + 12\langle -1 \rangle.$$

Applying invariants of bilinear forms to Theorem 11 gives more traditional counts, such as the following.

- When k is $\mathbf{C}$, or an algebraically closed field, applying Rank shows Theorem 1, i.e., that there are exactly 27 lines on a smooth cubic surface over $\mathbf{C}$.
- When $k = \mathbf{R}$, applying the signature shows Theorem 6. In more detail: a line L on a real cubic surface X is either defined over $\mathbf{R}$ or $\mathbf{C}$. If $k(L) = \mathbf{C}$, then $\mathrm{Type}(L) = \langle 1 \rangle$. Then $\mathrm{Tr}_{k(L)/k}\,\mathrm{Type}(L) = \langle 1 \rangle + \langle -1 \rangle$. If $k(L) = \mathbf{R}$, then $\mathrm{Type}(L) = \langle 1 \rangle$ if L is hyperbolic and $\mathrm{Type}(L) = \langle -1 \rangle$ if L is elliptic.

Over a finite field $k = \mathbf{F}_q$, a line on a smooth cubic surface is defined over some finite field extension $k(L) = \mathbf{F}_{q^d}$. Since $k(L)^*/(k(L)^*)^2 \cong \mathbf{Z}/2$ consists of two elements, it again makes sense to use Segre's elliptic/hyperbolic terminology, i.e., if $\mathrm{Type}(L) = \langle 1 \rangle$, then L is hyperbolic and if $\mathrm{Type}(L) = \langle u \rangle$, where u is the non-identity element of $k(L)^*/(k(L)^*)^2$, then L is elliptic. Applying the discriminant to Theorem 11 gives the following corollary.

Corollary 12. *Let X be a smooth cubic surface over the finite field $\mathbf{F}_q$, with q odd. Then*

$$|\{\textit{elliptic lines defined over } \mathbf{F}_{q^d} : d \textit{ is odd}\}| + |\{\textit{hyperbolic lines defined over } \mathbf{F}_{q^d} : d \textit{ is even}\}| \equiv 0 \mod 2$$

Many enumerative problems over $\mathbf{C}$ can be solved using degrees, characteristic classes, and intersection theory. A lot of wonderful examples are in [EH16] and [Ful98]. Such tools can be defined using homotopy theory. In the classical case, the homotopy theory topological spaces suffices, and to include the desired arithmetic information, we will use $\mathbf{A}^1$-homotopy theory. These notes will introduce enough $\mathbf{A}^1$-homotopy theory to do some first enumerative applications, and then give these applications, followed by project suggestions. This is a new direction of study, including contributions from Candace Bethea [BKW18], Marc Hoyois [Hoy14], Jesse Kass [KW16] [KW17], Marc Levine [Lev17b] [Lev17a] [Lev18a] [Lev18b], Padmavathi Srinivasan [SW18], Matthias Wendt [Wen18], and the lecturer.

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2. Lecture 2: User's guide to $\mathbf{A}^1$-homotopy theory for $\mathbf{A}^1$-enumerative geometry

2.1. **Spaces.** We will use Morel–Voevodsky's $\mathbf{A}^1$-homotopy theory of schemes. In this theory, we allow certain topological operations to be performed on schemes such as gluing, collapsing sub complexes, and finding small spheres around points of smooth schemes. The first two of these is accomplished by adding colimits into schemes. The last is the Purity Theorem of Morel–Voevodsky. It's proof requires properties of the étale or Nisnevich topology, as well as forcing $\mathbf{A}^1$ to be contractible, so it plays the role of the interval $[0, 1]$ in classical homotopy theory. Thus, we will add colimits to schemes in such a way that existing colimits coming from open covers remain colimits, and we will then force $\mathbf{A}^1$ to be contractible. These steps are all that is required to form the $\mathbf{A}^1$-homotopy

theory of schemes. For the present purposes, it will mostly suffice to view the existence of this theory as a license to treat schemes like topological spaces and smooth schemes like manifolds. In more detail and fairness, we have the following.

The *étale topology* is the Grothendieck topology on smooth k-schemes where a map $U = \coprod_\alpha U_\alpha \to X$ is an *étale cover* if $U \to X$ is étale (meaning for every u in U mapping to x in X the induced map on tangent spaces $T_u U \to T_x X$ is an isomorphism), and surjective on points.

The *Nisnevich topology* is defined so that covers are étale covers satisfying the additional property that for every x in X there is a u in U mapping to x such that the induced map on residue fields $k(x) \to k(u)$ is an isomorphism.

A homotopy theory could mean a simplicial model category. Such a category is enriched in simplicial sets, and has morphisms distinguished as weak equivalences, cofibrations, and fibrations. The category of simplicial sets will be denoted by $\mathbf{sSet}$ and can be thought of as the category of topological spaces. A homotopy theory could alternatively mean an infinity category or quasi-category, which is a simplicial set satisfying an inner horn filling condition. In both contexts there is an associated homotopy category, but one can do more things with the homotopy theory.

Let $\mathbf{Sm}_k$ denote the full subcategory of schemes over k with objects the smooth schemes over k. Then Yoneda embedding gives a functor $\mathbf{Sm}_k \to \mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{Set}) \to \mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{sSet})$ to the category of presheaves of simplicial sets on smooth schemes over k. The category $\mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{sSet})$ can be equipped with the structure of a simplicial model category of infinity category. The passage from $\mathbf{Sm}_k$ to $\mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{sSet})$ can be thought of as freely adjoining colimits to $\mathbf{Sm}_k$.

There is a formal process called *Bousfield localization* that allows us to force a chosen class of morphisms in a homotopy theory to be weak equivalences. We use this to make the colimits (or gluings) in $\mathbf{Sm}_k$ resulting from open (Nisnevich) covers remain colimits in the localization of $\mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{sSet})$. Given an open cover $U = \coprod_\alpha U_\alpha \to X$, we obtain a map $\mathrm{Cosk}_X^0 U \to X$ in $\mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{sSet})$ where $\mathrm{Cosk}_X^0 U$ is the Čech nerve of the cover. Bousfield localizing at these morphisms produces a new homotopy theory that we will denote $\mathbf{Sh}_k$ for "sheaves" and a localization functor

$$\mathrm{L}_{\mathrm{Nis}} : \mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{sSet}) \to \mathbf{Sh}_k.$$

We Bousfield localize one more so that the maps $X \times \mathbf{A}^1 \to X$ are weak equivalences for all smooth k-schemes X. We obtain the desired $\mathbf{A}^1$ homotopy theory, which we will denote $\mathbf{Spc}_k$ for "spaces" and a localization functor

$$\mathrm{L}_{\mathbb{A}^1} : \mathbf{Sh}_k \to \mathbf{Spc}_k.$$

In summary, we construct $\mathbf{A}^1$-homotopy theory by

$$\mathbf{Sm}_k \xrightarrow{\text{Yoneda}} \mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{sSet}) \xrightarrow{\mathrm{L}_{\mathrm{Nis}}} \mathbf{Sh}_k \xrightarrow{\mathrm{L}_{\mathbb{A}^1}} \mathbf{Spc}_k.$$

2.2. **Spheres, Thom spaces, and Purity.** Will follow [WW19, 2.3 and 2.4].

2.3. **Grothendieck–Witt group $\mathrm{GW}(k)$ and Milnor–Witt K-theory.** Will follow [WW19, 4.1 and 4.3].

2.4. **Degree.**

2.5. **Oriented Chow Groups or Chow–Witt Groups.** One can also define generalized cohomology theories and spectra in $\mathbf{A}^1$-homotopy theory. Motivic cohomology recovers the Chow group CH^i of a smooth scheme X by the formula $H^{2i}(X, \mathbf{Z}(i)) \cong \mathrm{CH}^i(X)$. As Chow groups are useful in classical intersection theory and enumerative geometry, we first say a word about them.

Let X be a smooth scheme over k of dimension d. Let $X^{(i)}$ denote the set of subvarieties of X of codimension i, i.e. reduced irreducible subschemes of dimension $d-i$. The group of cycles on X is the free abelian group generated by $X^{(i)}$. Rational equivalence is the equivalence relation generated by declaring $V \cap (X \times \{0\})$ equivalent to $V \cap (X \times \{1\})$ for any subvariety V of $X \times \mathbf{P}^1$, and the Chow group $\mathrm{CH}^i(X)$ is the quotient of the group of cycles of codimension i by rational equivalence. More information on Chow groups is available in, for example, [EH16].

Intersection theory over $\mathbf{C}$ is simplified by the existence of canonical orientations. Oriented Chow groups $\widetilde{\mathrm{CH}}^i$ input information about orientations to produce a more refined theory. They are also called Chow–Witt groups. They were introduced by Barge and Morel in [BM00] and further developed by Fasel [Fas08]. Elements of $\widetilde{\mathrm{CH}}^i$ can be represented by certain formal sums of subvarieties Z of codimenison i equipped with an element β of $\mathrm{GW}(k(Z))$.

In analogy with Bloch's formula $\mathrm{CH}^i \cong H^i(X, K_i^M)$, oriented Chow groups can be defined by

$$\widetilde{\mathrm{CH}}^i(X) = H^i(X, K_i^{MW}).$$

For example, $\widetilde{\mathrm{CH}}^0(\operatorname{Spec} k) \cong \mathrm{GW}(k)$. This cohomology can be computed by the Rost–Schmidt complex described in [Mor12, 5]. Let E be a field extension of k. For a 1-dimensional E-vector space Λ, let

$$K_n^{MW}(E; \Lambda) := K_n^{MW}(E) \otimes_{\mathbf{Z}[E^*]} \mathbf{Z}[\Lambda^*].$$

The Rost–Schmidt complex gives $H^i(X, K_i^{MW})$ as the kernel mod the image of a sequence (4)

$$\to \oplus_{z \in X^{(i-1)}} K_1^{MW}(k(z), \det_{k(z)} T_z X) \to \oplus_{z \in X^{(i)}} \mathrm{GW}(k(z), \det_{k(z)} T_z X) \to \oplus_{z \in X^{(i+1)}} K_{-1}^{MW}(k(z), \det_{k(z)} T_z X) \to$$

For a line bundle L on X, there is a sheaf $K_n^{MW}(L)$ on X constructed from the groups $K_n^{MW}(E; L(E))$, where E is the function field of a smooth scheme mapping to X. Using this sheaf, we define the oriented Chow groups $\widetilde{\mathrm{CH}}^i(X, L)$ twisted by a line bundle L by

$$\widetilde{\mathrm{CH}}^i(X, L) = H^i(X, K_i^{MW}(L)).$$

The Rost–Schmidt complex becomes

$$\to \oplus_{z\in X^{(i-1)}} K_1^{MW}(k(z), \det_{k(z)} T_zX \otimes_{k(z)} L(k(z))) \to \oplus_{z\in X^{(i)}} GW(k(z), \det_{k(z)} T_zX \otimes_{k(z)} L(k(z))) \to$$
$$\oplus_{z\in X^{(i+1)}} K_{-1}^{MW}(k(z), \det_{k(z)} T_zX \otimes_{k(z)} L(k(z))) \to$$

Since squares act trivially on K_n^{MW}, there is a canonical isomorphism

$$\widetilde{CH}^i(X, M \otimes L^{\otimes 2}) \cong \widetilde{CH}^i(X, M)$$

for line bundles M and L.

There are pullback and proper pushfoward maps on oriented Chow groups. For a map $f : X \to Y$ in $\mathbf{Sm}_k$ we have a pullback map $f^* : \widetilde{CH}^i(Y, L) \to \widetilde{CH}^i(X, f^*L)$. The canonical line bundle $\omega_{X/k}$ on a smooth scheme X is $\omega_{X/k} = \det T^*X$ the determinant of the cotangent bundle. When f is proper of relative dimension $r = \dim X - \dim Y$, there is a pushfoward map $f_* : \widetilde{CH}^i(X, \omega_{X/k} \otimes f^*L) \to \widetilde{CH}^{i-r}(Y, \omega_{Y/k} \otimes L)$.

There is a (non-commutative to pick up on orientations!) ring structure on oriented Chow groups $\oplus_i \widetilde{CH}^i(X, L^{\otimes i})$ [Fas07], giving an oriented intersection theory.

3. Lecture 3: Local degree

3.1. Local degree.

3.2. Eisenbud–Khimshiashvili–Levine signature formula.

3.3. $\mathbf{A}^1$-Milnor numbers.

4. Lecture 4: Euler class

Let X be a connected $\mathbf{R}$-manifold of dimension d and let $V \to X$ be a rank r vector bundle on X.

V is *oriented* by the choice of a *Thom class* u in $H^r(\mathrm{Th}(V), \mathbf{Z})$ such that for each x in X the pullback of u to the Thom space $\mathrm{Th}(V_x)$ of the fiber V_x of V at x generates $H^r(\mathrm{Th}(V_x), \mathbf{Z}) \cong \mathbf{Z}$. The manifold X is oriented if its tangent bundle TX is.

Given an oriented vector bundle $V \to X$, the *Euler class* $e(V)$ in $H^r(X, \mathbf{Z})$ is the pullback of u by any section $s : X \to V$, for example, s could be the zero section,

$$e(V) = S^*u.$$

When X is oriented of dimension $d = r$ equal to the rank of V, Poincaré duality provides an isomorphism $H^r(X, \mathbf{Z}) \cong \mathbf{Z}$, and we may view $e(V)$ as an integer.

$\mathbb{A}^1$-enumerative geometry: k field

<u>Goal</u> to record arithmetic information about geometric objects whose $\#$ is fixed over $\bar{k}$, (but not over k)

<u>tool</u>: $\mathbb{A}^1$-htpy thy

L1: User's guide to $\mathbb{A}^1$ htpy thy

$$S^n = \{(x_0, \dots, x_n) \mid \sum x_i^2 = 1\} = \mathbb{P}^n(\mathbb{R}) / \mathbb{P}^{n-1}(\mathbb{R})$$

$$\deg : [S^n, S^n] \longrightarrow \mathbb{Z}$$

pointed homotopy classes

Given $f : S^n \longrightarrow S^n$ and $p \in S^n$ s.t. $f^{-1}(y) = \{q_1, \dots, q_N\}$

$$\deg f = \sum_{i=1}^{N} \deg_{q_i} f$$

where $\deg_{q_i} f$ is the <u>local degree</u>:

Let $V \ni p$ be a small ball

Let $f^{-1}(V) \supset U \ni q_i$ be a small ball s.t. $f^{-1}(p) \cap U = q_i$

$$\begin{array}{ccc} U/\partial U \simeq S^n & & V/\partial V \simeq S^n \\ \wr\wr & & \wr\wr \\ U/(U - q_i) & \xrightarrow{\bar{f}} & V/(V - p) \end{array}$$

$$\deg_{q_i} f := \deg \bar{f}$$

Formula from differential topology

Let $(x_1, \dots, x_n)$ be oriented coordinates near q_i

Let $(y_1, \dots, y_n)$ ——— " ——— p

Then $f = (f_1, \dots, f_n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Let $J = \det\left(\frac{\partial f_i}{\partial x_j}\right)$

$$\deg_{q_i} f = \begin{cases} +1 & J(q_i) > 0 \\ -1 & J(q_i) < 0 \\ ? & J(q_i) = 0 \end{cases}$$

Eisenbud - H. Levine / Khimshiashvili
signature formula

Lannes / Morel: k field

degree for $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$
valued in $GW(k)$

$GW(k)$ = Grothendieck-Witt group of k

:= group completion under $\oplus$ of semi-ring $\otimes\oplus$
of non-degenerate, symmetric,
bilinear forms

generators : $\langle a \rangle \qquad a \in k^*/(k^*)^2$

$\langle a \rangle$ element associated to bilinear form

$$\langle a \rangle : k \times k \longrightarrow k$$
$$(x, y) \longmapsto axy$$

relations: (1) $\langle a \rangle \langle b \rangle = \langle ab \rangle$

(2) $\langle u \rangle + \langle v \rangle = \langle uv(u+v) \rangle + \langle u+v \rangle \qquad u+v \neq 0$

(3) $\langle u \rangle + \langle -u \rangle = \langle 1 \rangle + \langle -1 \rangle = \mathbb{H}$ ← hyperbolic form

Exercise: (2) $\Rightarrow$ (3) $\qquad$ char $k \neq 2$

$$W(k) = \frac{GW(k)}{\mathbb{Z}\mathbb{H}}$$

<u>Ex</u>: $GW(\mathbb{C}) \overset{\text{rank}}{\cong} \mathbb{Z}$

$(B : V \times V \to \mathbb{C}) \mapsto \dim V$

$\langle a \rangle \mapsto 1$

<u>Ex</u>: $GW(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$

Silvester's theorem

$B : V \times V \to \mathbb{C}$, there is a basis $\{v_1, \dots, v_r\}$ of V s.t.

Gram matrix $B(v_i, v_j)$ is $\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}$

Signature $B = \#1\text{'s} - \#(-1)\text{'s}$

$$\begin{tikzcd} GW(\mathbb{R}) \arrow[r, "\text{rank}\times\text{sign}"] \arrow[dr, "\cong"'] & \mathbb{Z}\times\mathbb{Z} \\ & \{(r,s)\in\mathbb{Z}\times\mathbb{Z} \mid r+s\equiv 0 \bmod 2\} \cong \mathbb{Z}\times\mathbb{Z} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

<u>Ex</u>: $GW(\mathbb{F}_q) \xrightarrow[\cong]{\text{rank}\times\text{disc}} \mathbb{Z}\times \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$

<u>Ex</u>: (Springer's theorem) K complete discrete valued field
k residue field

e.g. $K=\mathbb{Q}_p$ $K=\mathbb{F}_p((t))$, $k=\mathbb{F}_p$

Assume char $k\neq 2$

$$GW(K) \cong \frac{GW(k)\oplus GW(k)}{\mathbb{Z}(\mathbb{H},-\mathbb{H})}$$

Back to Lannes's formula:

$f:\mathbb{P}^1_k\to\mathbb{P}^1_k$ $p\in\mathbb{P}^1(k)$ $f^{-1}(p)=\{q_1,\dots,q_N\}$

s'pose $J(q_i)=f'(q_i)\neq 0$ $\forall i$

$$\deg f=\sum_{i=1}^{N}\langle J(q_i)\rangle$$

This doesn't depend on p!

<u>Exercise</u>: 1) $\deg\left(\mathbb{P}^1 \xrightarrow[z \mapsto az]{[x:y] \mapsto [ax:y]} \mathbb{P}^1\right) = \langle a \rangle$

2) $\deg\left(\mathbb{P}^1 \xrightarrow[z \mapsto z^2]{} \mathbb{P}^1\right) = \langle 1 \rangle + \langle -1 \rangle$

Cazanavé: studied naïve homotopy classes of maps $\mathbb{P}^1 \to \mathbb{P}^1$ (Karoubi–Villamayor)

$\mathbb{A}^1 = \operatorname{Spec} k[t]$ replaces $[0,1]$

<u>Def</u>: A naïve homotopy h between to maps of schemes $f, g : X \to Y$ is a map $h : X \times \mathbb{A}^1 \to Y$ s.t. $h|_{X \times \{0\}} = f$ and $h|_{X \times \{1\}} = g$

maps $\mathbb{P}^1 \to \mathbb{P}^1$ are rational functions f/g

$\deg\left(\frac{f}{g}\right) = \max(\deg f, \deg g)$ relatively prime

Bézout associated a bilinear form $\mathrm{Béz}\left(\frac{f}{g}\right)$ to f/g constructed as follows:

$$\frac{f(X)g(Y) - f(Y)g(X)}{X - Y} = \sum_{1 \le i,j \le \underset{\overset{\shortparallel}{n}}{\deg\left(\frac{f}{g}\right)}} B_{ij} X^{i-1} Y^{j-1}$$

$\mathrm{Béz}(f/g) : k^n \times k^n \longrightarrow k$

Gram matrix $(\text{Béz}(f/g)) = [B_{ij}]$

<u>Exercise</u>: (1) $\text{Béz}(\mathbb{P}^1 \xrightarrow{z \mapsto z^n} \mathbb{P}^1) = \begin{bmatrix} & & & 1 \\ & & \iddots & \\ & 1 & & \\ 1 & & & \end{bmatrix} \simeq \begin{cases} \frac{n}{2}\mathbb{H} & n \text{ even} \\ \frac{n-1}{2}\mathbb{H} + \langle 1 \rangle & n \text{ odd} \end{cases}$

(2) $\text{Béz}(\mathbb{P}^1 \xrightarrow{z \mapsto az^n} \mathbb{P}^1)$ $a \in k^*$

$$\boxed{\text{Béz}\left(\frac{f}{g}\right) = \deg\left(\frac{f}{g}\right)}$$

Cazanavé gave formula for addition of naive homotopy classes of maps and showed $[\mathbb{P}^1, \mathbb{P}^1]_N \to [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}$ is a group completion

<u>Morel</u>: $\deg^{\mathbb{A}^1} : [\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}]_{\mathbb{A}^1} \longrightarrow GW(k)$

To make sense of this we need: $\mathbb{P}^n/\mathbb{P}^{n-1}$

$[-,-]_{\mathbb{A}^1}$

$$\mathbb{P}^n/\mathbb{P}^{n-1} \text{ is } \operatorname{colim}\left(\begin{tikzcd} \mathbb{P}^{n-1} \arrow[r] \arrow[d] & \mathbb{P}^n \\ * & \end{tikzcd}\right) \simeq \mathbb{P}^n/\mathbb{P}^{n-1}$$

i.e. maps out of $\mathbb{P}^n/\mathbb{P}^{n-1}$ are maps out of $\mathbb{P}^n$ & $*$ which agree on $\mathbb{P}^{n-1}$

<u>ex</u>: $\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & U \cup V \end{tikzcd} \simeq \text{colim}$ or pushout

But colimits don't always exist in schemes

We want colimits and we want colimits of $\simeq$ schemes to produce $\simeq$ colimits. So colimit $\rightsquigarrow$ hocolim

Sm_k = smooth schemes$/k$

Simplicial sets or thick top spaces

$Pre(Sm_k)$ = Presheaves on Sm_k = Functors$(Sm_k^{op}, sSet)$

$$Sm_k \xrightarrow{\text{Yoneda}} Pre(Sm_k)$$ has colimits

$$X \longmapsto Map(-, X)$$

homotopy theory: Simplicial model category
or
∞-cat (= quasicategory)

notion of weak equivalence or homotopy equivalence

$$Sm_k \xrightarrow{\text{Yoneda}} Pre(Sm_k)$$

freely adding colimits.

<u>Problem</u>: We want colimits from "open covers" of schemes to be colimits.

Notion of an "open cover" of a scheme is a <u>Grothendieck topology</u>

Let τ be a Grothendieck topology

<u>Fix</u>: Bousfield localization imposes additional w.e.

$$\begin{tikzcd} Sm_k \arrow[r] & Pre(Sm_k) \arrow[r, shift right, "L_\tau"'] & Sh_k \arrow[l, shift right, "T"'] \arrow[r, shift right, "L_{\mathbb{A}^1}"'] & Sp^k_k \arrow[l, shift right, "T"'] \end{tikzcd}$$

$\mathrm{Cosk}^0 \coprod_\alpha U_\alpha \to X$ is a weak equivalence

$X \times \mathbb{A}^1 \to X$ is a weak equivalence

$\mathbb{A}^1$-homotopy thy

Topologies: Zariski, Nisnevich, étale

$\longrightarrow$ more open sets

<u>Def</u>: $f: X \to Y$ is <u>étale</u> at x if $(f^* \Omega_{Y/k})_x \xrightarrow{\sim} \Omega_{X/k,x}$

cotangent space

$\iff$ $\mathrm{Jac}\, f \neq 0$ in $k(x)$

<u>Def</u>: $\coprod_{\alpha \in A} U_\alpha \to X$ is an <u>étale cover</u> if it is étale and surjective

<u>Def</u>: $\coprod_{\alpha \in A} U_\alpha \to X$ is a <u>Nisnevich cover</u> if it is étale, surjective, and for every $x \in X$ $\exists$ $\alpha \in A$ and $y \in U_\alpha$ s.t. $y \mapsto x$ and $k(x) \cong k(y)$

<u>Facts</u>: • Any $Z \hookrightarrow X$ in Sm_k is Nisnevich locally equivalent to $\mathbb{A}^d_k \hookrightarrow \mathbb{A}^{d+c}_k$ (like étale top)

• Nisnevich coh can be computed using Čech cochains (like étale top)

• Alg K-thy satisfies Nisnevich descent (like Zariski top)

• Nis coh dim = Krull dim (like Zariski top)

Notation: $X \wedge Y = X \times Y / (X \times *) \cup (* \times Y)$

<u>ex</u>: In top, $S^n \wedge S^m \simeq S^{n+m}$

L2: Local degree and $\mathbb{A}^1$-Milnor numbers

Def Given pointed spaces X and Y, the <u>smash product</u> is $X \wedge Y = X \times Y / (X \times *) \cup (* \times Y)$

Ex: In classical alg top, $S^n \wedge S^m \simeq S^{n+m}$

Spheres:

$$\mathbb{G}_m = \operatorname{Spec} k[\tfrac{1}{z}, z] = \mathbb{A}^1 - \{0\}$$

$$S^{p+q\alpha} = (S^1)^{\wedge p} \wedge (\mathbb{G}_m)^{\wedge q} \simeq S^{p+q,q}$$

Ex:

$$\begin{tikzcd}
 & \mathbb{G}_m \arrow[r, "z \mapsto z"] \arrow[d] \arrow[dl, "z"'] & \mathbb{A}^1 \simeq * \arrow[d, "\cong"] \\
\frac{1}{z} & \mathbb{A}^1 \arrow[r, "\subseteq"'] & \mathbb{P}^1
\end{tikzcd}$$

$\mathbb{A}^1 \cong *$

pushout $\Rightarrow$ $\Sigma \mathbb{G}_m \simeq \mathbb{P}^1$, $\Sigma \mathbb{G}_m = S^1 \wedge \mathbb{G}_m$

Ex: $\mathbb{A}^n - \{0\} \simeq (S^1)^{\wedge n-1} \wedge (\mathbb{G}_m)^{\wedge n}$

pf: Induction and

$$\begin{tikzcd}
(\mathbb{A}^1-\{0\})\times(\mathbb{A}^{n-1}-\{0\}) \arrow[r] \arrow[d] & \mathbb{A}^1\times(\mathbb{A}^{n-1}-\{0\}) \arrow[d] & \simeq \mathbb{A}^{n-1}-\{0\} \\
(\mathbb{A}^1-\{0\})\times\mathbb{A}^n \arrow[r] \arrow[d, phantom, "\cong"] & \mathbb{A}^n-\{0\} & \\
\mathbb{A}^1-\{0\} & &
\end{tikzcd}$$

pushout

$$\begin{tikzcd}
X\times Y \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & \Sigma X\wedge Y = S^1\wedge X\wedge Y
\end{tikzcd}$$

pushout

$$\Rightarrow \quad \mathbb{A}^n-\{0\} \simeq \Sigma(\mathbb{A}^n-\{0\})\wedge(\mathbb{A}^1-\{0\})$$

<u>Ex</u>: $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq (S^1)^{\wedge n}\wedge(\mathbb{G}_m)^{\wedge n}$

<u>pf</u>: $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq \mathbb{P}^n/\mathbb{P}^n-\{0\} \simeq \mathbb{A}^n/\mathbb{A}^n-\{0\}$

$$\simeq */\mathbb{A}^n-\{0\} \simeq \operatorname{colim}\begin{tikzcd}
\mathbb{A}^n-\{0\} \arrow[r] \arrow[d] & * \\
*
\end{tikzcd}$$

$$\simeq \Sigma(\mathbb{A}^n-\{0\})$$

<u>Thom spaces</u> : $V \to X$ vector bundle

$X \hookrightarrow V$ zero section

trivial bundle

$$\mathrm{Thom}(V, X) = \mathrm{Th}(V) = X^V = V/V - X \cong_{\mathbb{A}^1} \frac{\mathbb{P}(V \oplus \mathcal{O})}{\mathbb{P}(V)}$$

<u>Purity</u>: (Morel-Voevodsky) Let $Z \hookrightarrow X$ be a closed immersion in Sm_k.

$$X/X - Z \cong_{\mathbb{A}^1} \mathrm{Th}(N_Z X)$$

where $N_Z X \to Z$ is the normal bundle

Morel: $\deg^{\mathbb{A}^1} : [\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \longrightarrow GW(k)$

$$\begin{tikzcd}
{[S^n, S^n]} \arrow[d, "\deg"] & {[\mathbb{P}^n_{\mathbb{R}}/\mathbb{P}^{n-1}_{\mathbb{R}}, \mathbb{P}^n_{\mathbb{R}}/\mathbb{P}^{n-1}_{\mathbb{R}}]} \arrow[l, "\mathbb{R}\text{-pts}"'] \arrow[r, "\mathbb{C}\text{-pts}"] \arrow[d, "\deg^{\mathbb{A}^1}"] & {[S^{2n}, S^{2n}]} \arrow[d, "\deg"] \\
\mathbb{Z} & GW(\mathbb{R}) \arrow[l, "\text{signature}"'] \arrow[r, "\text{rank}"] & \mathbb{Z}
\end{tikzcd}$$

Local $\mathbb{A}^1$-degree: following joint work with Jesse Kass
Thanks to M. Hoyois and F. Morel!

Suppose $f: \mathbb{A}^n \longrightarrow \mathbb{A}^n$, $x \in \mathbb{A}^n(k)$,

x is isolated in $f^{-1}(f(x))$, meaning $\exists$ $U \subset \mathbb{A}^n$ Zariski open, $x \in U$ s.t. $f^{-1}(f(x)) \cap U = x$

The <u>local</u> <u>$\mathbb{A}^1$-degree</u> $\deg_x f$ of f at x is defined to be the degree of

$$\mathbb{P}^n/\mathbb{P}^{n-1} \simeq \mathrm{Th}\, T_x\mathbb{A}^n \underset{\text{Purity}}{\simeq} U/U-\{x\} \xrightarrow{f} \mathbb{A}^n/\mathbb{A}^n-\{f(x)\} \simeq \mathbb{P}^n/\mathbb{P}^{n-1}$$

Now, drop the assumption that $k(x)=k$, but assume
$k(f(x)) = k$

<u>Def:</u> $\deg_x f$ is the degree of the composite

$$\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k/\mathbb{P}^n_k-\{x\} \simeq U/U-\{x\} \xrightarrow{\bar{f}} \mathbb{A}^n/\mathbb{A}^n-f(x) \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k$$

<u>Prop</u>: These two definitions agree

<u>Prop</u>: (global degree is sum of local degrees) Let $f: \mathbb{P}^n_k \longrightarrow \mathbb{P}^n_k$ be a finite map s.t. $f^{-1}(\mathbb{A}^n_k) = \mathbb{A}^n_k$

Let $\bar{f} : \mathbb{P}^n/\mathbb{P}^{n-1} \longrightarrow \mathbb{P}^n/\mathbb{P}^{n-1}$ denote the induced map. Then for any $y \in \mathbb{A}^n(k)$

$$\deg \bar{f} = \sum_{x \in f^{-1}(y)} \deg_x f$$

Computing $\deg_x f$:

In top, $f : \cancel{S^n \to S^n}$ $\quad \deg_x f = \begin{cases} +1 & \mathrm{Jac}\, f(x) > 0 \\ -1 & \mathrm{Jac}\, f(x) < 0 \\ ? & \mathrm{Jac}\, f(x) = 0 \end{cases}$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

<u>Eisenbud-Levine/Khimshiashvili Signature formula</u> :

$$\deg_0 f = \text{Signature } \omega^{EKL}$$

where ω^{EKL} is the isomorphism class of the following bilinear form:

$f = (f_1, \dots, f_n)$ $\qquad Q := \dfrac{\mathbb{R}[x_1, \dots, x_n]_0}{\langle f_1, \dots, f_n \rangle}$ finite dimensional complete intersection

$\Rightarrow Q$ Gorenstein, $\mathrm{Hom}_k(Q, k) \cong Q$ Better: distinguished iso, coming from distinguished socle element $\rightsquigarrow$ bilinear form

Explicitly: $\mathrm{Jac}\, f = \det\left(\dfrac{\partial f_i}{\partial x_j}\right)$

Choose $n : Q \longrightarrow k$ s.t. $n(\mathrm{Jac}\, f) = \dim_k Q$

$$\omega^{EKL} : Q \times Q \longrightarrow k \qquad \omega^{EKL}(a,b) = n(ab)$$

isomorphism class of ω^{EKL} does not depend on choice of n.

<u>Ex</u>: $f : \mathbb{A}^1 \to \mathbb{A}^1 \qquad f(z) = z^2$

<u>Eisenbud Q</u>: ω^{EKL} defined over a field k, and acts as "degree." Does ω^{EKL} have top interpretation?

<u>Thm</u>: (Kass-W) $\quad \deg_x f = \omega^{EKL}$ in $GW(k)$

reference for proof: "The local $\mathbb{A}^1$-Brouwer degree is the quadratic form of Eisenbud-Khimshiashvili-Levine."

<u>Exercises</u>: 1) $f : \mathbb{A}^2_k \to \mathbb{A}^2_k \qquad f(x,y) = (4x^3, 2y)$

char $k \neq 2$

Compute $\deg_0 f$

2) f étale at $0 \Rightarrow \deg_0 f = \langle \operatorname{Jac} f \rangle$

with descent data

f étale at $x \Rightarrow \deg_x f = \operatorname{Tr}_{k(x)/k} \langle \operatorname{Jac} f \rangle$

$\mathbb{A}^1$-Milnor #'s char $k \neq 2$

hypersurface singularity $p \in \{f=0\}$ bifurcates into nodes

$$\cong x_1^2 + \dots + x_n^2 = 0 \quad \text{over } k^s$$

in family $f(x_1, \dots, x_n) + a_1 x_1 + \dots + a_n x_n = t$
parametrized by t, fibers are smooth
or have nodes

$k = \mathbb{C}$ For any $(a_1, \dots, a_n)$ sufficiently close to 0

$$\#\text{ nodes in family} = \deg_p^{top} \operatorname{grad} f \quad \text{(Milnor)}$$

$$\overset{\shortparallel}{\mu_p} \quad \text{"Milnor \#"}$$

Over k? char $k \neq 2$

nodes at k-rat'l points:

$$\text{type}\left(x_1^2 + a x_2^2 = 0\right) := \langle a \rangle$$

$$\text{type}\left(\sum a_i x_i^2 = 0\right) := \langle 2^n \prod a_i \rangle$$

node at p with $k(p) = L$

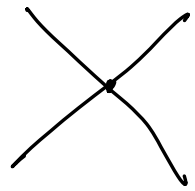
$$\text{type}\left(\sum a_i x_i^2 = 0\right) := \operatorname{Tr}_{L/k} \langle 2^n \prod a_i \rangle$$

Fact: $k \subseteq L$ is separable

Pictures $k = \mathbb{R}$ $\mathbb{R}$-rat'l nodes

$x_1^2 + x_2^2 = 0$

non-split = non-rat'l tgt directions



$x_1^2 - x_2^2 = 0$

split = rat'l tgt directions

<u>Thm</u> (Kass-W)

For generic $(a_1, \dots, a_n)$

$$\sum_{\substack{\text{nodes } p \\ \text{in family}}} \text{type}(p) = \deg_p^{\mathbb{A}^1} \operatorname{grad} f$$

$$\overset{\shortparallel}{\mu_p^{\mathbb{A}^1}} \quad \text{"}\mathbb{A}^1\text{-Milnor \#"}$$

<u>Ex</u> $f(x, y) = x^3 - y^2$ char $\neq 2, 3$

$p = (0, 0) \in \{ f = 0 \}$

$\operatorname{grad} f = (3x^2, -2y)$

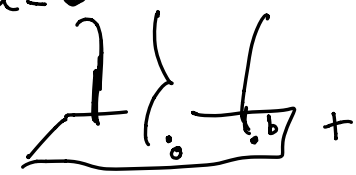
$\deg \operatorname{grad} f = \deg(3x^2) \deg(-2y)$

$$= \begin{array}{c|cc} & 1 & x \\ \hline 1 & 0 & \frac{1}{3} \\ x & \frac{1}{3} & 0 \end{array} \langle -2 \rangle = h \qquad \text{rank} = 2$$

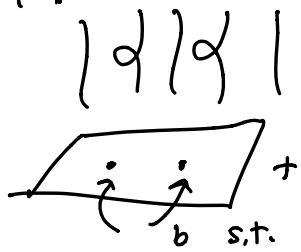
$= \langle 1 \rangle + \langle -1 \rangle$ So $\mu = 2$

family: $y^2 = x^3 + ax + t$

$a = 0$



$a \neq 0$



s.t. Disc $x^3+ax+t=0$

$\Leftrightarrow \quad -4a^3 - 27b^2 = 0$

bifurcates into 2 nodes

over $\mathbb{F}_5$: $\langle 1 \rangle = \langle -1 \rangle \Rightarrow$ can't bifurcate into a split & non-split node

over $\mathbb{F}_7$: $\langle 1 \rangle \neq \langle -1 \rangle \Rightarrow$ can't bifurcate into 2 split or 2 non-split nodes

<u>Exercise</u> : compute $\mu^{\mathbb{A}^1}$ for "ADE singularities"

singularity	equation
A_n	$x_1^2 + x_2^{n+1}$
D_n	$x_2(x_1^2 + x_2^{n-2})$
E_6	$x_1^3 + x_2^4$
E_7	$x_1(x_1^2 + x_2^3)$
E_8	$x_1^3 + x_2^5$

which ones are <u>not</u> multiples of $\mathbb{H}$?

More generally, one can view $e(V)$ as an integer when $V \to X$ is *relatively oriented*, which we will define to mean that there is a line bundle L and an isomorphism $\mathrm{Hom}(\det TX, \det V) \cong L^{\otimes 2}$. Okonek and Teleman introduced this notation in [OT14]. To see why such a generalization should exist, here are a few general remarks.

The Thom space $\mathrm{Th}(V)$ is locally an r-fold suspension of the base, so $\mathrm{Th}(V)$ can be viewed as a twisted shift of of X itself. Among the manifestations of this viewpoint is the Thom isomorphism between the cohomology of the Thom space and a twisted shift of the cohomology of the base: the Serre Spectral Sequence gives a natural isomorphism

$$H^*(X, \mathcal{O}(V)) \cong \widetilde{H}^{*+r}(\mathrm{Th}(V), \mathbf{Z}),$$

where $\mathcal{O}(V)$ is the local system whose fiber at x is $H^r(V_x, V_x - \{0\})$, called the *orientation sheaf* of V.

Some formal properties of orientation sheaves are as follows. Let $\det V$ denote the highest wedge power of the vector bundle V, so $\det V = \wedge^r V$ because V is rank r. For a linear map $A : \mathbf{R}^r \to \mathbf{R}^r$, the degree of the induced map of spheres $\mathbf{R}^r - \{0\} \to \mathbf{R}^r - \{0\}$ is the sign of the determinant of A. It follows that there is a natural isomorphism

$$\mathcal{O}(V) \cong \mathcal{O}(\det V).$$

Because determinants multiply under tensor product, we obtain a natural isomorphism $\mathcal{O}(V_1 \otimes V_2) \cong \mathcal{O}(V_1) \otimes \mathcal{O}(V_2)$. Because the square of any integer is positive, we obtain a natural isomorphism $\mathcal{O}(V_1) \otimes \mathcal{O}(V_1) \cong \mathbb{Z}$ between the tensor square of the orientation sheaf $\mathcal{O}(V_1)$ and the constant local system. Combining the above, we have a natural isomorphism $\mathcal{O}(V) \cong \mathcal{O}(\det V \otimes L^{\otimes 2})$ for any line bundle $L \to X$. It follows that a relative orientation of V induces an isomorphism $\mathcal{O}(V) \cong \mathcal{O}(TX)$.

A virtual vector bundle is a formal difference of vector bundles, and one can define the Thom spectrum of a virtual vector bundle, with similar properties to a Thom space, for instance there is a Thom isomorphism in this context. Furthermore, for virtual vector bundles Y, W over X, there is a canonical map $\mathrm{Th}(Y) \to \mathrm{Th}(Y \oplus W)$. We therefore have a map

$$\mathrm{Th}(-V) \to \mathrm{Th}(-V \oplus V) \simeq X$$

which induces a map on cohomology $H^0(X, \mathbf{Z}) \to H^0(\mathrm{Th}(-V), \mathbf{Z})$. Composing with the Thom isomorphism $H^{-(-r)}(X, \mathcal{O}(-V)) \cong H^0(\mathrm{Th}(-V), \mathbf{Z})$, we obtain a map

$$H^0(X, \mathbf{Z}) \to H^r(X, \mathcal{O}(-V)). \tag{5}$$

Define the image of 1 in $H^0(X, \mathbf{Z})$ under this map to be the Euler class $e(V)$ in $H^r(X, \mathcal{O}(-V))$.

Poincaré duality is roughly the statement that a compact manifold is self-dual, or more precisely, that the dual of a manifold X is a twisted shift of X: Precisely, the dual of X_+ is $\mathrm{Th}(-TX)$, where X_+ denotes X with a disjoint base point. This induces a Poincaré duality isomorphism $\widetilde{H}^d(X, \mathcal{O}(-TX)) \cong H_0(X, \mathbf{Z}) \cong \mathbf{Z}$. Therefore, if the orientation sheaves $\mathcal{O}(-V)$ and $\mathcal{O}(-TX)$ are isomorphic, which follows from the existence of a relative orientation (we saw this above provided we note that the above also implies $\mathcal{O}(-V) \cong \mathcal{O}(V)$), we may define view $e(V)$ as an integer.

This integer can be calculated with some calculus and a section σ of V such that all zeros of σ are isolated. Let p in X be such that $\sigma(p) = 0$. Choosing local coordinates of X near p and a local trivialization of V near p allows us to locally identify σ with a function

$$\sigma : \mathbf{R}^d \to \mathbf{R}^r.$$

When $d = r$, we may take the local degree of this function at the coordinates of p, obtaining an integer $\deg_p \sigma$. However, if we change the local trivialization by a linear function with negative determinant, we will change the sign of $\deg_p \sigma$. Therefore we must choose coordinates and a trivialization which are compatible with a relative orientation in the following sense.

Local coordinates around p give a distinguished local trivialization of TX. Taking the wedge product of a basis of vector fields, we obtain a distinguished section in $\det TX(U)$ for some neighborhood U of p. Similarly, a choice of local trivialization of V gives a distinguished section in $\det V(U)$, by possibly shrinking U. This in turn gives a distinguished section of $\mathrm{Hom}(\det TX, \det V)(U)$

Definition 13. *Local coordinates and a trivialization of* V *on an open neighborhood* U *of* p *are* compatible with the relative orientation *if the distinguished section of* $\mathrm{Hom}(\det TX, \det V)(U)$ *is the tensor square of a section in* $L(U)$.

When $\deg_p \sigma$ is computed with a choice of local coordinates and trivialization of V which is compatible with a fixed relative orientation the result is independent of the choices, an can be called the *local degree* or *index* of σ at p. The Euler class is the sum of the local terms

$$e(V) = \sum_{p:\sigma(p)=0} \deg_p \sigma$$

This discussion can be transported to $\mathbf{A}^1$-homotopy theory. The first Euler class in the context of $\mathbf{A}^1$-homotopy theory is due to Barge and Morel [BM00], and the technology has benefited from contributions of Morel, Jean Fasel, Aravind Asok, Marc Levine, Frédéric Déglise, Fangzhou Jin, and Adeel A. Khan, and Arpon Raksit. There will be further discussion in Section 4.3. The following point of view is from joint work with Jesse Kass [KW17, Section 4].

4.1. **Orientations.** Let X be a smooth scheme over k of dimension $d = r$. Let $V \to X$ be a rank r algebraic vector bundle.

Definition 14. $V \to X$ *is* oriented *by the data of a line bundle* L *on* X *and an isomorphism* $\det V \cong L^{\otimes 2}$.

Some authors use the terminology *weakly oriented* for this concept, reserving the term oriented for isomorphisms $\det V \cong \mathcal{O}$.

Definition 15. $V \to X$ *is* relatively oriented $\mathrm{Hom}(\det TX, \det V)$ *is oriented.*

Example 16. *Let* $X = \mathbf{P}^1$. *Let* $\mathcal{O}(-1)$ *denote the tautological bundle on* $\mathbf{P}^1$. *Then* $\det TX \cong TX \cong \mathcal{O}(2)$. *Therefore a line bundles* $\mathcal{O}(n)$ *is relatively orientable if and only if* n *is even.*

4.2. **Local indices and definition of the Euler class.** Let $V \to X$ be a relatively oriented rank r vector bundle on a smooth scheme over k of dimension $d = r$.

Suppose that σ is a section of V and that p is an isolated zero of σ, meaning that p is point of X such that $\sigma(p) = 0$, and such that there is an open set U containing p such that the only zero of σ in U is p.

We will define the *local index* or *degree* $\deg_p \sigma$ in $\mathrm{GW}(k)$ as follows.

Definition 17. *An étale map $\phi : U \to \mathbf{A}^d$ from a Zariski open neighborhood of p to $\mathbf{A}^d$ will be called* Nisnevich local coordinates around p *if the induced map of residue fields $k(\phi(p)) \to k(p)$ is an isomorphism.*

Nisnevich coordinates are guaranteed to exist if $d = 1$ [BKW18, Proposition 6] or if $k(p)$ is a separable extension of k and $d \geq 1$ [KW17, Lemma 18]. Let ϕ be Nisnevich coordinates around U. Let $\psi : V|_U \to \mathcal{O}_U^r$ be a local trivialization of V.

Definition 18. ϕ *and* ψ *are* compatible with the relative orientation *if the distinguished section in* $\mathrm{Hom}(\det TX, \det V)(U)$ *is the image of the tensor square of a section in* $L(U)$ *under the isomorphism* $\mathrm{Hom}(\det TX, \det V) \cong L^{\times 2}$ *coming from the relative orientation.*

Choose Nisnevich local coordinates $\phi : U \to \mathbf{A}^d$. After possibly shrinking U, it is possible to choose a trivialization $\psi : V|_U \to \mathcal{O}_U^r$ which is compatible with the relative orientation (multiplying the first coordinate by some element in $\mathcal{O}_U^*$ will suffice).

$\psi \circ \sigma|_U$ is then an element of $\mathcal{O}(U)^r$. As above, we wish to identify σ with a function $\mathbf{A}^d \to \mathbf{A}^r$, i.e., we wish for each of the r components of $\psi \circ \sigma$ to be in the image of $\phi^* : \mathcal{O}_{\mathbf{A}^d} \to \mathcal{O}_U$. When X is covered by opens of the form $\mathbf{A}^d$, as is the case when X is a Grassmannian, we can choose ϕ to be an isomorphism on local rings, and this is immediate. The general case can be made to work as well, however. We can add an element $G = (g_1, \ldots, g_r)$ of $\mathcal{O}_U^r$ to $\psi \circ \sigma$ so that $G + \psi \circ \sigma = \psi^*(F)$, with $F : \mathbf{A}^d \to \mathbf{A}^r$, and so that each g_i vanishes to a sufficiently high order at p. Then define

$$\deg_p \sigma = \deg_{\phi(p)} F$$

and this is independent of the choice of ϕ, ψ, and G [KW17, Corollary 29].

We then define the Euler Class $e(V, \sigma)$ in $\mathrm{GW}(k)$ of V with respect to the section σ (and the relative orientation):

Definition 19. $e(V, \sigma) = \sum_{p:\sigma(p)=0} \deg_p \sigma$

$e(V, \sigma)$ should be independent of σ in general. It is shown that $e(V, \sigma) = e(V, \sigma')$ when σ and σ' can be connected by a family parametrized by $\mathbf{A}^1$ of sections with only isolated zeros, giving a well-defined Euler class under the hypotheses of [KW17, Corollary 36].

Example 20. *Let 0 denote the origin of the distinguished copy of $\mathbf{A}^1$ in $\mathbf{P}^1$, and let n be an integer. Let $\mathcal{O}(2n \cdot 0) \to \mathbf{P}^1$ denote the line bundle associated to the locally free sheaf of meromorphic functions whose only poles are at 0 and these poles are of order no worse than $2n$. Then the section*

1 *has an isolated zero of order* $2n$ *at* 0*. For any chosen relative orientation,*

$$e(\mathcal{O}(2n \cdot 0)) = \sum_{p:1(p)=0} \deg_p 1 = \deg_0 x^{2n} = n(\langle 1\rangle + \langle -1\rangle).$$

4.3. **More perspectives on the Euler class.** Barge and Morel defined an Euler class in the oriented Chow groups of X [BM00]. Namely, let $p : V \to X$ be a rank r vector bundle on a smooth scheme X over k (X of dimension d not necessarily equal to r).

There is a canonical element $\langle 1\rangle$ in $\widetilde{\mathrm{CH}}^0(X)$. In the Rost–Schmidt complex (4), it is represented by $\langle 1\rangle$ in $\mathrm{GW}(k(X))$ under the inclusion of the summand corresponding to the codimension 0 scheme X. Let $\sigma : X \to V$ denote the zero section. Since σ is proper, there is a pushforward map $\sigma_* : \widetilde{\mathrm{CH}}^0(X, \omega_{X/k} \otimes \sigma^* p^* \omega_{X/k}) \to \widetilde{\mathrm{CH}}^r(\omega_{V/k} \otimes p^*\omega_{X/k})$. Since $p\sigma = 1$ and $\omega_{V/k} \cong p^*\omega_{X/k} \otimes \det V^*$, this pushforward can be identified with a map

$$\sigma_* : \widetilde{\mathrm{CH}}^0(X) \to \widetilde{\mathrm{CH}}^r(V, \det V^*).$$

The *Euler class* $e(V)$ in $\widetilde{\mathrm{CH}}^r(X, \det V^*)$ is then defined by $e(V) = (p^*)^{-1}\sigma_*\langle 1\rangle$ [AF16b]. The pushforward followed by the isomorphism $(p^*)^{-1}$ is analogous to the composition (5) defined using Thom spaces.

When V is relatively oriented, we have an isomorphism $\det V^* \cong \omega_{X/k} \otimes (L^*)^{\otimes 2}$, and therefore a pushforward map

$$\widetilde{\mathrm{CH}}^r(X, \det V^*) \to \widetilde{\mathrm{CH}}^{r-d}(\operatorname{Spec} k).$$

When $r = d$, we therefore obtain $e(V)$ in $\mathrm{GW}(k)$.

Remark 21. *This should be equal to the Euler class of Section 4.2.*

Morel has an Euler class constructed as the principal obstruction to a nonvanishing section [Mor12, 8.2]. This is known to agree with the Barge–Morel construction at least up to a unit in $\mathrm{GW}(k)$ when V is oriented [AF16a] [Lev17b].

Déglise, Jin and Khan have a construction using 6-functor formalism [DJK18].

There is another construction using Serre Duality of coherent sheaves. I learned about it because Mike Hopkins pulled it out of thin air after a talk I gave. Serre did something similar to Eva Bayer-Fluckiger and A. Raksit independently produced this construction as well. It appears in in work of M. Levine and A. Raksit [LR18] together with a proof that it agrees with the Barge–Morel construction.

4.4. **Lines meeting four lines in space.** with Padma Srinivasan [SW18]. See also the work of Matthias Wendt [Wen18].

5. Arizona Winter School Project Suggestions

There are many beautiful results in enumerative geometry, and it is not so far-fetched to suggest turning any such result into a project, where the classical count over $\mathbf{C}$ becomes an arithmetic count over a field. Some known difficulties with this perspective include the existence of orientations (some vector bundles which occur naturally in enumerative geometry are not relatively oriented), and obtaining enumerative descriptions of local contributions to Euler classes. The purpose of this section is to outline projects which (hopefully) are of a nature to be educational in the Arizona Winter School. Since these projects are intended to be tackled in a period of six days with some success, this list is not intended to be representative of the open problems in the field.

5.1. **Configurations over k.** Let $\mathcal{S} \to \mathrm{GW}(1,n)$ denote the tautological bundle on the Grassmannian of lines in $\mathbf{P}^n$. In [Wen18] and [SW18] the Euler class of $\wedge^2\mathcal{S}^* \to \mathrm{Gr}(1,3)$, and more generally, $\oplus_{i=1}^{2n-2}\mathcal{S} \to \mathrm{Gr}(1,n)$ is computed. This corresponds to an arithmetic count of the number of lines meeting $2n-2$ codimension 2 hyperplanes. However, these hyperplanes are all defined over k. One could ask more generally for a count of lines meeting a configuration of $2n-2$ codimension 2 hyperplanes, where the configuration is defined over k, but where the individual codimension 2 hyperplanes are not necessarily defined over k. In other words, we have $2n-2$ hyperplanes defined over $\overline{k}$, which are permuted by the $\mathrm{Gal}(\overline{k}/k)$-action, but where the $\mathrm{Gal}(\overline{k}/k)$-action is potentially non-trivial. An arithmetic count of these lines would correspond to the computation of an Euler class of the vector bundle whose fiber over $[W]$ in $\mathrm{GW}(1,n)$ is $(W^* \wedge W^*)^{2n-2}/S_{2n-2}$, where W denotes a linear subspace of dimension 2 of an $n+1$ dimensional vector space, S_{2n-2} denotes the symmetric group on $2n-2$ objects, and the action of S_{2n-2} on $(W^* \wedge W^*)^{2n-2}$ is by permutation.

Question 22. (1) *Is this vector bundle relatively orientable for certain n?*
(2) *If so, what is the Euler class?*
(3) *Can you give an enumerative interpretation of the local indices?*
(4) *If so, what is the resulting theorem?*

Question 23. *Can this be generalized to other counts of subspaces meeting a configuration over k? For example, can the count of balanced subspaces of* [Wen18, Section 9.2] *be generalized to where the subspaces are not defined over the ground field? Similarly, if we only require that the two quadrics of* [Wen18, Example 9.4] *are permuted by $\mathrm{Gal}(\overline{k}/k)$ instead of being individually defined over k, could we obtain a more general result?*

5.2. **Bitangents to a smooth plane quartic.** A bitangent to a curve in projective space is a line which is tangent to the curve at 2 points, counted with multiplicity. It is a classical theorem that there are 28 bitangents to a smooth degree 4 curve in $\mathbf{P}^2$ over $\mathbf{C}$. This is closely related to the 27 lines on a complex smooth cubic surface. Namely, let $S \subset \mathbf{P}^3_{\mathbf{C}}$ be a smooth cubic surface, and let p be a point of S not on any line. Projection π_p from p defines a rational map $\pi_p : \mathbf{P}^3_k \dashrightarrow \mathbf{P}^2_k$, and a map $\pi_p^{\mathrm{Bl}} : \mathrm{Bl}_p S \to \mathbf{P}^2$, which expresses S as a degree 2 cover of $\mathbf{P}^2$ branched over a quartic curve C. The images of the 27 lines on S and the exceptional divisor of the blow up give the 28 bitangents of C.

Question 24. *Is there an arithmetic count of the bitangents to a smooth plane quartic?*

One idea along these lines is as follows: perhaps a good local contribution for a bitangent line L is related to the two points of contact with the curve. If these points are defined over $k(L)[\sqrt{D}]$, then $\langle D \rangle$ in $GW(K(L))$ may be useful. Consider the involution I which appears in the arithmetic count of the lines on a cubic surface described above. Do the images of these fixed points have an independent interpretation as points on the bitangent, for example as points of contact with C?

5.3. **Eisenbud–Khimshaishvili–Levine form is $\mathbf{A}^1$-local degree over extensions of k.** The main result of [KW16] identifies the $\mathbf{A}^1$-local degree $\deg_0 f$ at 0 of a function $f : \mathbf{A}^n_k \to \mathbf{A}^n_k$ with an isolated zero at the origin with the bilinear form ω^{EKL} appearing in the Eisenbud–Levine–Khimshiashvili signature formula

$$\deg_0 f = \omega^{\mathrm{EKL}}.$$

Question 25. *Can this be generalized to equate the local degree $\deg_p f$ with ω^{EKL} when the residue field of p is not k?*

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PROBLEMS ABOUT FORMAL GROUPS AND COHOMOLOGY THEORIES
ARIZONA WINTER SCHOOL 2019

VESNA STOJANOSKA

1. Formal Group Laws

In this document, a formal group law over a commutative ring R is always commutative and one-dimensional. Specifically, it is a power series $F(x, y) \in R[[x, y]]$ satisfying the following axioms:

- $F(x, 0) = x = F(0, x)$,
- $F(x, y) = F(y, x)$, and
- $F(x, F(y, z)) = F(F(x, y), z)$.

Sometimes we denote $F(x, y)$ by $x +_F y$. For a non-negative integer n, the n-series of F is defined inductively by $[0]_F x = 0$, and $[n + 1]_F x = x +_F [n]_F x$.

(1) Show that for any formal group law $F(x, y) \in R[[x, y]]$, x has a formal inverse. In other words, there is a power series $i(x) \in R[[x]]$ such that $F(x, i(x)) = 0$. As a consequence, we can define $[-n]_F x = i([n]_F x)$.
(2) Check that the following define formal group laws:
   (a) $F(x, y) = x + y + uxy$, where u is any unit in R. Also show that this is isomorphic to the multiplicative formal group law over R.
   (b) $F(x, y) = \frac{x+y}{1+xy}$. *Hint:* Note that if $x = \tanh(u)$ and $y = \tanh(v)$, then $F(x, y) = \tanh(x + y)$.
(3) Suppose F, H are formal group laws over a complete local ring R, and we're given a morphism $f : F \to H$, i.e. $f(x) \in R[[x]]$ such that
$$f(F(x, y)) = H(f(x), f(y)).$$
Now let $\mathcal{C}$ be the category of complete local R-algebras and continuous maps between them; for $T \in \mathcal{C}$, denote by $\mathrm{Spf}(T) : \mathcal{C} \to \mathcal{S}et$ the functor $S \mapsto \mathrm{Hom}_{\mathcal{C}}(T, S)$.
   (a) Show that a formal group law F equips $\mathrm{Spf}\, R[[x]]$ with an abelian group structure. In other words, F determines a multiplication map $\mathrm{Spf}\, R[[x]] \times \mathrm{Spf}\, R[[x]] \to \mathrm{Spf}\, R[[x]]$ and a unit map $\mathrm{Spf}\, R \to \mathrm{Spf}\, R[[x]]$, making the usual diagrams commute. Denote the corresponding formal group by G_F.
   (b) Show that a morphism $f : F \to H$ as above corresponds exactly to a formal group homomorphism $G_F \to G_H$.
(4) Determine the p-series $[p]_{\mathbb{G}_m}(x)$ of the formal multiplicative group $\mathbb{G}_m$, and the height of $\mathbb{G}_m$.
(5) Using the p-series, show that if k is a field of characteristic p, there is no non-zero homomorphism from $\mathbb{G}_a$ to $\mathbb{G}_m$ over k.

(6) Let $g(x) \in R[[x]]$ be a power series of the form $g(x) = x + \sum_{i>1} b_i x^i$.
  (a) Show that there is a power series $g^{-1}(x) \in R[[x]]$ such that $g(g^{-1}(x)) = x = g^{-1}(g(x))$.
  (b) Show that $F_g(x, y) = g^{-1}(g(x) + g(y))$ is a formal group law.
  (c) If $R = \mathbb{Z}[e]/e^2$ and $g(x) = x + ex^n$, what is F_g?

(7) Let $R = \mathbb{Q}$, and let $g_n(x) = x + p^{-1}x^{p^n} + p^{-2}x^{p^{2n}} + \cdots$, and consider $F_n := F_{g_n}$ defined as in the previous exercise.
  (a) Assume you know that F_n is a formal group law defined ver $\mathbb{Z}$. (Can you prove that? – use Hazewinkel's functional equation lemma.) Calculate that $[p]_{F_n} \equiv x^{p^n} \mod p$. Conclude that the reductions Γ_n of F_n mod p for different n are all non-isomorphic.
  (b) Prove that $\phi(x) = ax$, for $a \in \bar{\mathbb{F}}_p$ is an endomorphism of Γ_n over $\bar{\mathbb{F}}_p$ if and only if $a \in \mathbb{F}_{p^n}$.
  (c) Prove that the endomorphism ring of Γ_n over $\mathbb{F}_p$ is the $\mathbb{Z}_p$-algebra generated by $S(x) = x^p$.
  (d) Prove that the endomorphism ring of Γ_n over $\bar{\mathbb{F}}_p$ is the $\mathbb{W}(\mathbb{F}_{p^n})$-algebra generated by $S(x)$.

(8) Let F be a formal group law over a torsion-free ring R. Show that the formal expression

$$f(x) = \int_0^x \frac{dt}{\frac{\partial F}{\partial y}(t, 0)} \in R_{\mathbb{Q}}[[x]]$$

satisfies the identity

$$f(F(x, y)) = f(x) + f(y),$$

so f is an isomorphism from F to $\mathbb{G}_a$ over $R_{\mathbb{Q}}$. In particular, any formal group law over a $\mathbb{Q}$-algebra is isomorphic to $\mathbb{G}_a$. *The power series $f(x)$ is called the logarithm of F and denoted* $\log_F(x)$.

(9) Suppose F is a formal group law over a torsion-free ring R. Show that the logarithm $\log_F(x)$ is the unique power series satisfying $\log_F [p]_F(x) = p \log_F x$.

(10) Let F be a formal group law over a torsion-free ring. Then its logarithm $\log_F(x)$ can be obtained as

$$\log_F(x) = \lim_{n\to\infty} p^{-n}[p^n]_F(x).$$

(11) Let F be a formal group law over R, and suppose the integer n is invertible in R. Show that the map $[n] : F \to F$ is an isomorphism.

(12) Suppose k is a perfect field of characteristic p, and suppose A is a complete local ring with maximal ideal $\mathfrak{m}$. The (p-typical) Witt vectors $\mathbb{W}(k)$ is a complete local ring with the following universal property: for any map of fields $i : k \to A/\mathfrak{m}$, there is a unique continuous map $\mathbb{W}(k) \to A$ which reduces to i modulo the maximal ideals. Show that this implies that the Lubin-Tate ring $E(k, \Gamma)$ classifying deformations of a formal group law Γ over k is a $\mathbb{W}(k)$-algebra.

(13) Try to prove directly that $\mathbb{Z}_p[\zeta]$, where ζ is a primitive $(p^n - 1)$-st root of unity, satisfies the above universal property to be the Witt vectors of $\mathbb{F}_{p^n}$.

(14) Let C be the elliptic curve over $\mathbb{F}_9$ defined by the equation $y^2 = x^3 - x$.
  (a) Show that the formal group law F_C of C has height 2, i.e. C is supersingular.

(b) Let $\tilde{C}$ be the elliptic curve over $\mathbb{W}(\mathbb{F}_9)[[u_1]]$ defined by the equation $y^2 = 4x^3 + u_1x^2 + 2x$. Show that the formal group law $F_{\tilde{C}}$ of $\tilde{C}$ is a universal deformation of the formal group law F_C of C.
(c) Suppose $(x, y) \to (\lambda^2 x, \lambda^3 y)$ defines an automorphism of C for some $\lambda \in \mathbb{F}_9^\times$. Show that λ has order 4, and conclude that $(\mathbb{F}_9^\times)^2$ is a subgroup of the automorphism group of C, and hence of the formal group law F_C.
(d) For $\lambda \in (\mathbb{F}_9^\times)^2$, compute the induced action on $\mathbb{W}(\mathbb{F}_9)[[u_1]]$.
(e) At the prime 2 instead of the 3, do a similar exercise for $y^2 + y = x^3$ over $\mathbb{F}_4$ and $y^2 + u_1xy + y = x^3$ over $\mathbb{W}(\mathbb{F}_4)[[u_1]]$.

2. Cohomology Theories and Formal Group Laws

Let $\mathcal{S}$ be the category of topological spaces, assumed to be compactly generated and weak Hausdorff. The assumption implies the existence of internal homs, aka mapping spaces. We study spaces up to homotopy, and in particular, any space is homotopy equivalent to a CW complex. Since things can be homotopic in multiple ways, and there are homotopies between homotopies etc, the correct way to encode all this is using model categories or infinity categories. Moreover, we really use the notion of a weak equivalence, i.e. an isomorphism on homotopy groups.

Denoting by $\mathcal{S}_*$ the category of pointed spaces, a (generalized, reduced) cohomology theory is a functor $\mathcal{S}_*^{op} \to \mathcal{A}b^{gr}$ to graded abelian groups, which is homotopy invariant, stable, exact, and additive. Examples include ordinary (singular) cohomology with arbitrary coefficients, complex and real K-theory, complex cobordism. Note that ordinary cohomology with commutative ring coefficients, as well as the other mentioned cohomology theories are multiplicative, i.e. the corresponding functors land in graded commutative rings.

(15) Write out what the axioms of a generalized cohomology theory mean.

Cohomology theories and natural transformations (aka cohomology operations) form a stable additive category. The category $\mathcal{S}p$ of spectra is a refinement of this, which allows for many natural constructions to be possible (eg. taking fibers or cofibers, localizations, etc.). At the end, spectra form a stable symmetric monoidal model category or ∞-category, whose tensor product is called the smash product (denoted $\wedge$), and the unit is the sphere spectrum S^0.

Suggested references: Adams's "Stable Homotopy and Generalized Homology" combined with the first sections of Bousfield-Friedlander's "Homotopy theory of Γ-spaces, spectra, and bisimplicial sets", or at the other end of the spectrum, Lurie's "Higher Algebra", with lots of options inbetween, a notable example being the appendix of Hill-Hopkins-Ravenel's "On the nonexistence of elements of Kervaire invariant one."

In the most naive definitions, a spectrum E is a sequence $\{E_n, \sigma_n\}_{n\in\mathbb{Z}}$ of pointed spaces E_n along with structure maps $\sigma_n : \Sigma E_n = E_n \wedge S^1 \to E_{n+1}$. A spectrum E defines a homology and cohomology theory for (pointed)

spaces X:

$$E_nX = \lim_{k\to\infty} \pi_{n+k}(X \wedge E_k), \qquad E^n(X) = \lim_{k\to\infty} [\Sigma^k X, E_{n+k}].$$

(These are really the reduced cohomology theories; The unreduced versions are the values on X_+, the space X with a disjoint base point.)

For any pointed space X, its suspension spectrum $\Sigma^\infty X$ consists of $\Sigma^n X$, with identities as the structure maps. The sphere spectrum S^0 is $\Sigma^\infty(*_+)$; the homotopy groups of a spectrum E are defined as

$$\pi_n E = \lim_{k\to\infty} \pi_{n+k} E_k.$$

Formal group laws in homotopy theory appear when studying complex orientable cohomology theories.

Suppose $\xi \to X$ is a (complex) vector bundle; its Thom space is defined as the quotient $Th\xi = D(\xi)/S(\xi)$ of the unit disc bundle modulo the unit sphere bundle in ξ. This construction is natural (for maps of vector bundles), and the projection defines a map $p : Th\xi \to X$.

(16) What is the Thom space of a trivial vector bundle?

A multiplicative cohomology theory E is complex oriented if for any complex vector bundle $\xi \to X$ of (complex) rank n, there is a class $u_\xi \in E^{2n}(Th\xi)$ such that:

- For each point $x \in X$, the composite
$$E^{2n}(Th\xi) \to E^{2n}(Th\xi|_x) \cong E^{2n}(S^{2n}) \cong E^0(S^0)$$
sends u_ξ to 1;
- For any map $f : Y \to X$, we have $u_{f^*\xi} = f^* u_\xi$; and
- If η is another complex vector bundle on X, then $u_{\xi\oplus\eta} = u_\xi u_\eta$.

(17) Show that $p^*(-)u_\xi$ determines an isomorphism $E^k(X) \to E^{k+2n}(Th\xi)$. This is called a Thom isomorphism.

(18) The classifying space for complex line bundles is $\mathbb{C}P^\infty \cong BU(1)$. Let γ be the tautological line bundle on $\mathbb{C}P^\infty$. Show that the zero section $\mathbb{C}P^\infty \to Th\gamma$ is a homotopy equivalence.

(19) Let $i : S^2 \cong \mathbb{C}P^1 \to \mathbb{C}P^\infty$ be the map classifying the tautological line bundle on $\mathbb{C}P^1$. Show that a complex orientation for E gives a "first Chern" class $c_1^E \in E^2(\mathbb{C}P^\infty)$, such that $i^*(c_1^E) = 1 \in E^0 \cong E^2(S^2) = E^2(\mathbb{C}P^1)$.

(20) Prove the splitting principle, i.e. show that for any vector bundle $\xi \to X$, there is a space Y and a map $f : Y \to X$, such that $f^*\xi$ is a sum of line bundles.

(21) Conclude that a complex oriented cohomology theory has a natural theory of Chern classes for complex vector bundles, so that if $f_L : X \to \mathbb{C}P^\infty$ classifies a line bundle L on X, then $c_1^E(L) = f_L^*(c_1^E)$, and moreover, if $\xi \simeq L_1 \oplus \cdots \oplus L_n$, then $c_k(\xi)$ is the k-th symmetric elementary function on $c_1^E(L_1), \ldots c_1^E(L_n)$.

We denote by $c^E(\xi)$ the "total Chern class" $1 + c_1^E(\xi) + c_2^E(\xi) + \cdots \in E^*(X)$; then $c^E(\xi_1 \oplus \xi_2) = c^E(\xi_1)c^E(\xi_2)$.

Formal group laws appear when we address the question of what is the Chern class of a tensor product of bundles, or specifically for line bundles. There is a multiplication map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu} \mathbb{C}P^\infty$, such that if L_1, L_2 are line bundles over X, then the composite

$$X \xrightarrow{L_1 \times L_2} \mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu} \mathbb{C}P^\infty$$

classifies the tensor $L_1 \otimes L_2$. So, to understand what $c_1^E(L_1 \otimes L_2)$ is, we must understand what $E^*\mu$ is.

(22) Show that if E is complex oriented, there is an isomorphism $E^*(\mathbb{C}P^\infty) \cong E^*[[x_E]]$. You will need to understand the cell structure of $\mathbb{C}P^\infty$ and use the Atiyah-Hirzebruch spectral sequence. Then conclude that $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[x_E, y_E]]$.

(23) For an oriented spectrum E, let $F_E(x, y) \in E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[x, y]]$ be $\mu^*(x)$. Show that this is a formal group law.

The complex cobordism spectrum MU consists of the spaces $MU_{2n} = MU(n) = Th\xi_n$, where ξ_n is the tautological rank n-bundle on $BU(n)$, and $MU_{2n+1} = \Sigma MU_{2n}$. The structure maps are constructed from the inclusions $BU(n) \to BU(n+1)$.

(24) Show that MU is complex orientable.

In fact, MU is the universal oriented cohomology theory; orientations on a ring spectrum E correspond to ring maps $MU \to E$. In line with the above correspondence between orientations and formal group laws, Quillen showed that MU carries the universal formal group law, and so MU_* is the Lazard ring.

(25) Show that the complex K-theory spectrum is complex orientable.

3. Landweber exactness

A number of cohomology theories can be constructed from MU, using the following criterion due to Landweber. For each prime p and non-negative integer n, let v_n be the coefficient of x^{p^n} in $[p]_{F_{MU}}$, where v_0 is understood to be p. Suppose R is an MU_*-module. Suppose for each p, n, the map $v_n : R/(p, v_1, \dots, v_{n-1}) \to R/(p, v_1, \dots, v_{n-1})$ is injective, i.e. $(p, v_1, \dots)$ is a regular sequence. Then the functor

$$X \mapsto MU_*(X) \otimes_{MU_*} R$$

is a homology theory. Note, this does not require that R be a flat MU_*-module, but rather is related to a flatness condition on the moduli stack of formal groups.

(26) Show that the map $MU_* \to \mathbb{Z}$ classifying the additive formal group law satisfies he Landweber exactness criterion. (The homology theory one obtains this way is ordinary homology with integer coefficients.)

(27) Show that the map $MU_* \to \mathbb{Z}[u^{\pm 1}]$ classifying the multiplicative formal group law group $F(x, y) = x + y + uxy$ satisfies the Landweber exactness

criterion. (The homology theory one obtains this way is the one represented by the complex K-theory spectrum.)

(28) Let k be a perfect field of characteristic p, and Γ a formal group law over k. Show that the map $MU_* \to E(k,\Gamma)[u^{\pm 1}]$ classifying the universal deformation of Γ makes $E(k,\Gamma)[u^{\pm 1}]$ into an MU_*-algebra which satisfies Landweber's criterion.

(29) Let C be an elliptic curve over R. Its formal group is classified by a map $MU_* \to R[u^{\pm 1}]$. Under what conditions on C does this satisfy the Landweber criterion?

4. Group actions

Another way to get new cohomologies from old is in the presence of group actions. For a finite group G, a spectrum with a G-action is best defined in the ∞-categorical land: it corresponds to a functor (of ∞-categories) $X : BG \to \mathcal{S}p$, where BG the category with one object and whose endomorphisms are G. Then there are associated spectra X^{hG} (homotopy fixed points), X_{hG} (homotopy orbits), and X^{tG} (Tate spectrum), sitting in a cofiber sequence

$$X_{hG} \to X^{hG} \to X^{tG}.$$

Moreover, there are spectral sequences

$$H_s(G, \pi_t X) \Rightarrow \pi_{t+s} X_{hG}$$
$$H^s(G, \pi_t X) \Rightarrow \pi_{t-s} X_{hG},$$
$$\hat{H}^s(G, \pi_t X) \Rightarrow \pi_{t-s} X_{tG},$$

compatible with the cofiber sequence above.

(30) The group C_2 acts on the complex K-theory spectrum by complex conjugation, with homotopy fixed points being KO, the spectrum classifying real K-theory. Compute the associated homotopy fixed point spectral sequence, and then also the Tate and homotopy orbit spectral sequences.

(31) Fixing a perfect field k and formal group law Γ over k, let G be a finite group of automorphisms of Γ over k. Then G acts on the spectrum $\mathbf{E}(k,\Gamma)$ from (28) above. In specific examples, can you compute anything about this action?

(32) In the world of ∞-categories, we can also take homotopy fixed points of groups acting on categories. Let Vect be the category of finite dimensional vector spaces over some field; then Vect has an action by C_2 given by $V \mapsto V^\vee$. Show that the homotopy fixed points Vect^{hC_2} is the category of non-degenerate symmetric bilinear forms.

The geometry of algebra and the algebra of geometry: model categories, infinity categories and spectra

The problems in this problem set are aimed at students who have seen the definitions of the objects under discussion, but may not have worked with them extensively. In the interest of getting students to the point where they can work on the problems, I recomment the following resources.

Category Theory and Model categories Dwyer and Spalinski, "Homotopy Theories and Model Categories," particularly sections 3 and 5. This also includes a quick introduction to category theory in section 2.

Simplicial sets Riehl, "A Liesurely Introduction to Simplicial Sets."

Spectra Schwede, *Symmetric Spectra*. Chapter 1, Sections 1, 4, 5.

These resources are chosen for their ease of reading and introductory nature. I have also tried to make sure that they are relatively short and allow students access to the problems as quickly as possible. They are definitely *not* comprehensive introductions to the subjects at hand.

Pedantic note

All categories are assumed to be locally small. Any category whose nerve we are taking is assumed to be small. Students attempting the problems should be careful to note where set-theoretic issues may arise, and discuss how they should be resolved.

The category **Top** of topological space is actually the category of compactly-generated Hausdorff spaces. (The category of all topological spaces is not cartesian-closed, and is therefore not well-behaved for the purposes of doing homotopy theory.)

1 Simplicial sets and nerves of categories

1.1. There exist two distinct maps $f, g\Delta^0 \longrightarrow \Delta^1$. Show that there exists a simplicial homotopy between these maps in one direction, but not the other. (In other words, there exists a simplicial homotopy from f to g, but not from g to f, or vice versa.)

1.2. Let $\mathcal{I}$ be the category $0 \longrightarrow 1$.

(a) Check that $N\mathcal{I} \cong \Delta^1$.

(b) Let $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a pair of functors, and suppose that $\alpha\colon F \Longrightarrow G$ is a natural transformation between them. Use the data of α to construct a simplicial homotopy from NF to NG.

(c) Conclude that if $F\colon \mathcal{C} \rightleftarrows \mathcal{D} :G$ is a pair of adjoint functors then $N\mathcal{C}$ is homotopy equivalent to $N\mathcal{D}$.

1.3. Let $\mathcal{O}_n$ be the full subcategory of Δ containing the objects $[0], \ldots, [n]$. Given any simplicial set $X\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ we then have its *n-th truncation* $\widetilde{X}\colon \mathcal{O}_n^{\mathrm{op}} \longrightarrow \mathbf{Set}$ given by precomposition. We define the category $s\mathbf{Set}_n \overset{\mathrm{def}}{=} \mathrm{Func}(\mathcal{O}_n^{\mathrm{op}}, \mathbf{Set})$, with natural transformations as the morphisms.

(a) Check that the n-th truncation is a functor $\tau_n\colon s\mathbf{Set} \longrightarrow s\mathbf{Set}_n$.

(b) Describe the left and right adjoints to the n-th truncation functor.

(c) Look up the definition of a monad/comonad.[1] Describe the monad and comonad constructed on $s\mathbf{Set}$ by these adjunctions. These are called the *n-skeleton* and *n-coskeleton* functors. Which is which?

(d) A simplicial set is said to be *n-skeletal* if its n-skeleton is equal to itself, and *n-coskeletal* if its n-coskeleton is equal to itself. Give an example of a simplicial set which is neither n-skeletal nor n-coskeletal.

(e) Let $\mathcal{C}$ be a category. Show that $N\mathcal{C}$ is 2-coskeletal.

1.4. Show that the functor $N\colon \mathbf{Cat} \longrightarrow s\mathbf{Set}$ is full and faithful. Can you give a condition for identifying objects contained in its essential image?[2]

1.5. For which categories $\mathcal{C}$ is $N\mathcal{C}$ a Kan complex?

1.6. Show that a filtered preorder is contractible.

1.7. Let X be a simplicial set which is k-coskeletal and for which, for all $n < k$, the lift

$$\begin{tikzcd} \Lambda^n_\ell \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

exists for all $0 \leq \ell \leq k$. Prove that X is a Kan complex.

1.8. Let $\mathcal{E}$ be a cocomplete category, and let $D\colon \Delta \longrightarrow \mathcal{E}$ be any functor. and

(a) Let $y\colon \Delta \longrightarrow s\mathbf{Set}$ be the Yoneda embedding. Describe y geometrically.

(b) Define $L\colon s\mathbf{Set} \longrightarrow \mathcal{E}$ to be the left Kan extension of D along y. Describe L in terms of coequalizers.

(c) When $\mathcal{E} = \mathbf{Top}$ and $D([n]) = \Delta^n$, describe L.

(d) Prove that L has a right adjoint R, and desribe R in the above situation.

(e) Now consider the case when $\mathcal{E} = \mathbf{Cat}$ and $D([n]) = 0 \longrightarrow \cdots \longrightarrow n$. Describe L and R; what are these usually called?

(f) Now suppose that $\mathcal{E} = s\mathbf{Set}$ and $D([n]) = N(P(n))$, where $P(n)$ is the partial order of subsets of $\{0, \ldots, n\}$. Describe L and R in this case; what are these usually called?

(g) Look up the definition of the homotopy coherent nerve. Can you construct a functor D that produces it?

1.9. Let K, L be two simplicial sets. We define $(K \times L)_n = K_n \times L_n$. Check that $|I \times I| \cong I^2$ explicitly. Prove that $|K \times L| \cong |K| \times |L|$.

1.10. Consider Quillen's Theorem A:

Theorem 1.1 (Quillen's Theorem A)**.** *Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor, and suppose that for all $D \in \mathcal{D}$, F/D is contractible. Then F is a homotopy equivalence.*

(a) Use Theorem A to show that if a category has an initial (resp. terminal) object then it is contractible.

(b) Use Theorem A to show that if $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ has a left (resp. right) adjoint then it is a homotopy equivalence.

(c) Use Theorem A to prove that $B\mathbf{N} \longrightarrow B\mathbb{Z}$ is a weak equivalence.

[1] I recommend Section 5.1 in Riehl's *Category Theory in Context*.

[2] This problem implies that it is completely natural to think of categories as spaces, as long as we are willing to consider simplicial sets as our model of spaces. Thus it is perfectly reasonable to state that "$\mathcal{C}$ is contractible" or "$F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a homotopy equivalence" in a non-ambiguous manner.

2 Model categories and simplicial categories

2.1. Let Ch_R be the category of bounded below chain complexes over a commutative ring R. (Here we grade chain complexes *homologically*, so differentials go down; you are of course welcome to do this problem with cohomologically graded chain complexes, and look at bounded-above chain complexes instead.) Define a model structure on Ch_R with

cofibrations injections with projective cokernel

fibrations surjections above degree 0

weak equivalences quasi-isomorphisms.

(a) Prove (*without* using the small object argument) that this gives a model structure on Ch_R. Prove that there exists an analogous model structure where the cofibrations are injections. (These are called, respectively, the *projective* and *injective* model structures.)

(b) Let M be an R-module. When is M cofibrant? What does the cofibrant replacement of $M[0]$ look like?

(c) Suppose we are given a Quillen pair $F\colon \mathcal{C} \rightleftarrows \mathcal{D} :G$. The *left derived functor of* F is given by composing F with a cofibrant replacement functor. Similarly, the *right derived functor of* G is given by composing G with a fibrant replacement functor. Let M be an R-module. Check that $\cdot \otimes M$ can be thought of a left Quillen functor[3] $\mathrm{Ch}_R \longrightarrow \mathrm{Ch}_{\mathbb{Z}}$ and describe its left derived functor. Can you do a similar construction for $\mathrm{Hom}(M, \cdot)$?

(d) Now modify this to produce a model structure on unbounded chain complexes $\mathrm{Ch}_R^{\geq 0}$.

2.2. Prove that a left Quillen functor preserves weak equivalences between cofibrant objects.

2.3. Let $\mathcal{C}$ be the following category, with $\mathcal{W}$ the marked subcategory:

$$\begin{tikzcd}
 & & B \arrow[dr, "\sim"] & & \\
\varnothing \arrow[r] & A \arrow[ur, "\sim"] \arrow[dr, "\sim"'] & & D \arrow[r] & * \\
 & & C \arrow[ur, "\sim"'] & &
\end{tikzcd}$$

Find all model structures on $\mathcal{C}$ with $\mathcal{W}$ as the weak equivalences.

2.4. Let $\mathcal{C}$ be a finite preorder closed under all limits and colimits; let $\mathcal{W}$ be a subcategory of $\mathcal{C}$. Prove that there exists a model structure on $\mathcal{C}$ with $\mathcal{W}$ as the weak equivalences if and only if the following two conditions hold:

- $\mathcal{W}$ satisfies 1-of-3: if f and g are composable morphisms such that $gf \in \mathcal{W}$ then $f, g \in \mathcal{W}$.
- Let $\mathcal{W}_c$ be those weak equivalences such that all pushouts are also weak equivalences, and let $\mathcal{W}_f$ be those weak equivalences such that all pullbacks are also weak equivalences. Every morphism in $\mathcal{W}$ factors as a morphism in $\mathcal{W}_c$ followed by a morphism in $\mathcal{W}_f$.

2.5. Show that all model structures on partial orders are Quillen equivalent to a model structure where all morphisms are fibrations and cofibrations and the weak equivalences are isomorphisms. Why doesn't this work for a general category?

2.6. (a) Show that in a model category $\mathcal{C}$, if there exists a zigzag of weak equivalences between two objects, then there exists a zigzag of length at most 4 between these two objects.

[3] How do you need to specify the model structures?

(b) Suppose that $\mathcal{C}$ is a partial order such that $\operatorname{Ho}\mathcal{C}$ is essentially large (not equivalent to a small category). Prove that the model structure on $\mathcal{C}$ is not cofibrantly generated.

2.7. Let $\mathcal{C}$ be a category. A *weak factorzation system* is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms in $\mathcal{C}$ such that

- ${}^{\boxslash}\mathcal{R} = \mathcal{L}$
- $\mathcal{L}^{\boxslash} = \mathcal{R}$
- For every morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$ there exists $f_L\colon A \longrightarrow X$ and $f_R\colon X \longrightarrow B$ with $f_L \in \mathcal{L}$ and $f_R \in \mathcal{R}$ such that $f_R f_L = f$.

(a) Show that in a weak factorization system $(\mathcal{L}, \mathcal{R})$, $\mathcal{L}$ is closed under retracts and pushouts and $\mathcal{R}$ is closed under retracts and pullbacks.

(b) Show that a model structure on $\mathcal{C}$ is given by a subcategory $\mathcal{W}$ of weak equivalences satisfying 2-of-3 together with two weak factorization systems $(\mathcal{L}, \mathcal{R})$ and $(\mathcal{L}', \mathcal{R}')$ such that $\mathcal{L} = \mathcal{L}' \cap \mathcal{W}$ and $\mathcal{R}' = \mathcal{R} \cap \mathcal{W}$.

(c) Classify all weak factorization systems on **Set**.

(d) Classify all model structures on **Set**.

2.8. A category $\mathcal{C}$ is a *simplicially enriched category*[4] if there exists a functor

$$\mathbf{Hom}\colon \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \longrightarrow s\mathbf{Set}$$

such that

- $\mathbf{Hom}(A, B)_0 = \operatorname{Hom}(A, B)$ for all $A, B \in \mathcal{C}$.
- $\mathbf{Hom}(A, -)$ has a left adjoint, generally denoted $\cdot \otimes A\colon s\mathbf{Set} \longrightarrow \mathcal{C}$ such that for all $K, L \in s\mathbf{Set}$, $(K \times L) \otimes A \cong K \otimes (L \otimes A)$.
- The functor $\mathbf{Hom}(-, B)$ has a left adjoint $B^{\cdot}\colon s\mathbf{Set} \longrightarrow \mathcal{C}^{\mathrm{op}}$.

(a) Prove that $s\mathbf{Set}$ is a simplicially enriched category with $\mathbf{Hom}(K, L)_n = \operatorname{Hom}(K \times \Delta^n, L)$.

(b) More generally, prove that for any simplicially enriched category $\mathcal{C}$, $\mathbf{Hom}(A, B)_n \cong \operatorname{Hom}(\Delta^n \otimes A, B)$.

(c) Prove that **Top** is a simplicially enriched category. What do the n-simplices in $\mathbf{Hom}(X, Y)$ represent?

(d) Prove that for fixed $K \in s\mathbf{Set}$, for any simplicially enriched category $\mathcal{C}$, there exists an adjuction $(K \otimes \cdot) \dashv \cdot^K$.

(e) Prove that for all $K, L \in s\mathbf{Set}$, $B^{K \times L} \cong (B^L)^K$.

2.9. Which objects in the category $s\mathbf{Cat}$ (of simplicial objects in **Cat**) arise from simplicially enriched categories?

2.10. (a) Show that a simplicially enriched category can be encoded as a functor $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}$.

(b) Which functors $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ can be obtained from simplicial categories?

2.11. Suppose that $\mathcal{C}$ is a simplicially enriched category and $X \in s\mathcal{C}$. We define

$$|X| = \operatorname{coeq}\left(\coprod_{f\colon [n] \longrightarrow [m]} \Delta^n \otimes X_m \rightrightarrows \coprod_{[n]} \Delta^n \otimes X_n \right).$$

[4] Often, by a horrible abuse of terminology, called a "simplicial category." You should be careful with this, however, since the term "simplicial category" can refer to the category Δ, a simplicially enriched category, or a simplicial object in the category of categories, three **completely distinct concepts**.

(a) Suppose that $\mathcal{C} = s\mathbf{Set}$. Prove that $|X| \cong \operatorname{diag} X$.

(b) Suppose that $\mathcal{C} = \mathbf{Top}$, and suppose that $X\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Top}$ has X_n discrete for all n. Thus we can factor $X\colon \Delta^{\mathrm{op}} \xrightarrow{X'} \mathbf{Set} \hookrightarrow \mathbf{Top}$, where the inclusion takes a set to a discrete topological space. Prove that $|X| \cong |X'|$.

(c) Suppose that X is constant. Prove that $|X| \cong X$.

(d) (*) Suppose that $X \longrightarrow Y$ is a map of bisimplicial sets such that $X_n \xrightarrow{\sim} Y_n$ is a weak equivalence for all n. Prove that the map $|X| \longrightarrow |Y|$ is a weak equivalence.[5]

2.12. We will now use the results of the previous problem to prove Quillen's Theorem A.

Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. We define the category $\mathcal{D}/F$ to have

objects $\bigcup_{D \in \mathcal{D}} \operatorname{ob} D/F$.

morphisms A morphism $(D, f\colon D \longrightarrow F(C)) \longrightarrow (D', f'\colon D' \longrightarrow F(C')$ is a pair of morphisms $g\colon C \longrightarrow C' \in \mathcal{C}$ and $g'\colon D' \longrightarrow D \in \mathcal{D}$ such that the square

$$\begin{tikzcd} D \arrow[r, "F(g)"] \arrow[d, "f"'] & D' \arrow[d, "f'"] \\ F(C) & F(C') \arrow[l, "g'"'] \end{tikzcd}$$

commutes.

(a) Let X be the bisimplicial set where $X_{p,q}$ is the set of pairs

$$(D_q \longrightarrow \cdots \longrightarrow D_0 \longrightarrow F(C_0), C_0 \longrightarrow \cdots \longrightarrow C_p).$$

The horizontal and vertical faces are induced from the nerves of $\mathcal{C}$ and $\mathcal{D}$, respectively. Use X to show that $\mathcal{D}/F$ is weakly equivalent to $\mathcal{C}$.

(b) Consider the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[d, "F"'] & \mathcal{D}/F \arrow[l] \arrow[r] \arrow[d] & \mathcal{D}^{\mathrm{op}} \arrow[d, equal] \\ \mathcal{D} & \mathcal{D}/1_{\mathcal{D}} \arrow[l] \arrow[r] & \mathcal{D}^{\mathrm{op}} \end{tikzcd}$$

where the middle vertical functor is induced by F. Use this and the result of part (a) to prove Quillen's Theorem A.

3 Spectra

For these problems, we use the category of symmetric spectra. Analogous results exist in most models of spectra.

3.1. Let $f\colon A \longrightarrow X$ be an inclusion of spaces. Prove that the sequence

$$A \longrightarrow X \longrightarrow X \cup_A CA$$

[5] This is a special case of a much more general phenomenon; however, to explore it in more detail we would need to discuss Reedy model category structures, which would take us far afield.

is a homotopy cofiber sequence by showing that any map $X \longrightarrow Y$ such that the precomposition $A \longrightarrow Y$ is null-homotopic factors through $X \cup_A CA$. Show that

$$X \longrightarrow X \cup_A CA \longrightarrow \Sigma A$$

is also a homotopy cofiber sequence. Use this to produce a long exact sequence in the cohomology represented by a spectrum E.

3.2. Show that the category of spectra is simplicial using the definition

$$\mathbf{Hom}(X, Y)_n = \mathrm{Hom}(X \wedge \Delta[n]_+, Y),$$

where for any pointed simplicial set K and spectrum E, $(E \wedge K)_n = E_n \wedge K$.

3.3. (a) Show that the category of chain complexes has a symmetric monoidal structure with unit given by $\mathbf{Z}[0]$ defined by

$$(C_\bullet \otimes D_\bullet)_n = \oplus_{i+j=n} C_i \otimes D_j.$$

Show that this product distributes over the direct sum of chain complexes, in the sense that

$$(C \oplus D) \otimes E \cong (C \otimes E) \oplus (D \otimes E).$$

(b) Show that $\wedge$ gives a symmetric monoidal structure to the category of pointed spaces and distributes over $\vee$.

(c) Consider the category of spectra as defined in Bousfield–Friedlander. We wish to construct a symmetric monoidal structure on this category with the smash product $\wedge$ as the symmetric monoidal structure. We can try to define such a smash product structure analogously to the tensor product on chain complexes by defining

$$(X \wedge Y)_n = \bigvee_{i+j=n} X_i \wedge Y_j.$$

Explain why this does *not* give a symmetric monoidal structure on spectra with $\mathbb{S}$ as the unit.

3.4. Let G be a finite group, considered as a category (i.e. the category has one object, G is the morphisms, and composition is multiplication). For any category $\mathcal{C}$, a *G-representation* in $\mathcal{C}$ is a functor $G \longrightarrow \mathcal{C}$.

(a) Let $\alpha\colon H \longrightarrow G$ be a homomorphism of groups. There is an induced functor

$$\mathrm{Res}_H^G\colon \mathrm{Rep}(G, \mathcal{C}) \longrightarrow \mathrm{Rep}(H, \mathcal{C}).$$

Show that this functor has a left adjoint; this is denoted Ind_H^G.

(b) Suppose that $\mathcal{C}$ is a symmetric monoidal category such that $\otimes$ commutes with coproducts. A *symmetric sequence* is a sequence

$$X_0, X_1, X_2, \ldots$$

such that $X_i \in \mathrm{Rep}(\Sigma_i, \mathcal{C})$. Show that the category of symmetric sequences in $\mathcal{C}$ has a symmetric monoidal structure with

$$(X \otimes Y)_n = \coprod_{i+j=n} \mathrm{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} (X_i \otimes Y_j).$$

(c) How does this definition relate to the smash product of symmetric spectra? (Hint: consider $\mathbb{S}$-modules.)

3.5. An *Ω-spectrum* is a spectrum each of whose structure maps induces a weak equivalence $X_n \longrightarrow \Omega X_{n+1}$. Look up the axioms of a generalized cohomology theory. Let E be an Ω-spectruc. Prove that the sequence of functors $h^n\colon Y \longmapsto [Y, E_n]$ is a generalized cohomology theory.[6]

[6] Brown's Representability Theorem gives the converse to this statement: every generalized cohomology theory is represented by a spectrum. However, the categories of cohomology theories and of spectra are *not* equivalent: the category of spectra is richer, and thus usually a more fruitful category to work with.

3.6. Let $X\colon \mathbf{FinSet}_* \longrightarrow \mathbf{Top}$ be a functor from the category of finite pointed sets to $\mathbf{Top}$. Write $S^1 \cong \Delta^1/\partial\Delta^1$, and define $S^n = S^1 \wedge \cdots \wedge S^1$. Note that this is a simplicial set which factors through the forgetful functor $\mathbf{FinSet}_* \longrightarrow \mathbf{Set}$; we can thus think of S^n as a functor $\Delta^{\mathrm{op}} \longrightarrow \mathbf{FinSet}_*$. Define

$$Y_n = |X \circ S^n|.$$

Note that Σ_n acts on Y_n by permuting the S^1-coordinates of S^n. Prove that Y is a symmetric spectrum.

3.7. Let R be a commutative ring spectrum. Let M be a right R-module spectrum and N a left R-module spectrum. We define

$$M \wedge_R N = \mathrm{coeq}(M \wedge R \wedge N \rightrightarrows M \wedge N).$$

There is a spectral sequence

$$E^2_{s,t} \cong \mathrm{Tor}^{\pi_* R}_{s,t}(\pi_* M, \pi_* N) \Longrightarrow \pi_*(M \wedge_R N).$$

Now let k be a field and R a k-module. Prove that $HR \wedge_{Hk} HR \cong H(R \otimes_k R)$, but $HR \wedge HR \ncong H(R \otimes_{\mathbb{Z}} R)$.

4 Replacing algebra with geometry

4.1. Check that the definition of group homology of G with coefficients in a left G-module M agrees with the homology of the bar construction $B(*, G, M)$ (considered as a simplicial set).

4.2. Let $\mathcal{V}$ be a closed symmetric monoidal category. A *category $\mathcal{C}$ enriched in $\mathcal{V}$* is

- a collection of objects $\mathrm{ob}\,\mathcal{C}$,
- for every $A, B \in \mathrm{ob}\,\mathcal{C}$, an object $\mathcal{C}(A, B) \in \mathcal{V}$,
- a *composition law*
$$\circ\colon \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, C)$$
for every triple $A, B, C \in \mathrm{ob}\,\mathcal{C}$, and
- a morphim $I \longrightarrow \mathcal{C}(A, A)$ called the *identity* for every $A \in \mathrm{ob}\,\mathcal{C}$. Here, I is the unit of the monoidal structure in $\mathcal{V}$.

(a) Write down associativity and unitality laws for composition.

(b) Check that an ordinary category is a category enriched over $\mathbf{Set}$, where $\otimes = \times$.

(c) Check that an abelian category is a category enriched over AbGp, with $\otimes$ being the tensor product of abelian groups. Why 't it work if we take the monoidal structure $(\mathrm{AbGp}, \oplus)$ instead of $(\mathrm{AbGp}, \otimes)$?

(d) Check that a simplicially enriched category is a category enriched over $s\mathbf{Set}$.

4.3. Let $X_\bullet$ be a simplcial abelian group. Define

$$C_n = \bigcap_{i=1}^{n} \ker d_i.$$

Prove that $X_\bullet \longmapsto C_\bullet$ gives a functor $s\mathbf{Set} \longrightarrow \mathrm{Ch}_{\mathbb{Z}}$. Construct an inverse functor and show that this gives an equivalence of categories.

4.4. Let R be a ring, and consider its Hochschild complex. Write down a simplicial set which has the Hochschild complex as its homology. Now use this simplicial set to produce an analogous spectrum whose geometric realization we can call $THH(R)$. (You'll need to replace R with HR.) Explain why $\pi_* THH(R) \ncong HH_*(R)$.

4.5. Use the simplicial structure on the category of spectra to explain why we can define $THH(R) \overset{\mathrm{def}}{=} S^1 \otimes HR$.

A LEISURELY INTRODUCTION TO SIMPLICIAL SETS

EMILY RIEHL

Abstract. Simplicial sets are introduced in a way that should be pleasing to the formally-inclined. Care is taken to provide both the geometric intuition and the categorical details, laying a solid foundation for the reader to move on to more advanced topological or higher categorical applications. The highlight, and the only possibly non-standard content, is a unified presentation of every adjunction whose left adjoint has the category of simplicial sets as its domain: geometric realization and the total singular complex functor, the ordinary nerve and its left adjoint, the homotopy coherent nerve and its left adjoint, and subdivision and extension are some of the many examples that fit within this framework.

1. Introduction

Simplicial sets, an extension of the notion of simplicial complexes, have applications to algebraic topology, where they provide a combinatorial model for the homotopy theory of topological spaces. There are functors between the categories of topological spaces and simplicial sets called the total singular complex functor and geometric realization, which form an adjoint pair and give a Quillen equivalence between the usual model structures on these categories.

More recently, simplicial sets have found applications in higher category theory because simplicial sets which satisfy a certain horn lifting criterion (see Definition 5.7) provide a model for $(\infty, 1)$-*categories*, in which every cell of dimension greater than one is invertible. These categories again provide a natural setting for homotopy theory; consequently, many constructions in this setting are motivated by algebraic topology as well.

These important applications will not be discussed here[1]. Instead, we aim to provide an elementary introduction to simplicial sets, accessible to anyone with a very basic familiarity with category theory. In §2, we give two equivalent definitions of simplicial sets, and in §3, we describe several examples. §4, which discusses several important adjunctions that involve the category of simplicial sets, is the heart of this paper. Rather than present the various adjunctions independently, as is commonly done, we use a categorical construction to show how all of these examples fit in the same general framework, greatly expediting the proof of the adjoint relationship. Finally, in §5, we define simplicial spheres and horns, which are the jumping off point for the applications to homotopy theory and higher category theory mentioned above.

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[1]Nor do we discuss the various model structures on the category of simplicial sets. See [DS95], [Hov99], or the original [Qui67] for an introduction to model categories and a description of the model structure relevant to topological spaces, and [Lur09] or [Rie08] for a description of the model structure relevant to $(\infty, 1)$-categories.

Because particular simplical sets called the standard simplices are represented functors, the Yoneda lemma plays an important role. We review several versions of this result in §3. The left adjoints in 4 are defined by means of a rather intricate colimit construction, though we give references for the more exotic colimits employed. Otherwise, comfort with the definitions of a category, functor, and natural transformation should suffice as prerequisites.

The standard references for the theory of simplicial sets and some of their many uses in algebraic topology are [GJ99] and [May67], though both sources quickly move on to more advanced topics. A wealth of excellent material can also be found in the classic [GZ67]. Less well known is a brief introduction given in [ML97, §7], which may serve better for the reader interested in only the most rudimentary definitions.

2. Definitions

Let $\mathbf{\Delta}$ be the category whose objects are finite, non-empty, totally ordered sets

$$[n] = \{0, 1, \ldots, n\}$$

and morphisms are order preserving functions. Equivalently, $\mathbf{\Delta}$ is the full subcategory of **Cat** whose objects are the posets defining finite, non-empty ordinals, regarded as categories in the usual manner; here, we denote the ordinal $n+1$ (above) by $[n]$ to emphasize connections with topology, which will be expounded upon below.

Definition 2.1. A *simplicial set* is a contravariant functor from $\mathbf{\Delta}$ to **Set**. More generally, for any category $\mathcal{C}$, a *simplicial object* in $\mathcal{C}$ is a functor $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$.

Let $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ be a simplicial set. It is standard to write X_n for the set $X[n]$ and call its elements *n-simplices*. We visualize an n-simplex as an n-dimensional tetrahedron with all of its lower-dimensional faces labeled by simplices of the appropriate dimension and whose $n+1$ vertices are ordered $0, \ldots, n$. Unlike an abstract simplicial complex [Hat02], the vertices (0-dimensional simplices) need not be distinct nor necessarily determine the simplex spanning them.

We write **sSet** for the category of simplicial sets, which is simply the functor category $\mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$. In particular, morphisms $f\colon X \to Y$ between simplicial sets are natural transformations: a map of simplicial sets consists of maps $X_n \to Y_n$ of n-simplices that commute with the actions by morphisms of $\mathbf{\Delta}$. In fact, it suffices to ask that these maps commute with the *face* and *degeneracy* maps defined in (2.3) below.

The combinatorial data of a simplicial set has a simpler presentation that exploits the fact that the category $\mathbf{\Delta}$ has a natural generating set of morphisms. For each $n \geq 0$ there are $n+1$ injections $d^i\colon [n-1] \to [n]$ called the *coface* maps and $n+1$ surjections $s^i\colon [n+1] \to [n]$ called the *codegeneracy* maps, defined as follows:

$$d^i(k) = \begin{cases} k, & k < i \\ k+1, & k \geq i \end{cases} \quad \text{and} \quad s^i(k) = \begin{cases} k, & k \leq i \\ k-1, & k > i \end{cases} \quad \text{for} \quad 0 \leq i \leq n.^2$$

The i-th coface map d^i misses the element i in the image, while the i-th codegeneracy map s^i sends two elements to i. These morphisms satisfy several obvious

[2] As is usual, the dependence on n is suppressed in this notation, as the codomain should be clear from the context.

relations:

$$\begin{aligned} d^j d^i &= d^i d^{j-1}, \quad i < j \\ s^j s^i &= s^i s^{j+1}, \quad i \leq j \\ s^j d^i &= \begin{cases} 1, & i = j, j+1 \\ d^i s^{j-1}, & i < j \\ d^{i-1} s^j, & i > j+1. \end{cases} \end{aligned} \tag{2.2}$$

It is not difficult to verify that every morphism of $\mathbf{\Delta}$ can be expressed as a composite of the coface and codegeneracy maps. If we impose further artificial requirements specifying the order in which the generating morphisms should occur, then each morphism of $\mathbf{\Delta}$ can be expressed uniquely as a composite of this particular form. Details are left to the reader (or see [GZ67] or [ML97, §7.5]).

If X is a simplicial set, we write

$$\begin{aligned} d_i &= Xd^i\colon\ X_n \to X_{n-1} && 0 \leq i \leq n \\ s_i &= Xs^i\colon\ X_n \to X_{n+1} && 0 \leq i \leq n \end{aligned} \tag{2.3}$$

and call these the *face* and *degeneracy* maps respectively. The d_i and s_i will then satisfy relations dual to (2.2). The face maps assign, to each $x \in X_n$, $n+1$ $(n-1)$-simplices $d_0(x), \ldots, d_n(x) \in X_{n-1}$. By convention, the face $d_i(x)$ is the one not containing the i-th vertex of x. The relation $d_i d_j = d_{j-1} d_i$ for all $i < j$ expresses the fact that, as faces of a common n-simplex, the i-th and $(j-1)$-th faces of $d_j(x)$ and $d_i(x)$ agree.

To each $x \in X_n$, the degeneracy maps associate $n+1$ $(n+1)$-simplices $s_0(x)$,..., $s_n(x) \in X_{n+1}$. The $(n+1)$-simplex $s_i(x)$ has x as its i-th and $(i+1)$-th faces; the intuition is that the projection that collapses the edge from the i-th to the $(i+1)$-th vertex to a point returns the n-simplex x. We will be more precise about these projections in the next section. We say a simplex $x \in X_n$ is *degenerate* if it is the image of some degeneracy map s_i and *non-degenerate* otherwise. By the Eilenberg-Zilber lemma [GZ67, pp 26-27], each degenerate $x \in X_n$ is uniquely expressible as $(X\epsilon)y$ for some non-degenerate $y \in X_m$ and epimorphism $\epsilon\colon [n] \twoheadrightarrow [m]$ in $\mathbf{\Delta}$.

In fact, the data of a simplicial set is completely specified by the sets X_n and the maps d_i, s_i in the sense of the following alternative definition:

Definition 2.4. A *simplicial set* X is a collection of sets X_n for each integer $n \geq 0$ together with functions $d_i\colon X_n \to X_{n-1}$ and $s_i\colon X_n \to X_{n+1}$ for all $0 \leq i \leq n$ and for each n satisfying the following relations:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, \quad i < j \\ s_i s_j &= s_{j+1} s_i, \quad i \leq j \\ d_i s_j &= \begin{cases} 1, & i = j, j+1 \\ s_{j-1} d_i, & i < j \\ s_j d_{i-1}, & i > j+1. \end{cases} \end{aligned} \tag{2.5}$$

This is the first definition given in [May67]. In practice, one usually specifies a simplicial set X by describing the sets X_n of n-simplices and then defining the required face and degeneracy maps. Mercifully, the required relations are often obvious, and even if they are not, it is still advisable to assert that they are, after privately verifying that they do in fact hold.

3. Yoneda and Examples

The simplest examples of simplicial sets are the *standard simplices* Δ^n for each $n \geq 0$, which are the represented functors for each object $[n] \in \mathbf{\Delta}$. Their universal properties will follow immediately from the Yoneda lemma, which we briefly review.

For any small category $\mathcal{C}$ and any object $c \in \mathcal{C}$, there is a *represented functor*

$$\mathcal{C}(-, c) \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

that takes an object $b \in \mathcal{C}$ to the set of arrows $\mathcal{C}(b, c)$. A morphism $g \colon a \to b$ in $\mathcal{C}$ induces a function $g^* \colon \mathcal{C}(b, c) \to \mathcal{C}(a, c)$ defined by pre-composition. This defines the functor $\mathcal{C}(-, c)$.

Furthermore, a morphism $f \colon c \to d$ in $\mathcal{C}$ defines a natural transformation

$$f_* \colon \mathcal{C}(-, c) \to \mathcal{C}(-, d)$$

defined pointwise by post-composition with f. This defines a functor

$$\mathcal{C} \hookrightarrow \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$$

called the *Yoneda embedding*. One version of the Yoneda lemma says that this functor is full and faithful, which means that natural transformations between represented functors uniquely correspond to morphisms between the representing objects.

Returning to simplicial sets, for each $[n] \in \mathbf{\Delta}$, we denote its image under the Yoneda embedding

$$y \colon \mathbf{\Delta} \hookrightarrow \mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}} = \mathbf{sSet} \quad \text{by} \quad \Delta^n := y[n] = \mathbf{\Delta}(-, [n]).$$

This is the simplicial set representing the *standard n-simplex*. From the definition, $\Delta^n_k = \mathbf{\Delta}([k], [n])$, i.e., k-simplices in Δ^n are maps $[k] \to [n]$ in $\mathbf{\Delta}$. The face and degeneracy maps d_i and s_i are given by pre-composition in $\mathbf{\Delta}$ by d^i and s^i, respectively. Explicitly:

$$d_i \colon \Delta^n_k \to \Delta^n_{k-1} \text{ is the function } [k] \xrightarrow{f} [n] \mapsto [k-1] \xrightarrow{d^i} [k] \xrightarrow{f} [n] \text{ and}$$

$$s_i \colon \Delta^n_k \to \Delta^n_{k+1} \text{ is the function } [k] \xrightarrow{f} [n] \mapsto [n+1] \xrightarrow{s^i} [k] \xrightarrow{f} [n].$$

The simplicial set Δ^n has a unique non-degenerate n-simplex, which corresponds to the identity map at $[n]$. More generally, the non-degenerate k-simplices of Δ^n are precisely the injective maps $[k] \to [n]$ in $\mathbf{\Delta}$.

The Yoneda lemma tells us that y is full and faithful, which means that simplicial set maps $f \colon \Delta^n \to \Delta^m$ correspond bijectively to morphisms $f \colon [n] \to [m]$ in $\mathbf{\Delta}$; as before, the functions $f \colon \Delta^n_k \to \Delta^m_k$ are defined by post-composition by f. We drop the convention employed above and use the same notation for both maps. For instance, the coface maps $d^i \colon \Delta^{n-1} \to \Delta^n$ play an important role in defining spheres and horns in §5.

A more robust version of the Yoneda lemma is the following.

Lemma 3.1 (Yoneda lemma)**.** *Let $c \in \mathcal{C}$ and $X \colon \mathcal{C}^{op} \to \mathbf{Set}$. Natural transformations $\mathcal{C}(-, c) \to X$ correspond bijectively to elements of the set Xc and the correspondence is natural in both variables.*

In our case, the Yoneda lemma tells us that for any simplicial set X, there is a natural bijective correspondence between n-simplices of X and morphisms $\Delta^n \to X$ in $\mathbf{sSet}$. More explicitly, an n-simplex x can be regarded as a map $x \colon \Delta^n \to X$ that

sends the unique non-degenerate n-simplex in Δ^n to x. Applying this notational convention, if x is an n-simplex, the $(n-1)$-simplex $d_i(x)$ equals xd^i:

$$x \in X_n \leftrightsquigarrow \Delta^n \xrightarrow{x} X \qquad d_i(x) \in X_{n-1} \leftrightsquigarrow \Delta^{n-1} \xrightarrow{d^i} \Delta^n \xrightarrow{x} X$$

This latter notation, which the author now prefers, can be a little tricky to become accustomed to but is then extremely convenient. Indeed, the author now prefers to think a simplicial set X as a graded set $\{X_n \mid n \in \mathbb{N}\}$ whose elements acted on the right by morphisms in $\mathbf{\Delta}$. This way, the natural notation for right action of maps in $\mathbf{\Delta}$ on simplices corresponds precisely to that of composition in the category **sSet** and one need only ever set notation for maps in $\mathbf{\Delta}$ and have the relations (2.2) in mind.

A simplicial set X has an associated *category of elements* $\mathbf{el}\, X$, in this case called the *category* of simplices, obtained by applying the Grothendieck construction to the functor $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. Objects of $\mathbf{el}\, X$ are simplices $x \in X_n$ for some n. A morphism $x \in X_n \to y \in X_m$ is given by a map $f\colon [n] \to [m]$ in $\mathbf{\Delta}$ such that $yf = x$. The *density theorem*, dual in some sense to the Yoneda lemma, says that any simplicial set is a colimit of standard simplices indexed by its category of elements

$$\mathrm{colim}_{x \in X_n} \Delta^n \cong X.$$

The map $\Delta^n \to X$ from the object in the colimit diagram indexed by $x \in X_n$ is, not surprisingly, $x\colon \Delta^n \to X$.

For example, the map $s^i\colon \Delta^{n+1} \to \Delta^n$ induced by the order-preserving surjection $s^i\colon [n+1] \to [n]$ that sends to elements to $i \in [n]$ is a form of projection: the standard simplex is mapped to one of lower dimension in such a way that the ordering of the vertices is preserved. Hence, precomposing $x\colon \Delta^n \to X$ with this yields an $(n+1)$-simplex xs^i whose i-th and $(i+1)$-th faces xs^id^i and xs^id^{i+1} equal x by (2.2).

In the remainder of this section, we will give two further examples of simplicial sets, which will be reintroduced in §4.

Example 3.2 (nerve of a category). Let $\mathcal{C}$ be any small category. Define the *nerve* of $\mathcal{C}$ to be the simplicial set $N\mathcal{C}$ defined as follows:

$N\mathcal{C}_0 = \mathrm{ob}\,\mathcal{C}$
$N\mathcal{C}_1 = \mathrm{mor}\,\mathcal{C}$
$N\mathcal{C}_2 = \{\text{pairs of composable arrows} \to\to \text{ in } \mathcal{C}\}$
$\vdots$
$N\mathcal{C}_n = \{\text{strings of } n \text{ composable arrows} \to\to \cdots \to \text{ in } \mathcal{C}\}$.

The degeneracy map $s_i\colon N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ takes a string of n composable arrows

$$c_0 \to c_1 \to \cdots \to c_i \to \cdots \to c_n$$

and obtains $n+1$ composable arrows by inserting the identity at c_i in the i-th spot. The face map $d_i\colon N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ composes the i-th and $i+1$-th arrows if $0 < i < n$, and leaves out the first or last arrow for $i = 0$ or n respectively. One can verify directly that these maps satisfy the relations from §2. However, we decline to do so because this will become obvious once we redefine the nerve of a category in §4.

Example 3.3 (total singular complex of a space). To begin, we define a *covariant* functor $\Delta\colon \mathbf{\Delta} \to \mathbf{Top}$ that sends $[n]$ to the standard topological n-simplex

$$\Delta_n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1, x_i \geq 0\}.$$

The morphisms $d^i\colon \Delta_{n-1} \to \Delta_n$ insert a zero in the i-th coordinate, while the morphisms $s^i\colon \Delta_{n+1} \to \Delta_n$ add the x_i and x_{i+1} coordinates. Geometrically, d^i inserts Δ_{n-1} as the i-th face of Δ_n and s^i projects the $n+1$ simplex Δ_{n+1} onto the n-simplex that is orthogonal to its i-th face.

Let Y be any topological space. We define a simplicial set SY by defining $SY_n = \mathbf{Top}(\Delta_n, Y)$ to be the set of continuous maps from the standard topological n-simplex to Y. Elements of this set are called n-simplices of Y in algebraic topology, which coincides with our terminology. Pre-composition by d^i induces a map of sets

$$d_i\colon \mathbf{Top}(\Delta_{n+1}, Y) \to \mathbf{Top}(\Delta_n, Y),$$

which obtains a singular n-simplex as the i-th face of a singular $n+1$-simplex. Similarly, pre-composition by s^i induces a map of sets

$$s_i\colon \mathbf{Top}(\Delta_{n-1}, Y) \to \mathbf{Top}(\Delta_n, Y),$$

whose image consists of the degenerate singular n-simplices, which are continuous functions $\Delta_n \to Y$ that factor through Δ_{n-1} along s^i. The morphisms d_i and s_i will satisfy (2.5) because the d^i and s^i satisfy the dual relations (2.2). This makes SY a simplicial set. It is called the *total singular complex* of a topological space.

In fact, both N and S are functors with a wide variety of applications. The functor S is implicit in the definition of singular homology, the n-th homology functor factoring as

$$H_n(-, \mathbb{Z})\colon \mathbf{Top} \xrightarrow{S} \mathbf{sSet} \xrightarrow{F_*} \mathbf{sAb} \xrightarrow{\sum_i (-1)^i d_i} \mathbf{Ch}_{\mathbb{Z}} \xrightarrow{H_n} \mathbf{Ab}.$$

The first functor associates the total singular complex to each space. The second maps this to the simplicial abelian group, with FSY_k the free abelian group on the k-simplices of SY. The third functor converts this into a chain complex with differentials defined to be the alternating sum of the face maps of the simplicial abelian group. The final functor takes homology of this chain complex, yielding the homology of the space Y.

The functor N can be used to model the classifying space of a group G. First, regard the group as a one object category (with the group elements as its morphisms), then take its nerve, and then its geometric realization, defined in §4. This construction is obviously functorial and also preserves finite products, making it a useful model in many contexts.

4. A unified presentation of all[3] adjunctions

In this section we will introduce several important pairs of adjoint functors with **sSet** the domain of the left adjoint. For all these examples, the same category-theoretic construction can be employed to define both left and right adjoints and to establish the adjunction. This construction works equally well with any small category in place of $\mathbf{\Delta}$, though we restrict ourselves to the level of generality needed here. After presenting the general proof, we will describe the examples of interest. First, recall

[3]whose left adjoint has domain **sSet**

Definition 4.1. An *adjunction* $L \dashv R$ consists of a pair of functors $L\colon \mathbf{sSet} \to \mathcal{E}$ and $R\colon \mathcal{E} \to \mathbf{sSet}$ together with hom-set bijections

$$\mathcal{E}(LX, e) \cong \mathbf{sSet}(X, Re)$$

for all $x \in \mathbf{sSet}$ and $e \in \mathcal{E}$, natural in both X and e.

Let $\mathcal{E}$ be any cocomplete, locally small category and $F\colon \mathbf{\Delta} \to \mathcal{E}$ any covariant functor. Using F, we may define a functor $R\colon \mathcal{E} \to \mathbf{sSet}$ by setting Re, for each $e \in \mathcal{E}$, to be the simplicial set with n-simplices

$$Re_n = \mathcal{E}(F[n], e),$$

the set of morphisms in $\mathcal{E}$ from $F[n]$ to e. As in Example 3.3, $d_i\colon Re_n \to Re_{n-1}$ is given by pre-composition by Fd^i and $s_i\colon Re_n \to Re_{n+1}$ is given by pre-composition by Fs^i. By functoriality of F, the Fd^i and Fs^i satisfy (2.2), so by contravariant functoriality of $\mathcal{E}(-, e)$, the d_i and s_i will satisfy (2.5). Thus, Re is a simplicial set. Levelwise post-composition defines a map of simplicial sets for each $e \to e' \in \mathcal{E}$ and makes R a functor.

The functor R will be the right adjoint of a functor $L\colon \mathbf{sSet} \to \mathcal{E}$, which is most concisely described as the left Kan extension of $F\colon \mathbf{\Delta} \to \mathcal{E}$ along the Yoneda embedding.

$$\begin{tikzcd}
 & \mathbf{sSet} \arrow[dr, dashed, "L"] \arrow[d, phantom, "\Uparrow\cong"] & \\
\mathbf{\Delta} \arrow[ur, hook, "y"] \arrow[rr, "F"'] & {} & \mathcal{E}
\end{tikzcd}$$

Explicitly, the value of L at a simplicial set X is a particular type of colimit called a coend. As is the case for any colimit, coends can also be described as a coequalizer between a pair of maps between certain coproducts, but this author believes the coend description makes it easier to understand the conditions imposed by the colimiting diagram.

For any set S and object $e \in \mathcal{E}$, the *copower* or *tensor* of e by S, denoted $S \cdot e$ is simply the coproduct $\coprod_S e$ of copies of e indexed by S. In particular, if X is a simplicial set, we may form copowers

$$X_m \cdot F[n]$$

for any $n, m \in \mathbb{N}$. A morphism $f\colon [n] \to [m]$ of $\mathbf{\Delta}$ induces a map

$$f_*\colon X_m \cdot F[n] \to X_m \cdot F[m],$$

which applies Ff to the copy of $F[n]$ in the component corresponding to $x \in X_m$ and includes it in the component corresponding to x in $X_m \cdot F[m]$, and also a map

$$f^*\colon X_m \cdot F[n] \to X_n \cdot F[n],$$

which maps the component corresponding to $x \in X_m$ to the component corresponding to $xf \in X_n$.

Definition 4.2. Consider the diagram whose objects are copowers $X_m \cdot F[n]$ for $m, n \in \mathbb{N}$ and whose arrows consist of morphisms f_* and f^* for each $f \in \operatorname{mor} \mathbf{\Delta}$. A *wedge* under this diagram is an object e of $\mathcal{E}$ together with maps $\gamma_n\colon X_n \cdot F[n] \to e$

such that the squares

$$\begin{tikzcd} X_m \cdot F[n] \arrow[r, "f_*"] \arrow[d, "f^*"'] & X_m \cdot F[m] \arrow[d, "\gamma_m"] \\ X_n \cdot F[n] \arrow[r, "\gamma_n"'] & e \end{tikzcd} \tag{4.3}$$

commute for each f. The *coend*, written $\int^n X_n \cdot F[n]$, is defined to be a universal wedge. Equivalently, $\int^n X_n \cdot F[n]$ is a coequalizer of the diagram

$$\coprod_{f\colon [n]\to[m]} X_m \cdot F[n] \underset{f^*}{\overset{f_*}{\rightrightarrows}} \coprod_{[n]} X_n \cdot F[n] \dashrightarrow \int^n X_n \cdot F[n]$$

On objects, the functor $L\colon \mathbf{sSet} \to \mathcal{E}$ is given by this coend:

$$LX \colon = \int^n X_n \cdot F[n].$$

If $\alpha\colon X \to Y$ is a map of simplicial sets, then α induces a wedge from the diagram (4.3) for X to the object LY; hence, the universal property of the coend defines a map

$$L\alpha\colon LX \to LY.$$

This defines the functor L on morphisms. Uniqueness of the universal property will imply that L is functorial, as is always the case when one uses a colimit construction to define a functor.

It follows from the Yoneda lemma (or really, just from the fact that y is full and faithful) that $L\Delta^n \cong F[n]$. The map $\Delta^n_m \cdot F[m] \to F[n]$ is defined to be Ff at the component corresponding to $f\colon [m] \to [n]$. It follows from functoriality of F that these define a wedge to $F[n]$. Given another wedge $\gamma_m\colon \Delta^n_m \cdot F[m] \to e$ the component of γ_n and the identity $1\colon [n] \to [n]$ defines a map $\gamma\colon F[n] \to e$.

At the component corresponding to f, $\Delta^n_m \cdot F[m] \to F[n] \to e$ is $\gamma \cdot Ff$, which equals γ_m at f because the γ_m form a wedge. This is obviously the only thing that works at the identity component.

It remains to show that L is left adjoint to R. Immediately from the definitions, we have hom-set isomorphisms

$$\mathbf{sSet}(\Delta^n, Re) \cong Re_n = \mathcal{E}(F[n], e) \cong \mathcal{E}(L\Delta^n, e), \tag{4.4}$$

the first by an application of the Yoneda lemma. A companion result to the Yoneda lemma, the density theorem mention in §3, says that every simplicial set is canonically a colimit of the standard simplices. Since L is defined by a colimit formula, it commutes with colimits. Hence the functor L is completely determined by its value on the Δ^n, and the full adjoint correspondence follows from (4.4).

We can also describe the adjoint correspondence more constructively. A morphism $\gamma\colon X \to Re$ of simplicial sets consists of maps $\gamma_n\colon X_n \to \mathcal{E}(F[n], e)$ for each n. We can use this to define morphisms $\gamma_n\colon X_n \cdot F[n] \to e$ in $\mathcal{E}$ by applying $\gamma_n(x)$ to the component of the copower corresponding to the element $x \in X_n$. We claim that the γ_n form a wedge under the diagram for X to the element e; we must show that (4.3) commutes for each $f \in \operatorname{mor} \mathbf{\Delta}$. Unravelling the definitions, this is equivalent

to commutativity of

$$\begin{tikzcd} X_m \arrow[r, "\gamma_m"] \arrow[d, "Xf"'] & \mathcal{E}(F[m], e) \arrow[d, "(Ff)^*"] \\ X_n \arrow[r, "\gamma_n"'] & \mathcal{E}(F[n], e) \end{tikzcd}$$

which is true by naturality of γ. The universal property of the coend, then gives us a map $\gamma^\flat \colon LX \to e$, which we use to define a map

$$\phi \colon \gamma \mapsto \gamma^\flat \colon \mathbf{sSet}(X, Re) \longrightarrow \mathcal{E}(LX, e).$$

Conversely, given an arrow $h \colon LX \to e$ in $\mathcal{E}$, we have maps

$$X_n \cdot F[n] \xrightarrow{\omega_n} LX \xrightarrow{h} e$$

for each n, where the ω_n are the maps of the coend wedge. We can use these to define functions

$$h_n^\sharp \colon X_n \to \mathcal{E}(F[n], e)$$

which take $x \in X_n$ to the map induced by $h\omega_n$ on the corresponding component of the coproduct. We claim that $h^\sharp \colon X \to Re$ is a map of simplicial sets. As noted above, the left hand square below commutes if and only if the right hand one does:

$$\begin{tikzcd} X_m \arrow[r, "h_m^\sharp"] \arrow[d, "Xf"'] & \mathcal{E}(F[m], e) \arrow[d, "(Ff)^*"] \\ X_n \arrow[r, "h_n^\sharp"'] & \mathcal{E}(F[n], e) \end{tikzcd} \qquad \begin{tikzcd} X_m \cdot F[n] \arrow[r, "f_*"] \arrow[d, "f^*"'] & X_m \cdot F[m] \arrow[d, "h\omega_m"] \\ X_n \cdot F[n] \arrow[r, "h\omega_n"'] & e \end{tikzcd}$$

So $h^\sharp$ is natural, since $h\omega$ is a wedge, and this defines a map

$$\psi \colon h \mapsto h^\sharp \colon \mathcal{E}(LX, e) \longrightarrow \mathbf{sSet}(X, Re).$$

It is easy to see that both ϕ and ψ are natural in X and e and are inverses. This shows that L is left adjoint to R, as desired.

Example 4.5. Post-composition turns the total singular complex defined in Example 3.3 into a functor $S \colon \mathbf{Top} \to \mathbf{sSet}$. Following the above prescription, its left adjoint $|-| \colon \mathbf{sSet} \to \mathbf{Top}$ is defined on objects by

$$|X| = \int^n X_n \times \Delta_n = \operatorname{colim} \left(\coprod_{f \colon [n] \to [m]} X_m \times \Delta_n \underset{f^*}{\overset{f_*}{\rightrightarrows}} \coprod_{[n]} X_n \times \Delta_n \right)$$

and is called the *geometric realization* of the simplicial set X. The copower $X_n \cdot \Delta_n$ of topological spaces is equivalent to the cartesian product $X_n \times \Delta_n$ where the set X_n is given the discrete topology. This coend is also knows as the *tensor product* of the functors $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X} \mathbf{Set} \xrightarrow{D} \mathbf{Top}$ and $\mathbf{\Delta} \xrightarrow{\Delta} \mathbf{Top}$, which is commonly denoted by $X \otimes_{\mathbf{\Delta}} \Delta$, ignoring the discrete topological space functor D. Such a tensor product can be defined more generally for any pair of functors $F \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ and $G \colon \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ has some monoidal product $\otimes$ to play the role of the cartesian product in **Top** and provided the desired copowers and coends exist in $\mathcal{D}$.

Example 4.6. The construction of the nerve of a category in Example 3.2 gives a functor $\mathbf{Cat} \to \mathbf{sSet}$. Let $F\colon \mathbf{\Delta} \to \mathbf{Cat}$ be the functor that sends $[n]$ to the category

$$\underset{\sim}{n} = \cdot \to \cdot \to \cdots \to \cdot$$

with $n+1$ objects and n generating non-identity arrows, as well as their composites and the requisite identities. Each order-preserving map in $\mathbf{\Delta}$ uniquely determines a functor between categories of this type by prescribing the object function, and the object functions of all such functors give rise to order preserving maps. Thus F is actually an embedding $\mathbf{\Delta} \hookrightarrow \mathbf{Cat}$. In this example, the right adjoint is commonly called the *nerve* functor $N\colon \mathbf{Cat} \to \mathbf{sSet}$, with

$$N\mathcal{C}_n = \mathbf{Cat}(\underset{\sim}{n}, \mathcal{C}),$$

or, more colloquially, the set of strings of n composable arrows in $\mathcal{C}$. The map $s_i\colon N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ inserts the appropriate identity arrow at the i-th place, and $d_i\colon N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ leaves off an outside arrow if $i = 0$ or n and composes the i-th and $(i+1)$-th arrows otherwise.

The left adjoint is typically denoted $\tau_1\colon \mathbf{sSet} \to \mathbf{Cat}$ for first truncation. Surprisingly, the image of a simplicial set X under this functor is completely determined by its 0-, 1-, and 2-simplices and the maps between them. This becomes obvious when we give an alternate description of τ_1. Given a simplicial set X, define $\operatorname{ob}\tau_1 X$ to be X_0. Morphisms in $\tau_1 X$ are freely generated by the set X_1 subject to relations given by elements of X_2 in the sense described below. The degeneracy map $s_0\colon X_0 \to X_1$ picks out the identity morphism for each object. The face maps $d_1, d_0\colon X_1 \to X_0$ assign a domain and codomain, respectively, to each arrow. To obtain $\tau_1 X$, we begin by taking the free graph on X_0 generated by the arrows X_1 and then impose the relation $h = gf$ if there exists a 2-simplex $x \in X_2$ such that $xd^2 = f, xd^0 = g$, and $xd^1 = h$:

$$\begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "h"'] & & 2 \end{tikzcd}$$

Composition is associative already in the free graph. The degenerate 2-simplices in the image of the maps s_0 and s_1 give witness to the fact that the identity arrows behave like identities with respect to composition. This makes $\tau_1 X$ a category.

It is not hard to verify explicitly that this definition of τ_1 gives a left adjoint to N; then the fact that left adjoints of a given functor must necessarily be naturally isomorphic gives an economical verification that this definition of τ_1 is compatible with the definition given above.

Example 4.7. This construction can also be used to show that **sSet** is *cartesian closed*, i.e., for every simplicial set Y, the functor $- \times Y\colon \mathbf{sSet} \to \mathbf{sSet}$ has a right adjoint. This is true for all categories of presheaves on a small category, and the construction of the right adjoint given here is the usual one.

Fix a simplicial set Y and let $F\colon \mathbf{\Delta} \to \mathbf{sSet}$ be given on objects by

$$[n] \mapsto \Delta^n \times Y$$

and on morphisms by $f \mapsto f \times 1_Y$. Recall that we mentioned that the functor L is defined to be the left Kan extension of F along the Yoneda embedding. In this

case, F is the composite of the Yoneda embedding with the functor $- \times Y$, so L is the functor $- \times Y$[4].

The right adjoint R is traditionally denoted $[Y, -]$ or $(-)^Y$ and sometimes referred to as an *internal hom*. For a simplicial set Z, the above construction gives

$$RZ_n = [Y, Z]_n = \mathbf{sSet}(\Delta^n \times Y, Z);$$

that is, n-simplices $[Y, Z]_n$ are natural transformations $\Delta^n \times Y \to Z$. $[Y, Z]$ is a simplicial set with face and degeneracy maps

$$d_i \colon [Y, Z]_n \to [Y, Z]_{n-1} \text{ given by pre-composition by } d^i \times 1 \colon \Delta^{n-1} \times Y \to \Delta^n \times Y$$
$$s_i \colon [Y, Z]_n \to [Y, Z]_{n+1} \text{ given by pre-composition by } s^i \times 1 \colon \Delta^{n+1} \times Y \to \Delta^n \times Y$$

These definitions give us the desired adjunction

$$\mathbf{sSet}(X \times Y, Z) \cong \mathbf{sSet}(X, [Y, Z]).$$

In fact, this bijection is natural in all three variables, which is easily verified.

There are several other examples of adjoint pairs of functors involving **sSet** that fit into the above template that we choose not to define here but list in case the reader may encounter these elsewhere. Descriptions of the first two can be found in [GJ99]; the third is defined in [Lur09].

Example 4.8. Groupoids can be regarded as categories where every morphism is an isomorphism; in fact the inclusion $\mathbf{Gpd} \hookrightarrow \mathbf{Cat}$ is reflective (i.e., has a left adjoint) and, more unusually, coreflective (has a right adjoint). Using inclusion and its left adjoint, the nerve functor for groupoids has the same form as above, yielding an adjunction

$$\Pi_1 \colon \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Gpd} \colon \mathbb{N}$$

The left adjoint Π_1 gives the fundamental groupoid of a simplicial set.

Example 4.9. Subdivision of a simplicial set gives a functor $\mathrm{sd} \colon \mathbf{sSet} \to \mathbf{sSet}$ such that $|\mathrm{sd}\, \Delta^n|$ is the barycentric subdivision of $|\Delta^n|$. This functor has a right adjoint

$$\mathrm{sd} \colon \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{sSet} \colon \mathrm{Ex}$$

Example 4.10. Let **sCat** denote the category of small categories enriched in **sSet** and simplicially enriched functors. The simplicial nerve of a simplicially enriched category (which is different from the nerve functor of Example 4.6) has a left adjoint

$$\mathbb{C} \colon \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{sCat} \colon \mathbb{N}$$

which, when restricted to the representable simplicial sets, can be regarded as a *simplicial thickening* or cofibrant replacement of the categories $\underset{\sim}{n}$. See [Rie11] for a considerably more detailed description of this adjunction.

[4]Alternatively, one can check directly that $X \times Y$ is a universal wedge under the diagram (4.3), with the limiting cone defined by the Yoneda lemma.

5. Spheres and Horns

We say a simplicial set Y is a subset of a simplicial set X if there is a monomorphism $Y \to X$, i.e., if $Y_n \subset X_n$ for all $[n] \in \mathbf{\Delta}$ and if

$$Xf\big|_{Y_n} = Yf$$

for all $f\colon [m] \to [n]$ in $\mathbf{\Delta}$. This second condition says that the subsets Y_n are closed under the right action by the face and degeneracy operations and furthermore that these operations agree with their definitions for X. In the presence of a simplicial set X, we often specify a simplicial subset by giving a set of *generators*, which will typically have the form of a subset $S \subset X_n$ for some n. The simplicial set *generated* by S is then the smallest simplicial subset of X that contains S. Its k-simplices will be those k-simplices of X that are in the image of S under the right action by some $f\colon [k] \to [n]$ in $\mathbf{\Delta}$.

There are a number of important simplicial subsets of the simplicial set Δ^n.

Definition 5.1. The i-th face $\partial_i \Delta^n$ of Δ^n is the simplicial subset generated by $d^i \in \Delta^n_{n-1}$. Equivalently,

$$\partial_i \Delta^n \cong \Delta^{n-1} \xrightarrow{d^i} \Delta^n.$$

Definition 5.2. The *simplicial n-sphere* $\partial\Delta^n$ is the simplicial subset of Δ^n given by the union of the faces $\partial_0\Delta^n, \dots, \partial_n\Delta^n$. Equivalently, it is the simplicial subset generated by $\{d^0, \dots, d^n\} \subset \Delta^n_{n-1}$. Alternatively, it is the colimit of the diagram consisting of the faces $\partial_i\Delta^n$ together with the morphisms that include each of the $(n-2)$-simplices that form *their* boundary into both of the faces in which each one is contained.

The sphere $\partial\Delta^n$ has the property that $(\partial\Delta^n)_k = \Delta^n_k$ for all $k < n$; all higher simplices of $\partial\Delta^n$ are degenerate. In otherwords, $\partial\Delta^n$ is the $(n-1)$-*skeleton* of Δ^n.

More generally, a *n-sphere* in X is a map $\partial\Delta^n \to X$ of simplical sets.

Definition 5.3. The *simplicial horn* Λ^n_k is the union of all of the faces of Δ^n except for the k-th face; equivalently, it is the simplicial subset of Δ^n generated by the set $\{d^0, \dots, d^{k-1}, d^{k+1}, \dots, d^n\}$. Alternatively, it can be described as a colimit in **sSet**, analogous to that for $\partial\Delta^n$ described above, except with the face $\partial_k\Delta^n$ left out of the colimiting diagram.

The horn Λ^n_k has the property that $(\Lambda^n_k)_j = \Delta^n_j$ for $j < n-1$ and $(\Lambda^n_k)_{n-1} = \Delta^n_{n-1} \backslash \{d^k\}$, with higher simplices again being degenerate.

More generally, a horn in X is a map $\Lambda^n_k \to X$ of simplicial sets.

Remark 5.4. For each of these simplicial sets, their geometric realization is the topological object suggested by their name; $|\partial_i\Delta^n|$ is the i-th face of the standard topological n-simplex $\Delta_n = |\Delta^n|$, $|\partial\Delta^n|$ is its boundary, and $|\Lambda^n_k|$ is the union of all faces but the k-th.

There are certain special types of simplicial sets, which are defined by various "horn filling" conditions. We give only the basic definitions and list a few sources for the interested reader.

Definition 5.5. A *Kan complex* is a simplicial set X such that every horn has a filler (which is not assumed to be unique). This means that for each horn $\Lambda^n_k \to X$

in X there exists an extension along the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ as shown

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

By the Yoneda lemma, the map $\Delta^n \to X$ identifies an n-simplex in X whose faces agree with those specified by the horn.

Lemma 5.6. *If X is a topological space, then SX is a Kan complex.*

Proof. By the adjunction (4.5), the diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & SX \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \quad \text{in } \mathbf{sSet} \text{ corresponds to} \quad \begin{tikzcd} |\Lambda^n_k| \arrow[r] \arrow[d, hook] & X \\ |\Delta^n| \arrow[ur, dashed] & \end{tikzcd} \quad \text{in } \mathbf{Top}.$$

A topological (n, k)-horn is a deformation retract of the standard n-simplex $\Delta_n = |\Delta^n|$, so the lift on the right hand side exists. The adjunct to this map gives us the desired lift on the left. $\square$

Kan complexes play an important role in studying the homotopy theory of **sSet**, which is closely linked to the homotopy theory of **Top** via the adjunction (4.5). For more details see [GJ99] or [May67].

Definition 5.7. A *quasi-category* or *∞-category* is a simplicial set X such that every *inner* horn, i.e., horn Λ^n_k with $0 < k < n$, has a filler.

Example 5.8. For any category $\mathcal{C}$, its nerve $N\mathcal{C}$ is a quasi-category. In fact, it is a quasi-category with the special property that every inner horn has an *unique* filler. Conversely, any quasi-category such that every inner horn has a unique filler is isomorphic to the nerve of a category.

We won't give formal proofs of these facts here (instead see [Lur09]) but we will at least provide some intuition for why the nerve of a category has a unique filler for horns $\Lambda^2_1 \to N\mathcal{C}$. This horn is often represented by the following picture:

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] & & x_2 \end{tikzcd} \quad \subset \quad \begin{tikzcd} & x_1 \arrow[dr] & \\ x_0 \arrow[ur] \arrow[rr] & & x_2 \end{tikzcd}$$

Here $f, g \in N\mathcal{C}_1$ are morphisms in $\mathcal{C}$ and $x_0, x_1, x_2 \in N\mathcal{C}_0$ are objects in $\mathcal{C}$. $fd_1 = x_0$ and $fd_0 = x_1$, colloquially, x_0 is the domain of f and x_1 is its codomain, and similarly for g. The essential point that this picture communicates is that if f and g are the generating 1-simplices of a horn $\Lambda^2_1 \to N\mathcal{C}$, then f and g are a composable pair of arrows in $\mathcal{C}$. The statement that this horn can be filled then simply expresses the fact that this pair necessarily has a composite gf. Composition is unique in a category, so this horn can be filled uniquely[5].

[5]However, in many higher categorical settings, composition is not required to be unique. This example gives a glimpse of why horn filling conditions and quasi-categories in particular are so useful for studying higher categories.

However, it will not necessarily be true that *outer* horns $\Lambda^2_0 \to N\mathcal{C}$ and $\Lambda^2_2 \to N\mathcal{C}$ will have fillers. Indeed, asking the $(2,2)$-horn depicted below has a filler is equivalent to asking that the arrow f have a right inverse, which is certainly not true in general.

$$\begin{tikzcd} & x_1 \arrow[dr, "f"] & \\ x_2 \arrow[rr, "1"'] & & x_2 \end{tikzcd} \subset \begin{tikzcd} & x_1 \arrow[dr] & \\ x_2 \arrow[ur] \arrow[rr] & & x_2 \end{tikzcd}$$

It turns out that if $\mathcal{C}$ is a *groupoid*, then these outer horns will have fillers. In fact *all* outer horns will have fillers, which says that the nerve of a groupoid is a Kan complex.

Quasi-categories were first defined by Boardman and Vogt in [BV73] under the name *weak Kan complexes*. In recent years, their theory has been developed extensively by André Joyal (see [Joy02] and, if you can find it, [Joy08]) and Jacob Lurie [Lur09].

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READING PROJECTS FOR MORROW 2019 AWS STUDY GROUP

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1. Introduction

This document is meant as a companion to Matthew Morrow's lecture notes "Topological Hochschild homology in arithmetic geometry"[1]. It consits of four "reading projects" each of which focuses on a key topic of the notes. The goal of the reading projects is to give a streamlined path to quickly approach interesting themes and theorems in Morrow's notes. This is not meant to suggest that reading the notes as they are is a bad idea, in fact they are already very streamlined. Rather, the hope is to focus on certain ideas and themes while reading the rest as well.

There is a consistent parallel between the theory of HH / HC^- / HP (and HC) in the rational setting and that of THH / TC^- / TP in positive characteristic. As a result the projects will often start in the characteristic 0 world (with results about HH and friends) before moving to characteristic p to discuss analogous results for THH, etc.

I use the currently most up to date (2nd March) numbering from the version of the notes updated 27th February.

2. Reading projects

Note: The introduction to each project uses very vague language, and is certainly up for discussion.

Reading project 1: The HKR-theorem.

Introduction: A recurring theme in the theory of (Topological) Hochschild homology is what happens when the input is *smooth*. The result is often related to differential geometry in the sense of algebraic de Rham cohomology. The Hochschild-Kostant-Rosenberg theorem states that in the smooth case the Hochschild homology groups are given by differential forms. If one restricts to the characteristic p case then there is a similar result for THH.

Goal: To understand the proof of Theorem 2.8, i.e.

Theorem (Hochschild-Kostant-Rosenberg)**.** *If A is a smooth k-algebra, then the natural maps $\Omega^n_{A/k} \to \mathrm{HH}_n(A/k)$ are isomorphisms of A-modules for all $n \geq 0$.*

Then move on to the analogous result in the topological setting, appearing as Theorem 5.6.

[1] I thank Sean Howe for suggesting that I make such a document. I have followed the style of a similar document that he made for AWS in 2017.

Theorem (Hesselholt's HKR)**.** *Let k be a perfect field of characteristic p and let R be a smooth k-algebra. Then the map of graded rings $\Omega^*_{R/k} \otimes_{\mathbb{F}_p} \mathrm{THH}_*(\mathbb{F}_p) \to \mathrm{THH}_*(R)$ is an isomorphism.*

Suggested reading guide:

(1) Section 2.1: This section contains the HKR theorem (Theorem 2.8), but the proof relies on a result about étale extensions from Section 2.4.
(2) Using the Tor-definition for HH, compute $\mathrm{HH}_*(k[x]/k)$ and check that you get the correct result, as predicted by HKR.
(3) Section 2.4, The cotangent complex and its basic properties proven in Proposition 2.27.
(4) Section 2.4, Prove Proposition 2.28, and apply it as sketched back in Remark 2.6
(5) Section 4.5, the definition of THH
(6) Section 5.1, Corollary 5.5(i). "$\mathrm{THH}(-)$ is a one-parameter deformation of $\mathrm{HH}(-/\mathbb{F}_p)$".
(7) Section 5.1, Proposition 5.6. Hesselholt's HKR theorem, using Corollary 5.5(i).

Reading project 2: The de Rham differential and S^1-actions.

Introduction: The HKR theorem (Theorem 2.8) relates Hochschild homology with differential *forms*. But differential forms assemble into a cochain complex, i.e. we have the de Rham differential $d : \Omega^n \to \Omega^{n+1}$. A natural question is then whether the Hochschild homology groups also admit such a structure. This leads to the theory of cyclic objects and/or mixed complexes. In fact the de Rham differential is a "shadow" of the circle action on Hochschild homology. From this point of view the theory of cyclic homology (and the two variants: negative/periodic cyclic homology) becomes very natural, as opposed to the *ad hoc* bicomplex constructions.

Goal: Get familiar with (various forms of) S^1-actions, and how they lead to differentials.

Suggested reading guide:

(1) Section 2.1. Go through definitions and the statement of HKR (Theorem 2.8). Skip past Remarks 2.6 and 2.7.
(2) Section 2.2. Focus on Remark 2.12.
(3) Section 2.2. Definition 2.17, Cyclic objects. Check that $\mathrm{HH}(A/k)$ (in the simplicial perspective from Remark 2.3) is a cyclic k-module.
(4) Section 2.2. Definition 2.19, Mixed complexes. Remark 2.21 shows how a cyclic object gives rise to a mixed complex.
(5) Pause and remind yourself what the singular homology $H_*(S^1, k)$ of the circle (as a topological group) is. Here k is any commutative ring. Remember to think about the product structure.
(6) Section 4.1. Lemma 4.3, in which the previous step is applied to show that mixed complexes are the same (up to homotopy) as "complexes with S^1-action".

(7) Section 4.2. On "crossed simplicial groups". Go throug example 4.5.(ii) in detail. Write out what Definition 4.4 says in this case and note that this agrees with the notion of cyclic objects.
(8) Section 4.3, where the statement $|\Lambda| \simeq BS^1$ is used to turn cyclic objects into (∞-categorical) S^1-actions.
(9) Go back to Section 4.1 and read the paragraphs after Lemma 4.3. They describe how (negative/periodic) cyclic homology arise from the algebraic S^1-action on Hochschild homology.

Alternative source: The last step of the reading guide shows how one can view negative cyclic homology as the (derived) fixed points of the circle action on the Hochschild complex. The cited paper by Hoyois gives further results in this direction.

Reading project 3: (T) HH **of** $\mathbb{F}_p$ **and perfectoid rings.**

Introduction: A fundamental calculation, due to Bökstedt, says that $\mathrm{THH}_*(\mathbb{F}_p)$ is a polynomial algebra with generator in degree two. The proof is not covered in the notes, since it requires some sophisticated topological methods. Rather, the approach is to see how (amazingly) far one can get using this as a black box. The corresponding result for $\mathrm{HH}_*(\mathbb{F}_p)$ is more approachable, and it is illuminating to consider the difference between the two.

Goal: Prove that $\mathrm{HH}_*(\mathbb{F}_p)$ is a divided power algebra on a single generator in degree 2.

Suggested reading guide:

(1) Section 2.1. Definitions.
(2) Section 2.4, starting with Example 2.32 and working backwards.
(3) The Krause-Nikolaus notes Proposition 2.6 gives the computation of $\mathrm{HH}_*(\mathbb{F}_p)$ using explicit resolutions.
(4) Compare this result with Bökstedt's result. For example consider the map $\mathrm{THH}_*(\mathbb{F}_p) \to \mathrm{HH}_*(\mathbb{F}_p)$ (e.g. following the remarks after Theorem 3.13 in the Krause-Nikolaus notes).

The perfectoid case: Using $\mathrm{THH}_*(\mathbb{F}_p) \simeq \mathbb{F}_p[u]$ as input, prove that for R perfectoid $\mathrm{THH}_*(R;\mathbb{Z}_p) \simeq R[u]$ where again u is a generator in degree two. This is Theorem 6.1 in Bhatt-Morrow-Scholze's second paper. It is a Nakayama lemma-type argument reducing to the characteristic p case and the characteristic zero case.

Alternative source: You might like to look at the notes by Höning in the Oberwolfach Report for a proof of Bökstedt's result. There is also a proof in the very recent paper by Hesselholt and Nikolaus "Topological cyclic homology", available at Hesselholt's website.

Reading project 4: Filtrations, descent, and crystalline cohomology.

Introduction: The work of Bhatt-Morrow-Scholze shows how one can use THH with its extra structure and nice properties to reconstruct crystalline cohomology. They also reconstruct other p-adic cohomology theories from THH allowing them to make comparisons between the resulting cohomologies. In this reading project we focus on the crystalline side.

Roughly this proceeds in two steps; first one uses the circle action on THH to construct TP. Secondly, one constructs a filtration on TP. This filtration exists because of a useful property of THH (and hence TP) namely *flat descent*. Now one shows that the associated graded pieces of the filtration on TP is given by complexes computing crystalline cohomology.

As usual there is an analogous story with HH (and HP).

Goal: Understand and prove Theorem 3.1 and Theorem 5.9. The latter states the following. Let k be a perfect field of characteristic p.

Theorem (BMS2)**.** *Let R be a smooth k-algebra. Then* $\mathrm{TP}(R)$ *admits a natural, complete, descending $\mathbb{Z}$-indexed filtration whose n^{th} associated graded pieces is given by*

$$gr^n \mathrm{TP}(R) \simeq R\Gamma_{crys}(R/W(k))[2n]$$

Suggested reading guide: We will try to prove statements about HH / HP and THH / TP in tandem.

(1) Propostions 2.28 (resp. Corollary 5.7) giving the HKR-filtration on HH (resp. THH).
(2) Using the HKR filtration prove that HH and HP satisfy flat descent, Lemma 3.7. Similarly, using the Hesselholt-HKR-filtration prove that THH and TP satisfy flat descent, Corollary 5.8.
(3) Section 3.1. on quasiregular semiperfect algebras and section 3.4. on the quasisyntomic site.
(4) Lemma 3.8. (resp. Lemma 5.10), dealing with filtrations on HP_0 (resp. TP_0).
(5) Theorem 3.1. (resp. Theorem 5.9), deduced from Lemma 3.8 (resp. Lemma 5.10).

Alternative sources: For additional details and perspectives on some of these results you might like to look at the Oberwolfach Report also cited in the notes. In particular the following sections are relevant:

- Niziol: proves flat descent for THH including a proof of flat descent for the cotangent complex.
- Česnavičius: identifies the associated graded pieces of the filtration on TP_0
- Elmanto: gives a proof of Theorem 5.9 using quasisyntomic descent.