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TOPOLOGY AND ARITHMETIC

Michael Hopkins
Lubin–Tate spaces: old and new questions

Jacob Lurie
Tamagawa numbers in the function field case

Matthew Morrow
Topological Hochschild homology
in arithmetic geometry

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$\mathbb{A}^1$*-enumerative geometry*

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1 Michael Hopkins: Lubin-Tate spaces: old and new questions

1.1 Lecture 1

Abelian Galois extensions of $\mathbb{Q}$ are $\mathbb{Q}(\zeta_n)$, where ζ_n is the nth root of 1.

Lubin-Tate: Get Galois extensions using formal groups. Formal group law over R. $F(x,y) = x +_F y = x + y + \cdots$

$x +_F 0 = 0 +_F x = x$ $x +_F y = y +_F x$ $(x +_F y) +_F z = x +_F (y +_F z)$

Lie variety over R Objects $\mathbb{A}^n$, $n = 0, 1, \ldots$ maps $\mathbb{A}^n \to \mathbb{A}^1$, $f(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n]$ $\mathbb{A}^n = A^n \to \prod \mathbb{A}^1$

Question: How many formal group laws are there? How to construct formal group laws?

Theorem 1.1 (Lazard). *$R \mapsto$ formal group laws over R, ring(L, R), $L = \mathbb{Z}[x_1, x_2, \ldots]$.*

Isomorphism $F \xrightarrow{g} G$ $g(x)$ $g(x +_F y) = g(x) +_F g(y)$

Universal isomorphism over $L[s_1, s_2, \ldots]$

Algebraic Topology

Cohomology theories E with Chern classes in complex line bunles, $V/X \to c_i(X) \in E^{2n}(X)$ $c_n(V\mathbb{C})_N = \sum_{i+j=n} c_i(Y)c_j(W)$

Not true in general $c_1(L_1 \otimes L_2) = c_1(L_1)$

Theorem 1.2 (Quillen). *For general, E, there exists a formal group law F $c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)) = c_1(L_1) +_F c_1(L_2)$*

$H^s(\mathfrak{m}_{FG}, \omega^*) \Rightarrow \pi_{2t-s}S^0 = \lim_{n\to\infty} \pi_{2t-s+n}S^n$ $\omega = \text{Lie } F^*$

Example 1.1. G_n $x + y$ G_m $x + y - xy = 1 - (1-x)(1-y)$

Are these isomorphic?

$g(x) = 1 - e^{-x}$ $g(x+y)$ $\overset{=}{\text{maybe}}=$ $g(x)g(x) - g(x)g(y)$ over $\mathbb{Q}$-algebra So isomorphic over rationals.

Are they isomorphic over $\mathbb{F}_p$? Are there even homomorphisms between them? Suppose $g : G_n \to G_m$ is one such $g(x + \cdots + x) = 1 - (1 - g(x))^p$ $0 = g(0) = g(x)^4 = g^0(x^p)$ so that $g = 0$ so no homomorphisms from additive group to multiplicative group.

Height:

$R = k$ field of char $p > 0$ $f : G_1 \to G_2$ Then there exists unique $g(x)$, $g'(0) \neq 0$, $y = p^a$ $f(x) = g(x^q)$ a is the height of f. Height of a formal group is by definition the height of mult by p height $G_n = \infty$ height $G_m = 1$

Theorem 1.3 (Dieudonne). *k perfect algebraically closed, any two formal groups of the same height are isomorphic.*

Lubin-Tate deformation spaces Γ, k field char $p > 0$, B complete local $\mathfrak{m}$-maximal local

A deformation of Γ to B $B \xrightarrow{r} B/\mathfrak{m} \xleftarrow{i} k$

(G, i, f), $G \xrightarrow{f} i^*\Gamma$ $\text{Deform}_\Gamma(B) \leftarrow$ groupoid

Theorem 1.4 (L-T). *$n =$ height of Γ π_0 $\text{Deform}_\Gamma(B) = \mathfrak{m}^{n-1}$*

We want to understand this set $\mathfrak{m}^{n-1}$ mod by automorphisms of Γ.

G_{univ} universal deformation $W[[u_1, \ldots, u_{n-1}]]$ W witt vectors of k Universal deformation $E_0 = W[[u_1 - u_{n-1}]]$ $E_* = W[[U_1 - u_{n-1}]][u, u^{-1}]$, $|u| = -2$

$\mathrm{Aut}\,\Gamma = S_n$, acts on E_* $UE_0 = E_{-2}$ sections of Lie G interested in $H^*(S_n; E_0)$, not the symmetric group $H^*(S_n; E_{2n})$

Question: Can one write down explicitly the action of $\mathrm{Aut}\,\Gamma$ on $W[[u_1 - u_{n-1}]]$.

Question: What is Pic(Lubin tate) $= H^1(\mathrm{Aut}\,\Gamma; E_0^*)$, conjectured answer enlists known $n = 2$, $p > 5$

Observation: $n = 2$, $H^*(S_n; W) \xrightarrow{\sim} H^*(S_n; E_0)$ $p > 3$ Shimomura $p \leq 3$ Beaudiy, Bobkova, Behrens, Wenn, … True for $n > 2$?

1.2 Lecture 2

Ex: G_m, $x+y-xy = 1-(1-x)(1-y)$ over $\mathbb{Z}_p$ $\operatorname{Aut} G_m = \mathbb{Z}_p^\times$ $\lambda \in \mathbb{Z}_p^\times$ $x \mapsto 1-(1-x)^\lambda$

Lubin-Tate ring $\mathbb{Z}_p$ $\operatorname{Aut} G_m$ acts trivially.

$E_0 = W[[u_1, \ldots, u_{n-1}]]$ $E_* = W[[u \cdots u_{n+1}[u^{\pm 1}]$, $|u| = -2$

$\mathbb{Z}_p[u^{1/2}]$ $\lambda \in \operatorname{Aut}(G_m) = \mathbb{Z}_p^\times$ How does λ act on u?

u^{-1} is an invariant differential on G_m u^{-1} is $dx + \cdots = (1-x)\,dx$

$1-(1-x)^\lambda = \lambda x + \text{hot}$ $u^{-1} \mapsto \lambda u^{-1}$.

$H^*(\mathbb{Z}_p^\times : \mathbb{Z}_p[u^{\pm 1}])$, $p > 2$ $\lambda^{-1} \in \mathbb{Z}_p^\times$ λ^{-1} generates $\mathbb{Z}_p^\times$ map $(\lambda^{-1})^{p-1} \neq 1 \mod p^2$

$H^1(\mathbb{Z}_p^\times : u^n \mathbb{Z}_p) = \mathbb{Z}_p/(\lambda^n - 1)$ $n \neq 0$

Makes sense for $\lambda \mapsto \lambda^n$ Replace by $\operatorname{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$

$n = 2$ Γ is formal group over k of height 2 $H^*(\operatorname{Aut}\Gamma; W[[u_1]]) = ?$

It turns out, $p > 3$ $\Lambda[x_1, x_3] = H^*(\operatorname{Aut}\Gamma; W) \xrightarrow{x} H^*(\operatorname{Aut}\Gamma; W[[u_1]])$

Dieudonne modules

k perfect field W ring of Witt vectors of k k acts on u, $u(x) = x^p$ W acts on u

Diuedonne-module: M free W-module of finite rank $F : u^*M \to M$ $F(am) = a^u F(m)$ for a$a \in W$,

$V : M \to u^*M$ $V(a^u m) = aUM$, $FU = UF = p$

Formal groups over $k \leftrightarrow$ diedonne modules

height $\leftrightarrow \dim_W M$ dim $\leftrightarrow \dim_k M/VM$

Ex: M basis $\gamma, U\gamma$ $F\gamma = V\gamma$

This $M \leftrightarrow$ height 2 formal group over k

$n \neq$

$\{\gamma, V\gamma, \ldots, V^{n-1}\gamma\}$ $F\gamma = V^{b-1}\gamma$

$\operatorname{Aut}\Gamma$, ht 2 $\gamma \to a\gamma + bV\gamma$ $V\gamma \to a^{\phi^{-1}}V\gamma + b^{\phi^{-1}}V^2\gamma = a^{\phi^{-1}}V\gamma + pb^{\phi^{-1}}\gamma$

$F\gamma \to a^\phi V\gamma + pb^\phi\gamma$ $a, b \in W\mathbb{F}_{p^2}$

$F\gamma = V\gamma$ $p\gamma = VF\gamma = V^2\gamma$

$\operatorname{Aut}(\Gamma) = \{ \begin{pmatrix} a & pb^{\phi^{-1}} \\ b & a^{\phi^{-1}} \end{pmatrix} : a, b \in W\mathbb{F}_{p^2} \}$.

Tapis de Cartier

$$\begin{tikzcd} M \arrow[r, dotted, "T"] \arrow[d] & W \arrow[d] \\ M/VM \arrow[r, "\sim"] & k \end{tikzcd}$$

lifts of Γ to W.

Dotted map W-linear specialization $\gamma \to 1$?

Bottom map $\gamma \mapsto 1$

$U_1(T) = T(V\gamma)$ $W[[u_1]]$

$\gamma \xrightarrow{y} a\gamma + bV\gamma$ $V\gamma \to a^{\phi^{-1}}V\gamma + pb^{\phi^{-1}}\gamma$ $y(u_1) = (a^{\phi^{-1}}u_1 + pb^{\phi^{-1}})/(bu_1 + a)$

Crystalline Approximation

$M \to E_{-2} = uW[[u_1 \cdots u_{n-1}]]$

$\gamma \to u$ $V\gamma \to uu_1 \vdots V^{n-1}\gamma \to uu_{n-1}$ $\operatorname{Aut}\Gamma$ equivalent.

G $k \leftarrow W \to u \otimes W$ $G \cong O_n$

Γ

$f(\gamma) = \log_G(x) = x + \cdots + u \otimes w[[x]]$ $f^{-1}(f(x) + f(y)) = x +_F y$

Ex: $\log_{G_m}(x) = \sum x^n/n$

$T : M \to W$ $f(x) = \sum T(F^n\gamma)x^{p^n}/p^n$ is the log of a formal ? over W
Ex: $G = G_m$ $M = ?\{\gamma\}$ $F\gamma = \gamma$
$\log = \sum x^{p^n}/p^n$
Ex: ht= 2 $T(\gamma) = 1$ $T(v\gamma) = 0$ $f(y) = \sum x^{p^{2n}}/p^n = l(x)$
$W[[w_1]]$ $T(\gamma) = 1$ $T(V\gamma) = W_1$
$f(x) = l(x) + w_1/pl(x^p)$ $f^{-1}(f(y) + f(y))$ does not have coefficients in $?[[w_1]]$
It does have coefficient in the divided power completion $w << w_1 >>$.

1.3 Lecture 3

1.4 Lecture 4

2 Jacob Lurie: Tamagawa numbers in the function field case

2.1 Lecture 1

Definition. q and q' are in the same genus if they are $\simeq$ mod N for all $N > 0$.

If q is a form over $\mathbb{Z}$ and R a commutative ring.

$$\{A \in \mathrm{GL}_n(R) : q \circ A = q\} = O_q(R) \supseteq O_q(\mathbb{R}) \supseteq O_q(\mathbb{Z})$$

a compact Lie group of dimension $n(n-1)/2$.

$$\mathrm{Mass}(q) = \sum_{q' \text{of genus } q} \frac{1}{|O_{q'}(\mathbb{Z})|},$$

where the sum is taken over equivalence classes of quadratic forms.

Definition (Unimodular). q is unimodular if nondegenerate mod p for all p

$x^2 + y^2 \equiv (x+y)^2 \mod 2$.
Mass Formula (Unimodular Case):
$8 \mid n$ $\mathrm{Mass}(q) =$ something else but

$$\mathrm{Mass}(q) = \sum_{q' \text{unimodular}} \frac{1}{|O_q(\mathbb{Z})|} = \frac{\zeta(n/2)\zeta(2)\zeta(4)\cdots\zeta(n-2)}{\mathrm{Vol}(S^1)\,\mathrm{Vol}(S^2)\cdots\mathrm{Vol}(S^{n-1})}$$

Example 2.1. $n = 8$

$$RHS = \frac{1}{2^{14}\cdot 3^5 \cdot 5^2 \cdot 7}$$

Then Mass-formula tells you there is a unique unimodular form in 3 variables.

Example 2.2. $n = 32$ RHS is approximately 40,000,000. Looking at left side, this implies there exists *a lot* of inequivalent unimodular forms in 32 variables.

Let q, q' are in the same genus. $q = q' \circ A_N$ for some $A_n \in \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z})$. WLOG $\{A_N\} = A \in \mathrm{GL}_n(\hat{\mathbb{Z}})$ $\hat{\mathbb{Z}} = \operatorname{proj\,lim} \mathbb{Z}/N\mathbb{Z} = \prod_p \mathbb{Z}_p$ $q = q' \circ A \Rightarrow q, q'$ are equivalent over $\mathbb{Z}_p$ for all p Then q, q' are equivalent over $\mathbb{Q}_p = \mathbb{Z}[1/p]$.

Hasse-Minkowski: Then $q = q' \circ B$, where $B \in \mathrm{GL}_n(\mathbb{Q})$ $q = q' \circ A = q \circ B^{-1} \circ A$ $B^{-1} \circ A \in O_q(\mathbb{Q})/O_q(A^{\mathrm{fin}})/O_q(\hat{\mathbb{Z}}))$ Want to count size of this.

$B^{-1} \circ A \in O_q(\mathbb{Q})/O_q(A)/O_q(\hat{\mathbb{Z}} \times \mathbb{R}))$

A has a natural topology that makes it into a locally compact ring containing $\mathbb{Q}$ as a discrete subring. This induces $O_q(A)$, which has the structure of a locally compact group with discrete subgroup $O_q(\mathbb{Q})$ and $O_q(\hat{Z} \times \mathbb{R})$, a compact open subgroup.

$O_q(\mathbb{Q})/O_q(\mathbb{A})$ acted on by $O_q(\hat{\mathbb{Z}} \times \mathbb{R})$

$$\#\text{ of orbits} = \frac{\mu(O_p(\mathbb{Q})\, O_q(\mathbb{A}))}{\mu(O_q(\hat{\mathbb{Z}} \times \mathbb{R}))}$$

Not quite correct.

$\mathrm{SO}_Q(A)$ has a canonical Haar measure called Tamagawa measure

$$2^k\,\mathrm{Mass}(q) = \frac{\mu(\mathrm{SO}_q(\mathbb{Q})/\,\mathrm{SO}_q(A))}{\mu(\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R}))}$$

$\mathrm{SO}_q(A) = \mathrm{SO}_q(\mathbb{R}) \times \prod_p^{\text{res}} \mathrm{SO}_q(\mathbb{Q}_p)$ $V_\mathbb{R}$ is the space of translation invariant topological forms on $\mathrm{SO}_q(\mathbb{R})$. $V_\mathbb{R} \supseteq V_\mathbb{Q}$ the space of translation invariant topological forms on $\mathrm{SO}_q(\mathbb{Q})$

$V_{\mathbb{Q}_p}$ the space of translation invariant topological forms on $\mathrm{SO}(\mathbb{Q}_p)$

$\mathrm{SO}_q(\mathbb{Q}_p)$ is a p-adic analytic Lie group.

$0 \neq \omega \in V_\mathbb{Q} \mapsto \mu_{\omega,\mathbb{R}}$

$0 \neq \omega \in V_\mathbb{Q} \mapsto \mu_{\omega,\mathbb{Q}_p}$

Tamagawa Measure

$$\mu_{\text{Tam}} = \prod_p \mu_{\omega,\mathbb{Q}_p} \times \mu_{\omega,\mathbb{R}}$$

independent of ω

$$\mathrm{Mass}(q) = 2^{-k} \frac{\mu_{\text{Tam}}(\mathrm{SO}_q(\mathbb{Q}) / \mathrm{SO}_q(\mathbb{R}))}{\mu_{\text{Tam}}(\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R}))}$$

$\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R}) = \mathrm{SO}_q(\mathbb{R}) \times \prod_p \mathrm{SO}_q(\mathbb{Z}_p)$ $\mu_{\text{Tam}}(\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R})) \stackrel{\text{def}}{=} \mu_{\omega,\mathbb{R}}(\mathrm{SO}_q(\mathbb{R})) \times \prod_p \mu_{\omega,\mathbb{Q}_p}(\mathrm{SO}_q(\mathbb{Z}_p))$

Mass Formula (Tamagawa-Weil Version) $\mu_{\text{Tam}}(\mathrm{SO}_q(\mathbb{Q}) / \mathrm{SO}_q(\mathbb{A})) = \mathbb{Z}$ SO_q has a two-sheeted double cover Spin_q

Equivalent: $\mu_{\text{Tam}}(\mathrm{Spin}_q(\mathbb{Q}) / \mathrm{Spin}_q(A)) = 1$

Conjecture (Weil)

Let G be a simply connected semisimple algebraic group over $\mathbb{Q}$ $\mu_{\text{Tam}}(G(\mathbb{Q})/G(\mathbb{A})) = 1$, where $G(\mathbb{Q})$ is τ_G, the Tamagawa number of G.

Now a theorem, proved by Weil in many cases, Langlands when split group, ...,

2.2 Lecture 2

$X \to \mathrm{Spec}(\mathbb{F}_q)$, where X smooth projective curve over $\mathbb{F}_q$, write K_X for the fraction field of X. A field which arrives this way is called a function field.

Number Fields $\mathbb{Q}$ prime numbers p and point at ∞ $\mathbb{Z}/p\mathbb{Z}$ $\mathbb{Z}_p$ $\mathbb{Q}_p$ or $\mathbb{R}$ $\mathbb{A}$ quadratic form q_0 over $\mathbb{Q}$ (SO_{q_0}) $\mathrm{SO}_{q_0}(\mathbb{Q}) \subseteq \mathrm{SO}_{q_0}(\mathbb{A})$ μ_{Tam} $\mu_{\mathrm{Tam}}(\mathrm{Spin}_{q_0}(\mathbb{Q})/\,\mathrm{Spin}_{q_0}(A))$ q quadratic form over $\mathbb{Z}$ $\mathrm{SO}_q(\mathbb{Z}/p\mathbb{Z})$ $\mathrm{Mass}(q) = \sum_{q' \text{quad at } q} \frac{1}{|O_q(\mathbb{Z})|}$ Mass Formula

Function Fields closed points $x \in X$ $k(x)$ field at x $\mathcal{O}_x$ complete local ring of X at $\mathcal{O}_x \cong k(x)[[z]]$ $K_a \sim k(x)((t))$ $\mathbb{A}_x = \prod^{\mathrm{res}}_{x \in X} K_x$ semisimple group G_0 over K_x $G_0(K_x) \subseteq G_0(A_x)$ μ_{Tam} $\mu_{\mathrm{Tam}}(G(K_x)/G_0(A_x)) = 1$ group scheme $G \to X$ (Ex: $G = X \times \mathrm{GL}_n$, $G = X \times \mathrm{SL}_n$) $G(X(x))$ $\sum_{\text{Prin G-bund P on X}} \frac{1}{|\,\mathrm{Aut}(P)|}$ Mass Formula $\sum_p \frac{1}{|\,\mathrm{Aut}(P)|} = q^D \prod_{x \in X} \frac{|??|}{|??|}$, where $d = \dim(G_0 Y K_x)$

$\mathrm{Bun}_G(X)$ the moduli stack of G-bundles

Maps: $\mathrm{Spec}\, R \to \mathrm{Bun}_G(X)$ similar G-bundle on $X \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}\, R$

Goal: Compute $\sum \frac{1}{|\,\mathrm{Aut}(S)|} =: |\,\mathrm{Bun}_G(X)(\mathbb{F}_q)|$

Digression Y algebraic variety over $\mathbb{F}_q$ $|Y(\mathbb{F}_q)|$ Idea: $\overline{Y} := Y \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}(\overline{F}_q)$ Think of $Y(\mathbb{F}_q) \subseteq \overline{Y}$ $\overline{Y} \xrightarrow{u} \overline{Y}$, where u is geometric frobenius

$$\overline{Y} \xrightarrow{u} \overline{Y}$$

$$\mathbb{P}^n \xrightarrow{u} \mathbb{P}^n$$

$[x_0 : \cdots : x_n]$, $[x_0^q : \cdots : x_n^q]$

$Y(\mathbb{F}_q)$ fixed points of u

Ideal (Weil) $|Y(\mathbb{F}_q)|$ should be $\sum(-1)^i \mathrm{Tr}(u \mid H^i(\overline{Y}))$

This is now a theorem of Grothendieck-Lefschetz Formula

Assume Y smooth of dimension d

$H^i_i(\overline{Y}) \sim H^{2d-i}(\overline{Y})^\vee$ poincare duality Not u-equivariant

$\sum(-1)^i \mathrm{Tr}(u^{-1} \mid H^i(\overline{Y})) = \frac{|Y(\mathbb{F}_q)|}{q^?}$

Idea apply this to $Y = \mathrm{Bun}_G(X)$

Definition. $Y = \mathrm{Bun}_G(x)$ satisfies the trace formula if

$$\frac{\sum \frac{1}{|\,\mathrm{Aut}(P)|}}{q^{\dim \mathrm{Bun}_G(X)} = \sum(-1)^i \frac{\mathrm{Tr}(u^{-1})}{|H^i(\overline{\mathrm{Bun}_G(x)})|} =: \mathrm{Tr}\,|u^{-1}|H^*(\overline{\mathrm{Bun}_G(X)})} \stackrel{?}{}$$

Weil's conjecture follows from two assertions

1. $\mathrm{Bun}_G X$ satisfies GL $\frac{\sum 1/|\,\mathrm{Aut}(P)|}{/} q^D = \mathrm{Tr}(u^{-1} \mid H^*(\overline{\mathrm{Bun}_G(X)})) =$

2 $\prod_{x \in X} \left(\frac{|G(k(X))|}{|K(x)|^d} \right)^{-1}$

First equality in 1. shown by theorem of Behrend in case G is a constant group, or everywhere semisimple.

Digression:

Let $x \in X$ be closed point. $\mathrm{Bun}_G(\{x\}) = BG_x$.

$\mathrm{Bun}_G(\{x\})(\mathbb{F}_q)$ is set of principle G-bundles on $\mathrm{Spec}(k(X))$ has one object, namely symmetry group is $G(K(x))$

$$\frac{|\mathrm{Bun}_G(\{x\})(\mathbb{F}_q)|}{q^{\dim \mathrm{Bun}_G(\{x\})}} = \frac{|k(x)|^d}{|G(k(x))|}$$

$\mathrm{Bun}_G(\{x\})$ satisfy GL trace formula

$$\frac{|k(x)|^d}{|G(k(x))|} = \mathrm{Tr}(u^{-1} \mid H^*(\overline{\mathrm{Bun}_G(\{x\})}))$$

$\mathrm{Tr}(u^{-1} \mid H^*(\mathrm{Bun}_G(X)) = \prod_{x \in X} \mathrm{Tr}(u^{-1} \mid H^*(\mathrm{Bun}_G(\{x\}))).$

$\mathrm{Bun}_G(X) = \prod_{x \in X}^{\mathrm{cont}} \mathrm{Bun}_G(\{x\})$

$H^?(\overline{\mathrm{Bun}_G(X)}) = \bigoplus_{x \in X}^{\mathrm{cont}} H^*(\mathrm{Bun}_G(\{x\}))$ Makes sense using theory of factorization homology

$\prod_{x \in X} \dfrac{1}{1 - 1/|k(x)|^2}$ $|\mathrm{SL}_2(\mathbb{F}_q)|/q^{\dim} = (q^3 - q)/q^3 = 1 - 1/q^2$

$\mathrm{Bun}_G(X) = \sqcup_{\ x \in \mathbb{Z}} \mathrm{Bun}_G^?(x)$

2.3 Lecture 3

$Y \to \operatorname{Spec}(\mathbb{F}_q)$ $|Y(\mathbb{F}_q)| = \sum_{y \in Y(\mathbb{F}_q) \text{iso classes}} \dfrac{1}{|\operatorname{Aut}(y)|}$

Definition. Y satisfies the G-L trace formula if

$$\frac{|Y(\mathbb{F}_q)|}{q^D} = \operatorname{Tr}(u^{-1} \colon H^*(\overline{Y})) := sum(-1)^i \operatorname{Tr}(u^{-1} \colon H^1(\overline{Y}))$$

For example, true if Y is a variety

Example: G linear algebraic group over $\mathbb{F}_q$ $Y = BG$ $\operatorname{Spec} R \to Y = BG$ equivalent to principle G-bundle on $\operatorname{Spec} R$

Example: $G = \mathbb{G}_m$ $Y = B\mathbb{G}_m$ $Y(\mathbb{F}_q)$ is the category of 1-dimensional vector spaces over $\mathbb{F}_q$ $|Y(\mathbb{F}_q)| = 1/(q-1)$

$D = \dim B|G_m$ $B\mathbb{G}_m = *//\mathbb{G}_m$ $\dim B\mathbb{G}_m = -1$

$$\frac{|Y(\mathbb{F}_q)|}{q^{\dim Y}} = \frac{q}{q-1}$$

RHS: $\operatorname{Tr}(u^{-1} \colon H^*(\overline{B\mathbb{G}_m}))$ $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ $B\mathbb{G}_m$, $B\mathbb{C}^*$, quotient of a contractible space by free action of $\mathbb{C}^*$

$\mathbb{C}^*$ acts freely on $V \setminus \{0\}$ V has dimension n $V \setminus \{0\} = S^{2n-1}$

If $\dim V = \infty$ $V \setminus \{0\}$ is contractible $B\mathbb{C}^* = (V \setminus \{0\})/\mathbb{C}^* =: \mathbb{CP}^\infty$

$H^*(\mathbb{CP}^\infty; A) \cong \Lambda[t]$, $\deg t = 2$

$H^*(\overline{B\mathbb{G}_m}) = \mathbb{Q}_\ell[t]$, where $\deg t = 2$

$u(t) = q^t$ $u(t^n) = q^n t^n$ $\operatorname{Tr}(u^{-1} \colon H^*(B\mathbb{G}_m)) = \sum_{n \geq 0} q^{-n} = q/(q-1)$

Conclusion: G-L is okay for $B\mathbb{G}_m$.

ℓ-adic homotopy:

$\overline{Y}$ is an algebraic-geometric object over an algebraically closed field $k = \overline{F}_q$ $y \in \overline{Y}(k)$

$\pi_1^{\text{et}}(\overline{Y}, y)$ profinite group. Assume $\overline{Y}$ is connected.

Then finite etale covers of $\overline{Y}$ are in correspondence with finite sets with continuous action $\pi_1^{\text{et}}(\overline{Y}, y)$.

$\pi_1^{\text{et}}(\overline{Y}, y)_\ell$ the maximal pro-ℓ quotient of $\pi_1^{\text{et}}(\overline{Y}, y)$, $\ell \neq 0$ in k

Artin-Mazur (ℓ-adic version)

Assume: $\pi_1(\overline{Y}, y)_\ell = 0$ if and only if $H^1_{\text{et}}(\overline{Y}, \mathbb{Z}/\ell) = 0$ Also: $H^n_{\text{et}}(\overline{Y}; \mathbb{Z}/\ell)$ finite.

To $\overline{Y}$, they associate a topological space Z (ℓ-adic homotopy type at y) with

1) Z is simply connected to $\pi_n Z$ is a finitely generated $\mathbb{Z}_\ell$-module for all n

2) $H^*_{\text{sing}}(Z; \mathbb{Z}/\ell) \simeq H^*_{\text{et}}(\overline{Y}; \mathbb{Z}/\ell)$

For each $n > 0$, $\pi_n(\overline{Y}) := \pi_n(Z)$ a finitely generated $\mathbb{Z}_\ell$-module

$\pi_n(\overline{Y})_{\mathbb{Q}_\ell} := \pi_n(\overline{Y})[1/\ell]$ finite dimensional vector space over $\mathbb{Q}_p$

Have a canonical pairing $b : \pi_n(\overline{Y})_{\mathbb{Q}_\ell} \times H^n(\overline{Y}) \to \mathbb{Q}_\ell$ $f : S^n \to Z$ $b(f, \eta) = f^*\eta \in H^n(S^n; \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$ $I = H^*_{\text{red}}(\overline{Y}) = \bigoplus_{n \geq 0} H^n(\overline{Y})$ $b : \pi_*(\overline{Y}) \times I \to \mathbb{Q}_\ell$ descends to a pairing $b : \pi_*(Y)_{\mathbb{Q}_\ell} \times I/I^2 \to \mathbb{Q}_\ell$, $\eta = \eta'\eta''$ $f^*(\eta) = f^*(\eta')f^*(\eta'')$

Assertion: If $H^*(\overline{Y})$ is polynomial ring (or even generators) then $\overline{b}$ is a perfect pairing.

$$\cdots \subseteq I^3 \subseteq I^2 \subseteq I \subseteq H^2(\overline{Y})$$

Ex: $\overline{Y} = \overline{B\mathbb{G}_m}$, this applies

Suppose $H(\overline{Y})$ is a polynomial ring $\overline{Y} = Y \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec}(\overline{F}_q)$

$\mathrm{Tr}(u^{-1}\colon H^*(\overline{Y})) := \sum(-1)^i \mathrm{Tr}(u^{-1}|H^i(\overline{Y}))$ $\pi_*\overline{Y}|_{\mathbb{Q}_\ell} \simeq (I/I^2)^\vee$ finite dimensional over $\mathbb{Q}_\ell$ u has complete eigenvalues $\lambda_1, \ldots, \lambda_n$ on $\pi_*(\overline{Y})_{\mathbb{Q}_\ell}$. u^{-1} has eigenvalues $\lambda_1, \ldots, \lambda_r$ $\mathrm{Tr}(u^{-1}|H^*(\overline{Y})) = \mathrm{Tr} * u^{-1}|\mathrm{gr}(H^*(\overline{Y})) = \mathrm{Tr}(u^{-1}|\operatorname{Sym}^*(I/I^2))$

$\mathrm{Tr}(u^{-1}|\operatorname{Sym}^*(I/I^2)) = \sum_{e_1,\ldots,e_n \geq 0} \lambda_1^{e_1}\lambda_2^{e_2}\cdots\lambda_n^{e_n} = \prod_{i=1}^n 1/(1-\lambda_i)$

$\mathrm{Tr}(u^{-1}|H^*(\overline{Y})) = (\det(1-u|(\pi_*\overline{Y})_{\mathbb{Q}_\ell}))^{-1}$

Ex $\overline{Y} = B\mathbb{G}_m$ $(\pi_*\overline{Y})_{\mathbb{Q}_\ell}$ 1-dimensional vector space acted on by $u = 1/q = \det(1-u) = 1 - 1/q$.

Ex: Let G be any connected linear algebraic group over $\mathbb{F}_q$ over $\mathbb{F}_q$, G-L trace formula for BG.

$$\frac{|BG(\mathbb{F}_q)|}{q^{\dim BG}} = \mathrm{Tr}(u^{-1}|H^*(\overline{BG}))$$

LHS is $q^{\dim G}/|G(\mathbb{F}_q)|$ RHS is $(\det(1-u|\pi_*(\overline{Y})_{\mathbb{Q}_\ell})^{-1}$

Steinberg's Formula $|G(\mathbb{F}_q)| = q^{\dim G}\det(1-u|\pi_*(\overline{BG})_{\mathbb{Q}_\ell})$

Ex: $G = \mathrm{GL}_n$ $H^*(\overline{BG}) = \mathbb{Q}_\ell[c_1, \ldots, c_n]$ $\pi_*(\overline{BG})_{\mathbb{Q}_\ell} = \mathbb{Q}_\ell\{e_1, \ldots, e_n\}$ $u(c_n) = q^i c_i$ $u(e_i) = q^{-i}e_i$

Steinberg $|\mathrm{GL}_n(\mathbb{F}_q)| = q^{n^2}(1-1/q)(1-1/q^2)\cdots(1-1/q^n)$

In general (not assuming $\overline{Y}$ is polynomial)

There is a spectral sequence $\operatorname{Sym}^*(\pi_*\overline{Y})^\vee_{\mathbb{Q}} \to H^*(\overline{Y})$

Gives some conclusion $\mathrm{Tr}(u^{-1}|H^*(\overline{Y})) = \det(1-u|\pi_*(\overline{Y}))^{-1} := \prod_i \det(1-u|\pi_1(\overline{Y}))^{(-1)^?}$ assuming everything converges. [For example, if $\pi_*(\overline{Y})_{\mathbb{Q}_\ell}$ is finite dimensional.]

This will apply when $Y = \mathrm{Bun}_G(X)$.

2.4 Lecture 4

2.5 Lecture 5

3 Matthew Morrow: Topological Hochschild homology in arithmetic geometry

3.1 Lecture 1

Goal: Classical Hochschild/cyclic homology Topological versions relations to alg/arith geometry

Today: Classical Theory Fix commutative base ring k For any k-algebra A have hochschild complex

$$\mathrm{HH}(A/k) := A \leftarrow A \otimes_k A \leftarrow A \otimes_k A \otimes_k A$$

given by maps $a_0 \otimes a_1 = a_0 a_1 - a_1 a_0$ $a_0 \otimes a_1 \otimes a_2 \mapsto a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 + a_2 a_0 \otimes a_1$

hochschild homology $\mathrm{HH}_n(A/k)$, $n \geq 0$, are homology of $\mathrm{HH}(A/k)$.

1. $\mathrm{HH}_0(A/k) = A/\langle ab - ba\rangle = A/[A,A] = A$ (if A is commutative)

2. If A is commutative then $\mathrm{HH}_1(A/k) = A \otimes_k A/\langle ab \otimes c - a \otimes bc + ac \otimes b \colon a, b, c, d\rangle$ (Leibniz ??? $\leftrightarrow a \otimes b$)$= \Omega^1_{A/k}$

3. $\mathrm{HH}_*(A/k) = \bigoplus_{n \geq 0} \mathrm{HH}(A/k)$ as commutative k-algebra (A-algebra) A commutative

4. 1.-s. then by universal property of $\Omega^*_{A/k} = \Lambda^*_A \Omega^1_{A/k}$

Theorem 3.1 (Hochschild-Kashent-Rosenberg, 60s)**.** *If A is smooth over k, then the maps $\epsilon_n : \Omega^n_{A/k} \to \mathrm{HH}_n(A/k)$ are isomorphisms.*

Philosophy (connes, feigir-Tsygs lodey-quillen) think of HH_* as generators of diff. forms (even if A is noncommutative).

To prove HKR, adopt homological perspective on HH.

Lemma 3.1. *For any flat k-algebra A, $\mathrm{HH}(A/k) \cong A \otimes_{A \otimes A^{op}} A$.*

Proof. Explicit isom. of complexes

$$\mathrm{HH}(A/k) \cong A \otimes_{A \otimes A^{\mathrm{op}}} \underbrace{[A \otimes_k A \leftarrow A \otimes_k \otimes_k A \otimes_k A]}_{\text{Bar complex}}$$

Bar complex is resolution of A by flat $A \otimes_k A^{\mathrm{op}}$ modules.

Corollary 3.1. $\mathrm{HH}_*(A/k) \cong \mathrm{Tor}_*^{A \otimes A^{op}}(A, A)$

Proof. (HKR thm) A smooth k-algebra must show that $\mathrm{HH}_*(A/k)$ is the exterior algebra on its degree 1 elements—this is well known for this graded algebra.

$$\mathrm{Tor}_*^B(C, C)$$

when $B \to C$ (surj?) has kernel is locally generated by a regular sequence $A \otimes_k A \to A$ (given $k \to A$ smooth).

Next: cyclic homology. $\mathrm{HH}(A/k) = A_{\mathbb{Z}/1} \leftarrow \underbrace{A \otimes_k A}_{\mathbb{Z}/2} \leftarrow \cdots$ where $\mathbb{Z}/n+1 C A^{\otimes n+1}$ generated $t_n : a_0 \otimes \cdots_n \mapsto a_n \otimes \cdots \otimes a_{n-1}$

Set 'norm' $N := \sum_{i=0}^n (-1)^n t_n : A^{\otimes n+1} \to A^{\otimes n+1}$

Extra dengenercy: $s : A^{\otimes n} \to A^{\otimes n+1}$ given by $a_0 \otimes a_n \mapsto 1 \otimes a_0 \otimes \cdots \otimes a_n$

Connes operator: $S : A^{\otimes n} \xrightarrow{N} A^{\otimes n} \xrightarrow{s} A^{\otimes n+1} \xrightarrow{1-(-1)^n t_n}$

Check: $B^2 = 0$, $Bb = -bB$, where b is the boundary map in HH. 'mixed complex or an algebraic S^1-complex.

i.e. $B : \mathrm{HH}(A/k) \to \mathrm{HH}(A/k)[-1]$

Idea: This refers to de Rham diff. commutative diagram

$$\begin{tikzcd} \mathrm{HH}_n(A/k) \arrow[r, "B"] & \mathrm{HH}_{n+1}(A/k) \\ \Omega^n_{A/k} \arrow[u, "\epsilon_n"] \arrow[r, "d"] & \Omega^{n+1}_{A/k} \end{tikzcd}$$

Def: hochschild complex

$$\begin{tikzcd} A^{\otimes 4} \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "B"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\ A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] & \\ A^{\otimes 2} \arrow[d, "b"] & A \arrow[l, "B"'] & & \\ A & & & \end{tikzcd}$$

right section $x \geq 0$ while left is ≤ 0

$\mathrm{HP}(A/K)$ (periodic cyclic homology) product totalization of this complex

$\mathrm{HC}(A/k)$ (cyclic homology) totalization of $x \geq 0$

$\mathrm{HC}^-(A/k)$ (negative cyclic homology) totalization of $x \leq 0$

$$0 \longrightarrow \mathrm{HH} \longrightarrow \mathrm{HC} \xrightarrow{s} \mathrm{HC}[2] \longrightarrow 0$$

$$0 \longrightarrow \mathrm{HC}^-[-2] \xrightarrow{s} \mathrm{HC}^- \longrightarrow \mathrm{HH} \longrightarrow 0$$

Norm sequence: $0 \longrightarrow \mathrm{HC}^- \longrightarrow \mathrm{HP} \longrightarrow \mathrm{HC}[s] \longrightarrow 0$

$$\mathrm{HP} \cong \operatorname{proj\,lim}(\cdot\,\mathrm{HC}[-4] \xrightarrow{s} \mathrm{HC}[-1] \xrightarrow{3} \mathrm{HC}) \to S : \mathrm{HP} \xrightarrow{\cong} \mathrm{HP}[z]$$

$\mathrm{HP}_n(A/k) \cong \mathrm{HP}_{n+z}(A/k)$.

coarse info about hh gives coarse info about $\mathrm{HP}, \mathrm{HC}^-, \mathrm{HC}$

Example: assume $\mathrm{HH}_{\mathrm{odd}}(A/k) = 0$, e.g. a perfectoidish. THen $\mathrm{HP}_0(A/k)$ is a complex filtered ring which ecnodes a lot of the above data. Here precisely $\mathrm{HP}_0(A/k)$ is a ring with filtered by ideals

$$\mathrm{Fil}^n \mathrm{HP}_0(A/k) = S^n(\mathrm{HC}^-_{2n}(A/k))$$

such that $\mathrm{HP}_0(A/k)/\mathrm{Fil}^n \cong \mathrm{HC}_{2n-2}(A/k)$ and $\mathrm{gr}^h \cong \mathrm{HH}_{2n}()$

3.2 Lecture 2

A brief review

A a k-algebra $\to$ Hochschild complex

$\mathrm{HH}(A/k) := A \overset{k}{\leftarrow} A^{\otimes 2} \overset{k}{\leftarrow} A^{\otimes 3} \overset{k}{\leftarrow} \cdots$

$\mathrm{HH}_n(A/k) \leftrightarrow \Omega^n_{A/k}$

Norm sequence: $0 \longrightarrow \mathrm{HC}^-(A/k) \longrightarrow \mathrm{HP}(A/k) \longrightarrow \mathrm{HC}(A/k)[2] \longrightarrow 0$

If $\mathrm{HH}_{odd}(A/k) = 0$, then also for $\mathrm{HC}^-, \mathrm{HP}, \mathrm{HC}$ and get $0 \longrightarrow \mathrm{HC}^-_{2n}(A/k) \longrightarrow \mathrm{HP}_{2n}(A/k) \longrightarrow \mathrm{HC}_{2n-2}(A/k) \to 0$

$\mathrm{HP}_{2n}(A/k) \cong \mathrm{HP}_0(A/k)$

$\mathrm{HP}_0(A/k)$ is filtered ring with associated graded $= \mathrm{HH}_{2*}(A/k)$

Main theorem about smooth algebras (loday-quillen, Feign-Tsyon,Connes) If A is a smooth k-algebra and $k \supseteq \mathbb{Q}$, then the norm sequence looks like $\prod_{i \in \mathbb{Z}} ?[2i]$, where ? is

$$0 \longrightarrow \Omega^{\geq i}_{A/k} \longrightarrow \Omega^{\cdot}_{A/k} \longrightarrow \Omega^{<i}_{A/k} \longrightarrow 0$$

,i.e. HP $\leftrightarrow$ de Rham cohomology HC^- $\leftrightarrow$ hodge filtration

Proof. Explicit map of chain complexes $\mathrm{HH}(A/k) \xrightarrow{\delta} [A \leftarrow \Omega^1_{A/k} \leftarrow \Omega^2_{A/k} \leftarrow \cdots]$ $a_0 \otimes \cdots \otimes a_n \mapsto \frac{1}{n!} a_0 da_1 \wedge \cdots \wedge da_n$ on $\mathrm{HH}_n(-)$, this splits $\epsilon : \Omega^n \to \mathrm{HH}_n$.

Because smooth algebra, maps are isomorphisms. Tells that δ is quism. So looking at double complex, replace up to quism columns by the 'dumb' complex here. Know b operator compatbile with de Rham so looks like sums of direct copies of de rham complexes. □

HC etc in characteristic p

Thm: R smooth over $\mathbb{F}_p$. Then the classical theorem is still true but the filtration is not naturally split, e.g. $\mathrm{HP}(R/\mathbb{F}_p)$ has a filtration whose graded pieces are $\Omega^i_{R/\mathbb{F}_p}[2i]$, $i \in \mathbb{Z}$.

'classical' proof: yoga if able ??

Today: Analysis if $\mathrm{HP}(R/\mathbb{F}_p)$ via perfectish map $-$ will generalize to topological case.

Idea: Don't study smooth algebra but instead quasiregular semiperfect (qrsp) $\mathbb{F}_p$-algebras $-$ big (non-noetherian) $-$ but homologically simple

Def: An $\mathbb{F}_p$-algebra A is qrsp if there exist

a perfect $\mathbb{F}_p$-algebra B, $B \xrightarrow{\cong} B$, $b \mapsto b^p$ a regular ideal $I \subseteq B$ I/I^2 is a finite projective B/I-module such that $B/I = A$.

eg of regular ideal: generated by a regular sequence

Examples: 1) $\mathbb{F}_p[t^{1/p^\infty}]/(t)$ 2) if R smooth $\mathbb{F}_p$-algebra then its perfection $R_{\mathrm{perf}} := \mathrm{inj}\lim R$, where limit over $x \mapsto x^p$ then $R_{\mathrm{perf}} \otimes_R \cdots \otimes_R R_{\mathrm{perf}}$ is qrsp

eg, $\mathbb{F}_p[t]_{\mathrm{perf}} \otimes_{\mathbb{F}_p[t]} \mathbb{F}_p[t]_{\mathrm{perf}} \cong \mathbb{F}_p[t^{1/p^\infty}] \otimes_{\mathbb{F}_p[t]} \mathbb{F}_p[t^{1/p^\infty}] = \mathbb{F}_p[t_1^{1/p^\infty}, t_2^{1/p^\infty}]/(t_1 - t_2)$

Technique: form qrsp to smooth all of our homology theories $F = \mathrm{HH}(-/\mathbb{F}_p), \mathrm{HC}(-/\mathbb{F}_p)$, etc, $\mathbb{F}_p$-algebra $\to D(\mathbb{F}_p)$

satisfy flat descent, meaning $S \to S'$ is a faithfully flat map of $\mathbb{F}_p$-algebras. then $F(S) \to \mathrm{Tot}(F(S') \overset{\to}{\to} F(S' \otimes_S S') \cdots)$

eg, R smooth over $\mathbb{F}_p$, then $R \to R_{\mathrm{perf}}$ is faithfully flat.

So: $F(R) \to \mathrm{Tot}(F(R_{\mathrm{perf}})) \to\to F(R_{\mathrm{perf}} \otimes_R \to R)$

all are qrsp.

Most understand The other homologies $\mathrm{HC}^-, \mathrm{HP}, \mathrm{HC}$ of any group qrs $\mathbb{F}_p$;agebar$\mathbb{Q}_p$

Let A be qrsp

Step 1: $\mathrm{HH}_{odd}(A/\mathbb{F}_p) = 0$ and $\mathrm{HH}_0(A)$ $\mathrm{HH}_k = I/I^2 I$, where $A = B/I$ $\mathrm{HH}_{2n} = \Gamma^n_A(I/I^2)$ nth divided power of I/I^2

$\cong \mathrm{Sym}^n_A(I/I^2)$ but mult. is twised by ntm $m!, n!/(n!m!)$

$\to \mathrm{HH}_{2*}(A/\mathbb{F}_p) \cong \Gamma^{A^*}(I/I^2)$ Key words; cotangent complex.

Step 2: $\mathrm{HP}_0(A/\mathbb{F}_p)$ is a filtered ring with associated graded $\Gamma^*_A(I/I^2)$ What is it?

ANswer: hcc divided power enveolppe of completed B onto A, regular defin. able $f^n/n!$ to surject to A.

Computation: $\mathrm{HP}(R/\mathbb{F}_p)$ (R smooth) is buitl from copies of

$$\mathrm{Tot}(\mathrm{HP}_0(R_{\mathrm{perf}}/\mathbb{F}_p)) \to\to \mathrm{HP}_0(R_{\mathrm{perf}} \otimes \cdots$$

, where R smooth, contributed by divided power enevelopes

ALso show up in theory of derived de Rham cohomolology (Bhatt)

so tot is $\cong \Omega^{\cdot}_{R/\mathbb{F}_[}$

3.3 Lecture 3

3.4 Lecture 4

3.5 Lecture 5

4 Kirsten Wickelgren: $\mathbb{A}^1$-enumerative geometry

4.1 Lecture 1

Enumerative Geometry: counts algebraic-geometric objects satisfying conditions over $\mathbb{C}$
Goal: To record information about fields of definition
Arithmetic count of the lines on a smooth cubic surface

Definition. Cubic surface is $\{(x,y,z)\colon f(x,y,z)=0\}$, where f is degree 3

Better: $X \subseteq \mathbb{P}^3 = \{[w,x,y,z]\}$, $[w,x,y,z]=[\lambda w,\lambda x,\lambda y,\lambda z]$, where $\lambda \in K^\times$ $X=\{[w,x,y,z]\colon f(w,x,y,z)=0\}$

Theorem 4.1 (Salmon, Cayley 1849). *Let X be a smooth, cubic surface over $\mathbb{C}$. Then X contains exactly 27 lines.*

Example 4.1. Fermat $f(w,x,y,z)=w^3+x^3+y^3+z^3$ $L=\{[S,-S,T,-T]\colon [S,T]\in\mathbb{P}^1\}$ $\lambda,\omega : \lambda^3=\omega^3=-1$
Lines $\{[S,\lambda S,T,\omega T]\colon S,T\in\mathbb{P}^1\}$ This produces $3\cdot 3\cdot 3=27$ llines

Modern proof: $\mathrm{Gr}(1,3)$, the Grassmannian parametrizing lines in $\mathbb{P}^3$, equivalently $W\subseteq\mathbb{C}^4$, $\dim W=2$
Let $S\to\mathrm{Gr}(1,3)$ be the tautological bundle, $S_W=W$ $\mathrm{Sym}^3 S^*\to\mathrm{Gr}(1,3)$ $\mathrm{Sym}^3 S^*_W$ is the cubic polynomial on W, i.e. $\mathrm{Sym}^3 W^*$ F determines element $\mathrm{Sym}^3(\mathbb{C}^4)^*$ then f determines a section of $\mathrm{Sym}^3 S^*$ by $\sigma_f(W)=f|_W$
Note: the line $\mathbb{P}W$ corresponding to W is in $X\Leftrightarrow\sigma_f(W)=0$.
Want: to count zeros of σ_f Euler class: $V\to M$ be a rank r $\mathbb{R}$-vector bundle on a dimension r $\mathbb{R}$-manifold M. Assume V is oriented
Choose a section σ with only isolated zeros. $\deg[S^{r-1},S^{r-1}]\to\mathbb{Z}$, homotopy classes of maps
$P\in M$, $\sigma(p)=0$ To define: $\deg_p\sigma\in\mathbb{Z}$
Here's how: choose local coordinates on M around p. There's a small ball around p with no other zeros. Choose local trivialization of V. Then σ can be identified with a function $\sigma:\mathbb{R}^r\to\mathbb{R}^r$ given by $0\mapsto 0$, $\sigma(\overline{B_0(1)}=0)\subset\mathbb{R}^r=0$ $S^{r-1}=OB_0(1)\xrightarrow{\overline{\sigma}}\partial B_0(1)=S^{r-1}$ given by $x\mapsto\sigma(x)/|\sigma(x)|$
Then $\deg_p\sigma=\deg(\overline{\sigma})$
Euler class $e(V)=\sum_{p:\sigma(p)=0}\deg_p\sigma$
Fact: X smooth then $\deg_p\sigma=1$, then number of lines on $X=e(\mathrm{Sym}^3 S^*)$. In particular, number of lines is independent of X $e(\mathrm{Sym}^3 S^*)=27$
Question: What about cubic surfaces over $\mathbb{R}$?
Segre in 20th century showed X can have 3, 7, 15, or 27 real lines.
Segre 1942 distinguished between different hyperbolic and elliptic real lines on X
Recall: L real line, $L\cong\mathbb{P}^1_{\mathbb{R}}$, $\mathrm{Aut}(L)\cong\mathbb{P}\mathrm{GL}_2(\mathbb{R})$ $I\leftrightarrow I=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ $z\mapsto\dfrac{az+b}{cz+d}$ $\mathrm{Fix}(I)=\{z\colon cz^2+(d-a)z+b=0\}$ either consists of 3 real points if and only if I hyperbolic. a ?? conjugate pair of points elliptic.
We associate an involution I to $L\subset X$ a real line on a real cubic surface.
$p\in L$ $T_pX\cap X=L\cup Q$ $Q\cap L=$ points q such that $T_qX=T_pX=\{p,p'\}$ $I(p)=p'$

Definition. L is elliptic/hyperbolic when I is

Alternatively, spin structure.

Example 4.2. Fermat cubic surface $x^3 + y^3 + z^3 = -1$ hyperbolic

Theorem 4.2 (Segre,Okonek,Teleman,...). *Number of hyperbolic lines $-$ number of elliptic lines $= 3$.*

$\mathbb{A}^1$-homotopy theory (due to Morel-Voemsky)

On smooth schemes over k, k a field. Morel deg: $[\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \to GW(k)$ where $GW(k)$ is the Grothendieck-Witt=group completion of semiring $\oplus, \otimes$ isomorphism classes of (nondegenerate symmetric) bilinear forms $B : V \times V$, finite dimensional k vector spaces

presentation: generators: $\langle a \rangle$, $a \in k^\times$ $\langle a \rangle : k \times k \to k$, $(x, y) \mapsto axy$ relations: $\langle ab^2 \rangle = \langle a \rangle$, $b \in k^\times$ $\langle a \rangle + \langle b \rangle = \langle a + b \rangle = \langle ab(a + b) \rangle$

Example 4.3. $GW(\mathbb{C}) \cong \mathbb{Z}$, rank, $B \mapsto \dim V$

Example 4.4. $GW(\mathbb{R}) \to \mathbb{Z} \times \mathbb{Z}$ by signatures $\times$ rank. Also iso to $\mathbb{Z} \times \mathbb{Z}$ $GW(\mathbb{F}_q) \to \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$ given by signatures $\times$ rank, isomorphism

There is an Euler class

$$e(V) = \sum_{p:\sigma(p)=0} \deg_p \sigma$$

R field char not 2. X a smooth cubic surface over k line $L \subseteq X$ is a closed point of $\mathrm{Gr}(1,3)$ $L = \{[a,b,c,d]S + [a',b',c',d']T \colon [S,T] \in \mathbb{P}^1\}$ $k(L) = k(a,b,c,d,a',b',c',d')$ $\mathbb{P}^1_{k(L)} \cong L \subseteq X_{k(L)} \subseteq \mathbb{P}^3_{k(L)}$ Given a line L on X, obtain involution $I \in \mathrm{Aut}(L) \cong \mathbb{P}\,\mathrm{GL}_2\, k(L)$ $\mathrm{Fix}(I)$ is either 2 $k(L)$ points or a conjugate pair of points in $k(L)[\sqrt{D}]$ for $D \in k(L)^*/(k(L)^*)^2$

Definition. $\mathrm{Type}(L) := \langle D \rangle \in GW(k(L))$

Equivalent to $D = ab - cd$, $I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\mathrm{Type}(L) = \langle -1 \rangle \deg I$

Theorem 4.3 (Kass-W.). *char $R \neq 2$, X smooth cubic surface*

$$\sum_{lines L of X} \mathrm{Tr}_{k(l)/k} \mathrm{Type}(L) = 15\langle 1 \rangle + 12\langle -1 \rangle$$

$\mathrm{Tr}_{k(L)/k} : GW(k(L)) \to GW(k)$ *given by* $(B : V \times V \to k(L)) \mapsto V \times V \xrightarrow{B} k(L) \to k$

$R = \mathbb{C}$, apply rank, number of lines is 27 $k = \mathbb{R}$ apply signature Number of hyperbolic lines $-$ number elliptic lines $= 3$

Corollary 4.1. *$k = \mathbb{F}_q$ Number of elliptic lines L with $k(L) = \mathbb{F}_{q^{2n+1}}$ plus number of hyperbolic lines with $k(L) = \mathbb{F}_{q^{2n+1}}$ is equivalent to 0.*

4.2 Lecture 2

User's guide to $\mathbb{A}^1$-homotopy theory

Want: $\mathbb{P}^n/\mathbb{P}^{n-1}$, colimit

Ex:

$$\begin{tikzcd} \mathbb{P}^{n-1} \arrow[r] \arrow[d] & \mathbb{P}^n \arrow[d] \\ * \arrow[r] & \mathbb{P}^n/\mathbb{P}^{n-1} \end{tikzcd}$$

Ex: Open sets U, V

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \\ V \arrow[r] & U \cup V \end{tikzcd}$$

Want to glue, crash schemes like topological spaces

treat smooth schemes like manifolds construction of $\mathbb{A}^1$-homotopy theory (Morel-Voensky)

Sm_R = smooth schemes$/k$ $\mathrm{Sm}_k \to \mathrm{Func}(\mathrm{Sm}_k^{\mathrm{op}}, \mathrm{sset})$ $Y \mapsto \mathrm{Mor}(-, Y)$

Homotopy theory can mean: simplical model category or ∞-category $\mathrm{Pre}(\mathrm{Sm}_k) = \mathrm{Func}(\mathrm{Sm}_k^{\mathrm{op}}, \mathrm{sset})$ freely adding relations

Problem: had colimits from 2 in Sm_k

Fix: force certain classes of maps to be weak equivalences. Bousfield localization

For an open cover $B = \sqcup_k U_k \to X$ force $\mathrm{cosk}_x^{\cdot} \sqcup_x U_x \overset{\sim}{\to} X$ $\mathrm{Pre}(\mathrm{Sm}_k) \xrightarrow{L\tau} \mathrm{Sh}_R$, τ a Grothendieck topology.

(more open sets left to right) Choices: Zariski topology, Niesnevich, étale topology

Def: $f : X \to Y$ (not necessarily smooth, Sm_k) is etale at x if $T_x X \xrightarrow{\sim} T_{f(x)} Y$

Def: $U = \sqcup_x U_x \to X$ is an etale cover if it is etale and surjective

Def: $U = \sqcup_x U_x$ is a Nisnevich cover if it is an etale cover and for every $x \in X$ there exists $u \in U$ such that $u \mapsto x, k(x) \xrightarrow{\sim} k(u)$

Nice properties: $Z \hookrightarrow X$ in Sm_k can often be viewed as $\mathbb{A}^d \to \mathbb{A}^n$

$\mathrm{Sm}_k \longrightarrow \mathrm{PSh}_k = \mathrm{Func}(\mathrm{Sm}_l^{\mathrm{op}}, \mathrm{sset}) \xrightarrow{L_?} \mathrm{Sh}_k \xrightarrow{L_?} \mathrm{Spc}_k$ where last arrow force $X \times \mathbb{A}^1 \xrightarrow{\sim} X$ Spc_k is $\mathbb{A}^1$-homotopy theory.

Spheres:

Def: Given pointed spaces X, Y $X \wedge Y := (X \times Y)/(X * *U * *Y)$

Ex: $S^n \wedge S^m = S^{n+m}$

Spheres: S^1, $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ $S^{p+q\alpha} = S^{p+q,q} = (S^1)^{\wedge p} \wedge (\mathbb{G}_m)^?$

Ex:

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{A}^1 \simeq * \arrow[d] \\ \mathbb{A}^1 \simeq * \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

then $\mathbb{P}^1 \simeq \Sigma \mathbb{G}_m = S^1 \wedge \mathbb{G}_m$

Ex: $\mathbb{A}^n \setminus \{0\} \simeq (S^1)^{\wedge n-1} \wedge (\mathbb{G}_m)^{\wedge n}$ induction and

$$\begin{tikzcd} (\mathbb{A}^{n-1} \setminus \{0\}) \times (\mathbb{A}^{n-1} \setminus \{0\}) \arrow[r] \arrow[d] & (\mathbb{A}^{n-1} \setminus \{0\}) \times \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^n \times \mathbb{A}^1 \setminus \{0\} \arrow[r] & \mathbb{A}^1 \setminus \{0\} \end{tikzcd}$$

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & \Sigma X \wedge Y \end{tikzcd}$$

Ex: $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq (S^1)^{\wedge n}(\mathbb{G}_m)^{\wedge n}$ $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq \mathbb{P}^n/\mathbb{P}^h \setminus \{0\} \simeq \mathbb{A}^n/\mathbb{A}^n \setminus \{0\} \simeq */\mathbb{A}^n \setminus \{0\} \simeq \Sigma(\mathbb{A}^n \setminus \{0\})$.

Thom Space: Let $V \to X$ algebraic vector space $\mathrm{Th}(V) = V/V - X \simeq \mathbb{P}(V \oplus \mathcal{O})/\mathbb{P}(V)$

$S \in$ sset $S \in \mathrm{Pre}(\mathrm{Sm}_k) = \mathrm{Func}(\mathrm{Sm}_k^{\mathrm{op}}\ \mathrm{sset})$

Purity Theorem: $Z \hookrightarrow X$ closed immersion in Sm_k $X/X - Z \simeq \mathrm{Th}(N_Z X)$

Ex: $\mathrm{Spec}\, k \hookrightarrow X$, where X is a smooth scheme U open neighborhood of z $U/U - z \simeq \mathbb{P}^n_{k(x)}/\mathbb{P}^{n-1}_{k(x)} \simeq \mathbb{P}^n/\mathbb{P}^{h-1} \wedge (\mathrm{Spec}\, k(z)+)$

Compare: z point on manifold, U small ball around z $\Sigma\partial U \simeq U/U - z$

$\mathrm{GW}(k), k^m_*(k)$:

$\mathrm{GW}(k)$ is group completion of isomorphism classes of symmetric, nondegenerate bilinear forms over k, $\otimes$ gives ring structure

Generators: $\langle a \rangle, a \in k^*$ $\langle a \rangle : k \times k \to k$ given by $(x, y) \mapsto axy$ relations: $\langle a, b^* \rangle = a$ $\langle a \rangle \langle b \rangle = \langle ab \rangle$ $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$

then $h := \langle 1 \rangle + \langle -1 \rangle = \langle a \rangle + \langle -a \rangle$ for all a.

Acted on by hyperbolic forms

rank: $\mathrm{GW}(k) \to \mathbb{Z}$ given by $B : V \times V \to k \mapsto \dim V$

Fundamental ideal: $I := \ker \mathrm{rank}$

$\mathrm{GW}(k) \supseteq I \supseteq I^2 \supseteq \cdots$ $K^M_i = \bigoplus_{i=0}^{\infty} \bigotimes_{j=1}^{i} k^*/\langle a \otimes (1 - a) \rangle$, Milnor K-theory groups

Milnor conjecture, theorem of Voedosky $1 \to \mathbb{Z}/? \to k^* \to k^* \to 1$ $k^* \to H^1_{\text{ét}}(k, /\mathbb{Z}/?)$ $I^n/I^{n+1} \xleftarrow{\sim} K^M_n(k) \xrightarrow{\sim} H^n_{\text{ét}}(k, \mathbb{Z}/?)$ with left map $a_0 \otimes \cdots \otimes a_n \mapsto (\langle 1 \rangle - \langle a_1 \rangle) \cdots (\langle 1 \rangle - \langle a_n \rangle)$

view maps $I^n \to I^n/I^{n+1}$ as invariants on $\mathrm{GW}(k)$ $n = 0$ rank $n = 1$ discriminant $n = 2$ hasse-witt invariant

$B : V \times V \to k$ disc $B = \det(B(v_i, v_j))$ $\{v_1, \ldots, v_n\}$ is a basis $n = 3$ Arason invariant

$K^{MW}_*(k)$ Milnor-Witt k-theory (hopkins-morel)

Generators: $[a]$ $a \in k^*$ $\deg 1, \deg -1$ η

relations: $\eta[a] = [a]\eta$ $[a][1 - a] = 0$ (Steinberg relation) $[ab] = [a] + [b] + \eta[a][b]$ $\eta h = 0$

$\mathrm{GW}(k) \cong K^{MW}_0(k)$ $\langle a \rangle \mapsto 1 + \eta[a]$ $h = \langle 1 \rangle + \langle -1 \rangle \mapsto h = 2 + \eta[-1]$

Degrees Theorem (Morel) $n \geq 2$

$[(S^1)^{\wedge n} \wedge \mathbb{G}_m^{\wedge j}, (S^1)^{\wedge n} \wedge \mathbb{G}_m^{\wedge ?}] \cong K^{MW}_{r-j}$ eg $[\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \cong \mathrm{GW}(k)$, $j = r = n$ $R = \mathbb{R}$

$$\begin{tikzcd} {[S^{2n}, S^{2n}]} \arrow[d, "\deg"] & {[\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}]} \arrow[l, "G-pts"'] \arrow[r, "R-pts"] \arrow[d, "\deg"] & {[S^n, S^n]} \arrow[d, "\deg"] \\ \mathbb{Z} & \mathrm{GW}(K) \arrow[l, "rank"] \arrow[r, "signature"] & \mathbb{Z} \end{tikzcd}$$

$\mathrm{GW}(?) K^{MW}_*(k) K^M_y(k)$ are global sections of sheaves.

Procedure for producing a sheaf K^{MW}_* from $K^{MW}_*(E)$, E finite type over k field.

plus data $V : E \to \mathbb{Z} \cup \{\infty\}$ valuation $\mathcal{O}_V = \{e \in E \colon v(e) \geq 0\}$ π uniformizer $v(\pi) = 1$ $k(V) := \mathcal{O}_V/\langle \pi \rangle$

$\partial^\pi_V : K^{MW}_*(E) \to K^{MW}_{*-1}(k(v)$ $\partial^\pi_V([\pi][a_1] \cdots [a_n]) = [\bar{a}_1] \cdots [\bar{a}_n]$ $q_i \in \mathcal{O}^*_V$ $\partial^\pi_V([a_1] \cdots [a_n]) = 0$

Correction: $\delta^\pi_x \eta = \eta \delta^\pi_V$ $\delta^\pi_x \eta = 0$

4.3 Lecture 3

Sheaves: K_*^{MW}, GW $GW(\operatorname{Spec} L \to \operatorname{Spec} L) =$ restruction $\mathcal{O}_K L$ or bilinear forms

Transfers: $K \subseteq L$ finite extension of finite schemes over R

$\operatorname{Tr}_{L/K} : GW(L) \to GW(K)$

geometric transfer, cohomological transfer, absolute transfer depends on generators, does not, twisted $\operatorname{Tr}_{L/K}(B : V \times V \to L) = V \times V \xrightarrow{B} \to L \xrightarrow{\operatorname{Tr}_{L/K}} K$ when $K \subseteq L$ is separable.

geometric: $L = k[z]/\langle f \rangle$

$\operatorname{Spec} L \overset{z}{\hookrightarrow} \mathbb{P}^1_k$ $\mathbb{P}^1_k \to \mathbb{P}^1_k/\mathbb{P}^1 \setminus \{z\}$ is a map $\operatorname{Tr}^{\text{geo}}_{L/K} GW(L) \to GW(K)$

CH: Chow groups $X \in \mathrm{Sm}_k$

$X^{(i)}$ is codimension i reduced, irreducible subschemes of X

$\mathrm{CH}^i(X) = \bigoplus_{X^{()}} \mathbb{Z}/\text{rational equivalence}$

$V \subset X \times \mathbb{P}^1$ $V \wedge (X \times \{0\}) \sim V \wedge (X \times \{1\})$

useful in enumerative geometry: chern classes, pushforward pullbacks, ring structure Bloch $\mathrm{CH}^i * X (= H^i(X, K_i^M)$

Oriented Chow groups or Chow-Witt group

$\tilde{\mathrm{CH}}^i(X) = H^i(X, K_i^{MW})$ elements are formal rank $Z \in X^{(i)}$ and

Barge-Morel

Computed by Rost-Schmidt complex

$\oplus_{Z \in X^{(i+1)}} K_1^{MV}(k(x), \det) \to \oplus_{z \in X^i} GW(k(z), \det_{k(z)} T_z X) \to \oplus_{z \in X^{(i+1)}} K_{-1}^{MW}(k(z), \det)$

Fasel, M. Levine: pullbacks $f : X \to Y$ pushforward, non-commutative ring structure

E field, Λ 1-dimensional E vector space $K_i^{MW}(E, \Lambda) = K_i^{MW}(E) \otimes \mathbb{Z}\, \mathbb{Z}[E]$ $\tilde{\mathrm{CH}}^i(X, L) = H^i(X, K_i^{MW}(i))$

$L \to X$ line bundle $f : X \to Y$ proper $\dim Y - \dim X = r$ $f_* \tilde{\mathrm{CH}}(X, \omega_{X/Z} \otimes f^* y)$ $\det TX \to \tilde{\mathrm{CH}}^{i-r}(Y, w_{Y/Z} BX)$

Degree via local degree

Algebraic topology

$f : S^n \to S^n$ $p \in S$ $\deg f = \sum \deg_q f$ $f^{-1}(p) = \{q_1, \ldots, q_n\}$

Differential topology formula for $\deg_{x_i} f$ choose coordinates $x_1, \ldots, x_n$ near q_i $y_1, \ldots, y_n$ near P $f : \mathbb{R}^n \to \mathbb{R}^n$

$\operatorname{Jac} f = \deg \frac{\partial f_i}{\partial x_i}$

$$\deg_{q_i} f = \begin{cases} 1, & \text{if } \operatorname{Jac} f > 0 \\ -1, & \text{if } \operatorname{Jac} f < 0 \end{cases}$$

$\mathbb{A}^1$-alg topology

Lanes/Morel: $f : \mathbb{P}^1 \to \mathbb{P}^1/k$ $p \in \mathbb{P}^1(k)$ $f^{-1}(p) = \{q_1, \ldots, q_n\}$ $\deg f = \sum (\operatorname{Jac}_{q_i} f) \in GW(k)$

this does not depend on P

Prop: (Global degree is a sum of local degrees)

$f : \mathbb{P}^n \to \mathbb{P}^n$ finite $f^{-1}(\mathbb{A}^n) = \mathbb{A}^n$ $\mathbb{P}^n/\mathbb{P}^{n-1} \xrightarrow{\overline{f}} \mathbb{P}^n/\mathbb{P}^{n-1}$

$\deg \overline{f} = \sum_{q \in f^{-1}(p)} \deg_q^{\mathbb{A}^1} f$ $p \in \mathbb{A}^n(k)$

where $\deg_q^{\mathbb{A}^1} f$ is degree of composite

$$\mathbb{P}^n/\mathbb{P}^{n-1} \cong U/U - q \to \mathbb{A}^n/\mathbb{A}^n - p \cong \mathbb{P}^n/\mathbb{P}^{n-1}$$

$k(q) = k$, $\mathrm{Th} N_p \mathbb{A}^*$

otherwise $\mathbb{P}^n/\mathbb{P}^{n-1} \to \mathbb{P}^n/\mathbb{P}^n - q$

If f is étale at q, then $\deg_q^{\mathbb{A}^1} f = \mathrm{Tr}_{k(q)/k}\langle \mathrm{Jac}\, f(q)\rangle$ and $k(q) \supseteq R$ separable.

A: Eisenbud-Lenine-Khinskdhflskhdfklashdgahsdgliadg lsdighasdgh Signature formula

$f : \mathbb{R}^n \to \mathbb{R}^n$ $0 \mapsto 0$ isolated zero

$\deg f =$ signature ω^{EKL}

ω^{EKL} is a bilinear form $Q = \mathbb{R}[x_1, \dots, x_n]_0 / \langle f_1, \dots, f_n\rangle$

$\mathrm{Jac}\, f \in Q$ pick any $\eta : Q \to \mathbb{R}$ $\mathbb{R}$-linear so that $\eta(\mathrm{Jac}\, f) = \dim Q$ $\omega^{EKL} : Q \times Q \to \mathbb{R}$ $(a, b) \mapsto \eta(ab)$

Q (Eisenbud): ω^{EKL} could be a degree even replacing $\mathbb{R}$ with K. DOes this have an interpratation?

Thm: (Kass-W.) $\deg_0^{\mathbb{A}^1} f = \omega^{EKL}$

Project: remove $k(x) = k$hypothesis

Ex: ω^{EKL} for $f(x) = x^2$ $Q = k[x]/\langle x^2\rangle$ basis $\{1, x\}$ $\mathrm{Jac}\, f = \partial x$ $\eta : k[x]/\langle x^1\rangle \to k$ $\eta(2x) = 2$ $\eta(1) = 0$

	1	x
1	0	1
x	1	0

$\omega^{EKL} = \langle 1\rangle + \langle -1\rangle$

$\mathbb{A}^1$-milnor numbers

joint with Jesse Kass

Def: A point p on a scheme X is a node if after base change to $k^?$ $\hat{\mathcal{O}}_{X,p} \cong k^s[[x_1, \dots, x_n]]/(x_1^2 + \cdots + x_n^2 + \mathrm{hot})$

Let X be a hypersurface $X = \{f = 0\} \subseteq \mathbb{A}^n$ $p \in X$ be a singularity As X is perturbed in a family P bifercates into nodes for $(a_1, \dots, a_n)$ have a family of hypersurfaces $f(x_1, \dots, x_n) + a_1 x_1 + \cdots + a_n x_n = f$ parametrized by $+$

Milnor $k = \mathbb{C}$ FOr any sufficiently small $(a_1, \dots, a_n)$ the family contains $\mu(P)$ nodes

$\mu(P) =$ milnor $\# = deg(gradf)(p)$

When k is not algebraically closed, nodes p contain arithmetic data

$R = \mathbb{R}$

nonsplit node, i.e. tangent directions not defiend over k

split node

Def: The type of a node $p \in \{f = 0\}$ $\deg_p^{\mathbb{A}^1} gradf$

Ex: Choose preimage of p after base change to $k(p)$

$\hat{\mathcal{O}}_{X,p} = k[[p]][[x_1, \dots, x_n]]/(a_1 x^2 + \cdots + a_n x^2 + \mathrm{hot})$

$\mathrm{type}(p) = \mathrm{Tr}_{k(p)/k}\langle 2^n a_1 a_2 \cdots a_n\rangle$, $k(p)/k$ always a seprable extension.

Ex: $\mathrm{type}(x^2 + y^2) = \langle a\rangle$

Def: p hypersurface singularity $p \in \{f = 0\}$ $M^{\mathbb{A}^1} = \deg_p gradf$

Thm: For generic $(a_1, \dots, a_n)$ Crass-W.

$\sum_{X nodes in family} type(X) = M^{\mathbb{A}^1}(p)$ in GW(k).

Ex: $f(x, y) = y^2 - x^2$ $gradf = (-3x^2, 2y)$ $M^{\mathbb{A}^1}(0) = \deg_0 gradf = \deg_0(x \mapsto -3x^1) \deg_0(y \mapsto 2y) = \langle -3\rangle(\langle 1\rangle + \langle -1\rangle)\langle 2\rangle = \langle -6\rangle + \langle 6\rangle = \langle 1\rangle + \langle -1\rangle = h$

Family parametrized by $+$ $y^2 = x^3 + ax + t$

$a = 0$

nodes occur when $x^3 + ax + t$ has a double root iff $= 27t^2 - 4a^3$

$\mathbb{F}_3 : \langle 1\rangle = \langle -1\rangle$ in a famiyl cant have one split and one nonsplit rational nodes

$\mathbb{F}_7 : \langle 1\rangle \neq \langle -1\rangle$ cant have 2 split or 2 nonsplit nodes.

4.4 Lecture 4