



Syracuse University

# MATH 701: Real Variables I

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## 0 Introduction

### 0.1 Course Description

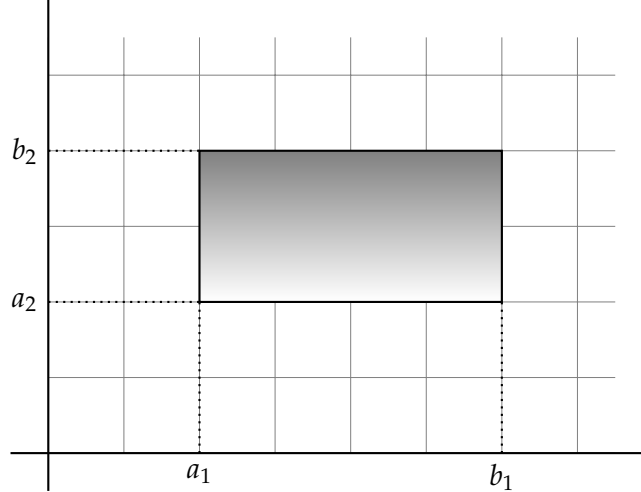
**MAT 701 Real Variables I:** Measure and integration, including basic theorems on integration and differentiation of sequences of functions; modes of convergence, product measures.

### 0.2 Disclaimer

These notes were taken in Fall 2018 in a course taught by Professor Leonid Kovalev. In some places, notation/material has been changed or added. Any errors in this text should be attributed to the typist – Caleb McWhorter – and not the instructor or any referenced text.

## 1 Outer Measure & Lebesgue Measure

We want to assign a notion of ‘size’ to sets. We denote this ‘size’ by  $\nu$ . Let  $\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$  denote ordinary Euclidean space. By an ‘interval’ in  $\mathbb{R}^n$ , we mean a set  $\{(a_1, b_1) \times (a_2, b_2) \times \dots \times (a_n, b_n) : a_i, b_i \in \mathbb{R}^n, a_i < b_i\}$ . By a closed interval in  $\mathbb{R}^n$ , we mean a set  $\{[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n] : a_i, b_i \in \mathbb{R}^n, a_i < b_i\}$ . We will often say ‘box’, which will always mean an open or closed interval in  $\mathbb{R}^n$ .



In defining a ‘size’ for sets, it makes sense to begin with a simple shape like a box. In the case of the plane, we know the a good notion of size is the area, and the area of a box  $[a_1, b_1] \times [a_2, b_2]$  is  $(b_1 - a_1) \cdot (b_2 - a_2)$ . We can immediately generalize this to  $\mathbb{R}^n$  as follows: if  $I \subset \mathbb{R}^n$  is an interval, then we define

$$\nu(I) := \prod_{j=1}^n (b_j - a_j) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$$

But the question remains, how do we generalize this to an arbitrary region  $E$ ? Given the above definition, it is natural to try to generalize to arbitrary sets  $E$  by approximating  $E$  by boxes, i.e. an open covering  $\{I_k\}$  of  $E$  by intervals (see Figure 1).

It then becomes clear that whatever the measure of  $E$  is, it should satisfy

$$\nu(E) \leq \sum_k \nu(I_k),$$

since the intervals cover  $E$ . After all, it would be strange indeed to allow  $E$  to have greater measure than its covering. Furthermore, if a notion of measure is going to be defined in terms of coverings, then the measure need be invariant of the choice of covering. These ideas are our guiding principles. So we take the following definition

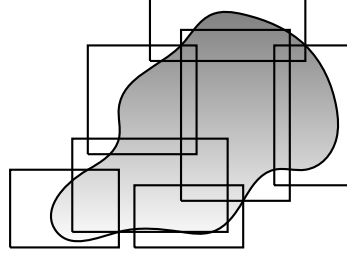


Figure 1: Approximating a set  $E$  by intervals.

**Definition (Outer Measure).** For a set  $E \subset \mathbb{R}^n$ , the outer measure (or exterior measure) of  $E$ , denoted  $|E|_e$ , is the function  $\nu : \mathbb{R}^n \rightarrow [0, \infty)$  given by

$$|E|_e := \inf_{E \subset \bigcup_k I_k} \sum \nu(I_k),$$

where the infimum is taken over all *countable* coverings  $\{I_k\}$  of  $E$ .

The coining of the term ‘outer measure’ is immediately obvious—we are measuring the size of a set via external objects, namely the open cover. However, less obvious is the need to restrict to countable coverings. The need to eliminate uncountable coverings is apparent, as then trouble arises defining the summation. But why not only allow finite coverings? For this consider the case  $E = \mathbb{Q} \cap [0, 1]$ . In any finite covering of  $[0, 1]$  by intervals, the intervals cannot all be pairwise disjoint. The reader will confirm, with a bit of thought, that if the intervals were all pairwise disjoint then there would be a rational number missed by the ‘covering.’ But this contradicts the fact that the collection was an open cover. The only possible open covering by intervals is the entire interval itself so that  $\inf \sum \nu(I_k) = 1$ . This violates the notation that there isn’t any ‘length’ or ‘area’ here since we have a sparse collection of points.

Furthermore, the same logic applies to the set  $E' = \mathbb{Q}^c \cap [0, 1]$ . So if the outer measure of  $E$  were 1, then this would be true too of  $E'$ . But clearly the outer measure of  $[0, 1]$  is 1. Now  $[0, 1] = E \cup E'$ , and  $E \cap E' = \emptyset$ . As  $1 + 1 \neq 1$ , this breaks countable subadditivity of the measure we are trying to define. By defining the outer measure in terms of countable covers, we obtain the expected answer  $|E|_e = 0$ .

**Example 1.1.** Let  $E = \mathbb{Q} \cap [0, 1]$  and  $\epsilon > 0$  be given. Since  $\mathbb{Q}$  is countable, so too is  $E$  countable. Enumerate the rationals in  $E$  as  $\{q_1, q_2, \dots, q_n, \dots\}$ . Now the set  $\{O_n\}_{n \in \mathbb{N}}$ , where  $O_n := (q_n - \frac{1}{2^{n+k+1}}, q_n + \frac{1}{2^{n+k+1}})$  and  $k \in \mathbb{N}$  is fixed, is a (countable) open covering of  $E$  by intervals. Furthermore, the  $O_n$  are pairwise disjoint. Choose  $k$  sufficiently large so that  $2^{-k} < \epsilon$ . The measure of this covering is then

$$\sum_{n=1}^{\infty} \nu(O_n) = \sum_{n=1}^{\infty} \left[ \left( q_n + \frac{1}{2^{n+k+1}} \right) - \left( q_n - \frac{1}{2^{n+k+1}} \right) \right] = \sum_{n=1}^{\infty} 2 \cdot \frac{1}{2^{n+k+1}} = \frac{1}{2^k} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2^k} < \epsilon.$$

◁

It is also important to note that our notion of measure preserves our notion of ‘size’ for intervals. Before proving this in the general case, we begin with the case for intervals in  $\mathbb{R}$ .

**Proposition 1.1.**  $|[a, b]|_e = b - a$ .

*Proof.* Clearly,  $|[a, b]|_e \leq b - a$ , so it remains to show that  $b - a \leq |[a, b]|_e$ . Suppose that  $[a, b]$  is a finite union of intervals of the form  $[c_j, d_j]$ . There exists  $j_1$  such that  $c_{j_1} \leq a$ . If  $d_{j_1} \geq b$ , then  $|[a, b]|_e \geq d_{j_1} - c_{j_1} \geq b - a$ , and we are done. Otherwise, it must be that  $d_{j_1} < b$ . There then exists  $j_2$  such that  $c_{j_2} \leq d_{j_1}$ . Continue this process inductively until one finally obtains  $d_{j_r} \geq b$ . But then taking the sum of these differences, one obtains a telescoping series

$$\underbrace{(d_{j_r} - c_{j_r})}_{\geq 0} + \underbrace{(d_{j_{r-1}} - c_{j_{r-1}})}_{\geq 0} + \cdots + \underbrace{(d_{j_1} - c_{j_1})}_{\geq 0} \geq b - a.$$

□

We now prove this for general intervals.

**Proposition 1.2.** For an interval  $I$ ,  $|I|_e = \nu(I)$ .

*Proof.* As  $I$  serves as an open covering for itself, we know that  $|I|_e \leq \nu(I)$ . It remains to show that  $\nu(I) \leq |I|_e$ . Suppose  $\{I_k\}_k$  is a countable open covering of  $I$ , and let  $\epsilon > 0$ . Every interval  $I$  is contained in the interior of a slightly larger interval  $I'$ , i.e.  $I \subseteq \text{Int}(I')$ , where  $\nu(I') - \nu(I) < \epsilon$ . Let  $I'_k$  denote an interval containing  $I_k$  in its interior with  $\nu(I'_k) \leq (1 + \epsilon)\nu(I_k)$ . We know that  $I \subset \bigcup_k \text{Int}(I'_k)$ . Now  $I$  is compact so that Heine Borel implies that there is an integer  $N$  such that  $I \subset \bigcup_{k=1}^N I'_k$ . We know  $\nu(I) \leq \sum_{k=1}^N \nu(I'_k)$ . But then

$$\nu(I) \leq (1 + \epsilon) \sum_{k=1}^N \nu(I_k) \leq (1 + \epsilon) \sum_{k=1}^{\infty} \nu(I_k).$$

It then follows that  $\nu(I) \leq |I|_e$ , completing the proof. □

More generally,  $\left| \bigcup_{k=1}^N I_k \right|_e = \sum_{k=1}^N \nu(I_k)$ , provided the  $I_k$  are non-overlapping intervals, i.e.  $\text{Int}(I_k) \cap \text{Int}(I_j) = \emptyset$ , see Figure 2. The proof of this is rather ugly—an exercise in making ‘obvious’ geometric facts obvious, and an exercise in bookkeeping—and we shall not concern ourselves with it.

As a final remark, observe that the exterior measure is a function to the nonnegative extended real line; that is, the exterior measure allows infinite values. Sets with an infinite exterior measure are considered measurable. For example, our intuition is that  $\mathbb{R}$  should have infinite length, and the reader routinely verifies that  $|\mathbb{R}|_e = \infty$ . The outer measure  $|\cdot|_e$  defined above does meet all our guiding principles as the following proposition verifies.

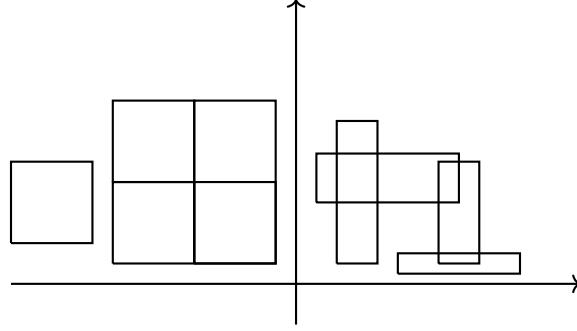


Figure 2: The intervals in Quadrant II are non-overlapping, while those in Quadrant I overlap.

**Proposition 1.3.**

(i)  $\nu(\emptyset) = 0$ .

(ii) *Monotonicity:* if  $E_1 \subset E_2$ , then  $|E_1|_e \leq |E_2|_e$ .

(iii) *Countable Subadditivity:*  $\left| \bigcup_{k=1}^{\infty} E_k \right|_e \leq \sum_{k=1}^{\infty} |E_k|_e$

*Proof.*

(i) This holds essentially by fiat.

(ii) This follows immediately from the fact that we are taking an infimum, and any open cover of  $E_2$  is an open cover of  $E_1$ .

(iii) Let  $E := \left| \bigcup_{k=1}^{\infty} E_k \right|_e$ . If any of the  $E_k$  have infinite exterior measure, the result is immediate. Assume then that  $|E_k|_e < \infty$  for all  $k$ . Choose  $\epsilon > 0$  and cover each  $E_k$  by intervals  $\{I_n\}$  such that  $\sum_n \nu(I_n) \leq |E_k|_e + \epsilon/2^k$ . Then  $E \subset \bigcup_{k,n} I_{k,n}$  and  $|E|_e \leq \sum_{k,n} \nu(I_{k,n}) = \sum_k \sum_n \nu(I_{k,n})$ . But then

$$|E|_e \leq \sum_k \left( |E_k|_e + \epsilon/2^k \right) = \epsilon + \sum_{k=1}^{\infty} |E_k|_e.$$

The result then follows by letting  $\epsilon$  tend to 0. □

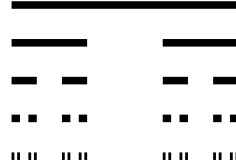
If one wants to generalize the notion of outer measures to spaces beyond  $\mathbb{R}^n$ , one can take the properties of Proposition 1.3 as the axioms for this abstract measure. Now we immediately have the following corollary:

**Corollary 1.1.** *Any subset of a set with outer measure zero has outer measure zero, and the countable union of sets with outer measure zero has outer measure zero. In particular, any countable set has outer measure zero.*

*Proof.* This follows immediately from the Countable Subadditivity of  $\nu$ .  $\square$

Now while Corollary 1.1 states that countable sets have outer measure zero, it need not be the case that uncountable sets need have positive measure.

**Example 1.2 (Cantor Set).** Begin with the closed unit interval  $C_0 := [0, 1]$ . From this interval, remove the middle third, i.e.  $(\frac{1}{3}, \frac{2}{3})$ , and label  $C_1 := [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ . Inductively construct  $C_n$  by removing the middle third of each closed subinterval of  $C_{n-1}$ . We define the Cantor Set by defining  $C := \lim_{n \rightarrow \infty} C_n$ . The first few stages of the construction of  $C$  are shown below. The fact that the Cantor set is uncountable follows from the fact that it is a nonempty compact set without isolated points.



What is  $|C|_e$ ? Note that  $C_n$  is the union of  $2^n$  intervals, each having length  $1/3^n$ , and that  $C \subset C_n$  for all  $n \in \mathbb{N}$ . But then the Cantor set is contained in a union of  $2^n$  intervals with length  $1/3^n$ , which has total length  $2^n \cdot 1/3^n = (2/3)^n$ . The fact that  $|C|_e = 0$  is then clear as  $\lim_{n \rightarrow \infty} (2/3)^n = 0$ .  $\triangleleft$

Note in general we do not have  $|\bigcup E_k|_e = \sum |E_k|_e$  even if the  $E_k$  are disjoint or even in the case of finite unions! Equality holds when the sets are, in a sense, ‘unentangled.’ By this, we mean that open coverings of one set tend to be disjoint from open coverings of the other set. If this is the case, the sets have to be covered separately, see the example on the left in Figure 3. However if the sets are ‘entangled’, then their open covers result in a great deal of ‘multiple-covering’ for the union. This excess covering allows one to more ‘efficiently’ cover the union—hence the smaller measure, see the example on the right in Figure 3.

There are many connections between Topology and Measure Theory, especially in the case of  $\mathbb{R}^n$ . Topology on  $\mathbb{R}^n$  is primarily interested in the structure of open and compact sets. This will prove useful for us since our notion of measure is defined in terms of open sets. As an example, take the following theorem.

**Theorem 1.1.** *For all  $E \subset \mathbb{R}^n$  and  $\epsilon > 0$ , there exists an open set  $G$  such that  $E \subset G$  and  $|G|_e \leq |E|_e + \epsilon$ .*



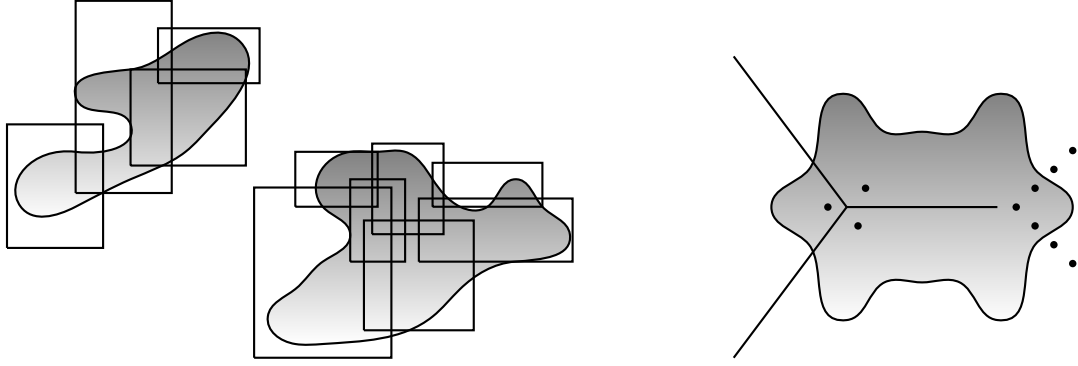


Figure 3: On the left, the two sets are ‘unentangled’ so that their opening coverings can be taken disjointly. On the right, the sets are the ‘blob’ and the line segments along with the points. These sets are ‘entangled’ and their open coverings will have ‘overlap.’

*Proof.* Every interval  $I$  is contained in the interior of a slightly larger interval  $I'$ , i.e.  $I \subseteq \text{Int}(I')$ , where  $\nu(I') - \nu(I) < \epsilon$ . Choose intervals  $I_k$  such that  $E \subset \bigcup_{k=1}^{\infty} I_k$  and  $\sum_k \nu(I_k) \leq |E|_e + \epsilon/2$ . Now find  $I'_k$  such that  $I_k \subset \text{Int}(I'_k)$  and  $\nu(I'_k) < \nu(I_k) + \epsilon/2^{k+1}$ . Let  $G = \bigcup_k \text{Int}(I'_k)$ . By construction,  $G$  is an open set containing  $E$ . To complete proof, observe

$$|G|_e \leq \sum_{k=1}^{\infty} \nu(I'_k) \leq \sum_{k=1}^{\infty} \nu(I_k) + \epsilon \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \leq |E|_e + \epsilon.$$

□

**Corollary 1.2.** *For every set  $E$ , there exists a  $G_\delta$  set  $G$  such that  $E \subset G$  and  $|E|_e = |G|_e$ .*

*Proof.* By Theorem 1.1, we know that for every  $k \in \mathbb{N}$ , there is an open set  $G_k \supset E$  with  $|G_k|_e \leq |E|_e + 1/k$ . Define  $G := \bigcap_{k=1}^{\infty} G_k$ . Then  $G$  is a  $G_\delta$  set and  $G \supset E$ . Finally,

$$|E|_e \leq |G|_e \leq |G_k|_e \leq |E|_e + \frac{1}{k},$$

so that  $|E|_e = |G|_e$ .

□

The set  $G_\delta$  referred to in Corollary 1.2 is a general type of set which will be of occasional interest and use for us:

**Definition ( $G_\delta$ -Set).** A  $G_\delta$  set is a countable intersection of open sets.

Notationally, the choice of ‘ $G$ ’ is because the set is open, and the choice of ‘ $\delta$ ’ stems from the fact we are using an intersection.

Theorem 1.1 says we can always approximate any set by an open set with approximately the same size, i.e. approximately the same exterior measure. However, this does not mean that  $|G \setminus E|_e \leq \epsilon$ . We do know that  $G = E \cup (G \setminus E)$ . By subadditivity, we have  $|G|_e \leq |E|_e + |G \setminus E|_e$ . But we do not know the measure of the second set. In fact, it is possible that  $|G \setminus E|_e$  could be very large. Now as it turns out for compact sets, it is sufficient to consider only finite coverings.

**Proposition 1.4.** *If  $K \subset \mathbb{R}^n$  is compact, then  $|K|_e = \inf \{\sum_k \nu(I_k) : \{I_k\} \text{ finite cover of } K\}$ .*

*Proof.* Let  $\epsilon > 0$ . By Theorem 1.1, every interval is contained some set  $\text{Int}(I')$ , where  $\nu(I') < \nu(I) + \epsilon$ . Given a countable cover  $I_k$  of  $K$ , choose  $I'_k$  such that  $I_k \subset \text{Int}(I'_k)$  and  $\nu(I'_k) < \nu(I_k) + \epsilon/2^k$ . Then  $\{\text{Int}(I'_k)\}_k$  is an open covering for  $K$ , so there exists a finite subcovering  $\{I_{k,n}\}_{n=1}^N$ . Therefore, we have  $K \subset \bigcup_j I'_{k,j}$ . By countable subadditivity,

$$\sum_{n=1}^N \nu(I'_{k,n}) < \epsilon + \sum_{n=1}^N \nu(I_k),$$

as desired. □

We now can define our notion of measurability with our notions of exterior measure firmly in place.

**Definition (Measurable).** A set  $E \subset \mathbb{R}^n$  is measurable if for all  $\epsilon > 0$ , there exists an open set  $G$  such that  $G \supset E$  and  $|G \setminus E|_e < \epsilon$ .

Essentially, a set is measurable if it can be well approximated by open sets. We choose the above notion of ‘closeness’ in order to obtain additivity of measures. Notice we also have mentioned the underlying topology via the use of ‘open.’ We are able to avoid invoking the underlying topology using greater abstraction, which shall come later. Note that we always have an open set such that  $|G|_e < |E|_e + \epsilon$  (c.f. Theorem 1.1), but this alone is weaker than the above definition; that is, if  $E$  is measurable then it satisfies the properties in Theorem 1.1. As a matter of notation, if  $E$  is measurable, we define  $|E| := |E|_e$ . The following propositions follow immediately from our definition.

**Proposition 1.5.** *Every open set is measurable.*

*Proof.* If  $E$  is an open set, choose  $G = E$ . □

**Proposition 1.6.** *If  $|E|_e = 0$ , then  $E$  is measurable.*

*Proof.* Choose  $G$  such that  $|G|_e < |E|_e + \epsilon = \epsilon$ . But then  $|G \setminus E|_e \leq |G|_e < \epsilon$ . □

**Proposition 1.7.** *A countable union of measurable sets is measurable, and*

$$|E| \leq \sum_k |E_k|.$$

*Proof.* Let  $\epsilon > 0$ . For each  $k$ , choose an open set  $G_k$  such that  $E_k \subset G_k$  and  $|G_k \setminus E_k|_e < \epsilon/2^k$ . Now  $G := \bigcup_k G_k$  is open, and  $E \subset G$ . Moreover since  $G \setminus E \subset \bigcup_k (G_k \setminus E_k)$ , we have

$$|G \setminus E|_e \leq \left| \bigcup_k (G_k \setminus E_k) \right|_e \leq \sum_k |G_k \setminus E_k|_e < \epsilon.$$

Therefore,  $\bigcup_k E_k$  is measurable. The fact that  $|\bigcup_k E_k| \leq \sum_k |E_k|$  follows from Proposition 1.3.  $\square$

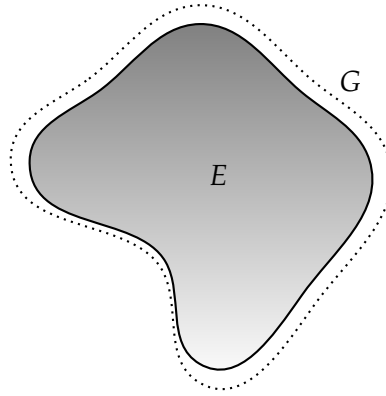


Figure 4: Open sets are not ‘fuzzy’ since they have a good notion of boundary. Similarly since  $G$  is not fuzzy and approximates  $E$ , it must be that  $E$  is not too close to being ‘fuzzy.’

**Proposition 1.8.** *All intervals are measurable.*

*Proof (Sketch).* We prove this only in the two dimensional case to avoid unnecessary complications. Given an interval  $I = [a, b]$ , choose  $I' = [a', b']$  such that  $I \subset \text{Int } I'$  and  $\nu(I') < \nu(I) + \epsilon$ . We need to show that  $|I' \setminus I|_e < \epsilon$ . Now  $I' \setminus I = [a', a) \cup (b, b']$ , and  $|I' \setminus I|_e \leq a - a' + b' - b = (b' - a') - (b - a) < \epsilon$ , as desired. Alternatively, one can use the fact that  $I$  is the union of its interior and its boundary, and that the boundary has measure zero. This alternative proof is left to the reader.  $\square$

As we have seen, open sets are easily seen to be measurable. But the case of closed sets is more complicated. For example in  $\mathbb{R}$ , every open set is the countable union of intervals of the form  $(a_k, b_k)$ . These intervals can even be taken to be disjoint. However, the same is

not true for closed sets—take the Cantor set for example, c.f. Example 1.2. We do have that in  $\mathbb{R}^n$ , every open set is a countable union of non-overlapping intervals. To prove this we shall make use of dyadic cubes.

A dyadic cube of generation zero,  $\mathcal{D}_0$ , are cubes with unit side lengths and integer vertices, i.e.  $\mathcal{D}_0 := \{[0, 1]^n + \tau : \text{fixed } \tau \in \mathbb{Z}^n\}$ . A generation one dyadic cube is  $\mathcal{D}_1 := \{\frac{1}{2}Q : Q \in \mathcal{D}_0\}$ . Generally,  $\mathcal{D}_n := \{\frac{1}{2}Q : Q \in \mathcal{D}_{n-1}\} = \{\frac{1}{2^n}Q : Q \in \mathcal{D}_0\}$ . Given  $\mathcal{D}_n$ , we say that  $\mathcal{D}_{n-1}$  is a parent of  $\mathcal{D}_n$ , and  $\mathcal{D}_i$ , where  $i < n$ , is an ancestor of  $\mathcal{D}_n$ . We say also that  $\mathcal{D}_{n+1}$  is a child of  $\mathcal{D}_n$ , and  $\mathcal{D}_j$ , where  $j > n$ , is a descendant of  $\mathcal{D}_n$ . One can allow  $n$  to be negative to create larger dyadic cubes. Define  $\mathcal{D} := \bigcup_{k=0}^{\infty} \mathcal{D}_k$ . If  $Q_1, Q_2$  are dyadic, then either  $Q_1 \subset Q_2$ ,  $Q_2 \subset Q_1$ , or they do not overlap. We now are in a position to prove the following lemma.

**Lemma 1.1.** *Every open set in  $\mathbb{R}^n$  is a countable union of non-overlapping intervals.*

*Proof.* Given an open set  $G$ , let  $\{I_k\}$  be all dyadic cubes that are contained in  $G$  and for which their parent is not contained in  $G$ . By the selection of the  $I$ 's, it follows that they are pairwise disjoint for if  $I_k$  and  $I_j$  overlap, then one contains the other, contradicting the selection process. Clearly, we have selected only countably many intervals. Now if  $x \in G$ , there exists  $r > 0$  such that there is an  $r$ -neighborhood of  $x$  contained in  $G$ . For sufficiently large  $n$ , all the cubes in  $\mathcal{D}_n$  have diameter less than  $r$ . But then there exists  $Q \in \mathcal{D}_n$  such that  $x \in Q$ . Note that  $Q \subset G$ . Now either  $\mathcal{D}_n$  is contained in  $G$  or it has an ancestor that is contained in  $G$ .  $\square$

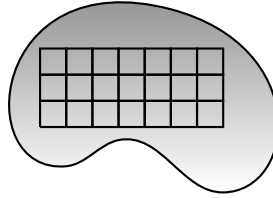


Figure 5: Tiling intervals inside an open set.

We can now make precise a discussion from earlier—if two sets are ‘unentangled’ then the measure of the union is the sum of the measures.

**Lemma 1.2.** *If  $A, B \subset \mathbb{R}^n$  and  $\text{dist}(A, B) > 0$ , then  $|A \cup B|_e = |A|_e + |B|_e$ .*

*Proof.* We know by subadditivity that  $|A \cup B|_e \leq |A|_e + |B|_e$ . It remains to show that  $|A|_e + |B|_e \leq |A \cup B|_e$ . Let  $\epsilon > 0$ , and choose intervals  $\{I_k\}$  such that  $A \cup B \subset \bigcup_k I_k$  and  $\sum_k |I_k| \leq |A \cup B|_e + \epsilon$ . Possibly partitioning each  $I_k$  into a finite number of subintervals, we may assume that  $\text{diam}(I_k) < \text{dist}(A, B)$ . ‘Sort’ the set  $\{I_k\}$  into two sets  $\{I'_k\}$  and  $\{I''_k\}$

which cover  $A$  and  $B$ , respectively. Then

$$|A|_e + |B|_e \leq \sum_k |I'_k| + \sum_k |I''_k| = \sum_k |I_k| \leq |A \cup B|_e + \epsilon.$$

Therefore,  $|A|_e + |B|_e \leq |A \cup B|_e$ , as desired.  $\square$

**Theorem 1.2.** *Every closed set  $A \subset \mathbb{R}^n$  is measurable.*

*Proof.* Given  $\epsilon > 0$ , we can choose  $G \supset A$  such that  $|G|_e < |A|_e + \epsilon$ . Now  $G \setminus A$  is open. By Lemma 1.1, we can write  $G \setminus A = \bigcup_{k=1}^{\infty} I_k$ , where  $\{I_k\}$  is a collection of non-overlapping open intervals. We want to show that  $\sum \nu(I_k) < \epsilon$ , which will imply that  $|G \setminus A|_e < \epsilon$ . It suffices to show that  $\sum_{k=1}^N \nu(I_k) < \epsilon$  for all  $N$ . Define  $K = \bigcup_{k=1}^N I_k$ . Since  $K$  is the finite union of compact sets in  $\mathbb{R}^n$ ,  $K$  is compact. Furthermore,  $K$  is disjoint from  $A$ . As the set  $A$  is closed and  $K$  is compact, it must be that  $\text{dist}(K, A) > 0$ . It then follows from Lemma 1.2 that

$$|K \cup A|_e = |K|_e + |A|_e = \sum \nu(I_k) + |A|_e \leq |G|_e < |A|_e + \epsilon.$$

$\square$

As it turns out, the measurability of a set is equivalent to the measurability of its complement. This is useful in circumstances where one set is easier to handle than the other.

**Theorem 1.3.**  *$E$  is measurable if and only if  $E^C$  is measurable.*

*Proof.* Suppose that  $E$  is measurable. For  $k \in \mathbb{N}$ , choose an open set  $G_k$  such that  $E \subset G_k$  and  $|G_k \setminus E|_e < 1/k$ . Now as  $G_k$  is open,  $G_k^C$  is closed, and hence measurable by Theorem 1.2. Let  $G := \bigcup_k G_k^C$ . Being the countable union of measurable sets,  $G$  is measurable, and  $G \subset E^C$ . Write  $E^C = G \cup Z$ , where  $Z = E^C \setminus G$ . Then  $Z \subset E^C \setminus G_k^C = G_k \setminus E$ , and therefore,  $|Z|_e < 1/k$  for all  $k$ . Hence,  $|Z|_e = 0$ , and so  $Z$  is measurable. But then  $E^C$  is the union of measurable sets, and is thus measurable. Finally, taking  $E$  to be  $E^C$  and rerunning the argument mutatis mutandis proves that if  $E^C$  is measurable, then  $E$  is measurable.  $\square$

**Corollary 1.3.** *If  $E_1, E_2$  are measurable, then  $E_1 \setminus E_2$  is measurable.*

*Proof.* We know that  $E_1 \setminus E_2 = E_1 \cap E_2^C$ . But  $E_2^C$  is measurable by Theorem 1.3. Therefore,  $E_1 \setminus E_2$  is measurable, being the intersection of measurable sets.  $\square$

Our definition of measurable is essentially that a set  $E$  is measurable if it can be ‘approximated well’ by open sets containing it. This is an exterior notion of measure. We can also define measurability instead by saying a set  $E$  is measurable if it can be ‘approximated well’ internally by closed sets, as the following proposition shows.

**Proposition 1.9.** *A set  $E \subset \mathbb{R}^n$  is measurable if and only if given  $\epsilon > 0$ , there exists a closed set  $F \subset E$  such that  $|E \setminus F|_e < \epsilon$ .*

*Proof.*  $E$  is measurable if and only if  $E^C$  is measurable, i.e. if and only if given  $\epsilon > 0$ , there exists an open set  $G$  with  $E^C \subset G$  and  $|G \setminus E^C|_e < \epsilon$ . But such an open set  $G$  exists, noting that  $G \setminus E^C = E \setminus F$ , if and only if  $G^C$  is closed,  $F \subset E$ , and  $|E \setminus F|_e < \epsilon$ .  $\square$

The previous theorems and propositions have shown that the complements of measurable sets are measurable, and countable unions and intersections of measurable sets are measurable. Since we shall make use of them in the future, note that if  $\{E_k\}$  is a collection of measurable sets, then so are  $\limsup$  and  $\liminf$ :

$$\limsup E_k = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k \quad \text{and} \quad \liminf E_k = \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} E_k,$$

though we shall not prove this. Now all of these previous theorems and propositions essentially follow because we are working in the context of a  $\sigma$ -algebra.

**Definition ( $\sigma$ -algebra).** A nonempty collection of sets  $\Sigma$  is called a  $\sigma$ -algebra if it satisfies

- (i)  $E^C \in \Sigma$  whenever  $E \in \Sigma$ .
- (ii)  $\bigcup_k E_k \in \Sigma$  whenever  $E_k \in \Sigma$  for all  $k$ .

Note that any collection of sets closed under countable unions is closed under countable intersections as  $[\bigcap_k U_k]^C = \bigcup_k U_k^C$ . The empty set and entire space are necessarily measurable sets. Generally, they are elements of any  $\sigma$ -algebra. We shall be particularly interested in two specific  $\sigma$ -algebras:

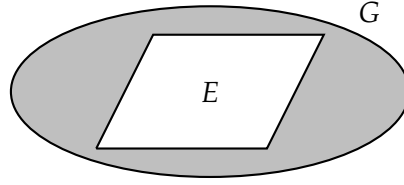
**Definition (Borel  $\sigma$ -algebra).** The Borel  $\sigma$ -algebra is the smallest  $\sigma$ -algebra containing all open subsets of  $\mathbb{R}^n$ .

**Definition (Lebesgue  $\sigma$ -algebra).** The Lebesgue  $\sigma$ -algebra is the  $\sigma$ -algebra containing all measurable sets.

Note that the Borel  $\sigma$ -algebra is *strictly* contained in the Lebesgue  $\sigma$ -algebra—though this is not simple to see.

## 1.1 Properties of Measurable Sets

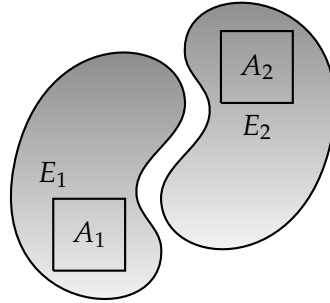
Our goal in this section is to examine measurable sets more closely. We begin by recalling that  $|E| = |E|_e$  whenever  $E$  is measurable, and  $|\bigcup_k E_k|_e \leq \sum_k |E_k|_e$ , i.e. countable subadditivity.



**Theorem 1.4** (Countable Subadditivity). *If  $\{E_k\}$  is a collection of disjoint, measurable sets, then*

$$\left| \bigcup_{n=1}^{\infty} E_k \right| = \sum_{k=1}^{\infty} |E_k|.$$

*Proof.* It is immediate that  $|\bigcup_{n=1}^{\infty} E_k| \leq \sum_{k=1}^{\infty} |E_k|$ , so that we need show  $\sum_{k=1}^{\infty} |E_k| \leq |\bigcup_{n=1}^{\infty} E_k|$ . Let  $E := \bigcup_k E_k$ . It suffices to show  $\sum_{k=1}^N |E_k| \leq |E|$  for all  $N$ . If  $\text{dist}(A, B) > 0$ , we know that  $|A \cup B|_e = |A|_e + |B|_e$ . In particular, this holds if  $A, B$  are disjoint compact sets. First, assume that each  $E_k$  is bounded. For each  $k \in \mathbb{N}$ , choose a closed subset  $A_k \subset E_k$



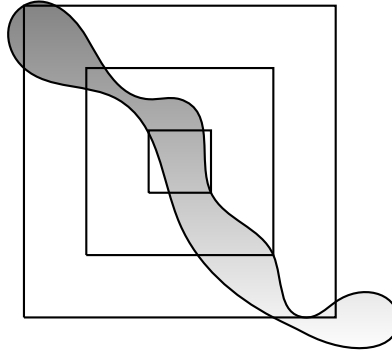
such that  $|E_k \setminus A_k|_e < \epsilon/2^k$ . Since  $A_k \subset E_k$  and  $E_k$  is bounded,  $A_k$  is a closed, bounded set. Therefore,  $A_k$  is compact. We also have

$$\left| \bigcup_{k=1}^N A_k \right| = \sum_{k=1}^N |A_k| \geq \sum_{k=1}^N |E_k| - \epsilon.$$

But as  $\bigcup_{k=1}^N A_k \subset E$ , we have  $|E| \geq \sum_{k=1}^N |E_k| - \epsilon$ . Letting  $\epsilon \rightarrow 0$  yields the desired inequality. This completes the proof in the case where each  $E_k$  is bounded.

Now suppose that some  $E_k$  may be unbounded. Let  $I_j := [-j, j]^n$  for  $j > 0$  and  $Q_0 := \emptyset$ . Define further  $Q_j := I_j \setminus I_{j-1}$  for  $n > 0$ . Consider the sets  $E_{k,j} := E_k \cap Q_j$  for  $j, k = 1, 2, \dots$ . Each of these sets are disjoint, bounded, and measurable. It is also routine to verify that

$$\begin{aligned} E_k &= \bigcup_j E_{k,j} = \bigcup_j (E_k \cap Q_j) \\ E &= \bigcup_k E_k = \bigcup_{k,j} E_{k,j} \end{aligned}$$



But then by the case above, we have

$$|E| = \left| \bigcup_k E_k \right| = \left| \bigcup_{k,j} E_{k,j} \right| = \sum_{k,j} |E_{k,j}| = \sum_k \sum_j |E_{k,j}| = \sum_k |E_k|.$$

□

**Corollary 1.4.** *If  $\{I_k\}$  is a sequence of nonoverlapping intervals, then  $|\bigcup_k I_k| = \sum_k |I_k|$ .*

*Proof.* It is immediate that  $|\bigcup_k I_k| \leq \sum |I_k|$ . We need prove the opposite inequality. As the  $I_k$  are disjoint, we know  $|\bigcup_k I_k| \geq |\bigcup_k \text{Int } I_k| = \sum_k |\text{Int } I_k| = \sum |I_k|$ . But then  $|\bigcup_k I_k| = \sum_k |I_k|$ . □

**Corollary 1.5.** *Suppose  $E_1$  and  $E_2$  are measurable,  $E_2 \subset E_1$ , and  $|E_2| < \infty$ . Then  $|E_1 \setminus E_2| = |E_1| - |E_2|$ .*

*Proof.* We know  $E_1 = E_2 \cup (E_1 \setminus E_2)$  and  $|E_1| = |E_2| + |E_1 \setminus E_2|$ . The result then follows as  $|E_2| < \infty$ . □

We end this section with an interaction between the Lebesgue measure and monotone sequences of sets.

**Proposition 1.10.**

(i) *If  $E_1 \subset E_2 \subset \dots$  are measurable, then  $|\bigcup_k E_k| = \lim_{k \rightarrow \infty} |E_k|$ ; that is, if  $E_k \nearrow E$ , then*

$$\lim_{k \rightarrow \infty} |E_k| = |E|.$$

(ii) *If  $E_1 \supset E_2 \supset \dots$  are measurable,  $|E_k| < \infty$  for some  $k$ , then  $|\bigcap_{k=1}^{\infty} E_k| = \lim_{k \rightarrow \infty} |E_k|$ ; that is, if*

$$E_k \searrow E \text{ and } |E_k| < \infty \text{ for some } k, \text{ then } \lim_{k \rightarrow \infty} |E_k| = |E|.$$



*Proof.*

- (i) Let  $E = \cup E_k$ . If  $|E_k| = \infty$  for some  $k$ , then  $\lim_{k \rightarrow \infty} |E_k|$  and  $|E|$  are infinite, and the result follows. Assume then that  $|E_k| < \infty$  for all  $k$ . We want to create a disjoint union. We shall express  $E$  as  $E = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_2) \cup \cdots \cup (E_k \setminus E_{k-1}) \cup \cdots$ . Define  $F_k := E_k \setminus E_{k-1}$ . Observe the sets  $F_k$  are measurable and pairwise disjoint. Then by Theorem 1.4,

$$|E| = \sum_{k=1}^{\infty} |F_k| = \lim_{n \rightarrow \infty} \sum_{k=1}^n |F_k| = \lim_{n \rightarrow \infty} \left| \bigcup_{k=1}^n F_k \right| = \lim_{n \rightarrow \infty} |E_n|.$$

- (ii) Let  $E = \cap E_k$ . Since considering tails of a sequence are sufficient, we may assume without loss of generality that  $|E_1| < \infty$ . We again want to create disjoint, measurable sets. Define  $F_k := E_k \setminus E_{k+1}$ . Observe that the sets  $F_k$  are measurable, pairwise disjoint, and that  $E_1 = E \cup \bigcup F_k$ . Then by Theorem 1.4,

$$|E_1| = |E| + \left| \bigcup F_k \right| = |E| + \sum_{k=1}^{\infty} |F_k| = |E| + \sum_{k=1}^{\infty} (|E_k| - |E_{k+1}|) = |E| + |E_1| - \lim_{k \rightarrow \infty} |E_k|.$$

But as  $|E_1| < \infty$ , we must then have  $|E| = \lim_{k \rightarrow \infty} |E_k|$ .

□

Note that the requirement that  $|E_k| < \infty$  for some  $k$  in the second part of Proposition 1.10 is necessary. For example, take  $E_k = \mathbb{R}^2 \setminus C_k$ , where  $C_k$  is the circle centered at the origin with radius  $k$ . Then  $|E_k| = \infty$  for all  $k$ , and  $E_k \searrow \emptyset$ . However,  $\lim_{k \rightarrow \infty} |E_k| = \infty$ , while  $|\emptyset| = 0$ .

## 1.2 Characterization of Measurable Sets

By definition,  $A$  is measurable if and only if for all  $\epsilon > 0$ , there exists an open set  $G \supset A$  such that  $|G \setminus A|_e < \epsilon$ . But this makes use of the underlying topology. Can we avoid this? After all, this ‘open set-ness’ is about continuity. Continuity is not even a requirement for integration, which is an end goal of ours. So we would like a description of measurability which avoids Topology entirely. Ironically, we shall first need more Topology.

Recall that  $G_\delta$  is a set which is the countable intersection of open sets,  $F_\sigma$  is the countable union of closed sets,  $G_{\delta,\sigma}$  are the countable intersection of closed sets.<sup>1</sup> One can even continue this to absurdity, i.e.  $G_{\delta\sigma\delta\sigma}$ . Furthermore, these are all subclasses of the Borel  $\sigma$ -algebra. One may wonder if all Borel created by these procedures, i.e. formed by countable union or intersections of open or closed sets? This turns out to not be the case, though it is

<sup>1</sup>Generally,  $G$  denotes open,  $F$  denotes closed,  $\delta$  is for countable intersections, and  $\sigma$  is for countable unions.

difficult to construct a set in this fashion which is not a Borel set. However, it is the case that measurable sets are ‘close’ to Borel sets, in the sense they differ by a set of measure zero, as the next results shows.

**Theorem 1.5.**  *$E$  is measurable if and only if  $E = H \setminus Z$  if and only if  $E = F \cup Z$ , where  $H$  is of type  $G_\delta$ ,  $F$  is of type  $F_\sigma$ , and  $|Z| = 0$ .*

*Proof.* It is sufficient to prove that each of the statements is equivalent to the measurability of  $E$ . If  $E$  can be represented as  $H \setminus Z$  or  $F \cup Z$ , then  $E$  is measurable, as  $H$ ,  $F$ , and  $Z$  are measurable. Now assume that  $E$  is measurable. For each  $k$ , choose an open set  $G_k$  such that  $E \subset G_k$  and  $|G_k \setminus E| < 1/k$ . Define  $H := \cap G_k$ . Clearly,  $H$  is of type  $G_\delta$ ,  $E \subset H$ , and  $H \setminus E \subset G_k \setminus E$  for each  $k$ . But then defining  $Z = H \setminus E$ , we see that  $|Z| = 0$ , as desired. Furthermore, if  $E$  is measurable, then so too is  $E^C$ . Now using the previous work, write  $E^C = (\cap G_k) \setminus Z$ , where the  $G_k$  are open sets and  $|Z| = 0$ . But then taking complements yields  $E = (\cup G_k^C) \cup Z$ . But as the sets  $G_k$  are open, the sets  $G_k^C$  are closed, implying  $F := \cup G_k^C$  is of type  $F_\sigma$ .  $\square$

**Remark.** This is different (in fact stronger) than  $E \subset H$ , where  $H$  is a  $G_\delta$  set with the same outer measure, i.e.  $|H|_e = |E|_e$ .

Theorem 1.5 essentially states that measurable sets are the Borel sets ‘plus or minus’ the measure zero sets. Note that we are still making Borel sets, which invoke Topology, so we still have not met our goal. However, we are now set up to prove Carathéodory’s Theorem, which we can use to meet our goal.

**Theorem 1.6 (Carathéodory).**  *$E$  is measurable if and only if for all  $A \subset \mathbb{R}^n$ ,  $|A \cap E|_e + |A \setminus E|_e = |A|_e$ .*

*Proof.* Suppose that  $E$  is measurable. Choose a  $G_\delta$  set  $H$  such that  $A \subset H$  and  $|H|_e = |A|_e$ . Since  $H = (H \cap E) \cup (H \setminus E)$ , and these are measurable sets, we have

$$|A|_e = |H| = |H \cap E| + |H \setminus E|.$$

But then  $|A|_e = |H| = |H \cap E| + |H \setminus E| \geq |A \cap E|_e + |A \setminus E|_e$ . By subadditivity, we obtain  $|A|_e \leq |A \cap E|_e + |A \setminus E|_e$ . Therefore,  $|A|_e = |A \cap E|_e + |A \setminus E|_e$ .

Now suppose that  $E$  satisfies the condition that  $|A \cap E|_e + |A \setminus E|_e = |A|_e$  for all  $A \subseteq \mathbb{R}^n$ . Let  $A$  be a  $G_\delta$  set such that  $E \subset A$  and  $|E|_e = |A|$ . Then on the left side of the equation, we have  $|E|_e + |A \setminus E|_e = |A| = |E|_e$ . Hence,  $|A \setminus E|_e = 0$ , so  $E$  is  $G_\delta$  minus a measure zero set. Notice that we have subtracted  $|E|_e$ ; hence, we need consider the case where  $|E|_e = \infty$ .

Assume that  $|E|_e = \infty$ . We know that  $E = \cup_{j=1}^\infty E_j$ , where  $E_j := E \cap [-j, j]^n$ .

So it suffices to prove that  $E_j$  has the property  $|A \cap E_j|_e + |A \setminus E_j|_e = |A|_e$  for all  $A \subseteq \mathbb{R}^n$  and  $j \in \mathbb{N}$ . [Since  $E_j$  is measurable, the countable union of the  $E_j$  is measurable.] Certainly,  $|A|_e \leq |A \cap E_j|_e + |A \cap E_j^c|_e$ . It remains to show  $|A|_e \geq |A \cap E_j|_e + |A \cap E_j^c|_e$ .

We have  $|A \cap E_j|_e + |A \cap E_j^c|_e = |A|_e$ . By construction,  $E_j$  has finite measure so that by the work above,  $E_j$  satisfies the property  $|A|_e = |A \cap E_j|_e + |A \cap E_j^c|_e$ .

We know also that  $Q_j := [-j, j]^n$  satisfies this property, so  $|A \cap E|_e = |A \cap E \cap Q|_e + |A \cap E \cap Q^c|_e$ . Similarly, we have  $|A \cap E^c|_e = |A \cap E^c \cap Q|_e + |A \cap E^c \cap Q^c|_e$ .

Combining these gives

$$|A|_e = |A \cap E \cap Q|_e + |A \cap E \cap Q^c|_e + |A \cap E^c \cap Q|_e + |A \cap E^c \cap Q^c|_e \geq |A \cap E_j|_e + |A \cap E_j^c|_e,$$

as desired.  $\square$

### 1.3 Continuous & Lipschitz Transformations

If  $E \subset \mathbb{R}^n$  is measurable, and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuous, is  $f(E)$  always measurable? The answer to this is no. We shall see this shortly.

**Lemma 1.3.** *There is a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and a set  $E$  with  $|E| = 0$ , such that  $|f(E)| > 0$ .*

*Proof (Sketch).*

**Lemma 1.4.** *If  $|A| > 0$ , then there exists  $B \subset A$  such that  $B$  is not measurable.*

We now can show that it is not necessarily that if  $E \subset \mathbb{R}^n$  is measurable, and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuous, that  $f(E)$  is always measurable. To see this, choose  $f$  and  $E$  as in Lemma 1.3. By Lemma 1.4, there exists a non-measurable subset of  $f(E)$ , say  $W$ . Let  $V := f^{-1}(W) \cap E$ . Observe that  $f(V) = W$ , and that  $V$  is measurable (being a subset of measure zero).

**Definition (Lipschitz).** We say that  $f$  is Lipschitz if there exists  $L$  such that  $|f(x) - f(y)| \leq L|x - y|$  for all  $x, y$  in the domain of  $f$ .

That is, Lipschitz functions are the special case of uniformly continuous functions where we can always choose  $\delta = \epsilon/L$ . Therefore, if  $f$  is Lipschitz then it is uniformly continuous, and hence also continuous. However, not all uniformly continuous functions are Lipschitz.

The Lipschitz condition can be thought of as the boundedness of secant lines for a given function.

Lipschitz maps preserve sets of measure zero.

**Theorem 1.7.** *If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is Lipschitz and  $|E| = 0$ , then  $|f(E)| = 0$ .*

*Proof.* For every  $\epsilon > 0$ , there exists a countable cover of  $E$  by intervals  $\{I_k\}$  so that  $\sum |I_k| < \epsilon$ .

$|E|_e = \inf\{\sum \nu(I_k) : \bigcup I_k \supset E, \text{ and } I_k \text{ cubes}\}.$

Tile interior of  $I'_k$  by dyadic cubes.

So we have covering with cubes with  $\sum \nu(I_k) < \epsilon$ .  $|f(I_k)| \leq C|I_k|$  by geometry. Note that  $C$  depends on  $L$  and  $n$ . Now cube has diameter at most  $L \cdot a\sqrt{n}$ . A cube of side length  $2La\sqrt{n}$  covers it.

But then  $|f(E)| \leq C \sum |I_k| \leq C\epsilon$ . Therefore,  $|f(E)| = 0$ .  $\square$

**Theorem 1.8.** *If  $E$  is measurable and  $f$  is Lipschitz, then  $f(E)$  is measurable.*

*Proof.* Since  $E$  is measurable, write  $E = H \cup Z$ , where  $H$  is  $F_\sigma$  and  $|Z| = 0$ . Now  $F_\sigma$  sets are the countable union of closed sets, which is equivalent to the countable union of compact sets.

Since  $f$  sends compact sets to compact sets,  $f$  preserves ' $F_\sigma$ -ness.' We know by Theorem 1.7 that if  $|E| = 0$ , then  $f(E) = 0$ . But then  $f(E) = f(H) \cup f(Z)$ , a union of  $F_\sigma$  set and a set of measure zero.  $\square$

Recall Cantor set maps to fat cantor set.

**Theorem 1.9.** *The continuous image of a Borel set is measurable.*

Not true that continuous map of Borel is Borel.

No explicit examples of non-measurable sets exist.

Axiom of Choice: For all collections of nonempty sets  $\{E_\alpha\}$ , one can choose an element from each  $E_\alpha$ , i.e. there exists a function  $f : \mathcal{A} \rightarrow \bigcup E_\alpha$  such that  $f(\alpha) \in E_\alpha$  for all  $\alpha$ . Said differently,  $\prod_{\alpha \in \mathcal{A}} E_\alpha \neq \emptyset$ .

Vitali Set: A nonmeasurable subset of  $[0, 1]$ . Equivalence relation  $x \sim y$  if  $x - y \in \mathbb{Q}$ . Let  $\{E_\alpha\}_{\alpha \in \mathcal{A}}$  be equivalence classes. Each  $E_\alpha$  is countable. Index set  $\mathcal{A}$  is uncountable. By AC, we can prove one element from each  $E_\alpha \cap [0, 1]$ . This gives a set  $V \subset [0, 1]$ . We claim that  $V$  is nonmeasurable. Indeed,  $V + \mathbb{Q} = \mathbb{R}$ , in that  $A \pm B = \{a \pm b : a \in A, b \in B\}$ . Meaning,  $\bigcup_{q \in \mathbb{Q}} (V + q) = \mathbb{R}$ . But then  $|V|_e > 0$  (otherwise, we get  $|\mathbb{R}|_e = 0$ ). But  $\bigcup_{q \in \mathbb{Q} \cap [0, 1]} (V + q) \subset [0, 2]$ . So we have a disjoint union of countably many translated copies of  $V$ . If  $V$  were measurable, we would obtain  $|[0, 2]| \geq \sum_{q \in \mathbb{Q} \cap [0, 1]} |V + q| = \infty$ .

**Lemma 1.5.** *If  $E \subset \mathbb{R}$  and  $|E| > 0$ , then there exists an interval  $I$  such that  $|E \cap I| > (1 - \epsilon)|I|$ .*

*Proof.* There exists an open set  $G$  such that  $G \supset E$  and  $|G| < \frac{|E|}{1 - \epsilon}$ . Now  $G = \bigcup (a_k, b_k)$ , disjoint and union contains set  $E$ . So  $|E| = \sum_k |E \cap (a_k, b_k)|$  and  $|G| = \sum_k |(a_k, b_k)|$ . If for all  $k$ ,  $|E \cap (a_k, b_k)| \leq (1 - \epsilon)(b_k - a_k)$ , then  $\sum_k$  yields  $|E| \leq (1 - \epsilon)|G|$ , contradiction.

**Lemma 1.6.** *If  $E \subset \mathbb{R}$  and  $|E| > 0$ , then  $E \setminus E$  contains a neighborhood of 0.*

*Proof.* We claim there exists  $\delta > 0$  such that if  $|x| < \delta$ , then  $E \cap (E + x) \neq \emptyset$ .  $x \in E \setminus E$  if and only if  $E + x$  intersects  $E$ .  $X = a - b$  if and only if  $h + x = a$ , where  $a, b \in E$ . Let  $I$  be an interval,  $|E \cap I| > \frac{2}{3}|I|$ . Let  $\delta = \frac{1}{3}|I|$ . If  $|x| < \delta$ .

$$|I \cup (I + \delta)| \leq \frac{4}{3}|I|.$$

If  $E$  and  $E + x$  were disjoint, then  $|(E \cap I) \cup (E \cap I + x)| = 2|E \cap I| > \frac{4}{3}|I| > |I \cup (I + x)|$ , a contradiction (since one is a subset of the other).  $\square$

**Theorem 1.10.** *If  $E \subset \mathbb{R}$ ,  $|E| > 0$ , then there exists  $E \subset E$  such that  $W$  is nonmeasurable.*

*Proof.* Since  $\bigcup_{q \in \mathbb{Q}} (V + q) = \mathbb{R}$ , where  $V$  Vitali set. We have  $\bigcup_{q \in \mathbb{Q}} [(V + q) \cap E] = E$ . So there exists  $q$  such that  $|(V + q) \cap E|_e > 0$ . But  $(V + q) \cap E$  is nonmeasurable as  $V + q$  has no measurable subsets of positive measure.

$((V + q) - (V + q) = V - V$  is disjoint from  $\mathbb{Q}$ .  $\square$

h	(*) is	Open	Borel	Measurable
	Open	Continuous	—	—
	Borel	Borel measurable	Borel measurable	—
	Measurable	Measurable	Measurable	?

## 2 Measurable Functions

When can we say that  $f \circ g$  is measurable? What data about  $f$  and  $g$  would ensure this? We know that if  $f$  is measurable and  $g$  is continuous does not work from previous discussion. One possibility ensuring this would be  $f$  is continuous and  $g$  is measurable as  $(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U))$ , as  $U$  is open  $f^{-1}(U)$  is open and  $g^{-1}$  of open is measurable (assuming  $g$  takes values in  $\mathbb{R}$  and  $f$  is continuous on some open set containing the range of  $g$ ). More generally, this will hold if  $f$  is Borel measurable on some Borel set containing the range of  $g$  and  $g$  measurable and real-valued.

**Example 2.1.** If  $f$  is measurable (and  $\mathbb{R}$ -valued) then  $f^2$  is measurable,  $f^2 = \phi \circ f$ , where  $\phi(t) = t^2$ . Similarly,  $|f|$  is measurable since  $|f| = \phi \circ f$ , where  $\phi(t) = |t|$ . Also,  $\text{sgn}(f) = \begin{cases} 1, & f = 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases}$  is measurable since sign is Borel measurable (simple to see by checking preimages). So is floor( $x$ ), where floor is supremum of integers which are at most  $x$ . As is ceilin( $x$ ), defined mutatis mutandis. [They are both monotone,]

**Theorem 2.1.** If  $f, g$  are measurable, then  $f \pm g$ ,  $fg$ , and  $f/g$  ( $g \neq 0$  a.e.) are measurable.

*Proof.*  $\{f + g > a\} = \bigcup_{b+c>a, b,c \in \mathbb{Q}} [\{f > b\} \cap \{g > c\}]$  hence measurable set (countable union of measurable sets). Just need to prove equality.  $\supset$  is obvious. Need only prove  $\subset$ . Suppose  $f(x) + g(x) > a$ . Let  $\epsilon = f(x) + g(x) - a$ . Use density of rationals to find  $b \in \mathbb{Q}$  such that  $f(x) - \epsilon/2 < b < f(x)$  and  $c \in \mathbb{Q}$  such that  $g(x) - \epsilon/2 < c < g(x)$ . But then  $b + c > f(x) + g(x) - \epsilon = a$ .

Now  $fg = \frac{1}{4} [(f + g)^2 - (f - g)^2]$  is then measurable. For quotient only need reciprocal. We know  $\frac{1}{g}$  is  $\phi \circ g$ , where  $\phi(t) = \frac{1}{t}$ , which is continuous on  $\mathbb{R} \setminus \{0\}$ . Now  $g$  is nonzero almost everywhere,  $\phi$  contains range of  $g$  (ignoring set where range is  $g = 0$ , since we can ignore for measurability).  $\square$

Another way could combine is max of functions. Better to generalize to countable. So countable sup/inf of measurable functions is measurable.

**Theorem 2.2.** The above statement, including infinite values.

*Proof.* IF  $f_k : E \rightarrow \overline{\mathbb{R}}$  is measurable for all  $k \in \mathbb{N}$ , then  $\sup_k f_k$  and  $\inf_k$  are measurable. So  $g(x) = \sup_{k \in \mathbb{N}} f_k(x)$  is measurable. Same for  $\inf$ .

$$\begin{aligned}\{g > a\} &= \bigcup_{k \in \mathbb{N}} \{f_k > a\} \\ \{g \geq a\} & \\ \{g < a\} & \\ \{g \leq a\} &\end{aligned}$$

First case done. Other follow similarly. HAND WAVE! For  $\inf$

$$\{\inf_k f_k < a\} = \bigcup_{k \in \mathbb{N}} \{f_k < a\}.$$

□

**Theorem 2.3.** Suppose  $f_k : E \rightarrow \overline{\mathbb{R}}$  are measurable, then  $\lim f_k$  is measurable.

*Proof.* Follows immediately from preceding result. [ $\lim = \limsup = \liminf$  when it exists.] □

Approximation: Given measurable  $f$ , find ‘simple’ functions  $f_k$  such that  $f_k \rightarrow f$  pointwise. Here by simple we mean finite range.

**Definition (Simple).**  $g$  is simple if its range is finite subset of  $\mathbb{R}$ . [Need measurable?]

Need this to build notion of integration.

**Theorem 2.4.** For all measurable functions, there exists  $f_k$  such that  $f_k \rightarrow f$  pointwise. If  $f \geq 0$ , we can take  $f_1 \leq f_2 \leq \dots$ .

*Proof.*  $2^{-k} \text{floor}(2^k f)$ . (floor to  $\frac{\text{integer}}{2^k}$ ). This converges monotonically since increasing  $k$  shortens the rounding distance, and then not as much difference so this converges to  $f$  uniformly. Range not finite but countable. So just need to distribute the values among many functions.

$$f_k(x) = \begin{cases} \frac{j-1}{2^k}, & \frac{j-1}{2^k} \leq f(x) < \frac{j}{2^k} \\ k, & f(x) \geq k \end{cases}$$

$f_k$  simple and  $f_k \rightarrow f$ . Now  $f_k = \min(k, \max(-k, 2^{-k} \text{floor}(2^k f)))$ .  $f \geq 0$ , then  $f_1 \leq f_2 \leq \dots$ . □

**Definition.**  $f : E \rightarrow \mathbb{R}$  is continuous at  $a \in E$  if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $x \in E$  and  $|x - a| < \delta$ , then  $|f(x) - f(a)| < \epsilon$ .

Why better than normal definition with limit? Because  $a$  can be a limit point not defined there. Take  $f(x) = 1/x^2$ , rather like allowing  $f(0) = \infty$ .

Semincontinuity means rather than bounding difference from above below, means one of inequality  $-\epsilon < f(x) - f(a) < \epsilon$  holds.

**Definition** (Upper Semicontinuous (usc)).  $f : E \rightarrow \overline{\mathbb{R}}$  is upper semicontinuous at  $a \in E$  if for all  $y > f(a)$ , there exists  $\delta > 0$  such that if  $|x - a| < \delta$ , then  $f(x) < y$ .

**Definition** (Lower Semicontinuous (lsc)).  $f : E \rightarrow \overline{\mathbb{R}}$  is lower semicontinuous at  $a \in E$  if for all  $y < f(a)$ , there exists  $\delta > 0$  such that if  $|x - a| < \delta$ , then  $f(x) > y$ .

**Example 2.2.**

$$f(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

This is usc (automatically because of the  $\infty$  value, why?) and lsc.

Both types of these functions are measurable.

Remark: lsc  $\rightarrow$  measurable usc  $\rightarrow$  measurable

*Proof.* if  $f$  usc on measurable  $E$ , then  $\{f > a\}$  is open in  $E$ . So  $\{f > a\} = E \cap U$ . Hence  $\{f > a\}$  is measurable.

For USC, use  $\{f < a\}$ .

BV function

Given  $f : [a, b] \rightarrow \mathbb{R}$ , define  $V(f; a, b) = \sup_P \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$  sup over all partitions  $P : a = x_0 < x_1 < \dots < x_n = b$ .  $V(x^2; -1, 1) = 2$  ( $P = \{-1, 0, 1\}$  gives the maximum value.)

**Definition.**  $f$  is BV on  $[a, b]$  if  $V(f; a, b) < \infty$

Interesting not called finite variation.

Something which is not BV is

$$f(x) = \begin{cases} \sin(1/x), & x \in (0, 1] \\ 0, & x = 0 \end{cases}$$

**Theorem 2.5.** If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  and  $\int_a^b |f'| dx$  (riemann integral), then  $V(f; a, b) = \int_a^b |f'| dx$ .



C.f. examples above.

*Proof.* Mean Value Theorem.

$$\sum_i |f(x_i) - f(x_{i-1})| = \sum_i |f'(\xi_i)|(x_i - x_{i-1})$$

$x_{i-1} < \xi_i < x_i$ , the above converges to integral.  $\square$

3. Subadditive wrt  $f$ :  $V(f + g; a, b) \leq V(f; a, b) + V(g; a, b)$ .

*Proof.* Triangle inequality.

$$|[f(x_i) + g(x_i)] - [f(x_{i-1}) + g(x_{i-1})]|$$

then use triangle splits into two and done.

Why not exactly like integral, i.e. strict inequality. Take nonconstant function  $f$  and then  $g = -f$ . Note  $f + g$  identically zero.

Sum of monotone functions are bounded variation.

**Theorem 2.6** (BV Decomposition). *If  $f$  is BV on  $[a, b]$ , then there exists increasing  $g, h$  on  $[a, b]$  such that  $f = g - h$ .*

*Proof.* Let  $g(x) = V(f; a, x)$ , increasing (monotonicity wrt interval). This forces  $h(x) := g(x) - f(x)$ . Why is  $h(x) \leq h(y)$  for  $x \leq y$ ?  $V(f; a, x) - f(x) \leq V(f; a, y) - f(y)$ . Why?  $f(y) - f(x) \leq V(f; a, y) - V(f; a, x) = V(f; x, y)$ , clearly this right side is  $\geq |f(x) - f(y)|$ .  $\square$

**Theorem 2.7** (Egorov's Theorem). *Suppose  $\{f_k\}$  is a collection of measurable functions,  $f_k : E \rightarrow \mathbb{R}$ ,  $|E| < \infty$ , and  $f_k \rightarrow f$  a.e., where  $f : E \rightarrow \mathbb{R}$ . Then for all  $\epsilon > 0$ , there exists a closed set  $E(\epsilon) \subset E$  such that  $f_k \rightarrow f$  uniformly on  $E(\epsilon)$  and  $|E \setminus E(\epsilon)| < \epsilon$ .*

*Proof.* We need to find a set on which  $\{f_k\}$  converges uniformly. It suffices to find, for each  $j \in \mathbb{N}$ ,  $E_j \subset E$  such that  $|E \setminus E_j| < \epsilon/2^j$ , and  $\sup_{E_j} |f - f_k| \leq 1/j$  for sufficiently large  $k$ . Indeed, letting  $E(\epsilon) := \bigcap_j E_j$ , then  $f_k \rightarrow f$  uniformly on  $E(\epsilon)$ , and  $|E \setminus E(\epsilon)| \leq \sum_j \epsilon/2^j < \epsilon$ .

To find  $E_j$ , consider the set  $G_m := \{x : |f_k(x) - f(x)| < 1/j \text{ for all } k \geq m\}$ . Since  $|f_k - f| \rightarrow 0$  a.e.,  $\bigcup_m G_m$  is almost  $E$ , i.e.  $|\bigcup_m G_m| = |E|$ . By continuity of measure for nested unions,  $|G_m| \rightarrow |E|$ . Therefore, there exists  $m$  such that  $|E \setminus G_m| < \epsilon/2^j$ . Defining  $E_j := G_m$  completes the proof.  $\square$

**Theorem 2.8** (Lusin's Theorem). *Let  $E$  be a measurable set. A function  $f : E \rightarrow \mathbb{R}$  is measurable if and only if for all  $\epsilon > 0$ , there exists a closed set  $E(\epsilon) \subset E$  such that  $f|_{E(\epsilon)}$  is continuous, and  $|E \setminus E(\epsilon)| < \epsilon$ .*

*Proof.* Suppose that  $f$  is a simple function, i.e.  $f = \sum_{k=1}^n a_k \chi_{E_k}$ , where  $E_1, \dots, E_n$  are disjoint measurable sets. Choose a closed subset  $E'_k \subset E_k$  such that  $|E_k \setminus E'_k| < \epsilon/n$ . Then  $E(\epsilon) := \bigcup_{k=1}^n E'_k$  so that  $|E \setminus E(\epsilon)| < \epsilon$ . It only remains to check that  $f|_{E(\epsilon)}$  is continuous, but this is immediate as the preimage of any set is closed.

Now suppose only that  $|E| < \infty$ . There is a collection of simple functions  $\{f_k\}$  such that  $f_k \rightarrow f$  pointwise. Choose some  $E(\epsilon/2^k)$  for  $f_k$  as in the case above. By Egorov's Theorem,  $f_k \rightarrow f$  uniformly on a set  $E(\epsilon)$ . Now  $f$  is continuous on  $E(\epsilon) \cap (\bigcap_k E_k(\epsilon/2^k))$ , which has compliment of measure at most  $\epsilon + \sum_k \epsilon/2^k = 2\epsilon$ .

Now suppose that  $|E| = \infty$ . Write  $E = \bigcup [E \cap \{|j| \leq |x| \leq j+1\}]$ . In each shell, we have  $E_j(\epsilon/2^j)$ , closed set such that  $f|_{E_j(\epsilon/2^j)}$  is continuous. Note that  $f$  continuous on  $\bigcup_j E_j(\epsilon/2^j)$ .  $\square$

The converse direction of Lusin's theorem is simple since  $f|_{E(1/j)}$  is continuous, then we have  $f$  Borel on  $\bigcup E(1/j)$  simply because  $\{x \in \bigcup E(1/j) : f > a\} = \bigcup \{x \in E(1/j) : f > a\}$ , countable union of Borel sets. and the rest has measure 0.

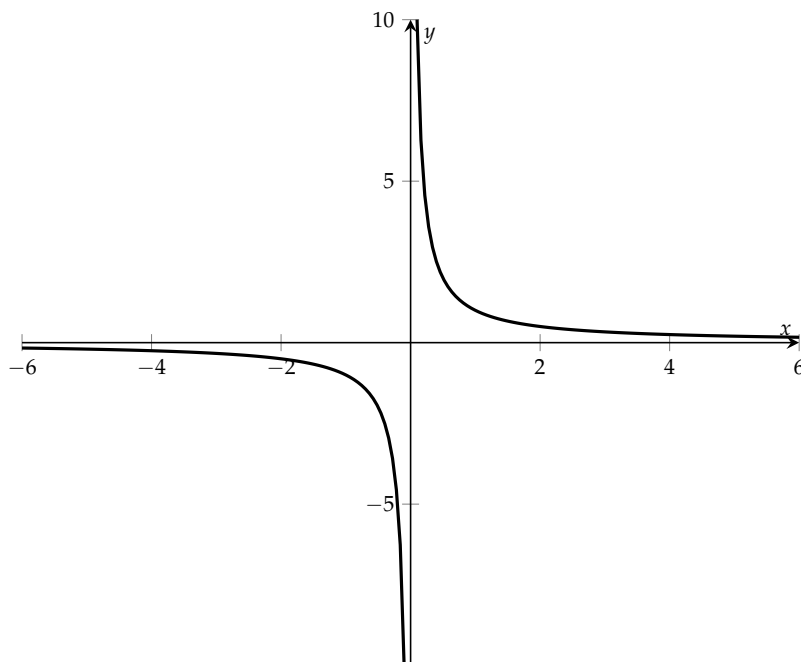
## 2.1 Convergence in Measure

Recall  $\limsup E_k := \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k = \{x : x \in E_k, \text{ infinitely many } k\}$ . We will not need  $\liminf$  quite as regularly as  $\limsup$ .

**Definition** (Convergence in Measure). Let  $\{f_k\}$  be a collection of measurable functions,  $f_k : E \rightarrow \mathbb{R}$ . We say  $f_k$  converges to  $f$  in measure if for all  $\epsilon > 0$ ,  $|\{ |f - f_k| > \epsilon \}| \rightarrow 0$ . We denote this by  $f_k \xrightarrow{m} f$ .

If we expand this definition out more, we have the following:  $f_k \xrightarrow{m} f$  if for  $\epsilon > 0$  and for all  $\epsilon_2 > 0$ , there exists  $N$  such that  $|\{ |f - f_k| > \epsilon \}| < \epsilon_2$  for all  $k \geq N$ . If this is possible, one may as well replace the double epsilon with a single epsilon for a more concise version. So the definition could be taken as follows:  $f_k \xrightarrow{m} f$  if for all  $\epsilon > 0$ , there exists  $N$  such that  $|\{ |f_k - f| > \epsilon \}| < \epsilon$  for all  $k \geq N$ .

**Example 2.3.**  $f_k(x) = \frac{1}{kx} \xrightarrow{m} 0$  on  $\mathbb{R}$ .



We know that  $\{|f_k| > \epsilon\} = \{|x| < \frac{1}{k\epsilon}\}$ . This set has measure  $2/(k\epsilon)$ , which tends to zero as  $k$  tends to infinity.  $\triangleleft$

**Example 2.4.**  $f_k(x) = \frac{x}{k} \xrightarrow{m} 0$  on  $\mathbb{R}$ .  $\{|f_k| > \epsilon\} = \{|\frac{x}{k}| > \epsilon\} = \{|x| > k\epsilon\}$ , which has infinite measure.  $\triangleleft$

**Example 2.5.** Convergence in measure does not imply pointwise convergence. Let  $f_k = \chi_{I_k}$ , where  $I_k$  are intervals.

$[0,1]$  measure 1  $[0,1/2]$ ,  $[1/2,1]$  measure 1/2  $[0,1/3]$ ,  $[1/3,2/3]$ ,  $[2/3,1]$  measure 1/3

For all  $\epsilon > 0$ ,  $\{|f_k| > \epsilon\} < |I_k| \rightarrow 0$ .

So the sequence converges to 0 in measure. However, it clearly does not converge to 0 in measure.

There are infinitely many times point part of the interval. That is for all  $x \in (0,1)$ , there are infinitely many  $k$  such that  $x \in I_k$  so that  $f_k(x) \not\rightarrow 0$ . This can be expressed as saying  $\limsup I_k = [0,1]$ .

**Theorem 2.9.** If  $|E| < \infty$ , then convergence a.e. implies convergence in measure.

*Proof.* Given  $\epsilon > 0$ , Egorov's Theorem gives  $E(\epsilon)$  such that  $|E \setminus E(\epsilon)| < \epsilon$ , and  $f_k \rightarrow f$  uniformly on  $E(\epsilon)$ . Then there exists  $N$  such that  $|f_k - f| < \epsilon$  on  $E(\epsilon)$  for all  $k \geq N$ . But then  $|\{|f_k - f| > \epsilon\}| < \epsilon$ , which implies convergence in measure.  $\square$

**Theorem 2.10.** If  $f_k \xrightarrow{m} f$ , then there is a subsequence  $\{f_{k_j}\}$  converging to  $f$  a.e..

*Proof.* Choose  $\epsilon = 1/2^j$ . Find  $N_j$  such that  $|\{ |f_k - f| > 1/2^j \}| < 1/2^j$  for all  $k \geq N$  (can make  $N_{j+1} > N_j$ ). We have  $\{f_{N_j}\}$ . We claim that  $f_{N_j} \rightarrow f$  a.e.. Let  $E_j = \{ |f_{N_j} - f| > 1/2^j \}$ . Since  $|E_j| < 1/2^j$ , we know  $\sum |E_j| < \infty$ . But then by HW 1, we know  $|\limsup E_j| = 0$ . For all  $x \notin \limsup E_j$ , we have  $x \in E_j$  only finitely many times. Then for sufficiently large  $j$ ,  $x \notin E_j$ , say  $x \notin E_j$  for  $j \geq j_0$ . But then  $|f_{N_j}(x) - f(x)| < 1/2^j \rightarrow 0$ .  $\square$

**Remark.** “Triangle Inequality” (quasi). Convergence in measure is not like convergence in metric space, though it is tempting to think of it in this way. We do have a replacement for a parallel triangle inequality. If  $|\{ |f - g| > \epsilon/2 \}| < \epsilon/2$ , and  $|\{ |g - h| > \epsilon/2 \}| < \epsilon/2$ , then  $|\{ |f - h| > \epsilon \}| < \epsilon$ . The proof is routine:  $|f - h| \leq |f - g| + |g - h|$ , so  $|f - h| > \epsilon$  then one of the right side must be at least  $\epsilon/2$ . Then  $\{ |f - h| > \epsilon \} \subset \{ |f - g| > \epsilon/2 \} \cup \{ |g - h| > \epsilon/2 \}$ . Their measures being at most  $\epsilon/2$ , the result then follows.

**Definition** (Cauchy in Measure). A collection of measurable functions  $\{f_k\}$  is said to be Cauchy in measure if for all  $\epsilon > 0$ , there exists  $N$  such that  $|\{ |f_j - f_k| > \epsilon \}| < \epsilon$  for all  $k, j \geq N$ .

As it turns out, this is equivalent to convergence in measure.

**Theorem 2.11.** Cauchy in measure is equivalent to convergence in measure.

*Proof (Sketch).*  $\leftarrow$  is immediate by “triangle inequality” measure remark. Now the other direction. There are three standard steps that apply in these types of arguments. The strategy being the important piece here. Step 1. Enough to obtain a convergent subsequence:  $f_1, f_2, f_3, \dots$  cauchy sequence of functions with convergent subsequence. Use triangle inequality to show must converge to limit of conv subsequence. Conclude  $f_k \rightarrow f$ . Notice this holds for any type of convergence in any space. Step 2: Every Cauchy sequence has a “geometric” subsequence. Meaning  $|\{ |f_{k_j} - f_{k_{j+1}}| > 1/2^j \}| < 1/2^j$ . This is true for any metric space and any convergence. Use  $\epsilon = 1/2^j$ , and get  $k_j$ . Step 3. Geometric trick implies convergent: given  $\epsilon > 0$ , let  $m$  be such that  $1/2^{m-1} < \epsilon$ . Let  $A = \bigcup_{j \geq m} \{ |f_{k_j} - f_{k_{j+1}}| > 1/2^j \}$ . Note  $|A| < \sum_{j=m}^{\infty} 1/2^j = 1/2^{m-1} < \epsilon$ . On  $E \setminus A$ , we have  $|f_{k_j} - f_{k_{j+1}}| \leq 1/2^j$  for all  $j \geq m$ . But this implies uniform convergence on the set  $E \setminus A$ . But then difference everywhere at most  $\epsilon$  on  $E \setminus A$  for sufficiently large  $j$ . Therefore,  $|\{ |f_{k_j} - f| > \epsilon \}| \leq |A| < \epsilon$ .  $\square$

Two approaches to  $\int_E f, f \geq 0$  on  $E$ .

First,  $\int_E f = \sup \{ \int_E g : g \text{ simple } g \leq f \}$ ,  $g = \sum_{k=1}^m a_k \chi_{E_k}$ , where  $E_k$  are disjoint, measurable sets. Here  $\int_E g = \sum a_k |E_k|$ .

Second,  $\int_E f = |R(f, E)|$ , where  $R(f, E) = \{ (x, y) \in \mathbb{R}^{n+1} : x \in E, 0 \leq y \leq f(x) \}$ .

First approach is common in many texts, we shall take the second.

So  $\int_E f$  exists if and only if  $R(f, E)$  is measurable.

Some advantages of the second approach are monotonicity and countable additivity over  $E$ : so  $0 \leq f \leq g$  and  $\int_E f, \int_E g$  exists, then  $\int_E f \leq \int_E g$  because  $R(f, E) \subset R(g, E)$ .

Also if  $E = \bigcup_{k \in \mathbb{N}} E_k$ , disjoint measurable sets, and  $\int_{E_k} f$  exists, then  $\int_E f$  exists and  $\int_E f = \sum_{k \in \mathbb{N}} \int_{E_k} f$  because  $R(f, E) = \bigcup R(f, E_k)$ , the union taken over the disjoint, measurable sets. Can show in other definition but takes work due to sup, whereas here we get it immediately. But we have the issue of existence and measurability.

The following main theorem.

**Theorem 2.12.** *If  $f: E \rightarrow [0, \infty]$  is measurable, then  $\int_E f$  exists (possibly  $\infty$ ).*

Idea of proof is from 4.1,  $f = \lim f_k$ ,  $f_1 \leq f_2 \leq \dots$  are simple. So we will show  $R(f, E)$  are measurable and take their union. Need Lemma 1,  $R(f_k, E)$  is measurable. Lemma 2  $\Gamma(f, E)$ , which needs to be defined (simply the graph  $\{(x, f(x)) : x \in E\}$ ) has measure 0. This is boundary top of graph of shaded area of graph, we care because  $\bigcup R(f_k, E)$  need not generally cover  $R(f, E)$ . It will surely cover  $R(f, E) \setminus \Gamma(f, E)$ , then  $y < f(x)$ , so there exists  $k$  such that  $y < f_k(x)$  so that  $(x, y) \in R(f_k, E)$ .

Both lemmas rely on

**Lemma 2.1.** *If  $E \subseteq \mathbb{R}^n$  is measurable, then  $E_a := E \times [a, b]$  is measurable in  $\mathbb{R}^{n+1}$  with measure  $|E_a| = |E| \cdot a$ , noting that  $0 \cdot \infty = 0$  in measure theory.*

*Proof.* If  $E$  is an interval, follows by volume formula. If  $E$  is general, then tile it by intervals, half open (so intervals disjoint).

If  $E$  is  $G_\delta$  of finite measure, write  $E = \bigcap G_k$ ,  $G_1 \supset \dots$  are open.

$$E_a = \bigcap_k (G_k \times [0, a]) \text{ so } |E_a| = \lim_k |G_k \times [0, a]| = a \lim_k |G_k| = a \cdot |E|.$$

What about general set?  $|E| < \infty$ ,  $E = H \setminus Z$ ,  $M$  is  $G_\delta$ ,  $|M| = |E|$ ,  $|H_a| = |H| \cdot a = |E| \cdot a$ . So  $|E_a| \leq |E| \cdot a$ .

Also for all  $\epsilon > 0$ , there exists closed  $F \subset E$  (hence  $G_\delta$ ),  $|F| > |E| - \epsilon$ .

$$|F_a| = a|F| > a|E| - a\epsilon$$

So  $E_a$  contains  $F_a$  closed of measure  $> |E_a| - a\epsilon$ . Hence  $E_a$  is measurable.  $|E_a| = a|E|$ .

Still assumes  $a < \infty$ , otherwise  $a = k$  and let  $k \rightarrow \infty$

□?

**Lemma 2.2.**  *$R(f_k, E) = \bigcup_j (E_j \times [0, a_j])$ ,  $E_j$  disjoint measurable sets, where  $f_k = \sum_j a_j \chi_{E_j}$ .*

Note that this shows that if  $f$  takes countably many values,  $f = a_k$  on  $E_k$ , then  $\int_E f = \sum a_k |E_k|$ .

**Lemma 2.3.**  $|\Gamma(f, E)| = 0$ . Let  $g = 2^{-j} \text{floor}(2^j f)$ , then  $g \leq f \leq g + 2^{-j}$ . Then  $R(g, E) \subset R(f, E) \subset \dots$ . Then  $\Gamma(f, E) \subset R(g + 2^{-j}, E) \setminus R(g - 2^{-j}, E)$ . This right side has measure at most  $2 \cdot 2^{-j} |E| \rightarrow 0$ , hence  $|\Gamma(f, E)| = 0$ .

We have not yet proved  $\int_E cf = c \int_E f$ , where  $c \geq 0$  and  $f \geq 0$ .

**Proposition 2.1.**

$$\int_E cf = c \int_E f$$

*Proof.* This is true for simple functions:  $\int c(\sum a_k \chi_{E_k}) = \int \sum ca_k \chi_{E_k} = \sum ca_k |E_k| = c \sum a_k |E_k| = c \int \sum a_k \chi_{E_k}$ . Then for general functions use the Monotone Convergence Theorem. For general  $f \geq 0$ , take simple  $f_k \geq 0$ ,  $f_k \nearrow f$ , then  $cf_k \nearrow cf$ , by MCT  $\int cf = \lim \int cf_k = \lim c \int f_k = c \int f$ .  $\square$

What if the convergence is not  $f_k \nearrow f$ ?

**Example 2.6.**  $\chi_{[k, \infty)} \rightarrow 0$  pointwise. But  $\int \chi_{[k, \infty)}$  are all infinite and certainly do not converge to 0. Note that  $\chi_{[k, \infty)} \searrow 0$ .

**Example 2.7.**  $k\chi_{(0, 1/k)}$ . But  $\int k\chi_{(0, 1/k)} = 1 \not\rightarrow 0$ .

**Example 2.8.**  $\frac{1}{k}\chi_{[0, k]} \rightarrow 0$  uniformly but  $\int \frac{1}{k}\chi_{[0, k]} = 1 \rightarrow 0$ .

What is the similarity of these three examples? The functions vanish but their integrals do not???? The general principle is that the area under the graph can disappear or 'escape' to infinity but does not appear from nowhere..... A precise statement is Fatou's Lemma. As a reminder,

**Definition** (limsup, liminf). Given a sequence  $\{f_k\}$ ,

$$\limsup_{k \rightarrow \infty} f_k = \lim_{n \rightarrow \infty} \sup_{k \geq n} f_k \quad \liminf_{k \rightarrow \infty} f_k = \lim_{n \rightarrow \infty} \inf_{k \geq n} f_k$$

**Example 2.9.** (i)  $\lim f_k(x)$  exists if and only if  $\limsup f_k(x) = \liminf f_k(x)$ .

(ii) The Piano Key Sequence (press one key going left to right):  $\chi_{[0, 1]}$ , then  $\chi_{[0, 1/2]}, \chi_{[1/2, 1]}$ , then  $\chi_{[0, 1/3]}, \chi_{[1/3, 2/3]}, \chi_{[2/3, 1]}$ . [Go in pascal triangle order]

$$\limsup f_k = \chi_{[0, 1]}$$

$$\liminf f_k = 0$$

**Remark.** Relation to sequences to sets  $\{E_k\}$ :  $\limsup E_k = \bigcap_{m=1}^{\infty} \bigcup_{k \geq m} E_k$ ,  $\liminf E_k = \bigcup_{m=1}^{\infty} \bigcap_{k \geq m} E_k$ .  $\limsup \chi_{E_k} = \chi_{\limsup E_k}$  and same for liminf. and sup like union and inf like intersection.

**Lemma 2.4** (Fatou).  $f_k \geq 0$  measurable.

$$\int_E \liminf f_k \leq \liminf \int_E f_k$$

*Proof.* Let  $g_m = \inf_{k \geq m} f_k$ . Then  $g_m \nearrow \liminf f_k$ . By MCT,  $\int g_m \rightarrow \int \liminf f_k$ . But  $f_k \geq g_m$  so that  $\int f_k \geq \int g_m$ , hence  $\liminf \int f_k \geq \lim \int g_m$ . [Rem: if  $a_k \geq b_k$  for all  $k$ , then  $\liminf a_k \geq \liminf b_k$ ,  $\limsup a_k \geq \limsup b_k$ .]  $\square$

**Theorem 2.13** (Lebesgue Dominated Convergence (DCT)). Suppose  $f_k \geq 0$  are measurable and there is  $\phi \geq 0$  measurable such that  $f_k \leq \phi$  ( $\phi$  dominates) and  $\int_E \phi < \infty$ . If  $f_k$  converges to  $f$  a.e., then  $\int_E f_k \rightarrow \int_E f$ .

*Proof.* Fatou's Lemma gives  $\int f \leq \liminf \int f_k$ . Also,  $\phi - f_k \geq 0$ . Then Fatou's Lemma applies again:  $\int \phi - f \leq \liminf \int (\phi - f_k)$ . Which is  $\int f \geq \limsup \int f_k$ . Then  $\int \phi - \int f \leq \liminf (\int \phi - \int f_k)$  cancel constant  $-\int f \leq \liminf (-\int f_k)$  then  $\int f \geq \limsup \int f_k$ . Comparing very first and last equations, we have  $\int f = \lim_{k \rightarrow \infty} \int f_k$ .  $\square$

**Example 2.10.**  $\frac{1}{k} \chi_{[0,k]} \rightarrow 0$  uniformly.

Try to cover with one function get something like  $1/x$ . This is not good enough for DCT since  $\int \frac{1}{x} = \infty$ . If  $f_k \leq \frac{1}{x^2}$  on  $[1, \infty)$ , this examples does not work. 'Best' you can do is  $\frac{1}{k^2} \chi_{[0,k]} \rightarrow 0 = \int \lim f_k$ .

**Example 2.11.**  $\lim \int_{[1,\infty)} \frac{\sin^{2k} x}{x^2} dx$ . Now  $\frac{\sin^{2k} x}{x^2} \rightarrow 0$  a.e. and is at most  $\frac{1}{x^2}$  in absolute value, so must be 0. Now if  $\lim \int_{[0,\infty)} \frac{\sin^{2k} x}{x^2} dx$ ,  $\frac{\sin^{2k} x}{x^2} \leq \frac{\sin^2 x}{x^2}$  and  $\int_{[0,1]} \frac{\sin^2 x}{x^2} \leq \int_{[0,1]} 1 \leq 1$ . so  $\int_{[0,\infty)} \frac{\sin^2 x}{x^2} < \infty$ .

### **3 Quiz Problems**



## 4 Quiz Solutions

## 5 Homework Problems

**Problem 1.** Given an arbitrary set  $A \subset \mathbb{R}$  and a number  $c > 0$ , let  $B = \{ca : a \in A\}$ . Prove that  $|B|_e = c |A|_e$ .

**Problem 2.** Suppose that a set  $A \subset \mathbb{R}$  is measurable. Prove that for every  $c > 0$  the set  $B = \{ca : a \in A\}$  is also measurable.

**Problem 3.** Given a sequence of continuous functions  $f_k : \mathbb{R} \rightarrow \mathbb{R}$ , let  $B$  be the set of all points  $x \in \mathbb{R}$  such that the sequence  $\{f_k(x)\}$  is bounded. Prove that  $B$  is a measurable set. [Hint: try to construct  $B$  from the sets  $\{x : |f_k(x)| \leq M\}$  by using countable unions and intersections.]

**Problem 4.** Given a sequence of continuous functions  $f_k : \mathbb{R} \rightarrow \mathbb{R}$ , let  $C$  be the set of all points  $x \in \mathbb{R}$  such that  $\lim_{k \rightarrow \infty} f_k(x) = 0$ . Prove that  $C$  is a measurable set.

**Problem 5.** Prove that the set

$$A = \{x \in \mathbb{R} : \exists k \in \mathbb{N} \text{ such that } |2^x - 2^k| \leq 1\}$$

is measurable and  $|A| < \infty$ .

**Problem 6.** Prove that the set

$$A = \{x \in [0, 1] : \forall q \in \mathbb{N} \exists p \in \mathbb{N} \text{ such that } |x - p/q| \leq 1/q^2\}$$

is measurable and  $|A| = 0$ .

**Problem 7.** Suppose  $E$  and  $Z$  are sets in  $\mathbb{R}^n$  such that  $E \cup Z$  is measurable and  $|Z| = 0$ . Prove that  $E$  is measurable.

**Problem 8.** Given a continuous function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , define  $\mathcal{M} = \{E \subset \mathbb{R}^n : f^{-1}(E) \text{ is Borel}\}$ .

(a) Prove that  $\mathcal{M}$  is a  $\sigma$ -algebra.

(b) Prove that if  $E$  is Borel, then  $f^{-1}(E)$  is Borel. [Hint: Use (a).]

**Problem 9.** Suppose  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a function with a continuous derivative. Prove that for every measurable set  $E$ , the set  $f(E)$  is also measurable. [Hint: although  $f$  need not be Lipschitz, its restriction to any bounded interval is Lipschitz.]

**Problem 10.** Given a set  $E \subset [0, \infty)$ , define a function  $f: [0, \infty) \rightarrow [0, \infty)$  by  $f(x) = |E \cap [0, x]|_e$ .

- (a) Prove that  $f$  is Lipschitz continuous.
- (b) Prove that for every number  $b$  with  $0 < b < |E|_e$  there exists a set  $F \subset E$  such that  $|F|_e = b$ .

**Problem 11.** Show that there exists a nested sequence of sets  $E_1 \supset E_2 \supset \cdots$  such that  $|E_1|_e < \infty$  and  $\bigcap_{k=1}^{\infty} E_k = \emptyset$  but  $\lim_{k \rightarrow \infty} |E_k|_e > 0$ ; that is, outer measure is not continuous under nested intersections. [Hint: Use the translates of the Vitali set.]

**Problem 12.** Show that for the standard middle-third Cantor set  $C \subset [0, 1]$ , the difference set  $C - C$  contains a neighborhood of 0. [Hint:  $C$  is the intersection of nested sets  $C_n$  where  $C_0 = [0, 1]$  and  $C_{n+1} = \frac{1}{3}C_n \cup (\frac{1}{3}C_n + \frac{2}{3})$ . Find  $C_n - C_n$  using induction.]

*Remark:* This shows that having  $|E| > 0$  is not necessary for  $E - E$  to contain a neighborhood of 0.

**Problem 13.** Suppose that  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is a function such that  $f(\mathbb{R}^n)$  is countable and  $f^{-1}(t)$  is measurable for every  $t \in \mathbb{R}$ . Prove that  $f$  is measurable.

**Problem 14.** Prove that without the assumption “ $f(\mathbb{R}^n)$  is countable” the statement in Problem 13 would not be true.

**Problem 15.** Suppose that  $f: \mathbb{R} \rightarrow \mathbb{R}$  is measurable, and  $g: \mathbb{R} \rightarrow \mathbb{R}$  is continuously differentiable with  $g' > 0$  everywhere. Prove that  $f \circ g$  is measurable.

**Problem 16.**

- (a) Suppose  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuous function such that  $f^2$  is measurable. Prove that  $f$  is measurable.
- (b) Prove that the statement in (a) is false if  $f$  is not assumed continuous.

**Problem 17.** Suppose that  $f : E \rightarrow \mathbb{R}$  is a measurable function, where  $E \subset \mathbb{R}^n$  is measurable.

- (a) Prove that there exists a Borel set  $H \subset E$  such that the restriction  $f|_H$  is Borel measurable and  $|E \setminus H| = 0$ .
- (b) If, in addition,  $E$  is a Borel set, prove that there exists a Borel measurable function  $g : E \rightarrow \mathbb{R}$  such that  $f = g$  a.e.. [Hint: For part (a), take a countable union of closed sets obtained from Lusin's Theorem.]

**Problem 18.** Suppose  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a function such that  $\phi(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Consider a sequence of measurable functions  $\{f_k\}$ ,  $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$ , such that  $|f_k(x)| \leq \phi(|x|)$  for every  $k$ , and  $f_k \rightarrow f$  a.e.. Prove that the conclusion of Egorov's Theorem holds in this situation; that is, for every  $\epsilon > 0$ , there exists a closed set  $E(\epsilon) \subset \mathbb{R}^n$  such that  $|\mathbb{R}^n \setminus E(\epsilon)| < \epsilon$  and  $f_k \rightarrow f$  uniformly on  $E(\epsilon)$ . [Hint: Follow the proof of Egorov's theorem.]

**Problem 19.** Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a function such that  $f(\mathbb{R}^n)$  is countable and  $f^{-1}(t)$  is measurable for every  $t \in \mathbb{R}$ . Prove that  $f$  is measurable.

**Problem 20.** Prove that without the assumption " $f(\mathbb{R}^n)$  is countable" the statement in Problem 19 would not be true.

## 6 Homework Solutions

**Solution Problem 1.** Given an arbitrary set  $A \subset \mathbb{R}$  and a number  $c > 0$ , let  $B = \{ca : a \in A\}$ . Prove that  $|B|_e = c |A|_e$ .

*Proof.* Given  $\epsilon > 0$ , let  $\{[s_k, t_k]\}$  be a countable cover of  $A$  such that

$$\sum_k (t_k - s_k) \leq |A|_e + \epsilon$$

The intervals  $[cs_k, ct_k]$  cover  $B$ , since every point of  $B$  is of the form  $ca$  where  $a \in A$  is covered by some interval  $[s_k, t_k]$ . Therefore,

$$|B|_e \leq \sum_k (ct_k - cs_k) = c \sum_k (t_k - s_k) \leq c |A|_e + \epsilon$$

Since  $\epsilon > 0$  was arbitrary, it follows that  $|B|_e \leq c |A|_e$ .

It remains to observe that  $A = c^{-1}B$ , which by the above implies  $|A|_e \leq c^{-1} |B|_e$ , i.e.,  $|B|_e \geq c |A|_e$ . Thus,  $|B|_e = c |A|_e$ .  $\square$

**Solution Problem 2.** Suppose that a set  $A \subset \mathbb{R}$  is measurable. Prove that for every  $c > 0$  the set  $B = \{ca : a \in A\}$  is also measurable.

*Proof.* Given  $\epsilon > 0$ , let  $G$  be an open set that contains  $A$  and satisfies  $|G \setminus A|_e < \epsilon/c$ . Since the function  $f(x) = x/c$  is continuous, the preimage of  $G$  under this function is also open. This preimage  $f^{-1}(G)$  is  $cG$ . Since  $A \subset G$ , it follows that  $B \subset cG$ . Moreover by Problem 1,

$$|(cG) \setminus B|_e = |c(G \setminus A)|_e = c |G \setminus A|_e < \epsilon.$$

Since  $\epsilon$  was arbitrary, this proves that  $B$  is measurable.  $\square$

**Solution Problem 3.** Given a sequence of continuous functions  $f_k : \mathbb{R} \rightarrow \mathbb{R}$ , let  $B$  be the set of all points  $x \in \mathbb{R}$  such that the sequence  $\{f_k(x)\}$  is bounded. Prove that  $B$  is a measurable set. [Hint: try to construct  $B$  from the sets  $\{x : |f_k(x)| \leq M\}$  by using countable unions and intersections.]

*Proof.* For  $k, m \in \mathbb{N}$ , let

$$A(k, m) = \{x \in \mathbb{R} : |f_k(x)| \leq m\} = f_k^{-1}([-m, m]).$$

Being the preimage of a closed set under a continuous function,  $A(k, m)$  is closed, and in particular measurable. Let

$$A := \bigcup_{m=1}^{\infty} \bigcap_{k=1}^{\infty} A(k, m),$$

which is also measurable, being obtained from measurable sets by countable set operations. We claim that  $A = B$ .

If  $x \in A$ , then there exists  $m \in \mathbb{N}$  such that  $|f_k(x)| \leq m$  for all  $k \in \mathbb{N}$ , which shows the sequence  $\{f_k(x)\}$  is bounded.

Conversely, if the sequence  $\{f_k(x)\}$  is bounded, then there exists  $m \in \mathbb{N}$  such that all elements of the sequence are at most  $m$  in absolute value. This means  $|f_k(x)| \leq m$  for all  $k$ , hence  $x \in A$ .  $\square$

**Solution Problem 4.** Given a sequence of continuous functions  $f_k: \mathbb{R} \rightarrow \mathbb{R}$ , let  $C$  be the set of all points  $x \in \mathbb{R}$  such that  $\lim_{k \rightarrow \infty} f_k(x) = 0$ . Prove that  $C$  is a measurable set.

*Proof.* For  $k, m \in \mathbb{N}$ , let

$$A(k, m) = \{x \in \mathbb{R} : |f_k(x)| < 1/m\} = f_k^{-1}((-1/m, 1/m)).$$

Being the preimage of an open set under a continuous function,  $A(k, m)$  is open, and in particular measurable. Let

$$A := \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} A(k, m).$$

This set is also measurable, being obtained from measurable sets by countable set operations. We claim that  $A = C$ .

Suppose  $x \in A$ . Given  $\epsilon > 0$ , pick  $m \in \mathbb{N}$  such that  $1/m \leq \epsilon$ . Since  $x \in \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} A(k, m)$ , there exists  $N$  such that  $x \in \bigcap_{k=N}^{\infty} A(k, m)$ , which means  $|f_k(x)| < 1/m$  for all  $k \geq N$ . Thus,  $|f_k(x)| < \epsilon$  for all  $k \geq N$ , which proves  $\lim_{k \rightarrow \infty} f_k(x) = 0$ .

Conversely, suppose  $x \in C$ . Given  $m \in \mathbb{N}$ , use the definition of the limit  $\lim_{k \rightarrow \infty} f_k(x) = 0$  to find  $N$  such that  $|f_k(x)| < 1/m$  for all  $k \geq N$ . The latter means  $x \in \bigcap_{k=N}^{\infty} A(k, m)$ . Therefore, for every  $m \in \mathbb{N}$  the inclusion  $x \in \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} A(k, m)$  holds. This means  $x \in A$ .  $\square$

**Solution Problem 5.** Prove that the set

$$A = \{x \in \mathbb{R} : \exists k \in \mathbb{N} \text{ such that } |2^x - 2^k| \leq 1\}$$

is measurable and  $|A| < \infty$ .

*Proof.* For each  $k \in \mathbb{N}$ , the inequality  $|2^x - 2^k| \leq 1$  is equivalent to  $\log_2(2^k - 1) \leq x \leq \log_2(2^k + 1)$ . Thus,  $A = \bigcup_{k=1}^{\infty} I_k$ , where  $I_k = [\log_2(2^k - 1), \log_2(2^k + 1)]$ . Begin an interval, each  $I_k$  is measurable. Hence,  $A$  is measurable. By countable subadditivity of measure,  $|A| \leq \sum_{k=1}^{\infty} |I_k|$ . It remains to show the series  $\sum_{k=1}^{\infty} |I_k|$  converges. This can be done by the

comparison test, limit comparison test, or the ratio test. We use the Limit Comparison Test with  $\sum_{k=1}^{\infty} 2^{-k}$  as a reference series:

$$\begin{aligned} \frac{|I_k|}{2^{-k}} &= \frac{\log_2(2^k + 1) - \log_2(2^k - 1)}{2^{-k}} \\ &= \frac{k + \log_2(1 + 2^{-k}) - (k + \log_2(1 - 2^{-k}))}{2^{-k}} \\ &= \frac{\log_2(1 + 2^{-k}) - \log_2(1 - 2^{-k})}{2^{-k}} \\ &= \frac{1}{\log 2} \left\{ \frac{\log(1 + 2^{-k})}{2^{-k}} + \frac{\log(1 - 2^{-k})}{-2^{-k}} \right\} \xrightarrow{k \rightarrow \infty} \frac{2}{\log 2} \end{aligned}$$

Here the last step is based on  $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$ . Since  $\sum_{k=1}^{\infty} 2^{-k}$  converges, so does  $\sum_{k=1}^{\infty} |I_k|$ .  $\square$

**Solution Problem 6.** Prove that the set

$$A = \{x \in [0, 1] : \forall q \in \mathbb{N} \exists p \in \mathbb{N} \text{ such that } |x - p/q| \leq 1/q^2\}$$

is measurable and  $|A| = 0$ .

*Proof.* Let  $A_q = \bigcup_{p=1}^{\infty} E(p, q)$ , where  $E(p, q) = \left[ \frac{p}{q} - \frac{1}{q^2}, \frac{p}{q} + \frac{1}{q^2} \right] \cap [0, 1]$ . This is a countable union of measurable sets  $E(p, q)$  (which are intervals, possibly empty), so it is measurable. Then the set  $A = \bigcap_{q \in \mathbb{N}} A_q$  is measurable too.

We have  $|E(p, q)| \leq 2/q^2$  by construction of  $E(p, q)$ . Also, when  $p > q + 1$ , we have  $\frac{p}{q} - \frac{1}{q^2} \geq 1 + \frac{1}{q} - \frac{1}{q^2} \geq 1$ , which implies  $E(p, q) = \emptyset$ . By subadditivity,

$$|A_q| \leq \sum_{p=1}^{\infty} |E(p, q)| \leq \sum_{p=1}^{q+1} \frac{2}{q^2} = \frac{2q+2}{q^2}$$

By monotonicity,  $|A| \leq |A_q|$  for each  $q$ . Since  $|A_q| \xrightarrow{q \rightarrow \infty} 0$ , it follows that  $|A| = 0$ .  $\square$

**Solution Problem 7.** Suppose  $E$  and  $Z$  are sets in  $\mathbb{R}^n$  such that  $E \cup Z$  is measurable and  $|Z| = 0$ . Prove that  $E$  is measurable.

*Proof.* Since  $Z \setminus E \subset Z$ , the monotonicity of outer measure implies  $|Z \setminus E|_e = 0$ , hence  $Z \setminus E$  is measurable. Then

$$E = (E \cup Z) \setminus (Z \setminus E)$$

is measurable, being the difference of two measurable sets. [This could be done with Carathéodory's Theorem or with the " $G_\delta$  minus a null set" theorem, but it is easier without.]  $\square$

**Solution Problem 8.** Given a continuous function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ , define  $\mathcal{M} = \{E \subset \mathbb{R}^n : f^{-1}(E) \text{ is Borel}\}$ .

- (a) Prove that  $\mathcal{M}$  is a  $\sigma$ -algebra.
- (b) Prove that if  $E$  is Borel, then  $f^{-1}(E)$  is Borel. [Hint: Use (a).]

*Proof.*

- (a) This proof does not involve  $f$  being continuous; the argument works for any map  $f$ . Taking preimages commutes with any set operations: for example,

$$f^{-1}(E^C) = \{x : f(x) \in E^C\} = \{x : f(x) \notin E\} = (f^{-1}(E))^C$$

and

$$f^{-1}\left(\bigcup_i E_i\right) = \{x : \exists i f(x) \in E_i\} = \bigcup_i f^{-1}(E_i).$$

If  $E \in \mathcal{M}$ , then  $f^{-1}(E^C) = f^{-1}(E)^C$  is the complement of a Borel set, and hence is Borel. Hence,  $E^C \in \mathcal{M}$ . Also, if  $E_k \in \mathcal{M}$  for each  $k \in \mathbb{N}$ , then

$$f^{-1}\left(\bigcup_k E_k\right) = \bigcup_k f^{-1}(E_k)$$

is the countable union of Borel sets, and hence is Borel. Therefore,  $\bigcup_k E_k \in \mathcal{M}$ .

The definition of a  $\sigma$ -algebra in the book also requires us to check that  $\mathcal{M}$  is nonempty: to do this, it suffices to notice that  $f^{-1}(\emptyset) = \emptyset$  is Borel, hence  $\emptyset \in \mathcal{M}$ .

- (b) Since  $f$  is continuous, the preimage of any open set under  $f$  is open, and is hence Borel. This means  $\mathcal{M}$  contains all open sets. By definition, the Borel  $\sigma$ -algebra  $\mathcal{B}$  is the *smallest*  $\sigma$ -algebra that contains all open sets. Thus,  $\mathcal{B} \subset \mathcal{M}$ , which by definition of  $\mathcal{M}$  means that  $f^{-1}(E)$  is Borel whenever  $E$  is Borel.

[It is tempting to approach statement (b) by "writing a Borel set  $E$  in terms of open/closed sets" and concluding that  $f^{-1}(E)$  can also be written in this way. But there is no such structural formula for Borel sets: one can only get the proper subclasses like  $G_\delta$ ,  $G_{\delta\sigma}$ ,  $G_{\delta\sigma\delta}$ , and so on. The whole story is complicated: see [Borel hierarchy](#) on Wikipedia].  $\square$



**Solution Problem 9.** Suppose  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a function with a continuous derivative. Prove that for every measurable set  $E$ , the set  $f(E)$  is also measurable. [Hint: although  $f$  need not be Lipschitz, its restriction to any bounded interval is Lipschitz.]

*Proof.* For each  $j \in \mathbb{N}$ , the set  $E_j = E \cap [-j, j]$  is measurable, as the intersection of two measurable sets is measurable. Since  $E = \bigcup_j E_j$ , it follows that  $f(E) = \bigcup_j f(E_j)$ . So it suffices to prove  $f(E_j)$  is measurable for every  $j$ .

The derivative  $f'$ , being continuous, is bounded on the interval  $[-j, j]$ . By the Mean Value Theorem,  $f$  is Lipschitz on  $[-j, j]$ : indeed,  $|f(a) - f(b)| \leq |a - b| \sup_{[-j, j]} |f'|$ . A technical detail arises: we only proved the measurability of images for Lipschitz functions on all of  $\mathbb{R}^n$ . To get around this, define

$$f_j(x) = \begin{cases} f(x), & x \in [-j, j] \\ f(-j), & x < -j \\ f(j), & x > j \end{cases}$$

Such an extended function  $f_j$  is Lipschitz continuous on all of  $\mathbb{R}$ . Indeed in each of the three closed intervals  $(-\infty, -j]$ ,  $[-j, j]$ ,  $[j, \infty)$ , the Lipschitz condition holds by construction. For arbitrary  $a < b$ , partition the interval  $[a, b]$  by the points  $\{-j, j\}$  should they lie there, and apply the Lipschitz continuity to each interval, and use the triangle inequality.

Conclusion:  $f_j(E_j)$ , which is the same as  $f(E_j)$ , is measurable.  $\square$

[Note: In fact, for every set  $E \subset \mathbb{R}^n$ , any Lipschitz function  $f: E \rightarrow \mathbb{R}^n$  can be extended to a Lipschitz function  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Therefore, when discussing the measurability of  $f(E)$ , it suffices to check that  $f$  is Lipschitz on the set  $E$ .

*Proof (Sketch).* It suffices to extend a real-valued Lipschitz function  $f: E \rightarrow \mathbb{R}$ , because the vector-valued case follows by extending each component. Let  $L$  be the Lipschitz constant of  $f$ , and define, for every  $x \in \mathbb{R}^n$ ,

$$F(x) = \inf_{a \in E} (f(a) + L|x - a|).$$

It is an exercise with the definition of  $\inf$  to prove that  $F$  is Lipschitz with constant  $L$ , and that  $F(x) = f(x)$  when  $x \in E$ .  $\square$

**Remark:** Extending a map  $f: E \rightarrow \mathbb{R}^n$  in the above fashion, one finds the Lipschitz constant of the extension is at most  $L\sqrt{n}$ , where  $L$  is the Lipschitz constant of the original map. There is a deeper extension theorem (due to Kirszbraun) according to which an extension with the same Lipschitz constant  $L$  exists.]

**Solution Problem 10.** Given a set  $E \subset [0, \infty)$ , define a function  $f: [0, \infty) \rightarrow [0, \infty)$  by  $f(x) = |E \cap [0, x]|_e$ .

- (a) Prove that  $f$  is Lipschitz continuous.
- (b) Prove that for every number  $b$  with  $0 < b < |E|_e$  there exists a set  $F \subset E$  such that  $|F|_e = b$ .

*Proof.*

- (a) We claim that  $0 \leq f(b) - f(a) \leq b - a$  for any  $a, b \in [0, \infty)$  such that  $a < b$ ; this yields the Lipschitz continuity with constant 1. On one hand,  $f(b) \geq f(a)$  by the monotonicity of outer measure:  $E \cap [0, a] \subset E \cap [0, b]$ . On the other,  $E \cap [0, b] \subset (E \cap [0, a]) \cup [a, b]$ , which implies

$$f(b) \leq |(E \cap [0, a]) \cup [a, b]|_e \leq |E \cap [0, a]|_e + |[a, b]| = f(a) + (b - a)$$

by subadditivity.

- (b) By Theorem 3.27 in the textbook, the outer measure is continuous under nested unions even if the sets are not measurable. Since  $E = \bigcup_{k \in \mathbb{N}} (E \cap [0, k])$ , it follows that

$$|E|_e = \lim_{k \rightarrow \infty} |E \cap [0, k]|_e = \lim_{k \rightarrow \infty} f(k).$$

Since  $b < |E|_e$ , by the definition of limit there exists  $k$  such that  $f(k) > b$ . Also,  $f(0) = |E \cap \{0\}| = 0$ . Applying the Intermediate Value Theorem to  $f$  on the interval  $[0, k]$  (which is possible since  $f$  is continuous by part (a)), we conclude that there exists  $x \in (0, k)$  such that  $f(x) = b$ . Then the set  $F = |E \cap [0, x]|$  meets the requirements.

□

**Solution Problem 11.** Show that there exists a nested sequence of sets  $E_1 \supset E_2 \supset \cdots$  such that  $|E_1|_e < \infty$  and  $\bigcap_{k=1}^{\infty} E_k = \emptyset$  but  $\lim_{k \rightarrow \infty} |E_k|_e > 0$ ; that is, outer measure is not continuous under nested intersections. [Hint: Use the translates of the Vitali set.]

*Proof.* Let  $V \subset [0, 1]$  be the Vitali set described in these notes: recall that  $|V|_e > 0$  and that the sets  $V + q$  are disjoint for all  $q \in \mathbb{Q}$ . Let

$$E_k = \bigcup_{j=k}^{\infty} \left( V + \frac{1}{j} \right).$$

Then  $E_1 \subset V + [0, 1] \subset [0, 2]$ , hence  $|E_1|_e \leq 2 < \infty$ .

Suppose  $x \in \bigcap_{k=1}^{\infty} E_k$ . This means that for each  $k \in \mathbb{N}$  there exists  $j \geq k$  such that  $x \in V + 1/j$ . In particular,  $x \in V + 1/j$  for infinitely many distinct values of  $j$ . But this is impossible as the sets  $V + 1/j$  are disjoint, a contradiction. Therefore,  $\bigcap_{k=1}^{\infty} E_k$  is empty.

The sets  $E_k$  are nested by construction, hence  $|E_k|_e$  is a non-increasing sequence. It is bounded from below by  $|V|_e$  because each  $E_k$  contains a translated copy of  $V$ . Thus,  $\lim_{k \rightarrow \infty} |E_k|_e \geq |V|_e > 0$ .  $\square$

**Solution Problem 12.** Show that for the standard middle-third Cantor set  $C \subset [0, 1]$ , the difference set  $C - C$  contains a neighborhood of 0. [Hint:  $C$  is the intersection of nested sets  $C_n$  where  $C_0 = [0, 1]$  and  $C_{n+1} = \frac{1}{3}C_n \cup (\frac{1}{3}C_n + \frac{2}{3})$ . Find  $C_n - C_n$  using induction.]

*Remark:* This shows that having  $|E| > 0$  is not necessary for  $E - E$  to contain a neighborhood of 0.

*Proof.* Recall that  $A - B = \{a - b : a \in A, b \in B\}$ . This definition implies that

$$(A_1 \cup A_2) - (B_1 \cup B_2) = \bigcup_{i,j=1}^2 (A_i - B_j) \quad (*)$$

Furthermore, for any  $t \in \mathbb{R}$  we have  $(A + t) - B = (A + B) + t$ ,  $A - (B + t) = (A - B) - t$ , and  $tA - tB = t(A - B)$ ; all these follow directly from the definition.

The equality  $C_0 - C_0 = [-1, 1]$  holds because, on one hand,  $|x - y| \leq 1$  when  $x, y \in [0, 1]$ , while on the other,  $C_0 - C_0 \supset [0, 1] - \{0, 1\} = [0, 1] \cup [-1, 0] = [-1, 1]$ .

Assume  $C_n - C_n = [-1, 1]$ . Use the relation  $C_{n+1} = \frac{1}{3}C_n \cup (\frac{1}{3}C_n + \frac{2}{3})$  and distribute the difference according to (\*) and other properties stated at the beginning:

$$\begin{aligned} C_{n+1} - C_{n+1} &= \left(\frac{1}{3}C_n - \frac{1}{3}C_n\right) \cup \left(\frac{1}{3}C_n - \frac{1}{3}C_n + \frac{2}{3}\right) \cup \left(\frac{1}{3}C_n - \frac{1}{3}C_n - \frac{2}{3}\right) \\ &= [-1/3, 1/3] \cup ([-1/3, 1/3] + 2/3) \cup ([-1/3, 1/3] - 2/3) \\ &= [-1/3, 1/3] \cup [1/3, 1] \cup [-1, -1/3] = [-1, 1] \end{aligned}$$

The set  $(\frac{1}{3}C_n + \frac{2}{3}) - (\frac{1}{3}C_n + \frac{2}{3})$  is not included above because it is the same as  $(\frac{1}{3}C_n - \frac{1}{3}C_n)$ . By induction,  $C_n - C_n = [-1, 1]$  for all  $n$ .

Since  $C \subset C_n$  for every  $n$ , it follows that  $C - C \subset [-1, 1]$ . To prove the reverse inclusion, fix  $a \in [-1, 1]$ . For each  $n$ , there exist  $x_n, y_n \in C_n$  such that  $x_n - y_n = a$ . Since all these numbers are contained in  $[0, 1]$ , we can pick a convergent subsequence  $\{x_{n_k}\}$ . So,  $x_{n_k} \rightarrow x$  and since  $x_{n_k} - y_{n_k} = a$ , we also have  $y_{n_k} \rightarrow y$  where  $y$  is such that  $x - y = a$ .

It remains to prove that  $x, y \in C$ . For each  $m \in \mathbb{N}$  we have  $x_{n_k}, y_{n_k} \in C_m$  for  $k \geq m$  by construction. Since  $C_m$  is compact, it follows that  $x, y \in C_m$ . And since this holds for every  $m \in \mathbb{N}$ , we have  $x, y \in C$ .  $\square$

**Solution Problem 13.** Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a function such that  $f(\mathbb{R}^n)$  is countable and  $f^{-1}(t)$  is measurable for every  $t \in \mathbb{R}$ . Prove that  $f$  is measurable.

*Proof.* Let  $B = f(\mathbb{R}^n)$ , a countable subset of  $\mathbb{R}$ . For any  $a \in \mathbb{R}$  we have

$$\{f > a\} = \bigcup_{b \in B, b > a} f^{-1}(b),$$

which is a countable union of measurable sets, and is hence measurable. The domain of  $f$ , which is  $\mathbb{R}^n$ , is also measurable. Thus  $f$  is measurable.  $\square$

**Solution Problem 14.** Prove that without the assumption “ $f(\mathbb{R}^n)$  is countable” the statement in Problem 13 would not be true.

*Proof.* The statement in Problem 13 is made for any  $n$ . To disprove it, it suffices to show it fails for some  $n$ . With massive loss of generality, let  $n = 1$ . Let  $V \subset [0, 1]$  be a Vitali set, and define  $f: \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} x + 1, & x \in V \\ -|x|, & x \notin V \end{cases}$$

By construction  $\{f > 0\} = V$ , which is nonmeasurable. Thus,  $f$  is nonmeasurable. On the other hand, for every  $t \in \mathbb{R}$  the set  $f^{-1}(t)$  is finite and therefore measurable. Indeed, if  $t$  is negative,  $f(x) = t$  holds for at most two values of  $x$ ; and when  $t \geq 0$ , there is at most one such value.  $\square$

[Remark: If we wanted to construct such an example on  $\mathbb{R}^n$  for every  $n$ , one way is to let

$$f(x_1, \dots, x_n) = \begin{cases} x_1 + 1, & \forall i \ x_i \in V; \\ -|x_1|, & \text{otherwise} \end{cases}$$

Then  $\{f > 0\} = V^n$  which is nonmeasurable, because on one hand,  $V^n + \mathbb{Q}^n = \mathbb{R}^n$  forces  $|V^n|_e > 0$ ; on the other,  $V^n + (\mathbb{Q} \cap [0, 1])^n$  is a bounded set containing infinitely many copies of  $V^n$ , which makes it impossible to have  $|V^n| > 0$ .

For every  $t \in \mathbb{R}$ , the preimage  $f^{-1}(t)$  consists at most two hyperplanes of the form  $\{x\} \times \mathbb{R}^{n-1}$ . So it is covered by countably many sets of the form  $\{x\} \times [-j, j]^{n-1}$ ,  $j \in \mathbb{N}$ . Here  $|\{x\} \times [-j, j]^{n-1}| = 0$  because this set is contained in a box of dimensions  $(\epsilon, 2j, \dots, 2j)$  whose volume can be arbitrarily small. In conclusion,  $|f^{-1}(t)| = 0$  for every  $t$ . Thus  $f$  is measurable.]

**Solution Problem 15.** Suppose that  $f: \mathbb{R} \rightarrow \mathbb{R}$  is measurable, and  $g: \mathbb{R} \rightarrow \mathbb{R}$  is continuously differentiable with  $g' > 0$  everywhere. Prove that  $f \circ g$  is measurable.

*Proof.* By the Mean Value Theorem,  $g$  is strictly increasing; therefore, it has an inverse  $h = g^{-1}$ . By the Inverse Function Theorem, the inverse function  $h$  is also continuously differentiable.

Given  $a \in \mathbb{R}$ , consider the set  $A = \{x : f(g(x)) > a\}$ . It can be written as  $\{x : g(x) \in B\}$ , where  $B = \{f > a\}$  is measurable; that is,  $A = h(B)$ . By Problem 9, the image of a measurable set under a continuously differentiable function is measurable. Therefore,  $A$  is measurable.  $\square$

### Solution Problem 16.

- (a) Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuous function such that  $f^2$  is measurable. Prove that  $f$  is measurable.
- (b) Prove that the statement in (a) is false if  $f$  is not assumed continuous.

*Proof.*

- (a) Since  $f$  is continuous, it is measurable.
- (b) Let  $n = 1$ , let  $V$  be a Vitali set, and define  $f(x) = 1$  when  $x \in V$  and  $f(x) = -1$  when  $x \notin V$ . Then  $f^2 \equiv 1$  is measurable, being continuous. But  $\{f > 0\} = V$  is not a measurable set, so  $f$  is not measurable.  $\square$

**Solution Problem 17.** Suppose that  $f : E \rightarrow \mathbb{R}$  is a measurable function, where  $E \subset \mathbb{R}^n$  is measurable.

- (a) Prove that there exists a Borel set  $H \subset E$  such that the restriction  $f|_H$  is Borel measurable and  $|E \setminus H| = 0$ .
- (b) If, in addition,  $E$  is a Borel set, prove that there exists a Borel measurable function  $g : E \rightarrow \mathbb{R}$  such that  $f = g$  a.e.. [Hint: For part (a), take a countable union of closed sets obtained from Lusin's Theorem.]

**Solution Problem 18.** Suppose  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a function such that  $\phi(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Consider a sequence of measurable functions  $\{f_k\}$ ,  $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$ , such that  $|f_k(x)| \leq \phi(|x|)$  for every  $k$ , and  $f_k \rightarrow f$  a.e.. Prove that the conclusion of Egorov's Theorem holds in this situation; that is, for every  $\epsilon > 0$ , there exists a closed set  $E(\epsilon) \subset \mathbb{R}^n$  such that  $|\mathbb{R}^n \setminus E(\epsilon)| < \epsilon$  and  $f_k \rightarrow f$  uniformly on  $E(\epsilon)$ . [Hint: Follow the proof of Egorov's theorem.]

**Solution Problem 19.** Suppose that  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is a function such that  $f(\mathbb{R}^n)$  is countable and  $f^{-1}(t)$  is measurable for every  $t \in \mathbb{R}$ . Prove that  $f$  is measurable.

**Solution Problem 20.** Prove that without the assumption “ $f(\mathbb{R}^n)$  is countable” the statement in Problem 19 would not be true.