Syracuse University

MATH 701:
Real Variables I

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0 Introduction

0.1 Course Description

MAT 701 Real Variables I: Measure and integration, including basic theorems on integration and differentiation of sequences of functions; modes of convergence, product measures.

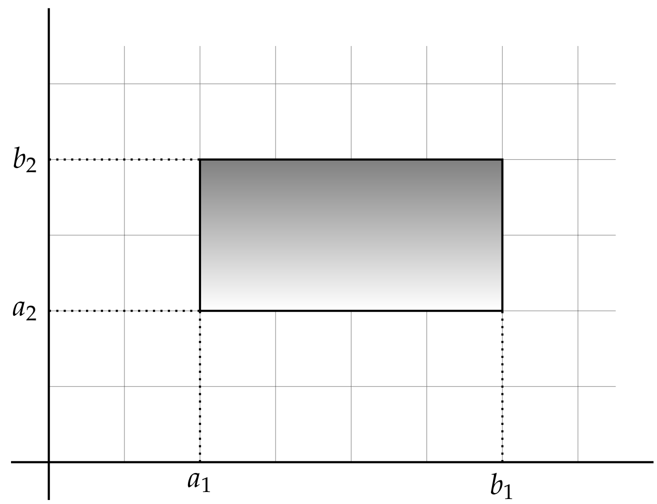
0.2 Disclaimer

These notes were taken in Fall 2018 in a course taught by Professor Leonid Kovalev. In some places, notation/material has been changed or added. Any errors in this text should be attributed to the typist – Caleb McWhorter – and not the instructor or any referenced text.

1 Outer Measure & Lebesgue Measure

1.1 Exterior Measure

We want to assign a notion of 'size' to sets. We denote this 'size' by ν. Let $\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$ denote ordinary Euclidean space. By an 'interval' in $\mathbb{R}^n$, we mean a set $\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) : a_i, b_i \in \mathbb{R}^n, a_i < b_i\}$. By a closed interval in $\mathbb{R}^n$, we mean a set $\{[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] : a_i, b_i \in \mathbb{R}^n, a_i < b_i\}$. We will often say 'box', which will always mean an open or closed interval in $\mathbb{R}^n$.



In defining a 'size' for sets, it makes sense to begin with a simple shape like a box. In the case of the plane, we know the a good notion of size is the area, and the area of a box $[a_1, b_1] \times [a_2, b_2]$ is $(b_1 - a_1) \cdot (b_2 - a_2)$. We can immediately generalize this to $\mathbb{R}^n$ as follows: if $I \subset \mathbb{R}^n$ is an interval, then we define

$$\nu(I) := \prod_{j=1}^{n} (b_j - a_j) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$$

But the question remains, how do we generalize this to an arbitrary region E? Given the above definition, it is natural to try to generalize to arbitrary sets E by approximating E by boxes, i.e. an open covering $\{I_k\}$ of E by intervals (see Figure 1).

It then becomes clear that whatever the measure of E is, it should satisfy

$$\nu(E) \leq \sum_k \nu(I_k),$$

since the intervals cover E. After all, it would be strange indeed to allow E to have greater measure than its covering. Furthermore, if a notion of measure is going to be defined in

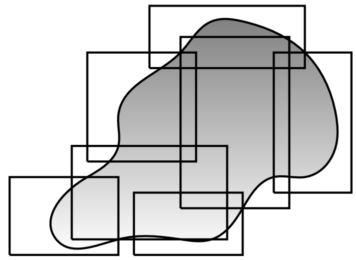


Figure 1: Approximating a set E by intervals.

terms of coverings, then the measure need be invariant of the choice of covering. These ideas are our guiding principles. So we take the following definition

Definition (Outer Measure)**.** For a set $E \subset \mathbb{R}^n$, the outer measure (or exterior measure) of E, denoted $|E|_e$, is the function $\nu : \mathbb{R}^n \to [0, \infty)$ given by

$$|E|_e := \inf_{E \subset \bigcup_k I_k} \sum \nu(I_k),$$

where the infimum is taken over all *countable* coverings $\{I_k\}$ of E.

The coining of the term 'outer measure' is immediately obvious—we are measuring the size of a set via external objects, namely the open cover. However, less obvious is the need to restrict to countable coverings. The need to eliminate uncountable coverings is apparent, as then trouble arises defining the summation. But why not only allow finite coverings? For this consider the case $E = \mathbb{Q} \cap [0, 1]$. In any finite covering of $[0, 1]$ by intervals, the intervals cannot all be pairwise disjoint. The reader will confirm, with a bit of thought, that if the intervals were all pairwise disjoint then there would be a rational number missed by the 'covering.' But this contradicts the fact that the collection was an open cover. The only possible open covering by intervals is the entire interval itself so that $\inf \sum \nu(I_k) = 1$. This violates the notation that there isn't any 'length' or 'area' here since we have a sparse collection of points.

Furthermore, the same logic applies to the set $E' = \mathbb{Q}^C \cap [0, 1]$. So if the outer measure of E were 1, then this would be true too of E'. But clearly the outer measure of $[0, 1]$ is 1. Now $[0, 1] = E \cup E'$, and $E \cap E' = \emptyset$. As $1 + 1 \neq 1$, this breaks countable subadditivity of the measure we are trying to define. By defining the outer measure in terms of countable covers, we obtain the expected answer $|E|_e = 0$.

Example 1.1. Let $E = \mathbb{Q} \cap [0, 1]$ and $\epsilon > 0$ be given. Since $\mathbb{Q}$ is countable, so too is E countable. Enumerate the rationals in E as $\{q_1, q_2, \ldots, q_n, \ldots\}$. Now the set $\{O_n\}_{n \in \mathbb{N}}$, where $O_n := (q_n - \frac{1}{2^{n+k+1}}, q_n + \frac{1}{2^{n+k+1}})$ and $k \in \mathbb{N}$ is fixed, is a (countable) open covering of E by intervals. Furthermore, the O_n are pairwise disjoint. Choose k sufficiently large so that $2^{-k} < \epsilon$. The measure of this covering is then

$$\sum_{n=1}^{\infty} \nu(O_n) = \sum_{n=1}^{\infty} \left[\left(q_n + \frac{1}{2^{n+k+1}} \right) - \left(q_n - \frac{1}{2^{n+k+1}} \right) \right] = \sum_{n=1}^{\infty} 2 \cdot \frac{1}{2^{n+k+1}} = \frac{1}{2^k} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2^k} < \epsilon.$$

◁

It is also important to note that our notion of measure preserves our notion of 'size' for intervals. Before proving this in the general case, we begin with the case for intervals in $\mathbb{R}$.

Proposition 1.1. $|[a, b]|_e = b - a$.

Proof. Clearly, $|[a, b]|_e \leq b - a$, so it remains to show that $b - a \leq |[a, b]|_e$. Suppose that $[a, b]$ is a finite union of intervals of the form $[c_j, d_j]$. There exists j_1 such that $c_{j_1} \leq a$. If $d_{j_1} \geq b$, then $|[a, b]|_e \geq d_{j_1} - c_{j_1} \geq b - a$, and we are done. Otherwise, it must be that $d_{j_1} < b$. There then exists j_2 such that $c_{j_2} \leq d_{j_1}$. Continue this process inductively until one finally obtains $d_{j_r} \geq b$. But then taking the sum of these differences, one obtains a telescoping series

$$\underbrace{(d_{j_r} - c_{j_r})}_{\geq 0} + \underbrace{(d_{j_{r-1}} - c_{j_{r-1}})}_{\geq 0} + \cdots + \underbrace{(d_{j_1} - c_{j_1})}_{\geq 0} \geq b - a.$$

□

We now prove this for general intervals.

Proposition 1.2. *For an interval I, $|I|_e = \nu(I)$.*

Proof. As I serves as an open covering for itself, we know that $|I|_e \leq \nu(I)$. It remains to show that $\nu(I) \leq |I|_e$. Suppose $\{I_k\}_k$ is a countable open covering of I, and let $\epsilon > 0$. Every interval I is contained in the interior of a slightly larger interval I', i.e. $I \subseteq \operatorname{Int}(I')$, where $\nu(I') - \nu(I) < \epsilon$. Let I'_k denote an interval containing I_k in its interior with $\ni (I'_k) \leq (1 + \epsilon)\nu(I_k)$. We know that $I \subset \bigcup_k \operatorname{Int}(I'_k)$. Now I is compact so that Heine Borel implies that there is an integer N such that $I \subset \bigcup_{k=1}^{N} I'_k$. We know $\nu(I) \leq \sum_{k=1}^{N} \nu(I'_k)$. But then

$$\nu(I) \leq (1 + \epsilon) \sum_{k=1}^{N} \nu(I_k) \leq (1 + \epsilon) \sum_{k=1}^{\infty} \nu(I_k).$$

It then follows that $\nu(I) \leq |I|_e$, completing the proof. □

More generally, $\left| \bigcup_{k=1}^{N} I_k \right|_e = \sum_{k=1}^{N} \nu(I_k)$, provided the I_k are non-overlapping intervals, i.e. $\operatorname{Int}(I_k) \cap \operatorname{Int}(I_j) = \emptyset$, see Figure 2. The proof of this is rather ugly—an exercise in making 'obvious' geometric facts obvious, and an exercise in bookkeeping—and we shall not concern ourselves with it.

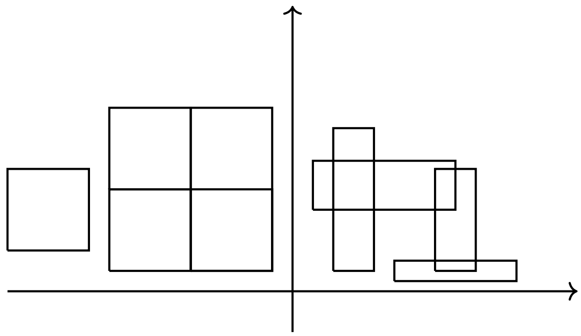


Figure 2: The intervals in Quadrant II are non-overlapping, while those in Quadrant I overlap.

As a final remark, observe that the exterior measure is a function to the nonnegative extended real line; that is, the exterior measure allows infinite values. Sets with an infinite exterior measure are considered measurable. For example, our intuition is that $\mathbb{R}$ should have infinite length, and the reader routinely verifies that $|\mathbb{R}|_e = \infty$. The outer measure $|\cdot|_e$ defined above does meet all our guiding principles as the following proposition verifies.

Proposition 1.3.

(i) $\nu(\emptyset) = 0$.

(ii) *Monotonicity: if* $E_1 \subset E_2$, *then* $|E_1|_e \leq |E_2|_e$.

(iii) *Countable Subadditivity:* $\left| \bigcup_{k=1}^{\infty} E_k \right|_e \leq \sum_{k=1}^{\infty} |E_k|_e$

Proof.
(i) This holds essentially by fiat.

(ii) This follows immediately from the fact that we are taking an infimum, and any open cover of E_2 is an open cover of E_1.

(iii) Let $E := |\bigcup_{k=1}^{\infty} E_k|_e$. If any of the E_k have infinite exterior measure, the result is immediate. Assume then that $|E_k|_e < \infty$ for all k. Choose $\epsilon > 0$ and cover each E_k by intervals $\{I_n\}$ such that $\sum_n \nu(I_n) \leq |E_k|_e + \epsilon/2^k$. Then $E \subset \bigcup_{k,n} I_{k,n}$ and $|E|_e \leq \sum_{k,n} \nu(I_{k,n}) = \sum_k \sum_n \nu(I_{k,n})$. But then

$$|E|_e \leq \sum_k \left(|E_k|_e + \epsilon/2^k \right) = \epsilon + \sum_{k=1}^{\infty} |E_k|_e.$$

The result then follows by letting ϵ tend to 0. $\square$

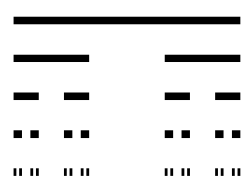
If one wants to generalize the notion of outer measures to spaces beyond $\mathbb{R}^n$, one can take the properties of Proposition 1.3 as the axioms for this abstract measure. Now we immediately have the following corollary:

Corollary 1.1. *Any subset of a set with outer measure zero has outer measure zero, and the countable union of sets with outer measure zero has outer measure zero. In particular, any countable set has outer measure zero.*

Proof. This follows immediately from the Countable Subadditivity of ν. $\square$

Now while Corollary 1.1 states that countable sets have outer measure zero, it need not be the case that uncountable sets need have positive measure.

Example 1.2 (Cantor Set)**.** Begin with the closed unit interval $C_0 := [0,1]$. From this interval, remove the middle third, i.e. $(\frac{1}{3}, \frac{1}{3})$, and label $C_1 := [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Inductively construct C_n by removing the middle third of each closed subinterval of C_{n-1}. We define the Cantor Set by defining $C := \lim_{n\to\infty} C_n$. The first few stages of the construction of C are shown below. The fact that the Cantor set is uncountable follows from the fact that it is a nonempty compact set without isolated points.



What is $|C|_e$? Note that C_n is the union of 2^n intervals, each having length $1/3^n$, and that $C \subset C_n$ for all $n \in \mathbb{N}$. But then the Cantor set is contained in a union of 2^n intervals with length $1/3^n$, which has total length $2^n \cdot 1/3^n = (2/3)^n$. The fact that $|C|_e = 0$ is then clear as $\lim_{n\to\infty}(2/3)^n = 0$. $\triangleleft$

Note in general we do not have $|\bigcup E_k|_e = \sum |E|_e$ even if the E_k are disjoint or even in the case of finite unions! Equality holds when the sets are, in a sense, 'unentangled.' By this, we mean that open coverings of one set tend to be disjoint from open coverings of the other set. If this is the case, the sets have to be covered separately, see the example on the left in Figure 3. However if the sets are 'entangled', then their open covers result in a great deal of 'multiple-covering' for the union. This excess covering allows one to more 'efficiently' cover the union—hence the smaller measure, see the example on the right in Figure 3.

There are many connections between Topology and Measure Theory, especially in the case of $\mathbb{R}^n$. Topology on $\mathbb{R}^n$ is primarily interested in the structure of open and compact sets. This will prove useful for us since our notion of measure is defined in terms of open sets. As an example, take the following theorem.

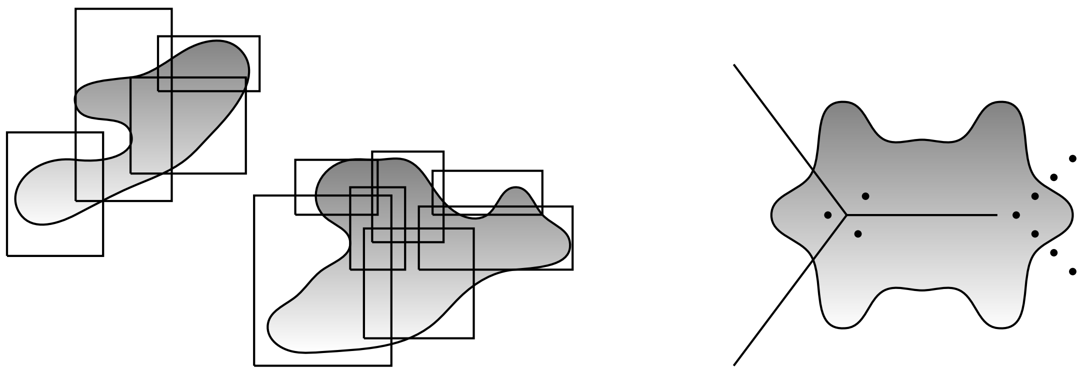


Figure 3: On the left, the two sets are 'unentangled' so that their opening coverings can be taken disjointly. On the right, the sets are the 'blob' and the line segments along with the points. These sets are 'entangled' and their open coverings will have 'overlap.'

Theorem 1.1. *For all $E \subset \mathbb{R}^n$ and $\epsilon > 0$, there exists an open set G such that $E \subset G$ and $|G|_e \leq |E|_e + \epsilon$.*

Proof. Every interval I is contained in the interior of a slightly larger interval I', i.e. $I \subseteq \text{Int}(I')$, where $\nu(I') - \nu(I) < \epsilon$. Choose intervals I_k such that $E \subset \bigcup_{k=1}^{\infty} I_k$ and $\sum_k \nu(I_k) \leq |E|_e + \epsilon/2$. Now find I'_k such that $I_k \subset \text{Int}(I'_k)$ and $\nu(I'_k) < \nu(I_k) + \epsilon/2^{k+1}$. Let $G = \bigcup_k \text{Int}(I'_k)$. By construction, G is an open set containing E. To complete proof, observe

$$|G|_e \leq \sum_{k=1}^{\infty} \nu(I'_k) \leq \sum_{k=1}^{\infty} \nu(I_k) + \epsilon \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \leq |E|_e + \epsilon.$$

□

Corollary 1.2. *For every set E, there exists a G_δ set G such that $E \subset G$ and $|E|_e = |G|_e$.*

Proof. By Theorem 1.1, we know that for every $k \in \mathbb{N}$, there is an open set $G_k \supset E$ with $|G_k|_e \leq |E|_e + 1/k$. Define $G := \bigcap_{k=1}^{\infty} G_k$. Then G is a G_δ set and $G \supset E$. Finally,

$$|E|_e \leq |G|_e \leq |G_k|_e \leq |E|_e + \frac{1}{k},$$

so that $|E|_e = |G|_e$. □

The set G_δ referred to in Corollary 1.2 is a general type of set which will be of occasional interest and use for us:

Definition (G_δ-Set)**.** A G_δ set is a countable intersection of open sets.

Notationally, the choice of 'G' is because the set is open, and the choice of 'δ' stems from the fact we are using an intersection.

Theorem 1.1 says we can always approximate any set by an open set with approximately the same size, i.e. approximately the same exterior measure. However, this does not mean that $|G \setminus E|_e \leq \epsilon$. We do know that $G = E \cup (G \setminus E)$. By subaddivitivity, we have $|G|_e \leq |E|_e + |G \setminus E|_e$. But we do not know the measure of the second set. In fact, it is possible that $|G \setminus E|_e$ could be very large. Now as it turns out for compact sets, it is sufficient to consider only finite coverings.

Proposition 1.4. *If $K \subset \mathbb{R}^n$ is compact, then $|K|_e = \inf \{\sum_k \nu(I_k) \colon \{I_k\}$ finite cover of $K\}$.*

Proof. Let $\epsilon > 0$. By Theorem 1.1, every interval is contained some set $\text{Int}(I')$, where $\nu(I') < \nu(I) + \epsilon$. Given a countable cover I_k of K, choose I'_k such that $I_k \subset \text{Int}(I'_k)$ and $\nu(I'_k) < \nu(I_k) + \epsilon/2^k$. Then $\{\text{Int}(I'_k)\}_k$ is an open covering for K, so there exists a finite subcovering $\{I_{k,n}\}_{n=1}^N$. Therefore, we have $K \subset \bigcup_j I'_{k,j}$. By countable subadditivity,

$$\sum_{n=1}^{N} \nu(I'_{k,n}) < \epsilon + \sum_{n=1}^{N} \nu(I_k),$$

as desired. □

1.2 The Lebesgue Measure

We now can define our notion of measurability with our notions of exterior measure firmly in place.

Definition (Measurable)**.** A set $E \subset \mathbb{R}^n$ is (Lebesgue) measurable if for all $\epsilon > 0$, there exists an open set G such that $G \supset E$ and $|G \setminus E|_e < \epsilon$.

Essentially, a set is measurable if it can be well approximated by open sets. We choose the above notion of 'closeness' in order to obtain additivity of measures. Notice we also have mentioned the underlying topology via the use of 'open.' We are able to avoid invoking the underlying topology using greater abstraction, which shall come later. Note that we always have an open set such that $|G|_e < |E|_e + \epsilon$ (c.f. Theorem 1.1), but this alone is weaker than the above definition; that is, if E is measurable then it satisfies the properties in Theorem 1.1. As a matter of notation, if E is measurable, we define $|E| := |E|_e$. The following propositions follow immediately from our definition.

Proposition 1.5. *Every open set is measurable.*

Proof. If E is an open set, choose $G = E$. □

Proposition 1.6. *If $|E|_e = 0$, then E is measurable.*

Proof. Choose G such that $|G|_e < |E|_e + \epsilon = \epsilon$. But then $|G \setminus E|_e \leq |G|_e < \epsilon$. □

Proposition 1.7. *A countable union of measurable sets is measurable, and*

$$|E| \leq \sum_k |E_k|.$$

Proof. Let $\epsilon > 0$. For each k, choose an open set G_k such that $E_k \subset G_k$ and $|G_k \setminus E_k|_e < \epsilon/2^k$. Now $G := \bigcup_k G_k$ is open, and $E \subset G$. Moreover since $G \setminus E \subset \bigcup_k (G_k \setminus E_k)$, we have

$$|G \setminus E|_e \leq \left| \bigcup_k (G_k \setminus E_k) \right|_e \leq \sum_k |G_k \setminus E_k|_e < \epsilon.$$

Therefore, $\bigcup_k E_k$ is measurable. The fact that $|\bigcup_k E_k| \leq \sum_k |E_k|$ follows from Proposition 1.3. □

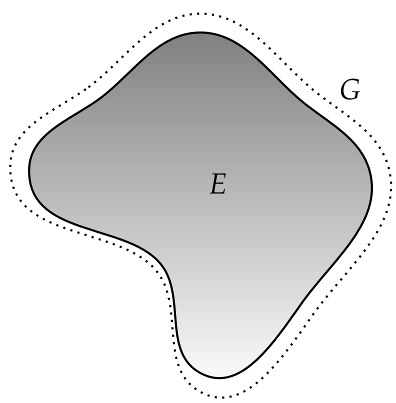


Figure 4: Open sets are not 'fuzzy' since they have a good notion of boundary. Similarly since G is not fuzzy and approximates E, it must be that E is not too close to being 'fuzzy.'

Proposition 1.8. *All intervals are measurable.*

Proof (Sketch). We prove this only in the two dimensional case to avoid unnecessary complications. Given an interval $I = [a, b]$, choose $I' = [a', b']$ such that $I \subset \operatorname{Int} I'$ and $v(I') < v(I) + \epsilon$. We need to show that $|I' \setminus I|_e < \epsilon$. Now $I' \setminus I = [a', a) \cup (b, b']$, and $|I' \setminus I|_e \leq a - a' + b' - b = (b' - a') - (b - a) < \epsilon$, as desired. Alternatively, one can use the fact that I is the union of its interior and its boundary, and that the boundary has measure zero. This alternative proof is left to the reader. □

As we have seen, open sets are easily seen to be measurable. But the case of closed sets is more complicated. For example in $\mathbb{R}$, every open set is the countable union of intervals of the form (a_k, b_k). These intervals can even be taken to be disjoint. However, the same is not true for closed sets—take the Cantor set for example, c.f. Example 1.2. We do have that in $\mathbb{R}^n$, every open set is a countable union of non-overlapping intervals. To prove this we shall make use of dyadic cubes.

A dyadic cube of generation zero, $\mathcal{D}_0$, are cubes with unit side lengths and integer vertices, i.e. $\mathcal{D}_0 := \{[0,1]^n + \tau\colon \text{fixed } \tau \in \mathbb{Z}^n\}$. A generation one dyadic cube is $\mathcal{D}_1 := \{\frac{1}{2}Q\colon Q \in \mathcal{D}_0\}$. Generally, $\mathcal{D}_n := \{\frac{1}{2}Q\colon Q \in \mathcal{D}_{n-1}\} = \{\frac{1}{2^n}Q\colon Q \in \mathcal{D}_0\}$. Given $\mathcal{D}_n$, we say that $\mathcal{D}_{n-1}$ is a parent of $\mathcal{D}_n$, and $\mathcal{D}_i$, where $i < n$, is an ancestor of $\mathcal{D}_n$. We say also that $\mathcal{D}_{n+1}$ is a child of $\mathcal{D}_n$, and $\mathcal{D}_j$, where $j > n$, is a descendant of $\mathcal{D}_n$. One can allow n to be negative to create larger dyadic cubes. Define $\mathcal{D} := \bigcup_{k=0}^{\infty} \mathcal{D}_k$. If Q_1, Q_2 are dyadic, then either $Q_1 \subset Q_2$, $Q_2 \subset Q_1$, or they do not overlap. We now are in a position to prove the following lemma.

Lemma 1.1. *Every open set in $\mathbb{R}^n$ is a countable union of non-overlapping intervals.*

Proof. Given an open set G, let $\{I_k\}$ be all dyadic cubes that are contained in G and for which their parent is not contained in G. By the selection of the I's, it follows that they are pairwise disjoint for if I_k and I_j overlap, then one contains the other, contradicting the selection process. Clearly, we have selected only countably many intervals. Now if $x \in G$, there exists $r > 0$ such that there is an r-neighborhood of x contained in G. For sufficiently large n, all the cubes in $\mathcal{D}_n$ have diameter less than r. But then there exists $Q \in \mathcal{D}_n$ such that $x \in Q$. Note that $Q \subset G$. Now either $\mathcal{D}_n$ is contained in G or it has an ancestor that is contained in G. □

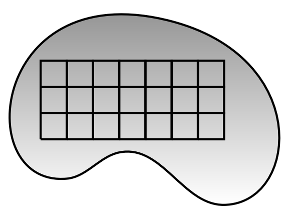


Figure 5: Tiling intervals inside an open set.

We can now make precise a discussion from earlier—if two sets are 'unentangled' then the measure of the union is the sum of the measures.

Lemma 1.2. *If $A, B \subset \mathbb{R}^n$ and $\operatorname{dist}(A, B) > 0$, then $|A \cup B|_e = |A|_e + |B|_e$.*

Proof. We know by subadditivity that $|A \cup B|_e \leq |A|_e + |B|_e$. It remains to show that $|A|_e + |B|_e \leq |A \cup B|_e$. Let $\epsilon > 0$, and choose intervals $\{I_k\}$ such that $A \cup B \subset \bigcup_k I_k$ and $\sum_k |I_k| \leq |A \cup B|_e + \epsilon$. Possibly partitioning each I_k into a finite number of subintervals, we may assume that $\operatorname{diam}(I_k) < \operatorname{dist}(A, B)$. 'Sort' the set $\{I_k\}$ into two sets $\{I'_k\}$ and $\{I''_k\}$ which cover A and B, respectively. Then

$$|A|_e + |B|_e \leq \sum_k |I'_k| + \sum_k |I''_k| = \sum_k |I_k| \leq |A \cup B|_e + \epsilon.$$

Therefore, $|A|_e + |B|_e \leq |A \cup B|_e$, as desired. □

Theorem 1.2. *Every closed set $A \subset \mathbb{R}^n$ is measurable.*

Proof. Given $\epsilon > 0$, we can choose $G \supset A$ such that $|G|_e < |A|_e + \epsilon$. Now $G \setminus A$ is open. By Lemma 1.1, we can write $G \setminus A = \bigcup_{k=1}^{\infty} I_k$, where $\{I_k\}$ is a collection of non-overlapping open intervals. We want to show that $\sum \nu(I_k) < \epsilon$, which will imply that $|G \setminus A|_e < \epsilon$. It suffices to show that $\sum_{k=1}^{N} \nu(I_k) < \epsilon$ for all N. Define $K = \bigcup_{k=1}^{N} I_k$. Since K is the finite union of compact sets in $\mathbb{R}^n$, K is compact. Furthermore, K is disjoint from A. As the set A is closed and K is compact, it must be that $\operatorname{dist}(K, A) > 0$. It then follows from Lemma 1.2 that

$$|K \cup A|_e = |K|_e + |A|_e = \sum \nu(I_k) + |A|_e \leq |G|_e < |A|_e + \epsilon.$$

□

As it turns out, the measurability of a set is equivalent to the measurability of its complement. This is useful in circumstances where one set is easier to handle than the other.

Theorem 1.3. *E is measurable if and only if E^C is measurable.*

Proof. Suppose that E is measurable. For $k \in \mathbb{N}$, choose an open set G_k such that $E \subset G_k$ and $|G_k \setminus E|_e < 1/k$. Now as G_k is open, G_k^C is closed, and hence measurable by Theorem 1.2. Let $G := \bigcup_k G_K^C$. Being the countable union of measurable sets, G is measurable, and $G \subset E^C$. Write $E^C = G \cup Z$, where $Z = E^C \setminus G$. Then $Z \subset E^C \setminus G_k^C = G_k \setminus E$, and therefore, $|Z|_e < 1/k$ for all k. Hence, $|Z|_e = 0$, and so Z is measurable. But then E^C is the union of measurable sets, and is thus measurable. Finally, taking E to be E^C and rerunning the argument mutatis mutandis proves that if E^C is measurable, then E is measurable. □

Corollary 1.3. *If E_1, E_2 are measurable, then $E_1 \setminus E_2$ is measurable.*

Proof. We know that $E_1 \setminus E_2 = E_1 \cap E_2^C$. But E_2^C is measurable by Theorem 1.3. Therefore, $E_1 \setminus E_2$ is measurable, being the intersection of measurable sets. □

Our definition is measurable is essentially that a set E is measurable if it can be 'approximated well' by open sets containing it. This is an exterior notion of measure. We can also define measurability instead by saying a set E is measurable if it can be 'approximated well' internally by closed sets, as the following proposition shows.

Proposition 1.9. *A set $E \subset \mathbb{R}^n$ is measurable if and only if given $\epsilon > 0$, there exists a closed set $F \subset E$ such that $|E \setminus F|_e < \epsilon$.*

Proof. E is measurable if and only if E^C is measurable, i.e. if and only if given $\epsilon > 0$, there exists an open set G with $E^C \subset G$ and $\left|G \setminus E^C\right|_e < \epsilon$. But such an open set G exists, noting that $G \setminus E^C = E \setminus F$, if and only if G^C is closed, $F \subset E$, and $|E \setminus F|_e < \epsilon$. □

The previous theorems and propositions have shown that the complements of measurable sets are measurable, and countable unions and intersections of measurable sets are measurable. Since we shall make use of them in the future, note that if $\{E_k\}$ is a collection of measurable sets, then so are lim sup and lim inf:

$$\limsup E_k = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \quad \text{and} \quad \liminf E_k = \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} E_k,$$

though we shall not prove this. Now all of these previous theorems and propositions essentially follow because we are working in the context of a σ-algebra.

Definition (σ-algebra)**.** A nonempty collection of sets Σ is called a σ-algebra if it satisfies

(i) $E^C \in \Sigma$ whenever $E \in \Sigma$.

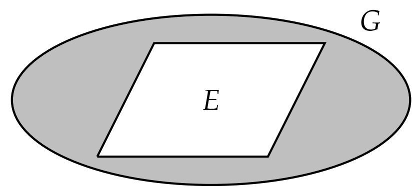
(ii) $\bigcup_k E_k \in \Sigma$ whenever $E_k \in \Sigma$ for all k.

Note that any collection of sets closed under countable unions is closed under countable intersections as $\left[\bigcap_k U_k\right]^C = \bigcup_k U_k^C$. The empty set and entire space are necessarily measurable sets. Generally, they are elements of any σ-algebra. We shall be particularly interested in two specific σ-algebras:

Definition (Borel σ-algebra)**.** The Borel σ-algebra is the smallest σ-algebra containing all open subsets of $\mathbb{R}^n$.

Definition (Lebesgue σ-algebra)**.** The Lebesgue σ-algebra is the σ-algebra containing all measurable sets.

Note that the Borel σ-algebra is *strictly* contained in the Lebesgue σ-algebra—though this is not simple to see.

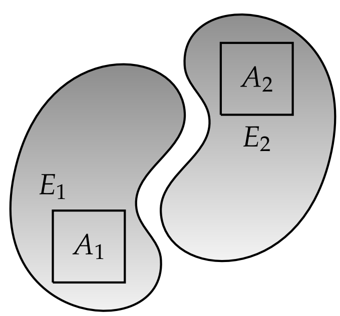


1.3 Properties of Measurable Sets

Our goal in this section is to examine measurable sets more closely. We begin by recalling that $|E| = |E|_e$ whenever E is measurable, and $|\bigcup_k E_k|_e \leq \sum_k |E_k|_e$, i.e. countable subadditivity.

Theorem 1.4 (Countable Subadditivity). *If $\{E_k\}$ is a collection of disjoint, measurable sets, then* $\left|\bigcup_{n=1}^{\infty} E_k\right| = \sum_{k=1}^{\infty} |E_k|$.

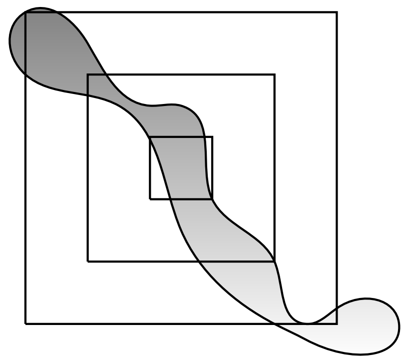
Proof. It is immediate that $|\bigcup_{n=1}^{\infty} E_k| \leq \sum_{k=1}^{\infty} |E_k|$, so that we need show $\sum_{k=1}^{\infty} |E_k| \leq |\bigcup_{n=1}^{\infty} E_k|$. Let $E := \bigcup_k E_k$. It suffices to show $\sum_{k=1}^{N} |E_k| \leq |E|$ for all N. If $\operatorname{dist}(A, B) > 0$, we know that $|A \cup B|_e = |A|_e + |B|_e$. In particular, this holds if A, B are disjoint compact sets. First, assume that each E_k is bounded. For each $k \in \mathbb{N}$, choose a closed subset $A_k \subset E_k$



such that $|E_k \setminus A_k|_e < \epsilon/2^k$. Since $A_k \subset E_k$ and E_k is bounded, A_k is a closed, bounded set. Therefore, A_k is compact. We also have

$$\left|\bigcup_{k=1}^{N} A_k\right| = \sum_{k=1}^{N} |A_k| \geq \sum_{k=1}^{N} |E_k| - \epsilon.$$

But as $\bigcup_{k=1}^{N} A_k \subset E$, we have $|E| \geq \sum_{k=1}^{N} |E_k| - \epsilon$. Letting $\epsilon \to 0$ yields the desired inequality. This completes the proof in the case where each E_k is bounded.



Now suppose that some E_k may be unbounded. Let $I_j := [-j, j]^n$ for $j > 0$ and $Q_0 := \emptyset$. Define further $Q_j := I_j \setminus I_{j-1}$ for $n > 0$. Consider the sets $E_{k,j} := E_k \cap Q_j$ for $j, k = 1, 2, \ldots$. Each of these sets are disjoint, bounded, and measurable. It is also routine to verify that

$$E_k = \bigcup_j E_{k,j} = \bigcup_j \left(E_k \cap Q_j\right)$$
$$E = \bigcup_k E_k = \bigcup_{k,j} E_{k,j}$$

But then by the case above, we have

$$|E| = \left|\bigcup_k E_k\right| = \left|\bigcup_{k,j} E_{k,j}\right| = \sum_{k,j} |E_{k,j}| = \sum_k \sum_j |E_{k,j}| = \sum_k |E_k|.$$

□

Corollary 1.4. *If $\{I_k\}$ is a sequence of nonoverlapping intervals, then $|\bigcup_k I_k| = \sum_k |I_k|$.*

Proof. It is immediate that $|\bigcup_k I_k| \leq \sum |I_k|$. We need prove the opposite inequality. As the I_k are disjoint, we know $|\bigcup_k I_k| \geq |\bigcup_k \operatorname{Int} I_k| = \sum_k |\operatorname{Int} I_k| = \sum |I_k|$. But then $|\bigcup_k I_k| = \sum_k |I_k|$. □

Corollary 1.5. *Suppose E_1 and E_2 are measurable, $E_2 \subset E_1$, and $|E_2| < \infty$. Then $|E_1 \setminus E_2| = |E_1| - |E_2|$.*

Proof. We know $E_1 = E_2 \cup (E_1 \setminus E_2)$ and $|E_1| = |E_2| + |E_1 \setminus E_2|$. The result then follows as $|E_2| < \infty$. □

We end this section with an interaction between the Lebesgue measure and monotone sequences of sets.

Proposition 1.10.

(i) If $E_1 \subset E_2 \subset \cdots$ *are measurable, then* $|\bigcup_k E_k| = \lim_{k\to\infty} |E_k|$*; that is, if* $E_k \nearrow E$*, then* $\lim_{k\to\infty} |E_k| = |E|$*.*

(ii) If $E_1 \supset E_2 \supset \cdots$ *are measurable,* $|E_k| < \infty$ *for some* k*, then* $|\bigcap_{k=1}^{\infty} E_k| = \lim_{k\to\infty} |E_k|$*; that is, if* $E_k \searrow E$ *and* $|E_k| < \infty$ *for some* k*, then* $\lim_{k\to\infty} |E_k| = |E|$*.*

Proof.

(i) Let $E = \cup E_k$. If $|E_k| = \infty$ for some k, then $\lim_{k\to\infty} |E_k|$ and $|E|$ are infinite, and the result follows. Assume then that $|E_k| < \infty$ for all k. We want to create a disjoint union. We shall express E as $E = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_2) \cup \cdots \cup (E_k \setminus E_{k-1}) \cup \cdots$. Define $F_k := E_k \setminus E_{k-1}$. Observe the sets F_k are measurable and pairwise disjoint. Then by Theorem 1.4,

$$|E| = \sum_{k=1}^{\infty} |F_k| = \lim_{n\to\infty} \sum_{k=1}^{\infty} |F_k| = \lim_{n\to\infty} \left| \bigcup_{k=1}^{n} F_k \right| = \lim_{n\to\infty} |E_n|.$$

(ii) Let $E = \cap E_k$. Since considering tails of a sequence are sufficient, we may assume without loss of generality that $|E_1| < \infty$. We again want to create disjoint, measurable sets. Define $F_k := E_k \setminus E_{k+1}$. Observe that the sets F_k are measurable, pairwise disjoint, and that $E_1 = E \cup \bigcup F_k$. Then by Theorem 1.4,

$$|E_1| = |E| + \left|\bigcup F_k\right| = |E| + \sum_{k=1}^{\infty} F_k = |E| + \sum_{k=1}^{\infty} \left(|E_k| - |E_{k-1}|\right) = |E| + |E_1| - \lim_{k\to\infty} |E_k|.$$

But as $|E_1| < \infty$, we must then have $|E| = \lim_{k\to\infty} |E_k|$.

$\square$

Note that the requirement that $|E_k| < \infty$ for some k in the second part of Proposition 1.10 is necessary. For example, take $E_k = \mathbb{R}^2 \setminus C_k$, where C_k is the circle centered at the origin with radius k. Then $|E_k| = \infty$ for all k, and $E_k \searrow \emptyset$. However, $\lim_{k\to\infty} |E_k| = \infty$, while $|\emptyset| = 0$.

1.4 Characterization of Measurable Sets

By definition, A is measurable if and only if for all $\epsilon > 0$, there exists an open set $G \supset A$ such that $|G \setminus A|_e < \epsilon$. But this makes use of the underlying topology. Can we avoid this?

After all, this 'open set-ness' is about continuity. Continuity is not even a requirement for integration, which is an end goal of ours. So we would like a description of measurability which avoids Topology entirely. Ironically, we shall first need more Topology.

Recall that G_δ is a set which is the countable intersection of open sets, F_σ is the countable union of closed sets, $G_{\delta,\sigma}$ are the countable intersection of closed sets.[1] One can even continue this to absurdity, i.e. $G_{\delta\sigma\delta\sigma}$. Furthermore, these are all subclasses of the Borel σ-algebra. One may wonder if all Borel created by these procedures, i.e. formed by countable union or intersections of open or closed sets? This turns out to not be the case, though it is difficult to construct a set in this fashion which is not a Borel set. However, it is the case that measurable sets are 'close' to Borel sets, in the sense they differ by a set of measure zero, as the next results shows.

Theorem 1.5. *E is measurable if and only if $E = H \setminus Z$ if and only if $E = F \cup Z$, where H is of type G_δ, F is of type F_σ, and $|Z| = 0$.*

Proof. It is sufficient to prove that each of the statements is equivalent to the measurability of E. If E can be represented as $H \setminus Z$ or $F \cup Z$, then E is measurable, as H, F, and Z are measurable. Now assume that E is measurable. For each k, choose an open set G_k such that $E \subset G_k$ and $|G_k \setminus E| < 1/k$. Define $H := \cap G_k$. Clearly, H is of type G_δ, $E \subset H$, and $H \setminus E \subset G_k \setminus E$ for each k. But then defining $Z = H \setminus E$, we see that $|Z| = 0$, as desired. Furthermore, if E is measurable, then so too is E^C. Now using the previous work, write $E^C = (\cap G_k) \setminus Z$, where the G_k are open sets and $|Z| = 0$. But then taking complements yields $E = (\cup G_k^C) \cup Z$. But as the sets G_k are open, the sets G_k^C are closed, implying $F := \cup G_k^C$ is of type F_σ. $\square$

Remark. This is different (in fact stronger) than $E \subset H$, where H is a G_δ set with the same outer measure, i.e. $|H|_e = |E|_e$.

Theorem 1.5 essentially states that measurable sets are the Borel sets 'plus or minus' the measure zero sets. Note that we are still making Borel sets, which invoke Topology, so we still have not met our goal. However, we are now set up to prove Carathéodory's Theorem, which we can use to meet our goal.

Theorem 1.6 (Carathéodory)**.** *E is measurable if and only if for all $A \subset \mathbb{R}^n$, $|A \cap E|_e + |A \setminus E|_e = |A|_e$.*

Proof. Suppose that E is measurable. Choose a G_δ set H such that $A \subset H$ and $|H|_e = |A|_e$. Since $H = (H \cap E) \cup (H \setminus E)$, and these are measurable sets, we have

$$|A|_e = |H| = |H \cap E| + |H \setminus E|.$$

[1]Generally, G denotes open, F denotes closed, δ is for countable intersections, and σ is for countable unions.

But then $|A|_e = |H| = |H \cap E| + |H \setminus E| \geq |A \cap E|_e + |A \setminus E|_e$. By subadditivity, we obtain $|A|_e \leq |A \cap E|_e + |A \setminus E|_e$. Therefore, $|A|_e = |A \cap E|_e + |A \setminus E|_e$.

Now suppose that E satisfies the condition that $|A \cap E|_e + |A \setminus E|_e = |A|_e$ for all $A \subseteq \mathbb{R}^n$. Let A be a G_δ set such that $E \subset A$ and $|E|_e = |A|$. Then on the left side of the equation, we have $|E|_e + |A \setminus E|_e = |A| = |E|_e$. Hence, $|A \setminus E|_e = 0$, so E is G_δ minus a measure zero set. Notice that we have subtracted $|E|_e$; hence, we need consider the case where $|E|_e = \infty$.

Assume that $|E|_e = \infty$. We know that $E = \bigcup_{j=1}^{\infty} E_j$, where $E_j := E \cap [-j, j]^n$. So it suffices to prove that E_j has the property $\left|A \cap E_j\right|_e + \left|A \setminus E_j\right|_e = |A|_e$ for all $A \subseteq \mathbb{R}^n$ and $j \in \mathbb{N}$. [Since each E_j is measurable and the countable union of the E_j is measurable, E would then be measurable.] Certainly by subadditivity, $|A|_e \leq \left|A \cap E_j\right|_e + \left|A \cap E_j^C\right|_e$. It remains to show $|A|_e \geq \left|A \cap E_j\right|_e + \left|A \cap E_j^C\right|_e$. By assumption for all $A \subset \mathbb{R}^n$, $|A \cap E|_e + |A \setminus E|_e = |A|_e$. We know also that $Q_j := [-j, j]^n$ and Q_j^C satisfy this property, so $|A \cap E|_e = |(A \cap E) \cap Q|_e + \left|(A \cap E) \cap Q^C\right|_e$. Similarly, we have $\left|A \cap E^C\right|_e = \left|(A \cap E^C) \cap Q\right|_e + \left|(A \cap E^C) \cap Q^C\right|_e$. Combing these gives

$$
\begin{aligned}
|A|_e &= \left|A \cap (E \cap Q_j)\right|_e + \left|A \cap (E \cap Q_j^C)\right|_e + \left|A \cap (E^C \cap Q_j)\right|_e + \left|A \cap (E^C \cap Q_j^C)\right|_e \\
&= \left|A \cap E_j\right|_e + \left|A \cap E \cap Q^C\right|_e + \left|A \cap E^C \cap Q\right|_e + \left|A \cap E_j^C\right|_e \\
&\geq \left|A \cap E_j\right|_e + \left|A \cap E_j^C\right|_e,
\end{aligned}
$$

as desired. □

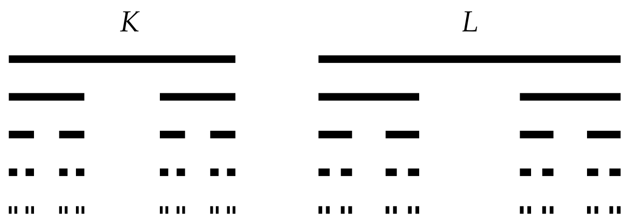
Note that in Theorem 1.6, there is no mention of openness or closed-ness. We can then take this as our definition of measurability, i.e. define measurable sets to be the sets which slit every set (independent of measurability) into pieces which are additive with respect to the outer measure. This is the one of the standard approaches when defining measures abstractly. Thus, we have completed our goal of finding a way to avoid invoking the underlying topology.

1.5 Continuous & Lipschitz Transformations

Here we shall investigate the interaction between continuous functions and measurability. We begin with an important question: if $E \subset \mathbb{R}^n$ is measurable and $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, is $f(E)$ always measurable? The general answer is no, but there are special cases where this does hold. For example, measures of sets are invariant under rotations of $\mathbb{R}^n$—though we will not prove this, see [WZ77, p. 36]. This also shows that the exterior measure (hence also the Lebesgue measure) is invariant under the choice of an orthogonal basis for $\mathbb{R}^n$. However, what other functions or transformations preserve measures? We first show that continuity alone will never be enough.

Lemma 1.3. *There is a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a set E with $|E| = 0$, such that $|f(E)| > 0$.*

Proof (Sketch). We construct a 'fat Cantor' set by following the same procedure as in the Cantor set, but instead removing the middle fourths of each interval. At each stage, 2^j intervals of length 4^{-j} are removed. Denote the ordinary Cantor set by K and the fat Cantor set by L. We construct a continuous function $f : [0,1] \to [0,1]$ such that $f(K) = L$.



Map K_j, the jth step in the construction of K, onto L_j in a piecewise linear fashion, i.e. stretch each interval in K_j to the corresponding interval in L_j. Clearly, f_j is continuous and $\sup |f_j - f_{j+1}| \leq 2^{-j}$. But then $f_j \to f$ uniformly so that f is continuous and $f(K) = L$. We can extend f to $\mathbb{R}$ by translation, i.e. define $f(x)$ for $x \notin [0,1]$ via $f(x) = f(a)$, where $a = x + n$ and $n \in \mathbb{Z}$ is an integer such that $x + n \in [0,1]$. Finally, recall that the ordinary Cantor set has measure 0, as was shown in Example 1.2, and it is routine to verify that $|L| > 0$. □

Before answer the question poised at the beginning of this section, we need two quick results, assuming the existence of a nonmeasurable set, which we shall prove in the next section.

Lemma 1.4. *Let E be a measurable subset of $\mathbb{R}^1$ with $|E| > 0$. Then the set of differences $\{d \colon d = x - y, x \in E, y \in E\}$ contains an interval centered at the origin.*

Proof. See [WZ77, p. 46]. □

Lemma 1.5. *Any set in $\mathbb{R}^1$ with positive outer measure contains a nonmeasurable set.*

Proof. Suppose $|A$ is a set with $|A|_e > 0$. Let E be the nonmeasurable Vitali set. For $q \in \mathbb{Q}$, let E_q denote the translation of E by q, i.e. $E_q = \{e + q \colon e \in E\}$. The sets E_q are disjoint, and $\cup E_r = \mathbb{R}$. But then it must be that $A = \cup(A \cap E_q)$. By Lemma 1.4, if $A \cap E_q$ is measurable then $|A \cap E_q| = 0$. But as $|A|_e > 0$, there must be some $q \in \mathbb{Q}$ such that $A \cap E_q$ is not measurable. □

We now sketch the idea why it need not be the case that $f(E)$ is measurable even when E is measurable. Choose f and E as in Lemma 1.3. By Lemma 1.5, there exists a non-measurable subset of $f(E)$, say W. Let $V := f^{-1}(W) \cap E$. Observe that $f(V) = W$, and that V is measurable (being a subset of measure zero). Therefore, the image of a measurable set under a continuous function need not be measurable, as claimed. The next obvious question is, if continuity is not sufficient to force the image of measurable sets to be measurable, is there a stronger condition that will? We recall a definition.

Definition (Lipschitz)**.** We say that f is Lipschitz if there exists L such that $|f(x) - f(y)| \leq L|x-y|$ for all x, y in the domain of f.

Lipschitz functions are the special case of uniformly continuous functions where we can always choose $\delta = \epsilon/L$. Therefore, if f is Lipschitz then f is uniformly continuous, and hence also continuous. However, not all uniformly continuous functions are Lipschitz, and certainly not all continuous functions are Lipschitz. The Lipschitz condition can be thought of as the boundedness of secant lines for a given function. Hence functions which are differentiable with bounded derivative on an interval are Lipschitz.

Example 1.3. The function $f(x) = x^{1/3}$ is Lipschitz on any interval not containing the origin as a limit point since $f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$ is bounded on such intervals. Indeed, if I is an interval containing the origin and $x \in I$, then

$$\frac{f(x) - f(0)}{x - 0} = \frac{\sqrt[3]{x}}{x} = \frac{1}{\sqrt[3]{x^2}}.$$

Taking x 'close' to 0 shows that the slopes of secant lines to $f(x)$ are unbounded. One can use the same process to verify this for any set containing 0 as a limit point. ◁

Example 1.4. The function $f : [0,1] \to [0,1]$ given by $f(x) = \sqrt{x}$ is uniformly continuous on $[0,1]$, being a continuous function on a compact interval. However, f is not Lipschitz continuous. If f were Lipschitz with constant L, then we would have $|\sqrt{x} - \sqrt{y}| \leq L|x-y|$ for all $x, y \in [0,1]$. Choose $x = 0$ and $y = \dfrac{1}{4k^2}$. We have $x, y \in [0,1]$ but

$$|\sqrt{x} - \sqrt{y}| = \frac{1}{2K} > \frac{1}{4K} = K|x-y|.$$

◁

Unlike general continuous functions or the function in Lemma 1.3, Lipschitz maps preserve sets of measure zero.

Theorem 1.7. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz and $|E| = 0$, then $|f(E)| = 0$.*

Proof. Recall that $|E|_e = \inf\{\sum v(I_k) \colon \bigcup I_k \supset E$, and I_k cubes$\}$. Since $|E| = 0$, for every $\epsilon > 0$, there exists a countable cover of E by intervals $\{I_k\}$ such that $\sum |I_k| < \epsilon$. Choose this covering to consist of dyadic cubes. By Lipschitz continuity, $f(I_k)$ is contained in an interval of width at most Lw_k, where L is the Lipschitz constant of f and w_k is the maximum width of the cube. Then $|f(I_k)| \leq L^d |I_k|$. Therefore, we have

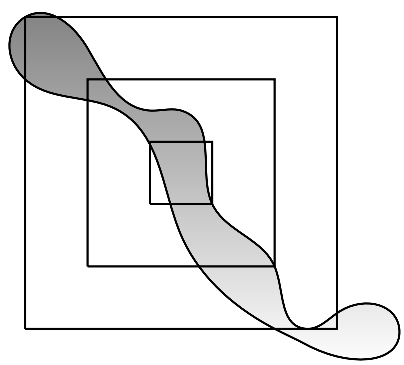
$$|f(E)| \leq L^d \sum |I_k| < L^d \epsilon.$$

The result then follows. □

Furthermore, Lipschitz functions preserve measurability, though not necessarily the value of the measure.

Theorem 1.8. *If E is measurable and f is Lipschitz, then $f(E)$ is measurable.*

Proof. Since E is measurable, write $E = H \cup Z$, where H is F_σ and $|Z| = 0$. Now F_σ sets are the countable union of closed sets, which is equivalent to the countable union of compact sets since any union of closed sets can be written as a countable union of compact sets, i.e. $F = \cup(F \cap [-j, j]^n)$. Now f is Lipschitz, hence continuous, so that f sends compacts sets to compact sets. Then it must be that f preserves 'F_σ-ness.' We know by Theorem 1.7, if $|E| = 0$ then $f(E) = 0$. But then $f(E) = f(H) \cup f(Z)$, a union of F_σ set and a set of measure zero, hence measurable. □



The proof of Theorem 1.8 involves a common trick: although a whole function may not be Lipschitz, 'pieces' of it may be, and these could be brought together to finish a result. This invoked the same underlying idea as in the proof of Theorem 1.4. Though Theorem 1.8 showed that Lipschitz functions preserve measurability, they do not generally preserve the measure. So while we cannot know that the measure of $f(E)$ is, we know that $f(E)$ is indeed measurable. We finish by commenting that in the special case of Lipschitz functions which are linear maps of $\mathbb{R}^n$, we can actually compute the measure of the resulting image.

Theorem 1.9. *Let T be a linear transformation of $\mathbb{R}^n$, and let E be a measurable set. Then $|T(E)| = \delta|E|$, where $\delta = |\det T|$.*

Proof. Let $\delta = |\det T|$. It is routine to verify that $|T(I)| = \delta|I|$ for any interval I. Now let $E \subseteq \mathbb{R}^n$ be a measurable subset, and $\epsilon > 0$. Choose intervals $\{I_k\}$ covering E with $\sum |I_k| < |E|_e + \epsilon$. Then

$$|T(E)|_e \leq \sum |T(I_k)| = \delta \sum |I_k| < \delta(|E|_e + \epsilon).$$

But then $|T(E)|_e \leq \delta\, |E|_e$.

It remains to show that $\delta\, |E|_e \leq |T(E)|_e$. Choose an open set $G \supset E$ with $|G \setminus E| < \epsilon$. Write G as a union of nonoverlapping intervals $\{I_k'\}$. By the work above, we may assume that $\delta > 0$. Since $T(I_k)$ are also nonoverlapping intervals,

$$|T(G)| = \sum |T(I_k)| = \delta \sum |I_k| = \delta|G|.$$

But as $E \subset G$, we have $\delta|E| \leq \delta|G| = |T(G)|$. Moreover,

$$|T(G)| \leq |T(E)| + |T(G \setminus E)| \leq |T(E)| + \delta\epsilon.$$

Combining inequalities yields $\delta|E| \leq |T(E)| + \delta\epsilon$, so that $\delta|E| \leq |T(E)|$. $\square$

1.6 A Nonmeasurable Set

Notice that 'all' the sets we have encountered thus far were measurable. Indeed, most sets you 'meet on the street' will be measurable. Nonmeasurable sets are true pathologies, which do not even exist depending on your logical axioms.

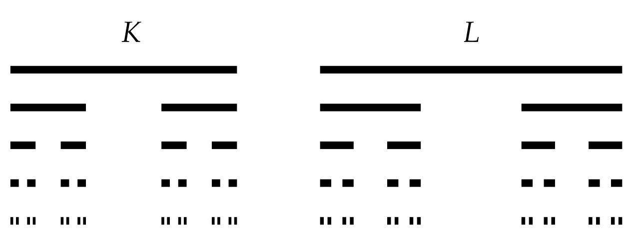
Recall the example from Lemma 1.3. We constructed a fat Cantor set, L. Note that removing fourths was not the only possible choice in the construction of L. Suppose at each stage we cut out lengths $\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots$. We want

$$\sum_{k=1}^{\infty} 2^{k-1}\epsilon_k = \frac{1}{2},$$

or any other number in $[0, 1)$ so that our final set L has positive measure. Clearly, choosing $\epsilon_k = \dfrac{1}{4^k}$ works. Furthermore,

$$|f_k - f_{k+1}| \leq 2^{-k}, |f_k^{-1} - f_{k+1}^{-1} \leq 2^{-k}$$

so that f is a homeomorphism.



Now when $W \subset F$ is nonmeasurable, $V = f^{-1}(W)$ is measurable but not Borel. If V were Borel, then $W = (f^{-1})^{-1}(V)$ would also be Borel, hence measurable, as the preimage of Borel sets are Borel, c.f. Problem 8. However, it need not be the case that the continuous image of a Borel set is measurable.

Theorem 1.10. *The continuous image of a Borel set is measurable.*

Though we shall not prove this here, this was shown by Lusin and his student Souslin in 1916, after reading a paper of Lebesgue and noting many errors including asserting that projections of Borel sets were Borel. Souslin's construction is a bit detailed, though not long, and we shall not pursue it here.

As mentioned previously, non-measurable sets are rare in practice, though they certainly do exist. To this day, no explicit examples of non-measurable sets exist. Indeed, their very existence hinges on the Axiom of Choice.

Axiom of Choice: For all collections of nonempty sets $\{E_\alpha\}$, one can choose an element from each E_α, i.e. there exists a function $f : \mathcal{A} \to \bigcup E_\alpha$ such that $f(\alpha) \in E_\alpha$ for all α. Said differently, $\prod_{\alpha \in \mathcal{A}} E_\alpha \neq \emptyset$.

The Axiom of Choice was once a point of great contention among mathematicians at the turn of the century as many counterintuitive results follow from its assumption. However generally today, the axiom is assumed without grudge. We shall assume it here. With this final ingredient, we can construct a nonmeasurable set.

Theorem 1.11 (Vitali Set)**.** *There exists a nonmeasurable set.*

Proof. Define an equivalence relation on $\mathbb{R}$ via $x \sim y$ if $x - y \in \mathbb{Q}$. Let $\{E_\alpha\}_{\alpha \in \mathcal{A}}$ be the set of equivalence classes. It is clear that the equivalence classes can be expressed $E_x := \{x + q \colon q \in \mathbb{Q}\}$. Furthermore, E_x and E_y are either identical or disjoint. There is one equivalence class consisting of all rational numbers, while the other equivalence classes consist entirely of irrational numbers, i.e. an irrational number translated by all possible rational numbers. Moreover since we can choose a representative of each equivalence class and all other elements of the class are the set of rational translates of this representative, each equivalence class is countable. However, the set of equivalence classes must be uncountable since their union is $\mathbb{R}$ and each class is countable.

Consider the sets $\{\mathcal{A}_\alpha\}_\alpha$, where $\mathcal{A}_\alpha := E_\alpha \cap [0, 1]$. By the Axiom of Choice, we can choose an element from each set A_α. Let $V \subset [0, 1]$ be the set of all the chosen elements. We claim that V is nonmeasurable. Now given two distinct points $x, y \in V$, we kow that $x - y \notin \mathbb{Q}$, for otherwise they would come from the same equivalence class and we have chosen unique elements from each set. Therefore, every pair of points from V differ by an irrational number. Now $\mathbb{R} = V + \mathbb{Q} = \{v + q \colon v \in V, q \in \mathbb{Q}\}$, so that $\cup_{q \in \mathbb{Q}}(V + q) = \mathbb{R}$. If $|V|_e - 0$, then so too is $|V + q|_e = 0$. This would then imply that $|\mathbb{R}|_e = 0$, a contradiction.

Therefore, $|V|_e > 0$. Now observe $\cup_{q\in(\mathbb{Q}\cap[0,1])}(V+q) \subset [0,2]$. Now as the sets $V+q$ are distinct, if V were measurable, so too would $V+q$ be measurable, and we would have $|[0,2]| \geq \sum_{q\in(\mathbb{Q}\cap[0,1])} |V+q| = \infty$, a contradiction. □

Notice that the sets $V+q$, if they were measurable, would necessarily have positive measure by the proof above. The proof then shows, in some sense, that these sets are so intertwined that one can fit infinitely many of them in an interval without having their measure go to zero.

Now we have stated that nonmeasurable sets are rare in practice. The next result shows that they are not, in principle, so rare. In fact, every measurable subset of $\mathbb{R}$ with positive measure contains a nonmeasurable set.

Lemma 1.6. *If $E \subset \mathbb{R}$ and $|E| > 0$, then there exists an interval I such that $|E \cap I| > (1-\epsilon)|I|$.*

Proof. There exists an open set G such that $G \supset E$ and $|G| < (1+\epsilon)|E|$. Express G as the union of disjoint, open intervals, i.e. $G = \cup(a_k, b_k)$. For each k, let $E_k := E \cap (a_k, b_k)$, and note that each E_k are measurable and $E \subset \cup E_k$. Therefore, we have $|E| = \sum_k |E_k|$ and $|G| = \sum_k |(a_k, b_k)|$. If it were the case that for each k, $(1+\epsilon)|E_k| \leq b_k - a_k$, then we would have

$$(1+\epsilon)|E| = (1+\epsilon)\sum |E_k| = \sum (1+\epsilon)|E_k| \leq \sum (b_k - a_k) = \sum |(a_k, b_k)| = |G|,$$

a contradiction. Therefore, there is some k_0 such that $b_{k_0} - a_{k_0} < (1+\epsilon)|E_k| = |E \cap (a_k, b_k)|$. Defining $I := (a_{k_0}, b_{k_0})$ completes the proof. □

Lemma 1.7. *If $E \subset \mathbb{R}$ and $|E| > 0$, then $E \setminus E$ contains a neighborhood of 0.*

Proof. We claim there exists $\delta > 0$ such that if $|x| < \delta$, then $E \cap (E+x) \neq \emptyset$. $x \in E \setminus E$ if and only if $E+x$ intersects E. $X = a - b$ if and only if $h + x = a$, where $a, b \in E$. Let I be an interval, $|E \cap I| > \frac{2}{3}|I|$. Let $\delta = \frac{1}{3}|I|$. If $|x| < \delta$.

$$|I \cup (I+\delta)| \leq \frac{4}{3}|I|.$$

If E and $E+x$ were disjoint, then $|(E\cap I) \cup (E \cap I + x)| = 2|E \cap I| > \frac{4}{3}|I| > |I \cup (I+x)|$, a contradiction (since one is a subset of the other). □

Theorem 1.12. *If $E \subset \mathbb{R}$, $|E| > 0$, then there exists $E \subset E$ such that W is nonmeasurable.*

Proof. Since $\bigcup_{q\in\mathbb{Q}}(V+q) = \mathbb{R}$, where V vitali set. We have $\bigcup_{q\in\mathbb{Q}}[(V+q)\cap E] = E$. So there exists q such that $|(V+q)\cap E|_e > 0$. But $(V+q)\cap E$ is nonmeasurable as $V+q$ has no measurable subsets of positive measure.

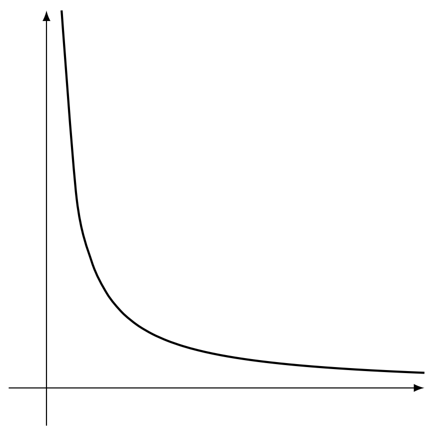
$((V+q) - (V+q) = V - V$ is disjoint from $\mathbb{Q}$. □

2 Lebesgue Measurable Functions

2.1 Measurable Functions

It is still our goal to build a generalized theory of integration for functions. However, there are two clear potential problems for functions which we might integrate:

- The graph of f could cover an infinite area and thus not be integrable in the traditional sense, e.g. $f(x) = \frac{1}{x}$ on the interval $(0,1)$.



- The graph of f could be so irregular that the 'area underneath' is not well defined enough so that such a question even makes sense, e.g.

$$\chi_V(x) = \begin{cases} 1, & x \in V \\ 0, & x \notin V \end{cases}$$

  where V is the Vitale set.

As in the case of the Riemann integral, only certain classes of functions will be integrable in the Lebesgue sense. It will be the goal of this section to define 'nice' classes of functions which address the issues above and which will later form a 'nice' class of functions with which to form an integration theory around.

Definition (Measurable Function)**.** Let $E \subseteq \mathbb{R}^n$ and $\overline{\mathbb{R}}$ denote the extended real line. We say that a function $f : E \to \overline{\mathbb{R}}$ is (Lebesgue) measurable if for all $a \in \mathbb{R}$, the set $\{f > a\} := \{x \in E \colon f(x) > a\}$ is measurable.

Note that we define $\{f = \infty\} := \bigcap_{k \in \mathbb{Z}} \{f > k\}$ and $\{f = -\infty\} := \bigcap_{k \in \mathbb{Z}} \{-f > k\}$. The reason for this choice of definition is that the sets $\{f > a\}$ describe the distribution for the values of f. In some sense, the 'smoother' f is, the smaller the variety of such sets one obtains from f. For example, when f is continuous, the sets $\{f > a\}$ are open sets since $f^{-1}((a,\infty))$ is an open set.

Furthermore, the measurability of a set should be linked in some way with the potential measurability of the functions on it and vice versa. If the set E is 'wild', less functions should be measurable on it. In fact, the measurability of E is equivalent to that of the set $\{f = -\infty\}$ since

$$E = \{f = -\infty\} \cup \left(\bigcup_{k=1}^{\infty} \{f > -k\} \right),$$

showing that if f is measurable, then E is measurable if and only if $\{f = -\infty\}$ is measurable.

Note that one could define Borel measurability similarly: $f : E \to \overline{\mathbb{R}}$ is Borel measurable, if for all $a \in \mathbb{R}$, the set $\{f > a\}$ is Borel. Thus, every Borel function is also measurable. We can of course define measurability of functions more generally: if M is a σ-algebra on $\mathbb{R}^n$, f is M-measurable if $\{f > a\} \in M$ for all $a \in \mathbb{R}$. For us, M will normally be the Lebesgue σ-algebra. Though what we do is in $\mathbb{R}^n$ with the Lebesgue σ-algebra, much of what we do holds, or has parallel results, in other spaces and σ-algebras.

Example 2.1. If M is the σ-algebra generated by $[0,1]$ and $[1,2]$. What functions f are M-measurable?

M is σ algebra generated by $[0,1]$ and $[1,2]$. What f are M-measurable?

f over $[0,1]$, $[1,2]$ constant functions, over complements. Not M-measurable, $\chi_{\{3\}}$, $f(x) = x$ or $f(x) = \sin x$.

Before continuing, we make an important definition since the following abbreviation will serve as a constant shortening of the words needed to state, prove, and discuss results later, so we define it now.

Definition (Almost Everywhere, a.e.)**.** We say that a property holds 'almost everywhere' if it holds everywhere except on a set of exceptional values with measure zero, e.g. f is finite almost everywhere signifies $|\{|f| = \infty\}| = 0$. We shall abbreviate this as 'a.e.'.

2.2 Properties of Measurable Functions

1. Continuous then measurable as $\{f > a\}$ are open.

2. Enough to check that $\{f > a\}$ is measurable for a in some dense set $\mathcal{D} \subset \mathbb{R}$. Why? For all real a, for all k, find $d_k \in \mathcal{D}$, $a < d_k < a + 1/k$. $\{f > a\} \cup_k \{f > d_k\}$.

3. Instead of $\{f > a\}$, it could be $\{f < a\}$ as $\{f \geq a\}$, $\{f \leq a\}$, can use any of these 4 ways to check measurability.

$$\begin{cases} \{f \geq a\} & = \bigcap_{k \in \mathbb{N}} \{f > a - 1/k\} \\ \{f > a\} & = \bigcup_{k \in \mathbb{N}} \{f \geq a + 1/k\} \end{cases}$$

Note that 2,3 are independent of the σ-algebra.

4. If f is measurable and $f = g$ a.e., then g is measurable (Borel fails this). Let $Z = \{f \neq g\}$, $|Z| = 0$. $\{f > a\} \setminus Z \subset \{g > a\} \subset \{f > a\} \cup Z$ but $\{f > a\} \setminus Z$ and $\{f > a\} \cup Z$ have the same measure.

Ex: $1/x$ is measurable on $\mathbb{R}$. Defined a.e. is okay.

5. iff f, then $f^{-1}(G)$ is measurable for all open $G \subset \mathbb{R}$.

(5b) also for all borel sets $G \subset \mathbb{R}$.

Proof. $G = \cup(a_k, b_k)$ so $f^{-1}(G) = \cup_k\{a_k < f < b_k\}$, measurable.

Proof. (5b) a homework generalization.

For all σ-algebra M, $\{E\colon f^{-1}(E) \in M\}$, where M Lebesgue σ-algebra, like M_f. By 5, M_f contains open sets, so it must contain Borel sets.

(5c) if f is finite a.e. and $f^{-1}(G)$ is measurable for all open $G \subset \mathbb{R}$, $\{f > a\} = f^{-1}((a, \infty))$.

Warning: the composition of measurable functions is not always measurable.

Recall C cantor set, F fat cantor set, $f(C) = F$, f homeomorphism $\mathbb{R} \to \mathbb{R}$. Let $W \subset F$ be nonmeasurable. Write χ_W as a composition $\chi_{g(W)} \circ g$

$\chi_{g(W)}$ is measurable, g is continuous (so measurable).

4. is what we gain for going Borel -> Lebesgue and this is the price we pay.

Rem: if f, g are Borel measurable, then $f \circ g$ is Borel measurable if for all $B \subset \mathbb{R}$ Borel we have $f^{-1}(B)$ is Borel, then $(f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B))$, each composition on right Borel.

(*) is	Open	Borel	Measurable
Open	Continuous	—	—
Borel	Borel measurable	Borel measurable	—
Measurable	Measurable	Measurable	?

When can we say that $f \circ g$ is measurable? What data about f and g would ensure this? We know that if f is measurable and g is continuous does not work from previous discussion. One possibility ensuring this would be f is continuous and g is measurable as $(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U))$, as U is open $f^{-1}(U)$ is open and g^{-1} of open is measurable (assuming g takes values in $\mathbb{R}$ and f is continuous on some open set containing the range of g). More generally, this will hold if f is Borel measurable on some Borel set containing the range of g and g measurable and real-valued.

Example 2.2. If f is measurable (and $\mathbb{R}$-valued) then f^2 is measurable, $f^2 = \phi \circ f$, where $\phi(t) = t^2$. Similarly, $|f|$ is measurable since $|f| = \phi \circ f$, where $\phi(t) = |t|$. Also, $\operatorname{sgn}(f) = \begin{cases} 1, & f = 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases}$ is measurable since sign is Borel measurable (simple to see by checking preimages). So is floor(x), where floor is supremum of integers which are at most x. As is ceilin(x), defined mutatis mutandis. [They are both monotone,]

Theorem 2.1. *If f, g are measurable, then $f \pm g$, fg, and f/g ($g \neq 0$ a.e.) are measurable.*

Proof. $\{f + g > a\} = \bigcup_{b+c>a, b,c\in\mathbb{Q}} [\{f > b\} \cap \{g > c\}]$ hence measurable set (countable union of measurable sets). Just need to prove equality. $\supset$ is obvious. Need only prove $\subset$. Suppose $f(x) + g(x) > a$. Let $\epsilon = f(x) + g(x) - a$. Use density of rationals to find $b \in \mathbb{Q}$ such that $f(x) - \epsilon/2 < b < f(x)$ and $c \in \mathbb{Q}$ such that $g(x) - \epsilon/2 < c < g(x)$. But then $b + c > f(x) + g(x)0\epsilon = a$.

Now $fg = \dfrac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$ is then measurable. For quotient only need reciprocal. We know $\frac{1}{g}$ is $\phi \circ g$, where $\phi(t) = \frac{1}{t}$, which is continuous on $\mathbb{R} \setminus \{0\}$. Now g is nonzero almost everywhere, ϕ contains range of g (ignoring set where range is $g = 0$, since we can ignore for measurability). $\square$

Another way could combine is max of functions. Better to generalize to countable. So countable sup/inf of measurable functions is measurable.

Theorem 2.2. *The above statement, including infinite values.*

Proof. IF $f_k : E \to \overline{R}$ is measurable for all $k \in \mathbb{N}$, then $\sup_k f_k$ and $\inf_k$ are measurable. So $g(x) = \sup_{k\in\mathbb{N}} f_k(x)$ is measurable. Same for inf.

$$\begin{aligned}
&\{g > a\} = \bigcup_{k\in\mathbb{N}} \{f_k > a\} \\
&\{g \geq a\} \\
&\{g < a\} \\
&\{g \leq a\}
\end{aligned}$$

First case done. Other follow similarly. HAND WAVE! For inf

$$\{\inf_k f_k < a\} = \bigcup_{k\in\mathbb{N}} \{f_k < a\}.$$

$\square$

Theorem 2.3. *Suppose $f_k : E \to \overline{R}$ are measurable, then are measurable.*

Proof. Follows immediately from preceding result. [lim= limsup=liminf when it exists.] $\square$

Approximation: Given measurable f, find 'simple' functions f_k such that $f_k \to f$ pointwise. Here by simple we mean finite range.

Definition (Simple)**.** g is simple if its range is finite subset of $\mathbb{R}$. [Need measurable?]

Need this to build notion of integration.

Theorem 2.4. *For all measurable functions, there exists f_k such that $f_k \to f$ pointwise. If $f \geq 0$, we can take $f_1 \leq f_2 \leq \cdots$.*

Proof. $2^{-k} floor(2^k f)$. (floor to $\frac{integer}{2^k}$). This converges monotonically since increasing k shortens the rounding distance, and then not as much difference so this converges to f uniformly. Range not finite but countable. So just need to distribute the values among many functions.

$$f_k(x) = \begin{cases} \frac{j-1}{2^k}, & \frac{j-1}{2^k} \leq f(x) < \frac{j}{2^k} \\ k, & f(x) \geq k \end{cases}$$

f_k simple and $f_k \to f$. Now $f_k = \min(k, \max(-k, 2^{-k} floor(2^k f)))$. $f \geq 0$, then $f_1 \leq f_2 \leq \cdots$. ☐

Definition. $f : E \to \mathbb{R}$ is continuous at $a \in E$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in E$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

Why better than normal definition with limit? Because a can be a limit point not not defined there. Take $f(x) = 1/x^2$, rather like allowing $f(0) = \infty$.

Semincontinuity means rather than bounding difference from above below, means one of inequality $-\epsilon < f(x) - f(a) < \epsilon$ holds.

Definition (Upper Semicontinuous (usc))**.** $f : E \to \overline{\mathbb{R}}$ is upper semicontinuous at $a \in E$ if for all $y > f(a)$, there exists $\delta > 0$ such that if $|x - a| < \delta$, then $f(x) < y$.

Definition (Lower Semicontinuous (lsc))**.** $f : E \to \overline{\mathbb{R}}$ is upper semicontinuous at $a \in E$ if for all $y > f(a)$, there exists $\delta > 0$ such that if $|x - a| < \delta$, then $f(x) > y$.

Example 2.3.

$$f(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

This is usc (automatically because of the ∞ value, why?) and lsc.

Both types of these functions are measurable.

Remark: lsc -> measurable usc-> measurable

Proof. if f usc on measurable E, then $\{f > a\}$ is open in E. So $\{f > a\} = E \cap U$. Hence $\{f > a\}$ is measurable.

For USC, use $\{f < a\}$.

BV function

Given $f : [a,b] \to \mathbb{R}$, define $V(f;a,b) = \sup_p \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$ sup over all partitions $P : a = x_0 < x_1 < \cdots < x_n = b$. $V(x^2; -1, 1) = 2$ (P={-1,0,1} gives the maximum value.)

Definition. f is BV on $[a,b]$ if $V(f;a,b) < \infty$

Interesting not called finite variation.

Something which is not BV is

$$f(x) = \begin{cases} \sin(1/x), & x \in (0,1] \\ 0, & x = 0 \end{cases}$$

Theorem 2.5. *If f is continuous on $[a,b]$ and differentiable on (a,b) and $\int_a^b |f'|\, dx$ (riemann integral), then $V(f;a,b) = \int_a^b |f'|\, dx$.*

C.f. examples above.

Proof. Mean Value Theorem.

$$\sum_i |f(x_i) - f(x_{i-1})| = \sum_i |f'(\xi_i)|(x_i - x_{i-1})$$

$x_{i-1} < \xi_i < x_i$. the above converges to integral. □

3. Subadditive wrt f: $V(f+g;a,b) \le V(f;a,b) + V(g;a,b)$.

Proof. Triangle inequality.

$$[f(x_i) + g(x_i)] - (f(x_{i-1}) + g(x_{i-1}))|$$

then use triangle splits into two and done.

Why not exactly like integral, i.e. strict inequality. Take nonconstant function f and then $g = -f$. Note $f + g$ identically zero.

Sum of monotone functions are bounded variation.

Theorem 2.6 (BV Decomposition)**.** *If f is BV on $[a,b]$, then there exists increasing g, h on $[a,b]$ such that $f = g - h$.*

Proof. Let $g(x) = V(f;a,x)$, increasing (mononticity wrt interval). This forces $h(x) := g(x) - f(x)$. Why is $h(x) \le h(y)$ for $x \le y$? $V(f;a,x) - f(x) \le V(f;a,y) - f(y)$. Why? $f(y) - f(x) \le V(f;a,y) - V(f;a,x) = V(f;x,y)$, clearly this right side is $\ge |f(x) - f(y)|$. □

Theorem 2.7 (Egorov's Theorem). *Suppose $\{f_k\}$ is a collection of measurable functions, $f_k : E \to \mathbb{R}$, $|E| < \infty$, and $f_k \to f$ a.e., where $f : E \to \mathbb{R}$. Then for all $\epsilon > 0$, there exists a closed set $E(\epsilon) \subset E$ such that $f_k \to f$ uniformly on $E(\epsilon)$ and $|E \setminus E(\epsilon)| < \epsilon$.*

Proof. We need to find a set on which $\{f_k\}$ converges uniformly. It suffices to find, for each $j \in \mathbb{N}$, $E_j \subset E$ such that $|E \setminus E_j| < \epsilon/2^j$, and $\sup_{E_j} |f - f_k| \leq 1/j$ for sufficiently large k. Indeed, letting $E(\epsilon) := \bigcap_j E_j$, then $f_k \to f$ uniformly on $E(\epsilon)$, and $|E \setminus E(\epsilon)| \leq \sum_j \epsilon/2^j < \epsilon$.

To find E_j, consider the set $G_m := \{x \colon |f_k(x) - f(x)| < 1/j \text{ for all } k \geq m\}$. Since $|f_k - f| \to 0$ a.e., $\bigcup_m G_m$ is almost E, i.e. $|\bigcup G_m| = |E|$. By continuity of measure for nested unions, $|G_m| \to |E|$. Therefore, there exists m such that $|E \setminus G_m| < \epsilon/2^j$. Defining $E_j := G_m$ completes the proof. □

Theorem 2.8 (Lusin's Theorem). *Let E be a measurable set. A function $f : E \to \mathbb{R}$ is measurable if and only if for all $\epsilon > 0$, there exists a closed set $E(\epsilon) \subset E$ such that $f\big|_{E(\epsilon)}$ is continuous, and $|E \setminus E(\epsilon)| < \epsilon$.*

Proof. Suppose that f is a simple function, i.e. $f = \sum_{k=1}^n a_k \chi_{E_k}$, where $E_1, \dots, E_n$ are disjoint measurable sets. Choose a closed subset $E'_k \subset E_k$ such that $|E_k \setminus E'_k| < \epsilon/n$. Then $E(\epsilon) := \bigcup_{k=1}^n E'_k$ so that $|E \setminus E(\epsilon)| < \epsilon$. It only remains to check that $f\big|_{E(\epsilon)}$ is continuous, but this is immediate as the preimage of any set is closed.

Now suppose only that $|E| < \infty$. There is a collection of simple functions $\{f_k\}$ such that $f_k \to f$ pointwise. Choose some $E(\epsilon/2^k)$ for f_k as in the case above. By Egorov's Theorem, $f_k \to f$ uniformly on a set $E(\epsilon)$. Now f is continuous on $E(\epsilon) \cap \left(\bigcap_k E_k(\epsilon/2^k)\right)$, which has compliment of measure at most $\epsilon + \sum_k \epsilon/2^k = 2\epsilon$.

Now suppose that $|E| = \infty$. Write $E = \bigcup [E \cap \{|j| \leq |x| \leq j+1\}]$. In each shell, we have $E_j(\epsilon/2^j)$, closed set such that $f\big|_{E_j(\epsilon/2^j)}$ is continuous. Note that f continuous on $\bigcup_j E_j(\epsilon/2^j)$. □

The converse direction of Lusin's theorem is simple since $f\big|_{E(1/j)}$ is continuous, then we have f Borel on $\bigcup E(1/j)$ simply because $\{x \in \bigcup E(1/j) \colon f > a\} = \bigcup\{x \in E(1/j) \colon f > a\}$, countable union of Borel sets. and the rest has measure 0.

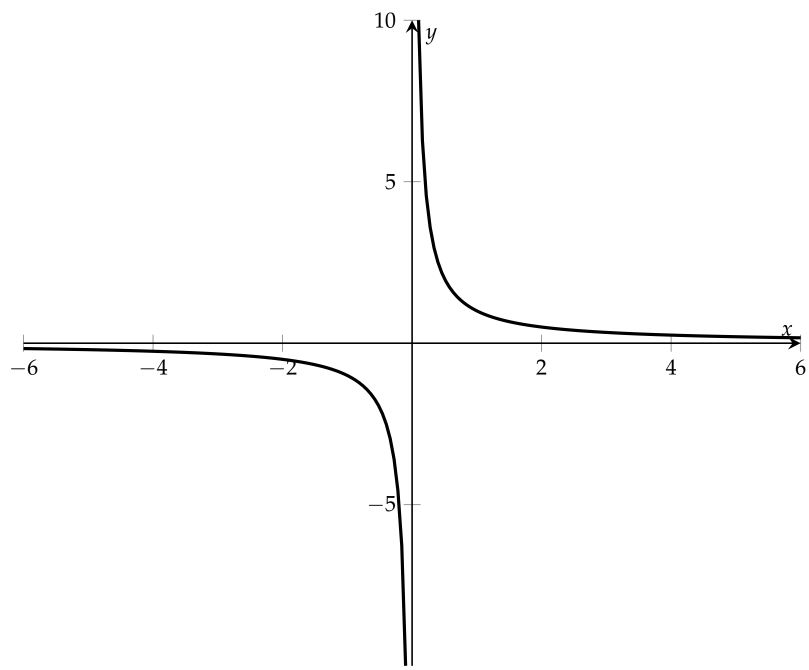
2.3 Convergence in Measure

Recall $\limsup E_k := \bigcap_{m=1}^\infty \bigcup_{k=m}^\infty E_k = \{x \colon x \in E_k, \text{ infinitely many } k\}$. We will not need lim inf quite as regularly as lim sup.

Definition (Convergence in Measure). Let $\{f_k\}$ be a collection of measurable functions, $f_k : E \to \mathbb{R}$. We say f_k converges to f in measure if for all $\epsilon > 0$, $|\{|f - f_k| > \epsilon\}| \to 0$. We denote this by $f_k \xrightarrow{m} f$.

If we expand this definition out more, we have the following: $f_k \xrightarrow{m} f$ if for $\epsilon > 0$ and for all $\epsilon_2 > 0$, there exists N such that $|\{|f - f_k| > \epsilon\}| < \epsilon_2$ for all $k \geq N$. If this is possible, one may as well replace the double epsilon with a single epsilon for a more concise version. So the definition could be taken as follows: $f_k \xrightarrow{m} f$ if for all $\epsilon > 0$, there exists N such that $|\{|f_k - f| > \epsilon\}| < \epsilon$ for all $k \geq N$.

Example 2.4. $f_k(x) = \dfrac{1}{kx} \xrightarrow{m} 0$ on $\mathbb{R}$.



We know that $\{|f_k| > \epsilon\} = \{|x| < \frac{1}{k\epsilon}\}$. This set has measure $2/(k\epsilon$, which tends to zero as k tends to infinity. ◁

Example 2.5. $f_k(x) = \frac{x}{k} \not\xrightarrow{m} 0$ on $\mathbb{R}$. $\{|f_k| > \epsilon\} = \{\dfrac{|x|}{k} > \epsilon\} = \{|x| > k\epsilon\}$, which has infinite measure. ◁

Example 2.6. Convergence in measure does not imply pointwise convergence. Let $f_k = \chi_{I_k}$, where I_k are intervals.

[0,1] measure 1 [0,1/2], [1/2,1] measure 1/2 [0,1/3], [1/3,2/3], [2/3,1] measure 1/3

For all $\epsilon > 0$, $\{|f_k| > \epsilon\} < |I_k| \to 0$.

So the sequence converges to 0 in measure. However, it clearly does not converge to 0 in measure.

There are infinitely many times point part of the interval. That is for all $x \in (0, 1)$, there are infinitely many k such that $x \in I_k$ so that $f_k(x) \not\to 0$. This can be expressed as saying $\limsup I_k = [0, 1]$.

Theorem 2.9. *If $|E| < \infty$, then convergence a.e. implies convergence in measure.*

Proof. Given $\epsilon > 0$, Egorov's Theorem gives $E(\epsilon)$ such that $|E \setminus E(\epsilon)| < \epsilon$, and $f_k \to f$ uniformly on $E(\epsilon)$. Then there exists N such that $|f_k - f| < \epsilon$ on $E(\epsilon)$ for all $k \geq N$. But then $|\{|f_k - f| > \epsilon\}| < \epsilon$, which implies convergence in measure. □

Theorem 2.10. *If $f_k \xrightarrow{m} f$, then there is a subsequence $\{f_{k_j}\}$ converging to f a.e..*

Proof. Choose $\epsilon = 1/2^j$. Find N_j such that $|\{|f_k - f| > 1/2^j\}| < 1/2^j$ for all $k \geq N$ (can make $N_{j+1} > N_j$). We have $\{f_{N_j}\}$. We claim that $f_{N_j} \to f$ a.e.. Let $E_j = \{|f_{N_j} - f| > 1/2^j\}$. Since $|E_j| < 1/2^j$, we know $\sum |E_j| < \infty$. But then by HW 1, we know $|\limsup E_j| = 0$. For all $x \notin \limsup E_j$, we have $x \in E_j$ only finitely many times. Then for sufficiently large j, $x \notin E_j$, say $x \notin E_j$ for $j \geq j_0$. But then $|f_{N_j}(x) - f(x)| < 1/2^j \to 0$. □

Remark. "Triangle Inequality" (quasi). Convergence in measure is not like convergence in metric space, though it is tempting to think of it in this way. We do have a replacement for a parallel triangle inequality. If $\{|f - g| > \epsilon/2\}| < \epsilon/2$, and $|\{|g - h| > \epsilon/2\}| < \epsilon/2$, then $|\{|f - h| > \epsilon\}| < \epsilon$. The proof is routine: $|f - h| \leq |f - g| + |g - h|$, so $|f - h| > \epsilon$ then one of the right side must be at least $\epsilon/2$. Then $\{|f - h| > \epsilon\} \subset \{|f - g| > \epsilon/2\} \cup \{|g - h| > \epsilon/2\}$. Their measures being at most $\epsilon/2$, the result then follows.

Definition (Cauchy in Measure)**.** A collection of measurable functions $\{f_k\}$ is said to be Cauchy in measure if for all $\epsilon > 0$, there exists N such that $|\{|f_j - f_k| > \epsilon\}| < \epsilon$ for all $k, j \geq N$.

As it turns out, this is equivalent to convergence in measure.

Theorem 2.11. *Cauchy in measure is equivalent to convergence in measure.*

Proof (Sketch). $\leftarrow$ is immediate by "triangle inequality" measure remark. Now the other direction. There are three standard steps that apply in these types of arguments. The strategy being the important piece here. Step 1. Enough to obtain a convergent subsequence: $f_1, f_2, f_3, \ldots$ cauchy sequence of functions with convergent subsequence. Use triangle inequality to show must converge to limit of conv subsequence. Conclude $f_k \to f$. Notice this holds for any type of convergence in any space. Step 2: Every Cauchy sequence has a "geometric" subsequence. Meaning $|\{|f_{k_j} - f_{k_{j+1}}| > 1/2^j\}| < 1/2^j$. This is true for any metric space and any convergence. Use $\epsilon = 1/2^j$, and get k_j. Step 3. Geometric trick implies convergent: given $\epsilon > 0$, let m be such that $1/2^{m-1} < \epsilon$. Let $\mathcal{A} = \bigcup_{j \geq m} \{|f_{k_j} - f_{k_{j+1}}| > 1/2^j\}$. Note $|\mathcal{A}| < \sum_{j=m}^{\infty} 1/2^j = 1/2^{m-1} < \epsilon$. On $E \setminus A$, we have $|f_{k_j} - f_{k_{j+1}}| \leq 1/2^j$ for all $j \geq m$. But this implies uniform convergence on the set $E \setminus A$. But then difference everywhere at most ϵ on $E \setminus A$ for sufficiently large j. Therefore, $|\{|f_{k_j} - f| > \epsilon\}| \leq |A| < \epsilon$. □

Two approaches to $\int_E f$, $f \geq 0$ on E.

First, $\int_E f = \sup\{\int_E g \colon g \text{ simple } g \leq f\}$, $g = \sum_{k=1}^m a_k \chi_{E_k}$, where E_k are disjoint, measurable sets. Here $\int_E g = \sum a_k |E_k|$.

Second, $\int_E = |R(f,E)|$, where $R(f,E) = \{(x,y) \in \mathbb{R}^{n+1} \colon x \in E, 0 \leq y \leq f(x)\}$.

First approach is common in many texts, we shall take the second.

So $\int_E f$ exists if and only if $R(f,E)$ is measurable.

Some advantages of the second approach are monotonicity and countable additivity over E: so $0 \leq f \leq g$ and $\int_E f$, $\int_E g$ exists, then $\int_E f \leq \int_E g$ because $R(f,E) < R(g,E)$.

Also if $E = \bigcup_{k \in \mathbb{N}} E_k$, disjoint measurable sets, and $\int_{E_k} f$ exists, then $\int_E f$ exists and $\int_E f = \sum_{k \in \mathbb{N}} \int_{E_k} f$ because $R(f,E) = \bigcup R(f,E_k)$, the union taken over the disjoint, measurable sets. Can show in other definition but takes work due to sup, whereas here we get it immediately. But we have the issue of existence and measurability.

The following main theorem.

Theorem 2.12. *If $f LE \to [0,\infty]$ is measurable, then $\int_E f$ exists (possibility ∞).*

Idea of proof is from 4.1, $f = \lim f_k$, $f_1 \leq f_2 \leq \cdots$ are simple. So we will show $R(f,E)$ are measurable and take their union. Need Lemma 1, $R(f_k,E)$ is measurable. Lemma 2 $\Gamma(f,E)$, which needs to be defined (simply the graph $\{(x, f(x) \colon x \in E\}$) has measure 0. This is boundary top of graph of shaded area of graph, we care because $\bigcup R(f_k,E)$ need not generally cover $R(f,E)$. It will surely cover $R(f,E) \setminus \Gamma(f,E)$, then $y < f(x)$, so there exists k such that $y < f_k(x)$ so that $(x,y) \in R(f_k,E)$.

Both lemmas rely on

Lemma 2.1. *If $E \subseteq \mathbb{R}^n$ is measurable, then $E_A := E \times [a,b]$ is measurable in $\mathbb{R}^{n+1}$ with measure $|E_a| = |E| \cdot a$, noting that $0 \cdot \infty = 0$ in measure theory.*

Proof. If E is an interval, follows by volume formula. If E is general, then tile it by intervals, half open (so intervals disjoint).

If E is G_δ of finite measure, write $E = \bigcap G_k$, $G_1 \supset \cdots$ are open.

$$E_a = \bigcap_k (G_k \times [0,a]) \text{ so } |E_a| = \lim |G_k \times [0,a]| = a \lim |G_k| = a \cdot |E|.$$

What about general set? $|E| < \infty$, $E = H \setminus Z$, M is G_δ, $|M| = |E|$, $|H_a| = |H| \cdot a = |E| \cdot a$. So $|E_a| \leq |E| \cdot a$.

Also for all $\epsilon > 0$, there exists closed $F \subset E$ (hence G_δ), $|F| > |E| - \epsilon$.

$$|F_a| = a|F| > a|E| - a\epsilon$$

So E_a contains F_a closed of measure $> |E_a|_e - a\epsilon$. Hence E_a is measurable. $|E_a| = a|E|$.

Still assumes $a < \infty$, otherwise $a = k$ and let $k \to \infty$ □?

Lemma 2.2. *$R(f_k, E) = \bigcup_j (E_j \times [0, a_j])$, E_j disjoint measurable sets, where $f_k = \sum_j a_j \chi_{E_j}$.*

Note that this shows that if f takes countably many values, $f = a_k$ on E_k, then $\int_E f = \sum a_k |E_k|$.

Lemma 2.3. *$|\Gamma(f, E)| = 0$. Let $g = 2^{-j} floor(2^j f)$, then $g \le f \le g + 2^{-j}$. Then $R(g, E) \subset R(f, E) \subset$. ???? Then $\Gamma(f, E) \subset R(g + 2^{-j}, E) \setminus R(g - 2^{-j}, E)$. This right side has measure at most $2 \cdot 2^{-j}|E| \to 0$, hence $|\Gamma(f, E)| = 0$.*

Last time: $\int_E f = |R(f, E)|$, $R(f, E) = \{(x, y) \in \mathbb{R}^{n+1} : 0 \le y \le f(x), x \in E\}$. $\Gamma(f, E) = \{(x, f(x)) \in \mathbb{R}^{n+1} : x \in E\}$.

$\int_E f = \sup\{\int g : g \le f, g \text{ simple}\}$, $g = \sum a_k \chi_{E_k} \to \sum g = \sum a_k |E_k|$, then $R(f, E) = (E \cap F) \times [0, 1]$.

Recall $R(f, E)$ is approximated by $R(f_k, E)$:

$\cup_k R(f_k, E) \subset R(f, E) \subset [\cup_k R(f_k, E)] \cup \Gamma(f, E)$.

Lemma: $|\Gamma(f, E)| = 0$ for all measurable f.

Proof. Write $E = \cup E_m$, $|E_m| < \infty$

Suffices $|\Gamma(f, E_m)| = 0$ for all $|\epsilon > 0$. Let $\mathcal{A}_k = \{x \in E_m : \epsilon k \le f(x) < \epsilon(k+1)\}$, $k = 0, 1, 2, \ldots$. Then $\Gamma(f, E_m) \subset \cup_k \mathcal{A}_k + [\epsilon_k, \epsilon(k+1)]$, then $|\Gamma(f, E_m)| \le \sum_k |\mathcal{A}_k| \epsilon = \epsilon |E_m|$, then $|\Gamma(f, E_m)| = 0$.

Theorem 2.13 (Monotone Convergence Theorem)**.** *If $f_k \ge 0$ are measurable on E and $f_1 \le f_2 \le \cdots$, then $\int_E \lim f_k = \lim \int_E f_k$.*

Proof. $f = \lim f_k$, then $R(f, E) = [\cup R(f_k, E)] \cup \Gamma(f, E)$ and $\cup R(f_k, E)$ are nested. So $|R(f, E)| = \lim |R(f_k, E)|$ by continuity of measure. □

Example 2.7. $\mathbb{Q} = \{q_1, q_2, \ldots\}$. Let $f_k = \chi_{\{q_1, \ldots, q_k\}}$. Then $f_k \nearrow f = \chi_{\mathbb{Q}}$. Monotone convergence theorem fails for Riemann integral, so $\int_{\chi_{\mathbb{Q}}} = \lim \int f_k = 0$.

Theorem 2.14. *The two definitions of $\int_E f$ agree:*

$$\int_E f = \sup\{\sum a_k |E_k| : g \le f, g = \sum a_k \chi_{E_k}\}$$

Proof. Note $\sum a_k |E_k| = \int_E g$ by measure of product. Since $f \ge g$, we get $\int f \ge \int g$ so $\ge$ holds. Reverse: pick $g_k \nearrow f$ pointwise. Then $\int g_k \to \int f$ so $\sup \int g_k \ge \int f$. □

Example 2.8. $f_1 \ge f_2 \ge \cdots$. Counterexample $f_k = \chi_{[k, \infty)}$. $\lim \int f_k = \infty$ but $\int \lim f_k = 0$. Similarly $\inf\{\int g : g \ge f, g \text{ simple}\} \ne \int f$ in general. if $g \ge f$ then $g > 0$ but being simple $g \ge c > 0$ then $\int g = \infty$.

Chebshev's Inequality: Suppose f is nonnegative measurable on E. For all $\alpha > 0$ $|\{x \in E \colon f(x) > \alpha\}| \leq \frac{1}{\alpha} \int_E f$.
Proof. Let $A = \{f > \alpha\}$. Then $f \geq \alpha\chi_A \to \int f \geq \int \alpha\chi_A = \alpha|A| \to |A| \leq \frac{1}{\alpha} \int f$.

Example 2.9. $f(x) = \dfrac{1}{\sqrt{x}}$ on $[0,1)$. Let $c = \int_{(0,1)} f \to$ Chevyshev $|\{f > \alpha\}| \leq \dfrac{c}{\alpha}$. In fact, $\{f > \alpha\} = (0, 1/\alpha^2) \to$ measure is $1/\alpha^2$, if $\alpha \geq 1$.

Additivity over functions: $\int_E (f+g) = \int_E f + \int_E g$, $f, g \geq 0$ measurable.
Proof. Use simple fucntions, $f_k \nearrow f$, $g_k \nearrow g$. So it suffices to prove that $\int (f_k + g_k) = \int f_k + \int g_k$. $f_k = \sum a_j \chi_{E_j}$, $g_k = \sum b_k \chi_{F_k}$. Let $H = E_i \cap F_j$ note $\cup E_i = \cup F_i = E$. Then $f_k = \sum_{i,j} a_i \chi_{H_{i,j}}$, $g_k = \sum b_j \chi_{H_{ij}}$. Then $f_k + g_k = \sum (a_i + b_j)\chi_{H_{ij}}$ then $\int (f_k + g_k) = \sum (a_i + b_j)|H_{ij}| = \sum a_i |H_{ij}| + \sum b_j |H_{ij}|$.
Countable subadditivity:
$\int \sum_{k=1}^{\infty} f_k = \sum_k \int_E f_k$, $f_k \geq 0$ measurable.
True for $\sum_{k=1}^{N}$ then $N \to \infty$ using MCT

Example 2.10. Let $\mathbb{Q} \cap [0,1] = \{q_1, \ldots\}$
$f_k = \dfrac{1}{\sqrt{|x - q_k|}}$ $\int_{[0,1]} f \leq c < \infty$
$f = \sum_{k=1}^{\infty} \dfrac{1}{k^2} \dfrac{1}{\sqrt{|x-q_k|}} \xrightarrow{\lim_{x \to q} f(x) = \infty \text{ for all } q \in \mathbb{Q}} \int f \leq c \sum \dfrac{1}{k^2} < \infty$ so $|\{f > \alpha\}| \leq \dfrac{1}{\alpha} C'$, what does this say about $\mathbb{Q}$?

We have not yet proved $\int_E cf = c \int_E f$, where $c \geq 0$ and $f \geq 0$.

Proposition 2.1.

$$\int_E cf = c \int_E f$$

Proof. This is true for simple functions: $\int c(\sum a_k \chi_{E_k}) = \int \sum c a_k \chi_{E_k} = \sum c a_k |E_k| = c \sum a_k |E_k| = c \int \sum a_k \chi_{E_k}$. Then for general functions use the Monotone Convergence Theorem. For general $f \geq 0$, take simple $f_k \geq 0$, $f_k \nearrow f$, then $cf_k \nearrow cf$, by MCT $\int cf = \lim cf_k = c \lim \int f_k = c \int f$. $\square$

What if the convergence is not $f_k \nearrow f$?

Example 2.11. $\chi_{[k,\infty)} \to 0$ pointwise. But $\int \chi_{[k,\infty)}$ are all infinite and certainly do not converge to 0. Note that $\chi_{[k,\infty)} \searrow 0$.

Example 2.12. $k\chi_{(0,1/k)}$. But $\int k\chi_{(0,1/k)} = 1 \not\to 0$.

Example 2.13. $\frac{1}{k}\chi_{[0,k]} \to 0$ uniformly but $\int \frac{1}{k}\chi_{[0,k]} = 1 \to 0$.

What is the similarity of these three examples? The functions vanish but their integrals do not???? The general principle is that the area under the graph can disappear or 'escape' to infinity but does not appear from nowhere.......... A precise statement is Fatou's Lemma. As a reminder,

Definition (limsup,liminf)**.** Given a sequence $\{f_k\}$,

$$\limsup_{k\to\infty} f_k = \lim_{n\to\infty} \sup_{k\geq n} f_k \liminf_{k\to\infty} f_k = \lim_{n\to\infty} \int_{k\geq n} f_k.$$

Example 2.14. (i) $\lim f_k(x)$ exists if and only if $\limsup f_k(x) = \liminf f_k(x)$.

(ii) The Piano Key Sequence (press one key going left to right): $\chi_{[0,1]}$, then $\chi_{[0,1/2]}, \chi_{[1/2,1]}$, then $\chi_{[0,1/3]}, \chi_{[1/3,2/3]}, \chi_{[2/3,1]}$. [Go in pascal triangle order]

$$\limsup f_k = \chi_{[0,1]}$$
$$\liminf f_k = 0$$

Remark. Relation to sequences to sets $\{E_k\}$: $\limsup E_k = \bigcap_{m=1}^{\infty} \bigcup_{k\geq m} E_k$, $\liminf E_k = \bigcup_{m=1}^{\infty} \bigcap_{k\geq m} E_k$. $\limsup \chi_{E_k} = \chi_{\limsup E_k}$ and same for liminf. and sup like union and inf like intersection.

Lemma 2.4 (Fatou)**.** *$f_k \geq 0$ measurable.*

$$\int_E \liminf f_k \leq \liminf \int_E f_k$$

Proof. Let $g_m = \inf_{k\geq m} f_k$. Then $g_m \nearrow \liminf f_k$. By MCT, $\int g_m \to \int \liminf f_k$. But $f_k \geq g_k$ so that $\int f_k \geq \int g_k$, hence $\liminf \int f_k \geq \lim \int g_k$. [Rem: if $a_k \geq b_k$ for all k, then $\liminf a_k \geq \liminf b_k$, $\limsup a_k \geq \limsup b_k$.] □

Theorem 2.15 (Lebesgue Dominated Convergence (DCT))**.** *Suppose $f_k \geq 0$ are measurable and there is $\phi \geq 0$ measurable such that $f_k \geq \phi$ (ϕ dominates) and $\int_E \phi < \infty$. If f_k converges to f a.e., then $\int_E f_k \to \int_E f$.*

Proof. Fatou's Lemma gives $\int f \leq \liminf \int f_k$. Also, $\phi - f_k \geq 0$. Then Fatou's Lemma applies again: $\int \phi - f \leq \liminf \int(\phi - f_k)$. Which is $\int f \geq \limsup \int f_k$. Then $\int \phi - \int f \leq \liminf(\int \phi - \int f_k)$ cancel constant $-\int f \leq \liminf(-\int f_k)$ then $\int f \geq \limsup f_k$. Comparing very first and last equations, we have $\int f = \lim_{k\to\infty} \int f_k$. □

Example 2.15. $\frac{1}{k}\chi_{[0,k]} \to 0$ uniformly.

Try to cover with one function get something like $1/x$. This is not good enough for DCT since $\int \frac{!}{x} = \infty$. If $f_k \leq \frac{1}{x^2}$ on $[1,\infty)$, this exmapes does not work. 'Best' you can do is $\frac{1}{k^2}\chi_{[0,k]} \to 0 = \int \lim f_k$.

Example 2.16. $\lim \int_{[1,\infty)} \frac{\sin^{2k} x}{x^2}\, dx$. Now $\frac{\sin^{2k} x}{x^2} \to 0$ a.e. and is at most $\frac{1}{x^2}$ in absolute value, so must be 0. Now if $\lim \int_{[0,\infty)} \frac{\sin^{2k} x}{x^2}\, dx$, $\frac{\sin^{2k} x}{x^2} \leq \frac{\sin^2 x}{x^2}$ and $\int_{[0,1]} \frac{\sin^2 x}{x^2} \leq \int_{[0,1]} 1 \leq 1$. so $\int_{[0,\infty)} \frac{\sin^2 x}{x^2} < \infty$.

Suppose that $f : E \to \overline{\mathbb{R}}$ is measurable. Define functions $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. These are both nonnegative functions with $f^+ - f^- = f$ and $f^+ + f^- = |f|$.

Definition (Integrable Function)**.** Suppose $f : E \to \overline{\mathbb{R}}$ is a measurable function, not necessarily nonnegative. We define $\int_E f$ to be

$$\int_E f := \int_E f^+ - \int_E f^-,$$

provided at least one of the integrals on the right is finite. We say that f is integrable on E if $\int_E f$ is finite, and we write $f \in L^1(E)$,

Remark. Note that $\int f$ is finite if and only if $\int f^+$ and $\int f^-$ are finite if and only if $\int |f|$ is finite. Therefore, all Lebesgue integrable functions with finite integral are automatically absolutely integrable. This means there are integrals which are (Improper) Riemann integrable which are not Lebesgue integrable. Take Example **??**, $\int_{[0,\infty)} \frac{\sin x}{x}\, dx$ does not exist as a Lebesgue integrable, i.e. $\frac{\sin x}{x} \notin L^1(0,\infty)$, while it is Riemann integrable on $(0,\infty)$.

We next examine a few properties of this integral. Not surprisingly, many of these properties resemble those of their primordial Riemann integral.

- Triangle Inequality: $\left|\int_E f\right| \leq \int_E |f|$ as $\left|\int f^+ - \int f^-\right| \leq \int f^+ + \int f^-$.
- Additivity of Domain: If $\{E_k\}_{k=1}^N$ is a collection of disjoint, measurable sets, then $\int_{\cup E_k} f = \sum_{k=1}^m \int_{E_k} f$ as $\int f^+$, $\inf f^-$ split using the properties of the integral for nonnegative functions.

  However, care is needed even in the countable additivity case. Consider $\int_0^\infty \sin x$, defining the E_k to be as below:

  We know that $\int_{E_k} f = 0$ for all k but that $\int_{\cup E_k} f$ does not exist.

Lemma 2.5. *Suppose that $\int_E f$ exists. Let $E = \bigcup_{k=1}^\infty E_k$ be a union of disjoint, measurable sets. Then $\int_{E_k} f$ exists for all k, and*

$$\int_E f = \sum_{k=1}^\infty \int_{E_k} f$$

Proof. By countable additivity for integrals of nonnegative functions, we know that $\int_E f^+ = \sum_k \int_{E_k} f^+$ and $\int_E f^- = \sum_k \int_{E_k} f^-$. Now observe

$$\int_E f = \sum_k \left(\int_{E_k} f^+ - \int_{E_k} f^- \right).$$

Since $\int_E f$ exists, at least one of $\int_E f^+$, $\int_E f^-$ is finite.

□

Additivity over f:

Suppose we have $\int_E (f+g) = \int_E f + \int_E g$, assuming that both integrals on the right are finite. Let $h := f+g$. There are eight possibilities for the signs of (f, g, h). This splits E into eight different parts. On each part, write a between f, g, h as addition of nonnegative functions, e.g. $f \geq 0$, $g < 0$, $h < 0$, then $f + g = h$ becomes $f + (-h) = (-g)$, all nonnegative here. Then

$$\int f + \int (-h) = \int (-g)$$

Note that we are making use of the fact that the functions are finite a.e..

Remark: if $f \in L^1(E)$, then $|f| < \infty$ a.e., since $\int |f| < \infty$.

Convergence:

There is little to say about convergence since everything follows from convergence of nonnegative functions.

For example, how does MCT apply? The answer is that it does not really apply. We need an integrable 'baseline', $\phi \in L^1(E)$, i.e. a function to play the role of the x-axis.

MCT for nonnegative functions:

If $\phi \leq f_1 \leq f_2 \leq \cdots$ or $\phi \geq f_1 \geq f_2 \geq \cdots$, then $\int f_k \to \int \lim f_k$.

Proof. Use MCT for $f_k - \phi$ or $\phi - f_k$, nonnegative, increasing. Then $\int (f_k - \phi) \to \int (f - \phi)$; hence, $\int f_k \to \int f$. □

However, there is one useful result here:

Theorem 2.16 (Dominated Convergence Theorem). *If $f_k \to f$ a.e. on E, and are measurable, and there exists $\phi \in L^1(E)$ such that $|f_k| \leq \phi$ for all k, then*

$$\int_E f_k \longrightarrow \int_E f.$$

Proof. Since $0 \leq \underbrace{f_k + \phi}_{\geq 0} \leq 2\phi$, DCT for nonnegative functions applies. $\int (f_k + \phi) \to \int (f + \phi)$. □

Example 2.17. Consider $\sum_{k=1}^{\infty} B_k \sin(kx)$ on the interval $(0, \pi)$. Assume that $C := \sum |B_k| < \infty$. What do we need to assume about B_k to obtain

$$\int_{(0,\pi)} f(x) = \sum_k \int_{(0,\pi)} B_k \sin(kx)?$$

Observe that C is a domination function. $f_k(x) = \sum_{j=1}^{k} B_j \sin(jx)$ satisfies $|f_k| \leq \phi$. Also, $f_k \to f$ pointwise.

Shorter proof: let $h = f + g$: $h^+ - h^- = f^+ - f^- + g^+ - g^-$ $h^+ + f^- + g^- = f^+ + g^+ + h^-$ Integrate, apply additivitivty for nonnegative. $\int h^+ + \int f^- + \int g^- = \int f^+ + \int g^+ + \int h^-$ all integrals finite then $\int h^+ - \int h^- = \int f^+ - \int f^- + \int g^+ - \int g^-$.

DCT: Works for continuous parameters.

If we have f_t (t real) and $|f_t| \leq \phi$, $\phi \in L^1(E)$ and f_t measurable and $f_t \to f$ a.e. as $t \to t_0$. Then $\lim_{t \to t_0} \int_E f_t = \int f$.

Proof. Sequential characterization of limits: Take any sequence $t_k \to t_0$ and apply DCT to f_{t_k}. Get $\int f_{t_k} \to \int f$. □

Example 2.18. Suppose $|E| < \infty$, $f \in L^1(E)$, $f > 0$ on E. a) $|int_E f^p \to |E|$ as $p \to 0^+$ b) if also $\log f \in l^1(E)$, then $\int_E \dfrac{f^p - 1}{p} \to \int_E \log f$ as $p \to 0^+$.

apf) $f^p \to 1$ as $p \to 0^+$. Need dominating function. Consider only $0 < p \leq 1$. Then $f^p \leq f$? if $f \geq 1$ but notice $f^p \leq 1$ if $0 < f < 1$. So $f^p \leq \max(f, 1) \leq f + 1$. This is simpler than the less mysterious max. So $f + 1 \in L^1(E)$ dominating function. Hence, $\int_E f^p \to \int_E 1 = |E|$.

bpf) derivative of $p \mapsto e^{p \log f}$ at $p = 0$ which is $\log f$. $\dfrac{f^p - 1}{p} \to \log f$. Slope of secant

Slope inc: $\dfrac{f^p - 1}{p} \leq \dfrac{f^1 - 1}{1} = f - 1$

$f \leq 1$: $\left\| \dfrac{f^p - 1}{p} \right| \leq |\log f|$.

If $f < 0$ is measurable and $f \in L^1(E)$, then

$$\int_E \frac{f^p - 1}{p} \to \int_E \log f \text{ as } p \to 0^+.$$

We need integrability to make this possible.

MCT applies: if $\phi \in L^1$ is a 'baseline' and $\phi \geq f_1 \geq f_2 \geq \cdots$ or $\phi \leq f_1 \leq f_2 \leq \cdots$, then $\int \lim f_k = \lim \int f_k$.

For all $p_k \searrow 0$, we want to show $f_k = \dfrac{f^{p_k} - 1}{p_k}$ is a decreasing sequence, $f_1 \geq f_2 \geq f_3 \geq \cdots$. The reason why is because of the shape of these things.

Now taking p_k which is smaller, so p_{k+1} is to left so the slope of the secant line is less because it is a convex function. The conclusion is the same for the decreasing version because this is also convex. 'Baseline' is then $f-1$.

2.4 Relation between the Lebesgue and Riemann-Stieltjes Integrals

Recall most important HW (5.1#1): if $f \geq 0$ measurable, then $f \in L^1$ if and only if $\sum_{j=-\infty}^{\infty} 2^j |\{f > 2^j\}| < \infty$. Let $\omega_f = |\{x \in E \colon f(x) > \alpha\}|$. Notice this is just notation for something we have already discussed at length. With this notation, we can say that

$$f \in L^1(E) \iff \sum_{j=-\infty}^{\infty} 2^j \omega_{|f|}(2^j) < \infty,$$

where f is measurable but not necessarily nonnegative. What is the significance of the '2' here? The proof of the exercise only makes use of the fact that multiplication by 2 preserves the inequality: $g = 2^j \chi_{\{2^j < f < 2^{j+1}\}}$ and then $g \leq f \leq 2g$. This is like the Cauchy Condensation Test. However, this works for any real number $\lambda > 1$. This allows us to then characterize the above as

$$\begin{aligned} f \in L^1(E) &\iff \sum_{j=-\infty}^{\infty} 2^j \omega_{|f|}(2^j) < \infty \\ &\iff \sum_{j=-\infty}^{\infty} \lambda^j \omega_{|f|}(\lambda^j) < \infty \end{aligned}$$

for any $\lambda > 1$. The generalization of this is as follows:

Definition (L^p). $f : E \to \overline{\mathbb{R}}$ belongs to $L^p(E)$ if $\int |f|^p < \infty$, where $0 < p < \infty$.

By the discussion above (or #1 of HW 5.1), we know that $f \in L^p$ if and only if $\sum \lambda^j |\{|f|^p > \lambda^j\}| < \infty$, i.e. $\sum \lambda^j |\{|f| > \lambda^{j/p}\}| < \infty$. If we choose $\lambda = 2^p$, then

$$f^p \in L^p \iff \sum_{j=-\infty}^{\infty} 2^{pj} |\{f > 2^j\}| < \infty.$$

Theorem 2.17. *If $|E| < \infty$ and $f : (a, b]$ is measurable, where $a, b \in \mathbb{R}$, then*

$$\int_E f = \int_a^b \alpha d(-\omega_f(\alpha)).$$

Proof. Consider $\mathcal{P}$: $a = x_0 < x_1 < \cdots < x_n = b$. Let $E_i = \{x_{i-1} < f \leq x_i\}$. Note E_i is a partition of E.

$$\sum_i x_{i-1} \chi_{E_i} \leq f \leq \sum_i x_i \chi_{E_i}.$$

But then

$$\sum_{i=1}^{n} x_{i-1}|E_i| \leq \int_E f \leq \sum_{i=1} n x_i |E_i| (*)$$

Here

$$|E_i| = -(\underbrace{\omega(x_i)}_{f>x_i} - \underbrace{\omega(x_{i-1})}_{f>x_{i-1}})$$

But then the left sum is $L(\alpha, P, -\omega)$ and the right one is $U(\alpha, P, -\omega)$. The sup of L and inf of U both go to $\int_a^b \alpha d(-\omega)$. Thus, $\int_E f = \int_a^b f \, d(-\omega)$. □

2.5 Relation between the Lebesgue and Riemann Integral

We want to compare $\int_{[a,b]} f$ and $R\int_a^b f$, where $a, b \in \mathbb{R}$ and f is bounded. Following shows we have generalized.

Theorem 2.18. *If $R\int_a^b f$ exists, then $\int_{[a,b]} f$ exists and they are equal.*

Proof. Riemann then Leb: If R integral exists, then there exists sequence of partitions $\{P_k\}$ such that $U(f, P_k)$ and $L(f, P_k)$ converge to $\int_a^b f$. Arrange $P_1 \subset P_2 \subset P_3 \subset \cdots$. Let g_k, h_k be step functions for these partitions. $g_k \leq f \leq h_k$ and $L(f, P_k) = \int g_k$, $U(f, P_k) = \int h_k$.

By MCT, $g_1 \leq g_2 \leq \cdots$ and $h_1 \geq h_2 \geq \cdots$. Then $\int g_k \to \int g$, where $g = \lim g_k$ and $\int h_k \to \int h$, where $h = \lim h_k$. So $\int g = \int h = R\int_a^b f$ since U and L converge to $\int_a^b f$. But we know that $g \leq f \leq h$. This means that we must have equality a.e.. But this completes the proof. □

The following characterization of Riemann integrals.

Theorem 2.19. *Riemann integral exists if and only if $|\mathcal{D}| = 0$, where $\mathcal{D}$ is the set of discontinuities of f.*

Proof. See text.

3 Repeated Integration

3.1 Fubini's Theorem

Given $f(x,y) = \dfrac{1}{\sqrt{xy}}$ on $[0,1] \times [0,1]$, we already have a concept of $\int_{[0,1]\times[0,1]} f$. But how does this relate to the integral

$$\int_0^1 \left(\int_0^1 \frac{1}{\sqrt{xy}}\, dy \right)\, dx?$$

We have the same issues with double series: $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$.

Example 3.1. Consider the infinite matrix with 1s on the main diagonal and -1 on the superdiagonal and zeros elsewhere. The sum of each row is 0 and the sum of column is 0 except the first and we obtain

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} \right) = 0$$
$$\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ij} \right) = 1$$

This is the same issues with integrals. For example, take $f = 1, -1, 0$ and consider

$$\int_0^{\infty} \left(\int_0^{\infty} f(x,y)\, dx \right)\, dy = 0$$
$$\int_0^{\infty} \left(\int_0^{\infty} f(x,y)\, dy \right)\, dx = 0$$

Note that $f \notin L^1([0,\infty) \times [0,\infty))$.

Theorem 3.1 (Fubini's Theorem)**.** *Let $I_1 \subseteq \mathbb{R}^n$, $I_2 \subseteq \mathbb{R}^m$ be intervals and $I = I_1 \times I_2$. Suppose that $f \in L^1(I)$. Then:*

(i) For almost every $x \in I_1$, the 'slice' function $y \mapsto f(x,y)$ is integrable on I_2.

(ii) The 'once-integrated' function $x \mapsto \int_{I_2} f(x,y)\, dy$ is integrable on I_1, and its value is $\int_I f$.

Why do we say for almost every x? A slice is defined on a set of measure zero. For the matrix example, one could choose a column and insert in a zero entry insert a nonmeasurable function or ∞. While the function becomes difficult, its integral is unchanged since its values were changed on a set of measure zero.

Recall that $\int_I f = \int_{\mathbb{R}^{n\times m}} f \cdot \chi_I$, etc.. So we can reduce the proof to $I_1 = \mathbb{R}^n$, $I_2 = \mathbb{R}^m$, and $I = \mathbb{R}^{n+m}$ by replacing f with $f\chi_I$. Let $\mathcal{F} := \{f \in L^1(\mathbb{R}^{n+m})\colon \text{Fubini holds for } f\}$. We prove that $\mathcal{F} = L^1$.

Properties of $\mathcal{F}$:

- $\mathcal{F}$ is a vector space: $f, g \in \mathcal{F}$, then $\alpha f + \beta g \in \mathcal{F}$.
- $\mathcal{F}$ is closed under monotone limits: if $f_k \in \mathcal{F}$ and $f_k \nearrow f$ or $f_k \searrow f$, where $f \in L^1(\mathbb{R}^{n \times m})$, then $f \in \mathcal{F}$.
- Squeeze Property: if $g \leq f \leq h$, where $g, h \in F$ and $g = h$ a.e., then $f \in \mathcal{F}$
- $\chi_I \in F$ for any closed bounded interval I.

To see why the second point holds:

Proof 2 $f_k \nearrow f$ supplies for every $x \in I_1$, x-slice of $f_k \nearrow x$-slice of f. For each k, there is $Z_k \subset I_1$ with $|Z_k| = 0$ such that the x-slice of f_k is integrable for $x \notin Z_k$. Let $Z = \cup_k Z_k$, then $|Z| = 0$. So MCT, $\int f_k(x, y)\, dy \nearrow \int f(x, y)\, dy$ (though this may be infinite) for $x \notin Z$. Again by MCT, $\int (\int f_k)\, dx \nearrow \int (\int f dy)\, dx$. Then finally, we have $\int_I f_k \to \int_{I_1} f$ by MCT, noting the last integral is finite. But then $\int (\int f dy)\, dx$ is finite since they are equal. But then Fubini's Theorem holds for f as well.

We now prove the second. For almost all x, the slice of g and h are both integrable, and since the slice of f is between, it is also integrable.

(Proof of 3???) Once integrated functions for g, f, h satisfy $\int g\, dy \leq \int f\, dy \leq \int h\, dy$ for almost all x. Integrate with respect to x

$$\left(\int g\, dy \right),$$

but then

$$\left(\int g\, dy \right)\, dx \leq \left(\int f\, dy \right)\, dx \leq \left(\int h\, dy \right)\, dx$$

But the left/right integrals are equal to $\int_I g$ and $\int_I h$, respectively. But these are equal since $g = h$ a.e.. Therefore, $\left(\int f\, dy \right)\, dx = \int_I f$.

Proof of 4: Slice function of χ_I is either $\equiv 0$ or χ_{I_2}, so $\int \chi_I = |I_2| \chi_{I_1}$. Then $\int \chi_I\, dy\, dx = |I_2||I_1| = |I| = \int \chi_I$. $\square$

4 Quiz Problems

Quiz 1. Suppose A is a closed proper subset of $[0,1]$ (so $A \subset [0,1]$ and $A \neq [0,1]$). Is $|A|_e < 1$?

Quiz 2. Suppose $A \subset \mathbb{R}$ is a set such that $A \cap [-k,k]$ is measurable for every $k \in \mathbb{N}$. Is it the case that A is measurable?

Quiz 3. Suppose $A_1 \supset A_2 \supset A_3 \supset \cdots$ is a nested sequence of measurable subsets of $\mathbb{R}^n$ such that $|A_1| < \infty$. Is it the case that $|A_k \setminus A_{k+1}| \to 0$ as $k \to \infty$?

Quiz 4. Suppose $E \subset \mathbb{R}^n$ is a set such that $|A \cap E|_e + \left|A \cap E^C\right|_e \leq |A|_e$ for every $A \subset \mathbb{R}^n$. Is it the case that E is measurable?

Quiz 5. Let $f(x)$ be the function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$$

Is it the case that f is a measurable function on $\mathbb{R}$?

Quiz 6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is measurable. Define

$$g(x) = \begin{cases} f(x), & |f(x)| \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Is g necessarily a measurable function on $\mathbb{R}$?

Quiz 7. Let $f(x)$ be the function given by

$$f(x) = \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Is f upper semicontinuous on $\mathbb{R}$?

Quiz 8. Suppose that $\{f_k : [0,1] \to \mathbb{R}\}$ is a collection of simple functions which converge uniformly to f. Is f necessarily a simple function?

Quiz 9. Suppose that $f : [0,\infty) \to [0,\infty)$ is measurable. Is it the case that

$$\int_{[0,\infty)} e^{-x/k} f(x)\, dx \longrightarrow \int_{[0,\infty)} f(x)\, dx$$

as k tends to infinity?

Quiz 10. Suppose that $f : [0,\infty) \to [0,\infty)$ is measurable and $\int_{[0,\infty)} f < \infty$. Is it true that $\int_{[k,\infty)} f(x)\, dx \to 0$ as $k \to \infty$?

5 Quiz Solutions

Solution Quiz 1. Suppose that $f : [0, \infty) \to [0, \infty)$ is measurable and $\int_{[0,\infty)} f < \infty$. Is it true that $\int_{[k,\infty)} f(x)\, dx \to 0$ as $k \to \infty$?

Proof. Let $f_k := f\chi_{[k,\infty)}$. Then goes to 0 as $k \to \infty$ and f dominates f_k and $f_k \to 0$.

Another idea: $f\chi_{(0,k)} \nearrow f$ by MCT, $\int \chi_{(0,k)} f \to \int f$.

Or

$R(f, E)$ has finite measure, have nested subsets from k to infinite areas, nested, contained, measure sets tend to measusre of intersection which is 0?

6 Homework Problems

Problem 1. Given an arbitrary set $A \subset \mathbb{R}$ and a number $c > 0$, let $B = \{ca\colon a \in A\}$. Prove that $|B|_e = c\,|A|_e$.

Problem 2. Suppose that a set $A \subset \mathbb{R}$ is measurable. Prove that for every $c > 0$ the set $B = \{ca\colon a \in A\}$ is also measurable.

Problem 3. Given a sequence of continuous functions $f_k : \mathbb{R} \to \mathbb{R}$, let B be the set of all points $x \in \mathbb{R}$ such that the sequence $\{f_k(x)\}$ is bounded. Prove that B is a measurable set. [*Hint: try to construct B from the sets $\{x\colon |f_k(x)| \le M\}$ by using countable unions and intersections.*]

Problem 4. Given a sequence of continuous functions $f_k\colon \mathbb{R} \to \mathbb{R}$, let C be the set of all points $x \in \mathbb{R}$ such that $\lim\limits_{k\to\infty} f_k(x) = 0$. Prove that C is a measurable set.

Problem 5. Prove that the set

$$A = \{x \in \mathbb{R}\colon \exists\, k \in \mathbb{N} \text{ such that } |2^x - 2^k| \le 1\}$$

is measurable and $|A| < \infty$.

Problem 6. Prove that the set

$$A = \{x \in [0,1]\colon \forall q \in \mathbb{N}\ \exists p \in \mathbb{N} \text{ such that } |x - p/q| \le 1/q^2\}$$

is measurable and $|A| = 0$.

Problem 7. Suppose E and Z are sets in $\mathbb{R}^n$ such that $E \cup Z$ is measurable and $|Z| = 0$. Prove that E is measurable.

Problem 8. Given a continuous function $f\colon \mathbb{R}^n \to \mathbb{R}^n$, define $\mathcal{M} = \{E \subset \mathbb{R}^n\colon f^{-1}(E) \text{ is Borel}\}$.

(a) Prove that $\mathcal{M}$ is a σ-algebra.

(b) Prove that if E is Borel, then $f^{-1}(E)$ is Borel. [*Hint: Use (a).*]

Problem 9. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a function with a continuous derivative. Prove that for every measurable set E, the set $f(E)$ is also measurable. [*Hint: although f need not be Lipschitz, its restriction to any bounded interval is Lipschitz.*]

Problem 10. Given a set $E \subset [0, \infty)$, define a function $f : [0, \infty) \to [0, \infty)$ by $f(x) = |E \cap [0, x]|_e$.

(a) Prove that f is Lipschitz continuous.

(b) Prove that for every number b with $0 < b < |E|_e$ there exists a set $F \subset E$ such that $|F|_e = b$.

Problem 11. Show that there exists a nested sequence of sets $E_1 \supset E_2 \supset \cdots$ such that $|E_1|_e < \infty$ and $\bigcap_{k=1}^{\infty} E_k = \varnothing$ but $\lim\limits_{k\to\infty} |E_k|_e > 0$; that is, outer measure is not continuous under nested intersections. [*Hint: Use the translates of the Vitali set.*]

Problem 12. Show that for the standard middle-third Cantor set $C \subset [0, 1]$, the difference set $C - C$ contains a neighborhood of 0. [*Hint: C is the intersection of nested sets C_n where $C_0 = [0, 1]$ and $C_{n+1} = \frac{1}{3}C_n \cup (\frac{1}{3}C_n + \frac{2}{3})$. Find $C_n - C_n$ using induction.*]

Remark: This shows that having $|E| > 0$ is not necessary for $E - E$ to contain a neighborhood of 0.

Problem 13. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a function such that $f(\mathbb{R}^n)$ is countable and $f^{-1}(t)$ is measurable for every $t \in \mathbb{R}$. Prove that f is measurable.

Problem 14. Prove that without the assumption "$f(\mathbb{R}^n)$ is countable" the statement in Problem 13 would not be true.

Problem 15. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is measurable, and $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with $g' > 0$ everywhere. Prove that $f \circ g$ is measurable.

Problem 16.

(a) Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that f^2 is measurable. Prove that f is measurable.

(b) Prove that the statement in (a) is false if f is not assumed continuous.

Problem 17. Suppose that $f : E \to \mathbb{R}$ is a measurable function, where $E \subset \mathbb{R}^n$ is measurable.

(a) Prove that there exists a Borel set $H \subset E$ such that the restriction $f_{|H}$ is Borel measurable and $|E \setminus H| = 0$.

(b) If, in addition, E is a Borel set, prove that there exists a Borel measurable function $g : E \to \mathbb{R}$ such that $f = g$ a.e.. [*Hint: For part (a), take a countable union of closed sets obtained from Lusin's Theorem.*]

Problem 18. Suppose $\phi : [0, \infty) \to [0, \infty)$ is a function such that $\phi(t) \to 0$ as $t \to \infty$. Consider a sequence of measurable functions $\{f_k\}$, $f_k : \mathbb{R}^n \to \mathbb{R}$, such that $|f_k(x)| \leq \phi(|x|)$ for every k, and $f_k \to f$ a.e.. Prove that the conclusion of Egorov's Theorem holds in this situation; that is, for every $\epsilon > 0$, there exists a closed set $E(\epsilon) \subset \mathbb{R}^n$ such that $|\mathbb{R}^n \setminus E(\epsilon)| < \epsilon$ and $f_k \to f$ uniformly on $E(\epsilon)$. [*Hint: Follow the proof of Egorov's theorem.*]

Problem 19. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a function such that $f(\mathbb{R}^n)$ is countable and $f^{-1}(t)$ is measurable for every $t \in \mathbb{R}$. Prove that f is measurable.

Problem 20. Prove that without the assumption "$f(\mathbb{R}^n)$ is countable" the statement in Problem 19 would not be true.

7 Homework Solutions

Solution Problem 1. Given an arbitrary set $A \subset \mathbb{R}$ and a number $c > 0$, let $B = \{ca \colon a \in A\}$. Prove that $|B|_e = c\,|A|_e$.

Proof. Given $\epsilon > 0$, let $\{[s_k, t_k]\}$ be a countable cover of A such that

$$\sum_k (t_k - s_k) \leq |A|_e + \epsilon$$

The intervals $[cs_k, ct_k]$ cover B, since every point of B is of the form ca where $a \in A$ is covered by some interval $[s_k, t_k]$. Therefore,

$$|B|_e \leq \sum_k (ct_k - cs_k) = c \sum_k (t_k - s_k) \leq c\,|A|_e + \epsilon$$

Since $\epsilon > 0$ was arbitrary, it follows that $|B|_e \leq c\,|A|_e$.

It remains to observe that $A = c^{-1}B$, which by the above implies $|A|_e \leq c^{-1}\,|B|_e$, i.e., $|B|_e \geq c\,|A|_e$. Thus, $|B|_e = c\,|A|_e$. □

Solution Problem 2. Suppose that a set $A \subset \mathbb{R}$ is measurable. Prove that for every $c > 0$ the set $B = \{ca \colon a \in A\}$ is also measurable.

Proof. Given $\epsilon > 0$, let G be an open set that contains A and satisfies $|G \setminus A|_e < \epsilon/c$. Since the function $f(x) = x/c$ is continuous, the preimage of G under this function is also open. This preimage $f^{-1}(G)$ is cG. Since $A \subset G$, it follows that $B \subset cG$. Moreover by Problem 1,

$$|(cG) \setminus B|_e = |c(G \setminus A)|_e = c\,|G \setminus A|_e < \epsilon.$$

Since ϵ was arbitrary, this proves that B is measurable. □

Solution Problem 3. Given a sequence of continuous functions $f_k : \mathbb{R} \to \mathbb{R}$, let B be the set of all points $x \in \mathbb{R}$ such that the sequence $\{f_k(x)\}$ is bounded. Prove that B is a measurable set. [*Hint: try to construct B from the sets $\{x \colon |f_k(x)| \leq M\}$ by using countable unions and intersections.*]

Proof. For $k, m \in \mathbb{N}$, let

$$A(k, m) = \{x \in \mathbb{R} \colon |f_k(x)| \leq m\} = f_k^{-1}([-m, m]).$$

Being the preimage of a closed set under a continuous function, $A(k, m)$ is closed, and in particular measurable. Let

$$A := \bigcup_{m=1}^{\infty} \bigcap_{k=1}^{\infty} A(k, m),$$

which is also measurable, being obtained from measurable sets by countable set operations. We claim that $A = B$.

If $x \in A$, then there exists $m \in \mathbb{N}$ such that $|f_k(x)| \leq m$ for all $k \in \mathbb{N}$, which shows the sequence $\{f_k(x)\}$ is bounded.

Conversely, if the sequence $\{f_k(x)\}$ is bounded, then there exists $m \in \mathbb{N}$ such that all elements of the sequence are at most m in absolute value. This means $|f_k(x)| \leq m$ for all k, hence $x \in A$. □

Solution Problem 4. Given a sequence of continuous functions $f_k \colon \mathbb{R} \to \mathbb{R}$, let C be the set of all points $x \in \mathbb{R}$ such that $\lim\limits_{k\to\infty} f_k(x) = 0$. Prove that C is a measurable set.

Proof. For $k, m \in \mathbb{N}$, let

$$A(k,m) = \{x \in \mathbb{R} \colon |f_k(x)| < 1/m\} = f_k^{-1}((-1/m, 1/m)).$$

Being the preimage of an open set under a continuous function, $A(k,m)$ is open, and in particular measurable. Let

$$A := \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} A(k,m).$$

This set is also measurable, being obtained from measurable sets by countable set operations. We claim that $A = C$.

Suppose $x \in A$. Given $\epsilon > 0$, pick $m \in \mathbb{N}$ such that $1/m \leq \epsilon$. Since $x \in \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} A(k,m)$, there exists N such that $x \in \bigcap_{k=N}^{\infty} A(k,m)$, which means $|f_k(x)| < 1/m$ for all $k \geq N$. Thus, $|f_k(x)| < \epsilon$ for all $k \geq N$, which proves $\lim\limits_{k\to\infty} f_k(x) = 0$.

Conversely, suppose $x \in C$. Given $m \in \mathbb{N}$, use the definition of the limit $\lim\limits_{k\to\infty} f_k(x) = 0$ to find N such that $|f_k(x)| < 1/m$ for all $k \geq N$. The latter means $x \in \bigcap_{k=N}^{\infty} A(k,m)$. Therefore, for every $m \in \mathbb{N}$ the inclusion $x \in \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} A(k,m)$ holds. This means $x \in A$. □

Solution Problem 5. Prove that the set

$$A = \{x \in \mathbb{R} \colon \exists k \in \mathbb{N} \text{ such that } |2^x - 2^k| \leq 1\}$$

is measurable and $|A| < \infty$.

Proof. For each $k \in \mathbb{N}$, the inequality $|2^x - 2^k| \leq 1$ is equivalent to $\log_2(2^k - 1) \leq x \leq \log_2(2^k + 1)$. Thus, $A = \bigcup_{k=1}^{\infty} I_k$, where $I_k = [\log_2(2^k - 1), \log_2(2^k + 1)]$. Begin an interval, each I_k is measurable. Hence, A is measurable. By countable subadditivity of measure, $|A| \leq \sum_{k=1}^{\infty} |I_k|$. It remains to show the series $\sum_{k=1}^{\infty} |I_k|$ converges. This can be done by the comparison test, limit comparison test, or the ratio test. We use the Limit Comparison Test with $\sum_{k=1}^{\infty} 2^{-k}$ as a reference series:

$$\begin{aligned}
\frac{|I_k|}{2^{-k}} &= \frac{\log_2(2^k+1) - \log_2(2^k-1)}{2^{-k}} \\
&= \frac{k + \log_2(1+2^{-k}) - (k + \log_2(1-2^{-k}))}{2^{-k}} \\
&= \frac{\log_2(1+2^{-k}) - \log_2(1-2^{-k})}{2^{-k}} \\
&= \frac{1}{\log 2}\left\{\frac{\log(1+2^{-k})}{2^{-k}} + \frac{\log(1-2^{-k})}{-2^{-k}}\right\} \xrightarrow{k\to\infty} \frac{2}{\log 2}
\end{aligned}$$

Here the last step is based on $\lim_{x\to 0} \dfrac{\log(1+x)}{x} = 1$. Since $\sum_{k=1}^{\infty} 2^{-k}$ converges, so does $\sum_{k=1}^{\infty} |I_k|$. $\square$

Solution Problem 6. Prove that the set

$$A = \{x \in [0,1] \colon \forall q \in \mathbb{N}\ \exists p \in \mathbb{N} \text{ such that } |x - p/q| \le 1/q^2\}$$

is measurable and $|A| = 0$.

Proof. Let $A_q = \bigcup_{p=1}^{\infty} E(p,q)$, where $E(p,q) = \left[\frac{p}{q} - \frac{1}{q^2}, \frac{p}{q} + \frac{1}{q^2}\right] \cap [0,1]$. This is a countable union of measurable sets $E(p,q)$ (which are intervals, possibly empty), so it is measurable. Then the set $A = \bigcap_{q\in\mathbb{N}} A_q$ is measurable too.

We have $|E(p,q)| \le 2/q^2$ by construction of $E(p,q)$. Also, when $p > q+1$, we have $\frac{p}{q} - \frac{1}{q^2} \ge 1 + \frac{1}{q} - \frac{1}{q^2} \ge 1$, which implies $E(p,q) = \varnothing$. By subadditivity,

$$|A_q| \le \sum_{p=1}^{\infty} |E(p,q)| \le \sum_{p=1}^{q+1} \frac{2}{q^2} = \frac{2q+2}{q^2}$$

By monotonicity, $|A| \le |A_q|$ for each q. Since $|A_q| \xrightarrow{q\to\infty} 0$, it follows that $|A| = 0$. $\square$

Solution Problem 7. Suppose E and Z are sets in $\mathbb{R}^n$ such that $E \cup Z$ is measurable and $|Z| = 0$. Prove that E is measurable.

Proof. Since $Z \setminus E \subset Z$, the monotonicity of outer measure implies $|Z \setminus E|_e = 0$, hence $Z \setminus E$ is measurable. Then

$$E = (E \cup Z) \setminus (Z \setminus E)$$

is measurable, being the difference of two measurable sets. [This could be done with Carathéodory's Theorem or with the "G_δ minus a null set" theorem, but it is easier without.] □

Solution Problem 8. Given a continuous function $f\colon \mathbb{R}^n \to \mathbb{R}^n$, define $\mathcal{M} = \{E \subset \mathbb{R}^n \colon f^{-1}(E) \text{ is Borel}\}$.

(a) Prove that $\mathcal{M}$ is a σ-algebra.

(b) Prove that if E is Borel, then $f^{-1}(E)$ is Borel. [*Hint: Use (a).*]

Proof.

(a) This proof does not involve f being continuous; the argument works for any map f. Taking preimages commutes with any set operations: for example,

$$f^{-1}(E^C) = \{x\colon f(x) \in E^C\} = \{x\colon f(x) \notin E\} = (f^{-1}(E))^C$$

and

$$f^{-1}\left(\bigcup_i E_i\right) = \{x\colon \exists i\, f(x) \in E_i\} = \bigcup_i f^{-1}(E_i).$$

If $E \in \mathcal{M}$, then $f^{-1}(E^C) = f^{-1}(E)^C$ is the complement of a Borel set, and hence is Borel. Hence, $E^C \, in \mathcal{M}$. Also, if $E_k \in \mathcal{M}$ for each $k \in \mathbb{N}$, then

$$f^{-1}\left(\bigcup_k E_k\right) = \bigcup_k f^{-1}(E_k)$$

is the countable union of Borel sets, and hence is Borel. Therefore, $\bigcup_k E_k \in \mathcal{M}$.

The definition of a σ-algebra in the book also requires us to check that $\mathcal{M}$ is nonempty: to do this, it suffices to notice that $f^{-1}(\emptyset) = \emptyset$ is Borel, hence $\emptyset \in \mathcal{M}$.

(b) Since f is continuous, the preimage of any open set under f is open, and is hence Borel. This means $\mathcal{M}$ contains all open sets. By definition, the Borel σ-algebra $\mathcal{B}$ is the *smallest* σ-algebra that contains all open sets. Thus, $\mathcal{B} \subset \mathcal{M}$, which by definition of $\mathcal{M}$ means that $f^{-1}(E)$ is Borel whenever E is Borel.

[It is tempting to approach statement (b) by "writing a Borel set E in terms of open/closed sets" and concluding that $f^{-1}(E)$ can also be written in this way. But there is no such structural formula for Borel sets: one can only get the proper subclasses like G_δ, $G_{\delta\sigma}$, $G_{\delta\sigma\delta}$, and so on. The whole story is complicated: see Borel hierarchy on Wikipedia).]

□

Solution Problem 9. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a function with a continuous derivative. Prove that for every measurable set E, the set $f(E)$ is also measurable. [*Hint: although f need not be Lipschitz, its restriction to any bounded interval is Lipschitz.*]

Proof. For each $j \in \mathbb{N}$, the set $E_j = E \cap [-j, j]$ is measurable, as the intersection of two measurable sets is measurable. Since $E = \bigcup_j E_j$, it follows that $f(E) = \bigcup_j f(E_j)$. So it suffices to prove $f(E_j)$ is measurable for every j.

The derivative f', being continuous, is bounded on the interval $[-j, j]$. By the Mean Value Theorem, f is Lipschitz on $[-j, j]$: indeed, $|f(a) - f(b)| \le |a - b| \sup_{[-j,j]} |f'|$. A technical detail arises: we only proved the measurability of images for Lipschitz functions on all of $\mathbb{R}^n$. To get around this, define

$$f_j(x) = \begin{cases} f(x), & x \in [-j, j] \\ f(-j), & x < -j \\ f(j), & x > j \end{cases}$$

Such an extended function f_j is Lipschitz continuous on all of $\mathbb{R}$. Indeed in each of the three closed intervals $(-\infty, -j]$, $[-j, j]$, $[j, \infty)$, the Lipschitz condition holds by construction. For arbitrary $a < b$, partition the interval $[a, b]$ by the points $\{-j, j\}$ should they lie there, and apply the Lipschitz continuity to each interval, and use the triangle inequality.

Conclusion: $f_j(E_j)$, which is the same as $f(E_j)$, is measurable. □

[Note: In fact, for every set $E \subset \mathbb{R}^n$, any Lipschitz function $f : E \to \mathbb{R}^n$ can be extended to a Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^n$. Therefore, when discussing the measurability of $f(E)$, it suffices to check that f is Lipschitz on the set E.

Proof (Sketch). It suffices to extend a real-valued Lipschitz function $f : E \to \mathbb{R}$, because the vector-valued case follows by extending each component. Let L be the Lipschitz constant of f, and define, for every $x \in \mathbb{R}^n$,

$$F(x) = \inf_{a \in E} (f(a) + L|x - a|).$$

It is an exercise with the definition of inf to prove that F is Lipschitz with constant L, and that $F(x) = f(x)$ when $x \in A$. □

Remark: Extending a map $f : E \to \mathbb{R}^n$ in the above fashion, one finds the Lipschitz constant of the extension is at most $L\sqrt{n}$, where L is the Lipschitz constant of the original map. There is a deeper extension theorem (due to Kirszbraun) according to which an extension with the same Lipschitz constant L exists.]

Solution Problem 10. Given a set $E \subset [0, \infty)$, define a function $f : [0, \infty) \to [0, \infty)$ by $f(x) = |E \cap [0, x]|_e$.

(a) Prove that f is Lipschitz continuous.

(b) Prove that for every number b with $0 < b < |E|_e$ there exists a set $F \subset E$ such that $|F|_e = b$.

Proof.

(a) We claim that $0 \leq f(b) - f(a) \leq b - a$ for any $a, b \in [0, \infty)$ such that $a < b$; this yields the Lipschitz continuity with constant 1. On one hand, $f(b) \geq f(a)$ by the monotonicity of outer measure: $E \cap [0, a] \subset E \cap [0, b]$. On the other, $E \cap [0, b] \subset (E \cap [0, a]) \cup [a, b]$, which implies

$$f(b) \leq |(E \cap [0, a]) \cup [a, b]|_e \leq |E \cap [0, a]|_e + |[a, b]| = f(a) + (b - a)$$

by subadditivity.

(b) By Theorem 3.27 in the textbook, the outer measure is continuous under nested unions even if the sets are not measurable. Since $E = \bigcup_{k \in \mathbb{N}} (E \cap [0, k])$, it follows that

$$|E|_e = \lim_{k \to \infty} |E \cap [0, k]|_e = \lim_{k \to \infty} f(k).$$

Since $b < |E|_e$, by the definition of limit there exists k such that $f(k) > b$. Also, $f(0) = |E \cap \{0\}| = 0$. Applying the Intermediate Value Theorem to f on the interval $[0, k]$ (which is possible since f is continuous by part (a)), we conclude that there exists $x \in (0, k)$ such that $f(x) = b$. Then the set $F = |E \cap [0, x]|$ meets the requirements.

□

Solution Problem 11. Show that there exists a nested sequence of sets $E_1 \supset E_2 \supset \cdots$ such that $|E_1|_e < \infty$ and $\bigcap_{k=1}^{\infty} E_k = \varnothing$ but $\lim_{k \to \infty} |E_k|_e > 0$; that is, outer measure is not continuous under nested intersections. [*Hint: Use the translates of the Vitali set.*]

Proof. Let $V \subset [0, 1]$ be the Vitali set described in these notes: recall that $|V|_e > 0$ and that the sets $V + q$ are disjoint for all $q \in \mathbb{Q}$. Let

$$E_k = \bigcup_{j=k}^{\infty} \left(V + \frac{1}{j} \right).$$

Then $E_1 \subset V + [0, 1] \subset [0, 2]$, hence $|E_1|_e \leq 2 < \infty$.

Suppose $x \in \bigcap_{k=1}^{\infty} E_k$. This means that for each $k \in \mathbb{N}$ there exists $j \geq k$ such that $x \in V + 1/j$. In particular, $x \in V + 1/j$ for infinitely many distinct values of j. But this is impossible as the sets $V + 1/j$ are disjoint, a contradiction. Therefore, $\bigcap_{k=1}^{\infty} E_k$ is empty.

The sets E_k are nested by construction, hence $|E_k|_e$ is a non-increasing sequence. It is bounded from below by $|V|_e$ because each E_k contains a translated copy of V. Thus, $\lim_{k\to\infty} |E_k|_e \geq |V|_e > 0$. $\square$

Solution Problem 12. Show that for the standard middle-third Cantor set $C \subset [0,1]$, the difference set $C - C$ contains a neighborhood of 0. [*Hint: C is the intersection of nested sets C_n where $C_0 = [0,1]$ and $C_{n+1} = \frac{1}{3}C_n \cup (\frac{1}{3}C_n + \frac{2}{3})$. Find $C_n - C_n$ using induction.*]

Remark: This shows that having $|E| > 0$ is not necessary for $E - E$ to contain a neighborhood of 0.

Proof. Recall that $A - B = \{a - b \colon a \in A, b \in B\}$. This definition implies that

$$(A_1 \cup A_2) - (B_1 \cup B_2) = \bigcup_{i,j=1}^{2} (A_i - B_j) \tag{*}$$

Furthermore, for any $t \in \mathbb{R}$ we have $(A+t) - B = (A+B) + t$, $A - (B+t) = (A-B) - t$, and $tA - tB = t(A - B)$; all these follow directly from the definition.

The equality $C_0 - C_0 = [-1,1]$ holds because, on one hand, $|x - y| \leq 1$ when $x, y \in [0,1]$, while on the other, $C_0 - C_0 \supset [0,1] - \{0,1\} = [0,1] \cup [-1,0] = [-1,1]$.

Assume $C_n - C_n = [-1,1]$. Use the relation $C_{n+1} = \frac{1}{3}C_n \cup (\frac{1}{3}C_n + \frac{2}{3})$ and distribute the difference according to (*) and other properties stated at the beginning:

$$\begin{aligned} C_{n+1} - C_{n+1} &= \left(\frac{1}{3}C_n - \frac{1}{3}C_n\right) \cup \left(\frac{1}{3}C_n - \frac{1}{3}C_n + \frac{2}{3}\right) \cup \left(\frac{1}{3}C_n - \frac{1}{3}C_n - \frac{2}{3}\right) \\ &= [-1/3, 1/3] \cup ([-1/3, 1/3] + 2/3) \cup ([-1/3, 1/3] - 2/3) \\ &= [-1/3, 1/3] \cup [1/3, 1] \cup [-1, -1/3] = [-1, 1] \end{aligned}$$

The set $\left(\frac{1}{3}C_n + \frac{2}{3}\right) - \left(\frac{1}{3}C_n + \frac{2}{3}\right)$ is not included above because it is the same as $\left(\frac{1}{3}C_n - \frac{1}{3}C_n\right)$. By induction, $C_n - C_n = [-1,1]$ for all n.

Since $C \subset C_n$ for every n, it follows that $C - C \subset [-1,1]$. To prove the reverse inclusion, fix $a \in [-1,1]$. For each n, there exist $x_n, y_n \in C_n$ such that $x_n - y_n = a$. Since all these numbers are contained in $[0,1]$, we can pick a convergent subsequence $\{x_{n_k}\}$. So, $x_{n_k} \to x$ and since $x_{n_k} - y_{n_k} = a$, we also have $y_{n_k} \to y$ where y is such that $x - y = a$.

It remains to prove that $x, y \in C$. For each $m \in \mathbb{N}$ we have $x_{n_k}, y_{n_k} \in C_m$ for $k \geq m$ by construction. Since C_m is compact, it follows that $x, y \in C_m$. And since this holds for every $m \in \mathbb{N}$, we have $x, y \in C$. $\square$

Solution Problem 13. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a function such that $f(\mathbb{R}^n)$ is countable and $f^{-1}(t)$ is measurable for every $t \in \mathbb{R}$. Prove that f is measurable.

Proof. Let $B = f(\mathbb{R}^n)$, a countable subset of $\mathbb{R}$. For any $a \in \mathbb{R}$ we have

$$\{f > a\} = \bigcup_{b \in B,\ b > a} f^{-1}(b),$$

which is a countable union of measurable sets, and is hence measurable. The domain of f, which is $\mathbb{R}^n$, is also measurable. Thus f is measurable. □

Solution Problem 14. Prove that without the assumption "$f(\mathbb{R}^n)$ is countable" the statement in Problem 13 would not be true.

Proof. The statement in Problem 13 is made for any n. To disprove it, it suffices to show it fails for some n. With massive loss of generality, let $n = 1$. Let $V \subset [0, 1]$ be a Vitali set, and define $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x + 1, & x \in V \\ -|x|, & x \notin V \end{cases}$$

By construction $\{f > 0\} = V$, which is nonmeasurable. Thus, f is nonmeasurable. On the other hand, for every $t \in \mathbb{R}$ the set $f^{-1}(t)$ is finite and therefore measurable. Indeed, if t is negative, $f(x) = t$ holds for at most two values of x; and when $t \geq 0$, there is at most one such value. □

[Remark: If we wanted to construct such an example on $\mathbb{R}^n$ for every n, one way is to let

$$f(x_1, \ldots, x_n) = \begin{cases} x_1 + 1, & \forall i\ x_i \in V; \\ -|x_1|, & \text{otherwise} \end{cases}$$

Then $\{f > 0\} = V^n$ which is nonmeasurable, because on one hand, $V^n + \mathbb{Q}^n = \mathbb{R}^n$ forces $|V^n|_e > 0$; on the other, $V^n + (\mathbb{Q} \cap [0, 1])^n$ is a bounded set containing infinitely many copies of V^n, which makes it impossible to have $|V^n| > 0$.

For every $t \in \mathbb{R}$, the preimage $f^{-1}(t)$ consists at most two hyperplanes of the form $\{x\} \times \mathbb{R}^{n-1}$. So it is covered by countably many sets of the form $\{x\} \times [-j, j]^{n-1}$, $j \in \mathbb{N}$. Here $|\{x\} \times [-j, j]^{n-1}| = 0$ because this set is contained in a box of dimensions $(\epsilon, 2j, \ldots, 2j)$ whose volume can be arbitrarily small. In conclusion, $|f^{-1}(t)| = 0$ for every t. Thus f is measurable.]

Solution Problem 15. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is measurable, and $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with $g' > 0$ everywhere. Prove that $f \circ g$ is measurable.

Proof. By the Mean Value Theorem, g is strictly increasing; therefore, it has an inverse $h = g^{-1}$. By the Inverse Function Theorem, the inverse function h is also continuously differentiable.

Given $a \in \mathbb{R}$, consider the set $A = \{x \colon f(g(x)) > a\}$. It can be written as $\{x \colon g(x) \in B\}$, where $B = \{f > a\}$ is measurable; that is, $A = h(B)$. By Problem 9, the image of a measurable set under a continuously differentiable function is measurable. Therefore, A is measurable. □

Solution Problem 16.

(a) Suppose $f \colon \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that f^2 is measurable. Prove that f is measurable.

(b) Prove that the statement in (a) is false if f is not assumed continuous.

Proof.

(a) Since f is continuous, it is measurable.

(b) Let $n = 1$, let V be a Vitali set, and define $f(x) = 1$ when $x \in V$ and $f(x) = -1$ when $x \notin V$. Then $f^2 \equiv 1$ is measurable, being continuous. But $\{f > 0\} = V$ is not a measurable set, so f is not measurable.

□

Solution Problem 17. Suppose that $f : E \to \mathbb{R}$ is a measurable function, where $E \subset \mathbb{R}^n$ is measurable.

(a) Prove that there exists a Borel set $H \subset E$ such that the restriction $f_{|H}$ is Borel measurable and $|E \setminus H| = 0$.

(b) If, in addition, E is a Borel set, prove that there exists a Borel measurable function $g : E \to \mathbb{R}$ such that $f = g$ a.e.. [*Hint: For part (a), take a countable union of closed sets obtained from Lusin's Theorem.*]

Solution Problem 18. Suppose $\phi \colon [0, \infty) \to [0, \infty)$ is a function such that $\phi(t) \to 0$ as $t \to \infty$. Consider a sequence of measurable functions $\{f_k\}$, $f_k : \mathbb{R}^n \to \mathbb{R}$, such that $|f_k(x)| \leq \phi(|x|)$ for every k, and $f_k \to f$ a.e.. Prove that the conclusion of Egorov's Theorem holds in this situation; that is, for every $\epsilon > 0$, there exists a closed set $E(\epsilon) \subset \mathbb{R}^n$ such that $|\mathbb{R}^n \setminus E(\epsilon)| < \epsilon$ and $f_k \to f$ uniformly on $E(\epsilon)$. [*Hint: Follow the proof of Egorov's theorem.*]

Solution Problem 19. Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}$ is a function such that $f(\mathbb{R}^n)$ is countable and $f^{-1}(t)$ is measurable for every $t \in \mathbb{R}$. Prove that f is measurable.

Solution Problem 20. Prove that without the assumption "$f(\mathbb{R}^n)$ is countable" the statement in Problem 19 would not be true.

References

[WZ77] R.K. Wheedon and A Zygmund. *Measure and Integral: An Introduction to Real Analysis*. 1977.