



Syracuse University

# MATH 701: Real Variables I

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## 0 Introduction

### 0.1 Course Description

**MAT 701 Real Variables I:** Measure and integration, including basic theorems on integration and differentiation of sequences of functions; modes of convergence, product measures.

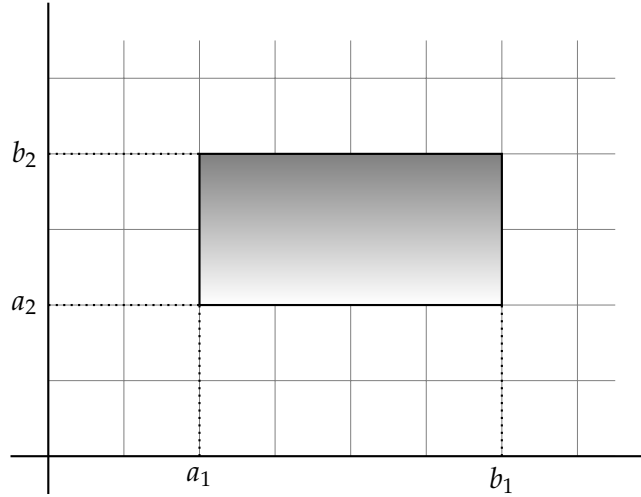
### 0.2 Disclaimer

These notes were taken in Fall 2018 in a course taught by Professor Leonid Kovalev. In some places, notation/material has been changed or added. Any errors in this text should be attributed to the typist – Caleb McWhorter – and not the instructor or any referenced text.

# 1 Outer Measure & Lebesgue Measure

## 1.1 Exterior Measure

We want to assign a notion of ‘size’ to sets. We denote this ‘size’ by  $\nu$ . Let  $\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$  denote ordinary Euclidean space. By an ‘interval’ in  $\mathbb{R}^n$ , we mean a set  $\{(a_1, b_1) \times (a_2, b_2) \times \dots \times (a_n, b_n) : a_i, b_i \in \mathbb{R}^n, a_i < b_i\}$ . By a closed interval in  $\mathbb{R}^n$ , we mean a set  $\{[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n] : a_i, b_i \in \mathbb{R}^n, a_i < b_i\}$ . We will often say ‘box’, which will always mean an open or closed interval in  $\mathbb{R}^n$ .



In defining a ‘size’ for sets, it makes sense to begin with a simple shape like a box. In the case of the plane, we know the a good notion of size is the area, and the area of a box  $[a_1, b_1] \times [a_2, b_2]$  is  $(b_1 - a_1) \cdot (b_2 - a_2)$ . We can immediately generalize this to  $\mathbb{R}^n$  as follows: if  $I \subset \mathbb{R}^n$  is an interval, then we define

$$\nu(I) := \prod_{j=1}^n (b_j - a_j) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$$

But the question remains, how do we generalize this to an arbitrary region  $E$ ? Given the above definition, it is natural to try to generalize to arbitrary sets  $E$  by approximating  $E$  by boxes, i.e. an open covering  $\{I_k\}$  of  $E$  by intervals (see Figure 1).

It then becomes clear that whatever the measure of  $E$  is, it should satisfy

$$\nu(E) \leq \sum_k \nu(I_k),$$

since the intervals cover  $E$ . After all, it would be strange indeed to allow  $E$  to have greater measure than its covering. Furthermore, if a notion of measure is going to be defined in

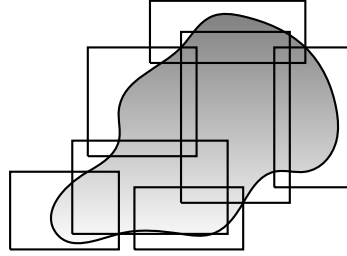


Figure 1: Approximating a set  $E$  by intervals.

terms of coverings, then the measure need be invariant of the choice of covering. These ideas are our guiding principles. So we take the following definition

**Definition (Outer Measure).** For a set  $E \subset \mathbb{R}^n$ , the outer measure (or exterior measure) of  $E$ , denoted  $|E|_e$ , is the function  $\nu : \mathbb{R}^n \rightarrow [0, \infty)$  given by

$$|E|_e := \inf_{E \subset \bigcup_k I_k} \sum \nu(I_k),$$

where the infimum is taken over all *countable* coverings  $\{I_k\}$  of  $E$ .

The coining of the term ‘outer measure’ is immediately obvious—we are measuring the size of a set via external objects, namely the open cover. However, less obvious is the need to restrict to countable coverings. The need to eliminate uncountable coverings is apparent, as then trouble arises defining the summation. But why not only allow finite coverings? For this consider the case  $E = \mathbb{Q} \cap [0, 1]$ . In any finite covering of  $[0, 1]$  by intervals, the intervals cannot all be pairwise disjoint. The reader will confirm, with a bit of thought, that if the intervals were all pairwise disjoint then there would be a rational number missed by the ‘covering.’ But this contradicts the fact that the collection was an open cover. The only possible open covering by intervals is the entire interval itself so that  $\inf \sum \nu(I_k) = 1$ . This violates the notation that there isn’t any ‘length’ or ‘area’ here since we have a sparse collection of points.

Furthermore, the same logic applies to the set  $E' = \mathbb{Q}^c \cap [0, 1]$ . So if the outer measure of  $E$  were 1, then this would be true too of  $E'$ . But clearly the outer measure of  $[0, 1]$  is 1. Now  $[0, 1] = E \cup E'$ , and  $E \cap E' = \emptyset$ . As  $1 + 1 \neq 1$ , this breaks countable subadditivity of the measure we are trying to define. By defining the outer measure in terms of countable covers, we obtain the expected answer  $|E|_e = 0$ .

**Example 1.1.** Let  $E = \mathbb{Q} \cap [0, 1]$  and  $\epsilon > 0$  be given. Since  $\mathbb{Q}$  is countable, so too is  $E$  countable. Enumerate the rationals in  $E$  as  $\{q_1, q_2, \dots, q_n, \dots\}$ . Now the set  $\{O_n\}_{n \in \mathbb{N}}$ , where  $O_n := (q_n - \frac{1}{2^{n+k+1}}, q_n + \frac{1}{2^{n+k+1}})$  and  $k \in \mathbb{N}$  is fixed, is a (countable) open covering of  $E$  by intervals. Furthermore, the  $O_n$  are pairwise disjoint. Choose  $k$  sufficiently large so

that  $2^{-k} < \epsilon$ . The measure of this covering is then

$$\sum_{n=1}^{\infty} \nu(O_n) = \sum_{n=1}^{\infty} \left[ \left( q_n + \frac{1}{2^{n+k+1}} \right) - \left( q_n - \frac{1}{2^{n+k+1}} \right) \right] = \sum_{n=1}^{\infty} 2 \cdot \frac{1}{2^{n+k+1}} = \frac{1}{2^k} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2^k} < \epsilon.$$

◁

It is also important to note that our notion of measure preserves our notion of ‘size’ for intervals. Before proving this in the general case, we begin with the case for intervals in  $\mathbb{R}$ .

**Proposition 1.1.**  $|[a, b]|_e = b - a$ .

*Proof.* Clearly,  $|[a, b]|_e \leq b - a$ , so it remains to show that  $b - a \leq |[a, b]|_e$ . Suppose that  $[a, b]$  is a finite union of intervals of the form  $[c_j, d_j]$ . There exists  $j_1$  such that  $c_{j_1} \leq a$ . If  $d_{j_1} \geq b$ , then  $|[a, b]|_e \geq d_{j_1} - c_{j_1} \geq b - a$ , and we are done. Otherwise, it must be that  $d_{j_1} < b$ . There then exists  $j_2$  such that  $c_{j_2} \leq d_{j_1}$ . Continue this process inductively until one finally obtains  $d_{j_r} \geq b$ . But then taking the sum of these differences, one obtains a telescoping series

$$\underbrace{(d_{j_r} - c_{j_r})}_{\geq 0} + \underbrace{(d_{j_{r-1}} - c_{j_{r-1}})}_{\geq 0} + \cdots + \underbrace{(d_{j_1} - c_{j_1})}_{\geq 0} \geq b - a.$$

□

We now prove this for general intervals.

**Proposition 1.2.** For an interval  $I$ ,  $|I|_e = \nu(I)$ .

*Proof.* As  $I$  serves as an open covering for itself, we know that  $|I|_e \leq \nu(I)$ . It remains to show that  $\nu(I) \leq |I|_e$ . Suppose  $\{I_k\}_k$  is a countable open covering of  $I$ , and let  $\epsilon > 0$ . Every interval  $I$  is contained in the interior of a slightly larger interval  $I'$ , i.e.  $I \subseteq \text{Int}(I')$ , where  $\nu(I') - \nu(I) < \epsilon$ . Let  $I'_k$  denote an interval containing  $I_k$  in its interior with  $\nu(I'_k) \leq (1 + \epsilon)\nu(I_k)$ . We know that  $I \subset \bigcup_k \text{Int}(I'_k)$ . Now  $I$  is compact so that Heine Borel implies that there is an integer  $N$  such that  $I \subset \bigcup_{k=1}^N I'_k$ . We know  $\nu(I) \leq \sum_{k=1}^N \nu(I'_k)$ . But then

$$\nu(I) \leq (1 + \epsilon) \sum_{k=1}^N \nu(I_k) \leq (1 + \epsilon) \sum_{k=1}^{\infty} \nu(I_k).$$

It then follows that  $\nu(I) \leq |I|_e$ , completing the proof. □

More generally,  $\left| \bigcup_{k=1}^N I_k \right|_e = \sum_{k=1}^N \nu(I_k)$ , provided the  $I_k$  are non-overlapping intervals, i.e.  $\text{Int}(I_k) \cap \text{Int}(I_j) = \emptyset$ , see Figure 2. The proof of this is rather ugly—an exercise in making ‘obvious’ geometric facts obvious, and an exercise in bookkeeping—and we shall not concern ourselves with it.



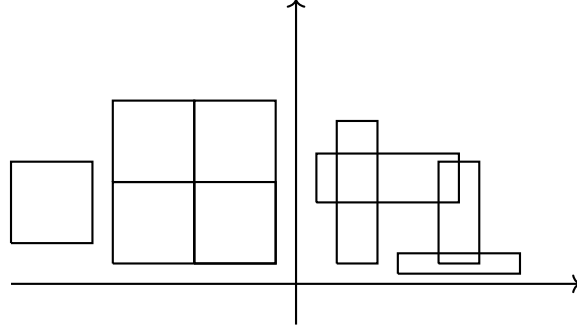


Figure 2: The intervals in Quadrant II are non-overlapping, while those in Quadrant I overlap.

As a final remark, observe that the exterior measure is a function to the nonnegative extended real line; that is, the exterior measure allows infinite values. Sets with an infinite exterior measure are considered measurable. For example, our intuition is that  $\mathbb{R}$  should have infinite length, and the reader routinely verifies that  $|\mathbb{R}|_e = \infty$ . The outer measure  $|\cdot|_e$  defined above does meet all our guiding principles as the following proposition verifies.

**Proposition 1.3.**

(i)  $\nu(\emptyset) = 0$ .

(ii) *Monotonicity:* if  $E_1 \subset E_2$ , then  $|E_1|_e \leq |E_2|_e$ .

(iii) *Countable Subadditivity:* 
$$\left| \bigcup_{k=1}^{\infty} E_k \right|_e \leq \sum_{k=1}^{\infty} |E_k|_e$$

*Proof.*

(i) This holds essentially by fiat.

(ii) This follows immediately from the fact that we are taking an infimum, and any open cover of  $E_2$  is an open cover of  $E_1$ .

(iii) Let  $E := \left| \bigcup_{k=1}^{\infty} E_k \right|_e$ . If any of the  $E_k$  have infinite exterior measure, the result is immediate. Assume then that  $|E_k|_e < \infty$  for all  $k$ . Choose  $\epsilon > 0$  and cover each  $E_k$  by intervals  $\{I_n\}$  such that  $\sum_n \nu(I_n) \leq |E_k|_e + \epsilon/2^k$ . Then  $E \subset \bigcup_{k,n} I_{k,n}$  and  $|E|_e \leq \sum_{k,n} \nu(I_{k,n}) = \sum_k \sum_n \nu(I_{k,n})$ . But then

$$|E|_e \leq \sum_k \left( |E_k|_e + \epsilon/2^k \right) = \epsilon + \sum_{k=1}^{\infty} |E_k|_e.$$

The result then follows by letting  $\epsilon$  tend to 0.  $\square$

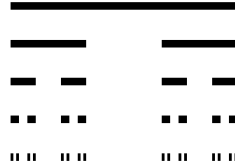
If one wants to generalize the notion of outer measures to spaces beyond  $\mathbb{R}^n$ , one can take the properties of Proposition 1.3 as the axioms for this abstract measure. Now we immediately have the following corollary:

**Corollary 1.1.** *Any subset of a set with outer measure zero has outer measure zero, and the countable union of sets with outer measure zero has outer measure zero. In particular, any countable set has outer measure zero.*

*Proof.* This follows immediately from the Countable Subadditivity of  $\nu$ .  $\square$

Now while Corollary 1.1 states that countable sets have outer measure zero, it need not be the case that uncountable sets need have positive measure.

**Example 1.2 (Cantor Set).** Begin with the closed unit interval  $C_0 := [0, 1]$ . From this interval, remove the middle third, i.e.  $(\frac{1}{3}, \frac{2}{3})$ , and label  $C_1 := [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ . Inductively construct  $C_n$  by removing the middle third of each closed subinterval of  $C_{n-1}$ . We define the Cantor Set by defining  $C := \lim_{n \rightarrow \infty} C_n$ . The first few stages of the construction of  $C$  are shown below. The fact that the Cantor set is uncountable follows from the fact that it is a nonempty compact set without isolated points.



What is  $|C|_e$ ? Note that  $C_n$  is the union of  $2^n$  intervals, each having length  $1/3^n$ , and that  $C \subset C_n$  for all  $n \in \mathbb{N}$ . But then the Cantor set is contained in a union of  $2^n$  intervals with length  $1/3^n$ , which has total length  $2^n \cdot 1/3^n = (2/3)^n$ . The fact that  $|C|_e = 0$  is then clear as  $\lim_{n \rightarrow \infty} (2/3)^n = 0$ .  $\triangleleft$

Note in general we do not have  $|\bigcup E_k|_e = \sum |E_k|_e$  even if the  $E_k$  are disjoint or even in the case of finite unions! Equality holds when the sets are, in a sense, ‘unentangled.’ By this, we mean that open coverings of one set tend to be disjoint from open coverings of the other set. If this is the case, the sets have to be covered separately, see the example on the left in Figure 3. However if the sets are ‘entangled’, then their open covers result in a great deal of ‘multiple-covering’ for the union. This excess covering allows one to more ‘efficiently’ cover the union—hence the smaller measure, see the example on the right in Figure 3.

There are many connections between Topology and Measure Theory, especially in the case of  $\mathbb{R}^n$ . Topology on  $\mathbb{R}^n$  is primarily interested in the structure of open and compact sets. This will prove useful for us since our notion of measure is defined in terms of open sets. As an example, take the following theorem.

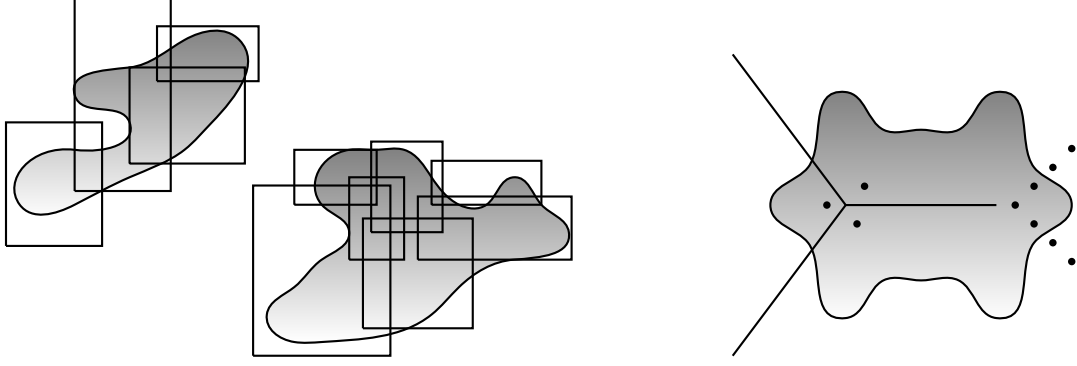


Figure 3: On the left, the two sets are ‘unentangled’ so that their opening coverings can be taken disjointly. On the right, the sets are the ‘blob’ and the line segments along with the points. These sets are ‘entangled’ and their open coverings will have ‘overlap.’

**Theorem 1.1.** *For all  $E \subset \mathbb{R}^n$  and  $\epsilon > 0$ , there exists an open set  $G$  such that  $E \subset G$  and  $|G|_e \leq |E|_e + \epsilon$ .*

*Proof.* Every interval  $I$  is contained in the interior of a slightly larger interval  $I'$ , i.e.  $I \subseteq \text{Int}(I')$ , where  $\nu(I') - \nu(I) < \epsilon$ . Choose intervals  $I_k$  such that  $E \subset \bigcup_{k=1}^{\infty} I_k$  and  $\sum_k \nu(I_k) \leq |E|_e + \epsilon/2$ . Now find  $I'_k$  such that  $I_k \subset \text{Int}(I'_k)$  and  $\nu(I'_k) < \nu(I_k) + \epsilon/2^{k+1}$ . Let  $G = \bigcup_k \text{Int}(I'_k)$ . By construction,  $G$  is an open set containing  $E$ . To complete proof, observe

$$|G|_e \leq \sum_{k=1}^{\infty} \nu(I'_k) \leq \sum_{k=1}^{\infty} \nu(I_k) + \epsilon \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \leq |E|_e + \epsilon.$$

□

**Corollary 1.2.** *For every set  $E$ , there exists a  $G_\delta$  set  $G$  such that  $E \subset G$  and  $|E|_e = |G|_e$ .*

*Proof.* By Theorem 1.1, we know that for every  $k \in \mathbb{N}$ , there is an open set  $G_k \supset E$  with  $|G_k|_e \leq |E|_e + 1/k$ . Define  $G := \bigcap_{k=1}^{\infty} G_k$ . Then  $G$  is a  $G_\delta$  set and  $G \supset E$ . Finally,

$$|E|_e \leq |G|_e \leq |G_k|_e \leq |E|_e + \frac{1}{k},$$

so that  $|E|_e = |G|_e$ .

□

The set  $G_\delta$  referred to in Corollary 1.2 is a general type of set which will be of occasional interest and use for us:

**Definition ( $G_\delta$ -Set).** A  $G_\delta$  set is a countable intersection of open sets.

Notationally, the choice of ‘ $G$ ’ is because the set is open, and the choice of ‘ $\delta$ ’ stems from the fact we are using an intersection.

Theorem 1.1 says we can always approximate any set by an open set with approximately the same size, i.e. approximately the same exterior measure. However, this does not mean that  $|G \setminus E|_e \leq \epsilon$ . We do know that  $G = E \cup (G \setminus E)$ . By subadditivity, we have  $|G|_e \leq |E|_e + |G \setminus E|_e$ . But we do not know the measure of the second set. In fact, it is possible that  $|G \setminus E|_e$  could be very large. Now as it turns out for compact sets, it is sufficient to consider only finite coverings.

**Proposition 1.4.** *If  $K \subset \mathbb{R}^n$  is compact, then  $|K|_e = \inf \{\sum_k \nu(I_k) : \{I_k\} \text{ finite cover of } K\}$ .*

*Proof.* Let  $\epsilon > 0$ . By Theorem 1.1, every interval is contained some set  $\text{Int}(I')$ , where  $\nu(I') < \nu(I) + \epsilon$ . Given a countable cover  $I_k$  of  $K$ , choose  $I'_k$  such that  $I_k \subset \text{Int}(I'_k)$  and  $\nu(I'_k) < \nu(I_k) + \epsilon/2^k$ . Then  $\{\text{Int}(I'_k)\}_k$  is an open covering for  $K$ , so there exists a finite subcovering  $\{I_{k,n}\}_{n=1}^N$ . Therefore, we have  $K \subset \bigcup_j I'_{k,j}$ . By countable subadditivity,

$$\sum_{n=1}^N \nu(I'_{k,n}) < \epsilon + \sum_{n=1}^N \nu(I_k),$$

as desired. □

## 1.2 The Lebesgue Measure

We now can define our notion of measurability with our notions of exterior measure firmly in place.

**Definition (Measurable).** A set  $E \subset \mathbb{R}^n$  is (Lebesgue) measurable if for all  $\epsilon > 0$ , there exists an open set  $G$  such that  $G \supset E$  and  $|G \setminus E|_e < \epsilon$ .

Essentially, a set is measurable if it can be well approximated by open sets. We choose the above notion of ‘closeness’ in order to obtain additivity of measures. Notice we also have mentioned the underlying topology via the use of ‘open.’ We are able to avoid invoking the underlying topology using greater abstraction, which shall come later. Note that we always have an open set such that  $|G|_e < |E|_e + \epsilon$  (c.f. Theorem 1.1), but this alone is weaker than the above definition; that is, if  $E$  is measurable then it satisfies the properties in Theorem 1.1. As a matter of notation, if  $E$  is measurable, we define  $|E| := |E|_e$ . The following propositions follow immediately from our definition.

**Proposition 1.5.** *Every open set is measurable.*

*Proof.* If  $E$  is an open set, choose  $G = E$ . □

**Proposition 1.6.** *If  $|E|_e = 0$ , then  $E$  is measurable.*

*Proof.* Choose  $G$  such that  $|G|_e < |E|_e + \epsilon = \epsilon$ . But then  $|G \setminus E|_e \leq |G|_e < \epsilon$ .  $\square$

**Proposition 1.7.** *A countable union of measurable sets is measurable, and*

$$|E| \leq \sum_k |E_k|.$$

*Proof.* Let  $\epsilon > 0$ . For each  $k$ , choose an open set  $G_k$  such that  $E_k \subset G_k$  and  $|G_k \setminus E_k|_e < \epsilon/2^k$ . Now  $G := \bigcup_k G_k$  is open, and  $E \subset G$ . Moreover since  $G \setminus E \subset \bigcup_k (G_k \setminus E_k)$ , we have

$$|G \setminus E|_e \leq \left| \bigcup_k (G_k \setminus E_k) \right|_e \leq \sum_k |G_k \setminus E_k|_e < \epsilon.$$

Therefore,  $\bigcup_k E_k$  is measurable. The fact that  $|\bigcup_k E_k| \leq \sum_k |E_k|$  follows from Proposition 1.3.  $\square$

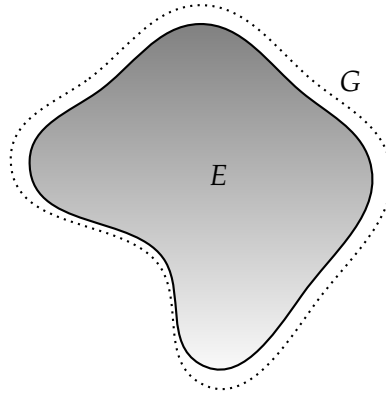


Figure 4: Open sets are not ‘fuzzy’ since they have a good notion of boundary. Similarly since  $G$  is not fuzzy and approximates  $E$ , it must be that  $E$  is not too close to being ‘fuzzy.’

**Proposition 1.8.** *All intervals are measurable.*

*Proof (Sketch).* We prove this only in the two dimensional case to avoid unnecessary complications. Given an interval  $I = [a, b]$ , choose  $I' = [a', b']$  such that  $I \subset \text{Int } I'$  and  $\nu(I') < \nu(I) + \epsilon$ . We need to show that  $|I' \setminus I|_e < \epsilon$ . Now  $I' \setminus I = [a', a] \cup (b, b']$ , and  $|I' \setminus I|_e \leq a - a' + b' - b = (b' - a') - (b - a) < \epsilon$ , as desired. Alternatively, one can use the fact that  $I$  is the union of its interior and its boundary, and that the boundary has

measure zero. This alternative proof is left to the reader.  $\square$

As we have seen, open sets are easily seen to be measurable. But the case of closed sets is more complicated. For example in  $\mathbb{R}$ , every open set is the countable union of intervals of the form  $(a_k, b_k)$ . These intervals can even be taken to be disjoint. However, the same is not true for closed sets—take the Cantor set for example, c.f. Example 1.2. We do have that in  $\mathbb{R}^n$ , every open set is a countable union of non-overlapping intervals. To prove this we shall make use of dyadic cubes.

A dyadic cube of generation zero,  $\mathcal{D}_0$ , are cubes with unit side lengths and integer vertices, i.e.  $\mathcal{D}_0 := \{[0, 1]^n + \tau : \text{fixed } \tau \in \mathbb{Z}^n\}$ . A generation one dyadic cube is  $\mathcal{D}_1 := \{\frac{1}{2}Q : Q \in \mathcal{D}_0\}$ . Generally,  $\mathcal{D}_n := \{\frac{1}{2^n}Q : Q \in \mathcal{D}_0\} = \{\frac{1}{2^n}Q : Q \in \mathcal{D}_0\}$ . Given  $\mathcal{D}_n$ , we say that  $\mathcal{D}_{n-1}$  is a parent of  $\mathcal{D}_n$ , and  $\mathcal{D}_i$ , where  $i < n$ , is an ancestor of  $\mathcal{D}_n$ . We say also that  $\mathcal{D}_{n+1}$  is a child of  $\mathcal{D}_n$ , and  $\mathcal{D}_j$ , where  $j > n$ , is a descendant of  $\mathcal{D}_n$ . One can allow  $n$  to be negative to create larger dyadic cubes. Define  $\mathcal{D} := \bigcup_{k=0}^{\infty} \mathcal{D}_k$ . If  $Q_1, Q_2$  are dyadic, then either  $Q_1 \subset Q_2$ ,  $Q_2 \subset Q_1$ , or they do not overlap. We now are in a position to prove the following lemma.

**Lemma 1.1.** *Every open set in  $\mathbb{R}^n$  is a countable union of non-overlapping intervals.*

*Proof.* Given an open set  $G$ , let  $\{I_k\}$  be all dyadic cubes that are contained in  $G$  and for which their parent is not contained in  $G$ . By the selection of the  $I$ 's, it follows that they are pairwise disjoint for if  $I_k$  and  $I_j$  overlap, then one contains the other, contradicting the selection process. Clearly, we have selected only countably many intervals. Now if  $x \in G$ , there exists  $r > 0$  such that there is an  $r$ -neighborhood of  $x$  contained in  $G$ . For sufficiently large  $n$ , all the cubes in  $\mathcal{D}_n$  have diameter less than  $r$ . But then there exists  $Q \in \mathcal{D}_n$  such that  $x \in Q$ . Note that  $Q \subset G$ . Now either  $\mathcal{D}_n$  is contained in  $G$  or it has an ancestor that is contained in  $G$ .  $\square$

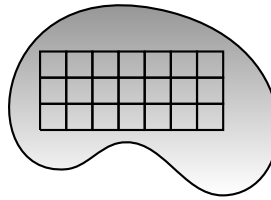


Figure 5: Tiling intervals inside an open set.

We can now make precise a discussion from earlier—if two sets are ‘unentangled’ then the measure of the union is the sum of the measures.

**Lemma 1.2.** *If  $A, B \subset \mathbb{R}^n$  and  $\text{dist}(A, B) > 0$ , then  $|A \cup B|_e = |A|_e + |B|_e$ .*

*Proof.* We know by subadditivity that  $|A \cup B|_e \leq |A|_e + |B|_e$ . It remains to show that  $|A|_e + |B|_e \leq |A \cup B|_e$ . Let  $\epsilon > 0$ , and choose intervals  $\{I_k\}$  such that  $A \cup B \subset \bigcup_k I_k$  and  $\sum_k |I_k| \leq |A \cup B|_e + \epsilon$ . Possibly partitioning each  $I_k$  into a finite number of subintervals, we may assume that  $\text{diam}(I_k) < \text{dist}(A, B)$ . ‘Sort’ the set  $\{I_k\}$  into two sets  $\{I'_k\}$  and  $\{I''_k\}$  which cover  $A$  and  $B$ , respectively. Then

$$|A|_e + |B|_e \leq \sum_k |I'_k| + \sum_k |I''_k| = \sum_k |I_k| \leq |A \cup B|_e + \epsilon.$$

Therefore,  $|A|_e + |B|_e \leq |A \cup B|_e$ , as desired.  $\square$

**Theorem 1.2.** *Every closed set  $A \subset \mathbb{R}^n$  is measurable.*

*Proof.* Given  $\epsilon > 0$ , we can choose  $G \supset A$  such that  $|G|_e < |A|_e + \epsilon$ . Now  $G \setminus A$  is open. By Lemma 1.1, we can write  $G \setminus A = \bigcup_{k=1}^{\infty} I_k$ , where  $\{I_k\}$  is a collection of non-overlapping open intervals. We want to show that  $\sum \nu(I_k) < \epsilon$ , which will imply that  $|G \setminus A|_e < \epsilon$ . It suffices to show that  $\sum_{k=1}^N \nu(I_k) < \epsilon$  for all  $N$ . Define  $K = \bigcup_{k=1}^N I_k$ . Since  $K$  is the finite union of compact sets in  $\mathbb{R}^n$ ,  $K$  is compact. Furthermore,  $K$  is disjoint from  $A$ . As the set  $A$  is closed and  $K$  is compact, it must be that  $\text{dist}(K, A) > 0$ . It then follows from Lemma 1.2 that

$$|K \cup A|_e = |K|_e + |A|_e = \sum \nu(I_k) + |A|_e \leq |G|_e < |A|_e + \epsilon.$$

$\square$

As it turns out, the measurability of a set is equivalent to the measurability of its complement. This is useful in circumstances where one set is easier to handle than the other.

**Theorem 1.3.**  *$E$  is measurable if and only if  $E^C$  is measurable.*

*Proof.* Suppose that  $E$  is measurable. For  $k \in \mathbb{N}$ , choose an open set  $G_k$  such that  $E \subset G_k$  and  $|G_k \setminus E|_e < 1/k$ . Now as  $G_k$  is open,  $G_k^C$  is closed, and hence measurable by Theorem 1.2. Let  $G := \bigcup_k G_k^C$ . Being the countable union of measurable sets,  $G$  is measurable, and  $G \subset E^C$ . Write  $E^C = G \cup Z$ , where  $Z = E^C \setminus G$ . Then  $Z \subset E^C \setminus G_k^C = G_k \setminus E$ , and therefore,  $|Z|_e < 1/k$  for all  $k$ . Hence,  $|Z|_e = 0$ , and so  $Z$  is measurable. But then  $E^C$  is the union of measurable sets, and is thus measurable. Finally, taking  $E$  to be  $E^C$  and rerunning the argument mutatis mutandis proves that if  $E^C$  is measurable, then  $E$  is measurable.  $\square$

**Corollary 1.3.** *If  $E_1, E_2$  are measurable, then  $E_1 \setminus E_2$  is measurable.*

*Proof.* We know that  $E_1 \setminus E_2 = E_1 \cap E_2^C$ . But  $E_2^C$  is measurable by Theorem 1.3. Therefore,  $E_1 \setminus E_2$  is measurable, being the intersection of measurable sets.  $\square$

Our definition is measurable is essentially that a set  $E$  is measurable if it can be ‘approximated well’ by open sets containing it. This is an exterior notion of measure. We can also define measurability instead by saying a set  $E$  is measurable if it can be ‘approximated well’ internally by closed sets, as the following proposition shows.

**Proposition 1.9.** *A set  $E \subset \mathbb{R}^n$  is measurable if and only if given  $\epsilon > 0$ , there exists a closed set  $F \subset E$  such that  $|E \setminus F|_e < \epsilon$ .*

*Proof.*  $E$  is measurable if and only if  $E^C$  is measurable, i.e. if and only if given  $\epsilon > 0$ , there exists an open set  $G$  with  $E^C \subset G$  and  $|G \setminus E^C|_e < \epsilon$ . But such an open set  $G$  exists, noting that  $G \setminus E^C = E \setminus F$ , if and only if  $G^C$  is closed,  $F \subset E$ , and  $|E \setminus F|_e < \epsilon$ .  $\square$

The previous theorems and propositions have shown that the complements of measurable sets are measurable, and countable unions and intersections of measurable sets are measurable. Since we shall make use of them in the future, note that if  $\{E_k\}$  is a collection of measurable sets, then so are  $\limsup$  and  $\liminf$ :

$$\limsup E_k = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k \quad \text{and} \quad \liminf E_k = \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} E_k,$$

though we shall not prove this. Now all of these previous theorems and propositions essentially follow because we are working in the context of a  $\sigma$ -algebra.

**Definition ( $\sigma$ -algebra).** A nonempty collection of sets  $\Sigma$  is called a  $\sigma$ -algebra if it satisfies

- (i)  $E^C \in \Sigma$  whenever  $E \in \Sigma$ .
- (ii)  $\bigcup_k E_k \in \Sigma$  whenever  $E_k \in \Sigma$  for all  $k$ .

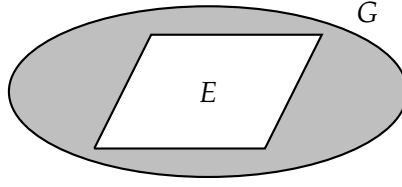
Note that any collection of sets closed under countable unions is closed under countable intersections as  $[\bigcap_k U_k]^C = \bigcup_k U_k^C$ . The empty set and entire space are necessarily measurable sets. Generally, they are elements of any  $\sigma$ -algebra. We shall be particularly interested in two specific  $\sigma$ -algebras:

**Definition (Borel  $\sigma$ -algebra).** The Borel  $\sigma$ -algebra is the smallest  $\sigma$ -algebra containing all open subsets of  $\mathbb{R}^n$ .

**Definition (Lebesgue  $\sigma$ -algebra).** The Lebesgue  $\sigma$ -algebra is the  $\sigma$ -algebra containing all measurable sets.

Note that the Borel  $\sigma$ -algebra is *strictly* contained in the Lebesgue  $\sigma$ -algebra—though this is not simple to see.





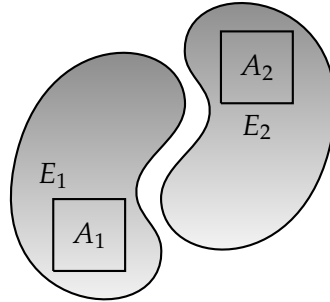
### 1.3 Properties of Measurable Sets

Our goal in this section is to examine measurable sets more closely. We begin by recalling that  $|E| = |E|_e$  whenever  $E$  is measurable, and  $|\bigcup_k E_k|_e \leq \sum_k |E_k|_e$ , i.e. countable subadditivity.

**Theorem 1.4** (Countable Subadditivity). *If  $\{E_k\}$  is a collection of disjoint, measurable sets, then*

$$\left| \bigcup_{n=1}^{\infty} E_k \right| = \sum_{k=1}^{\infty} |E_k|.$$

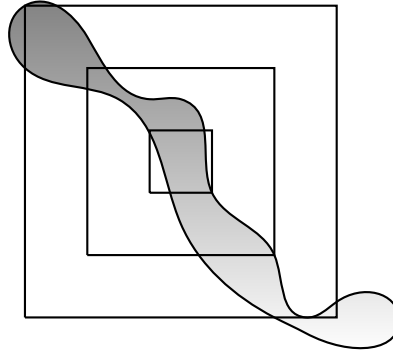
*Proof.* It is immediate that  $|\bigcup_{n=1}^{\infty} E_k| \leq \sum_{k=1}^{\infty} |E_k|$ , so that we need show  $\sum_{k=1}^{\infty} |E_k| \leq |\bigcup_{n=1}^{\infty} E_k|$ . Let  $E := \bigcup_k E_k$ . It suffices to show  $\sum_{k=1}^N |E_k| \leq |E|$  for all  $N$ . If  $\text{dist}(A, B) > 0$ , we know that  $|A \cup B|_e = |A|_e + |B|_e$ . In particular, this holds if  $A, B$  are disjoint compact sets. First, assume that each  $E_k$  is bounded. For each  $k \in \mathbb{N}$ , choose a closed subset  $A_k \subset E_k$



such that  $|E_k \setminus A_k|_e < \epsilon/2^k$ . Since  $A_k \subset E_k$  and  $E_k$  is bounded,  $A_k$  is a closed, bounded set. Therefore,  $A_k$  is compact. We also have

$$\left| \bigcup_{k=1}^N A_k \right| = \sum_{k=1}^N |A_k| \geq \sum_{k=1}^N |E_k| - \epsilon.$$

But as  $\bigcup_{k=1}^N A_k \subset E$ , we have  $|E| \geq \sum_{k=1}^N |E_k| - \epsilon$ . Letting  $\epsilon \rightarrow 0$  yields the desired inequality. This completes the proof in the case where each  $E_k$  is bounded.



Now suppose that some  $E_k$  may be unbounded. Let  $I_j := [-j, j]^n$  for  $j > 0$  and  $Q_0 := \emptyset$ . Define further  $Q_j := I_j \setminus I_{j-1}$  for  $j > 0$ . Consider the sets  $E_{k,j} := E_k \cap Q_j$  for  $j, k = 1, 2, \dots$ . Each of these sets are disjoint, bounded, and measurable. It is also routine to verify that

$$\begin{aligned} E_k &= \bigcup_j E_{k,j} = \bigcup_j (E_k \cap Q_j) \\ E &= \bigcup_k E_k = \bigcup_{k,j} E_{k,j} \end{aligned}$$

But then by the case above, we have

$$|E| = \left| \bigcup_k E_k \right| = \left| \bigcup_{k,j} E_{k,j} \right| = \sum_{k,j} |E_{k,j}| = \sum_k \sum_j |E_{k,j}| = \sum_k |E_k|.$$

□

**Corollary 1.4.** *If  $\{I_k\}$  is a sequence of nonoverlapping intervals, then  $|\bigcup_k I_k| = \sum_k |I_k|$ .*

*Proof.* It is immediate that  $|\bigcup_k I_k| \leq \sum |I_k|$ . We need prove the opposite inequality. As the  $I_k$  are disjoint, we know  $|\bigcup_k I_k| \geq |\bigcup_k \text{Int } I_k| = \sum_k |\text{Int } I_k| = \sum |I_k|$ . But then  $|\bigcup_k I_k| = \sum_k |I_k|$ . □

**Corollary 1.5.** *Suppose  $E_1$  and  $E_2$  are measurable,  $E_2 \subset E_1$ , and  $|E_2| < \infty$ . Then  $|E_1 \setminus E_2| = |E_1| - |E_2|$ .*

*Proof.* We know  $E_1 = E_2 \cup (E_1 \setminus E_2)$  and  $|E_1| = |E_2| + |E_1 \setminus E_2|$ . The result then follows as  $|E_2| < \infty$ . □

We end this section with an interaction between the Lebesgue measure and monotone sequences of sets.

**Proposition 1.10.**

- (i) If  $E_1 \subset E_2 \subset \dots$  are measurable, then  $|\bigcup_k E_k| = \lim_{k \rightarrow \infty} |E_k|$ ; that is, if  $E_k \nearrow E$ , then  $\lim_{k \rightarrow \infty} |E_k| = |E|$ .
- (ii) If  $E_1 \supset E_2 \supset \dots$  are measurable,  $|E_k| < \infty$  for some  $k$ , then  $|\bigcap_{k=1}^{\infty} E_k| = \lim_{k \rightarrow \infty} |E_k|$ ; that is, if  $E_k \searrow E$  and  $|E_k| < \infty$  for some  $k$ , then  $\lim_{k \rightarrow \infty} |E_k| = |E|$ .

*Proof.*

- (i) Let  $E = \bigcup E_k$ . If  $|E_k| = \infty$  for some  $k$ , then  $\lim_{k \rightarrow \infty} |E_k|$  and  $|E|$  are infinite, and the result follows. Assume then that  $|E_k| < \infty$  for all  $k$ . We want to create a disjoint union. We shall express  $E$  as  $E = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_2) \cup \dots \cup (E_k \setminus E_{k-1}) \cup \dots$ . Define  $F_k := E_k \setminus E_{k-1}$ . Observe the sets  $F_k$  are measurable and pairwise disjoint. Then by Theorem 1.4,

$$|E| = \sum_{k=1}^{\infty} |F_k| = \lim_{n \rightarrow \infty} \sum_{k=1}^n |F_k| = \lim_{n \rightarrow \infty} \left| \bigcup_{k=1}^n F_k \right| = \lim_{n \rightarrow \infty} |E_n|.$$

- (ii) Let  $E = \bigcap E_k$ . Since considering tails of a sequence are sufficient, we may assume without loss of generality that  $|E_1| < \infty$ . We again want to create disjoint, measurable sets. Define  $F_k := E_k \setminus E_{k+1}$ . Observe that the sets  $F_k$  are measurable, pairwise disjoint, and that  $E_1 = E \cup \bigcup F_k$ . Then by Theorem 1.4,

$$|E_1| = |E| + \left| \bigcup F_k \right| = |E| + \sum_{k=1}^{\infty} |F_k| = |E| + \sum_{k=1}^{\infty} (|E_k| - |E_{k+1}|) = |E| + |E_1| - \lim_{k \rightarrow \infty} |E_k|.$$

But as  $|E_1| < \infty$ , we must then have  $|E| = \lim_{k \rightarrow \infty} |E_k|$ .

□

Note that the requirement that  $|E_k| < \infty$  for some  $k$  in the second part of Proposition 1.10 is necessary. For example, take  $E_k = \mathbb{R}^2 \setminus C_k$ , where  $C_k$  is the circle centered at the origin with radius  $k$ . Then  $|E_k| = \infty$  for all  $k$ , and  $E_k \searrow \emptyset$ . However,  $\lim_{k \rightarrow \infty} |E_k| = \infty$ , while  $|\emptyset| = 0$ .

## 1.4 Characterization of Measurable Sets

By definition,  $A$  is measurable if and only if for all  $\epsilon > 0$ , there exists an open set  $G \supset A$  such that  $|G \setminus A|_e < \epsilon$ . But this makes use of the underlying topology. Can we avoid this?

After all, this ‘open set-ness’ is about continuity. Continuity is not even a requirement for integration, which is an end goal of ours. So we would like a description of measurability which avoids Topology entirely. Ironically, we shall first need more Topology.

Recall that  $G_\delta$  is a set which is the countable intersection of open sets,  $F_\sigma$  is the countable union of closed sets,  $G_{\delta,\sigma}$  are the countable intersection of closed sets.<sup>1</sup> One can even continue this to absurdity, i.e.  $G_{\delta\sigma\delta\sigma}$ . Furthermore, these are all subclasses of the Borel  $\sigma$ -algebra. One may wonder if all Borel created by these procedures, i.e. formed by countable union or intersections of open or closed sets? This turns out to not be the case, though it is difficult to construct a set in this fashion which is not a Borel set. However, it is the case that measurable sets are ‘close’ to Borel sets, in the sense they differ by a set of measure zero, as the next results shows.

**Theorem 1.5.**  *$E$  is measurable if and only if  $E = H \setminus Z$  if and only if  $E = F \cup Z$ , where  $H$  is of type  $G_\delta$ ,  $F$  is of type  $F_\sigma$ , and  $|Z| = 0$ .*

*Proof.* It is sufficient to prove that each of the statements is equivalent to the measurability of  $E$ . If  $E$  can be represented as  $H \setminus Z$  or  $F \cup Z$ , then  $E$  is measurable, as  $H$ ,  $F$ , and  $Z$  are measurable. Now assume that  $E$  is measurable. For each  $k$ , choose an open set  $G_k$  such that  $E \subset G_k$  and  $|G_k \setminus E| < 1/k$ . Define  $H := \cap G_k$ . Clearly,  $H$  is of type  $G_\delta$ ,  $E \subset H$ , and  $H \setminus E \subset G_k \setminus E$  for each  $k$ . But then defining  $Z = H \setminus E$ , we see that  $|Z| = 0$ , as desired. Furthermore, if  $E$  is measurable, then so too is  $E^C$ . Now using the previous work, write  $E^C = (\cap G_k) \setminus Z$ , where the  $G_k$  are open sets and  $|Z| = 0$ . But then taking complements yields  $E = (\cup G_k^C) \cup Z$ . But as the sets  $G_k$  are open, the sets  $G_k^C$  are closed, implying  $F := \cup G_k^C$  is of type  $F_\sigma$ .  $\square$

**Remark.** This is different (in fact stronger) than  $E \subset H$ , where  $H$  is a  $G_\delta$  set with the same outer measure, i.e.  $|H|_e = |E|_e$ .

Theorem 1.5 essentially states that measurable sets are the Borel sets ‘plus or minus’ the measure zero sets. Note that we are still making Borel sets, which invoke Topology, so we still have not met our goal. However, we are now set up to prove Carathéodory’s Theorem, which we can use to meet our goal.

**Theorem 1.6 (Carathéodory).**  *$E$  is measurable if and only if for all  $A \subset \mathbb{R}^n$ ,  $|A \cap E|_e + |A \setminus E|_e = |A|_e$ .*

*Proof.* Suppose that  $E$  is measurable. Choose a  $G_\delta$  set  $H$  such that  $A \subset H$  and  $|H|_e = |A|_e$ . Since  $H = (H \cap E) \cup (H \setminus E)$ , and these are measurable sets, we have

$$|A|_e = |H| = |H \cap E| + |H \setminus E|.$$

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<sup>1</sup>Generally,  $G$  denotes open,  $F$  denotes closed,  $\delta$  is for countable intersections, and  $\sigma$  is for countable unions.

But then  $|A|_e = |H| = |H \cap E| + |H \setminus E| \geq |A \cap E|_e + |A \setminus E|_e$ . By subadditivity, we obtain  $|A|_e \leq |A \cap E|_e + |A \setminus E|_e$ . Therefore,  $|A|_e = |A \cap E|_e + |A \setminus E|_e$ .

Now suppose that  $E$  satisfies the condition that  $|A \cap E|_e + |A \setminus E|_e = |A|_e$  for all  $A \subseteq \mathbb{R}^n$ . Let  $A$  be a  $G_\delta$  set such that  $E \subset A$  and  $|E|_e = |A|$ . Then on the left side of the equation, we have  $|E|_e + |A \setminus E|_e = |A| = |E|_e$ . Hence,  $|A \setminus E|_e = 0$ , so  $E$  is  $G_\delta$  minus a measure zero set. Notice that we have subtracted  $|E|_e$ ; hence, we need consider the case where  $|E|_e = \infty$ .

Assume that  $|E|_e = \infty$ . We know that  $E = \bigcup_{j=1}^{\infty} E_j$ , where  $E_j := E \cap [-j, j]^n$ . So it suffices to prove that  $E_j$  has the property  $|A \cap E_j|_e + |A \setminus E_j|_e = |A|_e$  for all  $A \subseteq \mathbb{R}^n$  and  $j \in \mathbb{N}$ . [Since each  $E_j$  is measurable and the countable union of the  $E_j$  is measurable,  $E$  would then be measurable.] Certainly by subadditivity,  $|A|_e \leq |A \cap E_j|_e + |A \cap E_j^C|_e$ . It remains to show  $|A|_e \geq |A \cap E_j|_e + |A \cap E_j^C|_e$ . By assumption for all  $A \subset \mathbb{R}^n$ ,  $|A \cap E|_e + |A \setminus E|_e = |A|_e$ . We know also that  $Q_j := [-j, j]^n$  and  $Q_j^C$  satisfy this property, so  $|A \cap E|_e = |(A \cap E) \cap Q_j|_e + |(A \cap E) \cap Q_j^C|_e$ . Similarly, we have  $|A \cap E^C|_e = |(A \cap E^C) \cap Q_j|_e + |(A \cap E^C) \cap Q_j^C|_e$ . Combining these gives

$$\begin{aligned} |A|_e &= |A \cap (E \cap Q_j)|_e + |A \cap (E \cap Q_j^C)|_e + |A \cap (E^C \cap Q_j)|_e + |A \cap (E^C \cap Q_j^C)|_e \\ &= |A \cap E_j|_e + |A \cap E \cap Q_j^C|_e + |A \cap E^C \cap Q_j|_e + |A \cap E_j^C|_e \\ &\geq |A \cap E_j|_e + |A \cap E_j^C|_e, \end{aligned}$$

as desired.  $\square$

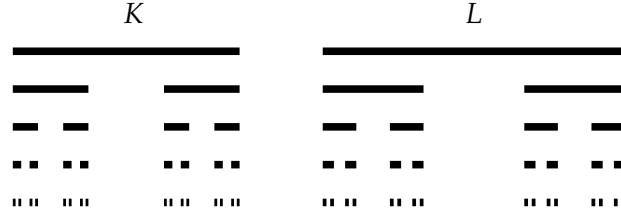
Note that in Theorem 1.6, there is no mention of openness or closed-ness. We can then take this as our definition of measurability, i.e. define measurable sets to be the sets which split every set (independent of measurability) into pieces which are additive with respect to the outer measure. This is the one of the standard approaches when defining measures abstractly. Thus, we have completed our goal of finding a way to avoid invoking the underlying topology.

## 1.5 Continuous & Lipschitz Transformations

Here we shall investigate the interaction between continuous functions and measurability. We begin with an important question: if  $E \subset \mathbb{R}^n$  is measurable and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuous, is  $f(E)$  always measurable? The general answer is no, but there are special cases where this does hold. For example, measures of sets are invariant under rotations of  $\mathbb{R}^n$ —though we will not prove this, see [WZ77, p. 36]. This also shows that the exterior measure (hence also the Lebesgue measure) is invariant under the choice of an orthogonal basis for  $\mathbb{R}^n$ . However, what other functions or transformations preserve measures? We first show that continuity alone will never be enough.

**Lemma 1.3.** *There is a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and a set  $E$  with  $|E| = 0$ , such that  $|f(E)| > 0$ .*

*Proof (Sketch).* We construct a ‘fat Cantor’ set by following the same procedure as in the Cantor set, but instead removing the middle fourths of each interval. At each stage,  $2^j$  intervals of length  $4^{-j}$  are removed. Denote the ordinary Cantor set by  $K$  and the fat Cantor set by  $L$ . We construct a continuous function  $f : [0, 1] \rightarrow [0, 1]$  such that  $f(K) = L$ .



Map  $K_j$ , the  $j$ th step in the construction of  $K$ , onto  $L_j$  in a piecewise linear fashion, i.e. stretch each interval in  $K_j$  to the corresponding interval in  $L_j$ . Clearly,  $f_j$  is continuous and  $\sup |f_j - f_{j+1}| \leq 2^{-j}$ . But then  $f_j \rightarrow f$  uniformly so that  $f$  is continuous and  $f(K) = L$ . We can extend  $f$  to  $\mathbb{R}$  by translation, i.e. define  $f(x)$  for  $x \notin [0, 1]$  via  $f(x) = f(a)$ , where  $a = x + n$  and  $n \in \mathbb{Z}$  is an integer such that  $x + n \in [0, 1]$ . Finally, recall that the ordinary Cantor set has measure 0, as was shown in Example 1.2, and it is routine to verify that  $|L| > 0$ .  $\square$

Before answer the question poised at the beginning of this section, we need two quick results, assuming the existence of a nonmeasurable set, which we shall prove in the next section.

**Lemma 1.4.** *Let  $E$  be a measurable subset of  $\mathbb{R}^1$  with  $|E| > 0$ . Then the set of differences  $\{d : d = x - y, x \in E, y \in E\}$  contains an interval centered at the origin.*

*Proof.* See [WZ77, p. 46].  $\square$

**Lemma 1.5.** *Any set in  $\mathbb{R}^1$  with positive outer measure contains a nonmeasurable set.*

*Proof.* Suppose  $A$  is a set with  $|A|_e > 0$ . Let  $E$  be the nonmeasurable Vitali set. For  $q \in \mathbb{Q}$ , let  $E_q$  denote the translation of  $E$  by  $q$ , i.e.  $E_q = \{e + q : e \in E\}$ . The sets  $E_q$  are disjoint, and  $\cup E_q = \mathbb{R}$ . But then it must be that  $A = \cup (A \cap E_q)$ . By Lemma 1.4, if  $A \cap E_q$  is measurable then  $|A \cap E_q| = 0$ . But as  $|A|_e > 0$ , there must be some  $q \in \mathbb{Q}$  such that  $A \cap E_q$  is not measurable.  $\square$

We now sketch the idea why it need not be the case that  $f(E)$  is measurable even when  $E$  is measurable. Choose  $f$  and  $E$  as in Lemma 1.3. By Lemma 1.5, there exists a

non-measurable subset of  $f(E)$ , say  $W$ . Let  $V := f^{-1}(W) \cap E$ . Observe that  $f(V) = W$ , and that  $V$  is measurable (being a subset of measure zero). Therefore, the image of a measurable set under a continuous function need not be measurable, as claimed. The next obvious question is, if continuity is not sufficient to force the image of measurable sets to be measurable, is there a stronger condition that will? We recall a definition.

**Definition (Lipschitz).** We say that  $f$  is Lipschitz if there exists  $L$  such that  $|f(x) - f(y)| \leq L|x - y|$  for all  $x, y$  in the domain of  $f$ .

Lipschitz functions are the special case of uniformly continuous functions where we can always choose  $\delta = \epsilon/L$ . Therefore, if  $f$  is Lipschitz then  $f$  is uniformly continuous, and hence also continuous. However, not all uniformly continuous functions are Lipschitz, and certainly not all continuous functions are Lipschitz. The Lipschitz condition can be thought of as the boundedness of secant lines for a given function. Hence functions which are differentiable with bounded derivative on an interval are Lipschitz.

**Example 1.3.** The function  $f(x) = x^{1/3}$  is Lipschitz on any interval not containing the origin as a limit point since  $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$  is bounded on such intervals. Indeed, if  $I$  is an interval containing the origin and  $x \in I$ , then

$$\frac{f(x) - f(0)}{x - 0} = \frac{\sqrt[3]{x}}{x} = \frac{1}{\sqrt[3]{x^2}}.$$

Taking  $x$  'close' to 0 shows that the slopes of secant lines to  $f(x)$  are unbounded. One can use the same process to verify this for any set containing 0 as a limit point.  $\triangleleft$

**Example 1.4.** The function  $f : [0, 1] \rightarrow [0, 1]$  given by  $f(x) = \sqrt{x}$  is uniformly continuous on  $[0, 1]$ , being a continuous function on a compact interval. However,  $f$  is not Lipschitz continuous. If  $f$  were Lipschitz with constant  $L$ , then we would have  $|\sqrt{x} - \sqrt{y}| \leq L|x - y|$  for all  $x, y \in [0, 1]$ . Choose  $x = 0$  and  $y = \frac{1}{4k^2}$ . We have  $x, y \in [0, 1]$  but

$$|\sqrt{x} - \sqrt{y}| = \frac{1}{2K} > \frac{1}{4K} = K|x - y|.$$

$\triangleleft$

Unlike general continuous functions or the function in Lemma 1.3, Lipschitz maps preserve sets of measure zero.

**Theorem 1.7.** If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is Lipschitz and  $|E| = 0$ , then  $|f(E)| = 0$ .

*Proof.* Recall that  $|E|_\epsilon = \inf\{\sum \nu(I_k) : \bigcup I_k \supset E, \text{ and } I_k \text{ cubes}\}$ . Since  $|E| = 0$ , for every  $\epsilon > 0$ , there exists a countable cover of  $E$  by intervals  $\{I_k\}$  such that  $\sum |I_k| < \epsilon$ . Choose

this covering to consist of dyadic cubes. By Lipschitz continuity,  $f(I_k)$  is contained in an interval of width at most  $Lw_k$ , where  $L$  is the Lipschitz constant of  $f$  and  $w_k$  is the maximum width of the cube. Then  $|f(I_k)| \leq L^d |I_k|$ . Therefore, we have

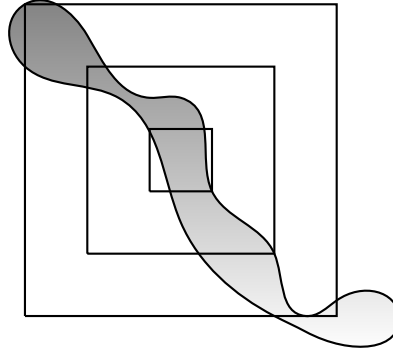
$$|f(E)| \leq L^d \sum |I_k| < L^d \epsilon.$$

The result then follows.  $\square$

Furthermore, Lipschitz functions preserve measurability, though not necessarily the value of the measure.

**Theorem 1.8.** *If  $E$  is measurable and  $f$  is Lipschitz, then  $f(E)$  is measurable.*

*Proof.* Since  $E$  is measurable, write  $E = H \cup Z$ , where  $H$  is  $F_\sigma$  and  $|Z| = 0$ . Now  $F_\sigma$  sets are the countable union of closed sets, which is equivalent to the countable union of compact sets since any union of closed sets can be written as a countable union of compact sets, i.e.  $F = \cup (F \cap [-j, j]^n)$ . Now  $f$  is Lipschitz, hence continuous, so that  $f$  sends compact sets to compact sets. Then it must be that  $f$  preserves ' $F_\sigma$ -ness.' We know by Theorem 1.7, if  $|E| = 0$  then  $f(E) = 0$ . But then  $f(E) = f(H) \cup f(Z)$ , a union of  $F_\sigma$  set and a set of measure zero, hence measurable.  $\square$



The proof of Theorem 1.8 involves a common trick: although a whole function may not be Lipschitz, 'pieces' of it may be, and these could be brought together to finish a result. This invoked the same underlying idea as in the proof of Theorem 1.4. Though Theorem 1.8 showed that Lipschitz functions preserve measurability, they do not generally preserve the measure. So while we cannot know that the measure of  $f(E)$  is, we know that  $f(E)$  is indeed measurable. We finish by commenting that in the special case of Lipschitz functions which are linear maps of  $\mathbb{R}^n$ , we can actually compute the measure of the resulting image.

**Theorem 1.9.** *Let  $T$  be a linear transformation of  $\mathbb{R}^n$ , and let  $E$  be a measurable set. Then  $|T(E)| = \delta |E|$ , where  $\delta = |\det T|$ .*



*Proof.* Let  $\delta = |\det T|$ . It is routine to verify that  $|T(I)| = \delta|I|$  for any interval  $I$ . Now let  $E \subseteq \mathbb{R}^n$  be a measurable subset, and  $\epsilon > 0$ . Choose intervals  $\{I_k\}$  covering  $E$  with  $\sum |I_k| < |E|_e + \epsilon$ . Then

$$|T(E)|_e \leq \sum |T(I_k)| = \delta \sum |I_k| < \delta(|E|_e + \epsilon).$$

But then  $|T(E)|_e \leq \delta |E|_e$ .

It remains to show that  $\delta |E|_e \leq |T(E)|_e$ . Choose an open set  $G \supset E$  with  $|G \setminus E| < \epsilon$ . Write  $G$  as a union of nonoverlapping intervals  $\{I'_k\}$ . By the work above, we may assume that  $\delta > 0$ . Since  $T(I_k)$  are also nonoverlapping intervals,

$$|T(G)| = \sum |T(I_k)| = \delta \sum |I_k| = \delta |G|.$$

But as  $E \subset G$ , we have  $\delta |E| \leq \delta |G| = |T(G)|$ . Moreover,

$$|T(G)| \leq |T(E)| + |T(G \setminus E)| \leq |T(E)| + \delta \epsilon.$$

Combining inequalities yields  $\delta |E| \leq |T(E)| + \delta \epsilon$ , so that  $\delta |E| \leq |T(E)|$ .  $\square$

## 1.6 A Nonmeasurable Set

Notice that ‘all’ the sets we have encountered thus far were measurable. Indeed, most sets you ‘meet on the street’ will be measurable. Nonmeasurable sets are true pathologies, which do not even exist depending on your logical axioms.

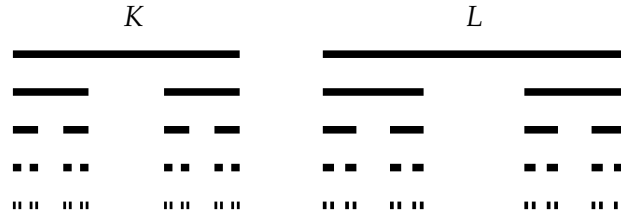
Recall the example from Lemma 1.3. We constructed a fat Cantor set,  $L$ . Note that removing fourths was not the only possible choice in the construction of  $L$ . Suppose at each stage we cut out lengths  $\epsilon_1, \epsilon_2, \dots, \epsilon_n, \dots$ . We want

$$\sum_{k=1}^{\infty} 2^{k-1} \epsilon_k = \frac{1}{2},$$

or any other number in  $[0, 1)$  so that our final set  $L$  has positive measure. Clearly, choosing  $\epsilon_k = \frac{1}{4^k}$  works. Furthermore,

$$|f_k - f_{k+1}| \leq 2^{-k}, |f_k^{-1} - f_{k+1}^{-1}| \leq 2^{-k}$$

so that  $f$  is a homeomorphism.



Now when  $W \subset F$  is nonmeasurable,  $V = f^{-1}(W)$  is measurable but not Borel. If  $V$  were Borel, then  $W = (f^{-1})^{-1}(V)$  would also be Borel, hence measurable, as the preimage of Borel sets are Borel, c.f. Problem 8. However, it need not be the case that the continuous image of a Borel set is measurable.

**Theorem 1.10.** *The continuous image of a Borel set is measurable.*

Though we shall not prove this here, this was shown by Lusin and his student Souslin in 1916, after reading a paper of Lebesgue and noting many errors including asserting that projections of Borel sets were Borel. Souslin's construction is a bit detailed, though not long, and we shall not pursue it here.

As mentioned previously, non-measurable sets are rare in practice, though they certainly do exist. To this day, no explicit examples of non-measurable sets exist. Indeed, their very existence hinges on the Axiom of Choice.

**Axiom of Choice:** For all collections of nonempty sets  $\{E_\alpha\}$ , one can choose an element from each  $E_\alpha$ , i.e. there exists a function  $f : \mathcal{A} \rightarrow \bigcup E_\alpha$  such that  $f(\alpha) \in E_\alpha$  for all  $\alpha$ . Said differently,  $\prod_{\alpha \in \mathcal{A}} E_\alpha \neq \emptyset$ .

The Axiom of Choice was once a point of great contention among mathematicians at the turn of the century as many counterintuitive results follow from its assumption. However generally today, the axiom is assumed without grudge. We shall assume it here. With this final ingredient, we can construct a nonmeasurable set.

**Theorem 1.11 (Vitali Set).** *There exists a nonmeasurable set.*

*Proof.* Define an equivalence relation on  $\mathbb{R}$  via  $x \sim y$  if  $x - y \in \mathbb{Q}$ . Let  $\{E_\alpha\}_{\alpha \in \mathcal{A}}$  be the set of equivalence classes. It is clear that the equivalence classes can be expressed  $E_x := \{x + q : q \in \mathbb{Q}\}$ . Furthermore,  $E_x$  and  $E_y$  are either identical or disjoint. There is one equivalence class consisting of all rational numbers, while the other equivalence classes consist entirely of irrational numbers, i.e. an irrational number translated by all possible rational numbers. Moreover since we can choose a representative of each equivalence class and all other elements of the class are the set of rational translates of this representative, each equivalence class is countable. However, the set of equivalence classes must be uncountable since their union is  $\mathbb{R}$  and each class is countable.

Consider the sets  $\{\mathcal{A}_\alpha\}_\alpha$ , where  $\mathcal{A}_\alpha := E_\alpha \cap [0, 1]$ . By the Axiom of Choice, we can choose an element from each set  $\mathcal{A}_\alpha$ . Let  $V \subset [0, 1]$  be the set of all the chosen elements. We claim that  $V$  is nonmeasurable. Now given two distinct points  $x, y \in V$ , we know that  $x - y \notin \mathbb{Q}$ , for otherwise they would come from the same equivalence class and we have chosen unique elements from each set. Therefore, every pair of points from  $V$  differ by an irrational number. Now  $\mathbb{R} = V + \mathbb{Q} = \{v + q : v \in V, q \in \mathbb{Q}\}$ , so that  $\bigcup_{q \in \mathbb{Q}} (V + q) = \mathbb{R}$ . If  $|V|_e = 0$ , then so too is  $|V + q|_e = 0$ . This would then imply that  $|\mathbb{R}|_e = 0$ , a contradiction.

Therefore,  $|V|_e > 0$ . Now observe  $\cup_{q \in (\mathbb{Q} \cap [0,1])} (V + q) \subset [0, 2]$ . Now as the sets  $V + q$  are distinct, if  $V$  were measurable, so too would  $V + q$  be measurable, and we would have  $|[0, 2]| \geq \sum_{q \in (\mathbb{Q} \cap [0,1])} |V + q| = \infty$ , a contradiction.  $\square$

Notice that the sets  $V + q$ , if they were measurable, would necessarily have positive measure by the proof above. The proof then shows, in some sense, that these sets are so intertwined that one can fit infinitely many of them in an interval without having their measure go to zero.

Now we have stated that nonmeasurable sets are rare in practice. The next result shows that they are not, in principle, so rare. In fact, every measurable subset of  $\mathbb{R}$  with positive measure contains a nonmeasurable set.

**Lemma 1.6.** *If  $E \subset \mathbb{R}$  and  $|E| > 0$ , then there exists an interval  $I$  such that  $|E \cap I| > (1 - \epsilon)|I|$ .*

*Proof.* There exists an open set  $G$  such that  $G \supset E$  and  $|G| < (1 + \epsilon)|E|$ . Express  $G$  as the union of disjoint, open intervals, i.e.  $G = \cup (a_k, b_k)$ . For each  $k$ , let  $E_k := E \cap (a_k, b_k)$ , and note that each  $E_k$  are measurable and  $E \subset \cup E_k$ . Therefore, we have  $|E| = \sum_k |E_k|$  and  $|G| = \sum_k |(a_k, b_k)|$ . If it were the case that for each  $k$ ,  $(1 + \epsilon)|E_k| \leq b_k - a_k$ , then we would have

$$(1 + \epsilon)|E| = (1 + \epsilon) \sum |E_k| = \sum (1 + \epsilon)|E_k| \leq \sum (b_k - a_k) = \sum |(a_k, b_k)| = |G|,$$

a contradiction. Therefore, there is some  $k_0$  such that  $b_{k_0} - a_{k_0} < (1 + \epsilon)|E_{k_0}| = |E \cap (a_{k_0}, b_{k_0})|$ . Defining  $I := (a_{k_0}, b_{k_0})$  completes the proof.  $\square$

**Lemma 1.7.** *If  $E \subset \mathbb{R}$  and  $|E| > 0$ , then  $E \setminus E$  contains a neighborhood of 0.*

*Proof.* We claim there exists  $\delta > 0$  such that if  $|x| < \delta$ , then  $E \cap (E + x) \neq \emptyset$ .  $x \in E \setminus E$  if and only if  $E + x$  intersects  $E$ .  $X = a - b$  if and only if  $h + x = a$ , where  $a, b \in E$ . Let  $I$  be an interval,  $|E \cap I| > \frac{2}{3}|I|$ . Let  $\delta = \frac{1}{3}|I|$ . If  $|x| < \delta$ .

$$|I \cup (I + \delta)| \leq \frac{4}{3}|I|.$$

If  $E$  and  $E + x$  were disjoint, then  $|(E \cap I) \cup (E \cap I + x)| = 2|E \cap I| > \frac{4}{3}|I| > |I \cup (I + x)|$ , a contradiction (since one is a subset of the other).  $\square$

**Theorem 1.12.** *If  $E \subset \mathbb{R}$ ,  $|E| > 0$ , then there exists  $E \subset E$  such that  $W$  is nonmeasurable.*

*Proof.* Since  $\cup_{q \in \mathbb{Q}} (V + q) = \mathbb{R}$ , where  $V$  is a Vitali set. We have  $\cup_{q \in \mathbb{Q}} [(V + q) \cap E] = E$ . So there exists  $q$  such that  $|(V + q) \cap E|_e > 0$ . But  $(V + q) \cap E$  is nonmeasurable as  $V + q$  has no measurable subsets of positive measure.

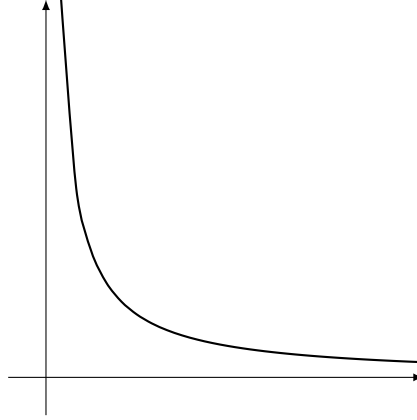
$((V + q) - (V + q)) = V - V$  is disjoint from  $\mathbb{Q}$ .  $\square$

## 2 Lebesgue Measurable Functions

### 2.1 Measurable Functions

It is still our goal to build a generalized theory of integration for functions. However, there are two clear potential problems for functions which we might integrate:

- The graph of  $f$  could cover an infinite area and thus not be integrable in the traditional sense, e.g.  $f(x) = \frac{1}{x}$  on the interval  $(0, 1)$ .



- The graph of  $f$  could be so irregular that the 'area underneath' is not well defined enough so that such a question even makes sense, e.g.

$$\chi_V(x) = \begin{cases} 1, & x \in V \\ 0, & x \notin V \end{cases}$$

where  $V$  is the Vitale set.

As in the case of the Riemann integral, only certain classes of functions will be integrable in the Lebesgue sense. It will be the goal of this section to define 'nice' classes of functions which address the issues above and which will later form a 'nice' class of functions with which to form an integration theory around.

**Definition** (Measurable Function). Let  $E \subseteq \mathbb{R}^n$  and  $\overline{\mathbb{R}}$  denote the extended real line. We say that a function  $f : E \rightarrow \overline{\mathbb{R}}$  is (Lebesgue) measurable if for all  $a \in \mathbb{R}$ , the set  $\{f > a\} := \{x \in E : f(x) > a\}$  is measurable.

Note that we define  $\{f = \infty\} := \bigcap_{k \in \mathbb{Z}} \{f > k\}$  and  $\{f = -\infty\} := \bigcap_{k \in \mathbb{Z}} \{-f > k\}$ . The reason for this choice of definition is that the sets  $\{f > a\}$  describe the distribution for the values of  $f$ . In some sense, the 'smoother'  $f$  is, the smaller the variety of such sets one

obtains from  $f$ . For example, when  $f$  is continuous, the sets  $\{f > a\}$  are open sets since  $f^{-1}((a, \infty))$  is an open set.

Furthermore, the measurability of a set should be linked in some way with the potential measurability of the functions on it and vice versa. If the set  $E$  is 'wild', less functions should be measurable on it. In fact, the measurability of  $E$  is equivalent to that of the set  $\{f = -\infty\}$  since

$$E = \{f = -\infty\} \cup \left( \bigcup_{k=1}^{\infty} \{f > -k\} \right),$$

showing that if  $f$  is measurable, then  $E$  is measurable if and only if  $\{f = -\infty\}$  is measurable.

Note that one could define Borel measurability similarly:  $f : E \rightarrow \overline{\mathbb{R}}$  is Borel measurable, if for all  $a \in \mathbb{R}$ , the set  $\{f > a\}$  is Borel. Thus, every Borel function is also measurable. We can of course define measurability of functions more generally: if  $M$  is a  $\sigma$ -algebra on  $\mathbb{R}^n$ ,  $f$  is  $M$ -measurable if  $\{f > a\} \in M$  for all  $a \in \mathbb{R}$ . For us,  $M$  will normally be the Lebesgue  $\sigma$ -algebra. Though what we do is in  $\mathbb{R}^n$  with the Lebesgue  $\sigma$ -algebra, much of what we do holds, or has parallel results, in other spaces and  $\sigma$ -algebras.

**Example 2.1.** If  $M$  is the  $\sigma$ -algebra generated by  $[0, 1]$  and  $[1, 2]$ . What functions  $f$  are  $M$ -measurable?

$M$  is  $\sigma$  algebra generated by  $[0, 1]$  and  $[1, 2]$ . What  $f$  are  $M$ -measurable?

$f$  over  $[0, 1]$ ,  $[1, 2]$  constant functions, over complements. Not  $M$ -measurable,  $\chi_{\{3\}}$ ,  $f(x) = x$  or  $f(x) = \sin x$ .

Before continuing, we make an important definition since the following abbreviation will serve as a constant shortening of the words needed to state, prove, and discuss results later, so we define it now.

**Definition** (Almost Everywhere, a.e.). We say that a property holds 'almost everywhere' if it holds everywhere except on a set of exceptional values with measure zero, e.g.  $f$  is finite almost everywhere signifies  $|\{f = \infty\}| = 0$ . We shall abbreviate this as 'a.e.'.

## 2.2 Properties of Measurable Functions

1. Continuous then measurable as  $\{f > a\}$  are open.

2. Enough to check that  $\{f > a\}$  is measurable for  $a$  in some dense set  $\mathcal{D} \subset \mathbb{R}$ . Why? For all real  $a$ , for all  $k$ , find  $d_k \in \mathcal{D}$ ,  $a < d_k < a + 1/k$ .  $\{f > a\} \cup_k \{f > d_k\}$ .

3. Instead of  $\{f > a\}$ , it could be  $\{f < a\}$  as  $\{f \geq a\}$ ,  $\{f \leq a\}$ , can use any of these 4 ways to check measurability.

$$\begin{cases} \{f \geq a\} &= \bigcap_{k \in \mathbb{N}} \{f > a - 1/k\} \\ \{f > a\} &= \bigcup_{k \in \mathbb{N}} \{f \geq a + 1/k\} \end{cases}$$

Note that 2,3 are independent of the  $\sigma$ -algebra.

4. If  $f$  is measurable and  $f = g$  a.e., then  $g$  is measurable (Borel fails this). Let  $Z = \{f \neq g\}$ ,  $|Z| = 0$ .  $\{f > a\} \setminus Z \subset \{g > a\} \subset \{f > a\} \cup Z$  but  $\{f > a\} \setminus Z$  and  $\{f > a\} \cup Z$  have the same measure.

Ex:  $1/x$  is measurable on  $\mathbb{R}$ . Defined a.e. is okay.

5. iff  $f$ , then  $f^{-1}(G)$  is measurable for all open  $G \subset \mathbb{R}$ .

(5b) also for all borel sets  $G \subset \mathbb{R}$ .

*Proof.*  $G = \cup(a_k, b_k)$  so  $f^{-1}(G) = \cup_k \{a_k < f < b_k\}$ , measurable.

*Proof.* (5b) a homework generalization.

For all  $\sigma$ -algebra  $M$ ,  $\{E: f^{-1}(E) \in M\}$ , where  $M$  Lebesgue  $\sigma$ -algebra, like  $M_f$ . By 5,  $M_f$  contains open sets, so it must contain Borel sets.

(5c) if  $f$  is finite a.e. and  $f^{-1}(G)$  is measurable for all open  $G \subset \mathbb{R}$ ,  $\{f > a\} = f^{-1}((a, \infty))$ .

Warning: the composition of measurable functions is not always measurable.

Recall  $C$  cantor set,  $F$  fat cantor set,  $f(C) = F$ ,  $f$  homeomorphism  $\mathbb{R} \rightarrow \mathbb{R}$ . Let  $W \subset F$  be nonmeasurable. Write  $\chi_W$  as a composition  $\chi_{g(W)} \circ g$

$\chi_{g(W)}$  is measurable,  $g$  is continuous (so measurable).

4. is what we gain for going Borel  $\rightarrow$  Lebesgue and this is the price we pay.

Rem: if  $f, g$  are Borel measurable, then  $f \circ g$  is Borel measurable if for all  $B \subset \mathbb{R}$  Borel we have  $f^{-1}(B)$  is Borel, then  $(f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B))$ , each composition on right Borel.

is \ (*)	Open	Borel	Measurable
Open	Continuous	—	—
Borel	Borel measurable	Borel measurable	—
Measurable	Measurable	Measurable	?

When can we say that  $f \circ g$  is measurable? What data about  $f$  and  $g$  would ensure this? We know that if  $f$  is measurable and  $g$  is continuous does not work from previous discussion. One possibility ensuring this would be  $f$  is continuous and  $g$  is measurable as  $(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U))$ , as  $U$  is open  $f^{-1}(U)$  is open and  $g^{-1}$  of open is measurable (assuming  $g$  takes values in  $\mathbb{R}$  and  $f$  is continuous on some open set containing the range of  $g$ ). More generally, this will hold if  $f$  is Borel measurable on some Borel set containing the range of  $g$  and  $g$  measurable and real-valued.

**Example 2.2.** If  $f$  is measurable (and  $\mathbb{R}$ -valued) then  $f^2$  is measurable,  $f^2 = \phi \circ f$ , where  $\phi(t) = t^2$ . Similarly,  $|f|$  is measurable since  $|f| = \phi \circ f$ , where  $\phi(t) = |t|$ . Also,  $\text{sgn}(f) = \begin{cases} 1, & f = 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases}$  is measurable since sign is Borel measurable (simple to see by checking

preimages). So is  $\text{floor}(x)$ , where floor is supremum of integers which are at most  $x$ . As is  $\text{ceilin}(x)$ , defined mutatis mutandis. [They are both monotone,]

**Theorem 2.1.** *If  $f, g$  are measurable, then  $f \pm g, fg$ , and  $f/g$  ( $g \neq 0$  a.e.) are measurable.*

*Proof.*  $\{f + g > a\} = \bigcup_{b+c>a, b,c \in \mathbb{Q}} [\{f > b\} \cap \{g > c\}]$  hence measurable set (countable union of measurable sets). Just need to prove equality.  $\supset$  is obvious. Need only prove  $\subset$ . Suppose  $f(x) + g(x) > a$ . Let  $\epsilon = f(x) + g(x) - a$ . Use density of rationals to find  $b \in \mathbb{Q}$  such that  $f(x) - \epsilon/2 < b < f(x)$  and  $c \in \mathbb{Q}$  such that  $g(x) - \epsilon/2 < c < g(x)$ . But then  $b + c > f(x) + g(x) - \epsilon = a$ .

Now  $fg = \frac{1}{4} [(f+g)^2 - (f-g)^2]$  is then measurable. For quotient only need reciprocal. We know  $\frac{1}{g}$  is  $\phi \circ g$ , where  $\phi(t) = \frac{1}{t}$ , which is continuous on  $\mathbb{R} \setminus \{0\}$ . Now  $g$  is nonzero almost everywhere,  $\phi$  contains range of  $g$  (ignoring set where range is  $g = 0$ , since we can ignore for measurability).  $\square$

Another way could combine is max of functions. Better to generalize to countable. So countable sup/inf of measurable functions is measurable.

**Theorem 2.2.** *The above statement, including infinite values.*

*Proof.* IF  $f_k : E \rightarrow \bar{\mathbb{R}}$  is measurable for all  $k \in \mathbb{N}$ , then  $\sup_k f_k$  and  $\inf_k f_k$  are measurable. So  $g(x) = \sup_{k \in \mathbb{N}} f_k(x)$  is measurable. Same for inf.

$$\{g > a\} = \bigcup_{k \in \mathbb{N}} \{f_k > a\}$$

$$\{g \geq a\}$$

$$\{g < a\}$$

$$\{g \leq a\}$$

First case done. Other follow similarly. HAND WAVE! For inf

$$\{\inf_k f_k < a\} = \bigcup_{k \in \mathbb{N}} \{f_k < a\}.$$

$\square$

**Theorem 2.3.** *Suppose  $f_k : E \rightarrow \bar{\mathbb{R}}$  are measurable, then are measurable.*

*Proof.* Follows immediately from preceding result. [ $\lim = \limsup = \liminf$  when it exists.]  $\square$

Approximation: Given measurable  $f$ , find ‘simple’ functions  $f_k$  such that  $f_k \rightarrow f$  pointwise. Here by simple we mean finite range.

**Definition (Simple).**  $g$  is simple if its range is finite subset of  $\mathbb{R}$ . [Need measurable?]

Need this to build notion of integration.

**Theorem 2.4.** For all measurable functions, there exists  $f_k$  such that  $f_k \rightarrow f$  pointwise. If  $f \geq 0$ , we can take  $f_1 \leq f_2 \leq \dots$ .

*Proof.*  $2^{-k} \text{floor}(2^k f)$ . (floor to  $\frac{\text{integer}}{2^k}$ ). This converges monotonically since increasing  $k$  shortens the rounding distance, and then not as much difference so this converges to  $f$  uniformly. Range not finite but countable. So just need to distribute the values among many functions.

$$f_k(x) = \begin{cases} \frac{j-1}{2^k}, & \frac{j-1}{2^k} \leq f(x) < \frac{j}{2^k} \\ k, & f(x) \geq k \end{cases}$$

$f_k$  simple and  $f_k \rightarrow f$ . Now  $f_k = \min(k, \max(-k, 2^{-k} \text{floor}(2^k f)))$ .  $f \geq 0$ , then  $f_1 \leq f_2 \leq \dots$ .  $\square$

**Definition.**  $f : E \rightarrow \mathbb{R}$  is continuous at  $a \in E$  if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $x \in E$  and  $|x - a| < \delta$ , then  $|f(x) - f(a)| < \epsilon$ .

Why better than normal definition with limit? Because  $a$  can be a limit point not defined there. Take  $f(x) = 1/x^2$ , rather like allowing  $f(0) = \infty$ .

Semincontinuity means rather than bounding difference from above below, means one of inequality  $-\epsilon < f(x) - f(a) < \epsilon$  holds.

**Definition (Upper Semicontinuous (usc)).**  $f : E \rightarrow \overline{\mathbb{R}}$  is upper semicontinuous at  $a \in E$  if for all  $y > f(a)$ , there exists  $\delta > 0$  such that if  $|x - a| < \delta$ , then  $f(x) < y$ .

**Definition (Lower Semicontinuous (lsc)).**  $f : E \rightarrow \overline{\mathbb{R}}$  is lower semicontinuous at  $a \in E$  if for all  $y < f(a)$ , there exists  $\delta > 0$  such that if  $|x - a| < \delta$ , then  $f(x) > y$ .

**Example 2.3.**

$$f(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

This is usc (automatically because of the  $\infty$  value, why?) and lsc.

Both types of these functions are measurable.

Remark: lsc  $\rightarrow$  measurable usc  $\rightarrow$  measurable

*Proof.* if  $f$  usc on measurable  $E$ , then  $\{f > a\}$  is open in  $E$ . So  $\{f > a\} = E \cap U$ . Hence  $\{f > a\}$  is measurable.

For USC, use  $\{f < a\}$ .



BV function

Given  $f : [a, b] \rightarrow \mathbb{R}$ , define  $V(f; a, b) = \sup_p \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$  sup over all partitions  $P : a = x_0 < x_1 < \dots < x_n = b$ .  $V(x^2; -1, 1) = 2$  ( $P = \{-1, 0, 1\}$  gives the maximum value. )

**Definition.**  $f$  is BV on  $[a, b]$  if  $V(f; a, b) < \infty$

Interesting not called finite variation.

Something which is not BV is

$$f(x) = \begin{cases} \sin(1/x), & x \in (0, 1] \\ 0, & x = 0 \end{cases}$$

**Theorem 2.5.** If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  and  $\int_a^b |f'| dx$  (riemann integral), then  $V(f; a, b) = \int_a^b |f'| dx$ .

C.f. examples above.

*Proof.* Mean Value Theorem.

$$\sum_i |f(x_i) - f(x_{i-1})| = \sum_i |f'(\xi_i)|(x_i - x_{i-1})$$

$x_{i-1} < \xi_i < x_i$ . the above converges to integral. □

3. Subadditive wrt  $f$ :  $V(f + g; a, b) \leq V(f; a, b) + V(g; a, b)$ .

*Proof.* Triangle inequality.

$$|f(x_i) + g(x_i) - (f(x_{i-1}) + g(x_{i-1}))|$$

then use triangle splits into two and done.

Why not exactly like integral, i.e. strict inequality. Take nonconstant function  $f$  and then  $g = -f$ . Note  $f + g$  identically zero.

Sum of monotone functions are bounded variation.

**Theorem 2.6** (BV Decomposition). If  $f$  is BV on  $[a, b]$ , then there exists increasing  $g, h$  on  $[a, b]$  such that  $f = g - h$ .

*Proof.* Let  $g(x) = V(f; a, x)$ , increasing (monotonicity wrt interval). This forces  $h(x) := g(x) - f(x)$ . Why is  $h(x) \leq h(y)$  for  $x \leq y$ ?  $V(f; a, x) - f(x) \leq V(f; a, y) - f(y)$ . Why?  $f(y) - f(x) \leq V(f; a, y) - V(f; a, x) = V(f; x, y)$ , clearly this right side is  $\geq |f(x) - f(y)|$ . □

**Theorem 2.7 (Egorov's Theorem).** Suppose  $\{f_k\}$  is a collection of measurable functions,  $f_k : E \rightarrow \mathbb{R}$ ,  $|E| < \infty$ , and  $f_k \rightarrow f$  a.e., where  $f : E \rightarrow \mathbb{R}$ . Then for all  $\epsilon > 0$ , there exists a closed set  $E(\epsilon) \subset E$  such that  $f_k \rightarrow f$  uniformly on  $E(\epsilon)$  and  $|E \setminus E(\epsilon)| < \epsilon$ .

*Proof.* We need to find a set on which  $\{f_k\}$  converges uniformly. It suffices to find, for each  $j \in \mathbb{N}$ ,  $E_j \subset E$  such that  $|E \setminus E_j| < \epsilon/2^j$ , and  $\sup_{E_j} |f - f_k| \leq 1/j$  for sufficiently large  $k$ . Indeed, letting  $E(\epsilon) := \bigcap_j E_j$ , then  $f_k \rightarrow f$  uniformly on  $E(\epsilon)$ , and  $|E \setminus E(\epsilon)| \leq \sum_j \epsilon/2^j < \epsilon$ .

To find  $E_j$ , consider the set  $G_m := \{x : |f_k(x) - f(x)| < 1/j \text{ for all } k \geq m\}$ . Since  $|f_k - f| \rightarrow 0$  a.e.,  $\bigcup_m G_m$  is almost  $E$ , i.e.  $|\bigcup_m G_m| = |E|$ . By continuity of measure for nested unions,  $|G_m| \rightarrow |E|$ . Therefore, there exists  $m$  such that  $|E \setminus G_m| < \epsilon/2^j$ . Defining  $E_j := G_m$  completes the proof.  $\square$

**Theorem 2.8 (Lusin's Theorem).** Let  $E$  be a measurable set. A function  $f : E \rightarrow \mathbb{R}$  is measurable if and only if for all  $\epsilon > 0$ , there exists a closed set  $E(\epsilon) \subset E$  such that  $f|_{E(\epsilon)}$  is continuous, and  $|E \setminus E(\epsilon)| < \epsilon$ .

*Proof.* Suppose that  $f$  is a simple function, i.e.  $f = \sum_{k=1}^n a_k \chi_{E_k}$ , where  $E_1, \dots, E_n$  are disjoint measurable sets. Choose a closed subset  $E'_k \subset E_k$  such that  $|E_k \setminus E'_k| < \epsilon/n$ . Then  $E(\epsilon) := \bigcup_{k=1}^n E'_k$  so that  $|E \setminus E(\epsilon)| < \epsilon$ . It only remains to check that  $f|_{E(\epsilon)}$  is continuous, but this is immediate as the preimage of any set is closed.

Now suppose only that  $|E| < \infty$ . There is a collection of simple functions  $\{f_k\}$  such that  $f_k \rightarrow f$  pointwise. Choose some  $E(\epsilon/2^k)$  for  $f_k$  as in the case above. By Egorov's Theorem,  $f_k \rightarrow f$  uniformly on a set  $E(\epsilon)$ . Now  $f$  is continuous on  $E(\epsilon) \cap (\bigcap_k E_k(\epsilon/2^k))$ , which has complement of measure at most  $\epsilon + \sum_k \epsilon/2^k = 2\epsilon$ .

Now suppose that  $|E| = \infty$ . Write  $E = \bigcup [E \cap \{|j| \leq |x| \leq j+1\}]$ . In each shell, we have  $E_j(\epsilon/2^j)$ , closed set such that  $f|_{E_j(\epsilon/2^j)}$  is continuous. Note that  $f$  continuous on  $\bigcup_j E_j(\epsilon/2^j)$ .  $\square$

The converse direction of Lusin's theorem is simple since  $f|_{E(1/j)}$  is continuous, then we have  $f$  Borel on  $\bigcup E(1/j)$  simply because  $\{x \in \bigcup E(1/j) : f > a\} = \bigcup \{x \in E(1/j) : f > a\}$ , countable union of Borel sets. and the rest has measure 0.

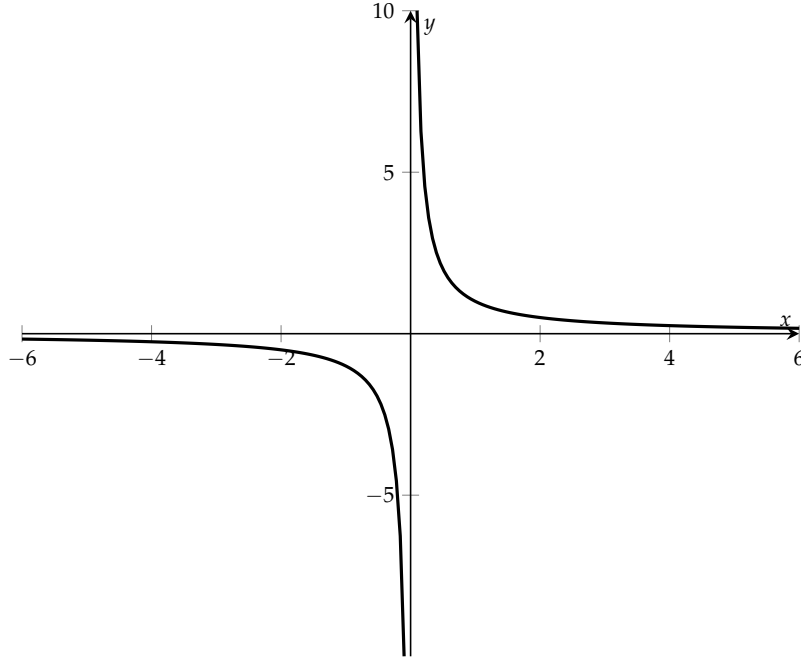
### 2.3 Convergence in Measure

Recall  $\limsup E_k := \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k = \{x : x \in E_k \text{ infinitely many } k\}$ . We will not need  $\liminf$  quite as regularly as  $\limsup$ .

**Definition (Convergence in Measure).** Let  $\{f_k\}$  be a collection of measurable functions,  $f_k : E \rightarrow \mathbb{R}$ . We say  $f_k$  converges to  $f$  in measure if for all  $\epsilon > 0$ ,  $|\{f - f_k| > \epsilon\}| \rightarrow 0$ . We denote this by  $f_k \xrightarrow{m} f$ .

If we expand this definition out more, we have the following:  $f_k \xrightarrow{m} f$  if for  $\epsilon > 0$  and for all  $\epsilon_2 > 0$ , there exists  $N$  such that  $|\{ |f - f_k| > \epsilon \}| < \epsilon_2$  for all  $k \geq N$ . If this is possible, one may as well replace the double epsilon with a single epsilon for a more concise version. So the definition could be taken as follows:  $f_k \xrightarrow{m} f$  if for all  $\epsilon > 0$ , there exists  $N$  such that  $|\{ |f_k - f| > \epsilon \}| < \epsilon$  for all  $k \geq N$ .

**Example 2.4.**  $f_k(x) = \frac{1}{kx} \xrightarrow{m} 0$  on  $\mathbb{R}$ .



We know that  $\{ |f_k| > \epsilon \} = \{ |x| < \frac{1}{k\epsilon} \}$ . This set has measure  $2/(k\epsilon)$ , which tends to zero as  $k$  tends to infinity.  $\triangleleft$

**Example 2.5.**  $f_k(x) = \frac{x}{k} \not\xrightarrow{m} 0$  on  $\mathbb{R}$ .  $\{ |f_k| > \epsilon \} = \{ \frac{|x|}{k} > \epsilon \} = \{ |x| > k\epsilon \}$ , which has infinite measure.  $\triangleleft$

**Example 2.6.** Convergence in measure does not imply pointwise convergence. Let  $f_k = \chi_{I_k}$ , where  $I_k$  are intervals.

$[0,1]$  measure 1  $[0,1/2]$ ,  $[1/2,1]$  measure 1/2  $[0,1/3]$ ,  $[1/3,2/3]$ ,  $[2/3,1]$  measure 1/3

For all  $\epsilon > 0$ ,  $\{ |f_k| > \epsilon \} < |I_k| \rightarrow 0$ .

So the sequence converges to 0 in measure. However, it clearly does not converge to 0 in measure.

There are infinitely many times point part of the interval. That is for all  $x \in (0,1)$ , there are infinitely many  $k$  such that  $x \in I_k$  so that  $f_k(x) \not\rightarrow 0$ . This can be expressed as saying  $\limsup I_k = [0,1]$ .

**Theorem 2.9.** If  $|E| < \infty$ , then convergence a.e. implies convergence in measure.

*Proof.* Given  $\epsilon > 0$ , Egorov's Theorem gives  $E(\epsilon)$  such that  $|E \setminus E(\epsilon)| < \epsilon$ , and  $f_k \rightarrow f$  uniformly on  $E(\epsilon)$ . Then there exists  $N$  such that  $|f_k - f| < \epsilon$  on  $E(\epsilon)$  for all  $k \geq N$ . But then  $|\{|f_k - f| > \epsilon\}| < \epsilon$ , which implies convergence in measure.  $\square$

**Theorem 2.10.** If  $f_k \xrightarrow{m} f$ , then there is a subsequence  $\{f_{k_j}\}$  converging to  $f$  a.e..

*Proof.* Choose  $\epsilon = 1/2^j$ . Find  $N_j$  such that  $|\{|f_k - f| > 1/2^j\}| < 1/2^j$  for all  $k \geq N$  (can make  $N_{j+1} > N_j$ ). We have  $\{f_{N_j}\}$ . We claim that  $f_{N_j} \rightarrow f$  a.e.. Let  $E_j = \{|f_{N_j} - f| > 1/2^j\}$ . Since  $|E_j| < 1/2^j$ , we know  $\sum |E_j| < \infty$ . But then by HW 1, we know  $|\limsup E_j| = 0$ . For all  $x \notin \limsup E_j$ , we have  $x \in E_j$  only finitely many times. Then for sufficiently large  $j$ ,  $x \notin E_j$ , say  $x \notin E_j$  for  $j \geq j_0$ . But then  $|f_{N_j}(x) - f(x)| < 1/2^j \rightarrow 0$ .  $\square$

**Remark.** "Triangle Inequality" (quasi). Convergence in measure is not like convergence in metric space, though it is tempting to think of it in this way. We do have a replacement for a parallel triangle inequality. If  $|\{|f - g| > \epsilon/2\}| < \epsilon/2$ , and  $|\{|g - h| > \epsilon/2\}| < \epsilon/2$ , then  $|\{|f - h| > \epsilon\}| < \epsilon$ . The proof is routine:  $|f - h| \leq |f - g| + |g - h|$ , so  $|f - h| > \epsilon$  then one of the right side must be at least  $\epsilon/2$ . Then  $\{|f - h| > \epsilon\} \subset \{|f - g| > \epsilon/2\} \cup \{|g - h| > \epsilon/2\}$ . Their measures being at most  $\epsilon/2$ , the result then follows.

**Definition** (Cauchy in Measure). A collection of measurable functions  $\{f_k\}$  is said to be Cauchy in measure if for all  $\epsilon > 0$ , there exists  $N$  such that  $|\{|f_j - f_k| > \epsilon\}| < \epsilon$  for all  $k, j \geq N$ .

As it turns out, this is equivalent to convergence in measure.

**Theorem 2.11.** Cauchy in measure is equivalent to convergence in measure.

*Proof (Sketch).*  $\leftarrow$  is immediate by "triangle inequality" measure remark. Now the other direction. There are three standard steps that apply in these types of arguments. The strategy being the important piece here. Step 1. Enough to obtain a convergent subsequence:  $f_1, f_2, f_3, \dots$  cauchy sequence of functions with convergent subsequence. Use triangle inequality to show must converge to limit of conv subsequence. Conclude  $f_k \rightarrow f$ . Notice this holds for any type of convergence in any space. Step 2: Every Cauchy sequence has a "geometric" subsequence. Meaning  $|\{|f_{k_j} - f_{k_{j+1}}| > 1/2^j\}| < 1/2^j$ . This is true for any metric space and any convergence. Use  $\epsilon = 1/2^j$ , and get  $k_j$ . Step 3. Geometric trick implies convergent: given  $\epsilon > 0$ , let  $m$  be such that  $1/2^{m-1} < \epsilon$ . Let  $A = \bigcup_{j \geq m} \{|f_{k_j} - f_{k_{j+1}}| > 1/2^j\}$ . Note  $|A| < \sum_{j=m}^{\infty} 1/2^j = 1/2^{m-1} < \epsilon$ . On  $E \setminus A$ , we have  $|f_{k_j} - f_{k_{j+1}}| \leq 1/2^j$  for all  $j \geq m$ . But this implies uniform convergence on the set  $E \setminus A$ . But then difference everywhere at most  $\epsilon$  on  $E \setminus A$  for sufficiently large  $j$ . Therefore,  $|\{|f_{k_j} - f| > \epsilon\}| \leq |A| < \epsilon$ .  $\square$

### 3 Lebesgue Integral

Two approaches to  $\int_E f, f \geq 0$  on  $E$ .

First,  $\int_E f = \sup\{\int_E g : g \text{ simple } g \leq f\}, g = \sum_{k=1}^m a_k \chi_{E_k}$ , where  $E_k$  are disjoint, measurable sets. Here  $\int_E g = \sum a_k |E_k|$ .

Second,  $\int_E f = |R(f, E)|$ , where  $R(f, E) = \{(x, y) \in \mathbb{R}^{n+1} : x \in E, 0 \leq y \leq f(x)\}$ .

First approach is common in many texts, we shall take the second.

So  $\int_E f$  exists if and only if  $R(f, E)$  is measurable.

Some advantages of the second approach are monotonicity and countable additivity over  $E$ : so  $0 \leq f \leq g$  and  $\int_E f, \int_E g$  exists, then  $\int_E f \leq \int_E g$  because  $R(f, E) \subset R(g, E)$ .

Also if  $E = \bigcup_{k \in \mathbb{N}} E_k$ , disjoint measurable sets, and  $\int_{E_k} f$  exists, then  $\int_E f$  exists and  $\int_E f = \sum_{k \in \mathbb{N}} \int_{E_k} f$  because  $R(f, E) = \bigcup R(f, E_k)$ , the union taken over the disjoint, measurable sets. Can show in other definition but takes work due to sup, whereas here we get it immediately. But we have the issue of existence and measurability.

The following main theorem.

**Theorem 3.1.** *If  $f|_E \rightarrow [0, \infty]$  is measurable, then  $\int_E f$  exists (possibility  $\infty$ ).*

Idea of proof is from 4.1,  $f = \lim f_k, f_1 \leq f_2 \leq \dots$  are simple. So we will show  $R(f, E)$  are measurable and take their union. Need Lemma 1,  $R(f_k, E)$  is measurable. Lemma 2  $\Gamma(f, E)$ , which needs to be defined (simply the graph  $\{(x, f(x)) : x \in E\}$ ) has measure 0. This is boundary top of graph of shaded area of graph, we care because  $\bigcup R(f_k, E)$  need not generally cover  $R(f, E)$ . It will surely cover  $R(f, E) \setminus \Gamma(f, E)$ , then  $y < f(x)$ , so there exists  $k$  such that  $y < f_k(x)$  so that  $(x, y) \in R(f_k, E)$ .

Both lemmas rely on

**Lemma 3.1.** *If  $E \subseteq \mathbb{R}^n$  is measurable, then  $E_a := E \times [a, b]$  is measurable in  $\mathbb{R}^{n+1}$  with measure  $|E_a| = |E| \cdot a$ , noting that  $0 \cdot \infty = 0$  in measure theory.*

*Proof.* If  $E$  is an interval, follows by volume formula. If  $E$  is general, then tile it by intervals, half open (so intervals disjoint).

If  $E$  is  $G_\delta$  of finite measure, write  $E = \bigcap G_k, G_1 \supset \dots$  are open.

$$E_a = \bigcap_k (G_k \times [0, a]) \text{ so } |E_a| = \lim |G_k \times [0, a]| = a \lim |G_k| = a \cdot |E|.$$

What about general set?  $|E| < \infty, E = H \setminus Z, M$  is  $G_\delta, |M| = |E|, |H_a| = |H| \cdot a = |E| \cdot a$ . So  $|E_a| \leq |E| \cdot a$ .

Also for all  $\epsilon > 0$ , there exists closed  $F \subset E$  (hence  $G_\delta$ ),  $|F| > |E| - \epsilon$ .

$$|F_a| = a|F| > a|E| - a\epsilon$$

So  $E_a$  contains  $F_a$  closed of measure  $> |E_a| - a\epsilon$ . Hence  $E_a$  is measurable.  $|E_a| = a|E|$ .

Still assumes  $a < \infty$ , otherwise  $a = k$  and let  $k \rightarrow \infty$

□?

**Lemma 3.2.**  $R(f_k, E) = \bigcup_j (E_j \times [0, a_j])$ ,  $E_j$  disjoint measurable sets, where  $f_k = \sum_j a_j \chi_{E_j}$ .

Note that this shows that if  $f$  takes countably many values,  $f = a_k$  on  $E_k$ , then  $\int_E f = \sum a_k |E_k|$ .

**Lemma 3.3.**  $|\Gamma(f, E)| = 0$ . Let  $g = 2^{-j} \text{floor}(2^j f)$ , then  $g \leq f \leq g + 2^{-j}$ . Then  $R(g, E) \subset R(f, E) \subset \dots$ . Then  $\Gamma(f, E) \subset R(g + 2^{-j}, E) \setminus R(g - 2^{-j}, E)$ . This right side has measure at most  $2 \cdot 2^{-j} |E| \rightarrow 0$ , hence  $|\Gamma(f, E)| = 0$ .

Last time:  $\int_E f = |R(f, E)|$ ,  $R(f, E) = \{(x, y) \in \mathbb{R}^{n+1} : 0 \leq y \leq f(x), x \in E\}$ .  $\Gamma(f, E) = \{(x, f(x)) \in \mathbb{R}^{n+1} : x \in E\}$ .

$\int_E f = \sup\{\int g : g \leq f, g \text{ simple}\}$ ,  $g = \sum a_k \chi_{E_k} \rightarrow \sum g = \sum a_k |E_k|$ , then  $R(f, E) = (E \cap F) \times [0, 1]$ .

Recall  $R(f, E)$  is approximated by  $R(f_k, E)$ :

$\bigcup_k R(f_k, E) \subset R(f, E) \subset [\bigcup_k R(f_k, E)] \cup \Gamma(f, E)$ .

Lemma:  $|\Gamma(f, E)| = 0$  for all measurable  $f$ .

*Proof.* Write  $E = \bigcup E_m$ ,  $|E_m| < \infty$

Suffices  $|\Gamma(f, E_m)| = 0$  for all  $|E_m| < \infty$ . Let  $\mathcal{A}_k = \{x \in E_m : \epsilon k \leq f(x) < \epsilon(k+1)\}$ ,  $k = 0, 1, 2, \dots$ . Then  $\Gamma(f, E_m) \subset \bigcup_k \mathcal{A}_k + [\epsilon k, \epsilon(k+1)]$ , then  $|\Gamma(f, E_m)| \leq \sum_k |\mathcal{A}_k| \epsilon = \epsilon |E_m|$ , then  $|\Gamma(f, E_m)| = 0$ .

**Theorem 3.2 (Monotone Convergence Theorem).** If  $f_k \geq 0$  are measurable on  $E$  and  $f_1 \leq f_2 \leq \dots$ , then  $\int_E \lim f_k = \lim \int_E f_k$ .

*Proof.*  $f = \lim f_k$ , then  $R(f, E) = [\bigcup R(f_k, E)] \cup \Gamma(f, E)$  and  $\bigcup R(f_k, E)$  are nested. So  $|R(f, E)| = \lim |R(f_k, E)|$  by continuity of measure.  $\square$

**Example 3.1.**  $\mathbb{Q} = \{q_1, q_2, \dots\}$ . Let  $f_k = \chi_{\{q_1, \dots, q_k\}}$ . Then  $f_k \nearrow f = \chi_{\mathbb{Q}}$ . Monotone convergence theorem fails for Riemann integral, so  $\int_{\mathbb{Q}} = \lim \int f_k = 0$ .

**Theorem 3.3.** The two definitions of  $\int_E f$  agree:

$$\int_E f = \sup\{\sum a_k |E_k| : g \leq f, g = \sum a_k \chi_{E_k}\}$$

*Proof.* Note  $\sum a_k |E_k| = \int_E g$  by measure of product. Since  $f \geq g$ , we get  $\int f \geq \int g$  so  $\geq$  holds. Reverse: pick  $g_k \nearrow f$  pointwise. Then  $\int g_k \rightarrow \int f$  so  $\sup \int g_k \geq \int f$ .  $\square$

**Example 3.2.**  $f_1 \geq f_2 \geq \dots$ . Counterexample  $f_k = \chi_{[k, \infty)}$ .  $\lim \int f_k = \infty$  but  $\int \lim f_k = 0$ . Similarly  $\inf\{\int g : g \geq f, g \text{ simple}\} \neq \int f$  in general. if  $g \geq f$  then  $g > 0$  but being simple  $g \geq c > 0$  then  $\int g = \infty$ .

**Chebyshev's Inequality:** Suppose  $f$  is nonnegative measurable on  $E$ . For all  $\alpha > 0$   
 $|\{x \in E: f(x) > \alpha\}| \leq \frac{1}{\alpha} \int_E f$ .

*Proof.* Let  $A = \{f > \alpha\}$ . Then  $f \geq \alpha \chi_A \rightarrow \int f \geq \int \alpha \chi_A = \alpha |A| \rightarrow |A| \leq \frac{1}{\alpha} \int f$ .

**Example 3.3.**  $f(x) = \frac{1}{\sqrt{x}}$  on  $[0, 1]$ . Let  $c = \int_{(0,1)} f \rightarrow$  Chebyshev  $|\{f > \alpha\}| \leq \frac{c}{\alpha}$ . In fact,  
 $\{f > \alpha\} = (0, 1/\alpha^2) \rightarrow$  measure is  $1/\alpha^2$ , if  $\alpha \geq 1$ .

**Additivity over functions:**  $\int_E (f + g) = \int_E f + \int_E g$ ,  $f, g \geq 0$  measurable.

*Proof.* Use simple functions,  $f_k \nearrow f$ ,  $g_k \nearrow g$ . So it suffices to prove that  $\int (f_k + g_k) = \int f_k + \int g_k$ .  $f_k = \sum a_j \chi_{E_j}$ ,  $g_k = \sum b_j \chi_{F_j}$ . Let  $H = E_i \cap F_j$  note  $\cup E_i = \cup F_i = E$ . Then  $f_k = \sum_{i,j} a_i \chi_{H_{ij}}$ ,  $g_k = \sum b_j \chi_{H_{ij}}$ . Then  $f_k + g_k = \sum (a_i + b_j) \chi_{H_{ij}}$  then  $\int (f_k + g_k) = \sum (a_i + b_j) |H_{ij}| = \sum a_i |H_{ij}| + \sum b_j |H_{ij}|$ .

**Countable subadditivity:**

$\int \sum_{k=1}^{\infty} f_k = \sum_k \int_E f_k$ ,  $f_k \geq 0$  measurable.

True for  $\sum_{k=1}^N$  then  $N \rightarrow \infty$  using MCT

**Example 3.4.** Let  $\mathbb{Q} \cap [0, 1] = \{q_1, \dots\}$

$$f_k = \frac{1}{\sqrt{|x - q_k|}} \int_{[0,1]} f \leq c < \infty$$

$$f = \sum_{k=1}^{\infty} \frac{1}{k^2} \frac{1}{\sqrt{|x - q_k|}} \xrightarrow{\lim_{x \rightarrow q} f(x) = \infty \text{ for all } q \in \mathbb{Q}} \int f \leq c \sum \frac{1}{k^2} < \infty \text{ so } |\{f > \alpha\}| \leq \frac{1}{\alpha} C',$$

what does this say about  $\mathbb{Q}$ ?

We have not yet proved  $\int_E cf = c \int_E f$ , where  $c \geq 0$  and  $f \geq 0$ .

**Proposition 3.1.**

$$\int_E cf = c \int_E f$$

*Proof.* This is true for simple functions:  $\int c(\sum a_k \chi_{E_k}) = \int \sum ca_k \chi_{E_k} = \sum ca_k |E_k| = c \sum a_k |E_k| = c \int \sum a_k \chi_{E_k}$ . Then for general functions use the Monotone Convergence Theorem. For general  $f \geq 0$ , take simple  $f_k \geq 0$ ,  $f_k \nearrow f$ , then  $cf_k \nearrow cf$ , by MCT  $\int cf = \lim \int cf_k = c \lim \int f_k = c \int f$ .  $\square$

What if the convergence is not  $f_k \nearrow f$ ?

**Example 3.5.**  $\chi_{[k, \infty)} \rightarrow 0$  pointwise. But  $\int \chi_{[k, \infty)}$  are all infinite and certainly do not converge to 0. Note that  $\chi_{[k, \infty)} \searrow 0$ .

**Example 3.6.**  $k \chi_{(0, 1/k)}$ . But  $\int k \chi_{(0, 1/k)} = 1 \not\rightarrow 0$ .

**Example 3.7.**  $\frac{1}{k} \chi_{[0, k]} \rightarrow 0$  uniformly but  $\int \frac{1}{k} \chi_{[0, k]} = 1 \rightarrow 0$ .

What is the similarity of these three examples? The functions vanish but their integrals do not???? The general principle is that the area under the graph can disappear or 'escape' to infinity but does not appear from nowhere..... A precise statement is Fatou's Lemma. As a reminder,

**Definition** (limsup, liminf). Given a sequence  $\{f_k\}$ ,

$$\limsup_{k \rightarrow \infty} f_k = \lim_{n \rightarrow \infty} \sup_{k \geq n} f_k \quad \liminf_{k \rightarrow \infty} f_k = \lim_{n \rightarrow \infty} \int_{k \geq n} f_k.$$

**Example 3.8.** (i)  $\lim f_k(x)$  exists if and only if  $\limsup f_k(x) = \liminf f_k(x)$ .

(ii) The Piano Key Sequence (press one key going left to right):  $\chi_{[0,1]}$ , then  $\chi_{[0,1/2]}, \chi_{[1/2,1]}$ , then  $\chi_{[0,1/3]}, \chi_{[1/3,2/3]}, \chi_{[2/3,1]}$ . [Go in pascal triangle order]

$$\limsup f_k = \chi_{[0,1]}$$

$$\liminf f_k = 0$$

**Remark.** Relation to sequences to sets  $\{E_k\}$ :  $\limsup E_k = \bigcap_{m=1}^{\infty} \bigcup_{k \geq m} E_k$ ,  $\liminf E_k = \bigcup_{m=1}^{\infty} \bigcap_{k \geq m} E_k$ .  $\limsup \chi_{E_k} = \chi_{\limsup E_k}$  and same for liminf. and sup like union and inf like intersection.

**Lemma 3.4** (Fatou).  $f_k \geq 0$  measurable.

$$\int_E \liminf f_k \leq \liminf \int_E f_k$$

*Proof.* Let  $g_m = \inf_{k \geq m} f_k$ . Then  $g_m \nearrow \liminf f_k$ . By MCT,  $\int g_m \rightarrow \int \liminf f_k$ . But  $f_k \geq g_m$  so that  $\int f_k \geq \int g_m$ , hence  $\liminf \int f_k \geq \lim \int g_m$ . [Rem: if  $a_k \geq b_k$  for all  $k$ , then  $\liminf a_k \geq \liminf b_k$ ,  $\limsup a_k \geq \limsup b_k$ .]  $\square$

**Theorem 3.4** (Lebesgue Dominated Convergence (DCT)). Suppose  $f_k \geq 0$  are measurable and there is  $\phi \geq 0$  measurable such that  $f_k \leq \phi$  ( $\phi$  dominates) and  $\int_E \phi < \infty$ . If  $f_k$  converges to  $f$  a.e., then  $\int_E f_k \rightarrow \int_E f$ .

*Proof.* Fatou's Lemma gives  $\int f \leq \liminf \int f_k$ . Also,  $\phi - f_k \geq 0$ . Then Fatou's Lemma applies again:  $\int \phi - f \leq \liminf \int (\phi - f_k)$ . Which is  $\int f \geq \limsup \int f_k$ . Then  $\int \phi - \int f \leq \liminf (\int \phi - \int f_k)$  cancel constant  $-\int f \leq \liminf (-\int f_k)$  then  $\int f \geq \limsup \int f_k$ . Comparing very first and last equations, we have  $\int f = \lim_{k \rightarrow \infty} \int f_k$ .  $\square$

**Example 3.9.**  $\frac{1}{k} \chi_{[0,k]} \rightarrow 0$  uniformly.

Try to cover with one function get something like  $1/x$ . This is not good enough for DCT since  $\int \frac{1}{x} = \infty$ . If  $f_k \leq \frac{1}{x^2}$  on  $[1, \infty)$ , this examples does not work. 'Best' you can do is  $\frac{1}{k^2} \chi_{[0,k]} \rightarrow 0 = \int \lim f_k$ .



**Example 3.10.**  $\lim \int_{[1,\infty)} \frac{\sin^{2k} x}{x^2} dx$ . Now  $\frac{\sin^{2k} x}{x^2} \rightarrow 0$  a.e. and is at most  $\frac{1}{x^2}$  in absolute value, so must be 0. Now if  $\lim \int_{[0,\infty)} \frac{\sin^{2k} x}{x^2} dx$ ,  $\frac{\sin^{2k} x}{x^2} \leq \frac{\sin^2 x}{x^2}$  and  $\int_{[0,1]} \frac{\sin^2 x}{x^2} \leq \int_{[0,1]} 1 \leq 1$ . so  $\int_{[0,\infty)} \frac{\sin^2 x}{x^2} < \infty$ .

Suppose that  $f : E \rightarrow \overline{\mathbb{R}}$  is measurable. Define functions  $f^+ = \max(f, 0)$  and  $f^- = \max(-f, 0)$ . These are both nonnegative functions with  $f^+ - f^- = f$  and  $f^+ + f^- = |f|$ .

**Definition (Integrable Function).** Suppose  $f : E \rightarrow \overline{\mathbb{R}}$  is a measurable function, not necessarily nonnegative. We define  $\int_E f$  to be

$$\int_E f := \int_E f^+ - \int_E f^-,$$

provided at least one of the integrals on the right is finite. We say that  $f$  is integrable on  $E$  if  $\int_E f$  is finite, and we write  $f \in L^1(E)$ ,

**Remark.** Note that  $\int f$  is finite if and only if  $\int f^+$  and  $\int f^-$  are finite if and only if  $\int |f|$  is finite. Therefore, all Lebesgue integrable functions with finite integral are automatically absolutely integrable. This means there are integrals which are (Improper) Riemann integrable which are not Lebesgue integrable. Take Example ??,  $\int_{(0,\infty)} \frac{\sin x}{x} dx$  does not exist as a Lebesgue integrable, i.e.  $\frac{\sin x}{x} \notin L^1(0, \infty)$ , while it is Riemann integrable on  $(0, \infty)$ .

We next examine a few properties of this integral. Not surprisingly, many of these properties resemble those of their primordial Riemann integral.

- Triangle Inequality:  $|\int_E f| \leq \int_E |f|$  as  $|\int f^+ - \int f^-| \leq \int f^+ + \int f^-$ .
- Additivity of Domain: If  $\{E_k\}_{k=1}^N$  is a collection of disjoint, measurable sets, then  $\int_{\cup E_k} f = \sum_{k=1}^m \int_{E_k} f$  as  $\int f^+$ ,  $\int f^-$  split using the properties of the integral for nonnegative functions.

However, care is needed even in the countable additivity case. Consider  $\int_0^\infty \sin x$ , defining the  $E_k$  to be as below:

We know that  $\int_{E_k} f = 0$  for all  $k$  but that  $\int_{\cup E_k} f$  does not exist.

**Lemma 3.5.** Suppose that  $\int_E f$  exists. Let  $E = \cup_{k=1}^\infty E_k$  be a union of disjoint, measurable sets. Then  $\int_{E_k} f$  exists for all  $k$ , and

$$\int_E f = \sum_{k=1}^\infty \int_{E_k} f$$

*Proof.* By countable additivity for integrals of nonnegative functions, we know that  $\int_E f^+ = \sum_k \int_{E_k} f^+$  and  $\int_E f^- = \sum_k \int_{E_k} f^-$ . Now observe

$$\int_E f = \sum_k \left( \int_{E_k} f^+ - \int_{E_k} f^- \right).$$

Since  $\int_E f$  exists, at least one of  $\int_E f^+$ ,  $\int_E f^-$  is finite. □

Additivity over  $f$ :

Suppose we have  $\int_E (f + g) = \int_E f + \int_E g$ , assuming that both integrals on the right are finite. Let  $h := f + g$ . There are eight possibilities for the signs of  $(f, g, h)$ . This splits  $E$  into eight different parts. On each part, write a between  $f, g, h$  as addition of nonnegative functions, e.g.  $f \geq 0, g < 0, h < 0$ , then  $f + g = h$  becomes  $f + (-h) = (-g)$ , all nonnegative here. Then

$$\int f + \int (-h) = \int (-g)$$

Note that we are making use of the fact that the functions are finite a.e..

Remark: if  $f \in L^1(E)$ , then  $|f| < \infty$  a.e., since  $\int |f| < \infty$ .

Convergence:

There is little to say about convergence since everything follows from convergence of nonnegative functions.

For example, how does MCT apply? The answer is that it does not really apply. We need an integrable 'baseline',  $\phi \in L^1(E)$ , i.e. a function to play the role of the  $x$ -axis.

MCT for nonnegative functions:

If  $\phi \leq f_1 \leq f_2 \leq \dots$  or  $\phi \geq f_1 \geq f_2 \geq \dots$ , then  $\int f_k \rightarrow \int \lim f_k$ .

*Proof.* Use MCT for  $f_k - \phi$  or  $\phi - f_k$ , nonnegative, increasing. Then  $\int (f_k - \phi) \rightarrow \int (f - \phi)$ ; hence,  $\int f_k \rightarrow \int f$ . □

However, there is one useful result here:

**Theorem 3.5** (Dominated Convergence Theorem). *If  $f_k \rightarrow f$  a.e. on  $E$ , and are measurable, and there exists  $\phi \in L^1(E)$  such that  $|f_k| \leq \phi$  for all  $k$ , then*

$$\int_E f_k \longrightarrow \int_E f.$$

*Proof.* Since  $0 \leq \underbrace{f_k + \phi}_{\geq 0} \leq 2\phi$ , DCT for nonnegative functions applies.  $\int (f_k + \phi) \rightarrow \int (f + \phi)$ . □

**Example 3.11.** Consider  $\sum_{k=1}^{\infty} B_k \sin(kx)$  on the interval  $(0, \pi)$ . Assume that  $C := \sum |B_k| < \infty$ . What do we need to assume about  $B_k$  to obtain

$$\int_{(0,\pi)} f(x) = \sum_k \int_{(0,\pi)} B_k \sin(kx)?$$

Observe that  $C$  is a domination function.  $f_k(x) = \sum_{j=1}^k B_j \sin(jx)$  satisfies  $|f_k| \leq C$ . Also,  $f_k \rightarrow f$  pointwise.

Shorter proof: let  $h = f + g$ :  $h^+ - h^- = f^+ - f^- + g^+ - g^-$ .  $h^+ + f^- + g^- = f^+ + g^+ + h^-$ . Integrate, apply additivity for nonnegative.  $\int h^+ + \int f^- + \int g^- = \int f^+ + \int g^+ + \int h^-$ . All integrals finite then  $\int h^+ - \int h^- = \int f^+ - \int f^- + \int g^+ - \int g^-$ .

DCT: Works for continuous parameters.

If we have  $f_t$  ( $t$  real) and  $|f_t| \leq \phi$ ,  $\phi \in L^1(E)$  and  $f_t$  measurable and  $f_t \rightarrow f$  a.e. as  $t \rightarrow t_0$ . Then  $\lim_{t \rightarrow t_0} \int_E f_t = \int_E f$ .

*Proof.* Sequential characterization of limits: Take any sequence  $t_k \rightarrow t_0$  and apply DCT to  $f_{t_k}$ . Get  $\int f_{t_k} \rightarrow \int f$ .  $\square$

**Example 3.12.** Suppose  $|E| < \infty$ ,  $f \in L^1(E)$ ,  $f > 0$  on  $E$ . a)  $\int_E f^p \rightarrow \int_E f$  as  $p \rightarrow 0^+$  b) if also  $\log f \in L^1(E)$ , then  $\int_E \frac{f^p - 1}{p} \rightarrow \int_E \log f$  as  $p \rightarrow 0^+$ .

apf)  $f^p \rightarrow 1$  as  $p \rightarrow 0^+$ . Need dominating function. Consider only  $0 < p \leq 1$ . Then  $f^p \leq f$  if  $f \geq 1$  but notice  $f^p \leq 1$  if  $0 < f < 1$ . So  $f^p \leq \max(f, 1) \leq f + 1$ . This is simpler than the less mysterious max. So  $f + 1 \in L^1(E)$  dominating function. Hence,  $\int_E f^p \rightarrow \int_E 1 = |E|$ .

bpf) derivative of  $p \mapsto \int_E f^p$  at  $p = 0$  which is  $\log f$ .  $\frac{f^p - 1}{p} \rightarrow \log f$ . Slope of secant

Slope inc:  $\frac{f^p - 1}{p} \leq \frac{f^1 - 1}{1} = f - 1$

$f \leq 1$ :  $\left| \frac{f^p - 1}{p} \right| \leq |\log f|$ .

If  $f < 0$  is measurable and  $f \in L^1(E)$ , then

$$\int_E \frac{f^p - 1}{p} \rightarrow \int_E \log f \text{ as } p \rightarrow 0^+.$$

We need integrability to make this possible.

MCT applies: if  $\phi \in L^1$  is a 'baseline' and  $\phi \geq f_1 \geq f_2 \geq \dots$  or  $\phi \leq f_1 \leq f_2 \leq \dots$ , then  $\int \lim f_k = \lim \int f_k$ .

For all  $p_k \searrow 0$ , we want to show  $f_k = \frac{f^{p_k} - 1}{p_k}$  is a decreasing sequence,  $f_1 \geq f_2 \geq f_3 \geq \dots$ . The reason why is because of the shape of these things.

Now taking  $p_k$  which is smaller, so  $p_{k+1}$  is to left so the slope of the secant line is less because it is a convex function. The conclusion is the same for the decreasing version because this is also convex. 'Baseline' is then  $f - 1$ .

### 3.1 Relation between the Lebesgue and Riemann-Stieltjes Integrals

Recall most important HW (5.1#1): if  $f \geq 0$  measurable, then  $f \in L^1$  if and only if  $\sum_{j=-\infty}^{\infty} 2^j |\{f > 2^j\}| < \infty$ . Let  $\omega_f = |\{x \in E: f(x) > \alpha\}|$ . Notice this is just notation for something we have already discussed at length. With this notation, we can say that

$$f \in L^1(E) \iff \sum_{j=-\infty}^{\infty} 2^j \omega_{|f|}(2^j) < \infty,$$

where  $f$  is measurable but not necessarily nonnegative. What is the significance of the '2' here? The proof of the exercise only makes use of the fact that multiplication by 2 preserves the inequality:  $g = 2^j \chi_{\{2^j < f < 2^{j+1}\}}$  and then  $g \leq f \leq 2g$ . This is like the Cauchy Condensation Test. However, this works for any real number  $\lambda > 1$ . This allows us to then characterize the above as

$$\begin{aligned} f \in L^1(E) &\iff \sum_{j=-\infty}^{\infty} 2^j \omega_{|f|}(2^j) < \infty \\ &\iff \sum_{j=-\infty}^{\infty} \lambda^j \omega_{|f|}(\lambda^j) < \infty \end{aligned}$$

for any  $\lambda > 1$ . The generalization of this is as follows:

**Definition** ( $L^p$ ).  $f : E \rightarrow \overline{\mathbb{R}}$  belongs to  $L^p(E)$  if  $\int |f|^p < \infty$ , where  $0 < p < \infty$ .

By the discussion above (or #1 of HW 5.1), we know that  $f \in L^p$  if and only if  $\sum \lambda^j |\{|f|^p > \lambda^j\}| < \infty$ , i.e.  $\sum \lambda^j |\{f > \lambda^{j/p}\}| < \infty$ . If we choose  $\lambda = 2^p$ , then

$$f^p \in L^1 \iff \sum_{j=-\infty}^{\infty} 2^{pj} |\{f > 2^j\}| < \infty.$$

**Theorem 3.6.** If  $|E| < \infty$  and  $f : (a, b]$  is measurable, where  $a, b \in \mathbb{R}$ , then

$$\int_E f = \int_a^b \alpha d(-\omega_f(\alpha)).$$

*Proof.* Consider  $\mathcal{P}$ :  $a = x_0 < x_1 < \dots < x_n = b$ . Let  $E_i = \{x_{i-1} < f \leq x_i\}$ . Note  $E_i$  is a partition of  $E$ .

$$\sum_i x_{i-1} \chi_{E_i} \leq f \leq \sum_i x_i \chi_{E_i}.$$

But then

$$\sum_{i=1}^n x_{i-1} |E_i| \leq \int_E f \leq \sum_{i=1}^n nx_i |E_i| (*)$$

Here

$$|E_i| = -(\underbrace{\omega(x_i)}_{f > x_i} - \underbrace{\omega(x_{i-1})}_{f > x_{i-1}})$$

But then the left sum is  $L(\alpha, P, -\omega)$  and the right one is  $U(\alpha, P, -\omega)$ . The sup of  $L$  and inf of  $U$  both go to  $\int_a^b \alpha d(-\omega)$ . Thus,  $\int_E f = \int_a^b f d(-\omega)$ .  $\square$

### 3.2 Relation between the Lebesgue and Riemann Integral

We want to compare  $\int_{[a,b]} f$  and  $R\int_a^b f$ , where  $a, b \in \mathbb{R}$  and  $f$  is bounded. Following shows we have generalized.

**Theorem 3.7.** *If  $R\int_a^b f$  exists, then  $\int_{[a,b]} f$  exists and they are equal.*

*Proof.* Riemann then Leb: If  $R$  integral exists, then there exists sequence of partitions  $\{P_k\}$  such that  $U(f, P_k)$  and  $L(f, P_k)$  converge to  $\int_a^b f$ . Arrange  $P_1 \subset P_2 \subset P_3 \subset \dots$ . Let  $g_k, h_k$  be step functions for these partitions.  $g_k \leq f \leq h_k$  and  $L(f, P_k) = \int g_k$ ,  $U(f, P_k) = \int h_k$ .

By MCT,  $g_1 \leq g_2 \leq \dots$  and  $h_1 \geq h_2 \geq \dots$ . Then  $\int g_k \rightarrow \int g$ , where  $g = \lim g_k$  and  $\int h_k \rightarrow \int h$ , where  $h = \lim h_k$ . So  $\int g = \int h = R\int_a^b f$  since  $U$  and  $L$  converge to  $\int_a^b f$ . But we know that  $g \leq f \leq h$ . This means that we must have equality a.e.. But this completes the proof.  $\square$

The following characterization of Riemann integrals.

**Theorem 3.8.** *Riemann integral exists if and only if  $|\mathcal{D}| = 0$ , where  $\mathcal{D}$  is the set of discontinuities of  $f$ .*

*Proof.* See text.

## 4 Repeated Integration

### 4.1 Fubini's Theorem

Given the function  $f(x, y) = \frac{1}{\sqrt{xy}}$  on the interval  $[0, 1] \times [0, 1]$ , we already have a concept of  $\int_{[0,1] \times [0,1]} f$ . But how does this integral relate to the integral

$$\int_0^1 \left( \int_0^1 \frac{1}{\sqrt{xy}} dy \right) dx?$$

One can ask the same questions and confront the same issues with infinite double summations, e.g.  $\sum_{i,j} a_{ij}$ .

**Example 4.1.** Consider the infinite matrix with 1s on the main diagonal and  $-1$  on the superdiagonal and zeros elsewhere. The sum of each row is 0 and, except for the first column, the sum of column is 0. This implies

$$\begin{aligned} \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} a_{ij} \right) &= 0 \\ \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} a_{ij} \right) &= 1 \end{aligned}$$

One can create a parallel example with integrals. For example, take  $f = 1, -1, 0$  and consider

$$\begin{aligned} \int_0^{\infty} \left( \int_0^{\infty} f(x, y) dx \right) dy &= 0 \\ \int_0^{\infty} \left( \int_0^{\infty} f(x, y) dy \right) dx &= 0 \end{aligned}$$

Note that  $f \notin L^1([0, \infty) \times [0, \infty))$ .

The goal of this section is to establish Fubini's Theorem, stated below, and its applications. This was proven by Guido Fubini in 1907. The theorem allows determines conditions under which it is possible to compute double integrals using iterated integrals. Consequently, this allows one to exchange the order of integration. In 1909, a student of Fubini named Leonida Tonelli established a similar theorem (Tonelli's Theorem) but applying to functions which are nonnegative instead of integrable.

**Theorem 4.1** (Fubini's Theorem). *Let  $I_1 \subseteq \mathbb{R}^n$ ,  $I_2 \subseteq \mathbb{R}^m$  be intervals, and  $I := I_1 \times I_2$ . Suppose that  $f \in L^1(I)$ . Then*

- (i) for almost every  $\mathbf{x} \in I_1$ , the ‘slice’ function  $\mathbf{y} \mapsto f(\mathbf{x}, \mathbf{y})$  is measurable and integrable on  $I_2$ .  
(ii) the ‘once-integrated’ function  $\mathbf{x} \mapsto \int_{I_2} f(\mathbf{x}, \mathbf{y}) \, d\mathbf{y}$  is measurable and integrable on  $I_1$ , and its value is  $\int_I f$ , i.e.

$$\iint_I f(\mathbf{x}, \mathbf{y}) \, d\mathbf{x}d\mathbf{y} = \int_{I_1} \left( \int_{I_2} f(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \right) d\mathbf{x}.$$

The statement ‘for almost every  $\mathbf{x}$ ’ is required as a slice may have ‘issues’ on some set of measure 0, which will not affect the integral. As a parallel issue, for example, consider the matrix in Example 4.1. One could instead insert for some zero entry a nonmeasurable function or the value  $\infty$ . So while the ‘slice’ function may be problematic, its integral is unchanged since its values were changed on a set of measure zero. Recall that

$$\int_I f = \int_{\mathbb{R}^{n+m}} f \cdot \chi_I.$$

By replacing  $f$  with  $f\chi_I$ , i.e. taking  $f \equiv 0$  outside of  $I$ , we may reduce the proof of Fubini’s Theorem to the case where  $I_1 = \mathbb{R}^n$ ,  $I_2 = \mathbb{R}^m$ , and  $I = \mathbb{R}^{n+m}$ . Let  $\mathcal{F} := \{f \in L^1(\mathbb{R}^{n+m}) : \text{Fubini holds for } f\}$ . To prove Fubini’s Theorem, we prove that  $\mathcal{F} = L^1$ . First, observe the following properties of  $\mathcal{F}$ :

**Proposition 4.1.** *Let  $\mathcal{F} := \{f \in L^1(\mathbb{R}^{n+m}) : \text{Fubini holds for } f\}$ . Then*

- (i)  $\mathcal{F}$  is a vector space:  $f, g \in \mathcal{F}$ , then  $\alpha f + \beta g \in \mathcal{F}$ .  
(ii)  $\mathcal{F}$  is closed under monotone limits: if  $f_k \in \mathcal{F}$  and  $f_k \nearrow f$  or  $f_k \searrow f$ , where  $f \in L^1(\mathbb{R}^{n+m})$ , then  $f \in \mathcal{F}$ .  
(iii) Squeeze Property: if  $g \leq f \leq h$  and  $g = h$  a.e., where  $g, h \in \mathcal{F}$ , then  $f \in \mathcal{F}$ .  
(iv)  $\chi_I \in \mathcal{F}$  for any closed bounded interval  $I$ .

*Proof.*

- (i) This is immediate from the fact that if  $f, g$  are measurable, then  $f + g$  and  $cf$  are measurable. Similarly, if  $f, g \in L(E)$ , then  $f + g, cf \in L(E)$ .  
(ii) By replacing  $f_k$  with  $-f_k$  for all  $k$ , we may assume without loss of generality that  $f_k \nearrow f$ . For each  $k$ , there is a set  $Z_k \subseteq \mathbb{R}^n$  with measure zero such that the  $\mathbf{x}$ -slice function of  $f_k$  is integrable for  $\mathbf{x} \notin Z_k$ . Let  $Z = \cup_k Z_k$ . Since  $Z$  is a countable union of sets with measure zero,  $Z$  has measure zero. If  $\mathbf{x} \notin Z$ , then the  $\mathbf{x}$ -slice function  $f_k(\mathbf{x}, \mathbf{y})$  is integrable for all  $k$ , and by the MCT applied to  $\{f_k(\mathbf{x}, \mathbf{y})\}$  as functions of  $\mathbf{y}$ , we have

$$h_k(\mathbf{x}) := \int f_k(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \nearrow \int f(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} =: h(\mathbf{x}).$$

By assumption,  $h_k(\mathbf{x})$  is integrable as a function of  $\mathbf{x}$ ,  $f_k \in L(I)$ , and  $\iint f_k(\mathbf{x}, \mathbf{y}) \, d\mathbf{x}d\mathbf{y} = \int h_k(\mathbf{x}) \, d\mathbf{x}$ . But then by the MCT,

$$\iint f(\mathbf{x}, \mathbf{y}) \, d\mathbf{x}d\mathbf{y} = \int h(\mathbf{x}) \, d\mathbf{x}.$$

As  $f \in L(I)$ , we have  $h \in L(I_1)$ , implying that  $h$  is finite almost everywhere. Therefore, Fubini's Theorem holds for  $f$  also.

We now prove the second. For almost all  $x$ , the slice of  $g$  and  $h$  are both integrable, and since the slice of  $f$  is between, it is also integrable.

(Proof of 3???) Once integrated functions for  $g, f, h$  satisfy  $\int g \, dy \leq \int f \, dy \leq \int h \, dy$  for almost all  $x$ . Integrate with respect to  $x$

$$\left( \int g \, dy \right),$$

but then

$$\left( \int g \, dy \right) dx \leq \left( \int f \, dy \right) dx \leq \left( \int h \, dy \right) dx$$

But the left/right integrals are equal to  $\int_I g$  and  $\int_I h$ , respectively. But these are equal since  $g = h$  a.e.. Therefore,  $\left( \int f \, dy \right) dx = \int_I f$ .

Proof of 4: Slice function of  $\chi_I$  is either  $\equiv 0$  or  $\chi_{I_2}$ , so  $\int \chi_I = |I_2|\chi_{I_1}$ . Then  $\int \chi_I \, dy \, dx = |I_2||I_1| = |I| = \int \chi_I$ .  $\square$

We are now in a position to prove Fubini's Theorem.

**Theorem 4.1** (Fubini's Theorem). *Let  $I_1 \subseteq \mathbb{R}^n$ ,  $I_2 \subseteq \mathbb{R}^m$  be intervals, and  $I := I_1 \times I_2$ . Suppose that  $f \in L^1(I)$ . Then*

- (i) *for almost every  $\mathbf{x} \in I_1$ , the 'slice' function  $\mathbf{y} \mapsto f(\mathbf{x}, \mathbf{y})$  is measurable and integrable on  $I_2$ .*
- (ii) *the 'once-integrated' function  $\mathbf{x} \mapsto \int_{I_2} f(\mathbf{x}, \mathbf{y}) \, d\mathbf{y}$  is measurable and integrable on  $I_1$ , and its value is  $\int_I f$ , i.e.*

$$\iint_I f(\mathbf{x}, \mathbf{y}) \, d\mathbf{x}d\mathbf{y} = \int_{I_1} \left( \int_{I_2} f(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \right) d\mathbf{x}.$$

*Proof.* We establish  $\mathcal{F} = L^1$ . We show this for  $G_\delta$  sets and sets of measure 0, which will be enough.

a. First, we show that  $X_I \in \mathcal{F}$  for any closed bounded interval. This does not necessary worked for closed. Indeed, open sets are union of closed sets.



Consider the sequence of characteristic functions on the boxes in the figure, say  $\chi_{I_k} \nearrow \chi_I$ . Now for a set which is not open or closed, squeeze this between an open set containing and a closed set inside it.

b. Now  $\chi_G \in \mathcal{F}$  if  $G$  is open and  $|G| < \infty$ . Tile  $G$  by half-open cubes  $Q_j$ ,  $\chi_G = \sum \chi_{Q_j}$ . Monotone limit of partials sums of 1 and 2 bullets above.

c.  $\chi_H \in \mathcal{F}$  if  $H$  is a  $G_\delta$  set with  $|H| < \infty$ . Indeed, there exists  $G$  open,  $H \subset G$ , and  $|G| < |H| + \infty$  so if  $H = \bigcap_{k=1}^\infty G_k$ ,  $G_k$ , with  $H = (G \cap G_1) \cap (G \cap G_1 \cap G_2) \cap (G \cap G_1 \cap G_2 \cap G_3) \cap \dots$  so it is a nested intersection of open sets of finite measure. Then use second bullet above.

d. Now  $\chi_Z \in \mathcal{F}$  if  $|Z| = 0$ . There exists a  $G_\delta$  set  $H$  such that  $Z \subset H$  and  $|H| = 0$ . Now  $0 \leq \chi_Z \leq \chi_H$  and  $\chi_H = 0$  a.e., then by squeeze (bullet 3) have the conclusion.

e. We now show  $\chi_X \in \mathcal{F}$  if  $|E| < \infty$ , any set of finite measure. Write  $E = H \setminus Z$ .  $H$  is  $G_\delta$ ,  $|Z| = 0$ . Then  $\chi_E = \chi_H \setminus \chi_Z$  and use linearity.

f. integrable simple functions  $\in \mathcal{F}$ . Then nonnegative integrable in  $\mathcal{F}$ . Then integrable in  $\mathcal{F}$  since  $f = f^+ - f^-$ .  $\square$

To use Fubini, we need  $f \in L^1$ . A lot of times, this is what we are trying to show! Fubini cannot be used to prove  $f \in L^1(I_1 \times I_2)$ . Take  $f(x, y) = \frac{1}{\sqrt{xy}}$  on  $I := [0, 1] \times [0, 1]$ . Cannot say  $\int_I f = \int_0^1 (\int_0^1 \frac{1}{\sqrt{xy}} dx) dy$  but need  $f$  integrable, which is what you are trying to show. For these purposes, we use Tonelli's Theorem.

**Theorem 4.2.** Same as Fubini Theorem,  $f \geq 0$ , measurable. Suppose  $f \geq 0$ , measurable on  $I_1 \times I_2$ . Then

a): For a.e.  $x \in I_1$ , the slice function  $y \mapsto f(x, y)$  is measurable on  $I_2$ . b): the once-integrated function  $x \mapsto \int_{I_2} f(x, y) dy$  is measurable on  $I_1$  and its integral is  $\int_I f$  (could be  $\infty$ ).

In the example above,  $\int_0^1 \frac{1}{\sqrt{xy}} dy = \frac{2}{\sqrt{x}}$  is measurable and  $\int_0^1 \frac{2}{\sqrt{x}} dx = 4$  so  $\int_I \frac{1}{\sqrt{xy}} = 4$ .

*Proof.* Let  $f_k = \min(f, k)\chi_{[-k, k]^n}$  integrable and  $f_k \nearrow f$ . Then Fubini applies. Spelled out.  $\int_{I_1 \times I_2} f_k = \int_{I_1} \left( \int_{I_2} f_k(x, y) dy \right) dx$ . The inner integral converges a.e.  $y$  to  $\int_{I_2} f(x, y) dy$  by MCT. Then left MCT  $\int_I f$ . And then MCT for outer integral. done.  $\square$

## 4.2 Applications of Fubini's Theorem

Let  $f: E \rightarrow [0, \infty]$  be measurable.  $\omega_f(y) = |\{x \in E: f(x) > y\}|$ . Then  $\int_E f = \int_0^\infty \omega_f(y) dy$ . *Proof.* Recall  $R(f, E) = \{(x, y) \in \mathbb{R}^{n+1}: x \in E, 0 \leq y \leq f(x)\}$ .  $\Gamma(f, E) = \{(x, y) \in \mathbb{R}^{n+1}: y = f(x)\}$ . Now  $|\Gamma| = 0$ ,  $g = \chi_{R \setminus \Gamma}$  so  $g(x, y) = 1$  if  $0 \leq y < f(x)$ , 0 otherwise.

$\int(\int g(x,y) dy) dx = \int(\int g(x,y) dx) dy$ . and  $\int f(x)\chi_E dx = \int \omega_f(y) dy$  so  $\int_E f = \int_0^\infty \omega_f(y) dy$ . This last integral is even Riemann integrable since it is monotone.

Generalizations:  $f$  measurable,  $\int_E |f|^p = p \int_0^\infty y^{p-1} \omega_{|f|}(y) dy$ .  $p > 0$ .

We know  $\int_E |f|^p = \int_0^\infty \omega_{|f|^p}(y) dy$ . Let  $y = z^p$ . Then this is  $p \int_0^\infty z^{p-1} \omega_{|f|^p}(z^p) dz = p \int_0^\infty z^{p-1} \omega_{|f|}(z) dz$  and  $\{|f|^p > z^p\} = \{|f| > z\}$ .  $\square$

Ex:  $f(x,y) = \frac{\sin xy}{(x^2 + y^2)^{3/2}} \in L^1(E)$ .  $E = \{(x,y) : x,y > 0\}$ .

Idea: estimate above by nonnegative function  $g$  and prove  $g \in L^1(E)$  by Tonelli. i.e.  $|f| \geq g$ .  $|f| \geq \frac{1}{(x^2 + y^2)^{3/2}}$  for sure but problems around origin but good at infinity. Have  $|f| \leq \frac{xy}{(x^2 + y^2)^{3/2}}$  good near 0. So use min of the two functions. Better: Use first when  $x^2 + y^2 \leq 1$ :  $g_1 = \frac{1}{(x^2 + y^2)^{3/2}} \chi_{\{x^2 + y^2 \geq 1\}}$  and  $\omega_{g_1}(\alpha) = \pi/(4\alpha^{2/3}) - \pi/4$  if  $\alpha < 1$  and 0 otherwise. Note that  $1/r^3 \cdot \chi_{r \geq 1}$  and  $\alpha < 1$   $1/r^3 \alpha$  and  $r = 1/\alpha^{1/3}$  and  $1/4$  if circle roots  $1/\alpha^{1/3}$ .

Slicing a subset of  $\mathbb{R}^{n+m}$ . Given a measurable set  $E \subset \mathbb{R}^{n+m}$ , consider  $E_x = \{y \in \mathbb{R}^m : (x,y) \in E\}$ .

- $E_x$  is measurable for almost all  $x$ .
- If  $|E| < \infty$ , then  $|E_x| < \infty$  for almost every  $x$ .
- If  $|E| = 0$ , then  $|E_x| = 0$  for almost every  $x$ .

All these things follow easily from applying Tonelli to  $\chi_E$  since almost every slice function is measurable. Slice fo  $\chi_E$  is  $\chi_{E_x}$  and  $\int_{\mathbb{R}^{n+m}} \chi_E < \infty$  then  $\int_{\mathbb{R}^n} |E_x| dx < \infty$ .

**Example 4.2.** Assuming the Continuum Hypothesis, there exists  $E \subset [0,1]^2$  such that every vertical slice  $E_x$  is countable and every horizontal slice  $E_y$  is co-countable, i.e.  $[0,1] \setminus E_y$  countable set. Using Tonelli

$$\int \left( \int \chi_E dy \right) dx = 0$$

since every vertical slice is countable. The other way

$$\int \left( \int \chi_E dx \right) dy = 1$$

This means  $E$  is not measurable in  $\mathbb{R}^2$  (otherwise these would be equal). Then let  $E = \{(x,y) : y \prec x\}$ .

**Products:** If  $E_1 \subset \mathbb{R}^n, E_2 \subset \mathbb{R}^m$ , are measurable, then  $E_1 \times E_2$  is measurable in  $\mathbb{R}^{n+m}$  and  $|E_1 \times E_2| = |E_1| |E_2|$ , taking  $0 \cdot \infty = 0$ . Assume  $E_1 \times E_2$  is measurable, then

$$|E_1 \times E_2| = \int \left( \int \chi_{E_1 \times E_2}(x, y) dy \right) dx = \int |E_2| \chi_{E_1}(x) dx.$$

It remains to prove that  $E_1 \times E_2$  is measurable. It suffices to prove if when  $|E_1| < \infty$  and  $|E_2| < \infty$ . In the general case,  $E_1 = \bigcup_{k=1}^{\infty} E_{1,k}$  and  $E_2 = \bigcup_{k=1}^{\infty} E_{2,k}$ , then  $E_1 \times E_2 = \bigcup_{j,k} E_{1,k} \times E_{2,j}$ .

There exists  $G_1 \supset E_1, G_2 \supset E_2$  such that  $|G_1 \setminus E_1| < \epsilon, |G_2 \setminus E_2| < \epsilon$ . Now  $G_1 \times G_2$  is open and  $E_1 \times E_2 \subset G_1 \times G_2$ . Now  $(G_1 \times G_2) \setminus (E_1 \times E_2) \subset [(G_1 \setminus E_1) \times G_2] \cup [G_1 \times (G_2 \setminus E_2)]$ . Both  $G_1 \setminus E_1$  and  $G_2 \setminus E_2$  have measure at most  $\epsilon$ . Now  $G_1, G_2$  by cumes  $I_k, Q_j$  with summed measure at most  $\epsilon$ . Then  $|I_k \times Q_j| = |I_k| |Q_j|$  so  $|(G_1 \setminus E_1) \times G_2| < \sum_{k,j} |I_k| |Q_j| < \epsilon |G_2|$ . Then just choose  $\epsilon$  sufficiently small.

**Convolution:**  $(f * g)(x) = \int_{\mathbb{R}^n} f(x-t)g(t) dt$ , where  $f, g$  are defined on  $\mathbb{R}^n$  are measurable. Now  $f * g = g * f$  by change of variable  $s = x - t$  so  $\int f(x-t)g(t) dt = \int f(s)g(x-s) ds$ , where  $s, t \in \mathbb{R}^n$ . In the general case,  $h = h^+ - h^-$ .

**Claim**  $\int h(x-t) dt = \int h(t) dt$ . **Proof:** if  $h \geq 0$ , then are measures of sets  $\{(t, y) : 0 \leq y \leq h(x-t)\}$  and  $\{(t, y) : 0 \leq y \leq h(t)\}$ . The first is obtained from the other by reflection  $t \mapsto -t$  and translation (add  $x$ ). These do not change measure or measurability.

**Example 4.3.** Ex: Convolution: Argument of  $\chi_{[a,b]} * \chi_{[c,d]}$  is only nonzero on  $\{x : a \leq x-t \leq b, c \leq t \leq d\}$  so  $\{x : x-b \leq t \leq x-a, c \leq t \leq d\}$ . so on  $\max(x-b, c) \leq t \leq \min(x-a, d)$ .

$$\chi_{[a,b]} * \chi_{[c,d]} = \int_{\mathbb{R}} \chi_{[a,b]}(x-t) \chi_{[c,d]}(t) dt = (\min(x-a, d) - \max(x-b, c))^+$$

So  $\chi_{[-1,1]} * \chi_{[-1,1]} = (\min(x+1, 1) - \max(x-1, -1))^+ = \dots$  grossly complicated (or just wrong). So better way  $|([-b, -a] + x) \cap [c, d]|$  so  $|([-1, 1] + x) \cap [-1, 1]|$ . Then get

**Theorem 4.3.** If  $f, g \in L^1(\mathbb{R}^n)$ , then  $f * g \in L^1(\mathbb{R}^n)$  and  $\int |f * g| \leq \int |f| \int |g|$ .

*Proof.* We know  $|f * g| \leq |f| * |g|$  by the triangle inequality. We can work with  $|f|$  and  $|g|$  so assume  $f, g \geq 0$ .  $\int (\int f(x-t)g(t) dt) dx$  by tonelli this is  $\int (\int f(x-t)g(t) dx) dt = \int g(t) (\int f(x-t) dx) dt$  where the inner integral is just  $\int f$  by translation. But then this final integral is  $\int f \cdot \int g$ .

But why  $f(x-t)g(t)$  measurable on  $\mathbb{R}^{2n}$ ? (need this for tonelli). Observe  $f(x)$  and  $g(t)$  are measurable on  $\mathbb{R}^{2n}$  since the preimages of  $> \text{something}$  is just  $\{(x, t) : f > a\} = \{x : f(x) > a\} \times \mathbb{R}$  since it does not depend on  $t$ . Then this is measurable. Use the linear transformation  $\phi : (x, t) \rightarrow (x-t, x+t)$  invertible linear, hence lipschitz. Use then HW 3.5: composition  $f \circ \phi$  is measurable,  $\phi^{-1}$  Lipschitz. Hence  $f(x-t)$  is measurable on  $\mathbb{R}^{2n}$ .

If  $f, g$  are probability density functions of  $X$  and  $Y$ , then  $f * g$  is the PDF of  $X + Y$ , assuming  $X, Y$  are independent.

### 4.3 Marcinkiewicz Integrals

In Analysis, one often has  $f * g$ , where one of the functions is not in  $L^1$ . These are so called “singular integrals.” The simplest example is the Hilbert transform

$$\int_{\mathbb{R}} \frac{f(x-t)}{t} dt$$

Let  $K$  be a compact set in  $\mathbb{R}^n$ , and  $\delta(y) := \text{dist}(y, K) = \min\{|x - y| : x \in K\}$ .

**Theorem 4.4.** *If  $B \supset K$  is a bounded open set, then for all  $\lambda > 0$ .*

$$\int \frac{\delta(y)^\lambda}{|x - y|^{n+\lambda}} dy < \infty$$

for a.e.  $x \in K$ .

*Proof.* It suffices to show  $\int_K \mu_\lambda(x) dx < \infty$ .

$$\int_K \mu_\lambda(x) dx = \int_K \int_{B \setminus K} \frac{\delta(y)^\lambda}{|x - y|^{n+\lambda}} dy dx = \int_{B \setminus K} \int_K \frac{\delta(y)^\lambda}{|x - y|^{n+\lambda}} dx dy.$$

But now observe  $|x - y| \geq \delta(y)$ . But then

$$\int_{B \setminus K} \int_K \frac{\delta(y)^\lambda}{|x - y|^{n+\lambda}} dx dy \leq \int_{B \setminus K} \left( \int_{|x-y| \geq \delta(y)} \frac{\delta(y)^\lambda}{|x - y|^{n+\lambda}} \right) dy \leq \int_{B \setminus K} C dx = C|B \setminus K| < \infty.$$

Using  $\int_{|x-y| \geq \delta} \frac{1}{|x - y|^{n+\lambda}} dx \leq \frac{C}{\delta^\lambda}$ . To prove this:

Recall  $\int |f|^p = p \int_0^\infty \alpha^{p-1} |\{|f| > \alpha\}| d\alpha = *$ . Use  $P = n + \lambda$ ,  $f = \frac{1}{|x - y|}$ .

$$|\{|x - y| \geq \delta : \frac{1}{|x - y|} > \alpha\}| = |B(y, 1/\alpha) \setminus B(y, \delta)| = C((1/\alpha)^n - \delta^n)^+$$

which is 0 if  $\alpha > 1/\delta$ . For  $*$ , note

$$* = (n + \lambda)^{1/\delta} \alpha^{n+\lambda-1} C(1/\alpha^n - \delta^n) d\alpha \leq C' \int_0^{1/\delta} \alpha^{\lambda-1} d\alpha = C'' \frac{1}{\delta^\lambda}$$

$B$  is needed as the integral is  $\sim \frac{1}{|y|^n}$  as  $y \rightarrow \infty$ . We could have integrate over  $B \setminus K$  since  $\delta \equiv 0$  on  $K$ . Also, the  $\delta$  is important for convergence. The almost always is also unavoidable: take  $K = \{0\}$  in  $\mathbb{R}$ . Then  $\delta(y) = |y|$ . Then  $\int_B \frac{|y|^\lambda}{|x - y|^{n+\lambda}} dy$  so at  $x = 0$ , we get  $\int_B \frac{1}{|y|^n} = \infty$ . Generally,  $\mu_\lambda(x) = \infty$  if  $x$  is an isolated point of  $K$ . If  $x$  is an interior point fo  $K$ ,  $\mu_\lambda(x) < \infty$  as  $\delta = 0$  in a neighborhood of  $x$ .

Question: in last part of proof, can we let  $\lambda = 0$ ? Specifically, is it true that  $\int_{B \setminus K} \frac{1}{|x - y|^n} dy < \infty$  for a.e.  $x \in K$ ? Counterexample?  $K$  is fat Cantor set?

## 5 $L^p$ Spaces

### 5.1 Introduction

Recall that a metric space is called complete if every Cauchy sequence converges. While most metric spaces are not complete, every metric space can be embedded into a complete space; that is, given a metric space  $X$ , there exists a space  $\tilde{X}$  such that there is an injection  $\iota : X \hookrightarrow \tilde{X}$ . The space  $\tilde{X}$  can be taken to be the set of equivalence classes of Cauchy sequences in  $X$ . Then  $X$  naturally embeds into  $\tilde{X}$  as the constant sequences. In fact using this construction,  $X$  is a dense subset of  $\tilde{X}$ .

#### Example 5.1.

- (i) The metric spaces  $\mathbb{R}^n$  and  $\mathbb{C}^n$  are complete metric spaces. This essentially follows from the Bolzano-Weierstrass Theorem.
- (ii) The space  $C[a, b]$  of continuous real-valued functions on  $[a, b]$  is a Banach space, and hence is complete.
- (iii)  $\mathbb{Q}$  is not a complete metric space. For example, it is routine to verify the sequence  $\{x_n\}$ , where  $x_0 = 1$  and  $x_{n+1} = \frac{x_n^2 + 2}{2x_n}$ , is Cauchy and that (using the Monotone Convergence Theorem)  $\lim_{n \rightarrow \infty} x_n = \sqrt{2} \notin \mathbb{Q}$ .
- (iv) The space of continuous functions on  $[0, 1]$  with metric  $d(f, g) = \int_0^1 |f - g| dx$  is not complete. [Here the integral should be taken as the Riemann integral.] To see this, take any sequence of functions converging to the Dirichlet function. More concretely, it is routine to verify that the sequence of functions  $\{f_n\}$ , where

$$f_n(x) := \begin{cases} 1, & x = 0 \\ 1 - nx, & 0 < x < \frac{1}{n} \\ 0, & \frac{1}{n} \leq x \end{cases}$$

is Cauchy. However,  $\lim_{n \rightarrow \infty} f_n(x)$  is the zero function  $[0, 1]$  except at  $x = 0$ , where the function is 1, which is not continuous. One can extend this example by considering the space of Riemann integrable functions on  $[0, 1]$  with the same metric  $d$ . Consider the sequence of compact sets  $\{F_k\}$  used in constructing the Fat Cantor Set  $F$ , c.f. Lemma 1.3. We know that  $\chi_{F_k}$  is Riemann integrable for all  $k$  and that  $\int |\chi_{F_k} - \chi_{F_j}| < \epsilon$  for all  $k, j \geq N$ , where  $N$  is taken sufficiently large. However, the function  $\chi_F$  is not Riemann integrable. Therefore,  $\lim \chi_{F_k}$  does not exist with respect to the given metric  $d$ .

◁

In any introductory course in Functional Analysis, a great deal of time is spent analyzing Banach spaces, which are a type of topological vector space.<sup>2</sup> These play roles in Analysis and the rest of Pure Mathematics, Statistics, Probability, Physics, Finance, Engineering, etc.. We want to build a theory of complete spaces based on integration. The spaces we will build, the “Lebesgue spaces”  $L^p$ , in doing so will serve as important examples in Functional Analysis.

## 5.2 Definition of $L^p$ and Examples

Suppose that  $E \subseteq \mathbb{C}^n$  or  $E \subseteq \mathbb{R}^n$  and let  $f : E \rightarrow \mathbb{C}$  or  $f : E \rightarrow \mathbb{R}$ . Then  $f = (f_1, \dots, f_n)$  is measurable if and only if each of the functions  $f_1, \dots, f_n$  are measurable. Equivalently, the sets  $\{a_j < f_j < b_j\} = f^{-1}(\prod_{j=1}^m (a_j, b_j))$  are measurable for all  $j$  and  $a_j, b_j \in \mathbb{R}$  if and only if for every countable union of open boxes  $U$ ,  $f^{-1}(U)$  is measurable.

**Definition** ( $L^p, 0 < p < \infty$ ). Let  $E \subseteq \mathbb{R}^n$  be measurable, and  $0 < p < \infty$ . Then we define

$$L^p(E) := \left\{ f : \int_E |f|^p < \infty \right\}$$

Again, we allow  $f$  to be complex-valued. In this case, if we write  $f = f_1 + if_2$  for measurable real-valued functions  $f_1, f_2$ , we have  $|f|^2 = f_1^2 + f_2^2$ . But then  $|f_1|, |f_2| \leq |f| \leq |f_1| + |f_2|$ . It follows then that  $f \in L^p(E)$  if and only if  $f_1, f_2 \in L^p(E)$ . We write

$$|f|_{p,E} := \left( \int_E |f|^p \right)^{1/p}, \quad 0 < p < \infty.$$

Therefore,  $L^p(E)$  is the set of equivalence classes of measurable functions  $f$  for which  $|f|_{p,E} < \infty$ . Finally, whenever the set  $E$  is clear from context, we will write  $L^p$  instead of  $L^p(E)$  and  $|f|_p$  instead of  $|f|_{p,E}$ . This this notation, we have  $L^1 = L$ .

**Remark.** Strictly speaking,  $L^p$  is a set of equivalence classes of functions  $f$  such that  $\int_E |f|^p < \infty$ ; that is, we say two functions  $f, g$  define the same equivalence class in  $L^p(E)$  if and only if  $f = g$  a.e. on  $E$ .

For  $c \in \mathbb{R}$  or  $c \in \mathbb{C}$  and  $f \in L^p(E)$ , it is clear that  $c \cdot f \in L^p(E)$ ; that is,  $c[f] = [cf]$ . Furthermore, if  $f, g \in L^p(E)$ , then

$$|f + g|^p \leq (|f| + |g|)^p \leq [2 \max(|f|, |g|)]^p = 2^p \max(|f|^p, |g|^p) \leq 2^p(|f|^p + |g|^p)$$

so that  $f + g \in L^p(E)$ . This shows that  $L^p$  is an  $\mathbb{R}$ , respectively  $\mathbb{C}$ , vector space. To define  $L^p(E)$  in the case where  $p = \infty$ , we will first need a definition.

<sup>2</sup>To be precise, a Banach space is a complete normed vector space.

**Definition** (Essential Supremum). Let  $f$  be a real or complex valued function on a measurable set  $E$  of positive measure. Then we define  $\sup_E \text{ess } f$  as follows: if  $|\{x \in E: f(x) > \alpha\}| > 0$  for all real  $\alpha$ , then we define  $\sup_E \text{ess } f = \infty$  and otherwise define  $\sup_E \text{ess } f := \inf\{\alpha: |\{x \in E: f(x) > \alpha\}| = 0\}$ .

The essential supremum is the generalization of the maximums to measurable functions; the difference being the fact that values of a function on sets of measure zero do not affect the essential supremum. As such, we say that  $f$  is essentially bounded (or simply bounded) if  $\sup_E \text{ess } |f| < \infty$ .

**Example 5.2.**

- (i) Take  $E = \mathbb{R}$  and consider  $f = \chi_{\mathbb{Q}}$ . Then  $\sup_{\mathbb{R}} \text{ess } f = 0$  since  $|\mathbb{Q}| = 0$  and  $f \equiv 0$  on  $\mathbb{R} \setminus \mathbb{Q}$ .
- (ii) Take  $E := [0, 1]$ , where  $F$  is the fat Cantor set, c.f. Lemma 1.3, and  $f = \chi_F$ . Then  $\sup_E \text{ess } \chi_F = 1$  by similar reasoning to (i).
- (iii) Let  $E = \mathbb{R}$  and define

$$f(x) = \begin{cases} -10, & x = -1 \\ 10, & x = 1 \\ \pi, & \text{otherwise} \end{cases}$$

Then  $\sup_E \text{ess } f = \pi$ . Notice that the largest value that  $f$  takes is 10 but it takes this value on a set of measure zero.

- (iv) Let  $E = \mathbb{R}$  and define

$$f(x) = \begin{cases} x^3 + x + 1, & x \in \mathbb{Q} \\ \arctan x, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

As in (i), since  $|\mathbb{Q}| = 0$ , we have  $\sup_E \text{ess } f = \frac{\pi}{2}$ .

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**Definition** ( $L^\infty(E)$ ). Let  $E \subseteq \mathbb{R}^n$  be a measurable set of positive measure. Then we define

$$L^\infty(E) = \{\text{meas } f : E \rightarrow \mathbb{C} : \sup_E \text{ess } |f| < \infty\}$$

First, note we write  $|f|_\infty = |f|_{\infty, E} = \sup_E \text{ess } |f|$ . The fact that  $f \in L^\infty(E)$  is equivalent to the fact that there exists  $M \in \mathbb{R}$  such that  $|f| \leq M$  a.e.. Thus,  $L^\infty(E)$  is the set of functions which are essentially bounded on  $E$ .

**Example 5.3.**

(i) Let  $E = \mathbb{R}$  and consider the function  $f = \frac{1}{|x|}$ . The function  $f$  is a.e. finite on  $E$  but it is not bounded so that  $f \notin L^\infty$ .

(ii) Consider the function  $f(x) = \frac{1}{\sqrt{x}}$  on the set  $E = (0, 1)$ . Then  $f \in L^p$  if

$$\int_{(0,1)} \left( \frac{1}{\sqrt{x}} \right)^p < \infty.$$

But this occurs if and only if  $p/2 < 1$ . Therefore,  $f \in L^p$  if and only if  $p < 2$ .

(iii) Let  $E = [1, \infty)$  and define  $f = \frac{1}{\sqrt{x}}$ . Then  $f \in L^p$  if and only if

$$\int_{[1,\infty)} \frac{1}{x^{p/2}} dx.$$

But this happens if and only if  $p > 2$  or  $p = \infty$ . Therefore,  $f \in L^p$  if and only if  $p \in [2, \infty]$ .

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### 5.3 Relations between $L^p$ spaces

Not all the  $L^p$  spaces are ‘created equally’—there are ‘nicer’ and ‘uglier’  $L^p$  spaces. Ranked, the  $L^p$  spaces are as follows:

1. (Nicest)  $L^2$
2.  $L^p, 1 < p < \infty, p \neq 2$
3.  $L^1$
4.  $L^\infty$
5. (Worst)  $L^p, 0 < p < 1$ .

The reasoning for placing  $L^2$  first is essentially because it is probably the most familiar to the reader. The reasoning for this ordering will make itself apparent in this section. The simplest relationship is between  $L^\infty$  and  $L^p$  for  $p < \infty$ .



**Proposition 5.1.** *If  $|E| < \infty$ , then  $|f|_\infty = \lim_{p \rightarrow \infty} |f|_p$ .*

*Proof.* Let  $M = |f|_\infty$ . If  $M' < M$ , then the set  $A = \{x \in E: |f(x)| > M'\}$  must have positive measure. Furthermore,

$$|f|_p \geq \left( \int_A |f|^p \right)^{1/p} \geq M'|A|^{1/p}.$$

However,  $|A|^{1/p} \rightarrow 0$  as  $p \rightarrow \infty$ . This proves that  $\liminf_{p \rightarrow \infty} |f|_p \geq M'$ . Therefore,  $\liminf_{p \rightarrow \infty} |f|_p \geq M$ . But we also have

$$|f|_p \leq \left( \int_E M^p \right)^{1/p} = M|E|^{1/p}.$$

Therefore,  $\limsup_{p \rightarrow \infty} |f|_p \leq M$ . □

Note that the requirement that  $|E| < \infty$  is necessary: the constant function  $f(x) = c$ ,  $c \neq 0$ , is in  $L^\infty$  but is not in  $L^p$  for any  $0 < p < \infty$ . Now we show that the  $L^p$  spaces are nested, and in a sense, form an interval.

**Proposition 5.2.** *If  $|E| < \infty$ , and  $f \in L^{p_2}$  then  $f \in L^{p_1}$  for all  $0 < p_1 < p_2$ . Furthermore, if  $f \in L^{p_1}$  and  $f \in L^{p_2}$ , then  $f \in L^p$  for all  $p_1 < p < p_2$ . That is,  $\{p: f \in L^p\}$  is an interval starting from 0.*

*Proof.* Let  $f \in L^{p_2}$ . Write  $E = E_1 \cup E_2$ , where  $E_1 := \{x: |f| \leq 1\}$  and  $E_2 := \{x: |f| > 1\}$ . Then we have

$$\int_E |f|^p = \int_{E_1} |f|^p + \int_{E_2} |f|^p, \quad 0 < p < \infty.$$

Now we have  $\int_{E_1} |f|^p \leq |E_1|$ . The integral  $\int_{E_2} |f|^p$  is an increasing function in  $p$ . As  $f \in L^{p_2}$  and  $p_1 < \infty$ , then  $f \in L^{p_1}$ . If  $p_2 = \infty$ , then  $f$  is a bounded function on a set of finite measure so that  $f \in L^{p_1}$ . For the second statement, suppose that  $p_2 < \infty$ . We have

$$\begin{aligned} \int_{E_1} |f|^p &\leq \int_{E_1} |f|^{p_1} < \infty \\ \int_{E_2} |f|^p &\leq \int_{E_2} |f|^{p_2} < \infty \end{aligned}$$

The result then follows. Now suppose that  $p_2 = \infty$ , and let  $M := \sup_E \text{ess } |f|$ . Proving  $f \in L^p$  except  $p_2 = \infty$  is a special case. let  $M = \sup_E \text{ess } |f|$ . The integral  $\int_{E_1} |f|^p = 0$ . It suffices to show the integral  $\int_{E_2} |f|^p$  is finite. We have

$$\int_{E_2} |f|^p \leq \int_{E_2} M^p \leq M^p |E_2| \leq M^p |\{ |f|^{p_2} > 1 \}| \leq M^p \int_{|f|^{p_2} > 1} |f|^{p_1} < \infty$$

□

Again, the requirement that  $|E| < \infty$  is necessary. For example,  $f(x) = x^{-1/p_1} \in L^{p_2}([1, \infty))$  for  $p_1 < p_2$  but  $f(x) \notin L^{p_1}([1, \infty))$ . Moreover, any nonzero constant is in  $L^\infty$  but not in  $L^{p_1}(E)$  if  $|E| = \infty$  and  $p_1 < \infty$ . A function may belong to  $L^{p_1}$  for  $p_1 < p_2$  and not belong to  $L^{p_2}$ . For example, if  $p_2 < \infty$  then  $f(x) = x^{-1/p_2} \in L^{p_1}((0, 1))$ , where  $p_1 < p_2$ , but  $f(x) \notin L^{p_2}((0, 1))$ . Furthermore,  $f(x) = -\log x \in L^{p_1}((0, 1))$  for  $p_1 < \infty$  but  $f(x) \notin L^\infty((0, 1))$ . For any set  $E$  (even with infinite measure), if  $f \in L^{p_1}(E)$ , then  $f \in L^{p_2}(E)$ , where  $p_1 < p_2$ . Finally, there are functions which are not in  $L^p$  for any  $p$ . For example, if  $f \equiv 1$  on  $\mathbb{R}$ , then  $f \notin L^p$  for all  $p$  except for  $p = \infty$ . If  $f(x) = x$  on  $\mathbb{R}$ , then  $f \notin L^p$  for all  $p \in (0, \infty]$ . One can find examples on finite intervals also. As a final remark for this section, if  $|E| < \infty$ , we have

$$\sup_E \text{ess } |f| = \lim_{p \rightarrow \infty} \left( \int_E |f|^p \right)^{1/p}$$

We are then able to write

$$\|f\|_{p,E} = \begin{cases} \left( \int_E |f|^p \right)^{1/p}, & 0 < p < \infty \\ \sup_E \text{ess } |f|, & 0 < p < 1 \end{cases}$$

## 5.4 Hölder and Minkowski Inequalities

In this section, we prove two important and powerful inequalities: Hölder's and Minkowski's Inequality. Hölder's inequality is about controlling products while Minkowski is about controlling sums. These are among the most useful tools in Analysis. We begin with a more basic inequality.

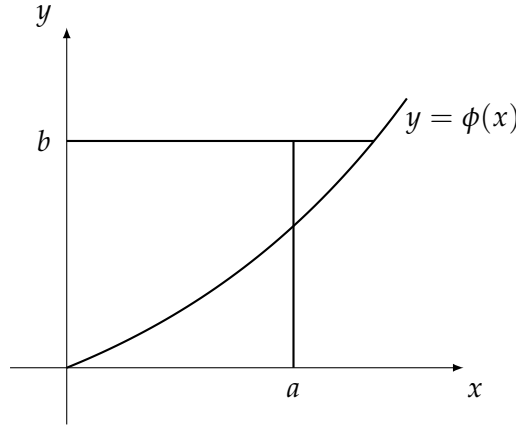
**Theorem 5.1** (Young's Inequality). *Let  $y = \phi(x)$  be continuous, real-valued, and strictly increasing for  $x \geq 0$  with  $\phi(0) = 0$ . If  $x = \psi(y)$  is the inverse of  $\phi$ , then for  $a, b > 0$ ,*

$$ab \leq \int_0^a \phi(x) \, dx + \int_0^b \psi(y) \, dy.$$

*with equality holds if and only if  $b = \phi(a)$ .*

*Proof.* The proof becomes immediate if we interpret each term as an area, and that the graph of  $\phi$  may also serve as the graph of  $\psi$  by interchange of the  $x$  and  $y$  axes. Equality

holds if and only if  $(a, b)$  lies on the graph of  $\phi$ .



□

Now taking  $\phi(x) = x^\alpha$  for  $\alpha > 0$ , the  $\psi(y) = y^{1/\alpha}$ , and Young's Inequality is

$$ab \leq \frac{a^{1+\alpha}}{1+\alpha} + \frac{b^{1+1/\alpha}}{1+1/\alpha},$$

Setting  $p = \alpha + 1$  and  $p' = 1 + 1/\alpha$ , we have

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'},$$

where  $a, b \geq 0$ ,  $1 < p < \infty$ , and  $\frac{1}{p} + \frac{1}{p'} = 1$ .

**Definition** (Conjugate Exponents). Numbers  $p, p' > 0$  with  $1/p + 1/p' = 1$  are called conjugate exponents. We make the convention that  $1/\infty = 0$  here.

With this terminology, the numbers 1 and  $\infty$  are conjugate. Other examples are the pairs  $\{1.1, 11\}$ ,  $\{1.5, 3\}$ , and  $\{2, 2\}$ , which is the only number which is self-conjugate.

Axioms of a norm:  $\|f\| \geq 0$ ,  $\|f\| = 0$  if and only if  $f = 0$ : Positivity  $\|\lambda f\| = |\lambda| \|f\|$  for  $\lambda \in \mathbb{C}$  or  $\mathbb{R}$  homogeneity  $\|f + g\| \leq \|f\| + \|g\|$  subadditivity (triangle inequality)

So  $\|f\|_p$  is a norm when  $1 \leq p \leq \infty$  (and not if  $0 < p < 1$ ).

For  $p = 1$  and  $p = \infty$ , this is easy:  $p = 1$ :  $\int |f + g| \leq \int (|f| + |g|) = \int |f| + \int |g|$ . For  $p = \infty$ ,  $|f| \leq M$  a.e.  $|g| \leq N$  a.e., then  $|f + g| \leq M + N$  a.e.. For  $1 < p < \infty$ , we need Hölder's inequality.

**Theorem 5.2** (Hölder's Inequality). *If  $1 \leq p \leq \infty$  and  $1/p + 1/p' = 1$ , then  $\|fg\|_1 \leq \|f\|_p \|g\|_{p'}$ , i.e.*

$$\begin{aligned} \int_E |fg| &\leq \left( \int_E |f|^p \right)^{1/p} \left( \int_E |g|^{p'} \right)^{1/p'}, \quad 1 < p < \infty \\ \int_E |fg| &\leq (\sup_E \text{ess } |f|) \int_E |g|. \end{aligned}$$

*Proof.* The last inequality is the case when  $p = \infty$ , and is routine. Suppose then that  $1 < p < \infty$ . If  $\|f\|_p = \|g\|_{p'} = 1$ , then

$$\int_E |fg| \leq \int_E \left( \frac{|f|^p}{p} + \frac{|g|^{p'}}{p'} \right) = \frac{\|f\|_p^p}{p} + \frac{\|g\|_{p'}^{p'}}{p'} = \frac{1}{p} + \frac{1}{p'} = 1 = \|f\|_p \|g\|_{p'}.$$

If  $\|f\|_p$  or  $\|g\|_{p'}$  is zero, then  $fg$  is zero a.e. in  $E$  and the result follows. We may then assume that neither  $\|f\|_p$  nor  $\|g\|_{p'}$  is zero. Setting

$$\tilde{f} = \frac{f}{\|f\|_p} \quad \text{and} \quad \tilde{g} = \frac{g}{\|g\|_{p'}}$$

we have  $\|\tilde{f}\|_p = \|\tilde{g}\|_{p'} = 1$ . But then in this case, the result follows by the work above.  $\square$

Notice outer exponents must always add to 1 in Hölder's Inequality. The case where  $p = 2$  is a classical result:

**Corollary 5.1** (Cauchy-Schwarz Inequality).

$$\int_E |fg| \leq \left( \int_E |f|^2 \right)^{1/2} \left( \int_E |g|^2 \right)^{1/2}$$

*Proof.* This follows immediately from Hölder's Inequality using the self conjugate exponents  $p = p' = 2$ .  $\square$

**Example 5.4.** Suppose  $f \in L^6([1, \infty])$ . Want to estimate the integral  $\int_1^\infty \frac{f(x)^2}{x} dx$ . Writing

$$\int_1^\infty \frac{f(x)^2}{x} dx = \int_1^\infty f(x)^2 \cdot \frac{1}{x} dx$$

We want to find a power  $p'$  such that the integral  $1/x^{p'}$  converges. However, we still want to make use of the given information. We have  $f^2$  and  $f \in L^6$ . Writing  $f^6 = (f^2)^3$ , we see

that  $p' = 3/2$  will work. So we have

$$\begin{aligned} \int_1^\infty \frac{f(x)^2}{x} dx &= \int f(x)^2 \cdot \frac{1}{x} dx \leq \left( \int_1^\infty (f^2)^3 \right)^{1/3} \left( \int_1^\infty \frac{1}{x^{3/2}} \right)^{2/3} \\ &= \left( \left( \int_1^\infty f^6 \right)^{1/6} \right)^2 \cdot 2^{2/3} \\ &= 2^{2/3} \|f\|_6^2. \end{aligned}$$

◁

There is a sort of partial converse to Hölder's Inequality, which gives a dual form of the  $L^p$  norm.

**Theorem 5.3.** *Let  $f$  be real-valued and measurable on  $E$ , and  $1 \leq p \leq \infty$  with conjugate exponent pair  $p'$ . Then*

$$\|f\|_p = \sup \int_E fg,$$

where the supremum is taken over all real-valued  $g$  with  $\|g\|_{p'} \leq 1$  and  $\int_E fg$  exists; that is,

$$\|f\|_p = \sup \left\{ \int_E fg : \|g\|_{p'} \leq 1, g \text{ real} \right\} = \sup \left\{ \frac{\int_E fg}{\|g\|_{p'}} : g \in L^{p'}, g \text{ real} \right\}.$$

*Proof.* The fact that the left side is larger than the right follows from Hölder's Inequality. It remains to prove the reverse inequality. If  $\|f\|_p = 0$ , then  $f \equiv 0$  a.e. on  $E$  and the result follows. If  $0 < \|f\|_p < \infty$ , by dividing by  $\|f\|_p$ , we may assume that  $\|f\|_p = 1$ . One routinely verifies that  $\|g\|_{p'} = 1$  and  $\int_E fg = 1$ , which completes this case.

Now suppose that  $\|f\|_p = \infty$ . Define  $f_k$  on  $E$  via

$$f_k(\mathbf{x}) = \begin{cases} \min(f(\mathbf{x}), k), & |\mathbf{x}| \leq k \\ 0, & |\mathbf{x}| > k \end{cases}$$

Now for all  $k$ ,  $f_k \in L^p$  and  $\|f_k\|_p \rightarrow \|f\|_p = \infty$ . By the work above, we have  $\|f_k\|_p = \int_E f_k g_k$  for some  $g_k \geq 0$  with  $\|g_k\|_{p'} = 1$ . But  $f \geq f_k$  so that

$$\int_E fg_k \geq \int_E f_k g_k \rightarrow \infty.$$

But it then follows that

$$\sup_{\|g\|_{p'}=1} \int_E fg = \infty = \|f\|_{p'}.$$

Now if  $f$  is allowed to be nonnegative, apply the work above to  $|f|$ . Then there exists  $g_k$  with  $\|g_k\|_{p'} = 1$  with

$$\|f\|_p = \lim \int_E |f| g_k = \lim \int_E f \tilde{g}_k,$$

where  $\tilde{g} := (\text{sgn } f)g_k$ . But then as  $\|\tilde{g}_k\|_{p'} = 1$ , the result follows. The cases  $p = 1$  and  $p = \infty$  are routine and left to the reader.  $\square$

Furthermore, we have another sort of ‘reverse Hölder’ result. Suppose that for  $f \in L^p$  and  $g \in L^q$ , we have  $\int_{\mathbb{R}} |fg| \leq C\|f\|_p\|g\|_q$  for some  $C$ , then  $p^{-1} + q^{-1} = 1$ . To see this, rescaling to  $f(\lambda x)$  and  $g(\lambda x)$ , we have

$$\int_{\mathbb{R}} f(\lambda x)g(\lambda x) dx = \frac{1}{\lambda} \int_{\mathbb{R}} fg.$$

Similarly,

$$\left( \int_{\mathbb{R}} |f(\lambda x)|^p \right)^{1/p} = \left( \frac{1}{\lambda} \right)^{1/p} \|f\|_p.$$

But then we have

$$\frac{1}{\lambda} \int |fg| \leq C \left( \frac{1}{\lambda} \right)^{1/p+1/q} \|f\|_p \|g\|_q.$$

However, this does not hold for all  $\lambda > 0$  unless  $p^{-1} + q^{-1} = 1$ . Note also the special case in Hölder’s Inequality when  $g \equiv 1$ :

$$\int_E |f| = \int_E |f \cdot 1| \leq \left( \int_E \|f\|^p \right)^{1/p} \left( \int_E 1^{p'} \right)^{1/p'}.$$

But then

$$\int_E |f| \leq |E|^{1/p'} \|f\|_p.$$

This is vacuous if  $|E| = \infty$ .

**Example 5.5.** Let  $1 < p \leq \infty$ . Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is  $C^1$  and  $f' \in L^p$ . Using Hölder’s Inequality with  $1/p'$ , there exists  $C$  such that  $|f(a) - f(b)| \leq C|a - b|^{1/p'}$ . We then have

$$|f(a) - f(b)| \leq \int_a^b |f'(x)| dx \leq (b - a)^{1/p'} \|f'\|_p.$$

Notice that Lipschitz continuity arises in Hölder’s Inequality in the case where  $p = \infty$ , so that  $p' = 1$ .  $\triangleleft$

**Theorem 5.4** (Minkowski's Inequality). *If  $1 \leq p \leq \infty$ , then  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ , i.e.*

$$\left( \int_E |f + g|^p \right)^{1/p} \leq \left( \int_E |f|^p \right)^{1/p} + \left( \int_E |g|^p \right)^{1/p}$$

*Proof.* The proof in the case of  $p = 1$  is immediate from the triangle inequality:  $|f + g| \leq |f| + |g|$ . Now if  $p = \infty$ , we know that  $|f| \leq \|f\|_\infty$ ,  $|g| \leq \|g\|_\infty$  a.e. in  $E$ . But then  $|f + g| \leq \|f\|_\infty + \|g\|_\infty$  a.e. in  $E$ . It then follows that  $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$ . For  $1 < p < \infty$ , we have

$$\begin{aligned} \|f + g\|_p^p &= \int_E |f + g|^p \\ &= \int_E |f + g|^{p-1} |f + g| \\ &\leq \int_E |f + g|^{p-1} |f| + \int_E |f + g|^{p-1} |g|. \end{aligned}$$

Then using Hölder's Inequality, Theorem 5.2, to  $|f + g|^{p-1}$ ,  $|g|$ , with exponents  $p/(p-1)$ ,  $p$ , respectively, we have

$$\int_E |f + g|^{p-1} |g| \leq \|f + g\|_p^{p-1} \|g\|_p.$$

The same holds for the other integral mutatis mutandis. But then  $\|f + g\|_p^p \leq \|f + g\|_p^{p-1} (\|f\|_p + \|g\|_p)$ . Dividing by  $\|f + g\|_p^{p-1}$ , the result then follows. [Note that if  $\|f + g\|_p = 0$ , the result is immediate, and if  $\|f + g\|_p = \infty$ , either  $\|f\|_p = \infty$  or  $\|g\|_p = \infty$ , the result is also immediate.]  $\square$

**Remark.** Minkowski's Inequality fails for  $0 < p < 1$ . For example, take  $E = (0, 1)$ , and  $f(x) = \chi_{(0, 1/2)}$ ,  $g(x) = \chi_{(1/2, 1)}$ . Then clearly  $\|f + g\|_p = 1$  while  $\|f\|_p + \|g\|_p = 2^{-1/p} + 2^{-1/p} = 2^{1-1/p} < 1$ .

Minkowski's Inequality also works for infinite sums as well.

**Theorem 5.5.** *If  $1 \leq p \leq \infty$ , then  $\|a_k + b_k\|_p \leq \|a_k\|_p + \|b_k\|_p$ ; that is,*

$$\left( \sum_k |a_k + b_k|^p \right)^{1/p} \leq \left( \sum_k |a_k|^p \right)^{1/p} + \left( \sum_k |b_k|^p \right)^{1/p}$$

For infinite series: if  $\sum_k \|f_k\|_p < \infty$ , then  $\sum f_k \in L^p$  and  $\|\sum f_k\|_p \leq \sum \|f_k\|_p$  for  $1 \leq p \leq \infty$ .

*Proof.* Minkowski then  $\|\sum_{k=1}^N |f_k|\|_p \leq \sum_k \|f_k\|_p$  then MCT allows  $N \rightarrow \infty$  here. So  $\sum |f_k| \in L^p$ . Hence,  $\sum f_k \in L^p$ .

## 5.5 Sequence Classes $\ell^p$

Sequence spaces are closely related to Functional Analysis, among many other areas of mathematics. Perhaps the most important of these spaces are the  $\ell^p$  spaces, which are special cases of  $L^p$  spaces for the counting measure on the natural numbers. These spaces are also important as they serve as classical examples of Banach spaces.

**Definition ( $\ell^p$ ).** If  $\{a_k\}$  is a real (or complex sequence), for  $0 < p < \infty$ , define

$$\|a_k\|_p = \left( \sum_k |a_k|^p \right)^{1/p},$$

while for  $p = \infty$ , define  $\|a\|_\infty = \sup_k |a_k|$ . Then we define  $\ell^p$  to be

$$\ell^p := \{ \{a_k\} : \|a_k\|_p < \infty \},$$

**Proposition 5.3.** If  $\{a_k\}$  is a real (or complex sequence), then  $\|a\|_p \leq \|a\|_q$  if  $p \geq q$ .

*Proof.* We can rescale so that  $\|a\|_1 = 1$ . Then  $|a_k| \leq 1$  for all  $k$ . Hence,  $\sum |a_k|^p \leq \sum |a_k|^q \leq 1$ . So  $\|a\|_q \leq 1$ . If  $p < \infty$ . The case  $p = \infty$  is easy.  $\square$

A sequence  $\{a_k\}$  can be represented by  $f = \sum_{k=1}^{\infty} a_k \chi_{[k, k+1)}$ . Then  $\|f\|_p = \|a\|_p$  as  $\int |f|^p = \sum_{k=1}^{\infty} \int_k^{k+1} |f|^p = \sum |a_k|^p$ . Given sequence  $a, b$ , can apply Holder/mink to corresponding functions and get  $\sum |a_k b_k| \leq \|a\|_p \|b\|_{p'}$  Holder  $\|a + b\|_p \leq \|a\|_p + \|b\|_p$  Mink

Ex:  $a_k = 1/k$  is in  $\ell^p \iff p > 1$ . More generally,  $a_k = 1/k^r \in \ell^p \iff p > 1/r$ , where  $r > 0$ .

For geometric sequences,  $a_k = 1/2^k \in \ell^p \iff$ , well for all  $p \in (0, \infty]$ .

$a_k = 1/(k^r \log^{2r}(k+1)) \iff p \geq 1/r$ .

Norm:  $\|x\| \geq 0$ ,  $\|x\| = 0$  iff  $x = 0$   $\|\lambda x\| = |\lambda| \|x\|$   $\|x + y\| \leq \|x\| + \|y\|$  Normed vector space:  $(V, \|\cdot\|)$ .

$1 \leq p \leq \infty$   $\ell^2$  norm  $L^p$   $L^p$  has norm  $\|f\|_p = (\int |f|^p)^{1/p}$   $\ell^p$  has norm  $(\sum |a_k|^p)^{1/p}$   
 $p = \infty$ , special.

Norms give metric which then have convergence, sequences, open sets, etc etc (all regular topology plus more, compact sets, close open convergence continuous functions, etc etc. So once have norm, do not need to define these terms as they come immediately from the metric immediately from the norm.

Now the main thing for today is the following: are these metric spaces complete?

**Definition (Banach Space).** A Banach Space is a complete normed linear space.

Complete means every Cauchy sequence converges.

Convergence in  $L^p$ :  $f_k \rightarrow f$  means  $\int |f_k - f|^p \rightarrow 0$  if  $p < \infty$  or essential sup  $|f_k - f| \rightarrow 0$  if  $p = \infty$ .



**Example 5.6.** Conversely,  $k^{-1}\chi_{[0,k]} \rightarrow 0$  pointwise but not in  $L^1$ .  $k\chi_{[0,1/k]} \rightarrow 0$  a.e. but not in  $L^1$ . The first converges in  $L^p$  for  $1 < p \leq \infty$  while the second does not converge in any  $L^p$  for  $1 \leq p \leq \infty$ .

So convergence pointwise and convergence in  $L^p$  are not related for all  $p$ .

**Lemma 5.1.** If  $f_k \rightarrow f$  in  $L^p$ , then  $f_k \xrightarrow{m} f$  (convergence in measure).

*Proof.*  $1 \leq p < \infty$ :  $|\{ |f_k - f| > \epsilon \}| = |\{ |f_k - f|^p > \epsilon^p \}| \leq 1/\epsilon^p \int |f_k - f|^p$  by Chebyshev and right integral  $\rightarrow 0$  as  $k \rightarrow \infty$ .  $p = \infty$ :  $f_k \rightarrow f$  in measure means  $f_k \rightarrow f$  uniformly on  $E \setminus Z$ ,  $|Z| = 0$  hence in measure.  $\square$

The converse false as previous example they converged in measure.

**Theorem 5.6.**  $L^p$  and  $\ell^p$ ,  $1 \leq p \leq \infty$  are Banach spaces.

$1 \leq p < \infty$ : Suppose  $\{f_k\}$  is Cauchy in  $L^p$ . Then  $\{f_k\}$  is Cauchy in measure.  $|\{ |f_k - f_j| > \epsilon \}| \rightarrow 0$  as  $k, j \rightarrow \infty$ . By Lemma, From 4.4, we have there exists  $f$  such that  $f_k \xrightarrow{m} f$ . there exists subsequence  $f_{k_i} \rightarrow f$  a.e. for a fixed  $i$ .

$$f_{k_i} - f_{k_j} \xrightarrow{a.e.} f_{k_i} - f \quad j \rightarrow \infty$$

By Fatou's Lemma:

$$\int |f_{k_i} - f|^p \leq \liminf_{j \rightarrow \infty} \int |f_{k_i} - f_{k_j}|^p$$

By cauchy. Conclude  $\int |f_{k_i} - f|^p \leq \epsilon$  for sufficiently large  $i$ . So  $f_{k_i} \rightarrow f$  in  $L^p$ . Hence  $f_k \rightarrow f$ . General fact: Cauchy sequence with convergent subsequence converges.  $\square$

Prove separability: separable if has countable dense subset.

**Theorem 5.7.**  $L^p$ ,  $\ell^p$  are separable if  $p < \infty$  not if  $p = \infty$ .

In  $L^p(\mathbb{R}^n)$ :  $\{\sum_{fin} (a_k + ib_k)\chi_{Q_k} : a_k, b_k \in \mathbb{Q}, Q_k \text{ dyadic cube}\}$ . is dense in  $L^p$ .

In  $L^p(E)$ , just restrict to  $E$  (for  $p < \infty$ ).

In  $\ell^p$   $\{(a_1, a_2, \dots, a_m, 0, 0, \dots)\}$  then real an imag. parts rational,  $m \in \mathbb{N}$  is dense for  $p < \infty$ .

$p = \infty$  Find countably many elements  $f_\alpha$  such that  $\|f_\alpha - f_\beta\|_\infty = 1$  for all  $\alpha \neq \beta$  in  $L^\infty(\mathbb{R}^n)$ :  $\chi_{[0,a]^n}$   $a > 0$  in  $\ell^\infty$  all  $\{0, 1\}$  sequences.  $L^\infty(E)$  in  $L^\infty(E)$ :  $|E| > 0$  consider  $\phi(a) = |E \cap [-a, a]^n|$ ,  $a > 0$  is continuous.

$$\phi(b) - \phi(a) \leq |[-b, b]^n \setminus [-a, a]^n| = (lb)^n - (la)^n \rightarrow 0$$

An inner product on a vector space  $V$ : a function  $\langle f, g \rangle \rightarrow \mathbb{C}$ ,  $f, g \in V$ . With  $\langle g, f \rangle = \overline{\langle f, g \rangle}$  (conjugate symmetry). and  $\langle \alpha f_1 + \beta f_2, g \rangle = \alpha \langle f_1, g \rangle + \beta \langle f_2, g \rangle$  (linear in first, hence second, argument).  $\langle f, f \rangle = 0$  and 0 if and only if  $f = 0$ .

**Example 5.7.**  $L^2(E)$  has  $\langle f, g \rangle = \int_E f \bar{g}$ .  $\ell^2$  has  $\langle a, b \rangle = \sum_k a_k \bar{b}_k$ . These are finite by Hölder,  $p = p' = 2$ .

General facts:  $\|f\| = \sqrt{\langle f, f \rangle}$  is a norm  $|\langle f, g \rangle| \leq \|f\| \|g\|$ . (abstract Cauchy-Schwarz)  
Hilbert space = complete inner product space.

$L^2, \ell^2$  are Hilbert spaces.

Orthogonality:  $f \perp g$  if  $\langle f, g \rangle = 0$ .

Pythagorean Theorem: if  $f \perp g$ , then  $\|f \pm g\|^2 = \|f\|^2 + \|g\|^2$ . Why?  $\langle f + g, f + g \rangle = \langle f, f \rangle + \langle f, g \rangle + \langle g, f \rangle + \langle g, g \rangle = \|f\|^2 + \|g\|^2$ .

More generally, if  $f_1, \dots, f_n$  are mutually orthogonal, then  $\|\sum f_k\|^2 = \sum \|f_k\|^2$ .

Orthonormal system:  $\{\phi_k\}$  in Hilbert space such that

$$\langle \phi_k, \phi_j \rangle = \begin{cases} 0, & k \neq j \\ 1, & k = j \end{cases}$$

**Example 5.8.** In  $\ell^2$ ,  $\phi_1 = (1, 0, 0, \dots)$ ,  $\phi_2 = (0, 1, 0, \dots)$ , ... in  $L^2$

**Example 5.9.**  $\phi_k = e^{2\pi i k x}$  on  $[0, 1]$  ( $k \in \mathbb{N}$  or  $k \in \mathbb{Z}$ ).

Why is it nice to have orthogonal systems? If  $\{\phi_k\}$  is an ONS, then

$$\left\| \sum_{k=1}^n c_k \phi_k \right\|^2 = \sum_{k=1}^n |c_k|^2.$$

How do we let  $n \rightarrow \infty$  here? If  $c \in \ell^2$ , i.e.  $\sum_{k=1}^{\infty} |c_k|^2 < \infty$ . Then  $\sum_1^{\infty} c_k \phi_k$  converges in  $L^2$  and  $\|\sum c_k \phi_k\| = \sum_{k=1}^{\infty} (c_k)^2$ .

*Proof.* We need for all  $\epsilon$  such that there exists

$$\left\| \sum_{k=m}^n c_k \phi_k \right\| < \epsilon$$

whenever  $m, n, \geq N$ . But

$$\left\| \sum_{k=m}^n c_k \phi_k \right\| = \sqrt{\sum_{k=m}^n |c_k|^2}$$

which is small as  $\sum |c_k|^2 < \infty$ .

Also, let  $s_n = \sum_1^n c_k \phi_k$ ,  $s_n \rightarrow s = \sum_1^{\infty} c_k \phi_k$ .  $\|s_n\| - \|s_n - s\| \leq \|s\| \leq \|s_n\| + \|s_n - s\|$ . On the right we have  $\sqrt{\sum_1^n |c_k|^2}$  and a term tending to 0, same on the left. Then by Squeeze Theorem we have  $\|s\| = \sqrt{\sum_1^{\infty} |c_k|^2}$ .

Reverse engineering process from  $f \in L^2$  to coefficients  $c_k$ . Fourier expansion. Note if  $f = \sum c_k \phi_k$  then  $c_k = \langle f, \phi_k \rangle$ .

Why?  $\langle f, \phi_j \rangle = \langle \sum c_k \phi_k, \phi_j \rangle = \sum c_k \langle \phi_k, \phi_j \rangle = c_j$ . Why equality? For finite ONS, use linearity. For infinite, use continuity of  $f \mapsto \langle f, g \rangle$ , c.f. hw 8.4. or by previous result  $\sum |c_k|^2 = \|f\|^2 < \infty$ .

Linear span of a set  $E$  = set of all finite linear combinations of elements of  $E$ .

**Theorem 5.8.** Given  $f \in L^2$  and  $\{\phi_k\}$  a ONS in  $L^2$ . TFAE

1)  $f = \sum c_k \phi_k$  (series converges in  $L^2$ ) 2)  $f \in \overline{\text{span } \{\phi_k\}}$  (closure in  $L^2$ )

1  $\rightarrow$  2: if  $f = \lim_{n \rightarrow \infty} \sum_1^n c_k \phi_k$ , then  $v_n = \sum^n c_k \phi_k$  is in the span since  $v_n \rightarrow f$  in  $L^2$ , we have  $f \in \text{closure of the span}$ .

2  $\rightarrow$  1: Let  $c_k = \langle f, \phi_k \rangle$  and  $s_n = \sum^n c_k \phi_k$ . Want to show that  $s_n \rightarrow f$ . Let  $M$  be the span of  $\{\phi_1, \dots, \phi_n\}$ .  $(s_n - f) \perp \text{span } (\phi_1, \dots, \phi_n)$ . because  $\langle s_n - f, \phi_k \rangle = c_k - c_k = 0$ . Then use Pythagorean Theorem: for any  $g$  in the span of  $\phi_1, \dots, \phi_n$ ,  $\|f - g\| \geq \|f - s_n\|$ . Now  $\|f - g\|^2 = \|f - s_n\|^2 + \|s_n - g\|^2$ . Since  $f$  is in the closure of the span up to  $\phi_n$  the distance between  $f$  and the span tends to 0. So once  $n$  sufficiently large, close to 0.  $\square$

**Corollary 5.2.** Since  $f - s_n \perp \phi_1, \dots, \phi_n$ ,  $\|f\|^2 = \sum^n |c_k|^2 + \|f - s_n\|^2$ .

This has two consequences: Bessel Inequality:  $\sum^\infty |c_k|^2 \leq \|f\|^2$ . Parseval's Identity:  $\sum |c_k|^2 = \|f\|^2$  if and only if  $s_n \rightarrow f$ .

## 6 Abstract Integration

### 6.1 Measure Spaces

So far, we only have had Lebesgue measure on  $\mathbb{R}^n$ . But there are other measures: length of curves in  $\mathbb{R}^2$ , surface area on surfaces, probability on probability space  $\Omega$   $P(\text{event}) = \text{measure of outcomes}$ .

**Definition** (Measure Space). A measure space is a triple,  $(X, \Sigma, \mu)$ , where  $X$  is a set,  $\Sigma$  is a  $\sigma$ -algebra on subsets of  $X$ , and  $\mu : \Sigma \rightarrow [0, \infty]$  is a countably additive function on disjoint sets.

**Remark.** One often also requires  $\mu(\emptyset) = 0$ . However, this does almost follow from our definition since  $\mu \geq 0$  and if  $\emptyset \neq A \subseteq X$ , then

$$\mu(A) = \mu(A \cup \emptyset) = \mu(A) + \mu(\emptyset)$$

so that  $\mu(\emptyset) = 0$  unless  $\mu(\emptyset) = \infty$ .

**Example 6.1.**

(i)  $(\mathbb{R}^n, \mathcal{L}, \mu)$ ,  $\mathcal{L}$  is the set of Lebesgue measurable set.  $\mu(E) = |E|$ .

(ii)  $(X, 2^X, \mu)$ ,  $X$  is any set,  $\mu(E)$  is the cardinality of the set.

(iii) Like (ii) but

$$\mu(E) = \begin{cases} 0, & E \text{ countable} \\ \infty, & E \text{ uncountable} \end{cases}$$

(iv)  $(\mathbb{R}^n, \mathcal{L}, \mu_w)$ , where

$$\mu_w(E) = \int_E w$$

where  $w \geq 0$  is measurable. One may say that  $w$  is the ‘weight’ or ‘density.’

**Non-Example 6.1.** like (ii) above but

$$\mu(E) = \begin{cases} 0, & |E| < \infty \\ \infty, & |E| \text{ infinite} \end{cases}$$

this is *not* a measure space since it is not countably additive.

**Definition** (Additive Set Function (ASF)). We say a function  $\phi : \Sigma \rightarrow \mathbb{R}$ , where  $\Sigma$  is a  $\sigma$ -algebra, is countably additive.

The difference between  $\phi$  and a measure is that it can be negative and it cannot be infinite. This is also called a *finite signed measure*.

**Example 6.2.**  $\phi(E) = \int_E f$ , where  $f \in L^1$ .

An advantage of ASF can pass to complement:  $\phi(E^C) = \phi(X) - \phi(E)$  and get results for  $\cap$  from  $\cup$ .

**Theorem 6.1.** *ASF is continuous on monotone sequences of sets, i.e.  $E_1 \subset E_2 \subset \dots$  then  $\phi(\cup E_k) = \lim \phi(E_k)$  or  $E_1 \supset E_2 \supset \dots$  then  $\phi(\cap E_k) = \lim \phi(E_k)$*

*Proof.* The second follows from the first by taking complements. Now

$$\bigcup_{k=1}^{\infty} E_k = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_2 \setminus E_1) \cup \dots$$

disjoint union so

$$\phi(\cup E_k) = \phi(E_1) + (\phi(E_2) - \phi(E_1)) + \dots$$

which telescopes so we have  $\lim_{k \rightarrow \infty} \phi(E_k)$ .  $\square$

For measures  $\mu$ , the first works but in the second statement above, we require  $\mu(E_1) < \infty$ .

**Theorem 6.2** ('Fatou's Lemma'). *if  $\phi \geq 0$  is an ASF (i.e. finite measure), then  $\phi(\liminf E_k) \leq \liminf \phi(E_k)$ . Hence by considering  $E_k^C$ ,  $\phi(\limsup E_k) \geq \limsup \phi(E_k)$ .*

*Proof.* Let  $M_k = \bigcap_{j \geq k} E_j$ , then  $\liminf E_k = \bigcup_{k=1}^{\infty} M_k$ .

$$\phi\left(\bigcup_{k=1}^{\infty} M_k\right) = \lim_{k \rightarrow \infty} \phi(M_k) = \liminf_{k \rightarrow \infty} \phi(E_k),$$

using  $M_k \subset E_k$ , so  $\phi(E_k) - \phi(M_k) = \phi(E_k \setminus M_k) \geq 0$ .

Jordan Decomposition for ASF

Every ASF is the difference of two nonnegative ASFs.

Application: Write ASF  $\phi = \phi_1 - \phi_2$ , both nonnegative. Conclude if  $E = \limsup E_k = \liminf E_k$ , then  $\phi(E) = \lim \phi(E_k)$ .

**Definition** (Upper Variation). Given an ASF  $\phi$ , upper variation  $\bar{V}(E) = \sup_{A \subset E} \phi(A)$ .

**Definition** (Lower Variation). Given an ASF  $\phi$ , lower variation  $\underline{V}(E) = -\inf_{A \subset E} \phi(A)$ .

**Definition** (Total Variation). Given an ASF  $\phi$ , total variation:  $V(E) = \bar{V}(E) + \underline{V}(E)$ .

How does this relate to Jordan decomposition? Idea of proof is to show  $\phi(E) = \bar{V}(E) - \underline{V}(E)$ .

where (needs to be proved)  $\bar{V}, \underline{V}$  are ASFs. Why this was

$$\int_{A \subset E, A \in \Sigma} \phi(A) = \inf_{B \subset E, B \in \Sigma} (\phi(E) - \phi(B)) = \phi(E) - \sup_{B \subset E, B \in \Sigma} \phi(B) = \phi(E) - \bar{V}(E)$$

Take  $A = E \setminus B$ . Hence,  $\phi(E) = \bar{V}(E) - \underline{V}(E)$ .

Clear total variation is ASF if the upper/lower are. So sketch proof that they are:

1) Countable subadditivity.  $A \subset \cup E_k$ , write as  $A = (A \cap E_1) \cup (A \cap E_2 \setminus E_1) \cup \dots$ . Get  $A = \cup A_k$ ,  $A_k \subset E_k$ .  $\phi(A_k) \leq \bar{V}(E_k)$  hence  $\phi(A) \leq \sum \bar{V}(E_k)$ . The upper follows the same.

2)  $V < \infty$ , which is done by contradiction.  $\bar{V}(E) \geq \sum \bar{V}(E_k)$  just pick  $A_k \subset E_k$ ,  $\phi(A_k) \geq \bar{V}(E_k) - \epsilon/2^k$ , use countable subadditivity of  $\phi$ .

## 6.2 Measurable Functions and Integrals

Repeat all previous work for abstract measure spaces. Recall  $(X, \Sigma, \mu)$  is a measure space.

**Definition.** A function  $f : X \rightarrow \bar{\mathbb{R}}$  is measurable if  $\{f > a\} \in \Sigma$  for all  $a \in \mathbb{R}$ .

Fur functions  $f : X \rightarrow \mathbb{R}$ , equivalent to  $f^{-1}(\text{Borel}) \in \Sigma$ .

**Example 6.3.** Let  $X = \mathbb{N}$ ,  $\Sigma = 2^X$ , then all functors are measurable.

Basic properties:  $cf$ ,  $f + g$ ,  $\sup_k f_k$ ,  $\inf_k f_k$ ,  $\lim_k f_k$  are measurable.

$\sum_{\text{finite}} a_k \chi_{E_k}$ , disjoint,  $a_k$  unique is measurable if and only if  $E_k \in \Sigma$ .

Every nonnegative measurable function  $f$  is the limit of  $f_k$  (simple),  $f_k \nearrow f$ ,  $f_k = \min(k, 2^{-k} \text{floor}(2^k f))$ , second piece round down to  $j/2^k$ :  $\{f_k = j/2^k\} = \{j/2^k \leq f < (j+1)/2^k\}$  or  $\emptyset$ .

What does not *not* generalize.

"continuous" then measurable. Continuity is not even defined on  $X$ .

Lusin's Theorem does not apply.

If  $f$  is measurable and  $f = g$  a.e. wrt  $\mu$  then  $g$  not necessarily measurable. (because subsets of measure zero no longer necessarily measurable). example:  $(\mathbb{R}, \mathcal{B}, |\cdot|)$ , Borel with Lebesgue measure. in Cantor set  $C$ , there is a non-Borel subset  $A$ .  $\chi_A \equiv 0$  a.e. but  $\chi_A$  is not measurable.

$X = \cup_{k=1}^{\infty} E_k$ ,  $E_k$  has finite measure. Example:  $\mathbb{R}$  with counting measure. ie  $\infty$  on infinite sets and finite otherwise.

**Definition.** If  $f \geq 0$  is measurable.  $\int_E f d\mu = \sup\{\sum_j a_j \mu(E_j)\}$  over simple functions  $\sum a_j \chi_{E_j} \leq f$ .

Can choose  $a_j = \inf_{E_j} f$ , so  $\int f d\mu = \sup_p L(f, P)$ , where  $P$  is partition of  $E$  into  $E_1, \dots, E_m$ ,  $L(f, P) = \sum_j \int_{E_j} f d\mu$ . Choosing finer partition  $P$ , get larger  $L(f, P)$ .

**Definition.** If  $f$  measurable and one of  $\int_E f^+ d\mu$ ,  $\int_E f^- d\mu$  is finite, define  $\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$ . if both finite,  $f \in L^1(E, \mu)$ .

**Example 6.4.**  $(\mathbb{N}, 2^{\mathbb{N}}, \text{counting})$ ,  $\int f d\mu = \sum_{k=1}^{\infty} f(k)$ , if abs convergent.  $L(f, P) = \sum_{k=1}^N f(k)$   
So integrals are series! Notice integral finite iff abs convergent.

**Example 6.5.**  $(\mathbb{R}, 2^{\mathbb{R}}, \delta_{x_0})$ .  $\delta_{x_0}$  the point mass at  $x_0$ , i.e. dirac delta

$$\delta_{x_0} = \begin{cases} 1, & x_0 \in A \\ 0, & x_0 \notin A \end{cases}$$

$\int_{\mathbb{R}} f d\delta_{x_0} = f(x_0)$  *Proof.* Suppose  $f$  nonnegative (otherwise, use  $f^+$  and  $-f^-$  and then combine later). Let  $P = \{E_1, \dots, E_m\}$  be a partition. One  $E_j$  contains  $x_0$ , then  $L(f, P) = \inf_{E_j} f \leq f(x_0)$ . Hence,  $\int f \leq f(x_0)$ . Also,  $P = \{\{x_0\}, \mathbb{R} \setminus \{x_0\}\}$  gives  $L(f, P) = f(x_0)$ .

We need another proof of MCT, since before used “volume definition” of  $\int f$  and this does not generalize.

**Lemma 6.1.** *if  $E = A \cup B$ , disjoint measurable, then  $\int_E f = \int_A f + \int_B f$  for nonnegative measurable  $f$ .*

*Proof.* Any partition  $P$  of  $E$  can be refined, ( $E_j$  replaced by  $E_j \cap A, E_j \cap B$ ) so we get  $P = P_A \cup P_B$ . Then  $L(f, P) = L(f_1, P_A) + L(f, P_B)$ . Hence,  $\int f \leq \int_A f + \int_B f$ . But  $P_A, P_B$  approx  $\int_A f, \int_B f$  part them into  $P$ , conclude  $\int_E f \geq \int_A f + \int_B f$ .  $\square$

**Example 6.6.**  $X = \mathbb{R}^2$ ,  $\Sigma = \text{Borel sets}$ ,  $\mu(E) = |\{x \in \mathbb{R}: (x, 0) \in E\}|$ .  $\int f d\mu = \int_{\mathbb{R}} f(x, 0) dx$ . Important example of integrating along a curve.

Our goal is now to prove the MCT (if  $f_k \nearrow f$ ,  $f_k \geq 0$  measurable, then  $\int f_k \rightarrow \int f$ ). We will need some preparation.

Subadditivity over  $f$ :  $\int (f + g) = \int f + \int g$ , where  $f, g \geq 0$  measurable. How do we prove this? For simple, easy:  $f = \sum a_k \chi_{E_k}$ ,  $g = \sum b_l \chi_{F_l}$ , rewrite as  $\sum_{k,l} a_k \chi_{E_k \cap F_l}$  and  $\sum_{k,l} b_l \chi_{E_k \cap F_l}$ . The first is integral  $\sum a_k \mu(E_k \cap F_l)$  and second is integral  $\sum b_l \mu(E_k \cap F_l)$ . But then for the sum:  $\sum (a_k + b_l) \chi_{E_k \cap F_l}$  converges to integral  $\sum (a_k + b_l) \mu(E_k \cap F_l)$ .

**Lemma 6.2.** *Suppose  $f_k, g \geq 0$  are simple,  $f_1 \leq f_2 \leq \dots \lim f_k \geq g$ . Then  $\lim \int f_k \geq \int g$ .*

*Proof.* Partition set  $E$  to make  $g$  constant (on each piece)  $g \equiv c$ . Given  $c' < c$ . Let  $A_k = \{f_k > c'\}$ . Then  $A_1 \subset A_2 \subset \dots$  and  $\bigcup A_k = E$ , so  $\mu(A_k) \rightarrow \mu(E)$ . But  $\int_E f_k \geq \int_{A_k} f_k \geq c' \mu(A_k)$ . So  $\lim \int f_k \geq c' \mu(E)$ . So  $\lim \int f_k \geq c \mu(E) = \int g$ .  $\square$

**Lemma 6.3.** *If  $f_k \nearrow f$ ,  $f_k$  simple nonnegative, then  $\int f_k \rightarrow \int f$ .*

*Proof.* For all simple  $g \leq f$ , we have  $\lim \int f_k \geq \int g$  by the previous lemma. Take sup over  $g$ , get  $\lim \int f_k \geq \int f$ . But reverse is immediate since  $f_k \nearrow f$  so  $f_k \leq f$ .  $\square$

Proof of  $\int (f + g) = \int f + \int g$ .

Take simple  $f_k \nearrow f$ ,  $g_k \nearrow g$ . Pass to limit in  $\int (f_k + g_k) = \int f_k + \int g_k$ .  $\square$

Then follows for general function by writing  $f^+$  and  $f^-$ .

Monotone Convergence Theorem (Two forms):

Sequence Form: if  $f_k$  are nonnegative, measurable, and  $f_k \nearrow f$ , then  $\int f_k \rightarrow \int f$ .

Series Form: If  $g_k \geq 0$  measurable, then  $\int \sum g_k = \sum \int g_k$ .

These are equivalent. [The duality of sequences and series in a sense.]

*Proof.* (Series Form since easier to prove)  $\int g \geq \sum_{k=1}^{\infty} \int g_k$  by considering  $g \geq \sum_{k=1}^N g_k$ .  $\int g \geq \sum_{k=1}^N \int g_k$ . Need  $\leq$ . Introduce simple  $g_{k,j} \nearrow g_k$  as  $j \rightarrow \infty$ .

$$\begin{array}{ccccccc} g_{11} & g_{12} & g_{13} & \cdots & \rightarrow & g_1 \\ g_{21} & g_{22} & g_{23} & \cdots & \rightarrow & g_2 \\ g_{31} & g_{32} & g_{33} & \cdots & \rightarrow & g_3 \end{array}$$

We use an “upper triangular sum”.  $s_j = \sum_{k=1}^j g_{k,j}$  which converges monotonically to  $g$ , i.e.  $s_j \nearrow g$ . By second lemma above,  $\int s_j \rightarrow \int g$ . Also,  $\sum_{k=1}^j g_k \geq s_j$ , hence  $\sum_{k=1}^{\infty} \int g_k \geq \int g$ .  $\square$

MCT implies Fatou which implies DCT.

Hölder, Minkowski:  $L^p(E, \mu)$  is a Banach space.

### 6.3 Absolute Continuity & Singular Sets

Now we move on to 10.3

Given a measure space  $(X, \Sigma, \mu)$  and an ASF  $\phi$ , we say

**Definition** (Absolutely Continuous).  $\phi$  is absolutely continuous with respect to  $\mu$  if  $\mu(A) = 0$ , then  $\phi(A) = 0$ .

**Definition** (Singular).  $\phi$  is singular with respect to  $\mu$  if there exists  $Z$  such that  $\mu(Z) = 0$  and  $\phi(A) = 0$  for all  $A \subset Z^C$ .

Equivalently, for total variation  $V$

AC:  $\mu(A) = 0$  then  $V(A) = 0$

S:  $\mu(Z) = 0$  and  $V(Z^C) = 0$ .

Main results: every ASF is the sum of AC and S. AC ASF has density  $\phi(A) = \int_A \omega$ .

Examples:

a)  $\mu$  = Lebesgue measure on  $\mathbb{R}$  and  $\phi(A) = \int_A f$ ,  $f \in L^1(\mathbb{R})$ . This is AC

b) same  $\mu$ ,  $\phi(A)$  is 1 if  $x_0 \in A$  and 0 if  $x_0 \notin A = \delta_{x_0}$ .

Singular: let  $Z = \{x_0\}$  then  $\mu(Z) = 0$ ,  $V(Z^C) = 0$ .

On  $[0, 1]$ , fix  $w \geq 0$  integrable. fix  $w \geq 0$ ,  $w \in L$ ,  $\mu_w(A) = \int_A w$ .

Is Lebesgue measure on  $[0, 1]$  AC wrt  $\mu_w$ ? No if  $|\{w = 0\}| > 0$

Yes if  $w > 0$  a.e.

Goal: Every ASF is AC + S: AC has ‘density’  $f$ ,  $\phi(A) = \int_A f$ .



Remark: for any  $E \subset \Sigma$ ,  $(E, \Sigma' = \{F \in \Sigma: F \subset E\}, \mu|_{\Sigma'})$  is a measure space (subspace of  $X$ ).

**Theorem 6.3.** (From  $0 - 0$  to  $\epsilon - \delta$  in AC)  $\phi$  is AC if and only if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $A$   $\mu(A) < \delta$  then  $|\phi(A)| < \epsilon$ .

*Proof.* Recall  $\phi = \bar{V} - \underline{V}$ , so  $|\phi(A)| \leq V(A)$ . Suppose there exists  $\epsilon > 0$  for all  $k$  there exists  $A_k$  with  $\mu(A_k) < 2^{-k}$  but  $V(A_k) \geq \epsilon$ . Let  $A = \limsup A_k$ . Then  $\mu(A) = 0$  (because  $\sum \mu(A_k) < \infty$ ) but  $V(A) \geq \limsup V(A_k) \geq \epsilon$ . (flipped Fatou). This contradicts AC.  $\square$

Ex: For infinite measures  $\mu, \nu$ , this is not true. For example,  $\mu$  is the Lebesgue measure on  $\mathbb{R}$  and  $\nu(A) = \int_A x^2 dx$ .  $\nu$  is AC wrt  $\mu$ . If  $\mu(A) = 0$ , then  $\nu(A) = 0$  but for all  $\delta > 0$ ,  $\int_{[n, n+\delta]} x^2 dx \rightarrow \infty$  as  $n \rightarrow \infty$ .

Hahn decomposition: if  $\phi$  is ASF, then there exists “positive set”  $P$  such that  $\phi \geq 0$  on subsets of  $P$  and  $\phi \leq 0$  on subsets of  $P^C$ .

*Proof.*  $P$  should maximize  $\phi$ . Recall  $\bar{V}(X) = \sup\{\phi(A): A \in \Sigma\}$  for all  $k$ . Choose  $A_k$  with  $\phi(A_k) > \bar{V}(X) - 2^{-k}$ . Let  $P = \liminf A_k$ . Note  $\underline{V}(A_k) = \bar{V}(A_k) - \phi(A_k)$ . Fatou’s lemma gives  $\underline{V}(A) \leq \liminf \underline{V}(A_k) = 0$ . Also,  $P^C = \limsup A_k^C$  and  $\bar{V}(A_k^C) = \bar{V}(X) - \bar{V}(A_k) < 1/2^k$ . Hence,  $\bar{V}(P^C) = 0$  [ $\sum \bar{V}(A_k^C) < \infty$ ]  $\square$

Ex: if  $\phi(A) = \int_A f$ , then  $P = \{f \geq 0\}$  or  $P = \{f > 0\}$ .

Compare to Jordan decomposition:  $\phi = \bar{V} - \underline{V}$ . Hahn makes this more precise.  $\bar{V}(A) = \phi(A \cap P)$ ,  $-\underline{V}(A) = \phi(A \setminus P)$  so  $\underbrace{\bar{V}(A)}_{\bar{V}} - \underbrace{(-\phi(A \setminus P))}_{\underline{V}}$ .

**Lemma 6.4** (Level Decomposition of ASF). Suppose  $\mu(X) < \infty$ ,  $\phi \geq 0$  an ASF, then for all  $a > 0$ , there exists partition  $X = Z \cup \bigcup_{k=1}^{\infty} E_k$ ,  $\mu(Z) = 0$

$$a(k-1)\mu(A) \leq \phi(A) \leq ak\mu(A)$$

for all  $A \subset E_k$

$$a(k-1) \leq \frac{\phi}{\mu} \leq ak$$

on  $E_k$

Ex: if  $\phi(A) = \int_A \frac{1}{\sqrt{x}} dx$  on  $X = [0, 1]$   $a = 1$   $Z = \emptyset$ ,  $E_1 = \emptyset$

*Proof.* WLOF,  $a = 1$  (else consider  $\phi/a$ ). Apply Hahn to  $\phi - k\mu$  ( $k \in \mathbb{N}$ ) get a set  $P_k$ ,  $\phi \geq k\mu$  on  $P_k$  and  $\phi \leq k\mu$  on  $P_k^C$ . Let  $Q_k = \bigcup_{m \geq k} P_m$  note  $\phi \geq k\mu$  on  $Q_k$ ,  $\phi \leq k\mu$  on  $Q_k$ . Let  $Z = \bigcap_k Q_k$  on this set  $\phi \geq k\mu$  so  $\mu(Z) = 0$ . Let  $E_1 = Q_1^C$  then  $\phi \leq \mu$  on  $E_1$ , for  $k > 1$ , let  $E_k = Q_{k-1} \setminus Q_k$ , then  $(k-1)\mu \leq \phi \leq k\mu$  as required.  $\square$

**Theorem 6.4** (Lebesgue Decomposition). *Let  $\phi$  be an ASF and  $\mu$  a  $\sigma$ -finite measure. Then there exists a unique additive set function  $\alpha, \sigma$  such that  $\phi = \alpha + \sigma$ ,  $\alpha$  is AC wrt  $\mu$ ,  $\sigma$  is S wrt  $\mu$ . Moreover, there exists  $f \in L^1(X, \mu)$  such that  $\alpha(A) = \int_A f$  and there exists  $Z$  such that  $\mu(Z) = 0$  and  $\sigma(A) = \phi(A \cap Z)$ .*

*Proof.* Uniqueness:  $\alpha + \sigma = \alpha' + \sigma'$ , then  $\alpha - \alpha' = \sigma' - \sigma$  so then both sides AC and S by equality, hence 0.

Existence of  $Z$ : comes from definition of singular. ASF  $\mu(Z) = 0$  and  $\phi = 0$  on  $Z^C$ . So  $\sigma(A) = \sigma(A \cap Z)$ . But  $\alpha(A \cap Z) = 0$  because  $\alpha$  is AC. Hence,  $\sigma(A) = \phi(A \cap Z)$ .

Hard parts existence of decomposition and existence of  $f$ .

Why do we care? Duality of functions spaces such as  $L^p$ . Dual space of  $L^p$  is  $\{\psi : L^p \rightarrow \mathbb{C} : \psi \text{ linear cont.}\}$ .  $\psi$  induces ASF  $\phi(E) = \psi(\chi_E)$ ,  $\phi(E \cup F) = \psi(\chi_E + \chi_F) = \phi(E) + \phi(F)$  if disjoint.

Why need to be  $\sigma$ -finite? ( $X = \bigcup_{k \in \mathbb{N}} E_k$ ,  $\mu(E_k) < \infty$ )

**Example 6.7.** Let  $\mu$  be counting measure on  $[0, 1]$  (gives number of elements of set).  $X = [0, 1]$ .  $\phi(A) = |A|$  is an ASF. If  $\phi = \alpha + \sigma$ , then  $\sigma(A) = 0$  for all  $A$  (because the only  $Z$  with  $\mu(Z) = 0$  is  $\emptyset$ ). Hence,  $\phi(A) = \int_A f d\mu$  for some  $f \in L^1(\mu)$ , which means  $\sum_{x \in [0, 1]} |f(x)| < \infty$ . But then  $f \equiv 0$  except on a countable set  $C$ . But then get  $\phi([0, 1] \setminus C) = 0$ , contradicting  $\phi(A) = |A|$ .

*Proof.* (LDT) Can reduce to  $\phi \geq 0$ . For general case  $\phi$ ,  $\phi = \bar{V} - \underline{V}$  so if  $\bar{V} = \alpha_1 + \theta_1$ ,  $\underline{V} = \alpha_2 + \theta_2$ , then  $\phi = (\alpha_1 - \alpha_2) + (\theta_1 - \theta_2)$ .

2) Reduce to  $\mu(X) < \infty$ . Else  $X = \bigcup_{\text{disjoint}} E_k$ , If  $\phi_k(A) = \phi(A)$  for  $A \subset E_k$  has LD,  $\alpha_k + \theta_k$ , then let  $\alpha(A) = \sum \alpha_k(A \cap E_k)$ ,  $\sigma(A) = \sum \sigma_k(A \cap E_k)$ .

3) Main case:  $\phi \geq 0$ ,  $\mu(X) < \infty$ . Use level decomposition with  $a = 2^{-m}$ . Obtain  $X = Z^m \cup \bigcup_{k=1}^{\infty} E_k^m$ ,  $\mu(Z^m) = 0$  and  $\phi(A)/\mu(A)$  is in  $[(k-1)2^{-m}, k2^{-m}]$  for all  $A \subset E_k^m$ . Let  $Z = \bigcup_{m=1}^{\infty} Z^m$ , this will be our  $Z$ . Need  $\phi(A \setminus Z) = \int_A f d\mu$ . Let  $f_m = \sum_k (k-1)2^{-m} \chi_{E_k^m}$ . How close is  $\int_A f_m$  to  $\phi(A \setminus Z)$ ?

$\phi$  is AC wrt  $\mu$  on each  $E_k^m$ , hence  $\phi(A \cap E_k^m) = \phi((A \setminus Z) \cap E_k^m)$ , null subsets of  $E^m$  are negligible. So  $\phi(A) = \phi(A \cap Z) + \phi(A \setminus Z) = \phi(A \cap Z) + \sum_k \phi(A \cap E_k^m)$ .

$\phi(A \cap E_k^m) \geq (k-1)2^{-m} \mu(A \cap E_k^m) = \int_{A \cap E_k^m} f_m$ .  $\phi(A \cap E_k^m) \leq k2^{-m} \mu(A \cap E_k^m) = \int_{A \cap E_k^m} f_m + 2^{-m} \mu(A \cap E_k^m)$ . Sum over  $k$

$$0 \leq \phi(A \setminus Z) - \int_A f_m d\mu \leq 2^{-m} \mu(X)$$

Remains to show  $f_m$  converge uniformly as  $m \rightarrow \infty$  (then  $\phi(A \setminus Z) = \int_A f d\mu$  with  $f = \lim f_m$ ).

Observe  $E_k^m$  is covered by  $E_j^{m+1}$  for  $j = 2k-2, 2k-1, 2k, 2k+1$  because

$$\left[ \frac{k-1}{2^m}, \frac{k}{2^m} \right] \cap \left[ \frac{j-1}{2^{m+1}}, \frac{j}{2^{m+1}} \right] = \emptyset$$

unless  $j$  is one of the values above. So if  $f_m = (k-1)2^{-m}$  then  $f_{m+1} = j2^{-m-1}$ , where  $2k-2 \leq j \leq 2k+1$ . hence  $|f_m - f_{m+1}| \leq 2^{-m}$ . We get uniformly convergent of  $\{f_m\}$ .  $\square$

Radon-Nikodym's Theorem:

Every AC ASF  $f$  wrt to  $\sigma$ -finite measure is of the form  $\phi(A) = \int_A f d\mu$  for some  $f \in L^1(\mu)$ .

Lebesgue decomposition for measures: Let  $\mu, \nu$  be  $\sigma$ -finite measures on  $(X, \Sigma)$ . Then there exists measurable  $f \geq 0$  and a measure  $\sigma$  such that  $\sigma$  is singular with respect to  $\mu$ .  $\nu(A) = \int_A f d\mu + \sigma(A)$ .

*Proof.* Let  $X = \bigcup E_j$ , disjoint.  $\nu(E_j) < \infty$ .  $\nu|_{E_j}$  is an ASF, so  $\nu(A) = \int f_j d\mu + \sigma_j(A)$  for  $A \subset E_j$ ,  $f_j \in L^1(E_j, \mu)$ . Note  $f_j \geq 0$ .

Let  $\sigma(A) = \sum_j \sigma(A \cap E_j)$  and  $f = \sum_j f_j \chi_{E_j}$  (gluing). Hence  $\nu(A) = \sum_j \nu(A \cap E_j) = \sum_j (\int_{A \cap E_j} f_j d\mu + \sigma_j(A \cap E_j)) = \int_A f d\mu + \sigma(A)$ .  $\square$

## 6.4 The Dual Space of $L^p$

**Definition.** Let  $B$  a normed vector space over  $\mathbb{R}$ . Define

$$\begin{aligned} B^* &= \{ \text{linear } \psi : B \rightarrow \mathbb{R} : \exists c \text{ } |\psi(v)| \leq c \|v\| \forall v \in B \} \\ &= \{ \text{bounded linear functionals} \} \end{aligned}$$

is the dual space of  $L^p$ .

**Theorem 6.5.** Let  $(X, \Sigma, \mu)$  be a  $\sigma$ -finite measure space,  $1 \leq p < \infty$ . Then there is a linear bijection between  $L^p(X, \mu)^*$  and  $L^{p'}(X, \mu)$  which preserves the norm.

Specifically, the bijection sends  $g \in L^p$  to  $\psi_g \in (L^p)^*$ ,  $\psi_g(f) = \int_X fg d\mu$ .

**Example 6.8.**  $(X, \Sigma, \mu) = (\mathbb{N}, 2^{\mathbb{N}}, \text{counting})$ , so  $L^p(X, \mu)$  is  $\ell^p$ : every  $\psi : \ell^p \rightarrow \mathbb{R}$  is  $\psi(x) = \sum_{k=1}^{\infty} x_k y_k$  for some  $y \in \ell^{p'}$ . Note so for  $\ell^\infty$ . One can define  $\psi(x) = \lim_{k \rightarrow \infty} x_k$  when exists, and extend to a bounded linear functional on  $\ell^\infty$ . Suppose we had  $y \in \ell^1$  such that  $\sum_1^{\infty} x_k y_k = \lim_{k \rightarrow \infty} x_k$  for all convergent sequences  $x$ . Problem is limit does not depend on any particular  $x_k$ . So can choose one and add one to it, etc. so we must have  $y_k = 0$ . But this is true for all of them so that we must have  $y = 0$  but then if  $x = 1$  for all  $k$  so sum is not equal to the limit, which is 1, a contradiction.

*Proof.* If  $g \in L^{p'}$ , then  $|\int fg| \leq \|f\|_p \|g\|_{p'}$ , so  $\psi_g$  is bounded:  $|\psi_g(f)| \leq \|g\|_{p'} \|f\|_p$ . So  $\|\psi_g\| \leq \|g\|_{p'}$ . This gives a linear map  $L^{p'} \rightarrow (L^p)^*$ . This map is injective: if  $\int fg = 0$  for all  $f \in L^p$ , then  $g \equiv 0$  a.e., LTR. Surjectivity:  $\sigma$ -finite case can be reduced to  $\mu(X) < \infty$  by  $X = \bigcup_1^{\infty} E_j$ ,  $E_1 \subset E_2 \subset E_3 \subset \dots$ .

We get  $g_j$  on  $E_j$  from the finite case ( $\mu(E_j) < \infty$ ). Glue them into one  $g$  which gives  $\psi(f) = \int_X fg$ . Finite case:  $\mu(X) < \infty$ . Define an ASF  $\phi(A) = \psi(\chi_A)$ . It is AC wrt

$\mu : \mu(A) = 0$  measure  $\chi_A = 0$  a.e., so  $\psi(\chi_A) = 0$ . By Radon-Nikodym, there exists  $g \in L^1(X, \mu)$  such that  $\phi(A) = \int_A g \, d\mu$  for all  $A \subset X$ . This means  $\psi(\chi_A) = \int g \chi_A \, d\mu$ .  $\psi(f) = \int g f \, d\mu$  for all simple  $f$ . Once we know that  $g \in L^p$ , we can conclude the proof as follows:  $\psi$  and  $\psi g$  are continuous on  $L^p$  and  $\psi = \psi g$  on a dense subset (simple functions), hence  $\psi = \psi g$ .

case  $p = 1, p' = \infty$ . Case  $1 < p < \infty, 1 < p' < \infty$  (textbook). for the first then,  $\phi(A) = \psi(\chi_A) \leq \|\psi\| \cdot \|\chi_A\|_1 = \|\psi\| \mu(A)$ . Hence,  $\phi(A)/\mu(A) \leq \|\psi\|$ . In the proof of Lebesgue decomposition, we will not have density  $> \|\psi\|$  because of the inequality just shown. Thus  $\|g\|_{p'} \leq \|\psi\|$ .  $\square$

## 7 Differentiation

Our goal is to ‘reverse engineer’  $f$  knowing  $F(E)$ .

### 7.1 Lebesgue Differentiation (LDT)

The idea is to consider  $\frac{1}{|Q|}F(Q)$ , the average of  $f$  over  $Q$ ,

$$\lim_{Q \searrow x} \frac{F(Q)}{|Q|},$$

where  $Q$  is a cube centered at  $x$  and  $|Q| \rightarrow 0$ .

**Theorem 7.1** (Lebesgue Differentiation Theorem (LDT)). *If  $f \in L^1(\mathbb{R}^n)$ , then for a.e.  $x \in \mathbb{R}^n$ ,*

$$f(x) = \lim_{Q \searrow x} \frac{F(Q)}{|Q|}.$$

**Example 7.1.**  $f = \chi_{[0,1]}$ .

Can we use Lusin’s Theorem to help prove this? That is, use: for all  $\epsilon > 0$ , there exists  $E$  such that  $f|_E$  is continuous and  $|E^c| < \epsilon$ . The answer is no as one cannot neglect  $(\int_{Q \setminus E} f)/|Q|$  as this can be large.

Can the limit not exist at all?

**Example 7.2.**  $f = \sum_{n=1}^{\infty} \chi_{[4^{-n}, 2 \cdot 4^{-n}]}$ .

Intervals disjoint. And sum clearly converges (Why?). Now let  $x = 0$ : then

$$\frac{1}{2 \cdot 4^{-k}} \int_{[-4^{-k}, 4^{-k}]} f = \frac{1}{2 \cdot 4^{-k}} \sum_{n=k+1}^{\infty} 4^{-n} = \frac{4^{-k-1} / (1 - 1/4)}{2 \cdot 4^{-k}} = \frac{1}{6}.$$

but

$$\frac{1}{4 \cdot 4^{-k}} \int_{[-2 \cdot 4^{-k}, 2 \cdot 4^{-k}]} f = \frac{1}{4 \cdot 4^{-k}} \sum_{n=k}^{\infty} 4^{-n} = \frac{4^{-k}(1 - 1/4)}{4 \cdot 4^{-k}} = \frac{1}{3}.$$

How to approximate  $L^1$  function by continuous functions.

**Theorem 7.2.** *The set  $C_c(\mathbb{R}^n)$  (sometimes called  $C_0$ ) (continuous functions with compact support) is dense in  $L^1(\mathbb{R}^n)$ .*

*Proof.* Let  $\mathcal{M} = \overline{C_c(\mathbb{R}^n)}$ . Also a linear subspace of  $L^1$ . Recall linear space of  $\{\chi_I : I \text{ interval}\}$  is dense in  $L^1$ . If we can show  $\chi_I \in \mathcal{M}$ , we are done. ( $\mathcal{M}$  is dense and closed, hence  $\mathcal{M} = L^1$ ). Given  $I$  (an interval) let  $f_k = [1 - k \text{dist}(x, I)]^+$  (positive so once negative call it zero,

$$\|f_k - \chi_I\| \leq |\{x: 0 < \text{dist}(x, I) < 1/k\}| \rightarrow 0.$$

But then  $\chi_I \in \mathcal{M}$ . □

Let us review the situation. Given  $f \in L^1$ , we have  $f_n \in C_c$  such that  $\|f_k - f\|_1 \rightarrow 0$  and  $\frac{1}{|Q|} \int f_k \rightarrow f_k(x)$  as  $Q \searrow x$  by continuity. We want to know  $\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q f$ . From what we know, we have

$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q f = \lim_{Q \searrow x} \lim_{k \rightarrow \infty} \frac{1}{|Q|} \int_Q f_k.$$

What we want to be able to do is interchange the two limits, as then using the continuity of  $f_k$ , we would have

$$\lim_{Q \searrow x} \lim_{k \rightarrow \infty} \frac{1}{|Q|} \int_Q f_k \stackrel{?}{=} \lim_{k \rightarrow \infty} \lim_{Q \searrow x} \frac{1}{|Q|} \int_Q f_k = \lim_{k \rightarrow \infty} f_k(x) = f(x).$$

a.e. after possibly passing to a subsequence. To make the interchange, need one of them to be uniform wrt the other. Uniformity means  $\frac{1}{|Q|} |\int_Q (f - f_k)|$  is small either for large  $k$  and all  $Q$  (independent of  $k$ ) or for small  $Q$  and all  $k$  (independent of  $Q$ ). The second is hopeless, as need uniform convergence for this. So we try the first approach. This leads to the idea of a maximal function.

**Definition** ((Hardy-Littlewood) Maximal Function). Give  $f \in L^1(\mathbb{R}^n)$ . Let

$$f^*(x) = \sup_{Q \text{ cent. at } x} \frac{1}{|Q|} \int_Q |f|,$$

the Hardy-Littlewood maximal function.

We will use this for  $f - f_k$ . Since  $\|f - f_k\|_1 < \epsilon$ , we will obtain  $(f - f_k)^* < \epsilon$  at ‘most points.’

**Example 7.3.**  $f = \chi_{[0,1]}$

$$f^* = \begin{cases} 1, & x \in (0, 1) \\ \frac{1}{2}, & x = 0, 1 \\ \frac{1}{2x}, & x > 1 \\ \frac{1}{2(1-x)}, & x < 0 \end{cases}$$

Note  $f^* \notin L^1(\mathbb{R})$ .

We want to show that  $f^*$  is ‘small’ if  $\|f\|_1$  is small. For this, we will need a lemma.

**Lemma 7.1** (Simple Covering Lemma (Vitali)). *Suppose  $E \subset \mathbb{R}^n$  has finite measure, i.e.  $|E| < \infty$ , and  $E \subset \cup_{Q \in \mathcal{K}} Q$ , where  $\mathcal{K}$  is a collection of cubes. Then there exists a finite disjoint subcollection  $\{Q_1, \dots, Q_N\} \subset \mathcal{K}$  and  $\sum_{j=1}^N |Q_j| \geq 5^{-n} |E|$ .*

*Proof.* Choose a compact set  $F \subset E$  with  $|F| > (1 - \epsilon)|E|$ . For each  $Q \in \mathcal{K}$ , choose  $Q'$  be a slightly larger open cube containing  $Q$ , i.e.  $|Q'| < (1 + \epsilon)|Q|$ . Then  $\{Q' : Q \in \mathcal{K}\}$  is an open cover of  $F$  since this collection covers  $E$ . Choose a finite subcovering  $\mathcal{K}' := \{Q'_1, \dots, Q'_M\}$  of  $F$ . By possibly relabeling, without loss of generality assume  $Q'_1$  be the largest cube in this collection. Again by possibly relabeling, let  $Q'_2$  be the largest cube in this collection disjoint from  $Q'_1$ . Continue this procedure. Since there are finitely many cubes in the collection, this procedure terminates. Say  $\{Q'_1, \dots, Q'_N\}$  were the cubes selected.

We claim  $\{3Q'_1, \dots, 3Q'_N\}$  cover  $F$ . Choose  $x \in F$ . We know that  $x \in Q'_j$  for some  $Q'_j \in \mathcal{K}'$ . If  $Q'_j \in \{Q'_1, \dots, Q'_N\}$ , we are done. If  $Q'_j \notin \{Q'_1, \dots, Q'_N\}$ , there must then exist  $\tilde{Q} \in \mathcal{K}'$  such that  $Q' \cap \tilde{Q} \neq \emptyset$ ,  $|\tilde{Q}| \geq |Q'|$ , and  $\tilde{Q} \in \{Q'_1, \dots, Q'_N\}$ . Since  $Q' \subset 3\tilde{Q}$ ,  $x$  is covered by  $\{Q'_1, \dots, Q'_N\}$ . Hence,

$$\sum |Q'_j| = 3^{-n} \sum |3Q'_j| \geq 3^{-n} |F| > 3^{-n} (1 - \epsilon) |E|.$$

But then  $\sum |Q_j| > 3^{-n} \left(\frac{1-\epsilon}{1+\epsilon}\right) |E| \geq 5^{-n} |E|$ . □

**Theorem 7.3** (Hardy-Littlewood).

$$|\{f^* > \alpha\}| \leq \frac{5^n}{\alpha} \|f\|_1$$

for all  $\alpha > 0$ .

*Proof.* Every  $x \in E$  is covered by  $Q$  with  $\frac{1}{|Q|} \int_Q |f| > \alpha$ . By the Simple Covering Lemma, choose disjoint cubes  $Q_1, \dots, Q_N$ . Then

$$\sum_j \int_{Q_j} |f| = \int_{\cup Q_j} |f| \leq \|f\|_1.$$

But  $\sum \int_{Q_j} |f| \geq \alpha \sum |Q_j| \geq \alpha 5^{-n} |E|$ . Hence,  $|E| \leq \frac{5^n}{\alpha} \|f\|_1$ . □

*Proof of LDT:* Given  $\epsilon > 0$ , choose  $g \in C_c(\mathbb{R}^n)$  such that  $\|f - g\|_1 < 5^{-n} \epsilon^2$ . Let  $A = \{(f - g)^* > \epsilon\}$ . Then by Hardy-Littlewood,  $|A| < \epsilon$ . Let  $B = \{|f - g| > \epsilon\}$ . Then  $|B| < \epsilon$ . For  $x \notin A \cup B$ ,

$$\frac{1}{|Q|} \int_Q |f - f(x)| \leq 2\epsilon + \underbrace{\frac{1}{|Q|} \int_Q |g - g(x)|}_{\rightarrow 0 \text{ as } Q \searrow x}$$

Hence,  $\limsup_{Q \searrow x} \frac{1}{|Q|} \int_Q |f - f(x)| \leq 2\epsilon$  for all  $x \notin A \cup B$ . Then

$$|\{x: \limsup_{Q \searrow x} \frac{1}{|Q|} \int_Q |f - f(x)| \geq 2\epsilon\}| \leq 2\epsilon.$$

But this set tends to 0 in measure so that we obtain  $\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f - f(x)| = 0$  a.e.  $x$ .

Then by this equation,  $\frac{1}{|Q|} \int_Q f \rightarrow f(x)$ .  $\square$

**Remark.** A point where (\*) holds is called a Lebesgue point of  $f$ .

**Example 7.4.**  $f(x) = \operatorname{sgn} x$ . Then  $x = 0$  is not a Lebesgue point.

It suffices for  $f$  to be locally integrable, meaning  $\int_{|x| < k} |f| < \infty$  for all  $k$ . Why? We only need to consider 'local sets' in (\*); that is, apply LDT to  $f\chi_{\{|x| \leq k\}}$  and conclude a.e. point of  $\{|x| < k\}$  is a Lebesgue point of  $f$ . In particular, we can apply to  $f = \chi_E$  since its integral is at most 1. LDT says a.e. point of  $E$  is a point of density,  $|Q \cap E|/|Q| \rightarrow 1$  as  $Q \searrow x$ .

Any cubes containing  $x$  can be allowed, meaning they do not need to be centered at  $x$ .

$$|Q'| \leq 2^n |Q| \frac{1}{|Q|} \int_Q |f - f(x)| \leq \frac{2^n}{|Q'|} \int_{Q'} |f - f(x)| \rightarrow 0.$$

FTC 1: If  $f \in L^1[a, b]$ , then  $\frac{d}{dx} \int_a^x f(t) dt = f(x)$  a.e.  $x$ .

## 7.2 Vitali Covering Lemma

**Definition** (Vitali Covering). A family of cubes  $\mathcal{K}$  covers a set  $E$  in the Vitali sense if every point of  $E$  is covered by arbitrarily small cubes  $Q \in \mathcal{K}$ .

**Lemma 7.2** (Vitali Covering Lemma). Suppose  $\mathcal{K}$  covers  $E$  in the Vitali sense,  $0 < |E| < \infty$ . Then for all  $\epsilon > 0$ , there exist disjoint  $Q_j \in \mathcal{K}$  such that  $|E \setminus \cup_j Q_j| = 0$  and  $\sum |Q_j| < (1 + \epsilon)|E|$ .

*Proof.* Recall the Simple Covering Lemma: disjoint  $Q_j$   $\sum |Q_j| \geq 5^{-n}|E|$ . Choose  $\epsilon < 1/25^{-n}$ . Choose open  $G \supset E$  such that  $|G| < (1 + \epsilon)|E|$ . Discard the cubes not contained in  $G$ , we still have a cover of  $E$ . The Simple Covering gives disjoint  $Q_{0,j}$  such that  $\sum_j |Q_{0,j}| \geq 5^{-n}|E|$ . Hence,  $|G \setminus \cup Q_{0,j}| < (1 + 5^{-n}/2)|E| - 5^{-n}|E| = (1 - 5^{-n}/2)|E|$ . Let  $r$  be this  $r < 1$ . So  $|E \setminus \cup Q_{0,j}| \leq r|E|$ , which is a finite union of closed cubes. Discard any cubes in  $\mathcal{K}$  that meet any  $Q_{0,j}$  remaining cubes cover  $E_1 = E \setminus \cup Q_{0,j}$  (because for all  $x \in E_1$ , there exists a small cube  $Q \in \mathcal{K}$  that contains  $x$  is disjoint from  $\cup Q_{0,j}$ ). Repeat for  $E_1$ , get cubes  $Q_{1,j}$  such that  $|E_1 \setminus \cup Q_{1,j}| \leq r|E_1|$  etc.. Call  $E_2$  the set on left, etc..  $E_1 \supset E_2 \supset E_3 \cdots$ . If  $|E_m| = 0$ , stop. Otherwise,  $Z = \cap E_m$  has  $|Z| = 0$  as  $|E_m| \leq r^m |E| \rightarrow 0$ .  $\square$



**Remark.** If finitely many cubes  $Q_j$  are used, we obtain

$$\left| E \setminus \bigcup_{j=1}^N Q_j \right| < \epsilon.$$

### 7.3 Differentiation of Monotone Functions

Given  $f : (a, b) \rightarrow \mathbb{R}$ , define four Dini numbers at each  $x \in (a, b)$

$$\begin{aligned}\overline{D}_+ f(x) &= \limsup_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h} \\ \overline{D}_- f(x) &= \limsup_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h} \\ \underline{D}_+ f(x) &= \liminf_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h} \\ \underline{D}_- f(x) &= \liminf_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h}\end{aligned}$$

If  $f$  is monotone, we want to prove these are equal and finite a.e.  $x$ .

**Example 7.5.**  $f(x) = \text{floor } x$ . At  $x = 1$

$$\begin{aligned}\overline{D}_+ &= 0 \\ \overline{D}_- &= \infty \\ \underline{D}_+ &= 0 \\ \underline{D}_- &= \infty\end{aligned}$$

If  $f$  is increasing, Dini numbers  $\geq 0$ .

**Example 7.6.** Cantor Lebesgue Staircase Function

For each  $m \in \mathbb{N}$ , the  $[0, 1] \setminus C$  contains  $2^m - 1$  intervals of size  $\geq 3^{-m}$ . [Why? because  $2^{m-1}$  holes of size  $3^{-m}$  so those  $\geq 3^{-m}$  are  $1 + 2 + 2^2 + \dots + 2^{m-1} = 2^m - 1$ .] Let  $f(x) = \frac{k}{2^m}$ ,  $k = 1, 2, \dots, 2^m - 1$  on those intervals. This defines  $f : [0, 1] \setminus C \rightarrow \mathbb{R}$ . If  $|x - y| \leq 3^{-m}$ , then  $|f(x) - f(y)| \leq 2^{-m}$  (exercise, uniformly contin). hence,  $f$  uniformly continuous. So has extension to the closure of its domain.

## 8 Quiz Problems

**Quiz 1.** Suppose  $A$  is a closed proper subset of  $[0, 1]$  (so  $A \subset [0, 1]$  and  $A \neq [0, 1]$ ). Is  $|A|_e < 1$ ?

**Quiz 2.** Suppose  $A \subset \mathbb{R}$  is a set such that  $A \cap [-k, k]$  is measurable for every  $k \in \mathbb{N}$ . Is it the case that  $A$  is measurable?

**Quiz 3.** Suppose  $A_1 \supset A_2 \supset A_3 \supset \cdots$  is a nested sequence of measurable subsets of  $\mathbb{R}^n$  such that  $|A_1| < \infty$ . Is it the case that  $|A_k \setminus A_{k+1}| \rightarrow 0$  as  $k \rightarrow \infty$ ?

**Quiz 4.** Suppose  $E \subset \mathbb{R}^n$  is a set such that  $|A \cap E|_e + |A \cap E^c|_e \leq |A|_e$  for every  $A \subset \mathbb{R}^n$ . Is it the case that  $E$  is measurable?

**Quiz 5.** Let  $f(x)$  be the function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$$

Is it the case that  $f$  is a measurable function on  $\mathbb{R}$ ?

**Quiz 6.** Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is measurable. Define

$$g(x) = \begin{cases} f(x), & |f(x)| \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Is  $g$  necessarily a measurable function on  $\mathbb{R}$ ?

**Quiz 7.** Let  $f(x)$  be the function given by

$$f(x) = \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Is  $f$  upper semicontinuous on  $\mathbb{R}$ ?

**Quiz 8.** Suppose that  $\{f_k : [0, 1] \rightarrow \mathbb{R}\}$  is a collection of simple functions which converge uniformly to  $f$ . Is  $f$  necessarily a simple function?

**Quiz 9.** Suppose that  $f : [0, \infty) \rightarrow [0, \infty)$  is measurable. Is it the case that

$$\int_{[0, \infty)} e^{-x/k} f(x) \, dx \longrightarrow \int_{[0, \infty)} f(x) \, dx$$

as  $k$  tends to infinity?

**Quiz 10.** Suppose that  $f : [0, \infty) \rightarrow [0, \infty)$  is measurable and  $\int_{[0, \infty)} f < \infty$ . Is it true that  $\int_{[k, \infty)} f(x) \, dx \rightarrow 0$  as  $k \rightarrow \infty$ ?

## 9 Quiz Solutions

**Solution Quiz 1.** Suppose that  $f : [0, \infty) \rightarrow [0, \infty)$  is measurable and  $\int_{[0, \infty)} f < \infty$ . Is it true that  $\int_{[k, \infty)} f(x) dx \rightarrow 0$  as  $k \rightarrow \infty$ ?

*Proof.* Let  $f_k := f\chi_{[k, \infty)}$ . Then goes to 0 as  $k \rightarrow \infty$  and  $f$  dominates  $f_k$  and  $f_k \rightarrow 0$ .

Another idea:  $f\chi_{(0, k)} \nearrow f$  by MCT,  $\int \chi_{(0, k)} f \rightarrow \int f$ .

Or

$R(f, E)$  has finite measure, have nested subsets from  $k$  to infinite areas, nested, contained, measure sets tend to measure of intersection which is 0?

**Solution Quiz 2.** Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is measurable and the function  $g(x, y) = f(x)f(y)^2$  is integrable on  $\mathbb{R}^2$ . Is  $f$  integrable on  $\mathbb{R}^2$ , i.e. is  $f \in L^1(\mathbb{R})$ ?

*Proof.* We do indeed have  $f \in L^1(\mathbb{R})$ . The function given by  $x \mapsto f(x)f(y)^2$  is integrable a.e. on  $y$ . There are two cases:  $f(y) = 0$  for almost all  $y$  or there exists  $y$  such that  $f$  is integrable and  $f(y) \neq 0$ . In either case,  $f$  is integrable.  $\square$

**Solution Quiz 3.** Let  $f(x) = 2$  and  $g(x) = 3$  for all  $x \in \mathbb{R}$ . Is the convolution  $f * g$  equal to 6 everywhere on  $\mathbb{R}$ ?

*Proof.* It is the case that  $f * g = 6$  as

$$(f * g)(x) = \int_{\mathbb{R}} f(x-t)g(t) dt = \int_{\mathbb{R}} 6 dt = \infty.$$

**Solution Quiz 4.** Suppose that  $f, g \in L^\infty(\mathbb{R}^n)$ . Is it the case that  $fg \in L^\infty(\mathbb{R}^n)$ ?

*Proof.* We have  $|fg| = |f| |g| \leq |f| \|g\|_\infty$  for almost all  $x \in \mathbb{R}^n$ . But then

$$\int |fg| \leq \int |f| \|g\|_\infty = \|g\|_\infty \int |f| = \|f\|_\infty \|g\|_\infty.$$

Alternatively,  $f, g \in L^\infty(\mathbb{R}^n)$  implies that  $f, g$  are bounded almost everywhere. Suppose  $|f| \leq M$  and  $|g| \leq N$  for almost all  $x \in \mathbb{R}^n$ . But then  $|fg| \leq MN$  for almost all  $x \in \mathbb{R}^n$ . Therefore,  $fg \in L^\infty(\mathbb{R}^n)$ . Note that it is not generally true that if  $f, g \in L^p$ , for some  $p < \infty$ , that  $fg \in L^p$ . For example,  $\frac{1}{x^r} \in L^p((0, 1))$  if and only if  $pr < 1$  if and only if  $p < 1/r$ . Then generally,  $(1/x^r)^2$  is not in the same  $L^p$ .  $\square$

**Solution Quiz 5.** Suppose that  $f, g : \mathbb{R} \rightarrow \mathbb{C}$  are measurable functions such that  $|f|^4$  and  $|g|^{4/3}$  are integrable. Prove or disprove:  $fg$  is integrable.

*Proof.* We know  $\int |fg| \leq \|f\|_4 \|g\|_{4/3}$  and  $\frac{1}{4} + \frac{3}{4} = 1$ . This fails if  $f, g$  are not measurable. Take  $g = 1$  and  $f$  a  $(-1)^n$  type of deal on a nonmeasurable set.  $\square$

**Solution Quiz 6.** Suppose that  $0 < p \leq \infty$  and that the sequence  $\{a_k\}$  belongs to  $\ell^p$ . Is it the case that every subsequence  $\{a_{k_j}\}$  belongs to  $\ell^p$ ?

*Proof.*  $p < \infty$ ,  $\sum |a_{k_j}|^p \leq \sum |a_k|^p$  and  $p = \infty$ ,  $a_k \leq M$  then  $a_{k_j} \leq M$ .

**Solution Quiz 7.** For  $k \in \mathbb{N}$ , let  $f_k = \sqrt{k}\chi_{[0,1/k]}$ . Is it the case that  $f_k \rightarrow 0$  in  $L^2(\mathbb{R})$ ?

*Proof.* This is not the case as

$$\|\sqrt{k}\chi_{[0,1/k]}\|_2 = \left( \int k\chi_{[0,1/k]} \right)^{1/2} = 1.$$

However, this does converge for  $p < 2$  as

$$\int_0^{1/k} k^{p/2} = k^{p/2-1} \rightarrow 0$$

if  $p < 2$ .  $\square$

## 10 Homework Problems

### Lebesgue Outer Measure

**Problem 1.** Prove that for every set  $E \subset \mathbb{R}^n$  and every  $\epsilon > 0$ , the Lebesgue outer measure  $|E|_e$  is equal to

$$\inf \left\{ \sum v(I_k) : E \subset \bigcup_{k=1}^{\infty} I_k, \text{ and } \text{diam } I_k < \epsilon \forall k \right\}$$

[Note: This is the same infimum as in the definition of  $|E|_e$  but with the additional requirement  $\text{diam } I_k < \epsilon$  for all  $k$ .]

**Problem 2.** Suppose that the sets  $E_k \subset \mathbb{R}^n$  are such that the series  $\sum_{k=1}^{\infty} |E_k|_e$  converges. Prove that the outer measure of the set

$$\limsup E_k := \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k$$

is zero.

### Measurable Sets

**Problem 3.** Given an arbitrary set  $A \subset \mathbb{R}$  and a number  $c > 0$ , let  $B = \{ca : a \in A\}$ . Prove that  $|B|_e = c|A|_e$ .

**Problem 4.** Suppose that a set  $A \subset \mathbb{R}$  is measurable. Prove that for every  $c > 0$  the set  $B = \{ca : a \in A\}$  is also measurable.

**Problem 5.** Given a sequence of continuous functions  $f_k : \mathbb{R} \rightarrow \mathbb{R}$ , let  $B$  be the set of all points  $x \in \mathbb{R}$  such that the sequence  $\{f_k(x)\}$  is bounded. Prove that  $B$  is a measurable set. [Hint: try to construct  $B$  from the sets  $\{x : |f_k(x)| \leq M\}$  by using countable unions and intersections.]

**Problem 6.** Given a sequence of continuous functions  $f_k : \mathbb{R} \rightarrow \mathbb{R}$ , let  $C$  be the set of all points  $x \in \mathbb{R}$  such that  $\lim_{k \rightarrow \infty} f_k(x) = 0$ . Prove that  $C$  is a measurable set.

### Properties of the Lebesgue Measure

**Problem 7.** Prove that the set

$$A = \{x \in \mathbb{R} : \exists k \in \mathbb{N} \text{ such that } |2^x - 2^k| \leq 1\}$$

is measurable and  $|A| < \infty$ .

**Problem 8.** Prove that the set

$$A = \{x \in [0, 1] : \forall q \in \mathbb{N} \exists p \in \mathbb{N} \text{ such that } |x - p/q| \leq 1/q^2\}$$

is measurable and  $|A| = 0$ .

**Problem 9.** Suppose  $E$  and  $Z$  are sets in  $\mathbb{R}^n$  such that  $E \cup Z$  is measurable and  $|Z| = 0$ . Prove that  $E$  is measurable.

**Problem 10.** Given a continuous function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , define  $\mathcal{M} = \{E \subset \mathbb{R}^n : f^{-1}(E) \text{ is Borel}\}$ .

- (a) Prove that  $\mathcal{M}$  is a  $\sigma$ -algebra.
- (b) Prove that if  $E$  is Borel, then  $f^{-1}(E)$  is Borel. [Hint: Use (a).]

### Lipschitz Transformations

**Problem 11.** Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function with a continuous derivative. Prove that for every measurable set  $E$ , the set  $f(E)$  is also measurable. [Hint: Although  $f$  need not be Lipschitz, its restriction to any bounded interval is.]

**Problem 12.** Given a set  $E \subset [0, \infty)$ , define a function  $f : [0, \infty) \rightarrow [0, \infty)$  by  $f(x) = |E \cap [0, x]|_e$ .

- (a) Prove that  $f$  is Lipschitz continuous.
- (b) Prove that for every number  $b$  with  $0 < b < |E|_e$  there exists a set  $F \subset E$  such that  $|F|_e = b$ .

### Nonmeasurable Sets

**Problem 13.** Show that there exists a nested sequence of sets  $E_1 \supset E_2 \supset \cdots$  such that  $|E_1|_e < \infty$  and  $\bigcap_{k=1}^{\infty} E_k = \emptyset$  but  $\lim_{k \rightarrow \infty} |E_k|_e > 0$ . That is, the outer measure is not continuous under nested intersections. [Hint: Use the translates of the Vitali set.]

**Problem 14.** Show that for the standard middle-third Cantor set  $C \subset [0, 1]$ , the difference set  $C - C$  contains a neighborhood of 0. [Hint:  $C$  is the intersection of nested sets  $C_n$ , where  $C_0 = [0, 1]$  and  $C_{n+1} = \frac{1}{3}C_n \cup (\frac{1}{3}C_n + \frac{2}{3})$ . Find  $C_n - C_n$  using induction.]

Remark: This shows that having  $|E| > 0$  is not necessary for  $E - E$  to contain a neighborhood of 0.

### Measurable Functions

**Problem 15.** Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a function such that  $f(\mathbb{R}^n)$  is countable, and  $f^{-1}(t)$  is measurable for every  $t \in \mathbb{R}$ . Prove that  $f$  is measurable.

**Problem 16.** Prove that without the assumption “ $f(\mathbb{R}^n)$  is countable” the statement in the previous problem would not be true.

**Problem 17.** Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is measurable, and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is continuously differentiable with  $g' > 0$  everywhere. Prove that  $f \circ g$  is measurable.

**Problem 18.** (a) Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuous function such that  $f^2$  is measurable. Prove that  $f$  is measurable.

(b) Prove that the statement in (a) is false if  $f$  is not assumed continuous.

### Semicontinuous Functions & Bounded Variation

**Problem 19.**

(a) Let  $E \subset \mathbb{R}^n$  be a set. Consider a sequence of lsc functions  $f_k : E \rightarrow \overline{\mathbb{R}}$  such that  $f_1 \leq f_2 \leq f_3 \leq \cdots$ . Prove that  $\lim_{k \rightarrow \infty} f_k$  is also an lsc function. [Note: The limit here is understood in the sense of the extended real line  $\overline{\mathbb{R}}$ , so it is assured to exist by monotonicity.]

(b) Give an example that shows (a) fails with “lsc” replaced by “usc.”

**Problem 20.** Fix  $a > 0$  and define  $f : [0, 1] \rightarrow \mathbb{R}$  so that  $f(1/k) = 1/k^a$  for  $k \in \mathbb{N}$ , and  $f(x) = 0$  for all other  $x$ . Prove that  $f$  is of bounded variation on  $[0, 1]$  when  $a > 1$ , and is not of bounded variation on  $[0, 1]$  when  $0 < a \leq 1$ .

### Egorov & Lusin’s Theorem

**Problem 21.** Suppose that  $f : E \rightarrow \mathbb{R}$  is a measurable function, where  $E \subset \mathbb{R}^n$  is measurable.

- (a) Prove that there exists a Borel set  $H \subset E$  such that the restriction  $f|_H$  is Borel measurable and  $|E \setminus H| = 0$ . [Hint: Take a countable union of closed sets obtained from Lusin’s theorem.]
- (b) If, in addition,  $E$  is a Borel set, prove that there exists a Borel measurable function  $g : E \rightarrow \mathbb{R}$  such that  $f = g$  a.e..

**Problem 22.** Suppose  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a function such that  $\phi(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Consider a sequence of measurable functions  $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $|f_k(x)| \leq \phi(|x|)$  for every  $k$ , and  $f_k \rightarrow f$  a.e.. Prove that the conclusion of Egorov’s Theorem holds in this situation: that is, for every  $\epsilon > 0$  there exists a closed set  $E(\epsilon) \subset \mathbb{R}^n$  such that  $|\mathbb{R}^n \setminus E(\epsilon)| < \epsilon$  and  $f_k \rightarrow f$  uniformly on  $E(\epsilon)$ . [Hint: Follow the proof of Egorov’s Theorem.]

### Convergence in Measure

**Problem 23.** Suppose that  $f, g, f_k, g_k$  ( $k \in \mathbb{N}$ ) are measurable functions on a set  $E \subset \mathbb{R}^n$  with values on  $\mathbb{R}$ , and that  $f_k \xrightarrow{m} f$  and  $g_k \xrightarrow{m} g$ . Prove that  $f_k + g_k \xrightarrow{m} f + g$ .

**Problem 24.**

- (a) In addition to the assumptions in the previous problem, suppose  $|E| < \infty$ . Prove that  $f_k g_k \xrightarrow{m} f g$ .
- (b) Show by an example that the assumption  $|E| < \infty$  cannot be omitted in (a). [Hint: You need some boundedness to control the terms in  $f_k g_k - f g = (f_k - f)g_k + f(g_k - g)$ . Consider that when  $\varphi : E \rightarrow \mathbb{R}$  is measurable,  $E = \bigcup_j \{|\varphi| \leq j\}$ .]



### Integral of Non-negative Functions

**Problem 25.** Suppose that  $f : E \rightarrow [0, \infty)$  is a measurable function, where  $E \subset \mathbb{R}^n$ . Prove that  $\int_E f$  is finite if and only if the series

$$\sum_{j=-\infty}^{\infty} 2^j |\{x \in E : f(x) > 2^j\}|$$

converges. [Hint: consider the sets  $E_k = \{2^k < f \leq 2^{k+1}\}$  and the function  $g(x) = \sum 2^k \chi_{E_k}$ . Compare  $\int_E g$  to the sum of series, and also to  $\int_E f$ .]

**Problem 26.** Let  $B = \{x \in \mathbb{R}^n : |x| < 1\}$ .

(a) Prove that  $\int_B |x|^{-p} dx$  is finite when  $0 < p < n$  and infinite when  $p \geq n$ .

(b) Prove that  $\int_{B^c} |x|^{-p} dx$  is finite when  $p > n$  and infinite when  $0 < p \leq n$ .

**Problem 27.**

(a) Suppose that  $f_k : E \rightarrow [0, \infty]$  (where  $E \subset \mathbb{R}^n$ ) are measurable functions such that  $\int_E f_k \rightarrow 0$  as  $k \rightarrow \infty$ . Prove that  $f_k \xrightarrow{m} 0$ .

(b) Give an example where  $f_k \xrightarrow{m} 0$  but  $\int_E f_k \not\rightarrow 0$ .

**Problem 28.** For  $k \in \mathbb{N}$  define  $f_k : [0, 1] \rightarrow [0, \infty]$  by

$$f_k(x) = \sum_{j=1}^k \chi_{I(j,k)}, \quad \text{where } I(j,k) = \left[ \frac{j}{k} - \frac{1}{k^3}, \frac{j}{k} + \frac{1}{k^3} \right].$$

Let  $f = \sum_{k=1}^{\infty} f_k$ . Prove that  $\int_{[0,1]} f < \infty$ .

**Problem 29.** Suppose that  $f : \mathbb{R}^n \rightarrow [0, \infty)$  is a measurable function such that  $\int_{\mathbb{R}^n} f < \infty$ . Also suppose  $\{E_k\}$  is a sequence of measurable sets  $E_k \subset \mathbb{R}^n$ . Let  $A = \limsup_{k \rightarrow \infty} E_k$  and

$B = \liminf_{k \rightarrow \infty} E_k$ . Prove that

$$\int_A f \geq \limsup_{k \rightarrow \infty} \int_{E_k} f$$

and

$$\int_B f \leq \liminf_{k \rightarrow \infty} \int_{E_k} f.$$

[Hint:  $\int_E f = \int_{\mathbb{R}^n} \chi_E f$ .]

**Problem 30.** Suppose that  $f : \mathbb{R}^n \rightarrow [0, \infty)$  is a measurable function such that  $\int_{\mathbb{R}^n} f < \infty$ . Prove that  $\int_{\mathbb{R}^n} e^{-k|x|} f(x) \rightarrow 0$  as  $k \rightarrow \infty$ .

### Integral of Measurable Functions

**Problem 31.** Prove that under the assumptions of the Lebesgue Dominated Convergence Theorem,  $\int_E |f_k - f| \rightarrow 0$  as  $k \rightarrow \infty$ .

**Problem 32.** Let  $f \in L^1(E)$ , where  $E \subset \mathbb{R}^n$  is a measurable set. Prove that

$$\lim_{k \rightarrow \infty} k \int_E \sin\left(\frac{f}{k}\right) = \int_E f.$$

**Problem 33.** Let  $f : E \rightarrow \mathbb{R}$  be a measurable function. Suppose that  $|E| < \infty$  and there exists a number  $p > 1$  such that

$$\limsup_{\alpha \rightarrow \infty} \alpha^p |\{x \in E : |f(x)| > \alpha\}| < \infty.$$

Prove that  $f \in L^1(E)$ . [Hint: use an exercise from Homework 25.]

**Problem 34.** Give an example of a sequence of integrable functions  $f_k : [0, 1] \rightarrow \mathbb{R}$  such that  $f_k \rightarrow f$  a.e.,  $\lim_{k \rightarrow \infty} \int_{[0,1]} f_k$  exists and is finite, but  $f$  is not integrable on  $[0, 1]$ . [Hint: approximate  $1/x$  by functions with integral 0.]

### Lebesgue, Riemann, & Riemann-Stieltjes Integrals

**Problem 35.** Determine the Riemann-Stieltjes integral  $\int \alpha d(-\omega_f(\alpha))$  corresponding to  $\int_E f$  where  $E = (0, 3)$  and  $f(x) = x + \lfloor x \rfloor$ . [You do not need to evaluate the integral. Here  $\lfloor x \rfloor$  is the greatest integer not exceeding  $x$ .]

**Problem 36.** Suppose  $E \subset [0, 1]$ . Prove that  $\chi_E$  is Riemann integrable on  $[0, 1]$  if and only if  $|\partial E| = 0$ .

### Fubini's & Tonelli's Theorem

**Problem 37.**

- (a) Suppose  $E \subset \mathbb{R}^2$  is a Borel set. For  $x \in \mathbb{R}$ , let  $E_x = \{y \in \mathbb{R} : (x, y) \in E\}$ . Prove that  $E_x$  is a Borel set in  $\mathbb{R}$ . [Hint: For a fixed  $x$ , prove that  $\{A \subset \mathbb{R}^2 : A_x \text{ is Borel in } \mathbb{R}\}$  is a  $\sigma$ -algebra that contains all open subsets of  $\mathbb{R}^2$ .]
- (b) Suppose  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a Borel measurable function. Prove that for every  $x \in \mathbb{R}$ , the function  $g(y) = f(x, y)$  is Borel measurable on  $\mathbb{R}$ . [Note: In contrast with Fubini's Theorem, this is no "a.e." here.]

**Problem 38.** Suppose  $f : [0, 1] \rightarrow \mathbb{R}$  is a measurable function such that the function  $g(x, y) = f(x) - f(y)$  is in  $L^1([0, 1]^2)$ . Prove that  $f \in L^1([0, 1])$ .

**Problem 39.** Prove that for any  $a > 0$  the function  $f(x, y) = e^{-xy} \sin x$  is in  $L^1(E)$ , where  $E = \{(x, y) \in \mathbb{R}^2, x > 0, y > a\}$ .

Remark: We proved in discussing Chapter 5.5 that if a Riemann integral  $\int_a^b h(x) dx$  exists (with  $a, b$  finite), then it is equal to the Lebesgue integral. This can be extended to improper Riemann integrals in two ways:

First, if  $h \geq 0$  and  $\int_a^b h(x) dx$  exists as an Improper Riemann Integral, then it is still equal to the Lebesgue integral, by the MCT (replace  $h$  with  $\min(h, k)\chi_{[-k, k]}$  and let  $k \rightarrow \infty$ .)

Second, if  $h \in L^1((a, b))$  and the improper Riemann integral  $\int_a^b h(x) dx$  exists, then the two integrals are equal. Indeed,  $\int_c^d h(x) dx$  is equal to the Lebesgue integral for any  $a < c < d < b$ , by the above result from Chapter 5.5. As  $c \rightarrow a$  or  $d \rightarrow b$ , we can pass to the limit in the Lebesgue Integral by the DCT ( $|h|$  is dominating), and in the Riemann integral, by the definition of an improper Riemann integral.

**Problem 40.** Apply Fubini's Theorem to the function  $f$  in the previous to prove that

$$\int_0^\infty \frac{e^{-ax} \sin x}{x} dx = \tan^{-1}(1/a).$$

[Hint: Integrate  $f$  in two different ways. You do not have to do the antiderivative  $\int e^{-xy} \sin x dx$  by hand; just look it up. Food for thought (not a part of the homework): how to let  $a \rightarrow 0$ ?]

### Application of Fubini's Theorem & Tonelli's Theorem

**Problem 41.** Suppose  $f \in L^1([0, 1])$ . Let  $g(x) = \int_{[x, 1]} \frac{f(t)}{t} dt$  for  $x \in (0, 1]$ . Prove that  $g \in L^1((0, 1])$  and  $\int_{(0, 1]} g = \int_{[0, 1]} f$ .

**Problem 42.** Prove that convolution is associative; that is, for  $f, g, h \in L^1(\mathbb{R}^n)$  we have  $(f * g) * h = f * (g * h)$ . [Note: We do not yet have the full change of variables formula, but we do have  $\int_{\mathbb{R}^n} f(x - y) dx = \int_{\mathbb{R}^n} f(x) dx$  as a consequence of the invariance of measure under translation.]

**Problem 43.** Suppose that  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz function. Let  $Z = \{x : g(x) = 0\}$  and suppose that  $\mathbb{R} \setminus Z$  is bounded. Let  $f(x) = 1/x^2$ . Prove that the convolution  $f * g$  is finite a.e. on  $Z$ .

**Problem 44.** Let  $C \subset [0, 1]$  be the standard “middle third” Cantor set. Let  $\delta(x) = \text{dist}(x, C)$ . For which positive numbers  $p$  is the function  $\delta^{-p}$  in  $L^1([0, 1])$ ? [Note: Although Tonelli could be applied here, it is easier to use the countable additivity of integral over the set of integration.]

### L<sup>p</sup> Classes

**Problem 45.** Prove that for any  $q \in (0, \infty]$ , there exist:

- (a) A function  $f : [2, \infty) \rightarrow \mathbb{R}$  such that  $f \in L^p([2, \infty)) \iff p > q$ .
- (b) A function  $f : [2, \infty) \rightarrow \mathbb{R}$  such that  $f \in L^p([2, \infty)) \iff p \geq q$ .

[Hint: Use a suitable power of  $x$ , with a logarithmic factor, if necessary. Recall that for nonnegative functions, the Improper Riemann Integral agrees with the Lebesgue Integral (Theorem 5.53 of the text).]

**Problem 46.** Prove that for any  $q \in (0, \infty]$ , there exist:

- a) A function  $f : (0, 1) \rightarrow \mathbb{R}$  such that  $f \in L^p((0, 1)) \iff p < q$ ;
- b) A function  $f : (0, 1) \rightarrow \mathbb{R}$  such that  $f \in L^p((0, 1)) \iff p \leq q$ .

Using this and the previous problem, show that for any interval  $J \subset (0, \infty]$ , there exists a function  $f$  on some set  $E \subset \mathbb{R}$  such that  $f \in L^p(E) \iff p \in J$ .

### Hölder & Minkowski's Inequality

**Problem 47.** Fix  $r \in (0, 1)$ .

(a) Suppose  $f \in L^p([2, \infty))$ , where  $1 \leq p < 1/(1 - r)$ . Prove that

$$\int_2^\infty \frac{|f(x)|}{x^r} dx < \infty.$$

(b) Show that the statement in (a) fails with  $p = 1/(1 - r)$ . [Hint:  $\int_2^\infty \frac{1}{x \log x} dx = \infty$ .]

**Problem 48.** Given any sequence  $\{x_1, x_2, \dots\}$  of real numbers, define

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2 \sqrt{|x - x_k|}}.$$

Prove that  $f \in L^p([0, 1])$  for  $0 < p < 2$ .

### Sequence Classes

**Problem 49.** Suppose  $1 \leq p \leq \infty$  and  $f \in L^p([1, \infty))$ . Define a sequence  $a$  by  $a_k = \int_k^{k+1} f$ ,  $k \in \mathbb{N}$ . Prove that  $a \in \ell^p$ .

**Problem 50.** Give an example of a continuous function  $f : [1, \infty) \rightarrow \mathbb{R}$  such that the sequence  $a$  defined in the previous problem is in  $\ell^1$ , but  $f \notin L^1(\mathbb{R})$ . [Hint:  $f$  should attain both positive and negative values so that there is some cancellation in  $\int_k^{k+1} f$ .]

### Banach Space Properties for $L^p$ and $\ell^p$

**Problem 51.** Suppose that  $p, p' \in [1, \infty]$  are conjugate exponents,  $f_k \rightarrow f$  in  $L^p(E)$ , and  $g_k \rightarrow g$  in  $L^{p'}(E)$ , where  $E$  is some measurable set. Prove that  $f_k g_k \rightarrow fg$  in  $L^1(E)$ .

**Problem 52.** Fix  $p \in [1, \infty]$ . Let  $D = \{a \in \ell^p : \forall k \in \mathbb{N} \ 0 \leq a_{k+1} \leq a_k\}$  be the set of all nonnegative non-increasing sequences in  $\ell^p$ . Prove that  $D$  is a closed subset of  $\ell^p$ .

### Additive Set Function Measures

**Problem 53.** Let  $(X, \Sigma, \mu)$  be a measure space. For  $A, B \in \Sigma$  let  $d(A, B) = \mu(A \triangle B)$ , where  $A \triangle B = (A \setminus B) \cup (B \setminus A)$  is the symmetric difference of  $A$  and  $B$ . Prove that  $d$  satisfies the triangle inequality:  $d(A, B) \leq d(A, C) + d(B, C)$  for  $A, B, C \in \Sigma$ .

**Problem 54.** Fix a function  $w \in L^1(\mathbb{R}^n)$  and define the additive set function  $\phi$  on the Lebesgue measurable subsets of  $\mathbb{R}^n$  by  $\phi(E) = \int_E w$ . Prove that the variations of  $\phi$  are given by  $\bar{V}(E) = \int_E w^+$ ,  $\underline{V}(E) = \int_E w^-$ , and  $V(E) = \int_E |w|$ .

### Hilbert Space Properties of $L^p$

**Problem 55.** For  $k \in \mathbb{N}$  let  $\phi_k(t) = \sqrt{1/\pi} \sin(kt)$ .

- (a) Prove that  $\{\phi_k : k \in \mathbb{N}\}$  is an orthonormal system in  $L^2([0, 2\pi])$ . [Hint: product-of-sines formula.]
- (b) Prove that the linear span of  $\{\phi_k\}$  is not dense in  $L^2([0, 2\pi])$ . [Hint: compute  $\langle f, \phi_k \rangle$  for the constant function  $f \equiv 1$ .]

**Problem 56.**

- (a) Prove that for every  $f \in L^2([0, 2\pi])$

$$\lim_{k \rightarrow \infty} \int_{[0, 2\pi]} f(t) \sin kt \, dt = 0.$$

- (b) Prove that (a) holds for every  $f \in L^1([0, 2\pi])$ ; this is known as the Riemann-Lebesgue Lemma. [Hint: Apply (a) to a simple function  $g$  such that  $\|f - g\|_1$  is small.]

### Measurable Functions & Integration

**Problem 57.** Let  $(X, \Sigma, \mu)$  be a measure space, and let  $f : X \rightarrow \mathbb{R}$  be a measurable function. For each Borel set  $E \subset \mathbb{R}$  define  $\nu(E) = \mu(f^{-1}(E))$ . Prove that  $\nu$  is a measure on the Borel  $\sigma$ -algebra of  $\mathbb{R}$ . [This measure called the pushforward of  $\mu$  under  $f$ .]

**Problem 58.** With the notation of the previous problem, prove that for every nonnegative Borel function  $g : \mathbb{R} \rightarrow [0, \infty)$  the function  $g \circ f$  is measurable on  $X$  and

$$\int_X (g \circ f) \, d\mu = \int_{\mathbb{R}} g \, d\nu.$$

[Hint: Begin with  $g = \chi_E$  and proceed toward more general  $g$ .]

### Absolutely Continuous Functions and Singular ASF

**Problem 59.** Let  $(X, \Sigma, \mu)$  be a measure space, and let  $f : X \rightarrow \overline{\mathbb{R}}$  be an integrable function; that is,  $f \in L^1(X, \mu)$ . Suppose that  $\int_A f = 0$  for every  $A \in \Sigma$ . Prove that  $f = 0$   $\mu$ -a.e.; that is,  $f = 0$  on  $X \setminus Z$ , where  $\mu(Z) = 0$ .

**Problem 60.** Let  $(X, \Sigma, \mu)$  be a measure space. Suppose that  $\phi_k : \Sigma \rightarrow \mathbb{R}$  is a singular ASF w.r.t  $\mu$ , for each  $k \in \mathbb{N}$ . Suppose further that  $\phi : \Sigma \rightarrow \mathbb{R}$  is an ASF such that  $\phi_k(A) \rightarrow \phi(A)$  for each  $A \in \Sigma$ . Prove that  $\phi$  is singular with respect to  $\mu$ .

**Problem 61.** For  $k \in \mathbb{N}$ , define  $b_k : [0, 1) \rightarrow \mathbb{R}$  by  $b_k(x) = 1$  if  $\lfloor 2^k x \rfloor$  is odd, and  $b_k(x) = 0$  otherwise. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{2b_k(x)}{3^k}.$$

Prove that:

- (a)  $f$  is a measurable function on  $[0, 1)$  with respect to the Lebesgue measure.
- (b)  $f([0, 1]) \subset C$ , where  $C$  is the standard middle-third Cantor set.

[Hint: You can use the following characterization of  $C$ ,

$$C = \{x \in [0, 1] : \text{dist}(3^m x, \mathbb{Z}) \leq 1/3 \text{ for } m = 0, 1, 2, \dots\}.$$

**Problem 62.** Let  $\sigma$  be the pushforward of the Lebesgue measure on  $[0, 1)$  under  $f$  from the previous problem; that is,  $\sigma(A) = |f^{-1}(A)|$  for Borel sets  $A \subset \mathbb{R}$ . Prove that:

- (a)  $\sigma$  is singular with respect to the Lebesgue measure on the Borel  $\sigma$ -algebra.
- (b)  $\sigma(\{p\}) = 0$  for every  $p \in \mathbb{R}$ . [Hint: Show that for distinct  $x, y \in [0, 1)$ , there exists  $k$  such that  $b_k(x) \neq b_k(y)$ . Deduce that  $f(x) \neq f(y)$ .]

## 11 Homework Solutions

**Solution Problem 1.** Prove that for every set  $E \subset \mathbb{R}^n$  and every  $\epsilon > 0$ , the Lebesgue outer measure  $|E|_e$  is equal to

$$\inf \left\{ \sum v(I_k) : E \subset \bigcup_{k=1}^{\infty} I_k, \text{ and } \text{diam } I_k < \epsilon \forall k \right\}$$

[Note: This is the same infimum as in the definition of  $|E|_e$  but with the additional requirement  $\text{diam } I_k < \epsilon$  for all  $k$ .]

*Proof.* Let  $S_1$  be the set of all sums  $\sum v(I_k)$ , where  $\{I_k\}$  is any countable cover of  $E$  by intervals  $I_k$ , and let  $S_2$  be the set of all sums  $\sum v(I_k)$ , where  $\{I_k\}$  is a countable cover of  $E$  by intervals  $I_k$  which satisfy  $\text{diam } I_k < \epsilon$  for all  $k$ . By definition,  $|E|_e = \inf S_1$ . We want to show  $|E|_e = \inf S_2$ . To prove this, we shall show  $S_2 = S_1$ .

The fact that  $S_2 \subset S_1$  is immediate from the definitions of both sets. Now suppose  $z \in S_1$ . By the definition of  $S_1$ , there exists a countable collection of intervals  $\{I_k\}$  such that  $E \subset \bigcup_k I_k$  and  $\sum_k v(I_k) = z$ . For each  $k$ , let  $L_k$  be the maximal side-length of  $I_k$ ; that is,  $L_k = \max_{j=1, \dots, n} \{b_j - a_j\}$ . For each  $k$ , choose an integer  $N_k$  sufficiently large so that  $L_k/N_k < \epsilon/\sqrt{n}$ . Dividing each edge  $[a_j, b_j]$  in  $I_k$  into equal 1-dimensional subintervals breaks  $[a_j, b_j]$  into  $N_k^n$  equal  $n$ -dimensional subintervals of  $I_k$  which cover  $I_k$ .

The length of each side was then reduced by a factor of  $N_k$ . Therefore, the volume (the product of these lengths) is  $v(I_k)/N_k^n$ . But then the sum of the volumes is  $v(I_k)$ . Each piece then has diameter at most

$$\sqrt{\sum_{j=1}^n ((b_j - a_j)/N_k)^2} \leq \sqrt{\sum_{j=1}^n (L_k/N_k)^2} < \sqrt{n} \cdot \frac{\epsilon}{\sqrt{n}} = \epsilon.$$

Therefore, the collection of all subintervals obtained after applying the above process to each  $k$  is a countable cover of  $E$ , and the sum of their volumes is exactly  $z$ . This shows that  $z \in S_2$ , completing the proof.  $\square$

**Solution Problem 2.** Suppose that the sets  $E_k \subset \mathbb{R}^n$  are such that the series  $\sum_{k=1}^{\infty} |E_k|_e$  converges. Prove that the outer measure of the set

$$\limsup E_k := \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k$$

is zero.



*Proof.* Let  $A := \limsup E_k$ . Since  $\sum_{k=1}^{\infty} |E_k|_e$  converges, the tail sums  $\sum_{k=m}^{\infty} |E_k|_e$  tend to zero as  $m \rightarrow \infty$ . Given  $\epsilon > 0$ , pick  $m$  such that  $\sum_{k=m}^{\infty} |E_k|_e < \epsilon$ . By the definition of  $A$ ,  $A \subset \bigcup_{k=m}^{\infty} E_k$ . The monotonicity and countable subadditivity of outer measure imply

$$|A|_e \leq \left| \bigcup_{k=m}^{\infty} E_k \right|_e \leq \sum_{k=m}^{\infty} |E_k|_e < \epsilon.$$

Since  $\epsilon$  was arbitrary, it follows that  $|A|_e \leq 0$ . Since  $|\cdot|_e \geq 0$ , we have  $|A|_e = 0$ .  $\square$

*Remark:* As mentioned in class, this problem can be solved purely on the basis of the three fundamental properties of outer measure. Two of them were mentioned above. The remaining one is  $|\emptyset|_e = 0$ . This property implies the outer measure cannot be negative, since  $\emptyset \subset A$  holds for every  $A$ .

**Solution Problem 3.** Given an arbitrary set  $A \subset \mathbb{R}$  and a number  $c > 0$ , let  $B = \{ca : a \in A\}$ . Prove that  $|B|_e = c |A|_e$ .

*Proof.* Given  $\epsilon > 0$ , let  $\{[s_k, t_k]\}$  be a countable cover of  $A$  such that  $\sum_k (t_k - s_k) \leq |A|_e + \epsilon$ . The intervals  $[cs_k, ct_k]$  cover  $B$ , since every point of  $B$  is of the form  $ca$ , where  $a \in A$  is covered by some interval  $[s_k, t_k]$ . Therefore,

$$|B|_e \leq \sum_k (ct_k - cs_k) = c \sum_k (t_k - s_k) \leq c |A|_e + \epsilon.$$

Since  $\epsilon > 0$  was arbitrary, it follows that  $|B|_e \leq c |A|_e$ . It remains to observe that  $A = c^{-1}B$ , which by the above implies  $|A|_e \leq c^{-1} |B|_e$ , i.e.  $|B|_e \geq c |A|_e$ . Thus,  $|B|_e = c |A|_e$ .  $\square$

**Solution Problem 4.** Suppose that a set  $A \subset \mathbb{R}$  is measurable. Prove that for every  $c > 0$  the set  $B = \{ca : a \in A\}$  is also measurable.

*Proof.* Given  $\epsilon > 0$ , let  $G$  be an open set that contains  $A$  and satisfies  $|G \setminus A|_e < \epsilon/c$ . Since the function  $f(x) = x/c$  is continuous, the preimage of  $G$  under this function is also open. This preimage  $f^{-1}(G)$  is  $cG$ . Since  $A \subset G$ , it follows that  $B \subset cG$ . Moreover, by the previous exercise

$$|(cG) \setminus B|_e = |c(G \setminus A)|_e = c |G \setminus A|_e < \epsilon.$$

Since  $\epsilon$  was arbitrary, this proves that  $B$  is measurable.  $\square$

**Solution Problem 5.** Given a sequence of continuous functions  $f_k : \mathbb{R} \rightarrow \mathbb{R}$ , let  $B$  be the set of all points  $x \in \mathbb{R}$  such that the sequence  $\{f_k(x)\}$  is bounded. Prove that  $B$  is a measurable set. [Hint: try to construct  $B$  from the sets  $\{x : |f_k(x)| \leq M\}$  by using countable unions and intersections.]

*Proof.* For  $k, m \in \mathbb{N}$ , let

$$A(k, m) = \{x \in \mathbb{R} : |f_k(x)| \leq m\} = f_k^{-1}([-m, m]).$$

Being the preimage of a closed set under a continuous function,  $A(k, m)$  is closed, and in particular measurable. Let

$$A := \bigcup_{m=1}^{\infty} \bigcap_{k=1}^{\infty} A(k, m),$$

which is also measurable, being obtained from measurable sets by countable set operations. We claim that  $A = B$ .

If  $x \in A$ , then there exists  $m \in \mathbb{N}$  such that  $|f_k(x)| \leq m$  for all  $k \in \mathbb{N}$ , which shows the sequence  $\{f_k(x)\}$  is bounded. Conversely, if the sequence  $\{f_k(x)\}$  is bounded, then there exists  $m \in \mathbb{N}$  such that all elements of the sequence are at most  $m$  in absolute value. This means  $|f_k(x)| \leq m$  for all  $k$ , hence  $x \in A$ .  $\square$

**Solution Problem 6.** Given a sequence of continuous functions  $f_k : \mathbb{R} \rightarrow \mathbb{R}$ , let  $C$  be the set of all points  $x \in \mathbb{R}$  such that  $\lim_{k \rightarrow \infty} f_k(x) = 0$ . Prove that  $C$  is a measurable set.

*Proof.* For  $k, m \in \mathbb{N}$ , let

$$A(k, m) = \{x \in \mathbb{R} : |f_k(x)| < 1/m\} = f_k^{-1}((-1/m, 1/m)).$$

Being the preimage of an open set under a continuous function,  $A(k, m)$  is open, and in particular measurable. Let

$$A := \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} A(k, m).$$

This set is also measurable, being obtained from measurable sets by countable set operations. We claim that  $A = C$ .

Suppose  $x \in A$ . Given  $\epsilon > 0$ , pick  $m \in \mathbb{N}$  such that  $1/m \leq \epsilon$ . Since  $x \in \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} A(k, m)$ , there exists  $N$  such that  $x \in \bigcap_{k=N}^{\infty} A(k, m)$ , which means  $|f_k(x)| < 1/m$  for all  $k \geq N$ . Thus,  $|f_k(x)| < \epsilon$  for all  $k \geq N$ , which proves  $\lim_{k \rightarrow \infty} f_k(x) = 0$ .

Conversely, suppose  $x \in C$ . Given  $m \in \mathbb{N}$ , use the definition of the limit  $\lim_{k \rightarrow \infty} f_k(x) = 0$  to find  $N$  such that  $|f_k(x)| < 1/m$  for all  $k \geq N$ . The latter means  $x \in \bigcap_{k=N}^{\infty} A(k, m)$ . Therefore, for every  $m \in \mathbb{N}$  the inclusion  $x \in \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} A(k, m)$  holds. This means  $x \in A$ .  $\square$

**Solution Problem 7.** Prove that the set

$$A = \{x \in \mathbb{R} : \exists k \in \mathbb{N} \text{ such that } |2^x - 2^k| \leq 1\}$$

is measurable and  $|A| < \infty$ .

*Proof.* For each  $k \in \mathbb{N}$ , the inequality  $|2^x - 2^k| \leq 1$  is equivalent to  $\log_2(2^k - 1) \leq x \leq \log_2(2^k + 1)$ . Thus,  $A = \bigcup_{k=1}^{\infty} I_k$ , where  $I_k = [\log_2(2^k - 1), \log_2(2^k + 1)]$ . Since  $I_k$  is an interval,  $I_k$  is measurable. Hence,  $A$  is measurable. By the countable subadditivity of the measure,  $|A| \leq \sum_{k=1}^{\infty} |I_k|$ . It remains to show the series  $\sum_{k=1}^{\infty} |I_k|$  converges. This can be done by the comparison test, limit comparison test, or the ratio test. Using the Limit Comparison Test with  $\sum_{k=1}^{\infty} 2^{-k}$  as a reference series, we have

$$\begin{aligned} \frac{|I_k|}{2^{-k}} &= \frac{\log_2(2^k + 1) - \log_2(2^k - 1)}{2^{-k}} \\ &= \frac{k + \log_2(1 + 2^{-k}) - (k + \log_2(1 - 2^{-k}))}{2^{-k}} \\ &= \frac{\log_2(1 + 2^{-k}) - \log_2(1 - 2^{-k})}{2^{-k}} \\ &= \frac{1}{\log 2} \left\{ \frac{\log(1 + 2^{-k})}{2^{-k}} + \frac{\log(1 - 2^{-k})}{-2^{-k}} \right\} \xrightarrow{k \rightarrow \infty} \frac{2}{\log 2}, \end{aligned}$$

where we have used the fact that  $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$ . Since  $\sum_{k=1}^{\infty} 2^{-k}$  converges, so does  $\sum_{k=1}^{\infty} |I_k|$ .  $\square$

**Solution Problem 8.** Prove that the set

$$A = \{x \in [0, 1] : \forall q \in \mathbb{N} \exists p \in \mathbb{N} \text{ such that } |x - p/q| \leq 1/q^2\}$$

is measurable and  $|A| = 0$ .

*Proof.* Let  $A_q = \bigcup_{p=1}^{\infty} E(p, q)$  where  $E(p, q) = \left[ \frac{p}{q} - \frac{1}{q^2}, \frac{p}{q} + \frac{1}{q^2} \right] \cap [0, 1]$ . This is measurable, being a countable union of measurable sets  $E(p, q)$  (which are intervals, possibly empty). Therefore, the set  $A = \bigcap_{q \in \mathbb{N}} A_q$  is also measurable.

We know  $|E(p, q)| \leq 2/q^2$  by the construction of  $E(p, q)$ . Also when  $p > q + 1$ , we have  $\frac{p}{q} - \frac{1}{q^2} \geq 1 + \frac{1}{q} - \frac{1}{q^2} \geq 1$ , which implies  $E(p, q) = \emptyset$ . By subadditivity, we have

$$|A_q| \leq \sum_{p=1}^{\infty} |E(p, q)| \leq \sum_{p=1}^{q+1} \frac{2}{q^2} = \frac{2q+2}{q^2}.$$

But then by monotonicity,  $|A| \leq |A_q|$  for each  $q$ . Since  $|A_q| \xrightarrow{q \rightarrow \infty} 0$ , it follows that  $|A| = 0$ .  $\square$

**Solution Problem 9.** Suppose  $E$  and  $Z$  are sets in  $\mathbb{R}^n$  such that  $E \cup Z$  is measurable and  $|Z| = 0$ . Prove that  $E$  is measurable.

*Proof.* Since  $Z \setminus E \subset Z$ , the monotonicity of outer measure implies  $|Z \setminus E|_e = 0$ , hence  $Z \setminus E$  is measurable. Then  $E = (E \cup Z) \setminus (Z \setminus E)$  is measurable, being the difference of two measurable sets.  $\square$

[This could be done with Carathéodory theorem or with the “ $G_\delta$  minus a null set” theorem, but it is easier without.]

**Solution Problem 10.** Given a continuous function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , define  $\mathcal{M} = \{E \subset \mathbb{R}^n : f^{-1}(E) \text{ is Borel}\}$ .

- (a) Prove that  $\mathcal{M}$  is a  $\sigma$ -algebra.
- (b) Prove that if  $E$  is Borel, then  $f^{-1}(E)$  is Borel. [Hint: Use (a).]

*Proof.*

- (a) The following argument does not make use of the continuity of  $f$  and works for any map  $f$ : Taking preimages commutes with any set operations. For example,

$$f^{-1}(E^c) = \{x : f(x) \in E^c\} = \{x : f(x) \notin E\} = (f^{-1}(E))^c$$

and

$$f^{-1}\left(\bigcup_i E_i\right) = \{x : \exists i f(x) \in E_i\} = \bigcup_i f^{-1}(E_i).$$

So if  $E \in \mathcal{M}$ , then  $f^{-1}(E^c) = f^{-1}(E)^c$  is the complement of a Borel set, hence is Borel. But then  $E^c \in \mathcal{M}$ . Also, if  $E_k \in \mathcal{M}$  for each  $k \in \mathbb{N}$ , then

$$f^{-1}\left(\bigcup_k E_k\right) = \bigcup_k f^{-1}(E_k)$$

is the countable union of Borel sets, hence is Borel. But then  $\bigcup_k E_k \in \mathcal{M}$ . The definition of a  $\sigma$ -algebra in the book also requires us to check that  $\mathcal{M}$  is nonempty. It suffices to notice that  $f^{-1}(\emptyset) = \emptyset$  is Borel, hence  $\emptyset \in \mathcal{M}$ .

- (b) Since  $f$  is continuous, the preimage of any open set under  $f$  is open, hence Borel. This means  $\mathcal{M}$  contains all open sets. By definition, the Borel  $\sigma$ -algebra  $\mathcal{B}$  is the *smallest*  $\sigma$ -algebra that contains all open sets. Thus  $\mathcal{B} \subset \mathcal{M}$ , which by definition of  $\mathcal{M}$  means that  $f^{-1}(E)$  is Borel whenever  $E$  is Borel.  $\square$

Remark: It is tempting to approach statement (b) by “writing a Borel set  $E$  in terms of open/closed sets” and concluding that  $f^{-1}(E)$  can also be written in this way. But there is no such structural formula for Borel sets: one can only get the proper subclasses like  $G_\delta$ ,  $G_{\delta\sigma}$ ,  $G_{\delta\sigma\delta}$ , and so on. The whole story is complicated: see [Borel hierarchy](#) on Wikipedia.

**Solution Problem 11.** Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function with a continuous derivative. Prove that for every measurable set  $E$ , the set  $f(E)$  is also measurable. [Hint: Although  $f$  need not be Lipschitz, its restriction to any bounded interval is.]

*Proof.* For each  $j \in \mathbb{N}$ , the set  $E_j = E \cap [-j, j]$  is measurable, as the intersection of two measurable sets. Since  $E = \bigcup_j E_j$ , it follows that  $f(E) = \bigcup_j f(E_j)$ . So it suffices to prove  $f(E_j)$  is measurable for every  $j$ .

The derivative  $f'$ , being continuous, is bounded on the interval  $[-j, j]$ . By the Mean Value Theorem,  $f$  is Lipschitz on  $[-j, j]$ ; Indeed,  $|f(a) - f(b)| \leq |a - b| \sup_{[-j, j]} |f'|$ . A technical detail arises: we only proved the measurability of images for Lipschitz functions on all of  $\mathbb{R}^n$ . To get around this, define

$$f_j(x) = \begin{cases} f(x), & x \in [-j, j] \\ f(-j), & x < -j \\ f(j), & x > j \end{cases}$$

Such extended function  $f_j$  is Lipschitz continuous on all of  $\mathbb{R}$ . Indeed, in each of three closed interval  $(-\infty, -j]$ ,  $[-j, j]$ ,  $[j, \infty)$  the Lipschitz condition holds by construction. For arbitrary  $a < b$ , partition the interval  $[a, b]$  by the points  $\{-j, j\}$  should they lie there. Then apply Lipschitz continuity to each interval, and use the Triangle Inequality. But then  $f_j(E_j)$ , which is the same as  $f(E_j)$ , is measurable, and the proof is complete.  $\square$

Note: In fact, for every set  $E \subset \mathbb{R}^n$ , any Lipschitz function  $f : E \rightarrow \mathbb{R}^n$  can be extended to a Lipschitz function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Therefore, when discussing the measurability of  $f(E)$  it suffices to check that  $f$  is Lipschitz on the set  $E$ .

*Proof (Sketch).* It suffices to extend a real-valued Lipschitz function  $f : E \rightarrow \mathbb{R}$ , because the vector-valued case follows by extending each component. Let  $L$  be the Lipschitz constant of  $f$ , and define, for every  $x \in \mathbb{R}^n$ ,

$$F(x) = \inf_{a \in E} (f(a) + L|x - a|).$$

It is an exercise with the definition of inf to prove that  $F$  is Lipschitz with constant  $L$ , and that  $F(x) = f(x)$  when  $x \in A$ .  $\square$

Remark: Extending a map  $f : E \rightarrow \mathbb{R}^n$  in the above fashion, one finds the Lipschitz constant of the extension is at most  $L\sqrt{n}$ , where  $L$  is the Lipschitz constant of the original map. There is a deeper extension theorem (due to Kirszbraun) according to which an extension with the same Lipschitz constant  $L$  exists.

**Solution Problem 12.** Given a set  $E \subset [0, \infty)$ , define a function  $f : [0, \infty) \rightarrow [0, \infty)$  by  $f(x) = |E \cap [0, x]|_e$ .

- (a) Prove that  $f$  is Lipschitz continuous.
- (b) Prove that for every number  $b$  with  $0 < b < |E|_e$  there exists a set  $F \subset E$  such that  $|F|_e = b$ .

*Proof.*

- (a) We claim that  $0 \leq f(b) - f(a) \leq b - a$  for any  $a, b \in [0, \infty)$  such that  $a < b$ ; this yields the Lipschitz continuity with constant 1. On one hand,  $f(b) \geq f(a)$  by the monotonicity of the outer measure:  $E \cap [0, a] \subset E \cap [0, b]$ . On the other,  $E \cap [0, b] \subset (E \cap [0, a]) \cup [a, b]$ , which implies

$$f(b) \leq |(E \cap [0, a]) \cup [a, b]|_e \leq |E \cap [0, a]|_e + |[a, b]| = f(a) + (b - a)$$

by subadditivity.

- (b) By Theorem 3.27 in the textbook, the outer measure is continuous under nested unions, even if the sets are not measurable. Since  $E = \bigcup_{k \in \mathbb{N}} (E \cap [0, k])$ , it follows that

$$|E|_e = \lim_{k \rightarrow \infty} |E \cap [0, k]|_e = \lim_{k \rightarrow \infty} f(k).$$

Since  $b < |E|_e$ , by the definition of limit there exists  $k$  such that  $f(k) > b$ . Also,  $f(0) = |E \cap \{0\}| = 0$ . Applying the Intermediate Value Theorem to  $f$  on the interval  $[0, k]$  (which is possible since  $f$  is continuous by part (a)), we conclude that there exists  $x \in (0, k)$  such that  $f(x) = b$ . Then the set  $F = |E \cap [0, x]|$  meets the requirements.  $\square$

**Solution Problem 13.** Show that there exists a nested sequence of sets  $E_1 \supset E_2 \supset \dots$  such that  $|E_1|_e < \infty$  and  $\bigcap_{k=1}^{\infty} E_k = \emptyset$  but  $\lim_{k \rightarrow \infty} |E_k|_e > 0$ . That is, the outer measure is not continuous under nested intersections. [Hint: Use the translates of the Vitali set.]

*Proof.* Let  $V \subset [0, 1]$  be the Vitali set described in class: recall that  $|V|_e > 0$  and that the sets  $V + q$  are disjoint for all  $q \in \mathbb{Q}$ . Let

$$E_k = \bigcup_{j=k}^{\infty} \left( V + \frac{1}{j} \right).$$

Then  $E_1 \subset V + [0, 1] \subset [0, 2]$ , hence  $|E_1|_e \leq 2 < \infty$ .

Suppose  $x \in \bigcap_{k=1}^{\infty} E_k$ . This means that for each  $k \in \mathbb{N}$ , there exists  $j \geq k$  such that  $x \in V + 1/j$ . In particular,  $x \in V + 1/j$  for infinitely many distinct values of  $j$ . But this is impossible as the sets  $V + 1/j$  are disjoint. This contradiction proves that  $\bigcap_{k=1}^{\infty} E_k$  is empty.

The sets  $E_k$  are nested by construction, hence  $|E_k|_e$  is a non-increasing sequence. It is bounded from below by  $|V|_e$  because each  $E_k$  contains a translated copy of  $V$ . Thus,  $\lim_{k \rightarrow \infty} |E_k|_e \geq |V|_e > 0$ .  $\square$

**Solution Problem 14.** Show that for the standard middle-third Cantor set  $C \subset [0, 1]$ , the difference set  $C - C$  contains a neighborhood of 0. [Hint:  $C$  is the intersection of nested sets  $C_n$ , where  $C_0 = [0, 1]$  and  $C_{n+1} = \frac{1}{3}C_n \cup (\frac{1}{3}C_n + \frac{2}{3})$ . Find  $C_n - C_n$  using induction.]

Remark: This shows that having  $|E| > 0$  is not necessary for  $E - E$  to contain a neighborhood of 0.

*Proof.* Recall that  $A - B = \{a - b : a \in A, b \in B\}$ . This definition implies that

$$(A_1 \cup A_2) - (B_1 \cup B_2) = \bigcup_{i,j=1}^2 (A_i - B_j). \quad (11.1)$$

Furthermore for any  $t \in \mathbb{R}$ , we have  $(A + t) - B = (A + B) + t$ ,  $A - (B + t) = (A - B) - t$ , and  $tA - tB = t(A - B)$ ; all these follow directly from the definition. The equality  $C_0 - C_0 = [-1, 1]$  holds because on one hand,  $|x - y| \leq 1$  when  $x, y \in [0, 1]$  while on the other,  $C_0 - C_0 \supset [0, 1] - \{0, 1\} = [0, 1] \cup [-1, 0] = [-1, 1]$ .

Assume  $C_n - C_n = [-1, 1]$ . Using the relation  $C_{n+1} = \frac{1}{3}C_n \cup (\frac{1}{3}C_n + \frac{2}{3})$  and distributing the difference according to (11.1) and other properties stated at the beginning gives

$$\begin{aligned} C_{n+1} - C_{n+1} &= \left( \frac{1}{3}C_n - \frac{1}{3}C_n \right) \cup \left( \frac{1}{3}C_n - \frac{1}{3}C_n + \frac{2}{3} \right) \cup \left( \frac{1}{3}C_n - \frac{1}{3}C_n - \frac{2}{3} \right) \\ &= [-1/3, 1/3] \cup ([-1/3, 1/3] + 2/3) \cup ([-1/3, 1/3] - 2/3) \\ &= [-1/3, 1/3] \cup [1/3, 1] \cup [-1, -1/3] = [-1, 1]. \end{aligned}$$

The set  $(\frac{1}{3}C_n + \frac{2}{3}) - (\frac{1}{3}C_n + \frac{2}{3})$  is not included above because it is the same as  $(\frac{1}{3}C_n - \frac{1}{3}C_n)$ . By induction,  $C_n - C_n = [-1, 1]$  for all  $n$ .

Since  $C \subset C_n$  for every  $n$ , it follows that  $C - C \subset [-1, 1]$ . To prove the reverse inclusion, fix  $a \in [-1, 1]$ . For each  $n$ , there exist  $x_n, y_n \in C_n$  such that  $x_n - y_n = a$ . Since all these numbers are contained in  $[0, 1]$ , we can pick a convergent subsequence  $\{x_{n_k}\}$ . So  $x_{n_k} \rightarrow x$  and since  $x_{n_k} - y_{n_k} = a$ , we also have  $y_{n_k} \rightarrow y$  where  $y$  is such that  $x - y = a$ .

It remains to prove that  $x, y \in C$ . For each  $m \in \mathbb{N}$  we have  $x_{n_k}, y_{n_k} \in C_m$  for  $k \geq m$  by construction. Since  $C_m$  is compact, it follows that  $x, y \in C_m$ . And since this holds for every  $m \in \mathbb{N}$ , we have  $x, y \in C$ .  $\square$

**Solution Problem 15.** Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a function such that  $f(\mathbb{R}^n)$  is countable, and  $f^{-1}(t)$  is measurable for every  $t \in \mathbb{R}$ . Prove that  $f$  is measurable.

*Proof.* Let  $B = f(\mathbb{R}^n)$ , a countable subset of  $\mathbb{R}$ . For any  $a \in \mathbb{R}$  we have

$$\{f > a\} = \bigcup_{b \in B, b > a} f^{-1}(b),$$

which is a countable union of measurable sets, hence measurable. The domain of  $f$ , which is  $\mathbb{R}^n$ , is also measurable. Thus,  $f$  is measurable.  $\square$

**Solution Problem 16.** Prove that without the assumption “ $f(\mathbb{R}^n)$  is countable” the statement in the previous problem would not be true.

*Proof.* The statement in the previous problem is made for any  $n$ . To disprove it, it suffices to show it fails for some  $n$ , for example  $n = 1$ . Let  $V \subset [0, 1]$  be a Vitali set, and define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} x + 1, & x \in V \\ -|x|, & x \notin V \end{cases}$$

By construction  $\{f > 0\} = V$ , which is nonmeasurable. Thus,  $f$  is nonmeasurable. On the other hand, for every  $t \in \mathbb{R}$  the set  $f^{-1}(t)$  is finite and therefore measurable. Indeed, if  $t$  is negative,  $f(x) = t$  holds for at most two values of  $x$ ; and when  $t \geq 0$ , there is at most one such value.  $\square$

**Remark:** If we wanted to construct such an example on  $\mathbb{R}^n$  for every  $n$ , one way is to let

$$f(x_1, \dots, x_n) = \begin{cases} x_1 + 1, & \forall i \ x_i \in V \\ -|x_1|, & \text{otherwise} \end{cases}$$

Then  $\{f > 0\} = V^n$  which is nonmeasurable, because on one hand,  $V^n + \mathbb{Q}^n = \mathbb{R}^n$  forces  $|V^n|_e > 0$ ; on the other,  $V^n + (\mathbb{Q} \cap [0, 1])^n$  is a bounded set containing infinitely many copies of  $V^n$ , which makes it impossible to have  $|V^n| > 0$ .



For every  $t \in \mathbb{R}$ , the preimage  $f^{-1}(t)$  consists at most two hyperplanes of the form  $\{x\} \times \mathbb{R}^{n-1}$ . So it is covered by countably many sets of the form  $\{x\} \times [-j, j]^{n-1}$ ,  $j \in \mathbb{N}$ . Here  $|\{x\} \times [-j, j]^{n-1}| = 0$  because this set is contained in a box of dimensions  $(\epsilon, 2j, \dots, 2j)$  whose volume can be arbitrarily small. In conclusion,  $|f^{-1}(t)| = 0$  for every  $t$ . Thus,  $f$  is measurable.

**Solution Problem 17.** Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is measurable, and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is continuously differentiable with  $g' > 0$  everywhere. Prove that  $f \circ g$  is measurable.

*Proof.* By the Mean Value Theorem,  $g$  is strictly increasing. Therefore,  $g$  has an inverse  $h = g^{-1}$ . By the Inverse Function Theorem, the inverse function  $h$  is also continuously differentiable.

Given  $a \in \mathbb{R}$ , consider the set  $A = \{x : f(g(x)) > a\}$ . It can be written as  $\{x : g(x) \in B\}$  where  $B = \{f > a\}$  is measurable. That is,  $A = h(B)$ . By Homework 11, the image of a measurable set under a continuously differentiable function is measurable. Thus,  $A$  is measurable.  $\square$

**Solution Problem 18.**

- (a) Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuous function such that  $f^2$  is measurable. Prove that  $f$  is measurable.
- (b) Prove that the statement in (a) is false if  $f$  is not assumed continuous.

*Proof.*

- (a) Since  $f$  is continuous, it is measurable.
- (b) Let  $n = 1$ , let  $V$  be a Vitali set, and define  $f(x) = 1$  when  $x \in V$  and  $f(x) = -1$  when  $x \notin V$ . Then  $f^2 \equiv 1$  is measurable, being continuous. But  $\{f > 0\} = V$  is not a measurable set, so  $f$  is not measurable.  $\square$

**Solution Problem 19.**

- (a) Let  $E \subset \mathbb{R}^n$  be a set. Consider a sequence of lsc functions  $f_k : E \rightarrow \overline{\mathbb{R}}$  such that  $f_1 \leq f_2 \leq f_3 \leq \dots$ . Prove that  $\lim_{k \rightarrow \infty} f_k$  is also an lsc function. [Note: The limit here is understood in the sense of the extended real line  $\overline{\mathbb{R}}$ , so it is assured to exist by monotonicity.]
- (b) Give an example that shows (a) fails with “lsc” replaced by “usc.”

*Proof.*

- (a) Recall the limit comparison property: if all terms of a sequence are at most  $M$ , then its limit (if it exists) is also at most  $M$ . Apply the contrapositive of this statement to  $f(x) = \lim_{k \rightarrow \infty} f_k(x)$ , and conclude that if  $f(x) > M$ , then there exists  $k$  such that  $f_k(x) > M$ . Thus,  $\{f > M\} \subset \bigcup_{k \in \mathbb{N}} \{f_k > M\}$ . The reverse inclusion is true as well, because for each  $k$ ,  $\{f_k > M\} \subset \{f > M\}$ , by virtue of  $f_k \leq f$ . In conclusion,  $\{f > M\} = \bigcup_{k \in \mathbb{N}} \{f_k > M\}$ . Each set on the right is open in  $E$  because  $f_k$  is lsc; therefore, the set on the left is also open. Since  $M$  is arbitrary, this shows  $f$  is lsc.
- (b) Let  $f_k = \chi_{[1/k, \infty)}$ , the domain being  $\mathbb{R}$ . This function is usc because any set of the form  $\{f_k < a\}$  is either  $\mathbb{R}$ ,  $\emptyset$ , or  $(-\infty, 1/n)$ , and all these sets are open in  $\mathbb{R}$ . Also,  $f_k \leq f_{k+1}$  because  $[1/k, \infty) \subset [1/(k+1), \infty)$ . But the limit  $f = \chi_{(0, \infty)}$  is not usc, since the set  $\{f < 1\} = (-\infty, 0]$  is not open.  $\square$

**Solution Problem 20.** Fix  $a > 0$  and define  $f : [0, 1] \rightarrow \mathbb{R}$  so that  $f(1/k) = 1/k^a$  for  $k \in \mathbb{N}$ , and  $f(x) = 0$  for all other  $x$ . Prove that  $f$  is of bounded variation on  $[0, 1]$  when  $a > 1$ , and is not of bounded variation on  $[0, 1]$  when  $0 < a \leq 1$ .

*Proof.* Suppose  $a > 1$ . Let  $0 = x_0 < x_1 < \cdots < x_n = 1$  be a partition of  $[0, 1]$ . By the Triangle Inequality,

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^n (|f(x_i)| + |f(x_{i-1})|) \leq 2 \sum_{i=1}^n |f(x_i)|.$$

[The second inequality holds because each value  $|f(x_i)|$  is repeated at most twice.] Ignoring any terms with  $f(x_i) = 0$ , we get a sum of the form

$$2 \sum_{k \in B} \frac{1}{k^a}, \quad \text{where } B = \mathbb{N} \cap \{1/x_i : i = 1, \dots, n\}.$$

Since  $a > 1$ , the sum  $S = \sum_{k \in \mathbb{N}} 1/k^a$  is finite. From  $2 \sum_{k \in B} \frac{1}{k^a} \leq 2S$ , it follows that  $V(f; 0, 1) \leq 2S$ , hence  $f$  is BV.

Now suppose  $0 < a \leq 1$ . For  $n \in \mathbb{N}$ , consider the partition

$$P_n = \left\{ 0, \frac{1}{n}, \frac{1}{n-1/2}, \frac{1}{n-1}, \frac{1}{n-3/2}, \dots, \frac{1}{3/2}, 1 \right\},$$

which can be described as  $P_n = \{0\} \cup \{1/(n - k/2) : k = 0, \dots, 2n - 2\}$ . The values of  $f$  at the points of  $P_n$  are

$$0, \frac{1}{n^a}, 0, \frac{1}{(n-1)^2}, 0, \dots, 0, \frac{1}{1^a}.$$

Summing the absolute values of the differences of consecutive terms here, we obtain

$$V(f; 0, 1) \geq 1 + 2 \sum_{k=2}^n \frac{1}{k^a}.$$

As  $n \rightarrow \infty$ , the right hand side tends to infinity because the series  $\sum_{k \in \mathbb{N}} 1/k^a$  diverges. Thus  $V(f; 0, 1) = \infty$ .  $\square$

**Solution Problem 21.** Suppose that  $f : E \rightarrow \mathbb{R}$  is a measurable function, where  $E \subset \mathbb{R}^n$  is measurable.

- (a) Prove that there exists a Borel set  $H \subset E$  such that the restriction  $f|_H$  is Borel measurable and  $|E \setminus H| = 0$ . [Hint: Take a countable union of closed sets obtained from Lusin's theorem.]
- (b) If, in addition,  $E$  is a Borel set, prove that there exists a Borel measurable function  $g : E \rightarrow \mathbb{R}$  such that  $f = g$  a.e..

*Proof.*

- (a) By Lusin's Theorem, for every  $k \in \mathbb{N}$ , there exists a closed set  $E_k \subset E$  such that  $|E \setminus E_k| < 1/k$ , and the restriction of  $f$  to  $E_k$  is continuous. Let  $H = \bigcup_{k \in \mathbb{N}} E_k$ . Then  $H$  is Borel, being a countable union of closed sets. Also,  $|E \setminus H| \leq |E \setminus E_k| < 1/k$  for every  $k$ , which implies  $|E \setminus H| = 0$ .

For every  $a \in \mathbb{R}$  and every  $k \in \mathbb{N}$  the set  $A_k = \{x \in E_k : f(x) > a\}$  is open in  $E_k$  because  $f|_{E_k}$  is continuous. Thus,  $A_k = E_k \cap G_k$  for some open set  $G_k$  in  $\mathbb{R}^n$ . Since both  $E_k$  and  $G_k$  are Borel, it follows that  $A_k$  is Borel. Then  $\{x \in H : f(x) > a\} = \bigcup_{k \in \mathbb{N}} A_k$  is Borel, which proves that  $f|_H$  is Borel measurable.

- (b) Let  $g(x) = f(x)$  for  $x \in H$  (with  $H$  as above) and  $g(x) = 0$  for  $x \in E \setminus H$ . Then  $f = g$  a.e. because  $|E \setminus H| = 0$ . If  $a \geq 0$ , then the set  $\{x \in E : f(x) > a\}$  is equal to  $\{x \in H : f(x) > a\}$  which is Borel by (a). If  $a < 0$ , then

$$\{x \in E : f(x) > a\} = \{x \in H : f(x) > a\} \cup (E \setminus H),$$

which is Borel as the union of two Borel sets. Thus,  $g$  is a Borel measurable function on  $E$ .  $\square$

**Solution Problem 22.** Suppose  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a function such that  $\phi(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Consider a sequence of measurable functions  $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $|f_k(x)| \leq \phi(|x|)$  for every  $k$ , and  $f_k \rightarrow f$  a.e.. Prove that the conclusion of Egorov's

Theorem holds in this situation: that is, for every  $\epsilon > 0$  there exists a closed set  $E(\epsilon) \subset \mathbb{R}^n$  such that  $|\mathbb{R}^n \setminus E(\epsilon)| < \epsilon$  and  $f_k \rightarrow f$  uniformly on  $E(\epsilon)$ . [Hint: Follow the proof of Egorov's Theorem.]

*Proof.* The proof of Egorov's theorem consists of two parts. Part 1 does not need the assumption  $|E| < \infty$ , and is included here unchanged for the sake of completeness.

Part 1: It suffices to find, for each  $j \in \mathbb{N}$ , a measurable set  $E_j$  such that  $|E_j^C| < \epsilon/2^j$  and  $\sup_{E_j} |f_k - f| \leq 1/j$  for all sufficiently large  $k$ . Indeed, if we can do this then the set  $F = \bigcap E_j$  satisfies  $|F^C| < \sum_{j \in \mathbb{N}} \epsilon/2^j = \epsilon$ . On this set,  $f_k$  converge uniformly to  $f$  since for every  $j$ , the inequality  $|f_k - f| \leq 1/j$  holds on  $F$  for all sufficiently large  $k$ . Since  $F$  is measurable, it contains a closed subset  $F'$  where  $|F \setminus F'|$  can be as small as we wish. So we can choose  $F'$  so that  $|(F')^C| < \epsilon$  as well.

Part 2: To find  $E_j$  as above, fix  $j$  and consider the sets  $G_m = \{x: |f_k(x) - f(x)| < 1/j \forall k \geq m\}$ . By construction, the set  $\bigcap_{m \in \mathbb{N}} G_m^C$  consists of points where  $f_k(x) \not\rightarrow f(x)$ , and thus has measure zero. We would like to conclude that  $|G_m^C| < \epsilon/2^j$  for some  $m$ , which provides the desired set  $E_j = G_m$ . In general this does not work because the continuity of measure for nested intersections requires finite  $|G_1^C|$ . But here we have help from the function  $\phi$ .

Since  $\phi(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ , there exists  $R$  such that  $\phi(x) < 1/(2j)$  on the set  $A_R = \{x: |x| > R\}$ . Hence  $|f_k| < 1/(2j)$  on  $A_R$  for every  $k$ , and consequently  $|f| \leq 1/(2j)$  a.e. on  $A_R$ . It follows that  $|f_k - f| < 1/j$  a.e. on  $A_R$ , which implies  $|G_1^C \cap A_R| = 0$ . Hence  $G_1^C \subseteq \{x: |x| \leq R\}$  and  $|G_1^C| < \infty$ . This allows us to apply the continuity of measure for nested intersections, and conclude that  $|G_m^C| \rightarrow 0$  as  $m \rightarrow \infty$ ; in particular there exists  $m$  such that  $|G_m^C| < \epsilon/2^j$ .  $\square$

Remark: The function  $\phi$  plays the role of a “dominating function” for this sequence, which can be informally described as a function that mitigates the effects of “escaping to infinity.” We will see more of this idea in Chapter 5 of the text.

**Solution Problem 23.** Suppose that  $f, g, f_k, g_k$ , where  $k \in \mathbb{N}$ , are measurable functions on a set  $E \subset \mathbb{R}^n$  with values on  $\mathbb{R}$ , and that  $f_k \xrightarrow{m} f$  and  $g_k \xrightarrow{m} g$ . Prove that  $f_k + g_k \xrightarrow{m} f + g$ .

*Proof.* Recall that  $f_k \rightarrow f$  in measure if and only if for every  $\epsilon > 0$  there exists  $N$  such that  $|\{|f_k - f| > \epsilon\}| < \epsilon$  for all  $k \geq N$ . Given  $\epsilon > 0$ , choose  $N_1$  such that  $|\{|f_k - f| > \epsilon/2\}| < \epsilon/2$  for all  $k \geq N_1$ . Also choose  $N_2$  such that  $|\{|g_k - g| > \epsilon/2\}| < \epsilon/2$  for all  $k \geq N_2$ . By the Triangle Inequality,  $|(f_k + g_k) - (f + g)| \leq |f_k - f| + |g_k - g|$ . Hence,

$$\{|(f_k + g_k) - (f + g)| > \epsilon\} \subset \{|f_k - f| > \epsilon/2\} \cup \{|g_k - g| > \epsilon/2\},$$

which implies  $|\{(f_k + g_k) - (f + g)| > \epsilon\}| < \epsilon/2 + \epsilon/2 = \epsilon$  for all  $k \geq \max\{N_1, N_2\}$ . Thus,  $f_k + g_k \rightarrow f + g$ .  $\square$

### Solution Problem 24.

- (a) In addition to the assumptions in the previous problem, suppose  $|E| < \infty$ . Prove that  $f_k g_k \xrightarrow{m} f g$ .
- (b) Show by an example that the assumption  $|E| < \infty$  cannot be omitted in (a). [Hint: You need some boundedness to control the terms in  $f_k g_k - f g = (f_k - f)g_k + f(g_k - g)$ . Consider that when  $\varphi : E \rightarrow \mathbb{R}$  is measurable,  $E = \bigcup_j \{|\varphi| \leq j\}$ .]

*Proof.*

- (a) Fix  $\epsilon > 0$ . For  $m \in \mathbb{N}$ , let  $A_m = \{x \in E : |f(x)| \leq m, |g(x)| \leq m\}$ . Since both  $f$  and  $g$  are finite ( $\mathbb{R}$ -valued) functions, it follows that  $\bigcup_{m \in \mathbb{N}} A_m = E$ . Using the continuity of measure together with  $|E| < \infty$ , we conclude that there exists  $m$  such that  $|E \setminus A_m| < \epsilon/3$ . Be sure that  $m > \epsilon$ , by taking larger  $m$  if necessary. Because  $f_k \xrightarrow{m} f$ , there exists  $K_1$  such that  $|\{ |f_k - f| > \epsilon/(4m) \}| < \epsilon/3$  for all  $k \geq K_1$ . Furthermore because  $g_k \xrightarrow{m} g$ , there exists  $K_2$  such that  $|\{ |g_k - g| > \epsilon/(2m) \}| < \epsilon/3$  for all  $k \geq K_2$ . On the set where  $|f_k - f| \leq \epsilon/(4m)$ ,  $|g_k - g| \leq \epsilon/(4m)$ ,  $|f| \leq m$ , and  $|g| \leq m$  we have  $|g_k| \leq m + \epsilon/(4m) < 2m$  (since  $m > \epsilon$ ) and therefore

$$|f_k g_k - f g| \leq |f_k - f| |g_k| + |f| |g_k - g| \leq \frac{\epsilon}{4m} \cdot 2m + m \cdot \frac{\epsilon}{2m} = \epsilon.$$

This means

$$\{|f_k g_k - f g| > \epsilon\} \subset (E \setminus A_m) \cup \{|f_k - f| > \epsilon/(2m)\} \cup \{|g_k - g| > \epsilon/(4m)\},$$

and the measure of this union is less than  $\epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$ .

- (b) Let  $f_k(x) = 1/k$  and  $g_k(x) = x$  on  $\mathbb{R}$ . Then  $f_k \rightarrow 0$  uniformly, hence in measure. Also,  $g_k$  converges to  $x$ , being a constant sequence. But  $f_k(x)g_k(x) = x/k$  does not converge to 0 in measure because

$$|\{ |x/k| > \epsilon \}| = |\{ |x| > k\epsilon \}| = \infty$$

for every  $k$ .  $\square$

**Solution Problem 25.** Suppose that  $f : E \rightarrow [0, \infty)$  is a measurable function, where  $E \subset \mathbb{R}^n$ . Prove that  $\int_E f$  is finite if and only if the series

$$\sum_{j=-\infty}^{\infty} 2^j |\{x \in E : f(x) > 2^j\}|$$

converges. [Hint: consider the sets  $E_k = \{2^k < f \leq 2^{k+1}\}$  and the function  $g(x) = \sum 2^k \chi_{E_k}$ . Compare  $\int_E g$  to the sum of series, and also to  $\int_E f$ .]

**Note:** The convergence of a doubly-infinite series  $\sum_{j=-\infty}^{\infty} c_j$  means that both  $\sum_{j=0}^{\infty} c_j$  and  $\sum_{j=1}^{\infty} c_{-j}$  converge. In case of nonnegative terms the convergence is equivalent to partial sums  $\sum_{j=-N}^N c_j$  being bounded.

*Proof.* Let  $E_k$  and  $g$  be as above. By construction,  $f = 0$  if and only if  $g = 0$ , and  $g < f \leq 2g$  on each set  $E_k$ , which together cover  $\{f > 0\}$ . Thus,  $g \leq f \leq 2g$ . Since  $g$  has countable range, its integral is computed (using countable additivity over the domain) as  $\int_E g = \sum_{k=-\infty}^{\infty} 2^k |E_k|$ . For the same reason,  $\int_E 2g = \sum_{k=-\infty}^{\infty} 2^{k+1} |E_k|$ . [Note that although it's true that  $\int 2g = 2 \int g$  for general measurable  $g$ , we don't need this fact from Chapter 5.2 here.] These facts together with inequalities  $g \leq f \leq 2g$  yield that  $\int_E f < \infty$  if and only if  $\sum_{k=-\infty}^{\infty} 2^k |E_k| < \infty$ .

Let  $F_k = \{x \in E : f(x) > 2^k\}$ . It remains to prove that

$$\sum_{k=-\infty}^{\infty} 2^k |F_k| < \infty \iff \sum_{k=-\infty}^{\infty} 2^k |E_k| < \infty. \quad (11.2)$$

To this end, note that  $F_k = \bigcup_{j=k}^{\infty} E_j$ , which is a disjoint union. Hence

$$\begin{aligned} \sum_{k=-\infty}^{\infty} 2^k |F_k| &= \sum_{k=-\infty}^{\infty} \sum_{j=k}^{\infty} 2^k |E_j| \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^j 2^k |E_j| \\ &= \sum_{j=-\infty}^{\infty} 2^{j+1} |E_j| \end{aligned}$$

which proves (11.2). □

**Solution Problem 26.** Let  $B = \{x \in \mathbb{R}^n : |x| < 1\}$ .

(a) Prove that  $\int_B |x|^{-p} dx$  is finite when  $0 < p < n$  and infinite when  $p \geq n$ .

(b) Prove that  $\int_{B^c} |x|^{-p} dx$  is finite when  $p > n$  and infinite when  $0 < p \leq n$ .

Note: These integrals are Lebesgue integrals, and we do not yet have anything like the Fundamental Theorem of Calculus for such integrals. Use the previous problem. You can also use the fact that the measure of a ball of radius  $R$  is  $C_n R^n$  for some constant  $C_n$  that depends on  $n$ .

*Proof.*

(a) Let  $f = |x|^{-p}$  and  $F_k = \{x \in \mathbb{R}^n : f(x) > 2^k\}$ . Note that  $F_k = \{x : |x| < 2^{-k/p}\}$ , so  $|F_k| = C_n 2^{-kn/p}$ .

By the previous problem,  $\int_B f < \infty \iff \sum_{k=-\infty}^{\infty} 2^k |B \cap F_k| < \infty$ . When  $k \leq 0$ , we have  $B \subset F_k$ , hence  $|B \cap F_k| = |B|$ . The geometric series  $\sum_{k \leq 0} 2^k |B| = 2|B|$  converges regardless of  $p$ . For  $k \geq 1$ ,  $F_k \subset B$ , so  $|B \cap F_k| = |F_k|$ , hence

$$\sum_{k=1}^{\infty} 2^k |B \cap F_k| = C_n \sum_{k=1}^{\infty} 2^{k(1-n/p)},$$

which is a geometric series that converges if and only if  $1 - n/p < 0$ . This proves (a).

(b) By the previous problem,  $\int_{B^c} f < \infty \iff \sum_{k=-\infty}^{\infty} 2^k |B^c \cap F_k| < \infty$ . As noted above,  $F_k \subset B$  when  $k \geq 1$ , which implies  $B^c \cap F_k = \emptyset$ . When  $k \leq 0$ , we have  $B \subset F_k$ , hence  $|B^c \cap F_k| = |F_k| - |B|$ . Since the sum  $\sum_{k \leq 0} 2^k |B| = 2|B|$  converges regardless of  $p$ , it remains to consider the convergence of  $\sum_{k \leq 0} 2^k |F_k|$ . Writing  $j = -k$ , we arrive at

$$\sum_{k \leq 0} 2^k |F_k| = \sum_{j=0}^{\infty} 2^{-j} C_n 2^{jn/p} = C_n \sum_{j=0}^{\infty} 2^{j(n/p-1)},$$

which is a geometric series that converges if and only if  $n/p - 1 < 0$ . This proves (b).  $\square$

### Solution Problem 27.

(a) Suppose that  $f_k : E \rightarrow [0, \infty]$  (where  $E \subset \mathbb{R}^n$ ) are measurable functions such that  $\int_E f_k \rightarrow 0$  as  $k \rightarrow \infty$ . Prove that  $f_k \xrightarrow{m} 0$ .

(b) Give an example where  $f_k \xrightarrow{m} 0$  but  $\int_E f_k \not\rightarrow 0$ .

*Proof.*

(a) For any  $\epsilon > 0$ , Chebyshev's Inequality yields

$$|\{f_k > \epsilon\}| \leq \frac{1}{\epsilon} \int_E f_k \rightarrow 0,$$

which means  $f_k \xrightarrow{m} 0$ .

(b) Either of  $f_k = k\chi_{(0,1/k)}$  or  $g_k = k^{-1}\chi_{(0,k)}$  works. (Or even  $h_k \equiv 1/k$ ). Indeed,  $\{f_k \neq 0\} \rightarrow 0$ , and  $g_k, h_k$  converge to zero uniformly (which is stronger than convergence in measure). Yet  $\int f_k = 1$ ,  $\int g_k = 1$ , and  $\int h_k = \infty$ .  $\square$

**Solution Problem 28.** For  $k \in \mathbb{N}$ , define  $f_k : [0, 1] \rightarrow [0, \infty]$  by

$$f_k(x) = \sum_{j=1}^k \chi_{I(j,k)}, \quad \text{where } I(j,k) = \left[ \frac{j}{k} - \frac{1}{k^3}, \frac{j}{k} + \frac{1}{k^3} \right].$$

Let  $f = \sum_{k=1}^{\infty} f_k$ . Prove that  $\int_{[0,1]} f < \infty$ .

*Proof.* Note that  $\int_E \chi_F = |E \cap F|$  by the formula for the integral of a simple function. Applying this (and the additivity of the integral) to  $f_k$  yields

$$\int_{[0,1]} f_k = \sum_{j=1}^k |I(j,k) \cap [0,1]| \leq \sum_{j=1}^k |I(j,k)| = \sum_{j=1}^k \frac{2}{k^3} = \frac{2}{k^2}.$$

By the countable additivity over nonnegative functions (Theorem 5.16),

$$\int_{[0,1]} f = \sum_{k=1}^{\infty} \int_{[0,1]} f_k \leq \sum_{k=1}^{\infty} \frac{2}{k^2} < \infty.$$

[One can also say that partial sums converge to  $f$  in an increasing way, but this argument was already made in the proof of Theorem 5.16].  $\square$

**Solution Problem 29.** Suppose that  $f : \mathbb{R}^n \rightarrow [0, \infty)$  is a measurable function such that  $\int_{\mathbb{R}^n} f < \infty$ . Also suppose  $\{E_k\}$  is a sequence of measurable sets  $E_k \subset \mathbb{R}^n$ . Let  $A = \limsup_{k \rightarrow \infty} E_k$  and  $B = \liminf_{k \rightarrow \infty} E_k$ . Prove that

$$\int_A f \geq \limsup_{k \rightarrow \infty} \int_{E_k} f$$

and

$$\int_B f \leq \liminf_{k \rightarrow \infty} \int_{E_k} f.$$



[Hint:  $\int_E f = \int_{\mathbb{R}^n} \chi_E f$ .]

*Proof.* The second inequality follows from Fatou's Lemma, using the fact (discussed in class) that  $\chi_B = \liminf_{k \rightarrow \infty} \chi_{E_k}$ :

$$\int_B f = \int_{\mathbb{R}^n} \chi_B f = \int_{\mathbb{R}^n} \liminf_{k \rightarrow \infty} \chi_{E_k} f \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} \chi_{E_k} f = \liminf_{k \rightarrow \infty} \int_{E_k} f.$$

To prove the inequality for  $\int_A f$ , note that  $f - \chi_{E_k} f \geq 0$ , and apply Fatou's Lemma to this sequence:

$$\int_{\mathbb{R}^n} \liminf_{k \rightarrow \infty} (f - \chi_{E_k} f) \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} (f - \chi_{E_k} f).$$

Expand both sides, recalling that  $\liminf_{k \rightarrow \infty} (-a_k) = -\limsup_{k \rightarrow \infty} a_k$ , and using the assumption that  $\int_{\mathbb{R}^n} f$  is finite yields

$$\int_{\mathbb{R}^n} f - \int_{\mathbb{R}^n} \limsup_{k \rightarrow \infty} \chi_{E_k} f \leq \int_{\mathbb{R}^n} f - \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} \chi_{E_k} f.$$

Canceling  $\int_{\mathbb{R}^n} f$ , we get

$$\int_{\mathbb{R}^n} \limsup_{k \rightarrow \infty} \chi_{E_k} f \geq \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} \chi_{E_k} f,$$

which is precisely  $\int_A f \geq \limsup_{k \rightarrow \infty} \int_{E_k} f$ . □

**Solution Problem 30.** Suppose that  $f : \mathbb{R}^n \rightarrow [0, \infty)$  is a measurable function such that  $\int_{\mathbb{R}^n} f < \infty$ . Prove that  $\int_{\mathbb{R}^n} e^{-k|x|} f(x) \rightarrow 0$  as  $k \rightarrow \infty$ .

*Proof.* For all  $x \neq 0$  we have  $e^{-k|x|} f(x) \rightarrow 0$  as  $k \rightarrow \infty$ ; thus, the functions converge a.e. to 0. Also,  $f$  is a dominating function, since its integral is finite and  $e^{-k|x|} f(x) \leq f(x)$  for all  $x$  and all  $k$ . By the Dominated Convergence Theorem,

$$\int_{\mathbb{R}^n} e^{-k|x|} f(x) \rightarrow \int_{\mathbb{R}^n} 0 = 0.$$

□

**Solution Problem 31.** Prove that under the assumptions of the Lebesgue Dominated Convergence Theorem,  $\int_E |f_k - f| \rightarrow 0$  as  $k \rightarrow \infty$ .

*Proof.* By assumption, there is an integrable dominating function for  $\{f_k\}$ , call it  $\varphi$ . By passing to the limit,  $|f| \leq \varphi$  a.e., which implies  $|f_k - f| \leq |f_k| + |f| \leq 2\varphi$  a.e.. Note that  $\int_E 2\varphi = 2 \int_E \varphi < \infty$ . Since  $f_k \rightarrow f$  a.e., it follows that  $|f_k - f| \rightarrow 0$  a.e.. By the DCT,

$$\int_E |f_k - f| \rightarrow \int_E 0 = 0.$$

□

**Solution Problem 32.** Let  $f \in L^1(E)$ , where  $E \subset \mathbb{R}^n$  is a measurable set. Prove that

$$\lim_{k \rightarrow \infty} k \int_E \sin\left(\frac{f}{k}\right) = \int_E f.$$

*Proof.* By the Mean Value Theorem,  $\sin t = \sin t - \sin 0 = (\cos \xi) t$  for some  $\xi$  between 0 and  $t$ . This implies two things:

- (a)  $|\sin t| \leq |t|$  because  $|\cos \xi| \leq 1$
- (b) as  $t \rightarrow 0$ , we have  $(\sin t)/t \rightarrow 1$  because  $\cos \xi \rightarrow 1$ .

Applying (a) to  $\sin\left(\frac{f}{k}\right)$ , we find that

$$k \left| \sin\left(\frac{f}{k}\right) \right| \leq k \frac{|f|}{k} = |f|,$$

which means  $|f|$  is a dominating function for the sequence  $f_k = k \sin(f/k)$ .

Applying (b), we find that

$$f_k = \frac{\sin(f/k)}{f/k} f \rightarrow f, \quad k \rightarrow \infty.$$

By the Dominated Convergence Theorem,  $\int_E f_k \rightarrow \int_E f$ .

□

**Solution Problem 33.** Let  $f : E \rightarrow \mathbb{R}$  be a measurable function. Suppose that  $|E| < \infty$  and there exists a number  $p > 1$  such that

$$\limsup_{\alpha \rightarrow \infty} \alpha^p |\{x \in E : |f(x)| > \alpha\}| < \infty.$$

Prove that  $f \in L^1(E)$ . [Hint: use an exercise from Homework 25.]

*Proof.* Let  $M = \limsup_{\alpha \rightarrow \infty} \alpha^p |\{x \in E : |f(x)| > \alpha\}|$ . By definition, this means

$$M = \lim_{\beta \rightarrow \infty} \sup_{\alpha \geq \beta} \alpha^p |\{x \in E : |f(x)| > \alpha\}|.$$

Thus, there exists  $\beta$  such that  $\sup_{\alpha \geq \beta} \alpha^p |\{x \in E : |f(x)| > \alpha\}| \leq M + 1$ . Choose an integer  $m$  such that  $2^m \geq \beta$ . Then for  $j \geq m$ , we have

$$2^{jp} |\{x \in E : |f(x)| > 2^j\}| \leq M + 1.$$

Hence,

$$\sum_{j=m}^{\infty} 2^j |\{x \in E : |f(x)| > 2^j\}| \leq \sum_{j=m}^{\infty} 2^j \frac{M+1}{2^{jp}} = (M+1) \sum_{j=m}^{\infty} 2^{(1-p)j},$$

which converges because  $2^{1-p} < 1$ .

Also,

$$\sum_{j=-\infty}^{m-1} 2^j |\{x \in E : |f(x)| > 2^j\}| \leq \sum_{j=-\infty}^{m-1} 2^j |E| = 2^m |E| < \infty,$$

by summing a geometric series. Thus,  $\sum_{j=-\infty}^{\infty} 2^j |\{x \in E : |f(x)| > 2^j\}|$ , which by Homework 25 implies  $|f| \in L^1(E)$ . Hence,  $f \in L^1(E)$ .  $\square$

**Solution Problem 34.** Give an example of a sequence of integrable functions  $f_k : [0, 1] \rightarrow \mathbb{R}$  such that  $f_k \rightarrow f$  a.e.,  $\lim_{k \rightarrow \infty} \int_{[0,1]} f_k$  exists and is finite, but  $f$  is not integrable on  $[0, 1]$ . [Hint: approximate  $1/x$  by functions with integral 0.]

*Proof.* We know that  $\int_{[0,1]} \frac{1}{x} = \infty$  from Homework 26. Let  $C_k = \int_{(1/k,1]} \frac{1}{x}$  which is finite because the function is bounded by  $k$  on this finite interval. Define

$$f_k = -kC_k\chi_{[0,1/k)} + \frac{1}{x}\chi_{(1/k,1]}.$$

Then  $f_k$  is integrable (sum of two integrable functions) and  $\int_{[0,1]} f_k = -kC_k|[0, 1/k)| + C_k = 0$ . On the other hand, for every  $x > 0$  we have  $f_k(x) = 1/x$  for all  $k$  such that  $k > 1/x$ ; thus,  $f_k \rightarrow 1/x$  a.e..  $\square$

**Solution Problem 35.** Determine the Riemann-Stieltjes integral  $\int \alpha d(-\omega_f(\alpha))$  corresponding to  $\int_E f$  where  $E = (0, 3)$  and  $f(x) = x + \lfloor x \rfloor$ . [You do not need to evaluate the integral.]

Here  $\lfloor x \rfloor$  is the greatest integer not exceeding  $x$ .]

*Proof.* Note that  $f(x) = x$  on  $(0, 1)$ ,  $f(x) = x + 1$  on  $[1, 2)$  and  $f(x) = x + 2$  on  $[2, 3)$ . Hence, for any  $\alpha \in \mathbb{R}$ , the set  $\{x \in E: f(x) > \alpha\}$  is equal to

$$\{x \in (0, 1): x > \alpha\} \cup \{x \in [1, 2): x + 1 > \alpha\} \cup \{x \in [2, 3): x + 2 > \alpha\}.$$

The set  $(a, b) \cap (c, \infty)$  can be expressed as  $\max(a, c) < x < b$ , so its measure is  $(b - \max(a, c))^+$ . This makes it possible to write  $\omega_f(\alpha)$  as

$$\omega_f(\alpha) = (1 - \max(\alpha, 0))^+ + (2 - \max(\alpha, 1))^+ + (3 - \max(\alpha, 2))^+.$$

Since the range of  $f$  is  $(0, 5)$ , the desired Riemann-Stieltjes integral is  $\int_0^5 \alpha d(-\omega_f(\alpha))$  with  $\omega_f$  given by the above formula.  $\square$

Note: One can rewrite  $\omega_f$  in other ways. For example,

$$\omega_f(\alpha) = \begin{cases} 3 - \alpha, & 0 \leq \alpha \leq 1 \\ 2, & 1 \leq \alpha \leq 2 \\ 4 - \alpha, & 2 \leq \alpha \leq 3 \\ 1, & 3 \leq \alpha \leq 4 \\ 5 - \alpha, & 4 \leq \alpha \leq 5 \end{cases}$$

**Solution Problem 36.** Suppose  $E \subset [0, 1]$ . Prove that  $\chi_E$  is Riemann integrable on  $[0, 1]$  if and only if  $|\partial E| = 0$ .

*Proof.* More generally, we claim that for any set  $E \subset X$  in a metric space  $(X, d)$  the boundary  $\partial E$  coincides with the set of discontinuities of the characteristic function  $\chi_E$ . Indeed for  $a \in X$  to be a point of continuity for  $\chi_E$ , we must have for every  $\epsilon > 0$  some  $\delta > 0$  such that  $d(x, a) < \delta \implies |\chi_E(x) - \chi_E(a)| < \epsilon$ . By using this with  $\epsilon = 1$  and recalling that  $\chi_E$  takes only the values 0, 1, we conclude that  $a$  is a point of continuity for  $\chi_E$  if and only if  $\chi_E$  is constant in some neighborhood of  $a$ . The latter means exactly one of two things:  $\chi_E = 0$  in a neighborhood of  $a$  (so,  $a$  is an interior point of  $E^C$ ), or  $\chi_E = 1$  in a neighborhood of  $a$  (so,  $a$  is an interior point of  $E$ ). It remains to recall that  $\partial E$  is the set of all points that are neither interior for  $E$  nor for  $E^C$ .

Applying the above with  $X = \mathbb{R}$ , we conclude that the set of discontinuities of  $\chi_E$  on  $\mathbb{R}$  is  $\partial E$ . When  $\chi_E$  is restricted to  $[0, 1]$ , the discontinuities at 0 and 1 may disappear (e.g., the restriction on  $\chi_{[0, 1/2]}$  to  $[0, 1]$  is continuous at 0), but the two-point set has measure zero anyway. In conclusion,  $|\partial E| = 0$  if and only if the restriction of  $\chi_E$  to  $[0, 1]$  is continuous

a.e.. Since  $\chi_E$  is bounded, the latter property is equivalent to Riemann integrability by Theorem 5.54 of the text.  $\square$

### Solution Problem 37.

- (a) Suppose  $E \subset \mathbb{R}^2$  is a Borel set. For  $x \in \mathbb{R}$ , let  $E_x = \{y \in \mathbb{R} : (x, y) \in E\}$ . Prove that  $E_x$  is a Borel set in  $\mathbb{R}$ . [Hint: For a fixed  $x$ , prove that  $\{A \subset \mathbb{R}^2 : A_x \text{ is Borel in } \mathbb{R}\}$  is a  $\sigma$ -algebra that contains all open subsets of  $\mathbb{R}^2$ .]
- (b) Suppose  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a Borel measurable function. Prove that for every  $x \in \mathbb{R}$ , the function  $g(y) = f(x, y)$  is Borel measurable on  $\mathbb{R}$ . [Note: In contrast with Fubini's Theorem, this is no "a.e." here.]

*Proof.*

- (a) Let  $M = \{A \subset \mathbb{R}^2 : A_x \text{ is Borel in } \mathbb{R}\}$ . Note that  $\emptyset, \mathbb{R}^2 \in M$ , since their slices are  $\emptyset$  and  $\mathbb{R}$ , respectively. For an arbitrary  $A \in M$ , we have  $(A^c)_x = (A_x)^c$ , and since the complement of a Borel set is Borel,  $A^c \in M$ . Also, for any countable family  $A_k \in M$ ,  $(\bigcup_k A_k)_x = \bigcup_k (A_k)_x$  is Borel, which means  $A \in M$ . Thus,  $M$  is a  $\sigma$ -algebra.

For any open set  $A \subset \mathbb{R}$  the intersection of  $A$  with any set  $B$  is open as a subset of  $B$  (MAT 601; one can also see this as the definition of subspace topology in MAT 661). Therefore,  $A_x$  is open in  $\mathbb{R}$  for every open set  $A \subset \mathbb{R}^2$ . This implies that  $M$  contains all open sets; and being a  $\sigma$ -algebra, it contains all Borel sets. In other words,  $E_x$  is Borel in  $\mathbb{R}$  whenever  $E$  is Borel in  $\mathbb{R}^2$ .

- (b) For any  $a \in \mathbb{R}$  the set  $\{y \in \mathbb{R} : g(y) > a\}$  is the  $x$ -slice of the set  $\{(u, v) \in \mathbb{R}^2 : f(u, v) > a\}$ . The latter set is Borel, hence the former is also Borel by part (a). This shows  $g$  is Borel measurable.  $\square$

**Remark:** A shorter proof of both (a) and (b) is to observe that, for a fixed  $x$ , the function  $h(y) = (x, y)$  is a continuous map from  $\mathbb{R}$  into  $\mathbb{R}^2$ , and therefore is Borel measurable (in the sense that the preimage of any Borel set is Borel). In class, we proved that the composition of Borel measurable functions is Borel measurable. Therefore, if  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is Borel measurable, then the composition  $f \circ h$  is Borel measurable; this composition is exactly  $g$ . This proves (b). Part (a) follows by applying (b) to  $f = \chi_E$  and noting that  $g = \chi_{E_x}$ .

**Solution Problem 38.** Suppose  $f : [0, 1] \rightarrow \mathbb{R}$  is a measurable function such that the function  $g(x, y) = f(x) - f(y)$  is in  $L^1([0, 1]^2)$ . Prove that  $f \in L^1([0, 1])$ .

*Proof.* By Fubini's Theorem for almost every  $x \in [0, 1]$ , the slice-function  $y \mapsto g(x, y)$  is integrable. Fix such an  $x$ . Then  $f(x)$  is a finite constant, hence integrable on  $[0, 1]$  as well (with respect to  $y$ ). By linearity,  $f(y) = f(x) - g(x, y)$  is integrable on  $[0, 1]$ .  $\square$

**Solution Problem 39.** Prove that for any  $a > 0$  the function  $f(x, y) = e^{-xy} \sin x$  is in  $L^1(E)$ , where  $E = \{(x, y) \in \mathbb{R}^2, x > 0, y > a\}$ .

**Remark:** We proved in discussing Chapter 5.5 that if a Riemann integral  $\int_a^b h(x) dx$  exists (with  $a, b$  finite), then it is equal to the Lebesgue integral. This can be extended to improper Riemann integrals in two ways:

First, if  $h \geq 0$  and  $\int_a^b h(x) dx$  exists as an Improper Riemann Integral, then it is still equal to the Lebesgue integral, by the MCT (replace  $h$  with  $\min(h, k)\chi_{[-k, k]}$  and let  $k \rightarrow \infty$ .)

Second, if  $h \in L^1((a, b))$  and the improper Riemann integral  $\int_a^b h(x) dx$  exists, then the two integrals are equal. Indeed,  $\int_c^d h(x) dx$  is equal to the Lebesgue integral for any  $a < c < d < b$ , by the above result from Chapter 5.5. As  $c \rightarrow a$  or  $d \rightarrow b$ , we can pass to the limit in the Lebesgue Integral by the DCT ( $|h|$  is dominating), and in the Riemann integral, by the definition of an improper Riemann integral.

*Proof.* The function  $f$  is continuous and therefore measurable on  $E$ . Since  $|\sin x| \leq x$  for  $x \geq 0$ , we have  $|f| \leq g$  where  $g(x, y) = xe^{-xy}$  is also continuous, hence measurable. It suffices to prove  $g \in L^1(E)$ , which can be done using Tonelli's Theorem:

$$\int_E g = \int_0^\infty \left( \int_a^\infty xe^{-xy} dy \right) dx = \int_0^\infty e^{-ax} dx = \frac{1}{a} < \infty.$$

$\square$

**Solution Problem 40.** Apply Fubini's Theorem to the function  $f$  in the previous to prove that

$$\int_0^\infty \frac{e^{-ax} \sin x}{x} dx = \tan^{-1}(1/a).$$

[Hint: Integrate  $f$  in two different ways. You do not have to do the antiderivative  $\int e^{-xy} \sin x dx$  by hand; just look it up. Food for thought (not a part of the homework): how to let  $a \rightarrow 0$ ?]

*Proof.* By the previous problem, Fubini's Theorem applies to  $f$ . On one hand,

$$\int_E f = \int_0^\infty \left( \int_a^\infty e^{-xy} \sin x dy \right) dx = \int_0^\infty \frac{e^{-ax} \sin x}{x} dx$$

On the other,

$$\begin{aligned}
 \int_E f &= \int_a^\infty \left( \int_0^\infty e^{-xy} \sin x \, dx \right) dy \\
 &= \int_a^\infty \left( \int_0^\infty e^{-xy} \sin x \, dx \right) dy \\
 &= \int_a^\infty \left( -e^{-xy} \frac{y \sin x + \cos x}{y^2 + 1} \Big|_{x=0}^{x=\infty} \right) dy \\
 &= \int_a^\infty \frac{1}{y^2 + 1} dy = \frac{\pi}{2} - \tan^{-1} a = \tan^{-1}(1/a),
 \end{aligned}$$

using the facts that  $e^{-xy} \rightarrow 0$  as  $x \rightarrow \infty$  (with  $y > 0$ ), and that  $\tan^{-1} x \rightarrow \pi/2$  as  $x \rightarrow \infty$ .  $\square$

**Solution Problem 41.** Suppose  $f \in L^1([0, 1])$ . Let  $g(x) = \int_{[x, 1]} \frac{f(t)}{t} dt$  for  $x \in (0, 1]$ . Prove that  $g \in L^1((0, 1])$  and  $\int_{(0, 1]} g = \int_{[0, 1]} f$ .

*Proof.* Let  $h(x, t) = f(t)/t$  if  $0 \leq x \leq t \leq 1$  and  $h(x, t) = 0$  otherwise. This is a measurable function on  $[0, 1] \times [0, 1]$ , because:

- The function  $(x, t) \mapsto f(t)$  is measurable, as discussed in class: level sets are products of the level sets of  $f$  with  $[0, 1]$ ;
- $1/t$  is continuous a.e., hence measurable.
- The characteristic function of the closed set  $\{(x, t) : 0 \leq x \leq t \leq 1\}$  is measurable.

Thus, Tonelli's Theorem applies to  $|h|$ . Using this fact and the fact that  $f \in L^1([0, 1])$ , we have

$$\int_{[0, 1] \times [0, 1]} |h| = \int_{[0, 1]} \int_{[0, t]} \frac{|f(t)|}{t} dx dt = \int_{[0, 1]} |f(t)| dt < \infty. \quad (11.3)$$

Thus,  $h \in L^1([0, 1] \times [0, 1])$ , which means Fubini's Theorem applies to  $h$ . Similar to (11.3), we get

$$\int_{[0, 1] \times [0, 1]} h = \int_{[0, 1]} \int_{[0, t]} \frac{f(t)}{t} dx dt = \int_{[0, 1]} f(t) dt < \infty. \quad (11.4)$$

However, the same integral is also equal to

$$\int_{[0, 1] \times [0, 1]} h = \int_{[0, 1]} \int_{[x, 1]} \frac{f(t)}{t} dt dx = \int_{[0, 1]} g(x) dx. \quad (11.5)$$

From (11.4) and (11.5), the result follows.  $\square$

**Solution Problem 42.** Prove that convolution is associative; that is, for  $f, g, h \in L^1(\mathbb{R}^n)$  we have  $(f * g) * h = f * (g * h)$ . [Note: We do not yet have the full change of variables formula, but we do have  $\int_{\mathbb{R}^n} f(x - y) dx = \int_{\mathbb{R}^n} f(x) dx$  as a consequence of the invariance of measure under translation.]

*Proof.* Since  $f, g, h \in L^1$ , the convolutions are in  $L^1$  as well. Using the commutativity of convolution,  $(f * g) * h = (g * f) * h$  which can be written as (with all integrals over  $\mathbb{R}^n$ )

$$\int (g * f)(x - t) h(t) dt = \int \left( \int f(s) g(x - t - s) ds \right) h(t) dt. \quad (11.6)$$

The convolution  $f * (g * h)$  can be written as  $(g * h) * f$ , which is

$$\int (g * h)(x - t) f(t) dt = \int \left( \int g(x - t - s) h(s) ds \right) f(t) dt. \quad (11.7)$$

In (11.7), relabel  $t$  as  $s$  and  $s$  as  $t$  — this is not using any theorem, just changing the labels. Then the desired equality of (11.6) and (11.7) becomes

$$\int \left( \int f(s) g(x - t - s) h(t) ds \right) dt \stackrel{?}{=} \int \left( \int f(s) g(x - t - s) h(t) dt \right) ds. \quad (11.8)$$

The measurability of each of the functions  $(s, t) \mapsto f(s)$ ,  $(s, t) \mapsto g(x - t - s)$ ,  $(s, t) \mapsto h(t)$ , follows as in the proof of the commutativity of convolution (the composition of a measurable function with a linear transformation is measurable).

Since the convolution  $(|f| * |g|) * |h|$  is in  $L^1$ , it is finite a.e.. Let  $x$  be such that  $(|f| * |g|) * |h|$  is finite at  $x$ . This means that

$$\int \left( \int |f(s)| |g(x - t - s)| |h(t)| ds \right) dt < \infty.$$

By Tonelli's Theorem, the function  $(s, t) \mapsto f(s)g(x - t - s)h(t)$  is in  $L^1(\mathbb{R}^{2n})$ . Hence, Fubini's Theorem can be applied to the integrals in (11.8), meaning they are equal. Thus,  $(f * g) * h = f * (g * h)$  a.e. in  $\mathbb{R}^n$ .  $\square$

Note: It is not clear whether  $(f * g) * h = f * (g * h)$  holds in the stricter sense of both convolutions having the same domain and being identically equal on that domain. This is true when  $f, g, h \geq 0$ , since then we can apply Tonelli's Theorem directly to (11.8).

**Solution Problem 43.** Suppose that  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz function. Let  $Z = \{x : g(x) = 0\}$  and suppose that  $\mathbb{R} \setminus Z$  is bounded. Let  $f(x) = 1/x^2$ . Prove that the convolution  $f * g$  is finite a.e. on  $Z$ .



*Proof.* Since  $f$  is continuous,  $Z$  is closed. Since  $\mathbb{R} \setminus Z$  is bounded, there exists  $b > 0$  such that  $\mathbb{R} \setminus Z \subset [-b, b]$ . Let  $B = (-b-1, b+1)$ ,  $K = Z \cap B$ , and  $\delta(y) = \text{dist}(y, K)$ . The Marcinkiewitz Integral,

$$M_1(x) = \int_B \frac{\delta(y)}{(x-y)^2} dy$$

is finite a.e. on  $K$ . Since  $g = 0$  on  $K$ , the Lipschitz property implies  $|g(y)| \leq L\delta(y)$ . Also, since  $g = 0$  on  $B^c$ , we have

$$\int_{\mathbb{R}} \frac{|g(y)|}{(x-y)^2} dy = \int_B \frac{|g(y)|}{(x-y)^2} dy \leq LM_1(x) < \infty$$

for a.e.  $x \in K$ . By the Integral Triangle Inequality,  $f * g$  is finite a.e. on  $K$ .

It remains to consider  $x \in Z \setminus B$ . We have  $|x-y| \geq 1$  for every  $y \in \mathbb{R} \setminus Z$  since  $y \in [-b, b]$  and  $x \notin (-b-1, b+1)$ . Thus,

$$\int_{\mathbb{R}} \frac{|g(y)|}{(x-y)^2} dy \leq \int_{[-b, b]} |g(y)| dy < \infty$$

is finite, as an integral of a continuous function over a bounded set.  $\square$

**Solution Problem 44.** Let  $C \subset [0, 1]$  be the standard “middle third” Cantor set. Let  $\delta(x) = \text{dist}(x, C)$ . For which positive numbers  $p$  is the function  $\delta^{-p}$  in  $L^1([0, 1])$ ? [Note: although Tonelli could be applied here, it is easier to use the countable additivity of integral over the set of integration.]

*Proof.* The set  $[0, 1] \setminus C$  is the union of intervals  $I_{k,j}$ , where for each  $k \in \mathbb{N}$ , we have  $2^{k-1}$  intervals of length  $1/3^k$ . Say  $I_{k,j} = (a, b)$ ; then for  $x \in (a, b)$ , we have  $\delta(x) = \min(x-a, b-x)$ . By symmetry and substitution,

$$\int_a^b \delta^{-p} = 2 \int_0^{(b-a)/2} t^{-p} dt.$$

This immediately rules out  $p \geq 1$ , when the above integral diverges. For  $0 < p < 1$ , it evaluates to

$$\frac{2}{1-p} \left( \frac{b-a}{2} \right)^{1-p} = \frac{2^p}{1-p} |I_{k,j}|^{1-p}.$$

Sum this over  $k, j$ , recalling that  $|I_{k,j}| = 1/3^k$  and that there are  $2^{k-1}$  such intervals:

$$\sum_{k,j} \int_{I_{k,j}} \delta^{-p} = \frac{2^p}{1-p} \sum_{k \in \mathbb{N}} \frac{2^{k-1}}{3^{(1-p)k}}.$$

This is a geometric series with ratio  $2/3^{1-p}$ , so it converges if and only if  $2 < 3^{1-p}$ , which is equivalent to  $p < 1 - \log 2 / \log 3$ .  $\square$

**Solution Problem 45.** Prove that for any  $q \in (0, \infty]$ , there exist:

(a) A function  $f : [2, \infty) \rightarrow \mathbb{R}$  such that  $f \in L^p([2, \infty)) \iff p > q$ .

(b) A function  $f : [2, \infty) \rightarrow \mathbb{R}$  such that  $f \in L^p([2, \infty)) \iff p \geq q$ .

[Hint: Use a suitable power of  $x$ , with a logarithmic factor, if necessary. Recall that for nonnegative functions, the Improper Riemann Integral agrees with the Lebesgue Integral (Theorem 5.53 of the text).]

*Proof.*

(a) If  $q = \infty$ , we need  $f$  such that  $f \in L^p([2, \infty)) \iff p > \infty$ . Since  $p > \infty$  is false for all  $p$ , this means  $f \notin L^p([2, \infty))$  for all  $p$ . This is achieved by choosing  $f(x) = x$ , since this function tends to infinity as  $x \rightarrow \infty$ , and so do all of its positive powers.

If  $0 < q < \infty$ , let  $f(x) = x^{-1/q}$ . Then

$$\int_2^\infty |f|^p = \int_2^\infty x^{-p/q},$$

which converges iff  $p/q > 1$ ; that is, iff  $p > q$ . Here and below, we use the fact (Theorem 5.53 of the text) that the convergence of an Improper Riemann Integral of a nonnegative function implies its Lebesgue Integrability.

(b) If  $q = \infty$ , let  $f(x) \equiv 1$ . Then  $f \in L^\infty$  but  $f^p \equiv 1$  is never integrable, so  $f \in L^p$  when  $p < \infty$ .

If  $0 < q < \infty$ , let  $f(x) = (x \log^2 x)^{-1/q}$ . When  $p > q$ ,

$$\int_2^\infty |f|^p = \int_2^\infty x^{-p/q} \log^{-2p/q} x \leq \log^{-2p/q} 2 \int_2^\infty x^{-p/q} < \infty.$$

When  $p = q$ , the antiderivative of  $x^{-1} \log^{-2} x$  is  $C - 1/\log x$  (check by differentiation). Since the antiderivative has a finite limit as  $x \rightarrow \infty$ , the integral converges.

When  $p < q$ , we have

$$\int_2^M |f|^p = \int_2^M x^{-p/q} \log^{-2p/q} x \geq \log^{-2p/q} M \int_2^M x^{-p/q} = \log^{-2p/q} M \frac{M^{1-p/q} - 2^{1-p/q}}{1-p/q}.$$

L'Hôpital's rule implies  $\lim_{x \rightarrow \infty} \frac{x^\epsilon}{\log x} = \infty$  for every  $\epsilon > 0$ . Therefore,  $(\log M)^{-2p/q} M^{1-p/q} \rightarrow \infty$  as  $M \rightarrow \infty$ , and the integral diverges.  $\square$

Remark: One can avoid logarithms in all these examples by using piecewise constant functions such as

$$f(x) = \sum_{j \in \mathbb{N}} j^{-2/q} 2^{-j/q} \chi_{[2^j, 2^{j+1})}.$$

Indeed,  $\int_2^\infty |f|^p$  is

$$f(x) = \sum_{j \in \mathbb{N}} j^{-2p/q} 2^{-jp/q} 2^j = \sum_{j \in \mathbb{N}} j^{-2p/q} 2^{(1-p/q)j},$$

which quickly shows convergence for  $p \geq q$  and divergence for  $p \leq q$ . When  $p \neq q$ , the ratio test yields this result; when  $p = q$ , the sum is  $\sum j^{-2} < \infty$ . A similar example works for the next problem, using intervals  $(2^{-j-1}, 2^{-j}]$  instead.

**Solution Problem 46.** Prove that for any  $q \in (0, \infty]$ , there exist:

- (a) A function  $f : (0, 1) \rightarrow \mathbb{R}$  such that  $f \in L^p((0, 1)) \iff p < q$ ;
- (b) A function  $f : (0, 1) \rightarrow \mathbb{R}$  such that  $f \in L^p((0, 1)) \iff p \leq q$ .

Using this and the previous problem, show that for any interval  $J \subset (0, \infty]$ , there exists a function  $f$  on some set  $E \subset \mathbb{R}$  such that  $f \in L^p(E) \iff p \in J$ .

*Proof.*

- (a) If  $q = \infty$ , let  $f(x) = \log x$ . Then  $f \notin L^\infty((0, 1))$  but for every  $p \in (0, \infty)$  we have  $\lim_{x \rightarrow 0} (\log x) / x^{p/2} = 0$  by L'Hôpital, hence  $(\log x) / x^{p/2}$  is bounded on  $(0, 1)$ . This means  $|\log x|^p \leq C / x^{1/2}$  for some constant  $C$ , and since  $\int_0^1 C / x^{1/2} < \infty$ , we have  $\log x \in L^p((0, 1))$ .

If  $0 < q < \infty$ , let  $f(x) = x^{-1/q}$ . Then

$$\int_0^1 |f|^p = \int_0^1 x^{-p/q},$$

which converges if and only if  $p/q < 1$ ; that is, if and only if  $p < q$ .

- (b) If  $q = \infty$ , let  $f(x) = 1$ , which is in  $L^p$  for all  $p \in (0, \infty]$ .

If  $0 < q < \infty$ , let  $f(x) = (x \log^2 x)^{-1/q} \chi_{(0, 1/2)}$ , where the cut-off function  $\chi_{(0, 1/2)}$  is needed to avoid a problem with  $\log x \rightarrow 0$  as  $x \rightarrow 1$ . When  $p < q$ ,

$$\int_0^1 |f|^p = \int_0^{1/2} x^{-p/q} |\log x|^{-2p/q} \leq \log^{-2p/q} 2 \int_0^{1/2} x^{-p/q} < \infty.$$

When  $p = q$ , the antiderivative of  $x^{-1} \log^{-2} x$  is  $C - 1 / \log x$  (check by differentiation). Since the antiderivative has a finite limit as  $x \rightarrow 0$ , the integral converges.

When  $p > q$ , we have

$$\begin{aligned}\int_{\delta}^{1/2} |f|^p &= \int_{\delta}^{1/2} x^{-p/q} |\log x|^{-2p/q} \geq \log^{-2p/q}(1/\delta) \int_{\delta}^{1/2} x^{-p/q} \\ &= \log^{-2p/q}(1/\delta) \frac{(1/2)^{1-p/q} - \delta^{1-p/q}}{1-p/q}.\end{aligned}$$

l'Hôpital's rule implies  $\lim_{x \rightarrow \infty} \frac{x^{\epsilon}}{\log x} = \infty$  for every  $\epsilon > 0$ . Therefore,  $(1/\delta)^{p/q-1} \log^{-2p/q}(1/\delta) \rightarrow \infty$  as  $\delta \rightarrow 0$ , and the integral diverges.  $\square$

**Solution Problem 47.** Fix  $r \in (0, 1)$ .

(a) Suppose  $f \in L^p([2, \infty))$ , where  $1 \leq p < 1/(1-r)$ . Prove that

$$\int_2^{\infty} \frac{|f(x)|}{x^r} dx < \infty.$$

(b) Show that the statement in (a) fails with  $p = 1/(1-r)$ . [Hint:  $\int_2^{\infty} \frac{1}{x \log x} dx = \infty$ .]

*Proof.*

(a) Let  $g(x) = 1/x^r$  and  $p' = p/(p-1)$ . By Hölder's Inequality  $\int_2^{\infty} |fg| \leq \|f\|_p \|g\|_{p'}$  so it remains to show  $\|g\|_{p'} < \infty$ . If  $p = 1$ , then  $p' = \infty$  and  $\|g\|_{\infty} = 1 < \infty$ . Otherwise,  $1 < p < 1/(1-r)$  implies  $1-r < 1/p < 1$ , hence  $1 < 1/p' < r$ . Then  $\int_2^{\infty} |g|^{p'} = \int_2^{\infty} x^{-rp'} < \infty$  because  $rp' > 1$ .

(b) Let  $f(x) = \frac{1}{x^{1-r} \log x}$ . Then

$$\int_2^{\infty} \frac{|f(x)|}{x^r} dx = \int_2^{\infty} \frac{1}{x \log x} dx = \infty.$$

(Using again the fact that improper Riemann integral of a nonnegative function is equal to its Lebesgue integral.) On the other hand,  $f \in L^p$  with  $p = 1/(1-r)$ :

$$\int_2^{\infty} |f(x)|^p dx = \int_2^{\infty} \frac{1}{x \log^p x} dx < \infty$$

because  $p > 1$ .  $\square$

**Solution Problem 48.** Given any sequence  $\{x_1, x_2, \dots\}$  of real numbers, define

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2 \sqrt{|x - x_k|}}.$$

Prove that  $f \in L^p([0, 1])$  for  $0 < p < 2$ .

*Proof.*

Lemma: For  $0 < p < 2$  there exists a number  $C_p \in (0, \infty)$  such that

$$\int_0^1 |x - a|^{-p/2} dx \leq C_p$$

for all  $a \in \mathbb{R}$ . Assume the lemma for now; its proof appears below.

When  $1 \leq p < 2$ , Minkowski's Inequality for infinite series yields

$$\|f\|_p \leq \sum_{k=1}^{\infty} \left( \int_0^1 \frac{1}{k^{2p} |x - x_k|^{p/2}} \right)^{1/p} \leq C_p^{1/p} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

This proves that  $f \in L^p([0, 1])$ . For  $0 < p < 1$ , use the relation between  $L^p$  spaces on a set of finite measure:  $f \in L^1([0, 1]) \implies f \in L^p([0, 1])$  for any  $p \in (0, 1)$ .

Proof of Lemma, Version 1: If  $a \in [0, 1]$ , then by translation

$$\int_0^1 |x - a|^{-p/2} dx = \int_{-a}^{1-a} |x|^{-p/2} dx \leq \int_{-1}^1 |x|^{-p/2} dx. \quad (11.9)$$

So we can use  $C_p = \int_{-1}^1 |x|^{-p/2} dx$  which is finite because  $p/2 < 1$ . If  $a < 0$ , then  $|x - a| = x - a > x$  for all  $x \in [0, 1]$ . Hence,  $\int_0^1 |x - a|^{-p/2} dx \leq \int_0^1 x^{-p/2} dx \leq C_p$  by (11.9). If  $a > 1$ , then  $|x - a| = a - x > 1 - x$  for all  $x \in [0, 1]$ , hence  $\int_0^1 |x - a|^{-p/2} dx \leq \int_0^1 |x - 1|^{-p/2} dx \leq C_p$  by (11.9).

Proof of Lemma, Version 2: Let  $I = [-a, 1 - a]$  and  $J = [-1/2, 1/2]$ . It suffices to prove that

$$\int_I |x|^{-p/2} dx \leq \int_J |x|^{-p/2} dx,$$

because then  $C_p = \int_J |x|^{-p/2} dx$  suffices. Note that

$$|x|^{-p/2} \geq 2^{p/2} \text{ on } J, \text{ and } |x|^{-p/2} \leq 2^{p/2} \text{ on } J^c. \quad (11.10)$$

By canceling out the integral over  $I \cap J$  (which may be empty) and using (11.10), we obtain

$$\begin{aligned} \int_J |x|^{-p/2} dx - \int_I |x|^{-p/2} dx &= \int_{J \setminus I} |x|^{-p/2} dx - \int_{I \setminus J} |x|^{-p/2} dx \\ &\geq \int_{J \setminus I} 2^{p/2} dx - \int_{I \setminus J} 2^{p/2} dx \\ &= 2^{p/2} (|J \setminus I| - |I \setminus J|) = 0, \end{aligned}$$

where the last step follows from  $|I| = |J|$ .

The second proof is longer, but it gives the best possible bound  $C_p$ , and this idea generalizes to other sets and functions.  $\square$

**Solution Problem 49.** Suppose  $1 \leq p \leq \infty$  and  $f \in L^p([1, \infty))$ . Define a sequence  $a$  by  $a_k = \int_k^{k+1} f, k \in \mathbb{N}$ . Prove that  $a \in \ell^p$ .

*Proof.*

Case  $p = \infty$ : By the definition of  $L^\infty$ , there exists  $M \in \mathbb{R}$  such that  $|f| \leq M$  a.e. on  $[1, \infty)$ . Hence

$$|a_k| \leq \int_k^{k+1} |f| \leq \int_k^{k+1} M = M$$

for every  $k$ , which yields  $\|a\|_\infty \leq M < \infty$ .

Case  $1 \leq p < \infty$ : Note that  $\|\chi_{[k, k+1)}\|_{p'} = 1$  for any  $k$  and any  $p'$ : for  $p' < \infty$  this is because  $\int_k^{k+1} 1 = 1$ , and for  $p' = \infty$  this is clear from the definition of the norm. By Hölder's inequality,

$$\int |f \chi_{[k, k+1)}| \leq \left( \int_k^{k+1} |f|^p \right)^{1/p} \|\chi_{[k, k+1)}\|_{p'} = \left( \int_k^{k+1} |f|^p \right)^{1/p}.$$

Therefore, using the countable additivity of integral over the domain of integration, we have

$$\sum_{k=1}^{\infty} |a_k|^p \leq \sum_{k=1}^{\infty} \int_k^{k+1} |f|^p = \int_1^{\infty} |f|^p < \infty.$$

$\square$

**Solution Problem 50.** Give an example of a continuous function  $f : [1, \infty) \rightarrow \mathbb{R}$  such that the sequence  $a$  defined in the previous problem is in  $\ell^1$ , but  $f \notin L^1(\mathbb{R})$ . [Hint:  $f$  should attain both positive and negative values so that there is some cancellation in  $\int_k^{k+1} f.$ ]

*Proof.* Let  $f(x) = \sin(2\pi x)$ . Then

$$a_k = \int_k^{k+1} \sin(2\pi x) dx = \left. \frac{-1}{2\pi} \sin(2\pi x) \right|_k^{k+1} = 0$$

for every  $k$ , so  $a \in \ell^1$ .

On the other hand, using symmetry properties of the sine function ( $\sin(t + \pi) = -\sin t$ ), we get

$$\begin{aligned} \int_k^{k+1} |\sin(2\pi x)| dx &= 2 \int_k^{k+1/2} \sin(2\pi x) dx \\ &= \left. \frac{-1}{\pi} \sin(2\pi x) \right|_k^{k+1/2} \\ &= \frac{-1}{\pi} (-1 - 1) = \frac{2}{\pi}. \end{aligned}$$

Therefore, using the countable additivity of integral over the domain of integration, we have

$$\int_1^\infty |f| = \sum_{k=1}^\infty \int_k^{k+1} |f| = \sum_{k=1}^\infty \frac{2}{\pi} = \infty.$$

This shows  $f \notin L^1([1, \infty))$ . □

**Solution Problem 51.** Suppose that  $p, p' \in [1, \infty]$  are conjugate exponents,  $f_k \rightarrow f$  in  $L^p(E)$ , and  $g_k \rightarrow g$  in  $L^{p'}(E)$ , where  $E$  is some measurable set. Prove that  $f_k g_k \rightarrow f g$  in  $L^1(E)$ .

*Proof.* In any metric space, a convergent sequence is bounded; hence,  $\{\|f_k\|_p\}$ , which is  $\{d(f_k, 0)\}$  in terms of the metric  $d$  on  $\ell^p$ , is a bounded sequence. (One can also say that  $\|f_k\|_p \leq \|f\|_p + \|f - f_k\|_p$  where the second term tends to zero, hence is bounded. But I wanted to emphasize that we can bring concepts from metric space theory, such as bounded sequences, into the study of  $\ell^p$  and  $L^p$  spaces.) Choose  $M$  such that  $\|f_k\|_p \leq M$  for all  $k$ . By the Triangle Inequality and Hölder's inequality,

$$\begin{aligned} \|f_k g_k - f g\|_1 &= \|f_k g_k - f_k g + f_k g - f g\|_1 \\ &\leq \|f_k g_k - f_k g\|_1 + \|f_k g - f g\|_1 \\ &\leq \|f_k\|_p \|g_k - g\|_{p'} + \|f_k - f\|_p \|g\|_{p'} \\ &\leq M \|g_k - g\|_{p'} + \|f_k - f\|_p \|g\|_{p'} \rightarrow 0, \end{aligned}$$

where the convergence to 0 follows from  $\|g_k - g\|_{p'} \rightarrow 0$  and  $\|f_k - f\|_p \rightarrow 0$ . □

**Solution Problem 52.** Fix  $p \in [1, \infty]$ . Let  $D = \{a \in \ell^p : \forall k \in \mathbb{N} \ 0 \leq a_{k+1} \leq a_k\}$  be the set of all nonnegative non-increasing sequences in  $\ell^p$ . Prove that  $D$  is a closed subset of  $\ell^p$ .

*Proof.* Suppose that  $a^{(j)}$  is a sequence of elements of  $D$  such that  $a^{(j)} \rightarrow a$  in  $\ell^p$ . Our goal is to prove that  $a \in D$ .

The definition of  $\ell^p$  norm (either a sum or a sup) implies  $|b_k| \leq \|b\|_p$  for every index  $k$  and any sequence  $b \in \ell^p$ . In our case, this yields  $|a_k^{(j)} - a_k| \leq \|a^{(j)} - a\|_p \rightarrow 0$ , which means  $a_k = \lim_{j \rightarrow \infty} a_k^{(j)}$ . It then follows that:

- $a_k \geq 0$ , by letting  $j \rightarrow \infty$  in the inequality  $a_k^{(j)} \geq 0$ .
- $a_k \geq a_{k+1}$ , by letting  $j \rightarrow \infty$  in the inequality  $a_k^{(j)} \geq a_{k+1}^{(j)}$ .

[This is using the comparison property of limits: if both sides of a non-strict inequality have limits, the inequality holds for the limits as well.] Thus,  $a \in D$ .  $\square$

Remark: There is no need to prove that  $a \in \ell^p$ , this is a part of the assumption " $a^{(j)} \rightarrow a$  in  $\ell^p$ ."

**Solution Problem 53.** Let  $(X, \Sigma, \mu)$  be a measure space. For  $A, B \in \Sigma$  let  $d(A, B) = \mu(A \triangle B)$ , where  $A \triangle B = (A \setminus B) \cup (B \setminus A)$  is the symmetric difference of  $A$  and  $B$ . Prove that  $d$  satisfies the triangle inequality:  $d(A, B) \leq d(A, C) + d(B, C)$  for  $A, B, C \in \Sigma$ .

*Proof.* We claim

$$A \triangle B \subset (A \triangle C) \cup (B \triangle C). \quad (11.11)$$

To prove (11.11), let  $x \in A \triangle B$ . Then either  $x \in A \setminus B$  or  $x \in B \setminus A$ ; we may assume  $x \in A \setminus B$ , because the other case is handled by relabeling  $A$  and  $B$ . Consider two cases:

Case 1:  $x \in C$ , then we have  $x \in C \setminus B$ , hence  $x \in B \triangle C$ .

Case 2:  $x \notin C$ , then  $x \in A \setminus C$ , hence  $x \in A \triangle C$ . In either case (11.11) holds, completing the proof of the claim.

Since  $\mu$  is a measure, it is monotone with respect to inclusion and subadditive (p.243 of the textbook). Therefore, (11.11) implies

$$\mu(A \triangle B) \leq \mu((A \triangle C) \cup (B \triangle C)) \leq \mu(A \triangle C) + \mu(B \triangle C),$$

which was to be proved.  $\square$



**Solution Problem 54.** Fix a function  $w \in L^1(\mathbb{R}^n)$  and define the additive set function  $\phi$  on the Lebesgue measurable subsets of  $\mathbb{R}^n$  by  $\phi(E) = \int_E w$ . Prove that the variations of  $\phi$  are given by  $\bar{V}(E) = \int_E w^+$ ,  $\underline{V}(E) = \int_E w^-$ , and  $V(E) = \int_E |w|$ .

*Proof.* Fix a measurable set  $E$ . For any arbitrary measurable set  $A \subset E$ , we have

$$\int_A w = \int_A w^+ - \int_A w^- \leq \int_A w^+ \leq \int_E w^+,$$

using the fact that  $w^+, w^- \geq 0$ . Thus,  $\bar{V}(E) \leq \int_E w^+$ . To prove the reverse inequality, observe that the set  $P = \{x \in \mathbb{E} : w(x) \geq 0\}$  satisfies  $\int_P w^+ = \int_E w^+$  (because  $w^+ \equiv 0$  on  $E \setminus P$ ) and  $\int_P w^- = 0$  (because  $w^- \equiv 0$  on  $P$ ). Thus,

$$\int_P w = \int_P w^+ - \int_P w^- = \int_E w^+,$$

completing the proof of  $\bar{V}(E) = \int_E w^+$ .

Applying the previous result to  $-w$ , we obtain

$$\sup_{A \subset E} \int_A (-w) = \int_E (-w)^+.$$

Since  $(-w)^+ = w^-$ , this yields

$$\underline{V}(E) = - \inf_{A \subset E} \int_A w = \sup_{A \subset E} \int_A (-w) = \int_E w^-.$$

Finally,  $V(E) = \bar{V}(E) + \underline{V}(E) = \int_E (w^+ + w^-) = \int_E |w|$ . □

**Solution Problem 55.** For  $k \in \mathbb{N}$  let  $\phi_k(t) = \sqrt{1/\pi} \sin(kt)$ .

- (a) Prove that  $\{\phi_k : k \in \mathbb{N}\}$  is an orthonormal system in  $L^2([0, 2\pi])$ . [Hint: product-of-sines formula.]
- (b) Prove that the linear span of  $\{\phi_k\}$  is not dense in  $L^2([0, 2\pi])$ . [Hint: compute  $\langle f, \phi_k \rangle$  for the constant function  $f \equiv 1$ .]

*Proof.*

- (a) By the product of sines formula,

$$\phi_k(t)\phi_j(t) = \frac{1}{2\pi} (\cos((k-j)t) - \cos((k+j)t)).$$

The integral  $\int_0^{2\pi} \cos mt \, dt$ , with  $m \in \mathbb{Z}$ , is equal to  $2\pi$  when  $m = 0$  and is 0 otherwise, because the antiderivative  $m^{-1} \sin mt$  is  $2\pi$ -periodic. Hence,

$$\langle \phi_k, \phi_j \rangle = \int_0^{2\pi} \phi_k(t) \phi_j(t) \, dt = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}$$

which means  $\{\phi_k\}$  is an orthonormal system.

- (b) For every  $k \in \mathbb{N}$  the integral  $\int_0^{2\pi} 1 \phi_k \, dt$  is 0 because the antiderivative of  $\phi_k$  is  $-\sqrt{1/k} \cos(kt)$  which is  $2\pi$ -periodic. Thus, all Fourier coefficients  $c_k = \langle 1, \phi_k \rangle$  are zeros. Recall from class that the following are equivalent for an orthonormal system in  $L^2$  (and in Hilbert spaces in general):

- (i) The closure of its linear span contains  $f$ .
- (ii)  $\sum c_k \phi_k = f$  (the Fourier series converges to  $f$  in  $L^2$ )
- (iii)  $\sum |c_k|^2 = \|f\|_2^2$  (Parseval's Identity holds)

The conclusion follows by observing that (ii) fails here (or, that (ii) fails).  $\square$

#### Solution Problem 56.

- (a) Prove that for every  $f \in L^2([0, 2\pi])$

$$\lim_{k \rightarrow \infty} \int_{[0, 2\pi]} f(t) \sin kt \, dt = 0.$$

- (b) Prove that (a) holds for every  $f \in L^1([0, 2\pi])$ ; this is known as the Riemann-Lebesgue Lemma. [Hint: Apply (a) to a simple function  $g$  such that  $\|f - g\|_1$  is small.]

*Proof.*

- (a) We know from the previous problem that the functions  $\phi_k(t) = \sqrt{1/\pi} \sin(kt)$  form an orthonormal system in  $L^2([0, 2\pi])$ . The integral  $\int_{[0, 2\pi]} f(t) \sin kt \, dt$  is  $\sqrt{\pi} c_k$ , where  $c_k = \langle f, \phi_k \rangle$ . Bessel's Inequality  $\sum |c_k|^2 \leq \|f\|^2$  implies  $c_k \rightarrow 0$ , which proves (a).
- (b) Given  $\epsilon > 0$ , pick a simple function  $g$  such that  $\|f - g\|_1 < \epsilon/2$  (such  $g$  exists by the density of simple functions in  $L^p$  for  $1 \leq p < \infty$ , Section 8.4). Since  $g$  is a bounded function on a bounded interval, it belongs to all  $L^p$  spaces. In particular,  $g \in L^2$ . By part (a), there exists  $N$  such that

$$\left| \int_{[0, 2\pi]} g(t) \sin kt \, dt \right| < \frac{\epsilon}{2} \quad \forall k \geq N.$$

By Hölder's inequality (which is just a comparison of integrals in this case),

$$\left| \int_{[0,2\pi]} (f(t) - g(t)) \sin kt \, dt \right| \leq \|f - g\|_1 \|\sin kt\|_\infty < \frac{\epsilon}{2}$$

for all  $k$ . Thus,

$$\left| \int_{[0,2\pi]} f(t) \sin kt \, dt \right| < \epsilon \quad \forall k \geq N,$$

which means  $\int_{[0,2\pi]} f(t) \sin kt \, dt \rightarrow 0$  by definition.  $\square$

**Solution Problem 57.** Let  $(X, \Sigma, \mu)$  be a measure space, and let  $f : X \rightarrow \mathbb{R}$  be a measurable function. For each Borel set  $E \subset \mathbb{R}$  define  $\nu(E) = \mu(f^{-1}(E))$ . Prove that  $\nu$  is a measure on the Borel  $\sigma$ -algebra of  $\mathbb{R}$ . [This measure called the pushforward of  $\mu$  under  $f$ .]

*Proof.* Recall that a real-valued function is measurable if and only if the preimages of all Borel sets are measurable. Thus,  $\nu$  is well-defined. Also,  $\nu(E) \geq 0$  for every Borel  $E$  because  $\mu \geq 0$ .

Given disjoint Borel sets  $E_k \subset \mathbb{R}$ , observe that  $f^{-1}(E_k)$  are also disjoint, since taking pre-images commutes with all set operations. Hence

$$\begin{aligned} \nu\left(\bigcup_k E_k\right) &= \mu\left(f^{-1}\left(\bigcup_k E_k\right)\right) \\ &= \mu\left(\bigcup_k f^{-1}(E_k)\right) \\ &= \sum_k \mu\left(f^{-1}(E_k)\right) \\ &= \sum_k \nu(E_k), \end{aligned}$$

which proves the countable additivity of  $\nu$ .  $\square$

**Solution Problem 58.** With the notation of the previous problem, prove that for every nonnegative Borel function  $g : \mathbb{R} \rightarrow [0, \infty)$  the function  $g \circ f$  is measurable on  $X$  and

$$\int_X (g \circ f) \, d\mu = \int_{\mathbb{R}} g \, d\nu.$$

[Hint: Begin with  $g = \chi_E$  and proceed toward more general  $g$ .]

*Proof.* For every Borel set  $E \subset \mathbb{R}$  we have  $(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E))$ , where  $g^{-1}(E)$  is Borel by assumption and therefore  $f^{-1}(g^{-1}(E))$  is measurable. This shows that  $g \circ f$  is measurable.

If  $g = \chi_E$  for some Borel set  $E \subset \mathbb{R}$ , then  $g \circ f = \chi_{f^{-1}(E)}$ . Hence,

$$\int_X (g \circ f) d\mu = \mu(f^{-1}(E)) = \nu(E) = \int_{\mathbb{R}} g d\nu.$$

By the linearity of integrals, the equality  $\int_X (g \circ f) d\mu = \int_{\mathbb{R}} g d\nu$  extends from characteristic functions to all simple functions.

Given a general Borel function  $g : \mathbb{R} \rightarrow [0, \infty)$ , let  $g_k \nearrow g$  be an approximating sequence of simple Borel functions, for example  $g_k = \min(k, 2^{-k} \lfloor 2^k g \rfloor)$ . By definition of measurability,  $g_k$  is measurable in whatever  $\sigma$ -algebra  $g$  is measurable, in this case Borel. Therefore,  $\int_X (g_k \circ f) d\mu = \int_{\mathbb{R}} g_k d\nu$  holds by the preceding case. Note that  $g_k \circ f \nearrow g \circ f$ . Letting  $k \rightarrow \infty$  and using the Monotone Convergence Theorem, we obtain  $\int_X (g \circ f) d\mu = \int_{\mathbb{R}} g d\nu$ .  $\square$

**Solution Problem 59.** Let  $(X, \Sigma, \mu)$  be a measure space, and let  $f : X \rightarrow \overline{\mathbb{R}}$  be an integrable function; that is,  $f \in L^1(X, \mu)$ . Suppose that  $\int_A f = 0$  for every  $A \in \Sigma$ . Prove that  $f = 0$   $\mu$ -a.e.; that is,  $f = 0$  on  $X \setminus Z$ , where  $\mu(Z) = 0$ .

*Proof.* For  $k \in \mathbb{N}$ , let  $E_k = \{x \in X : f(x) \geq 1/k\}$ . Then

$$0 = \int_{E_k} f d\mu \geq \int_{E_k} \frac{1}{k} d\mu = \frac{1}{k} \mu(E_k),$$

which shows  $\mu(E_k) = 0$ . Taking the union over  $k$ , we obtain  $\mu(\{f > 0\}) = 0$ . By applying this argument to  $-f$  we get  $\mu(\{f < 0\}) = 0$ . Thus  $f = 0$   $\mu$ -a.e..  $\square$

**Solution Problem 60.** Let  $(X, \Sigma, \mu)$  be a measure space. Suppose that  $\phi_k : \Sigma \rightarrow \mathbb{R}$  is a singular ASF w.r.t  $\mu$ , for each  $k \in \mathbb{N}$ . Suppose further that  $\phi : \Sigma \rightarrow \mathbb{R}$  is an ASF such that  $\phi_k(A) \rightarrow \phi(A)$  for each  $A \in \Sigma$ . Prove that  $\phi$  is singular with respect to  $\mu$ .

*Proof.* By the definition of a singular ASF, for each  $k$  there exists a set  $Z_k \subset X$  such that  $\mu(Z_k) = 0$  and  $\phi_k(A) = 0$  for all  $A \subset Z_k^C$ . Let  $Z = \bigcup_k Z_k$ . Then  $\mu(Z) = 0$  by countable additivity. Also, for any set  $A \subset Z^C$  we have  $A \subset Z_k^C$  for all  $k$ , hence  $\phi_k(A) = 0$  for all  $k$ , hence  $\phi(A) = 0$ . This shows  $\phi$  is singular w.r.t  $\mu$ .  $\square$

**Solution Problem 61.** For  $k \in \mathbb{N}$ , define  $b_k : [0, 1) \rightarrow \mathbb{R}$  by  $b_k(x) = 1$  if  $\lfloor 2^k x \rfloor$  is odd, and  $b_k(x) = 0$  otherwise. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{2b_k(x)}{3^k}.$$

Prove that:

- (a)  $f$  is a measurable function on  $[0, 1]$  with respect to the Lebesgue measure.
- (b)  $f([0, 1]) \subset C$ , where  $C$  is the standard middle-third Cantor set.

[Hint: You can use the following characterization of  $C$ ,

$$C = \{x \in [0, 1] : \text{dist}(3^m x, \mathbb{Z}) \leq 1/3 \text{ for } m = 0, 1, 2, \dots\}.$$

*Proof.*

- (a) Each term of the sum, namely  $\frac{2b_k(x)}{3^k}$ , has countably many discontinuities (specifically, the points  $x$  such that  $2^k x$  is an integer). By the converse part of Lusin's Theorem, it is measurable. Every partial sum of the series is measurable as the sum of measurable function. Finally, the series converges for every  $x$ , as its  $k$ th term is at most  $2/3^k$ . The limit of a sequence of measurable functions is measurable.
- (b) Given  $m \in \{0, 1, 2, \dots\}$ , observe that

$$3^m f(x) = \sum_{k=1}^m 2b_k(x)3^{m-k} + \sum_{k=m+1}^{\infty} \frac{2b_k(x)}{3^{k-m}},$$

where the first sum is an integer, call it  $q$ . The second (tail) sum is nonnegative and does not exceed  $\sum_{k=m+1}^{\infty} \frac{2}{3^{k-m}} = \frac{2/3}{1-1/3} = 1$ . Thus,  $q \leq 3^m f(x) \leq q + 1$ .

To refine this further, consider two cases. If  $b_{m+1}(x) = 0$ , then the tail sum is at most

$$\sum_{k=m+2}^{\infty} \frac{2}{3^{k-m}} = \frac{2/9}{1-1/3} = \frac{1}{3}.$$

Hence,  $q \leq 3^m f(x) \leq q + 1/3$ , proving that  $|3^m f(x) - q| \leq 1/3$ .

If  $b_{m+1}(x) = 1$ , then the tail sum is at least  $\frac{2b_{m+1}(x)}{3^{m+1-m}} = \frac{2}{3}$ . Hence,  $q + 2/3 \leq 3^m f(x) \leq q + 1$ . This implies  $|3^m f(x) - (q + 1)| \leq 1/3$ . In either case,  $\text{dist}(3^m f(x), \mathbb{Z}) \leq 1/3$ . This proves  $f(x) \in C$ .  $\square$

**Solution Problem 62.** Let  $\sigma$  be the pushforward of the Lebesgue measure on  $[0, 1]$  under  $f$  from the previous problem; that is,  $\sigma(A) = |f^{-1}(A)|$  for Borel sets  $A \subset \mathbb{R}$ . Prove that:

- (a)  $\sigma$  is singular with respect to the Lebesgue measure on the Borel  $\sigma$ -algebra.

- (b)  $\sigma(\{p\}) = 0$  for every  $p \in \mathbb{R}$ . [Hint: Show that for distinct  $x, y \in [0, 1)$ , there exists  $k$  such that  $b_k(x) \neq b_k(y)$ . Deduce that  $f(x) \neq f(y)$ .]

*Proof.*

- (a) We know  $|C| = 0$  from earlier in the semester. Also,  $\sigma(\mathbb{R} \setminus C) = |f^{-1}(\mathbb{R} \setminus C)| = |\emptyset| = 0$  by part (b) of the previous problem. Thus,  $\sigma$  is singular.
- (b) Suppose  $x, y$  are distinct points in  $[0, 1)$ . Without loss of generality,  $x < y$ . For sufficiently large integers  $k$ , we have  $2^k(y - x) \geq 1$ , hence  $\lfloor 2^k y \rfloor > \lfloor 2^k x \rfloor$  (adding 1 to a number increases its integer part by 1). Let  $m$  be the smallest integer such that  $\lfloor 2^m y \rfloor \neq \lfloor 2^m x \rfloor$ . Then  $\lfloor 2^{m-1} x \rfloor = \lfloor 2^{m-1} y \rfloor$ ; call this number  $q$ . Since  $2^{m-1}x, 2^{m-1}y \in [q, q+1)$ , it follows that  $2^m x, 2^m y \in [2q, 2q+2)$ . Therefore,  $\lfloor 2^m x \rfloor, \lfloor 2^m y \rfloor \in \{2q, 2q+1\}$ . Since these are distinct and  $y > x$ , we conclude that  $\lfloor 2^m y \rfloor = 2q+1$  and  $\lfloor 2^m x \rfloor = 2q$ . Thus,  $b_m(y) = 1$  and  $b_m(x) = 0$ . Also note that  $b_k(x) = b_k(y)$  for all  $k < m$  by the minimality of  $m$ .

The difference  $f(y) - f(x)$  can be estimated from below as follows:

$$\begin{aligned}
 f(y) - f(x) &= \sum_{k=1}^{m-1} \frac{2(b_k(y) - b_k(x))}{3^k} + \frac{2}{3^m} + \sum_{k=m+1}^{\infty} \frac{2(b_k(y) - b_k(x))}{3^k} \\
 &= 0 + \frac{2}{3^m} + \sum_{k=m+1}^{\infty} \frac{2(b_k(y) - b_k(x))}{3^k} \\
 &\geq \frac{2}{3^m} + \sum_{k=m+1}^{\infty} \frac{2(0 - 1)}{3^k} \\
 &= \frac{2}{3^m} - \frac{2/3^{m+1}}{1 - 1/3} \\
 &= \frac{1}{3^m} > 0.
 \end{aligned}$$

This shows that  $f$  is strictly increasing. In particular it is injective, which implies that  $\sigma(\{p\}) = |f^{-1}(p)| = 0$  for all  $p$ , where the set  $f^{-1}(p)$  has either 0 or 1 elements.  $\square$

## References

- [WZ77] R.K. Wheedon and A Zygmund. *Measure and Integral: An Introduction to Real Analysis*. 1977.