Syracuse University

MAT 738:
Algebraic Geometry

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0.1 Course Description

MAT 738 Introduction to Algebraic Geometry: The study of the zeros of polynomials. Classical algebraic varieties in affine and projective space, followed by introduction to modern theory of sheaves, schemes, and cohomology.

0.2 Disclaimer

These notes were taken in Spring 2019 in a course taught by Professor Steven Diaz. In some places, notation/material has been changed or added. Any errors in this text should be attributed to the typist – Caleb McWhorter – and not the instructor or any referenced text.

0.3 Conventions

Although some results in these notes can be generalized to arbitrary fields, we assume throughout that all fields mentioned are algebraically closed, unless otherwise noted. When arbitrary rings are mentioned, they are assumed to be commutative and possess an identity. All ring homomorphisms $\phi : R \to S$ are assumed to have the property that $\phi(1_R) = 1_S$. For a ring R, the whole ring is considered to be an ideal, but not a prime ideal.

1 Affine Varieties

1.1 Introduction

One could define Algebraic Geometry as the study of solutions to systems of polynomial equations. The early history of Algebraic Geometry was focused on what we will eventually know as affine varieties, especially the simplest cases plane algebraic curves, e.g. lines, circles, parabolas, ellipses, hyperbolas, and cubic curves. The development of Algebraic Geometry was slow, due to cumbersome language, notation, varying approaches, and especially varying or ineffective definitions. However motived by work in various fields including Complex Analysis, (Algebraic) Topology, Number Theory, and especially Commutative Algebra, Algebraic Geometry developed rapidly. Modern Algebraic Geometry is due largely to the development of the theory of sheaves and schemes by Grothendieck and Serre, but is also due to the contribution of many others such as Zariski, Čech, Leray, Cartan, et al..

1.2 Affine Varieties

Let k be a fixed algebraically closed field.[1] Often, we will focus on the case where $k = \mathbb{C}$.

Definition (Affine n-space)**.** Denote by k^n the set of ordered n-tuples of elements of k. Then define $\mathbb{A}^n_k := k^n$, called affine n-space over k. Note that this is also denoted $\mathbb{A}^n(k)$ or simply $\mathbb{A}^n$ when k is understood.

For ease of notation, we often denote the polynomial ring $k[x_1, \dots, x_n]$ simply by 'A'. We can think of elements of $A := k[x_1, \dots, x_n]$ as being functions $f : \mathbb{A}^n \to k$ via $(k_1, \dots, k_n) \mapsto f(k_1, \dots, k_n)$, i.e. via evaluation; that is, given $f \in A$ and $P = (k_1, \dots, k_n)$, we have $f(P) = f(k_1, \dots, f_n) \in k$. This allows us to consider the vanishing set of the polynomial f.

Definition (Zero Set)**.** For $f \in A := k[x_1, \dots, x_n]$, define $Z(f) := \{P \in \mathbb{A}^n \mid f(P) = 0\}$, called the zero set of f. For $T \subseteq A$, we define the zeros of T, denoted $Z(T)$, by

$$Z(T) := \bigcap_{f \in T} Z(f) = \{P \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } f \in T\}.$$

If $T = \{f_1, \dots, f_r\}$, we will often write $Z(T) = Z(f_1, \dots, f_r)$.

Note the underlying field k does matter here. For example, $Z(x^2 + 1) = \emptyset$ if $k = \mathbb{R}$ but if $k = \mathbb{C}$, then $Z(x^2 + 1) = \{\pm i\}$. As another example in $\mathbb{R}^2$, $Z(y - x^2, x - y^2)$ consists of two points, namely the points $(0, 0), (1, 1)$ of intersection, see Figure 1. However if $k = \mathbb{C}^2$, then $Z(y - x^2, x - y^2)$ consists of four points: $(0, 0), (1, 1), (-\zeta_3, \zeta_3^2), (\zeta_3^2, -\zeta_3)$, where ζ_3 is a primitive cube root of unity.

[1] Note that much of elementary Algebraic Geometry, the requirement that k be algebraically closed is unnecessary, and k being an arbitrary field, or even simply an integral domain would suffice.

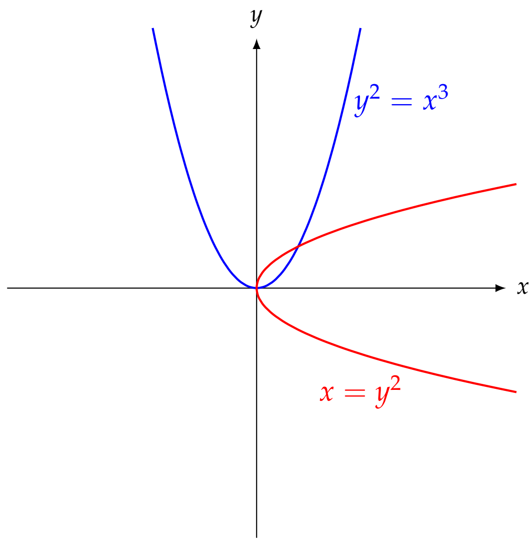

Figure 1: If $k = \mathbb{R}^2$, then the set $Z(y - x^2, x - y^2)$ is the intersection of $y = x^2$ with $x = y^2$.

We now have enough to define the basic building block of Algebraic Geometry.

Definition (Algebraic Set)**.** A subset $Y \subseteq \mathbb{A}^n$ is called an algebraic set if and only if there exists a subset $T \subseteq A$ such that $Y = Z(T)$.

That is, a set of points Y is an algebraic set if Y is a set of common zeros for a collection of polynomials T. This is the start of the bridge between Algebraic Geometry, describing geometric objects using algebraic terms, as some of the following examples shall being to show (though one will have to wait for coordinate rings and much later material to see the deeper connections).

Example 1.1.

(i) The emptyset, $\varnothing$, is algebraic since $\varnothing = Z(1)$, where $f = 1$ is the constant polynomial $f = 1$. Furthermore, $\mathbb{A}^n$ is an algebraic set since $\mathbb{A}^n = Z(0)$, where $f = 0$ is the zero polynomial.

(ii) Any single point $P = (k_1, \ldots, k_n) \in \mathbb{A}^n$ is an algebraic set since $\{P\} = Z(x_1 - k_1, \ldots, x_n - k_n)$. In fact, any *finite* collection of points is algebraic, c.f. Proposition 1.1. [One should check that you can write down an explicit polynomial to confirm this.]

(iii) The set $\{(x, y) \colon y - f(x) = 0\}$, where $f(x)$ is a polynomial, is trivially algebraic. But this is precisely the graph of the function $y = f(x)$. For example, the cubic $y = x^3$ is algebraic (meaning its graph). This easily generalizes to $y = f(x_1, \ldots, x_n)$, where $f(x_1, \ldots, x_n)$ is a polynomial. Furthermore considering the graph of $Ax^2 + By^2 +$

$Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$, we see that the graph of every quadratic surface an algebraic surface, e.g. the cone, ellipsoid, cylinder, hyperboloid, etc..

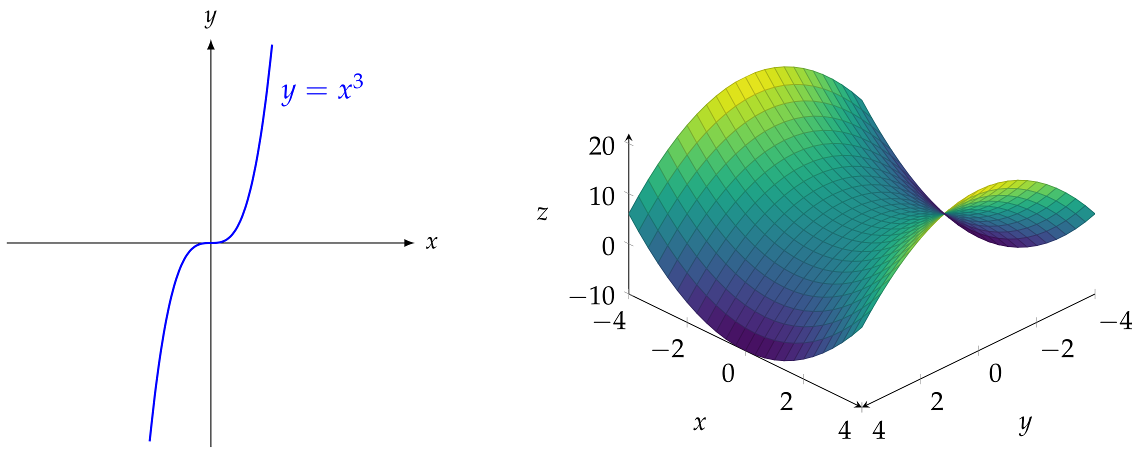

Figure 2: One the left, the algebraic set given by the curve $y = x^3$ (not including axes). On the right, the algebraic set given by $z = x^2 - y^2 + 6$ (not including axes).

(iv) A 'line glued to a circle' is an algebraic set, i.e. the image in Figure 3 (excluding the axes) is an algebraic set since it represents the set $Z((x^2 + y^2 - 1)(y - 2)) = \{(x, y) \colon (x^2 + y^2 - 1)(y - 2)\}$. Why? If $(x^2 + y^2 - 1)(y - 2) = 0$, then either $x^2 + y^2 = 1$, in which case the point lies on the circle, or $y - 2 = 0$ in which case the point (x, y) lies on the line $y = 2$. In fact, even the coordinate axes are algebraic sets because together they are $Z(xy)$, by a similar reasoning.

(v) The set of all $n \times n$ matrices can be identified with the set $\mathbb{C}^{n^2}$. This space contains many subsets of interest. For example, the matrices of determinant 1, $\mathrm{SL}_n(\mathbb{C})$, forms an affine algebraic variety in $\mathbb{C}^{n^2}$ because it is the vanishing set of the polynomial given by $\Delta - 1$, where

$$\Delta(x_{ij}) = \det \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

is the determinant. In fact, a determinantal variety is the set of all matrices (considered as a subset of $\mathbb{C}^{n^2}$) of rank at most k. For $k \geq n$, the determinantal variety is the whole space $\mathbb{C}^{n^2}$. But for $k < n$, the rank of a matrix A is at most k if and only if all its $(k+1) \times (k+1)$ subdeterminants vanish. But as these subdeterminants are

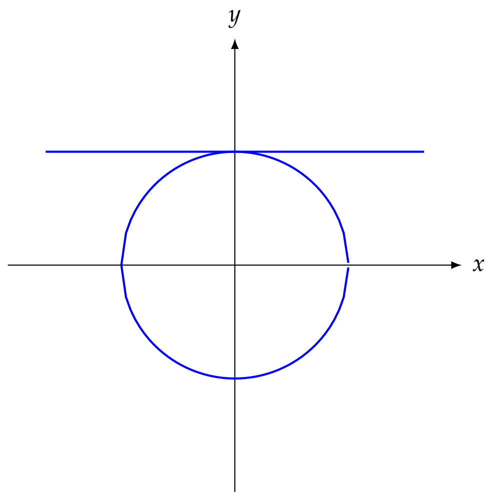

Figure 3: The algebraic set given by a 'line glued to a circle.' [Axes not included.]

polynomials in the variables x_{ij}, the set of matrices of rank at most k is an affine algebraic variety. ◁

Of course, not all sets are algebraic. Not even all curves in $\mathbb{R}^2$ are algebraic, as the following exercise demonstrates.

Exercise.

(a) Prove that in $\mathbb{A}^2_{\mathbb{R}}$, the graph of $y = \sin x$ is not an algebraic set, i.e. prove the set $\{(x, y) : y = \sin x\}$ is not algebraic. [*Hint: The graph intersects the x-axis at infinitely many points, but single variable polynomials have finitely many roots.*]

(b) Prove that in $\mathbb{A}^2_{\mathbb{R}}$, the graph of $y = e^x$ is not an algebraic set, i.e. prove the set $\{(x, y) : y = e^x\}$ is not an algebraic set. [*Hint: e^x grows 'faster' than any polynomial, i.e.* $\lim_{x\to\infty} \frac{e^x}{x} = \infty$.]

(c) Prove that the open ball in the usual Euclidean topology on $\mathbb{C}^n$ is not an algebraic set by showing that every affine algebraic variety in $\mathbb{C}^n$ is closed in the Euclidean topology. [*Hint: Polynomials are continuous functions from $\mathbb{C}^n$ to $\mathbb{C}$, so their zero sets are closed.*] Explain then why the set of invertible matrices, $\mathrm{GL}_n(\mathbb{C})$ is not an affine algebraic variety.

Note that if $N \subseteq M$, then $Z(M) \subseteq Z(N)$, since any polynomial which vanishes on all of M certainly vanishes on all of N. Combining this with the Hilbert Basis Theorem: if R is noetherian, then $R[x_1, \ldots, x_n]$ is noetherian, we obtain the following result.

Proposition 1.1.

(a) *Let $T \subseteq A$ and J be the ideal generated by T, then $Z(T) = Z(J)$. In particular, every algebraic set in $\mathbb{A}^n$ is of the form $Z(J)$ for some ideal J of A.*

(b) *Every algebraic set in $\mathbb{A}^n$ is of the form $Z(T)$ for some finite set $T \subseteq A$.*

Proof.

(a) Since T generates J, we know that $T \subseteq J$ which immediately implies $Z(J) \subseteq Z(T)$. For the reverse inclusion, let $P \in Z(T)$ so that for all $f \in T$, $f(P) = 0$. If $g \in J$, we have $g = \sum_{i=1}^m a_i f_i$, where $a_i \in A$, $f_i \in T$. But then we have

$$g(P) = \sum_{i=1}^m a_i(P) f_i(P) = \sum_{i=1}^m a_i(P) \cdot 0 = 0.$$

Therefore, $Z(T) = Z(J)$. By definition, $Y \subseteq \mathbb{A}^n$ is algebraic if and only if there is $T \subseteq A$ with $Y = Z(T)$. Taking $J := \langle T \rangle$, the second claim follows.

(b) Since k is a field, we know that $A = k[x_1, \ldots, x_n]$ is noetherian by the Hilbert Basis Theorem. Recall that every ideal in a noetherian ring is finitely generated (in fact this is equivalent to the definition). By (a), we know that $Y \subseteq \mathbb{A}^n$ is algebraic if and only if $Y = Z(J)$ for some ideal J of A. But as A is noetherian, we can find a finite number of generators, say $T := \{f_1, \ldots, f_r\}$, for J. Therefore, $Y = Z(J) = Z(\langle T \rangle) = Z(T) = Z(f_1, \ldots, f_r)$, as desired. □

Our goal is to build an underlying topological structure with which to work. This will form the basis for much of what is to come later. But first, we will need a proposition.

Proposition 1.2.

(a) *The empty set and whole space are algebraic sets.*

(b) *The finite union of algebraic sets is algebraic.*

(c) *The intersection of any algebraic sets is algebraic.*

Proof.

(a) Recall from Example 1.1 that $\emptyset = Z(1)$ and $\mathbb{A}^n = Z(0)$ are algebraic sets.

(b) By induction, it suffices to prove that the union of two algebraic sets is algebraic. Suppose that S_1, S_2 are algebraic sets. Then by Proposition 1.1, we know that $S_1 = Z(T_1)$ and $S_2 = Z(T_2)$ for some $T_1, T_2 \subseteq A$. We need show that $S_1 \cup S_2 = Z(T_1) \cup Z(T_2) = Z(T)$ for some set T. We shall show that $Z(T_1) \cup Z(T_2) = Z(T_1 T_2)$, i.e.

$T = T_1T_2 = \{t_1t_2 \colon t_1 \in T_1, t_2 \in T_2\}$ works, so that again by Proposition 1.1, $S_1 \cup S_2$ is algebraic.

To see that $Z(T_1) \cup Z(T_2) \subseteq Z(T_1T_2)$, let $P \in Z(T_1) \cup Z(T_2)$. But then $P \in Z(T_1)$ or $Z(T_2)$. Without loss of generality, suppose $P \in Z(T_1)$. Any element of T_1T_2 is of form fg, where $f \in T_1, g \in T_2$. But then $P \in Z(T_1T_2)$.

To see that $Z(T_1T_2) \subseteq Z(T_1) \cup Z(T_2)$, choose $P \in Z(T_1T_2)$. We will show that if $P \notin Z(T_1)$ then $P \in Z(T_2)$. So suppose $P \notin Z(T_1)$. Then there is $f \in T_1$ with $f(P) \neq 0$. Choose any $g \in T_2$. Now $fg \in T_1T_2$ so $(fg)(P) = 0$. But $0 = (fg)(P) = f(P)g(P)$. We know $f(P) \neq 0$ wheneverk is an integral domain, so that we must have $g(P) = 0$. As $g \in T_2$ was arbitrary, then $P \in Z(T_2)$.

(c) Suppose $\{S_i\}_{i \in \mathcal{I}}$ is a collection of algebraic sets. Then for each $i \in \mathcal{I}$, we know that $S_i = Z(T_i)$ for some $T_i \subseteq A$. But then we have

$$\bigcap_{i \in \mathcal{I}} S_i = \bigcap_{i \in \mathcal{I}} Z(T_i) = Z\left(\bigcup_{i \in \mathcal{I}} T_i\right),$$

where the last equality follows since $P \in Z(\bigcup_{i \in \mathcal{I}} T_i)$ if and only if every polynomial in every T_i vanishes at P, i.e. $P \in \bigcap_{i \in \mathcal{I}} Z(T_i)$. □

Notice that Proposition 1.2 shows that the algebraic sets are precisely the closed sets in some topology, called the Zariski topology on $\mathbb{A}^n$.

Definition (Zariski Topology)**.** The Zariski topology on $\mathbb{A}^n$ is defined by taking the open sets to be the complements of algebraic sets.

Example 1.2. What is the Zariski topology on $\mathbb{A}^1$? Since k is a field, $A = k[x]$ is a PID, so every ideal in A is principal. But by Propositon 1.1, every algebraic set Y is of the form $Y = Z(J)$ for some ideal J. But then $Y = Z(f)$ for some polynomial. Since k is algebraically closed, we can factor $f(x)$ as $f(x) = c(x-a_1)\cdots(x-a_n)$, where $c, a_1, \ldots, a_n \in k$. Therefore, $Z(f) = \{a_1, \ldots, a_n\}$. We know also from Example 1.1 that any finite set in $\mathbb{A}^n$ is algebraic. The closed sets in the Zariski topology are then $\varnothing$, $\mathbb{A}^1$, and all finite subsets. Therefore, the Zariski topology on $\mathbb{A}^1$ consists of $\varnothing$, $\mathbb{A}^1$, and the complements of finite subsets. But then every set which is not the whole set or $\varnothing$ contain all but finitely many elements of k. In particular, every two open sets intersect so that the Zarisiki topology on $\mathbb{A}^1$ is (extremely) not Hausdorff. Note that one did not need the fact that $k[x]$ a PID. Any algebraic set is if the form $\bigcap_{i \in \mathcal{I}} Z(f_i)$, where $f_i \in k[x]$. However, f_i can only be 0, in which case $Z(f_i) = \mathbb{A}^1$, a nonzero constant, in which case $Z(f_i) = \varnothing$, or a nonconstant polynomial in which case $Z(f_i)$ consists on finitely many points. In particular, the intersection consists of finitely many points. ◁

When studying objects geometrically, one breaks them into their constituent pieces and studies them individually, as well as how they fit together. So when studying geometric objects from an algebraic or topological structure, we will also want to break pieces into smaller structures.

Definition (Irreducible)**.** A nonempty subset Y of a topological space X is irreducible if and only if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets each of which is closed in Y (in the induced topology). The empty set is not considered to be irreducible. A set which is not irreducible is reducible.

Example 1.3. Whenever k is infinite, e.g. when k is algebraically closed, $\mathbb{A}^1$ is irreducible as the only proper closed subsets are finite, so $\mathbb{A}^1$ could not be written as a finite union of closed sets. $\triangleleft$

Example 1.4. In $\mathbb{A}^2$, the set $Z(xy)$ is reducible as we can write $Z(xy) = Z(x) \cup Z(y)$, i.e. for xy to vanish either x must vanish or y must vanish. This should be expected since the coordinate axes can be broken up into the x-axis and y-axis individually, c.f. Example 1.1(iv). $\triangleleft$

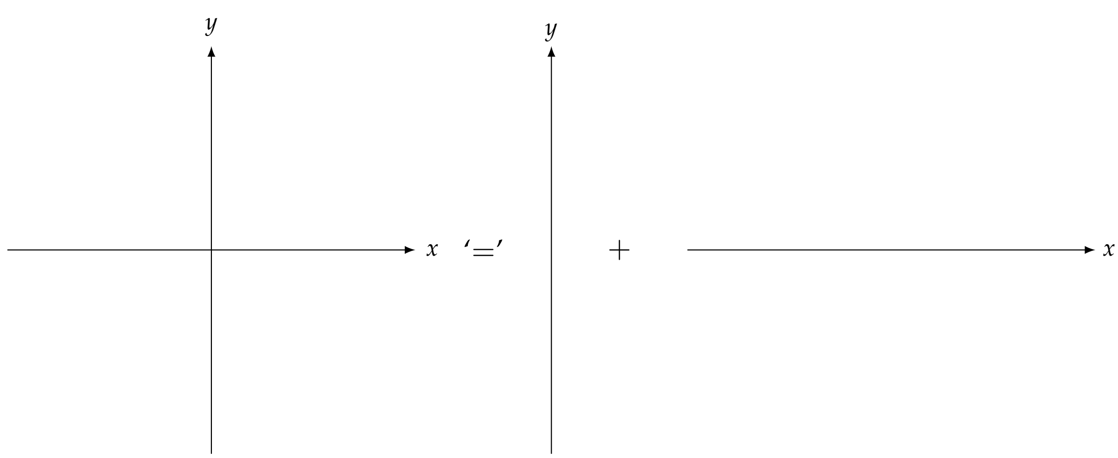

Figure 4: The decomposition $Z(xy) = Z(x) \cup Z(y)$.

Example 1.5.

(i) Any nonempty open subset of an irreducible space is irreducible and dense. This confirms our intuition that Zariski open sets are very 'big'.

(ii) If Y is an irreducible subset of X, then its closure $\overline{Y}$ in X is also irreducible.

(iii) If Y is irreducible and U_1, U_2 are nonempty open subsets of Y, then $U_1 \cap U_2 \neq \emptyset$. Why? If the intersection were not empty, then $Y \setminus U$ would be a closed proper subset of Y containing U_2, which contradicts the fact that Y_2 is dense.

(iv) If Y is irreducible and U is nonempty and open in Y, then U is both dense and irreducible. Why? If $\overline{U} \subsetneq Y$, then $Y \setminus U \subsetneq Y$ so that $Y = \overline{U} \cup (Y \setminus U)$, contradicting the irreducibility of Y. Now if U were reducible, then there would be closed subsets Y_1, Y_2 of Y with $U = (U \cap Y_1) \cup (U \cap Y_2)$. Each $U \cap Y_i$ is a proper subset of U and $U \subsetneq Y_i$.

Definition (Affine Algebraic Variety)**.** An affine algebraic variety is an irreducible closed subset of $\mathbb{A}^n$ (with the induced topology). This is also sometimes referred to as simply an affine variety. An open subset of an affine variety is a quasi-affine variety.

Notice all the proceeding definitions and examples connected subsets of $\mathbb{A}^n$ and ideals in A. Before moving forward, we first need to examine this correspondence more closely. Notice before one began with a collection of polynomials and then obtained a collection of points. One can go in the other direction as well, which gives the following definition.

Definition ($I(Y)$)**.** Let $Y \subseteq \mathbb{A}^n$ be a subset. Define the ideal of Y, denoted $I(Y)$, by $I(Y) := \{f \in A \colon f(P) = 0 \text{ for all } P \in Y\}$.

Of course, one should actually check that this is an ideal.

Proposition 1.3. *Let $Y \subseteq \mathbb{A}^n$ be any subset. Then $I(Y)$ is an ideal.*

Proof. We know that $0 \in I(Y)$ since 0 vanishes everywhere. Now suppose that $f, g \in I(Y)$ and $h \in A$. For any $P \in Y$,

$$\begin{aligned} (f+g)(P) &= f(P) + g(P) = 0 + 0 = 0 \\ (hf)(P) &= h(P)f(P) = h(P)(0) = 0 \end{aligned}$$

so that $f + g, hf \in I(Y)$. $\square$

We then have two functions between two different important sets, namely

$$\left\{ \text{ Subsets of } \mathbb{A}^n \text{ } \right\} \underset{Z(-)}{\overset{I(-)}{\rightleftarrows}} \left\{ \begin{array}{c} \text{subsets of} \\ k[x_1, \ldots, x_n] \end{array} \right\}$$

To find the relationship between this connection, we will need the following result.

Proposition 1.4.

(i) If $T_1 \subseteq T_2$ are subsets of A, then $Z(T_1) \supseteq Z(T_2)$.

(ii) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{A}^n$, then $I(Y_1) \supseteq I(Y_2)$.

(iii) For any two subsets Y_1, Y_2 of $\mathbb{A}^2_1$, we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(iv) *For any ideal $J \subseteq A$, $I(Z(J)) = \sqrt{J}$, where $\sqrt{J} := \{f \in A\colon f^r \in J$ for some $r > 0\}$ is the radical of J.*

(v) *For any subset $Y \subseteq \mathbb{A}^n$, $Z(I(Y)) = \overline{Y}$, the closure of Y in the Zariski topology.*

Proof.

(i) If $P \in Z(T_2)$, then $f(P) = 0$ for all $f \in T_2$. But then $f(P) = 0$ for all $f \in T_1$, showing that $P \in Z(T_1)$.

(ii) If $f \in I(Y_2)$, then $f(P) = 0$ for all $P \in Y_2$. But then $f(P) = 0$ for all $P \in Y_1$, showing that $f \in I(Y_1)$.

(iii) We know that $f \in I(Y_1 \cup Y_2)$ if and only if $f(P) = 0$ for all $P \in Y_1 \cup Y_2$ if and only if $f(P) = 0$ for all $P ß n Y_1$ and $f(P) = 0$ for all $P \in Y_2$ if and only if $f \in I(Y_1)$ and $f \in I(Y_2)$ if and only if $f \in I(Y_1) \cap I(Y_2)$.

(iv) This follows immediately from Hilbert's Nullstellensatz, c.f. Theorem 1.1: $f \in \sqrt{J}$ if and only if $f^r \in J$ for some $r > 0$ if and only if $f^r(P) = 0$ for all $P \in Z(J)$, where the last "if and only if" follows from the definition of $Z(-)$ and the Nullstellensatz.

(v) We know that $Z(I(Y))$ is closed because it is algebraic. We know also that $Z(I(Y)) \supseteq Y$ because if $P \in Y$, then $f(P) = 0$ for all $f \in I(Y)$, showing $P \in Z(I(Y))$. Now if W is any closed set, say $W = Z(J)$, then $I(W) \supseteq J$ because if $f \in J$, then $f(P) = 0$ for all $P \in Z(J) = W$, showing $f \in I(W)$. Thus, $Z(I(W)) \subseteq Z(J) = W$. We know that $Y = W$ and $Z(I(W)) \supseteq W$. But then $W = Z(I(W))$.

    Now suppose W is any closed subset containing Y. From (ii), we have $I(W) \subseteq I(Y)$. But then from (i), we know that $Z(I(W)) \supseteq Z(I(Y))$. Then from (iii), we know $W = Z(I(W))$ so that $W \supseteq Z(I(Y))$. Therefore, $Z(I(Y))$ is a closed set containing Y and contained in any closed set containing Y. Therefore, $Z(I(Y)) = \overline{Y}$. □

As a reminder for the reader,

Theorem 1.1 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field, J be an ideal in $A = k[x_1, \dots, x_n]$, and $f \in A$ be a polynomial vanishing at all points of $Z(J)$. Then $f^r \in J$ for some $r > 0$.*

Proof. See any 'standard' introductory text on Commutative Algebra. □

Note for Hilbert's Nullstellensatz, one does need k to be algebraically closed, as the following examples show:

Example 1.6.

(i) If $k = \mathbb{R}$ and $J = (x^2 + x + 1) \subseteq \mathbb{R}[x]$, then $Z(J) = \emptyset$. Now 1 vanishes at all points of $\emptyset = Z(J)$, but no power of 1 is in $(x^2 + x + 1)$.

(ii) If $k = \mathbb{R}$ and $J = (x^2 + y^2) \subseteq \mathbb{R}[x, y]$, then $Z(J) = \{(0, 0)\}$. Now x vanishes at $(0, 0)$ but no power of x is in $(x^2 + y^2)$. $\triangleleft$

As a result, we can now see the connection between the the subsets of $\mathbb{A}^n$ and the subsets of $k[x_1, \ldots, x_n]$ that we had discussed earlier.

$$\begin{array}{ccc}
\left\{ \text{Subsets of } \mathbb{A}^n \right\} & \underset{Z(-)}{\overset{I(-)}{\rightleftarrows}} & \left\{ \begin{array}{c} \text{subsets of} \\ k[x_1, \ldots, x_n] \end{array} \right\} \\
\text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\
\left\{ \begin{array}{c} \text{Closed subsets} \\ \text{of } \mathbb{A}^n \end{array} \right\} & \underset{Z(-)}{\overset{I(-)}{\underset{\sim}{\rightleftarrows}}} & \left\{ \begin{array}{c} \text{Radical ideals} \\ \text{of } k[x_1, \ldots, x_n] \end{array} \right\}
\end{array}$$

Corollary 1.1. *There is a one-to-one (inclusion-reversing) correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals in A given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. Furthermore, an algebraic set is irreducible if and only if its ideal is a prime ideal.*

Proof. All of this was shown in Proposition 1.4 with the exception of the final statement. Now suppose that Y is irreducible. We need show $I(Y)$ is prime. Suppose that $fg \in I(Y)$. Then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$. But then $Y = Y \cap Z(fg) = (Y \cap Z(f)) \cup (Y \cap Z(g))$, both of which are closed subsets of Y (in the subspace topology). Since Y is irreducible, Y must be one of the sets in this union. Without loss of generality, assume $Y = Y \cap Z(f)$. But then $f \in I(Y)$, as desired. Now suppose that $\mathfrak{p}$ is a prime ideal, and $Z(\mathfrak{p}) = Y_1 \cup Y_2$. Then we must have $\mathfrak{p} = I(Y_1) \cap I(Y_2)$. But then $\mathfrak{p} = I(Y_1)$ or $\mathfrak{p} = I(Y_2)$. Without loss of generality, assume $\mathfrak{p} = I(Y_1)$. Then we have $Z(\mathfrak{p}) = Z(I(Y_1)) = Y_1$. Therefore, $Z(\mathfrak{p})$ is irreducible. $\square$

This gives the most complete diagram as

$$\begin{array}{ccc}
\left\{ \text{Subsets of } \mathbb{A}^n \right\} & \underset{Z(-)}{\overset{I(-)}{\rightleftarrows}} & \left\{ \begin{array}{c} \text{subsets of} \\ k[x_1, \ldots, x_n] \end{array} \right\} \\
\text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\
\left\{ \begin{array}{c} \text{Closed subsets} \\ \text{of } \mathbb{A}^n \end{array} \right\} & \underset{Z(-)}{\overset{I(-)}{\underset{\sim}{\rightleftarrows}}} & \left\{ \begin{array}{c} \text{Radical ideals} \\ \text{of } k[x_1, \ldots, x_n] \end{array} \right\} \\
\text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\
\left\{ \begin{array}{c} \text{Subvarieties} \\ \text{of } \mathbb{A}^n \end{array} \right\} & \underset{Z(-)}{\overset{I(-)}{\underset{\sim}{\rightleftarrows}}} & \left\{ \begin{array}{c} \text{Prime ideals} \\ \text{of } k[x_1, \ldots, x_n] \end{array} \right\}
\end{array}$$

Example 1.7.

(i) $\mathbb{A}^n$ is irreducible because it corresponds to the zero ideal in A, which is prime.

(ii) If f is an irreducible polynomial in $A = k[x, y]$ (a UFD), then f generates a prime ideal (irreducibles are primes in a UFD). Then it must be that $Y = Z(f)$ is irreducible. This is called the affine curve defined by the equation $f(x, y) = 0$. If the degree of f is d, we say that Y is a curve of degree d. Generally, if f is an irreducible polynomial in $A = k[x_1, \ldots, x_n]$, the affine variety $Y = Z(f)$ is called a hypersurface for $n > 3$ (or simply a surface if $n = 3$).

(iii) A maximal ideal $\mathfrak{m}$ of $A = k[x_1, \ldots, x_n]$ corresponds to a minimal irreducible set (by 'inclusion-reversingness') of $\mathbb{A}^n$, which must be a point, say $P = (a_1, \ldots, a_n)$. Then every maximal ideal of A is of the form $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$. Note that we can express $\{P\} = Z(\langle x_1 - a_1, \ldots, x_n - a_n\rangle)$, $\langle x_1 - a_1, \ldots, x_n - a_n\rangle$ is maximal as $k[x_1, \ldots, x_n]/\langle x_1 - a_1, \ldots, x_n - a_n\rangle \cong k$ is a field, and that this result requires that k be algebraically closed. $\triangleleft$

Note that polynomials in more than one variable can have infinitely many roots. For example, $f(x, y) = xy$ has zeros whenever $x = 0$ or $y = 0$. Clearly, $Z(xy) = \{(x, 0)\ x \in \mathbb{R}\} \cup \{(0, y) \mid y \in \mathbb{R}\}$, the coordinate axes. In fact in the finite field case, there are nonzero polynomials which are everywhere zero, the trivial example being $f(x) = \prod_{i=1}^{p^n}(x - k_i)$, where $k = \{k_1, \ldots, k_{p^n}\}$. But in the case where k is infinite, there will always exist a point where the polynomial does not have a root.

Proposition 1.5. *Let k be an infinite field and $0 \neq f \in k[x_1, \ldots, x_n]$. Then there exists a point $P \in \mathbb{A}^n$ such that $f(P) \neq 0$.*

Proof. We proceed by induction on n. The case where $n = 1$ is routine since any polynomial in a single variable has finitely many roots by degree considerations. Assume then that the result is true for n, and let $f \in k[x_1, \ldots, x_{n+1}]$. Write f as $a_m x_{n+1}^m + a_{m-1}x_{n+1}^{n-1} + \cdots + a_1 x_{n+1} + a_0$, where $a_i \in k[x_1, \ldots, x_n]$. Since f is nonzero, some $a_i \neq 0$. Recalling $k[x_1, \ldots, x_{n+1}] = k[x_1, \ldots, x_n][x_{n+1}]$ and using the induction hypothesis, there exists $Q = (b_1, \ldots, b_n)$, where $b_i \in k$, such that $a_i(Q) \neq 0$. Now $f(b_1, \ldots, b_n, x_{n+1})$ is a nonzero polynomial in a single variable, which can have only finitely many roots. Because k is infinite, there must then be $b_{n+1} \in k$ so that $f(b_1, \ldots, b_n, b_{n+1}) \neq 0$, as desired. $\square$

Definition ((Affine) Coordinate Ring)**.** If $Y \subseteq \mathbb{A}^n$ is an algebraic set, we define the affine coordinate ring, denoted $A(Y)$, to be $A/I(Y)$.

One can think of elements of $A(Y)$ to be functions from $Y \to k$: if $Y \subseteq \mathbb{A}^n, f \in A = k[x_1, \ldots, x_n[$, then f certainly gives a function $f : \mathbb{A}^n \to k$ via restriction. Note that $f, g \in k[x_1, \ldots, x_n]$ induce the same function from $Y \to k$ if and only if $f - g \in I(Y)$. Hence, $A(Y)$ is the ring of all functions $f : Y \to k$ that are restrictions of polynomials.

Example 1.8. Let $Y \subseteq \mathbb{A}^2$ be the set given by $xy = 1$, i.e. $Z(xy-1)$. It is routine to verify that $\sqrt{\langle xy-1\rangle} = \langle xy-1\rangle$. But then we have

$$k[x,y]/(xy-1) \cong k[x,x^{-1}],$$

the ring of Laurent polynomials. The isomorphism is easy to prove when one realizes the relation $xy-1=0$ implies that $xy=1$ so that we can take $y = x^{-1}$. $\triangleleft$

Remark. We know that $A(Y)$ is an integral domain if and only if $I(Y)$ is prime if and only if Y is irreducible. Now $A(Y)$ is finitely generated as a k-algebra because it is a quotient of $k[x_1,\ldots,x_n]$, which is finitely generated as a k-algebra (one can take the image of the generators to be a generating set). On the other hand, any finitely generated k-algebra, say B, which is a domain is the affine coordinate ring of some variety. Why? Consider a map $\psi : k[x_1,\ldots,x_n] \to B$ is a surjection with kernel I. Then taking $Y = Z(I)$, we have $A(Y) = B$.

To study the topology of the affine varieties we have defined, we will need to introduce an important class of topological spaces, which is just the topological variant of its algebraic cousin.

Definition (Noetherian)**.** A topological space X is called noetherian if it satisfies the ascending chain condition on open subsets (every chain of open subsets stabilizes), i.e. for any chain of open subsets $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_r \subseteq \cdots$, there exists an n such that $U_n = U_{n+1} = \cdots$. Equivalently, a topological space X is noetherian if it satisfies the descending chain condition on closed subsets.

Example 1.9. The topological space $\mathbb{A}^n$, with the Zariski topology, is noetherian as given a set of closed subsets $Y_1 \supseteq Y_2 \supseteq \cdots$, applying $Y(-)$ (noting it is inclusion-reversing), we have $I(Y_1) \subseteq I(Y_2) \subseteq \cdots$. But this is an ascending chain of ideals in $A = k[x_1,\ldots,x_n]$. As A is noetherian (by the Hilbert Basis Theorem), the chain of ideals must stabilize, say at n. Finally as $Y_i = Z(I(Y_i))$, so too must the chain $Y_1 \supseteq Y_2 \supseteq \cdots$ stabilize at n. $\triangleleft$

Example 1.10. The topological space $\mathbb{R}^n$, with the usual Euclidean topology, is not noetherian. Let $\{a_n\}$ be a strictly monotone decreasing set with $a_n \to 0$. Letting Y_i be the closed sphere centered at the origin with radius a_i, we see that $\{Y_i\}$ is a chain of closed subsets that does not stabilize. $\triangleleft$

Proposition 1.6. *In a noetherian topological space, every nonempty set of closed subsets has a minimal element.*

Proof. Let S be a nonempty collection of closed subsets. Choose $Y_1 \in S_1$. If Y_1 is minimal, we are done. If not, choose $Y_2 \subseteq Y_1$. If Y_2 is minimal, we are done. If not, choose Y_3 such that $Y_3 \subseteq Y_2$. Continuing in this process, we construct a chain $Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \cdots$. Since the space is noetherian, this chain must stabilize, say at n. But then Y_n must be a minimal element of S. $\square$

This is equivalent to the algebraic property that every collection of submodules (or ideals) must contain a maximal element.

Proposition 1.7. *In a noetherian topological space X, every nonempty closed subset Y can be expressed a finite union $Y = Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets of Y. If requires $Y_i \supsetneq Y_j$ for $i \neq j$ (which is possible), the Y_i are uniquely determined up to reordering. In this case, the Y_i are called the irreducible components of Y.*

Proof. Let S be the set of all nonempty closed subsets of X that cannot be expressed as a finite union of irreducible closed sets. We wish to show that $S = \emptyset$. Suppose that $S \neq \emptyset$. Because X is noetherian, S must have a minimal element, say Y. Because $Y \in S$, Y cannot be irreducible. Thus, we can write $Y = A \cup B$, where $A, B \subsetneq Y$ are nonempty and closed in Y. But by the minimality of Y, A, B must be the finite union of irreducibles. Suppose $A = A_1 \cup \cdots \cup A_s$ and $B = B_1 \cup \cdots \cup B_t$ irreducible, where the A_i and B_j are irreducible closed subsets of A, B, respectively. But then the A_i, B_j are closed in Y, and we have $Y = A_1 \cup \cdots \cup A_s \cup B_1 \cup \cdots \cup B_t$, which contradicts the fact that $Y \in S$. Therefore, every nonempty closed subset Y of X has such a decomposition. Suppose $Y = Y_1 \cup \cdots \cup Y_s$, where the Y_i are irreducible. If $Y_i \supseteq Y_j$ for $i \neq j$, you could drop Y_j from the expression. If so, then do just that. We can then assume that $Y_i \not\supseteq Y_j$ for $i \neq j$.

It remains to show that this expression is unique. Assume $Y = Y_1 \cup \cdots \cup Y_s = Y_1' \cup \cdots \cup Y_t'$ as expressions as in the theorem. If $i = j$, $Y_i \not\supseteq Y_j$ and $Y_i' \not\supseteq Y_j'$. Then $Y_1 = Y_1 \cap Y = Y_1 \cap (Y_1' \cup \cdots \cup Y_t') = (Y_1 \cap Y_1') \cup \cdots \cup (Y_1 \cap Y_t')$. But Y_1 is irreducible, so these cannot all be closed proper subsets of Y_1. Therefore, $Y_1 = Y_1 \cap Y_i'$ for some i, i.e. $Y_1 \subseteq Y_i'$. Mutatis mutandis, $Y_i' \subseteq Y_j$ for some j. Then $Y_1 \subseteq Y_i' \subseteq Y_j$. Then $j = 1$ and $Y_1 = Y_i'$. One then reindexes so that $Y_1 = Y_1'$.

We claim that $\overline{Y \setminus Y_1} = Y_2 \cup \cdots \cup Y_s$ and $\overline{Y \setminus Y_1} = Y_2' \cup \cdots \cup Y_t'$. Now $Y_2 \cup \cdots \cup Y_s$ is a closed set containing $Y \setminus Y_1$. Let W be any closed set containing $Y \setminus Y_1$. Fix any $i \geq 2$, then $W \supseteq Y_i \setminus Y_1$. Then $Y_i = (Y_i \cap W) \cup (Y_i \cap Y_1)$. But as Y_i is irreducible, either $Y_i = Y_i \cap W$ or $Y_i = Y_i \cap Y_1$. If $Y_i = Y_i \cap Y_1$ then $Y_i \subseteq Y_1$, a contradiction. It must then be that $Y_i = Y_i \cap W$ so that $Y_i \subseteq W$. But then $Y_2 \cup \cdots \cup Y_s \subseteq W$.

Therefore, we have shown $Y_2 \cup \cdots \cup Y_s = Y_2' \cup \cdots \cup Y_t'$. Clearly, this process must terminate: without loss of generality assume $s \leq t$. We then have $Y_{s+1}' \cup \cdots \cup Y_t' = \emptyset$ and $Y_i = Y_i'$ for all i, as desired. $\square$

Corollary 1.2. *Every algebraic set in $\mathbb{A}^n$ can be uniquely expressed as a union of varieties, no one containing another.*

Definition (Dimension)**.** If X is a topological space, we say that the dimension of X, denoted $\dim X$, is the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset$

$\cdots \subset Z_n$ of distinct irreducible closed subsets of X. We define the dimension of an affine or quasi-affine variety to be the its dimension as a topological space.

Example 1.11. The dimension of $\mathbb{A}^1$ is 1 because the only irreducible closed subsets of $\mathbb{A}^1$ are single points or the whole space. ◁

Remark. In general, it is not a simple feat to compute the dimension of a space directly from its definition. One can find lower bounds by finding an example of a chain of length n. However, it is difficult to prove that there is no longest chain or that even a given chain is *the* longest. This approach generally only works if one has a strong description of what the irreducible closed sets in the space are. Note that a maximal chain does not have to be a maximum chain. Meaning that just because a chain cannot be refined to a longer chain does not mean there does not exist some other longer chain of sets.

There is also the notion of dimension for rings.

Definition ((Krull) Dimension)**.** If A is a ring, the height of a prime ideal $\mathfrak{p}$ is the supremum of all n such that there exists a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$ of distinct prime ideals. We define the (Krull) dimension of A to be the supremum of all the heights of all prime ideals.

In special cases, the relationship between this ring theoretic definition of dimension and the topological definition gives us a much more practical method of computing the dimension of a space.

Proposition 1.8. *If Y is an affine algebraic set, then the dimension of Y is the dimension of its affine coordinate ring $A(Y)$.*

Proof. If $Y \subseteq \mathbb{A}^n$ is an affine algebraic set, then the irreducible closed subsets of Y correspond to prime ideals of $A = k[x_1, \ldots, x_n]$ containing $I(Y)$. But these correspond to prime ideals in $A(Y)$. Therefore, $\dim Y$ is the length of the longest chain of prime ideals in $A(Y)$, which is precisely its dimension as a ring. □

Generally, tools from Commutative Algebra are the typical approach to computing the dimension of a space.

Theorem 1.2. *Let k be a field, and let B be an integral domain which is a finitely generated k-algebra. Then*

(i) the dimension of B is equal to the transcendence degree of the quotient field $K(B)$ of B over k

(ii) for any prime ideal $\mathfrak{p}$ in B, we have

$$\dim B = \operatorname{height} \mathfrak{p} + \dim B/\mathfrak{p}$$

Proof. See Matsumura [Mat70, Chapter 5 §14], or in the case where k is algebraically closed, Atiyah-Macdonald [AM69, Chapter 11]. $\square$

Proposition 1.9. *The dimension of $\mathbb{A}^n$ is n.*

Proof. By Proposition 1.8, the dimension of the polynomial ring $k[x_1, \ldots, x_n]$ is n. But then the result follows from (i) in Theorem 1.2 because $k(x_1, \ldots, x_n)$ has transcendence degree n. $\square$

Remark. If $Y \subseteq X$ is a closed subset, we know that $\dim Y \leq \dim X$. But in the cases where $\dim X < \infty$, this gives another approach to computing $\dim Y$ by trying the following: find lower bounds on the $\dim Y$ by finding chains in Y. Find upper bounds on the $\dim Y$ by using the fact that you know $\dim X$ and can find chains in X starting at Y. Then one tries to 'squeeze' these chains together.

Proposition 1.10. *If Y is a quasi-affine variety, then* $\dim Y = \dim \overline{Y}$.

Proof. If $Z_0 \subset \cdots \subset Z_n$ is a sequence of distinct closed irreducible subsets of Y, then $\overline{Z}_0 \subset \cdots \subset \overline{Z}_n$ is a sequence of distinct closed irreducible subsets of $\overline{Y}$. Thus, we must have $\dim Y \leq \dim \overline{Y}$. In particular, the dimension of Y must be finite. Therefore, there exists a maximal chain in Y, say $Z_0 \subset \cdots \subset Z_n$, where $n = \dim Y$. Since the chain is maximal, $Z_0 = \{P\}$, where P is a point, because if Z_0 contained at least two points, say P, Q then $\{P\} \subset Z_0$ would contradict the maximality of the chain. The point P corresponds to some maximal ideal $\mathfrak{m}$ of the affine coordinate ring $A(\overline{Y})$. Each $\overline{Z}_i$ corresponds to a prime ideal contained in $\mathfrak{m}$, so that the height $\mathfrak{m} = n$. On the other hand as P is a point in affine space, $A(\overline{Y})/\mathfrak{m} \cong k$. But then $n = \dim A(\overline{Y}) = \dim \overline{Y}$. Therefore, $\dim Y = \dim \overline{Y}$. $\square$

Note this is something special about varieties because, generally, there are topological spaces X and dense open sets U with $\dim U < \dim X$, c.f. [Har77, Exercise 1.10].

Example 1.12. Any nonempty open subset of an irreducible space is irreducible and dense. Suppose $U \subseteq Y$ is an open set, where Y is irreducible. To prove density, note that (assuming U was nonempty) if $\overline{U} \neq Y$, then we could write $Y = \overline{U} \cup U^C$, contradicting the irreducibility of Y. To see the irreducibility of U, suppose that $U = A \cup B$ with A, B closed in Y. Because U is dense in Y, we know $Y = \overline{U} = \overline{A \cup B} = \overline{A} \cup \overline{B}$. As Y is irreducible, we must have $Y = \overline{A}$ or $Y = \overline{B}$. Without loss of generality, assume that $Y = \overline{A}$. Therefore, $A = \overline{A} \cap U = Y \cap U = U$. But then U is irreducible. $\triangleleft$

Proposition 1.11. *If Y is an irreducible subsets of X, then its closure in X, $\overline{Y}$, is irreducible.*

Proof. Assume $\overline{Y}$ is not irreducible. There exist closed subsets Y_1, Y_2 of X with $\overline{Y} \not\subseteq Y_1$, $\overline{Y} \not\subseteq Y_2$, and $\overline{Y} = (\overline{Y} \cap Y_1) \cup (\overline{Y} \cap Y_2)$. As $Y \subseteq \overline{Y}$, we have $Y = (Y \cap Y_1) \cup (Y \cap Y_2)$. But Y

irreducible, so $Y = Y \cap Y_1$ or $Y = Y \cap Y_2$. Without loss of generality, suppose $Y = Y \cap Y_1$. But then $Y \subseteq Y_1$, which as Y_1 closed, $\overline{Y} \subseteq Y_1$, a contradiction. □

Theorem 1.3 (Krull Hauptidealsatz)**.** *Let A be a noetherian ring, and let $f \in A$ be an element which is neither a zero divisor nor a unit. Then every minimal prime ideal $\mathfrak{p}$ containing f has height 1.*

Proof. See Atiyah-Macdonald [AM69, p.122]. □

Proposition 1.12. *A noetherian integral domain A is a UFD if and only if every ideal of height one is principal.*

Proof. See Matsumura [Mat70, p.141] or Bourbaki [Bou65, Chapter 7 §3]. □

Proposition 1.13. *A variety $Y \subseteq \mathbb{A}^n$ has dimension $n-1$ if and only if it is the zero set, $Z(f)$, for some irreducible polynomial $f \in k[x_1, \ldots, x_n]$.*

Proof. Suppose that $Y = Z(f)$. If f is irreducible, we know that $Z(f)$ is a variety. Now $\mathfrak{p} = (f)$ is certainly the minimal prime of f. Then by Theorem 1.2, $Z(f)$ has dimension 1. But again by Theorem 1.2, $\operatorname{height} \mathfrak{p} + \dim k[x_1, \ldots, x_n] = \dim k[x_1, \ldots, n]$. Then by Proposition 1.8, $\dim Y = \dim A(Y) = n - 1$.

Now suppose Y has dimension $n-1$. Observe $\dim k[x_1, \ldots, x_n] = n - 1$, and $\dim A = n$. Then Y corresponds to a prime ideal of height 1, say $\mathfrak{p}$. Now the polynomial ring A is a UFD, so by Proposition 1.12, $\mathfrak{p}$ is necessarily generated by an irreducible polynomial, say f. But then we have shown $Y = Z(f)$, as desired. □

2 Projective Varieties

2.1 Introduction

Graded rings first.

Definition (Graded Ring)**.** Let R be a ring. A grading of R is an expression of the additive group $(R, +)$ as an internal direct sum $R = \oplus_{i=1}^{\infty} R_i$ with the property that if $a \in R_d, b \in R_e$, then $ab \in R_{e+d}$, written $R_e \cdot R_d \subseteq R_{de}$. A graded ring is a ring together with a given grading. An element of R_d is said to be homogeneous of degree d.

Example 2.1. Let $R = k[x_1, \ldots, x_n]$. $R_i = \{f \in R\colon \text{all monomials of } f \text{ having deg } i\} \cup \{0\}$, is homogenous of every degree. $\mathbb{C}[x, y]$, $x^2 + 4xy - 17y^2$ is homogeneous of degree 2. $x^3 - xy$ is not homogeneous. ◁

Remark. You can replace $\oplus_{i=0}^{\infty} R_i$ with $\oplus_{i \in M} R_i$, where M is any monoid. We say the case of $\mathbb{Z}$ is $\mathbb{Z}$-grading.

Let R be a graded ring. Each $f \in R$ has a unique expression $f = f_0 + \cdots + f_d$, where f_i is homogeneous of degree i—simply the definition of a direct sum.

Definition and prop (homogeneous graded ideal)

Proposition 2.1. *Let R be a graded ring and $I \subseteq R$ be an ideal. The following are equivalent:*

(i) I can be generated by homogeneous elements

(ii) $I = \oplus_{i=0}^{\infty} (I \cap R_i)$ as a group

(iii) given $f \in R$, write $f = f_0 + \cdots + f_d$, where f_i is homogeneous of degree i. Then $f \in I$ if and only if all $f_i \in I$.

Proof. (b) if and only if (c) definition of direct sum. In (c), note that all $f_i \in I$ certainly $f \in I$. The only things to check is $f \in I$, then $f_i \in I$. (c) to (a): write every $f \in I$ as $f = f_0 + \cdots + f_d$, where f_i is homogeneous of degree i. Certainly, I is generated by all the f_i that appear as f varies over all $f \in I$. (a) to (c): Say I is generated by $\{f_\alpha\}$, $\alpha \in A$, where f_α is homogenous of degree d_α. Choose $F \in I$, then $F = \sum_{i=1}^{n} a_i f_{\alpha_i}$. Write $F = F_0 + \cdots + F_d$, where F_j is homogeneous of degree j. What is F_j in terms of the expression?

$$F_j = \sum_{i=1}^{n} ((\deg j - \deg(f_{\alpha_i})) \text{ piece of } a_i) f_{\alpha_i \in I}.$$

□

Proposition 2.2. *Let R be a graded ring.*

(i) if I, J are homogeneous ideals, then so are $I+J$, IJ, $I \cap J$, and $\sqrt{J}$

(ii) let I be a homogeneous ideal. Then I is prime if and only if for any two homogeneous elements $f, g \in R$, if $fg \in I$ then either $f \in I$ or $g \in I$.

(iii) if $f, g \in R$, and integral domain, with f, g nonzero, then fg is homogeneous if and only if f, g are homogeneous.

Proof.
$f = f_0 + \cdots + f_d$, $g = g_0 + \cdots + g_e$ $fg = (f_d g_e) + (f_{d-1} g_e + f_d g_{e-1}) + \cdots$ If all $f_i \in I$, then $F \in I$ and we are done. If all $g_i \in I$, then $g \in I$ and we are done. So assume that $f_i \notin I$ and some $g_j \notin I$. Let s be the largest number such that $f_s \notin I$ and let t be the largest number such that $g_t \notin I$. We examine the degree $s+t$ piece of fg as $fg \in I$ and I is homogeneous, it is in I. It will have the form $f_s g_t +$ terms of the form $f_i g_j$ with either $i > s$ or $j > t$. All of these extra terms are in I, the whole sum is in the ideal. Thus, $f_s g_t \in I$ then either $f_s \in I$ or $g_t \in I$, a contradiction. □

For a homogeneous ideal I of R, we often denote $I \cap R_i$ by I_i. It then follows that

$$R/I \cong \frac{\oplus R_i}{\oplus I_i} \cong \oplus_{i=0}^{\infty} R_i / I_i.$$

Thus, R/I is naturally a graded ring with $(R/I)_i = R_i / I_i$. Now suppose R is a noetherian graded ring and $I \subset R$ is a homogeneous ideal. We know I can be generated by homogeneous ideal because it is homogeneous. Because R is noetherian, I can be generated by finitely many elements. Can it be generated by finitely many homogeneous elements? Indeed, this is the case. If you set up the proof of the ACC then every ideal is f.g., you can prove ACC then every set of generators for an ideal has a finite subset that generates.

Now for projective space, you may have heard 'parallel lines meet at infinity'.

The basic idea is to 'compactify' $\mathbb{A}^n$ by adding 'points at infinity'. This is similar to the way $\mathbb{R}^1$ is compactified to S^1 or $\mathbb{R}^2$ to S^2. This is not at all apparent from the initial definition of porjective space, but will eventually become clear.

Consider $\mathbb{A}^{n+1}$ with coordinates $x_0, x_1, \ldots, x_n$. On $\mathbb{A}^{n+1} \setminus \{(0, \ldots, 0)\}$, we define the following equivalence relation: $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$ if and only if there exists $\lambda \in k^\times$ such that $(a_0, \ldots, a_n) = (\lambda b_0, \ldots, \lambda b_n)$. The equivalence class of $(a_0, \ldots, a_n)$ is the one-dimensional subspace of $\mathbb{A}^{n_1}$ spanned by $(a_0, \ldots, a_n)$—minus the origin, which is the line through the origin and $(a_0, \ldots, a_n)$ minus the origin.

Definition. Projective n-space over k, denoted $\mathbb{P}^n_k$ or $\mathbb{P}^n(k)$ or $\mathbb{P} = \mathbb{A}^{n+1}_k \setminus \{(0, \ldots, 0)\} / \sim$. A point $P \in \mathbb{P}$ is an equivalence class of some point $(a_0, \ldots, a_n) \in \mathbb{A}^{n+1} \setminus \{(0, \ldots, 0)\}$ and is denoted $[a_0, \ldots, a_n]$. The a_i are called the homogeneous coordinates of the point P. They are well defined up to a nonzero constant multiple. $\mathbb{P}$ is also the set of lines through the origin in k^{n+1}, the set of one-dimensional subspaces of k^{n+1}.

Denote the polynomial ring $k[x_0, \dots, x_n]$ by S. For $f \in S$ and $O \in \mathbb{P}^n$, $f(P)$ is not well defined because the coordinates of P are not well defined. But suppose $f \in S$ is homogeneous: $f(\lambda a_0, \dots, \lambda a_n) = \lambda^{\deg f} f(a_0, \dots, a_n)$. $f(P)$ still not well defined since $\lambda \in k^\times$, whether $f(P) = 0$ is well defined. So we define for a homogeneous $f \in S$, we define the zeros of f, denoted $Z(f)$ by $Z(f) = \{P \in \mathbb{P}^n : f(P) = 0\}$. For a set of homogeneous elements $T \subseteq S$, we define the zeros of T, denoted $Z(T)$ by $Z(T) = \cap_{f \in I} Z(f) = \{P \in \mathbb{P}^n : f(P) = 0 \text{ for all } f \in T\}$. A set $Y \subseteq \mathbb{P}^n$ is called algebraic if and only if $Y = Z(T)$ for some subset $T \subseteq S$ of homogeneous elements. If $I \subseteq S$ is a homogeneous ideal, we define $Z(I) = \{P \in \mathbb{P}^n : f(P) = 0 \text{ for all homogeneous } f \in I\} = \cap_{f \in I, f homog} Z(f)$.

Proposition 2.3. *Let $T \subseteq S$ be a set of homogeneous elements, and let $I \subseteq S$ be the homogenous ideal generated by T. Then $Z(T) = Z(I)$. Thus, every algebraic set in $\mathbb{P}^n$ is of the form $Z(I)$ for a homogeneous ideal and $Z(\{(f_1, \dots, f_r)\})$ for finitely many homogeneous f_i, $Z(f_1, \dots, f_r)$.*

The algebraic subsets of $\mathbb{P}^n$ satisfy the properties needed to be the closed subsets of a topology on $\mathbb{P}^n$, it is the Zariski topology.

Definition. A projective algebraic variety (or simply projective variety) is an irreducible algebraic set in $\mathbb{P}^n$, with the induced topology. An open subset of a projective variety is a quasi-projective variety. The dimension of a variety or quasi-projective variety is its dimension as a topological space. If Y is any subset of $\mathbb{P}^n$, we define the homogeneous ideal of Y in S, denoted $I(Y)$, generated by $\{f \in S : f \text{ homogeneous and } f(P) = 0 \text{ for all } P \in Y\}$. If Y is an algebraic set, we define the set of homogeneous coordinate ring of Y to be $S(Y) := S/I(Y)$.

Note that $I(Y)$ is a homogeneous ideal so $S(Y)$ is a graded ring. These are similar to old algebraic sets with one funny difference. S is clearly a homogeneous ideal of S. $Z(S) = \emptyset$, $S = k[x_0, \dots, x_n]$ $(x_0, \dots, x_n)$ is also clearly a homogeneous ideal of S. It is often called S_+ because it contains all homogeneous elements of positive degree $Z(S_+) = \emptyset$. This is the only thing which goes awry. S_+ is sometimes called the irrelevant maximal ideal since it removed from the correspondence we saw in the affine case.

We now work on showing $\mathbb{P}^n$ is $\mathbb{A}^{n+1}$ with points added at ∞. In $\mathbb{P}^n$, consider closed sets of the form $Z(f)$ with f a nonconstant polynomial These are called hypersurfaces. When f is linear, it is called a hyperplane. Of particular interest is when $f = x_i$ for some i. Set $H_i = Z(x_i) = \{[a_0, \dots, a_n] \mid a_i = 0\}$ and $U_i = \mathbb{P}^n \setminus H_i$, $i = 0, \dots, n$, U_i open. The U_i are an open cover of $\mathbb{P}^n$

$$(\cup_i^n U_i)^C = \cap_0^n U_i^C = \cap_0^n H_i = \{[a_0, \dots, a_N] : a_i = 0 \text{ all } i\} = \emptyset.$$

$P = [a_0, \dots, a_n] \in U_i$ if and only if $P = [a_0/a_i, \dots, a_{i-1}/a_i, 1, a_{i+1}/a_i, \dots, a_n/a_i]$. Each point in U_i has a unique set of homogeneous coordinates such that the ith coordinate is 1. Define a map $\phi_i : \mathbb{A}^n \to U_u$ by $\phi_i(a_1, \dots, a_n) = [a_1, a_2, \dots, 1, \dots, a_n]$. Moreover, this is clearly a bijection.

Proposition 2.4. *The map ϕ_i is a homeomorphism of U_i with its induced topology to $\mathbb{A}^n$ with its Zariski topology.*

The proof is based on homogenization and dehomogenization of polynomials. To make notation simpler, we assume $i = 0$. Let $S = k[x_0, \ldots, x_n]$, $A = k[x_1, \ldots, x_n]$, and S^h denote the homogeneous elements of S. There is a ring homomorphism $\alpha : S \to A$, defined as evaluation at $x_0 = 1$: $\alpha(f(x_0, \ldots, x_n)) = f(1, x_1, \ldots, x_n)$. Since α is a ring homomorphism

$$\alpha(f+g) = \alpha(f) + \alpha(g)$$
$$\alpha(fg) = \alpha(f)\alpha(g),$$

called dehomogenization because even if f is homogeneous, $\alpha(f)$ may not be: $\alpha(x_0^2 + x_0x_1) = 1 + x_1$.

Homogenization: $\beta : A \to S^h$: $\beta(f(x_1, \ldots, x_n)) = x_0^{\deg f} f(x_1/x_0, \ldots, x_n/x_0)$, $\beta(f)$ is homogeneous of degree $\deg f$. Look at a monomial of f: $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, $i_1 + \cdots + i_n \leq \deg f$.

$$x_0^{\deg f}(x_1/x_0)^{i_1} \cdots (x_n/x_0)^{i_n} = x_0^{\deg f - (i_1 + \cdots + i_n)} x_1^{i_1} \cdots x_n^{i_n}$$

has degree $\deg f$.

$\beta(fg) = \beta(f)\beta(g)$. $\beta(fg) = x_0^{\deg f + \deg g} f(x_1/x_0, \ldots, x_n/x_0) g(x_1/x_0, \ldots, x_n/x_0) = x_0^{\deg f} f(x_1/x_0, \ldots, x_n/x_0) x_0^{\deg g} g(x_1/x_0, \ldots, x_n/x_0) = \beta(f)\beta(g)$. $\beta(f+g)$ might not equal $\beta(f) + \beta(g)$.

If $\deg f \neq \deg g$, $\beta(f) + \beta(g)$ will not even be homogeneous.

If $\deg f = \deg g = \deg(f+g)$, then $\beta(f+g) = \beta(f) + \beta(g)$.

$$\alpha(\beta(f)) = \alpha(x_0^{\deg f} f(x_1/x_0, \ldots, x_n/x_0) = 1^{\deg f} f(x_1/x_0, \ldots, x_n/x_0) = f$$

$\beta(\alpha(f)) = \beta(\alpha(x_0^l f))$. Assume you have a homogeneous F that is not divisible by x_0.

$$F = \sum a_{i_0 \cdots i_n} x_0^{i_0} \cdots x_n^{i_n}$$

where $i_0 + \cdots + i_n = d$.

$\alpha(F) = \sum a_{i_0 \cdots i_n} x_1^{i_0} \cdots x_n^{i_n}$ and in at least one case $i_1 + \cdots + i_n = d$.

$\beta(\alpha(F)) = x_0^d a_{i_0 \cdots i_n}((x_1/x_0)^{i_0} \cdots (x_n/x_0)^{i_n} = F$. Consider a point $(a_1, \ldots, a_n) \in \mathbb{A}^n$. $\phi_0(a_1, \ldots, a_n) = [1, a_1, \ldots, a_n] \in U_0 \subseteq \mathbb{P}^n$.

$f \in k[x_1, \ldots, x_n]$ $F \in k[x_0, \ldots, x_n]$ F homogeneous $f(a_1, \ldots, a_n) = 1^{\deg f} f(a_1/1, \ldots, a_n/1) = \beta(f)(1, a_1, \ldots, a_n)$. $F(1, a_1, \ldots, a_n) = \alpha(F)(a_1, \ldots, a_n)$. f vanishes at $P \in \mathbb{A}^n$ if and only if $\beta(f)$ vanishes on $\phi_0(P) \in U_0$ F vanishes at $\phi_0(P) \in U_0$ if and only if $\alpha(F)$ vanishes at $P \in \mathbb{A}^n$.

It is now easy to see that $\phi_0 : \mathbb{A}^n \to U_0 \subseteq \mathbb{P}^n$ is a homeomorphism.

Suppose $Y \subseteq \mathbb{A}^n$ is closed. $Y = Z(T)$, $T \subseteq k[x_1, \ldots, x_n]$. Set $\beta(T) = \{\beta(f) \mid f \in T\}$, $\phi_0(Y) = Z(\beta(T)) \cap U_0$. $W \subseteq U_0$ closed in U_0 $W = U_0 \cap \overline{W}$, $\overline{W}$ closed in $\mathbb{P}^n$ $\overline{W} = Z(T)$, $T \subseteq S^h$ $\alpha(T) = \{\alpha(F) \mid F \in T\}$ $\phi_0^{-1}(W) = Z(\alpha(T))$

Corollary 2.1. *If Y is a projective (respectively quasi-projective) variety, then Y is covered by the open sets $U_i \cap Y$, $i = 0, \ldots, n$, which are homeomorphic to affine (respectively quasi-affine) varieties with the map ϕ_i.*

$\mathbb{A}^n \cong U_i \subseteq \mathbb{P}^n$, $U_i = \mathbb{P}^n \setminus H_i$, $H_i = Z(x_i) = \{[a_0, \ldots, 0, \ldots, a_n \mid \text{not all } a_i = 0\} \cong \mathbb{P}^{n-1}$, $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$. Parallel lines meet at infinity.

$\mathbb{A}^2$ coordinates x, y $Ax + By + C = 0$ at least one of $A, B \neq 0$. $AB + Cy + D = 0$, $C \neq D$.

Homogenize $Z(A(X/Z) + B(Y/Z) + C = 0)$ $Ax + By + CZ = 0 - Ax + BY + DZ = 0$ so $(C - D)Z = 0$ so $Z = 0$. If $Z = 0$, still have $Ax + By = 0$.

$x = \lambda B$, $y = -\lambda A$, $[\lambda B, -\lambda A, 0]$ for $\lambda \neq 0$. One point in $\mathbb{P}^2$. $[B, -A, 0]$ it is in H_2. $x_0 \iff x, x_1 \iff y, x_2 \iff z$.

Slope of $Ax + By + C = 0$ is the ratio $-A/B$. So each slope determines what point at infinity the line goes to.

Looking at a curve in different patches and seeing how they fit together. Hyperbola: $xy - 1 = 0$ homogeneize that $xy - z^2 = 0$ $z = 1$, $xy - 1 = 0$ $y = 1$ $x - z^2 = 0$ $x = 1$ $y - z^2 = 0$

$B(1/2, 2), [1/2, 2, 1]$ $C[-1, -1, 1], (-1, -1)$ $D[-1/2, -2, 1], (-1/2, -2)$ $A[1, 1, 1], (1, 1)$

$y - z^2$ (plot) $E[1, 0, 0](0, 0)$ $A(1, 1)[1, 1, 1]$ $B(4, 2)[1/2, 2, 1] = [1, 4, 2]$ $C[-1, -1, 1] = [1, 1, -1]$ $D[-1/2, -2, 1] = [1, 4, -2]$

$x - z^2 = 0$ (plot) $F(0, 0)$ $A(1, 1)[1, 1, 1]$ $B(1/4, 1/2), [1/2, 2, 1] = [1/4, 1, 1/2]$ $C(1, -1)[-1, -1, 1] = [1, 1, -1]$ $D(1/4, -1/2)[-1/2, -2, 1] = [1/4, 1, -1/2]$

$\rho_d(P) = [M_0(P), M_1(P), \ldots, M_N(P)]$ First note that the map is well defined. $\rho_d([\lambda a_0, \ldots, \lambda a_n]) = \lambda^d \rho_d([a_0, \ldots, a_n])$ because all of the M_i are homogeneous of degree d.

$f : \mathbb{P}^2 \to \mathbb{P}^2$ $f[(x_0, x_1, x_2]) = [x_0 x_1 x_0 x_2, x_0^2]$ $[0, 0, 1] \to [0, 0, 0]$, bad

But not a problem above since one of a_i is nonzero. The exercise asks you to check various things. ρ_d is one-to-one, $\rho_d(\mathbb{P}^n)$ is a closed subset of $\mathbb{P}^N$, if we give $\rho_d(\mathbb{P}^n)$ its induced topology as a subset of $\mathbb{P}^N$ then $\rho_d : \mathbb{P}^n \to \rho_d(\mathbb{P}^n)$ is a homeomorphism. It seems reasonable to call it an embedding.

Example 2.2. $n = 1, d = 2$ $[a_0, a_1] \mapsto [a_0^2, a_0 a_1, a_1^2]$, labeled X, Y, Z. The image is contained in $Z(XZ - Y^2)$. In fact,.... ◁

Example 2.3 (Segre Embedding)**.** Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^n$ be the map defined by sending the ordered pair $[a_0, \ldots, a_r] \times [b_0, \ldots, b_s] \mapsto [\ldots, a_i b_j, \ldots]$ all $a_i b_j$. $N = rs + r + s = (r+1)(s+1) - 1$. This is well defined. $[\lambda a_0, \ldots, \lambda a_r] \times [\mu b_0, \ldots, \mu b_s] = \lambda\mu[\cdots a_i b_j \cdots]$. ◁

Example 2.4. $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ $[a_0, a_1] \times [b_0, b_1] \mapsto [a_0 b_0, a_0 b_1, a_1 b_0, a_1 b_1]$ labeled $WXYZ$. image lies on $WZ - XY = 0$ in fact $=$. Book asks you to show ψ is one-to-one and image is closed. ◁

The book does not ask you to show that ψ is a homeomorphism onto its image when you give the image the induced topology. Why? We have not defined the Zariski topology on $\mathbb{P}^r \times \mathbb{P}^s$.

It is an exercise to show that if we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two $\mathbb{A}^1$'s. Hint: In $\mathbb{A}^2$ with coordinates x, y, $Z(x-y)$ is closed in $\mathbb{A}^2$ with the Zariski topology, but not the product topology.

There are three ways to define the Zariski topology on $\mathbb{P}^r \times \mathbb{P}^s$.

1. $\mathbb{P}^r$ is covered by U_i's with the U_i homeomorphic to $\mathbb{A}^r$. $\mathbb{P}^s$ is covered by U_j's with each U_j homeomorphic to $\mathbb{A}^s$. $\mathbb{P}^r \times \mathbb{P}^s$ will be covered by $U_i \times U_j$'s $\mathbb{P}^s$ is covered by U_j's with each U_j homeomorphic to $\mathbb{A}^s$. $\mathbb{P}^r \times \mathbb{P}^s$ will be covered by $U_i \times U_j$'s. Give $U_i \times U_j$ the Zariski topology of $\mathbb{A}^{r+s}$, $U_i \times U_k \cap U_k \times U_l$. The topology induced from $U_i \times U_j$ is the same as the topology induced from $U_k \times U_l$. So you can glue to get a topology on $\mathbb{P}^r \times \mathbb{P}^s$. $X \subseteq \mathbb{P}^r \times \mathbb{P}^s$ closed if and only if $X \cap U_i \times U_j$ closed all i, j.

2. on $\mathbb{P}^r$ take homogeneous coordinates $x_0, \ldots, x_r$ and on $\mathbb{P}^s$ take homogeneous coordinates $y_0, \ldots, y_s$. A polynomial $f \in k[x_0, \ldots, x_r, y_0, \ldots, y_s]$ is said to be bihomogeneous of bidegree d, e if and only if every monomial of f has degree d in the x_i's and degree e in the y_j's. In this case, $f(\lambda a_0, \ldots, \lambda a_r, \ldots, \mu b_0, \ldots, \mu b_r) = \lambda^d \mu^e f(a_0, \ldots, a_r, b_0, \ldots, b_r)$ whether f vanishes on $(P, Q) \in \mathbb{P}^r \times \mathbb{P}^s$ is well defined. Define $Y \subseteq \mathbb{P}^r \times \mathbb{P}^s$ is closed if and only if $Y = Z(T)$ for some subset $T \subseteq k[x_0, \ldots, y_s]$ of bihomogeneous polynomials.

3. Give $\mathbb{P}^r \times \mathbb{P}^s$ the unique topology it must have for the Segre map $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \psi(\mathbb{P}^r \times \mathbb{P}^s) \subseteq \mathbb{P}^N$ to be a homeomorphism when $\psi(\mathbb{P}^r \times \mathbb{P}^s)$ is given the induced topology as a subset of $\mathbb{P}^N$.

All three give the same topology.

Note: any affine variety is also a quasi-affine variety. Same thing holds for projective and quasi-projective.

Definition (Regular)**.** Let Y be a quasi-affine variety in $\mathbb{A}^n$. A function $f : Y \to k$ is regular at a point $P \in Y$ if and only if there is an open neighborhood U with $P \in U \subseteq Y$ and polynomials $g, h \in A = k[x_1, \ldots, x_n]$ such that h is nowhere zero on U and $f = g/h$ on U. We say that f is regular on Y if and only if it is regular at every point of Y.

"A function is regular if and only if it is locally a rational function."

Note: It is important to remember this is not iff continuous does not imply regular. $f : \mathbb{A}^1 \to \mathbb{A}^1$ every bijection is continuous.

Lemma 2.1. *A regular function is continuous when k is identified with $\mathbb{A}^1_k$ with its Zariski topology.*

Proof. It is enough to show that $f^{-1}(C)$ is closed for any closed set C. First, note that $f^{-1}(\emptyset) = \emptyset$. Now the (nonempty) closed sets of $\mathbb{A}^1_k$ are a finite collection of points. Because of this and the fact that a finite union of closed sets is closed, it suffices to show that $f^{-1}(a)$ is a closed set, i.e. a finite collection of points, where $a \in \mathbb{A}^1_k$.

This can be checked locally: a subset Z of a topological space Y is closed if there exists an open cover of Y such that $Z \cap U$ is closed in U for every U in cover. Let U be an open set on which f can be represented $f = g/h$, where g, h polynomials on U and $h \neq 0$. The collection of all such U's form an open cover of $\mathbb{A}^1_k$. Observe $f^{-1}(a) \cap U = \{P \in U : g(P)/h(P) = a\}$. But $g(P)/h(P) = a$ if and only if $(g - ah)(p) = 0$. This means $f^{-1}(a) \cap U = Z(g - ah) \cap U$, which is a closed set. Therefore, $f^{-1}(a)$ is closed in Y, as desired. □

Functions on projective spaces are more tricky. $P \in \mathbb{P}^n$, $f \in k[x_0, \ldots, x_n]$. $f(P)$ is not well defined even if f if homogeneous. $f, g \in k[x_0, \ldots, x_n]$ both continuous functions of the same degree d. also assume $g(P) \neq 0$. $P = [a_0, \ldots, a_n]$

$$\frac{f(\lambda a_0, \ldots, \lambda a_n)}{g(\lambda a_0, \ldots, \lambda a_n)} = \frac{\lambda^d f(a_0, \ldots, a_n)}{\lambda^d g(a_0, \ldots, a_n)} = \frac{f(a_0, \ldots, a_n)}{g(a_0, \ldots, a_n)}.$$

Therefore, $f/g(P)$ is well defined.

Def: Let Y be a quasi projective variety in $\mathbb{P}^n$. A function $f : Y \to k$ is regular at a point $P \in Y$ iff there is an open neighborhood Y with $P \in U \subseteq Y$ and homogeneous polynomials $g, h \in S = k[x_0, \ldots, x_n]$ of the same degree such that h is nowhere zero on Y and $f = g/h$ on Y. We say f is regular iff it is regular at every point.

Proposition 2.5. *Let $Y \subseteq \mathbb{P}^n$ be a quasi-projective variety. $f : Y \to k$ a function. $P \in Y$ and assume $P \in U_i = \mathbb{P}^n \setminus Z(x_i)$. Then when we think of U_i as $\mathbb{A}^n$, $Y \cap U_i$ is a quasi-affine variety and by restriction we get a function $f|_{Y \cap U_i} : Y \cap U_i \to k$. Then f is regular at P under the projective definition iff $f|_{Y \cap U_i}$ is regular at P under the affine definition.*

Proof. Assume f is regular at P under the projective definition. Thus, we have an open set U with $P \in U \subseteq Y$ and homogeneous polynomials $g, h \in S = k[x_0, \ldots, x_n]$ of the same degree such that h is nowhere zero on U and $f = g/h$ on U. On $U_0 \cap U$, which is open in $U_0 \cap Y$

$$\frac{g(a_0, \ldots, a_n)}{h(a_0, \ldots, a_n)} = \frac{g(1, a_1/a_0, \ldots, a_n/a_0)}{h(1, a_1/a_0, \ldots, a_n/a_0)}.$$

But $g(1, x_1, \ldots, x_n)h(1, x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$, $h \neq 0$ on $U_0 \cap Y$ so $f|_{U_0 \cap Y}$ is regular at P under the affine defintion.

Assume $f|_{U_0 \cap Y}$ is regular at P under the affine definition. There is an open neighborhood U with $P \in U \subseteq U_0 \cap Y$ and polynomials $g, h \in k[x_1, \ldots, x_n]$ such that h is nowhere zero on U and $f = g/h$ on U. Note that U is open in Y not just $U_0 \cap Y$. Let G and H be the homogenizations of g and h. If they do not have the same degree multiply the one of lower degree by an appropriate power of x_0 to make them both have the same degree. Now call them $\overline{G}, \overline{H}$. For any point $P = [1, a_1, \ldots, a_n] \in U$,

$$f(P) = \frac{g(P)}{h(P)} = \frac{\overline{G}(1, a_1, \ldots, a_n)}{\overline{H}(1, a_1, \ldots, a_n)} = \frac{\overline{G}(\lambda, \lambda a_1, \ldots, \lambda a_n)}{\overline{H}(\lambda, \lambda a_1, \ldots, \lambda a_n)}$$

Fact: X a variety f, g regular functions on X. Suppose $f = g$ on some nonempty open $U \subseteq X$. Then $f = g$ on all X. Proof $Z(f - g)$ is closed in X and since in an irreducible space nonempty opens are dense it is also dense. $Z(f - g) = X$. $\square$

Definition. Let k be a fixed algebraically closed field. A variety over k is an affine, quasi-affine, projective, or quasi-projective variety. If X and Y are two varieties, a morphism $\phi : X \to Y$ is a continuous map such that for every open $V \subseteq Y$ and for every regular function $f : V \to k$, the function $f \circ \varphi : \varphi^{-1}(V) \to k$ is regular. A morphism $\varphi : X \to Y$ is a isomorphism if and only if there exists a morphism $\psi : Y \to X$ such that $\varphi \circ \psi = 1_Y$ and $\psi \circ \varphi = 1_X$.

Note that an isomorphism is bijective and bicontinuous (hence a homeomorphism). The converse does not hold. There exists bijective bicontinuous function that are morphisms in one direction but not in the other.

Example 2.5. $X = \mathbb{A}^1_t$, $Y = Z(Y^2 - X^3) \subseteq \mathbb{A}^2_{X,Y}$ $\varphi : X \to Y$, $\varphi(t) = (t^2, t^3)$ One can check that φ is a morphism, one-to-one, onto, φ^{-1} exists. Check that it is continuous. φ^{-1} is not a morphism. There exists a regular function $f : X \to k$ such that $f \circ \varphi^{-1} :? \to ?$ is not regular. f is locally a rational function but $f \circ \varphi^{-1}$ is not. Problem at the cusp.

Alternate definition of a morphism:

Proposition 2.6. *Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be quasi-projective varieties and $f : X \to Y$ a function. Then f is a morphism if and only if the following condition is satisfied: given any $P \in X$, there exists a neighborhood U of P, $P \in U \subseteq X$, such that $f(U)$ is contained in one of the affine opens $U_i \subseteq \mathbb{P}^m$.*

Denote this set by U_0. Now think of $f : U \to \mathbb{A}^m$. We require further that there exist m regular functions $g_i : U \to k$ such that for all $Q \in U$

$$f(Q) = (g_1(Q), \dots, g_m(Q))$$

Further since regular functions are locally rational by shrinking U, you may assume each g_i is a rational function.

A morphism is a function locally given by tuples of rational functions. We have shown polynomials are continuous in the Zariski topology. Easily follows rational function are continuous where defined. A composition of rational functions is rational.

Example 2.6. Consider the Veronese $\nu : \mathbb{P}^2 \to \mathbb{P}^2$ given by $\nu([x_0, x_1]) = [x_0^2, x_0x_1, x_1^2]$. It is a morphism. $y \mapsto (y, y^2)$. On $U_0 \subseteq \mathbb{P}^1$, $\nu(U_0) \subseteq U_0'$

$$U(1, x_1/x_0) = [1, x_1/x_0, (x_1/x_0)^2]$$

$U_1 \subseteq \mathbb{P}^1$, $\nu(U_1) \subseteq U_2'$ $\nu(x_0/x_1, 1) = [(x_0/x)^2, x_0/x, 1]$ $\nu(y) = [y^2, y]$.

Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be quasi-projective varieties and $f : X \to Y$ be a function. Then f is a morphism if and only if the following condition is satisfied: given any $P \in X$, there is an open neighborhood U of P. $P \in U \subseteq X$ and m polynomials. $F_0, \ldots, F_m \in k[x_0, \ldots, x_n]$ all homogeneous of the same degree such that for all $Q \in U$, $f(Q) = [F_0(Q), F_1(Q), \ldots, F_m(Q)]$.

Example 2.7. We can find the inverse of the Veronese embedding. $\nu : \mathbb{P}^1 \to \mathbb{P}^2$ $\nu([x_0, x_1]) = px_0^2, x_0x_1, x_1^2]$ labeled (Y_0, Y_1Y_2) On $U_0 \subseteq \mathbb{P}^2$, $\nu^{-1}[Y_0, Y_1, Y_2] = [Y_0, Y_1]$ On $U_2 \cap \nu(\mathbb{P}^1) \subseteq \mathbb{P}^2$ $\nu^{-1}[Y_0, Y_1, Y_2] = [Y_1, Y_2]$ U_0 and U_2 cover $\nu(\mathbb{P}^1)$

U_0: $[x_0, x_1] \mapsto [x_0, x_0x_1, x_1^2] \mapsto [x_0^2, x_0x_1] = [x_0, x_1]$ since $x_0 \neq 0$.

U_1: $[x_0, x_1] \mapsto [x_0, x_0, x_1, x_1^2] \mapsto [x_0x_1, x_1^2] = [x_0, x_1]$ since $x_1 \neq 0$.

No the other way:

U_0: $[Y_0, Y_1, Y_2] \mapsto [Y_0, Y_1] \mapsto [Y_0^2, Y_0Y_1, Y_1^2]$ only working on $\nu(\mathbb{P}^1)$, $Y_0Y_2 = Y_1^2$ $[Y_0^2, Y_0Y_1, Y_0Y_2] = [Y_0, Y_1, Y_2]$

U_1: $[Y_0, Y_1, Y_2] \to [Y_1, Y_2] \to [Y_1^2, Y_1Y_2, Y_2^2] = [Y_0Y_2, Y_1Y_2, Y_2^2] = [Y_0, Y_1, Y_2]$ since $Y_2 \neq 0$.

$\nu(\mathbb{P}^1) \cap U_0 \cap U_2$ on $\mathbb{P}^2$ $[Y_0, Y_1, Y_2] \to [Y_0, Y_1] = [Y_0Y_2, Y_1, Y_2] = [Y_1^2, Y_1Y_2] = [Y_1, Y_2]$ using $Y_0Y_2 = Y_1^2$.

Definition (Local Ring of P on Y)**.** Let Y be a variety. We denote by $\mathcal{O}(Y)$ the ring of all regular functions on Y. If P is a point of Y, we define the local ring of P on Y, denoted $\mathcal{O}_{P,Y}$ or simply $\mathcal{O}_P$, to be the ring of germs of regular functions on Y near P, i.e. equivalence classes of pairs $\langle U, f\rangle$, where U is an open neighborhood of P in Y and f is a regular function on U with the equivalence relation $\langle U, f\rangle \sim \langle V, g\rangle$ if and only if $f = g$ on $U \cap V$.

The ring operations on $\mathcal{O}_{P,Y}$ are given below, though we leave it to the reader that these operations are well defined (make use of the fact that Y is a variety, and hence Y is irreducible).

$$\langle U, f\rangle + \langle V, g\rangle = \langle U \cap V, f + g\rangle$$
$$\langle U, f\rangle\langle V, g\rangle = \langle U \cap V, fg\rangle$$

There is more to check. For instance, we have not shown that $\mathcal{O}_{P,Y}$ is a local ring in the 'regular' sense of the word, i.e. a ring possessing a unique maximal ideal, or that the relation is indeed an equivalence relation. Again, we leave it to the reader to verify these facts as they are routinely checked.

Notice that for $\langle U, f\rangle \in \mathcal{O}_{P,Y}$, $f(P)$ is well defined because P is contained in every U. The evaluation map $\mathrm{ev} : \mathcal{O}_{P,Y} \to k$ given by $\langle U, f\rangle \mapsto f(P)$ is easily seen to be a ring homomorphism. For surjectivity, observe that $\langle Y, a\rangle\mathcal{O}_{P,Y}$ is the constant function a for $a \in k$. The kernel is a maximal ideal, $\mathfrak{m}_{P,Y} = \{\langle U, f\rangle \colon f(P) = 0\}$. Moreover, $\mathcal{O}_{P,Y}/\mathfrak{m}P, Y \cong k$.

Remark. For any commutative noetherian ring R, we say that R is a local ring if and only if it has a unique maximal ideal, $\mathfrak{m}$. We say that $R/\mathfrak{m}$ is the residue field of R.

Proposition 2.7. *$\mathcal{O}_{P,Y}$ is a local ring, i.e. $\mathfrak{m}_{P,Y}$ the unique maximal idea.*

Proof. Recall that a ring is local if and only if it has a unique maximal ideal if and only if the only proper maximal ideal consists of all non-units of the ring. Suppose that $\langle U, f\rangle \notin \mathfrak{m}_{P,Y}$. Then $f(P) \neq 0$. By possibly restricting to a smaller open subset, $P \in U_1 \subseteq U$, we may write $f = g/h$, where $g, h \in k[x_1, \ldots, x_n]$, $h \neq 0$ on U_1, and $g(P) \neq 0$. Let $U_2 = U_1 \setminus Z(g)$. We know that $P \in U_2$ and h/g is regular on U_2. But then $\langle U_2, h/g\rangle \in \mathcal{O}_{P,Y}$. But

$$\langle U_2, h/g\rangle \cdot \langle U, f\rangle = \langle U_2, h/g\rangle\langle U_2, g/h\rangle = \langle U_2, 1\rangle,$$

so that $\langle U, f\rangle$ is invertible in $\mathcal{O}_{P,Y}$. Therefore, $\langle U, f\rangle$ is not contained in any proper ideal. This shows that $\mathfrak{m}_{P,Y}$ is the unique maximal ideal. □

Note that the above proof did not show that $\mathcal{O}_{P,Y}$ was indeed noetherian, but we shall see this later. In fact, one does not need commutative or noetherian to define local, merely the unique maximal ideal. However, commutative algebraists typically, when using the term 'local ring', intend for this to mean commutative, noetherian, and with unique maximal ideal.

Definition. If Y is a variety, we define the function field of $K(Y)$ of Y as follows: an element of $K(Y)$ is an equivalence class of pairs $\langle U, f\rangle$, where U is a nonempty open subset of Y, f is a regular function on U, and we define $\langle U, f\rangle \sim \langle V, g\rangle$ if and only if $f = g$ on $U \cap V$. We define

$$\begin{aligned}\langle U, f\rangle + \langle V, g\rangle &= \langle U \cap V, f + g\rangle \\ \langle U, f\rangle\langle V, g\rangle &= \langle U \cap V, fg\rangle.\end{aligned}$$

The elements of $K(Y)$ are called rational functions on Y.

One should check that $K(Y)$ is indeed a field. Note that one needs Y to be irreducible so that any two open sets intersect. Otherwise, the addition and multiplication described above might not make sense. We have natural ring homomorphisms

$$k \hookrightarrow \mathcal{O}(Y) \hookrightarrow \mathcal{O}_{P,Y} \hookrightarrow K(Y),$$

with maps $a \mapsto$ constant function a, $f \mapsto \langle U, f\rangle$, and $\langle U, f\rangle\langle U, f\rangle$, respectively. It is routine to verify that these are injective ring homomorphisms. Furthermore, $\mathcal{O}(Y)$, $\mathcal{O}_{P,Y}$, and $K(Y)$ are all k-algebras. We often think of them as subrings.

Now let $f : X \to Y$ be a morphism of varieties. Since by definition under a morphism, regular functions pull back to regular functions, it is easy to see we get the following ring homomorphism: $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. If $P \in X$, $f(P) \in Y$. Now $f^* : \mathcal{O}_{f(P),Y} \to \mathcal{O}_{P,X}$ is given by $\langle U, g\rangle \mapsto \langle f^{-1}(U), f^*(g)\rangle$ and is a k-algebra homomorphism. When f is an isomorphism, these are both isomorphisms. Furthermore, $(f^{-1})^* = (f^*)^{-1}$, and $\mathcal{O}(Y), \mathcal{O}_{P,Y}$ are isomorphic invariants. What about the map $f^* : K(Y) \to K(X)$? This is not always defined: if $f(X)$ is contained in a proper closed subset of Y, there will be some open $U \subseteq Y$ such that $f^{-1}(U) = \varnothing$.

We now give a brief review of localization in commutative rings.

Definition. Let R be a ring and $S \subseteq R$. We say S is multiplicatively closed if and only if $1 \in S$ and if $a, b \in S$, then $ab \in S$.

Given a ring R and a multiplicatively closed set $S \subseteq R$, we define the localization of R (with respect to S), denoted $S^{-1}R$ or R_S as follows: as a set $S^{-1}R$ is the set of equivalence classes of elements r/s, where $r \in R, s \in S$, under the equivalence relation

$$\frac{r}{s} \sim \frac{a}{t} \Leftrightarrow = \exists w \in S(SUCHTHAT)w(rt - sa) = 0.$$

We make $S^{-1}R$ into a ring by defining

$$\frac{r}{s} + \frac{a}{t} = \frac{rt + as}{st}$$
$$\frac{r}{s} \cdot \frac{a}{t} = \frac{ra}{st}.$$

One of the most common examples of such an S and a ring is when R is an integral domain and S is the prime ideal $S = R \setminus \{0\}$. In this case $S^{-1}R$ is the fraction field of R. More generally for any integral domain R, if $\mathfrak{p}$ is a prime ideal, then $S := R \setminus \mathfrak{p}$ is a multiplicatively closed subset (as one easily verifies). In this special case, $S^{-1}R$ is denoted $R_\mathfrak{p}$ instead of $R_S = R_{R\setminus\mathfrak{p}}$, called the localization of R at $\mathfrak{p}$. As another typical example, if $f \in R$ is not nilpotent, then $S = \{1, f, f^2, \ldots\}$ is a multiplicatively closed subset of R. In this case, $S^{-1}R$ is denoted R_f, called the localization of R at f.

Remark. If f is prime, then (f) is prime. However, $R_f \neq R_{(f)}$.

Theorem 2.1. *Let $Y \subseteq \mathbb{A}^n$ be an affine variety with affine coordinate ring $A(Y)$. Then*

1. $\mathcal{O}(Y) \cong A(Y)$
2. *for each point $P \in Y$, let $\mathfrak{m}_P \subseteq A(Y)$ be the ideal of functions vanishing at P. Then $P \mapsto \mathfrak{m}_P$ gives a one-to-one correspondence between the points of Y and the maximal ideals of $A(Y)$*
3. *for each P, $\mathcal{O}_{P,Y} \cong A(Y)_{\mathfrak{m}_P}$ and $\dim \mathcal{O}_{P,Y} = \dim Y$*
4. *$K(Y)$ is isomorphic to the quotient field of $A(Y)$, and hence $K(Y)$ is a finitely generated field extension of k with transcendence degree $\dim Y$.*

Proof.

(a) We have an injective homomorphism $\alpha : A(Y) \to \mathcal{O}(Y)$. Start with the homomorphism $f : k[x_1, \ldots, x_n] \to \mathcal{O}(Y)$. A regular function need to be locally rational (but polynomials are globally rational). The kernel of the map is $I(Y)$. Now

$$\alpha' : A(Y) = k[x_1, \ldots, x_n]/I(Y) \longrightarrow \mathcal{O}(Y)$$

is injective. We prove surjectivity later.

(b) The points of $\mathbb{A}^n$ are in bijective correspondence with the maximal ideals of $k[x_1, \ldots, x_n]$. [We know that the maximal ideals are points $P = (a_1, \ldots, a_n)$ and $I(P) = (x_1 - a_1, \ldots, x_n - a_n)$, the polynomials vanishing at P.] Now $P \in Y$ if and only if $I(P) \supseteq I(Y)$. But $I(P)$ is maximal. The points of Y are then in bijective correspondence with the maximal ideals of $k[x_1, \ldots, x_n]$ containing $I(Y)$ which are in correspondence with the maximal ideals of $A(Y)$, which are $k[x_1, \ldots, x_n]/I(Y)$.

(c) For each point P, there is a natural map $A(Y)_{\mathfrak{m}_P} \to \mathcal{O}_{P,Y}$

$a/b \in A(Y)_{\mathfrak{m}_P}$, where $a, b \in A(Y)$ and $b(P) \neq 0$. Replace a, b by polynomials that represent them. Now $Y \setminus Z(b)$ is an open neighborhood of P in Y.

$a/b \mapsto \langle Y \setminus Z(b), a/b \rangle \in \mathcal{O}_{P,Y}$ is injective as α is injective.

For surjective, $\langle U, f \rangle \in \mathcal{O}_{P,Y}$ represented by $\langle V, g/h \rangle$, g, h polynomials, $h \neq 0$ on V, $g/h \in A(Y)$.

This gives $\mathcal{O}_{P,Y} \cong A(Y)_{\mathfrak{m}_P}$.

$S : A \to C$ Dense: For any $c \in C$, there is an $a \in A$ such that Sa is isomorphic to c. On objects, surjective up to isomorphism.

$X \to A(X)$

On objects, surjective up to isomorphism.

$X \to A(X)$, X affine variety.

affine varieties $\mapsto$ finitely generated integral domains over k.

Bijection: $\alpha : \mathrm{Hom}(X, Y) \overset{\sim}{\mathrm{Hom}} (A(Y), \mathcal{O}(X) \cong A(X))$, where used $\mathcal{O}(X) \cong A(X)$ is X is affine.

full and faithful come from being a bijection.

dense: Remark 1.4.6. Showed that any f.g. integral domain over k was isomorphic to $A(X)$ for some X.

Automorphisms of $\mathbb{P}^n$.

Let M be an invertible $(n+1) \times (n+1)$ matrix with entries in k. M induces a map $m : \mathbb{P}^n \to \mathbb{P}^n$ via

$$[a_0, \ldots, a_n] \to M \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} \quad [\lambda a_0, \ldots, \lambda a_n] \mapsto \lambda M \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$$

Since M is invertible, the only $[a_0, \ldots, a_n]$ mapping to $[0, \ldots, 0]$ is $[0, \ldots, 0]$. It's a morphism because if you multiply out $M \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$, you can see each entry is a homogeneous linear polynomial in the a_i.

The inverse morphism is multiplication by M^{-1}.

That is an automorphism.

If $D = \begin{bmatrix} d & & \\ & d & \\ & & \ddots \end{bmatrix}$, then $d \neq 0$. M and DM induce the same map:
$DM \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} = dM \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$.

$\mathrm{GL}_k(n+1)$ group of all $(n+1) \times (n+1)$ invertible matrices operation matrix multiplication general linear group.

The set of scalar matrices is a normal subgroup.

Define $PGL_k(n+1) = \mathrm{GL}_k(n+1)/scalarmatrices$. Projective general linear group.

It can be shown that $\mathrm{Aut}(\mathbb{P}^n_k) \cong PGL_k(n+1)$.

Linear Varieties (Subspaces) in $\mathbb{P}^n$

$\mathbb{P}^n = \mathbb{A}^{n+1} \setminus \{0\}/\sim$, where $\sim (a_0, \ldots, a_n) \sim \lambda(a_0, \ldots, a_n)$, where $\lambda \in k^\times$ which is lines through origin.

Go back to thinking of $\mathbb{A}^{n+1}$ as k^{n+1} a vector space.

If $V \subseteq k^{n+1}$ is any subspace of dimension $m+1$. Let $X = \{P \in \mathbb{P}^n : thelinetowhichPcorrespondsliesinV\}$

$X \cong \mathbb{P}^m$, $V \cong k^{m+1}$

X is called a linear subvariety of $\mathbb{P}^n$, sometimes it is called a linear subspace. This is confusing because $\mathbb{P}^n$ is not a vector space.

$V \subseteq k^{n+1}$ is the zero locus of $n+1-(m+1)$ homogeneous linear equations.

X will also be the zero set of these equations.

In fact, they generate its ideal. By change of variables, you assume those equations $x_0 = \cdots = x_{(n+1)-(m+1)-1} = 0$. Then not hard to see X is called an m-plane.

$X \cap U_i \subseteq \mathbb{A}^n$.

It really is a plane.

Some important exercises in Hartshorne 3.1e., 3.14, 3.15, 21

4. Rational Maps

Definition. Let X and Y be varieties. A rational map

Let's convince ourselves that the statement is true.

We have $\langle U, \varphi_U \rangle$ such that $\varphi_U(U)$ is dense in Y. This means that for any nonempty open $V \subset U$, $\varphi_U(U) \cap V \neq \emptyset$. Now suppose we have another pair $\langle W, \varphi_W \rangle$. We need to show that for any open $V \subseteq Y$, $\varphi_U(U) \cap V \neq \emptyset$ and since morphisms are continuous, φ_U^{-1} is nonempty open in U and hence open in X. This means $\varphi_U^{-1}(V) \cap W \neq \emptyset$ because X is irreducible. [Recall all nonempty open subsets meet.] Choose $P \in \varphi_U^{-1}(V) \cap W$, note $P \in U \cap W$. By definition, φ_U and φ_W agree on $U \cap W$.

$$\varphi_W(P) = \varphi_U(P) \in V.$$

Show $\varphi_W(W) \cap V \neq \emptyset$.

A rational map might not be defined at some points of X. There may be some points of X not in any U such that $\langle U, \varphi_U \rangle$ is in the rational map.

Example: Projection from a point: $\varphi : \mathbb{P}^n \setminus \{P\} \to \mathbb{P}^{n-1}$. $\langle \mathbb{P}^n \setminus \{P\}, \varphi \rangle$. Sometimes rational maps are denoted by dotted arrows. In a talk, Fulton once said thats because some points (parts of the arrow) do not make it and fall out. This means we cannot always compose rational maps:

$$X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$$

where φ, ψ are rational maps. It could be that $\varphi(X) = U\langle U, \varphi \rangle \varphi_U(U)$ is contained in the set of points where ψ is not defined. This will not happen if φ is dominant.

In fact, you can check that if φ and ψ are dominant then so is $\psi \circ \varphi$. So you can make a category the objects are varieties and the morphisms are dominant rational maps.

Definition. A birational map $\varphi : X \to Y$ is a rational map that admits an inverse, namely a rational map $\psi : Y \to X$ such that $\psi \circ \varphi = 1_X$ and $\varphi \circ \psi = 1_X$ as rational maps. (Equal to identity where they are defined). If there is a birational map from X to Y, we say that there are birationally equivalent or birational or birationally isomorphic.

What is a birational morphism? That is a morphism that has a rational inverse. It is a birational isomorphism.

Examples: $\mathbb{A}^n$ and $\mathbb{P}^n$ are birationally isomorphic. (They are not isomorphic.) $\varphi : \mathbb{A}^n \to \mathbb{P}^n$ given by $(a_1, \ldots, a_n) \mapsto (1, a_1, \ldots, a_n)$ and $\psi : \mathbb{P}^n \to \mathbb{A}^n$ defined on U_0 given by $(1, a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n)$.

2) $X = Z(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2$ X is birationally isomorphic to $\mathbb{A}^1$. $\varphi : \mathbb{A}^1 \to X$ $\varphi(t) = (t^2 - 1, t(t^2 - 1))$ $(t(t^2-1))^2 - (t^2-1)^2(t^2 - 1 + 1)$ $t^2(t^2-1)^2 - (t^2-1)^2t^2 = 0$ $\varphi(\mathbb{A}^1) \subseteq X$ $\psi : X \to \mathbb{A}^1$ $\psi(x, y) = y/x$, defined where $x \neq 0$ $\psi \circ \varphi(t) = \psi(t^2 - 1, t(t^2 - 1)) = \frac{t(t^2-1)}{t^2-1} = 1$ away from $t = \pm 1$ $\varphi \circ \psi = \varphi(y/x) = (y^2/x^2 - 1, y/x(y^2/x^2 - 1))$ but on X $y^2/x^2 = x + 1$ as long as $x \neq 0 = (x + 1 - 1, y/x(x + 1 - 1)) = (x, y)$ away from $x = 0$.

Geometric interpretation of the map

Projecting from the origin to the line $x = 1$.

line $y = tx$ through the origin. It meets $x = 1$ at point $(1, t)$. It meets $y^2 - x^2(x+1)$ at $t^2x^2 - x^2(x+1) = 0$

$$x^2(t^2 - (x+1)) = 0$$

so $x = 0$ twice.

$x = t^2 - 1$ $y = t(t^2 - 1)$

Remark: Clearly a morphism is a rational map. Thus, X is isomorphic to Y implies X is birationally isomorphic to Y. Those examples show reverse does not hold.

$\mathcal{O}(\mathbb{A}^n) = k[x_1, \ldots, x_n]$ $\mathcal{O}(\mathbb{P}^n) = k$

$\mathbb{A}^n$ not isomorphic to $\mathbb{P}^n$ $A(\mathbb{A}^1) = k[x]$ $A(X) = k[x, y]/(y^2 - x^2(x+1))$

$k[x]$ is a PID

In $A(X)$, look at ideal generated by $\overline{x}, \overline{y}$. Anything in $(y^2 - x^2(x+1))$ has all its monomials of degree at least $(\overline{x}, \overline{y}) = (h(\overline{x}, \overline{y}))$ $h$would have to have no constant term because $h \in (\overline{x}, \overline{y})$.

$\overline{x}$ and $\overline{y}$ would have to be constant multiples of degree 1 term. Birational is a weaker equivalence relation than isomorphic. Ideally, you would like to classify varieties up to isomorphism. This is **extremely** difficult. You break it into steps. First, classify up to birational isomorphism, then for each birational equivalence class, try to classify up to isomorphism.

Confusing terminology.

When we say X is an affine variety, we mean X is a closed irreducible set of some affine space, say $\mathbb{A}^n$.

When we say a variety X is affine, we mean that X is isomorphic to an affine variety

Example: $\mathbb{A}^1 \setminus \{0\}$ not an affine variety but it is affine.

$\mathbb{A}^1_t \setminus \{0\} \cong X = Z(xy-1) \subseteq \mathbb{A}^2_{x,y}$

$\varphi : \mathbb{A}^1_t \setminus \{0\} \to X$ given by $t \mapsto (t, 1/t)$ $\psi : X \to \mathbb{A}^1 \setminus \{0\}$ given by $(x,y) \mapsto x$ $\psi(\varphi(t)) = \psi(t, 1/t) = t$ $\varphi(\psi(x,y)) = \varphi(x) = (x, 1/x)$ But on $xy - 1 = 0$, $y = 1/x$ so $(x, 1/x) = (x, y)$.

Lemma 4.2: Let Y be a hypersurface in $\mathbb{A}^n$ given by the equation $f(x_1, \ldots, x_n) = 0$. Then $\mathbb{A}^n \setminus Y$ is isomorphic to the hypersurface H in $\mathbb{A}^{n+1}$ given by $x_{n+1}f - 1 = 0$. In particular, $\mathbb{A}^n \setminus Y$ is affine and its coordinate ring is $k[x_1, \ldots, x_n]_f$. In previous example, $f = x$ so $x_{n+1}f$ becomes $xy = 1$.

Pf: $\varphi : \mathbb{A}^n \setminus Y \to H$ given by $(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 1/f)$

$1/f \cdot f - 1 = 0$ $\psi : H \to \mathbb{A}^n \setminus Y$ $\psi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$ Check $\varphi \circ \psi = 1$, $\psi \circ \varphi = 1$

$k[x_1, \ldots, x_{n+1}]/(x_{n+1}f - 1) = A(H) \cong k[x_1, \ldots, x_n]_f$

as $x_{n+1}f - 1 = 0$ $x_{n+1}f = 1$ $f = 1/x_{n+1}$.

$k[x_1, \ldots, x_{n+1}] \to k[x_1, \ldots, x_n]_f$ given by $x_i \mapsto x_i$, $1 \le i \le n$ $x_{n+1} \mapsto 1/f$.

Obviously, a surjective ring homomorphism. Show kernel is $(x_{n+1}f - 1)$. Kernel contains the ideal is easy.

$X = Z(\{x_i y_i = x_j y_i \mid i, j = 1, \ldots, n\})$

5) X is covered by open sets each isomorphic to $\mathbb{A}^n$, $\mathbb{A}^n \times \mathbb{P}^{n-1}$ is covered by $\mathbb{A}^n \times y_i \cong \mathbb{A}^n \times \mathbb{A}^{n-1} \cong \mathbb{A}^{2n-1}$. Set $V_i = X \cap \mathbb{A}^n \times U_i$. On V_i, $y_i = 1$, $x_i y_j = x_j y_i$ becomes $x_i y_j = x_j$, $j = 1, \ldots, n$. The points of X are of the form $(x_i y_1, x_i y_2, \ldots, x_i, \ldots, x_i, y_n) \times (y_1, y_2, \ldots, 1, \ldots, y_n)$. Notice that this satisfies all the other equations $x_s y_t = x_t y_s$, $x_s = x_i y_s$, $x_t = x_i y_t$, $x_i y_s y_2 = x_i y_t y_s$

$\mathbb{A}^n \to V_i$ given by $(y_1, \ldots, x_i, \ldots, y_n) \mapsto (x_i y_1, x_i y_2, \ldots, x_i, \ldots, x_i, y_n) \times (y_1, y_2, \ldots, 1, \ldots, y_n)$

Note that on V_i, $\varphi^{-1}(0)$ is given by $x_i = 0$

This again shows that $\varphi^{-1}(0)$ fits nicely into X.

Def: If is a closed subvariety of $\mathbb{A}^n$ passing through 0, we define the blowing-up of Y at the point 0 to by $\tilde{Y} = \overline{\varphi^{-1}(Y \setminus \{0\})}$, where $\varphi : X \to \mathbb{A}^n$ is the blowing up of $\mathbb{A}^n$ at 0.

We denote by $\varphi : \tilde{y} \to Y$, the restrictionof φ to $\tilde{Y}$. To blow-up any other point P on $\mathbb{A}^n$, make a change of variables sending P to 0. Blow-up points of $\mathbb{P}^n$ by going to an affine patch. $\tilde{Y}$ is also sometimes called the proper transform of Y.

Ex (4.9.1):

Let Y be the plane cubic curve given by the equation $y^2 = x^2(x+1)$ Blow-up of $\mathbb{A}^2$ at (0,0) is covered by two $\mathbb{A}^2$'s.

One one 1 $x = x, y = xy, \varphi^{-1}(0), x = 0$ other 2 $x = xy, y = y, \varphi^{-1}(0), y = 0$

$x^2y^2 = x^2(x+1)$ divide out x^2, $y^2 = x+1$

2 $y^2 = x^2y^2(xy+1)$ 1 $= x^2(xy+1)$ $1/x^2 = xy+1$ $1/x^2 - 1) = xy$ $y = (1/x^2 - 1)/x = (1-x^2)/x^3$

A Theorem prove by the classical Italian algebraic geometers late 1800s early 1900s. Assume $k = \mathbb{C}$ and let X, Y be nonsingular projective surfaces (surfaces means dimension 2, nonsingular means that they are complex manifolds). Let $\varphi : X \to Y$ be a birational isomorphism. Then there exists a commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, shift left, "\Psi"] \arrow[d, "f"] & \tilde{Y} \arrow[l, shift left, "\Psi^{-1}"] \arrow[d, "g"] \\ X \arrow[r, shift left, "\varphi"] & Y \arrow[l, shift left, "\varphi^{-1}"] \end{tikzcd}$$

where ψ is an isomorphism of nonsingular complex projective surfaces and f, g are both finite sequences of blow-ups. Every birational isomorphism of nonsingular projective surfaces can be factored into a finite sequence of blow-ups followed by a finite sequence of blow-downs.

1990, Mori got a Fields medal for sort of generalizing this to three-folds.

Example: Consider $Q = Z(xy - zw) \subseteq \mathbb{P}^3$. Q is the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$. $(s,t) \times (u,v) \mapsto (su, sv, tu, tv)$, which we label (w, x, y, z). How do we know that Q is birationally isomorphic to $\mathbb{P}^2$.

Why? Know $U_i \subseteq \mathbb{P}^2$ and $U_i \not\cong \mathbb{A}^2$. But know $\mathbb{P}^1 \times \mathbb{P}^1 \supseteq U_i \times U_j$ and $U_i \times U_j \cong \mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$. Isomorphic open sets so birationally iso.

Two families of lines on Q $P \times \mathbb{P}^1$, vary P $\mathbb{P}^1 \times R$, vary R

Fix s, t vary u, v $tw - sy = 0$ $tx - sz = 0$ last two interesction of two planes = line

$tw - sy = 0$ $tsu - stu = 0$

$tx - sz = 0$ $tsv - stv = 0$

fix u, v, vary s, t

$vw - ux = 0$ $vy - uz = 0$

Consider $P = [1,0,0,0] \in Q$ Project from P onto $w = 0$ $\varphi([w,x,y,z]) = [x,y,z]$

Projecting we get a morphism $\varphi : Q \setminus P \to \mathbb{P}^2 = \{w = 0\}$.

That is a birational isomorphism. $\varphi^{-1}[x,y,z] = [xy, xz, yz, z^2]$

φ is defined except when $x = y = z = 0$. φ^{-1} is defined except when

$z = 0$ and eitehr $x = 0$ or $y = 0$

$[0,*,0]$, $[*,0,0]$ two points, $[0,1,0]$, $[1,0,0]$

$[x,y,z] \to [xy, xz, yz, z^2] \to [xz, yz, z^2] = [x,y,z]$ if $z \neq 0$

$[w,x,y,z] \to [x,y,z] \to [xy, xz, yz, z^2]$ but on Q, $xy = wz$, $= [wz.xz.yz.z^2] = [w,x,y,z]$ if $z \neq 0$

Since φ is projection, the two lines on Q through P

These two lines are $[1,0] \times [u,v] \to [u,v,0,0]$, $[s,t] \times [1,0] \to [s,0,t,0]$.

Where do they map to in $\mathbb{P}^2$? $[v,0,0] \to [1,0,0]$ $[0,t,0] = [0,1,0]$ Those are the two points at which φ^{-1} is undefined.

Look at where the two familes of lines on Q land in $\mathbb{P}^2$

$$\begin{tikzcd} (s,t) \times (u,v) \arrow[r] \arrow[dr] & (su, sv, tu, tv) \arrow[d, "\varphi"] \\ & (\underbrace{sv}_{x}, \underbrace{tu}_{y}, \underbrace{tv}_{z}) \end{tikzcd}$$

Fix s,t and let u,v vary. $tx - sz = 0$ all these lines go through $[0,1,0]$ Fix u,v, $[1,0,0]$

Need to blow-up $[1,0,0]$ and $[0,1,0]$ so the lines separate out.

The line through $[1,0,0]$ and $[0,1,0]$ is in both families but the families on Q are disjoint.

Blow down that line. Each family of lines is now missing a line. It gets replaced by the other $\varphi^{-1}(0)$.

Normalization of Varieties

Let X be an affine variety with affine coordinate ring $A(X) = k[x_1, \ldots, x_n]/I(X)$. Let B be the integral closure of $A(X)$ in its fraction field Frac $A(X)$. It can be proven that B is still a finitely generated k-algebra. So it must correspond to some affine $\tilde{X}$. The containment $A(X) \subseteq B$ must correspond to a morphism $\tilde{X} \to X$. $\tilde{X}$ is called the normalization of X. For general X, cover X with affine sets, doing normalization to each of them. Then glue them back together to get $\tilde{X}$.

Properties of $f : \tilde{X} \to X$

1. f is projective (proper) 2. f induces an isomorphism between $X \setminus \operatorname{Sing} X$ and $f^{-1}(X \setminus \operatorname{Sing} X)$ 3. The singularities of $\tilde{X}$ have codimension at least 2. Thus for curves, you have a desingularization. For higher dimensions, often, $\tilde{X}$ is still singular.

Intersections in Projective Space

Recall the following result from linear algebra. Let V be a vector space of dimension n and let U, W be subspaces of dimensions r, s. Then $\dim(U \cap W) \geq r + s - n$. There are similar results for intersections of varieties in affine and projective space.

Proposition 2.8 (Affine Dimension Theorem)**.** *Let X, Y, Z be varieties of dimensions r, s that are closed subsets in $\mathbb{A}^n$. Then every irreducible component W of $Y \cap Z$ has dimension at least $r + s - n$. This includes the possibility that $Y \cap Z$ is empty.*

Proof. We proceed in several steps. First, suppose that Z is a hypersurface defined by an equation $f = 0$. This is Exercise I.18, $r = n - 1$, $n - 1 + s - n = s - 1$

Now for the general case. We consider the product $Y \times Z \subseteq \mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n}$, which is a variety of dimension $r + s$. Exercise I.3.15. Let Δ be the diagonal $\{P \times P \colon P \in \mathbb{A}^n\} \subseteq \mathbb{A}^{2n}$. Then $\mathbb{A}^n$ is isomorphic to Δ be the map $P \to P \times P$.

Δ is closed in $\mathbb{A}^{2n}$. Take coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. x_i coordinate on first $\mathbb{A}^n$ and y_i coordinate on second $\mathbb{A}^n$. $\Delta = Z(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$ $\varphi : \mathbb{A}^n \to \mathbb{A}^{2n}$ $\varphi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, x_1, \ldots, x_n)$ morphism because it is given by polynomials. $\varphi^{-1}(x_1, \ldots, x_n, x_1, \ldots, x_n) := (x_1, \ldots, x_n)$ also clearly a morphism.

Under this isomorphism, $Y \cap Z$ corresponds to $(Y \times Z) \cap \Delta$

$(a_1, \ldots, a_n) \in Y \cap Z$ if and only if $(a_1, \ldots, a_n, a_1, \ldots, a_n) \in (Y \times Z) \cap \Delta$

Since Δ has dimension $n \cong \mathbb{A}^n$.

And since $\underbrace{r + s - n}_{\mathbb{A}^n} = \underbrace{(r + s)}_{Y \times Z} + \underbrace{n}_{\Delta} - 2n$

We reduce to proving the result for the two varieties $Y \times Z$ and Δ in $\mathbb{A}^{2n}$

Recall $\Delta = Z(x_1 - y_1, \ldots, x_n - y_n)$. Apply the first case n times. Dimension goes down by at most 1 for each. $\square$

Theorem 2.2 (Projective Dimension Theorem)**.** *Let Y, Z be varieties of dimensions r, s that are closed subsets in $\mathbb{P}^n$. Then every irreducible component of $Y \cap Z$ has dimension at least $r + s - n$. Furthermore, if $r + s - n \geq 0$, then $Y \cap Z$ is nonempty.*

Proof. The first statement follows from the previous result since $\mathbb{P}^n$ is covered by affine n-spaces. For the second result, let $C(Y)$ and $C(Z)$ be the cones over Y, Z in $\mathbb{A}^{n+1}$, Exercise I.2.10. (The cone over a projective variety. Let $Y \subseteq \mathbb{P}^n$ be a nonempty algebraic set. and let $\theta : \mathbb{A}^{n+1} \setminus \{(0, \ldots, 0)\} \to \mathbb{P}^n$ be the map which sends the point with affine coordinates $(a_0, \ldots, a_n)$ to the point with homogeneous coordinates $[a_0, \ldots, a_n]$. We define the cone over Y to be ... (copy rest of exercise)).

Then $C(Y), C(Z)$ have dimensions $r + 1, s + 1$, respectively. Furthermore, $C(Y) \cap C(Z)$ is nonempty because it contains the origin $P = (0, \ldots, 0)$. By the Affine Dimension Theorem $C(Y) \cap C(Z)$ has dimension at least $(r + 1) + (s + 1) - (n + 1) = r + s - n + 1 > 0$. $C(Y) \cap C(Z)$ must have points other than $(0, \ldots, 0)$. These correspond to points of $Y \cap Z$. $\square$

But we want to do better than just understanding how the dimensions work.

Recall the following: Let $f \in k[x]$ be a nonconstant polynomial of degree d. Then f has d roots (because k is assumed alg. closed) and the roots counted with multiplicity.

Suppose you want a similar statement about polynomials in two variables. To get a finite number of points, you want to intersect two polynomial curves.

X two lines meet at one point line conic (give ellipise and line crossing) meet at two points.

Two conics (2 ellipses) 4 points Line & cubic 3 points.

f,g # points of intersection $=\deg f \deg g$ Need assumptions.

f, g no common factors k algebraically closed $y = x^2 + 1$ and $y = 0$, parabola and line do not meet but algebraic closure do meet.

Multiplicity of course in the counting. $xy = 1$ asymptote $y = x^2, x = 0$ than (0,0,..,0)

These coorespond to points of $Y \cap Z$.

Definition. A numerical polynomial is a polynomial $P(z) \in \mathbb{Q}[z]$ such that $P(n) \in \mathbb{Z}$ for all $n \gg 0$, $n \in \mathbb{Z}$.

Proposition 2.9. *(a) If $P \in \mathbb{Q}[z]$ is a numerical polynomial, then there are integers $c_0, \ldots, c_r$ such that*

$$P(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \cdots + c_r,$$

where

$$\binom{z}{r} = \frac{1}{r!}z(z-1)\cdots(z-r+1)$$

is the binomial coefficient function. In particular, $P(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

(b) If $f : \mathbb{Z} \to \mathbb{Z}$ is any function and if there exists a numerical polynomial $Q(z)$ such that the difference function $\Delta f = f(n+1) - f(n)$ (like the derivative) is equal to $Q(n)$ for all $n \gg 0$, then there exists a $P(z)$ such that $f(n) = P(n)$ for all $n \gg 0$.

Proof.

(a) By induction on the degree of P, the case of degree 0 being obvious (degree 0 means that $P(z) = c_r$, $c_r \in \mathbb{Q}$. But $P(n) \in \mathbb{Z}$ for $n \gg 0$, $P(n) = c_r$ for all n so that $c_r \in \mathbb{Z}$.

Since $\binom{z}{r} = \dfrac{z^r}{r!} +$ l.ot., we can express any polynomial $P \in \mathbb{Q}[z]$ of degree r in the above form with unique $c_0, \ldots, c_r \in \mathbb{Q}$. (Start from z^rm work down.) For any polynomial P, we define the difference function by

$$\Delta\binom{z}{r} = \binom{z}{r-1}.$$

This is Pascal's triangle identity $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$. Now $\Delta P(z) = c_0\binom{z}{r-1} + c_1\binom{z}{r-2} + \cdots + c_{r-1}$. $\Delta P(n) \in \mathbb{Z}$ for all $n \gg 0$ since $P(n) \in \mathbb{Z}$ for all $n \gg 0$.

$\Delta P(z)$ is a numerical polynomial. By induction, $c_0, \ldots, c_{r-1} \in \mathbb{Z}$. Claim $\binom{z}{r} \in \mathbb{Z}$ for all $z \in \mathbb{Z}$. Now $z \geq r$ because it is a binomial coefficient. For $0 \leq z < r$, it is 0. For $z < 0$, $-z + r - 1 \geq r$ so that $\binom{-z+r-1}{r} \in \mathbb{Z}$ by the previous case.

$$\binom{-z+r-1}{r} = \frac{1}{r!}(-z+r-1)(-z+r-2)\cdots = \frac{1}{r!}(-z+r-1-r)(-z+r-1-(r+1)) = (-1)^r\binom{z}{r}$$

Then $P(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \cdots + c_{r-1}\binom{z}{1} + c_r$. For $z \in \mathbb{Z}$, we know every term up to c_r is in $\mathbb{Z}$. By assumption, $P(n) \in \mathbb{Z}$ for $n \gg 0$. This gives $c_r \in \mathbb{Z}$. The $P(n)$ is always in $\mathbb{Z}$ for $n \in \mathbb{Z}$.

(b) Using (a), write $Q = c_0\binom{z}{r} + \cdots + c_r$, where $c_0, \ldots, c_r \in \mathbb{Z}$. Let $P(z) = c_0\binom{z}{r+1} + \cdots + c_r\binom{z}{1}$, $\Delta P = Q$, $\Delta(f - P) = 0$ for all $n \gg 0$. But $\Delta(f - P) = \Delta(f) - \Delta(P) = Q$. $(f - P)(n) = c_{r+1}$ for all $n \gg 0$. $f = P + c_{r+1}$ for all $n \gg 0$. □

Definition. Let S be a ring. A grading of S is an expression of the additive group $(S, +)$ as an internal direct sum $S = \bigoplus_{i=0}^{\infty}$ with the property that if $a \in S_e$ and $b \in S_d$, then $ab \in S_{e+d}$, written $S_e \cdot S_d \subseteq S_{e+d}$. A graded ring is a ring together with a given grading on it. An ideal $I \subseteq S$ is called graded or homogeneous if $I = \bigoplus_{i=0}^{\infty}(I \cap S_i)$ as a group.

Now let S be a graded ring and M an S-module. We say that M is a graded S-module if and only if M as a group can be expressed as an internal direct sum $M = \bigoplus_{i=-\infty}^{\infty} M_i$ such that if $f \in S_e$ and $M \in M_d$, then $f_m \in M_{e+d}$, written $S_e \cdot M_d \subseteq M_{e+d}$.

Example 2.8. If S is a graded ring, it is a graded module over itself. Any homogeneous ideal of S is a graded module over S. Any quotient of S by a homogeneous ideal is a graded module over S.

Let S be a graded ring and M a graded S-module. We define the twisted module $M(l)$ by setting $M(l)_i := M_{l+i}$. Say $a \in S_e$, $b \in M(l)_d$, then $b \in M_{l+d}$ so $ab \in M_{l+l+d} = M(l)_{l+d}$ so this is still a grading.

Definition. Let S be a graded ring and M, N graded modules over S. An s-module homomorphism $\varphi : M \to N$ is said to be graded or homogeneous of degree d if and only if for all e, $\varphi(M_e) \subseteq N_{d+e}$. Two graded modules are considered isomorphic if and only if they are isomorphic via an isomorphism that is graded of degree 0.

The most interesting graded homomorphisms are those graded of degree 0. In fact, usually when one says the homomorphism is graded without saying of degree d, it is implied that the degree is 0. This is where twisting comes in. Any homomorphism that is graded of degree d can be made into one graded of degree 0. $\varphi : M \to N$ graded of degree d, $\varphi : M(-d) \to N$ or $\varphi : M \to N(d)$ are graded of degree 0.

$\varphi(M(-d)_e) = \varphi(M_{-d+e}) \subseteq N_{d-d+e} = N_e$ $\varphi(M_e) \subseteq N_{d+e} = N(d)_e$

Definition. Let A be a ring and M an A-module. The length of M over A, denoted $\ell_A(M)$, is equal to the length of the longest strictly increasing sequence of submodules of M.

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_d = M.$$

The length is the number of steps.

Example 2.9. If A is a field, this is just a vector space dimension.

Proposition 2.10. *If M is a graded S-module, we define the annihilator of M,* $\operatorname{Ann}(M) = \{s \in S \colon sM = 0\}$*. This is a homogeneous ideal in S.*

Proof. ideal: $(s+t)m = sm + tm$, $sm = 0, tm = 0$ then $(s+t)m = 0$ and $(as)m = a(sm) = 0$. Homogeneous: $s = s_0 + \cdots + s_d \in \operatorname{Ann} M$, need to show $s_i \in M$.

$s \in \operatorname{Ann} M \iff sm = 0$ for all $m \in M \iff sm_i = 0$ for all $m_i \in M$ such that m_i homogeneous . The forward direction obvious, reverse any $m \in M$ is sum of $\sum m_i$.

So we want to show $s_j m_i = 0$ for $0 \leq j \leq d$. and m_i homogeneous of degree i.

$sm_i = 0$ then $s_0 m_i + s_1 m_i + \cdots + s_d m_i = 0$ but each term in sum is homogeneous of different degree then $s_j m_i = 0$ for all j. □

Proposition 2.11. *Let M be a finitely generated graded module over a noetherian graded ring S. Then there exists a filtration*

$$0 = M^0 \subsetneq M^1 \subsetneq \cdots \subsetneq M^r = M$$

by graded submodules, such that for each i,

$$M^i / M^{i-1} \cong (S/\mathfrak{p}_i)(l_i),$$

where $\mathfrak{p}_i$ is a homogeneous prime ideal and $l_i \in \mathbb{Z}$. The filtration is not unique but for any such filtration, we have

(a) if $\mathfrak{p}$ is a homogeneous ideal of S, then $\mathfrak{p} \supseteq$ $\operatorname{Ann}(M)$ if and only if $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some i. In particular, the minimal elements of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ are the minimal primes of M, i.e. the primes which are minimal containing $\operatorname{Ann} M$.

(b) for each minimal prime of M, the number of times which $\mathfrak{p}$ occurs in the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ is equal to the length of $M_{\mathfrak{p}}$ over the local ring $S_{\mathfrak{p}}$ (and hence is independent of the filtration).

Proof. See Hartshorne

Definition. If $\mathfrak{p}$ is a minimal prime of a graded S-module M, then the multiplicity of M at $\mathfrak{p}$, denoted $\mu_{\mathfrak{p}}(M)$, to be the length of $M_{\mathfrak{p}}$ over $S_{\mathfrak{p}}$.

Now say $S = k[x_1, \ldots, x_n]$ and M a graded S-module. Notice this makes each M_i a vector space over k. Define the Hilbert function of M, denoted φ_M, as

$$\varphi_M(l) := \dim_k M_l.$$

$\varphi_M : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$.

Proposition 2.12.

(a) If $\emptyset \neq Y \subseteq \mathbb{P}^n$, then the degree f Y is a positive integer.

(b) *Let $Y = Y_1 \cup Y_2$, where Y_1, Y_2 have the same dimension and where* $\dim(Y_1 \cap Y_2) < r$. *Then* $\deg Y = \deg Y_1 + \deg Y_2$.

(c) $\deg \mathbb{P}^n = 1$

(d) *If $H \subseteq \mathbb{P}^n$ is a hypersurface whose ideal is generated by a homogeneous polynomial of degree d, then* $\deg H = d$. *In other words, this definition is consistent with the degree of hypersurface defined earlier.*

Proof.

(a) As $\emptyset \neq Y$, P_Y is a polynomial of degree $r = \dim Y \geq 0$. It 'looks like'

$$c_0\binom{z}{r} + c_1\binom{z}{r-1} + \cdots,$$

where $c_i \in \mathbb{Z}$. Then $\deg Y = c_0 \in \mathbb{Z}$ is positive for $l \gg 0$ as $P_Y(l) = \varphi_{S/I}(l) = \dim_k(S/I)_l \geq 0$ as $Y \neq \emptyset$. $I(Y) \neq (x_0, \ldots, x_n)$. In fact, $\dim(S/I)_l > 0$.

(b) Let I_1, I_2 be the ideals of Y_1, Y_2. Then $I = I_1 \cap I_2$ is the ideal of Y. We have an exact sequence

$$0 \longrightarrow S/I \longrightarrow S/I_1 \oplus S/I_2 \longrightarrow S/(I_1 + I_2) \longrightarrow 0$$

We check that the map $S/I_1 \oplus S/I_2 \longrightarrow S/(I_1 + I_2) \longrightarrow 0$ given by $(f + I_1, g + I_2) \mapsto (f + g) + (I_1 + I_2)$ is well defined. Replace f by $f + f_1$, $f_1 \in I$, g by $g + g_1$, $g_1 \in I_2$. Then $f + g$ is replaced by $(f + g) + (f_1 + g_1)$ with $f_1 + g_1 \in I_1 + I_2$, proving the map is well defined. It is simple to see that the map is a homomorphism. To see that the map is onto, note that $f + (I_1 + I_2)$ is mapped to $(f + I_1, 0 + I_2)$. What is the kernel of this map? We know that $(f + g) + (I_1 + I_2) = 0$ if and only if $f + g \in I_1 + I_2$ if and only if $f + g = f_1 + g_1$, where $f_1 \in I_1$ and $g_1 \in I_2$. But then $f - f_1 = g_1 - g$ and as f is defined up to elements of I_1 and g is only defined up to elements of I_2. Then we may assume $f = -g$. Then the homomorphism $S \longrightarrow S/I_1 \oplus S/I_2$ is defined by $f \mapsto (f + I_1, -f + I_2)$ surjects onto the kernel. What is the kernel of this map? We know $(f + I_1, -f + I_2) = 0$ if and only fi $f \in I_1$ and $-f \in I_2$ if and only if $f \in I_1 \cap I_2$. Thus, $S/(I_1 \cap I_2)$ maps isomorphically onto the kernel. All the maps have degree 0. $Z(I_1 + I_2) = Y_1 \cap Y_2$, which has dimension less than r.

$$\varphi_{S/I_1)} + \varphi_{S/I_2} = \varphi_{S/I_1 \oplus S/I_2} = \varphi_{S/I} + \varphi_{I_1 + I_2}.$$

The first starts in degree r. The $\varphi_{S/I}$ starts in degree r, and $\varphi_{I_1+I_2}$ startsin degree $r - 1$. Look at leading coefficients and obtain the result.

(c) $S(\mathbb{P}^n) = k[x_0, \ldots, x_n]$, $P_{\mathbb{P}^n}(z) = \binom{z+n}{n}$ counting monomials.

$$n! = \frac{(z+n)(z+n-1)\cdots(z+n-n+1)}{n!} = z^n + \cdots$$

(d) Let f be a generator of degree d. Exact sequence

$$0 \longrightarrow S(-d) \xrightarrow{\times f} S \longrightarrow S/(f) \longrightarrow 0$$

$S(-d)$ makes the map of degree 0. Taking Hilbert polynomials, we have $P_H(z) = \binom{z+n}{n} - \binom{z-d+n}{n}$.

$\dfrac{1}{n!}(z+n)(z+n-1)(z+n-2)\cdots(z+n-n+1) - \dfrac{1}{n!}(z-d+n)(z-d+n-1)(z-d+n-2)\cdots(z-$

The z^n terms cancel. Look at z^{n-1} terms. How do you get a z^{n-1}? Take z in all factors except where you take a constant.

$\dfrac{1}{n!}z^{n-1}(n+(n-1)+(n-2)+2+1) - \dfrac{1}{n!}z^{n-1}((-d+n)+(-d+n-1)+(-d+n-2)\cdots(-d+1)) =$

H has dimension $n-1$ so degree is $(n-1)! \cdot \dfrac{d}{(n-1)!} = d$.

Let $Y \subseteq \mathbb{P}^n$ be a projective variety of dimension r. Let H be a hypersurface not containing Y. Then by (7.2), $Y \cap H = Z_1 \cup \cdots \cup Z_s$, where Z_j are varieties of dimension $r-1$. Let $\mathfrak{p}$ be the homogeneous prime ideal of Z_j. We define the intersection multiplicity of Y and H along Z_j to be $i(Y, H; Z_j) := \mu_{\mathfrak{p}_i}(S/I_Y + I_H)$, where I_Y and I_H are homogeneous ideals of Y and H. The module $M = S/I_y + I_H$ has annihilator $I_Y + I_H$ and $Z(I_Y + I_H) = Y \cap H = Z_1 \cup \cdots \cup Z_s$, so $\mathfrak{p}_j$ is a minimal prime of M.

Minimal prime corresponds to maximal irreducible subvariety = irreducible component.

$\mu_{\mathfrak{p}}(M) = l_{S_{\mathfrak{p}}}(M_{\mathfrak{p}})$ minimal prime of M makes finite.

Theorem 2.3. *Let Y be a variety of dimension ≥ 1 in $\mathbb{P}^n$, and let H be a hypersurface not containing Y. Let $Z_1, \ldots, Z_s$ be the irreducible components of $Y \cap H$. Then*

$$\sum_{j=1}^{s} i(Y, H; Z_j) \deg Z_j = \deg Y \deg H.$$

Proof. Say $I(H) = (f)$ and $\deg f = d$, where $d = \deg H$. We have an exact sequence

$$0 \longrightarrow S/I_Y(-d) \xrightarrow{\times f} S/I_Y \longrightarrow M \longrightarrow 0,$$

where $M = S/I_Y + I_H$. Multiplication by f is injective because f does not vanish on Y, which is irreducible so I_Y prime. $f(g + I_Y) = 0$. Now $fg + I$ if and only if $fg \in I_Y$ but $f \notin I_Y$ and I_Y prime. $g \in I_Y$.

$$P_M(z) = P_Y(z) - P_Y(z-d)$$

Our result comes from comparing the leading coefficients of both sides of this equation. Let Y have dimension r degree. Then $P_Y(z) = \frac{e}{r!}z^r + \cdots$ on the right.

$$\frac{e}{r!}z^r + \cdots - (e/r!(z-d)^r + \cdots)$$

z^{r-1}'s cancel. Need to look at z^{r-1}

$$\frac{e}{r!}z^r + \alpha z^{r-1} + [\frac{e}{r!}(z-d)^r + \alpha(z-d)^{r-1} + \cdots$$

$$\alpha z^{r-1} \alpha z^{r-1} + \frac{der}{r!}z^{r-1} = \frac{de}{(r-1)!}z^{r-1} + \cdots$$

P_M side by (7.4) we have a filtration

$$0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^q = M,$$

whose quotients M^i/M^{i-1} are of the form $(S/q_i)(l_i)$. Hence,

$$P_M = \sum_{i=1}^{q} P_i,$$

where P_i is the Hilbert polynomial of $(S/q_i)(l_i)$

$$0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow M^2/M^1 \longrightarrow 0$$

$P_{M^2} = P_{M^1} + P_{M^2/M^1}$

$M^1 \cong (S/\mathfrak{p}_i)(l_i)$, $M^2/M^1 \cong (S/\mathfrak{p}_j)(l_j)$

If $Z(q_i)$ is a projective variety of dimension r_i and deg f_i, then

$$P_i(z) = \frac{f_i}{r_i!}z^{r_i} + \cdots$$

shift l_i does not change leading coefficient. We only care about the P_i where $r_i = r - 1$. Those are the minimal primes that correspond to the irreducible components Y_i's.

Each of these occurs $\mu_{\mathfrak{p}_j}(M) = l_{S_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j})$ times, when you localize M at $\mathfrak{p}_j$, all the other primes disappear. The number of times that min appears is the length.

coeff z^{r-1}

$$\sum_{j=1}^{s} \frac{i(Y, H; Z_j) \deg Z_j}{(r-1)!}$$

$\square$

"When you intersect a variety with a hypersurface, the degree of the intersection is the product of the degrees."

Example 2.10. Let G be a fixed abelian group and X a topological space. For any open $U \subseteq X$, set $\mathcal{F}(U)$ to be all the functions $f : U \to G$, using the convention that $\mathcal{F}(\emptyset) = 0$. For $V \subseteq U$ an open set, define $\rho_{UV}(f) = f|_V$ to be restriction of domain. Now $\mathcal{F}(U)$ is a group under pointwise addition: $f(x) + g(x)$, and restriction of functions is a group homomorphisms (if $W \subseteq V \subseteq U$, restricting from U to V then V to W is the same as restricting from U to W). There are several special cases of this:

(i) $G = \mathbb{R}$ and $\mathcal{F}(U)$ is the set of all continuous functions. Note that the restriction of continuous functions are continuous and the sum of continuous functions is continuous.

(ii) X is a differentiable manifold and $\mathcal{F}(U)$ is the set of all differentiable functions $f : U \to \mathbb{R}$.

(iii) X is a complex manifold and $\mathcal{F}(U)$ is the set of all holomorphic functions $f : U \to \mathbb{C}$

(iv) X is a variety over k and $\mathcal{F}(U)$ is the set of all regular functions $f : U \to k$. This presheaf is denoted $\mathcal{O}$ or $\mathcal{O}_X$ and is called the structure sheaf on X or the sheaf of regular functions on X.

Remark. In (i)–(iv), $\mathcal{F}(U)$ is actually a ring and the restriction maps are ring homomorphisms. We say these are presheaves of rings. You can have presheaves of lots of things: groups, ring, fields, modules, vector spaces, etc. —even nonalgebraic things like sets. Perhaps the best way to define this is with category theory: given a topological space X, we can make a category Top(X) as follows: objects are open subsets of X, the morphisms are inclusions of open sets. For open sets U, V, if $V \subseteq U$, $\mathrm{Hom}(V, U)$ has one element, the inclusion $V \subseteq U$, and if $V \not\subseteq U$, then $\mathrm{Hom}(V, U) = \emptyset$. For any category $\mathcal{C}$, we say that $\mathcal{C}$ has a terminal object T if and only if there is an object $T \in \mathrm{obj}\, \mathcal{C}$ such that for any object S in $\mathcal{C}$, $\mathrm{Hom}(S, T)$ consists of a single element. In other words, every object has a unique morphism to T. For example, Ab, the category of abelian groups, the zero group is the terminal object. These not not exist for every category. Now let X be a topological space and $\mathcal{C}$ be a category with a terminal object T. A presheaf of $\mathcal{C}$'s on X is an arrow reversing functor $F : Top \to \mathcal{C}$ such that $F(\emptyset) = T$.

Terminology: $\mathcal{F}$ a presheaf on X, $\mathcal{F}(U)$ is called the sections of $\mathcal{F}$ over U. This is sometimes denoted $\Gamma(U, \mathcal{F})$. The maps ρ_{UV} are called restriction homomorphisms. We write $s|_V$ instead of $\rho_{UV}(s)$ if $s \in \mathcal{F}(U)$. The maps ρ_{UV} relate local and global information. A sheaf is a presheaf that satisfies additional conditions that tie local and global together more tightly.

Definition. A preasheaf $\mathcal{F}$ on a topological space X is a sheaf if and only if it satisfies the following supplementary conditions:

- if U is an open set, $\{V_i\}$ is an open covering of U, and if $s \in \mathcal{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i, then $s = 0$, i.e. a section which is everywhere locally zero is globally zero.

- if U is an open set, $\{V_i\}$ an open covering of U, and we have $s_i \in \mathcal{F}(V_i)$ for each i with $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each i. (Gluing), i.e. a sections which appear to be glue-able, are glue-able.

Note that the conditions in the first imply the s in the last is unique: s, t both satisfy $s|_{V_i} = s_i$ for all i, $t|_{V_i} = s_i$ for all i, $(s-t)|_{V_i} = 0$ for all i so that by the third condition, $s - t = 0$, i.e. $s = t$.

Example 2.11. A basic example is $\mathcal{F}(U)$ is the set of all functions $f : U \to G$. We need check the conditions, but but are trivial. The other examples follow by set theory for the second condition, check whether a function is continuous differentiable, holomorphic, reg. locally etc.

Example 2.12. A non-example. Let X be a topological space and G a fixed group.

For $U \neq \emptyset$, $\mathcal{F}(U) = G$, $\mathcal{F}(\emptyset) = 0$. For any $V \subseteq U$, $V \neq \emptyset$, $\rho_{UV} \to G$ is the identity, $\rho_{U\emptyset} = 0$. This is called the constant presheaf.

For $U \neq \emptyset$, $\mathcal{G}(U) = G$, $\mathcal{G}(\emptyset) = 0$. All restriction maps, except ρ_{UU}, are the zero map. Note $\rho_{UU} = 1_U$.

These are presheaves which are not sheaves.

If X has any open set U with an open covering of U by $\{U_i\}$ with no $U_i = U$ and G has at least two elements, then $\mathcal{G}$ violates both the conditions above:

for the first, choose $s \neq 0$, $s \in \mathcal{G}(U) = G$. $\rho_{UU_i}(s) = 0$ for all i, $s \neq 0$ but restricts to 0 everywhere.

for the second, $g \in G$, $g \neq 0$. Let $s_i \in \mathcal{G}(U_i)$, $s_i = g$, $s_i|_{U_i \cap U_j} = 0$. Should be $s \in \mathcal{G}(U)$ with $s|_{U_i} = s_i$ but $s|_{U_i} = 0$ and $s_i \neq 0$.

Now $\mathcal{F}$ will satisfy the first. only thing that can ever restrict to 0 is the zero function. But for the last one, if X has two open sets U_1, U_2 with $U_1 \cap U_2 = \emptyset$, $U_1, U_2 \neq \emptyset$ and G has at least two elements, then $\mathcal{F}$ violates the last point. $\{U_1, U_2\}$ is an open cover of $U_1 \cup U_2$. Choose $g_1, g_2 \in G$, $g_1 \neq g_2$, $s_1 \in \mathcal{F}(U_1), s_1 = g_1$, $s_2 \in \mathcal{F}(U_2), s_2 = g_2$, $s_1|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2} = 0$. Because the restriction from $\mathcal{F}(U_1 \cup U_2) \to \mathcal{F}(U_1)$ or $\mathcal{F}(U_2)$ is the identity. No element of $\mathcal{F}(U_1 \cup U_2)$ could restrict to s_1 on U_1 and s_2 on U_2.

There is a modified version of the constant presheaf that is a sheaf.

Let X be a topological space. We define the constant sheaf $\mathcal{A}$ on X determined by A (an abelian group) as follows: Give A the discrete topology, for any open set $U \subseteq X$, let $\mathcal{A}(U)$ be the set of all continuous maps $f : U \to A$. The restriction maps are just restriction of functions. This is a sheaf. If U is connected, $\mathcal{A}(U) = A$.

Definition. If $\mathcal{F}$ is a presheaf on X and if P is a point on X, we define the stalk $\mathcal{F}_P$ of $\mathcal{F}$ at P to be the direct limit of the groups $\mathcal{F}(U)$ for all opens U containing P via the restriction maps ρ. If $\mathcal{F}$ is actually a sheaf, we also define the stalk in this way.

Thus, an element of $\mathcal{F}_P$ is represented by a pair $\langle U, s\rangle$, where U is an open set containing P, $s \in \mathcal{F}(U)$, $\langle U, s\rangle \sim \langle V, t\rangle$ if and only if there is an open set $W \subseteq V \cap U$, $P \in W$ with $s|_W = t|W$.

$\mathcal{F}_P$ germs of sections at P, X a variety, $P \in X$, $(\mathcal{O}_X)_{\mathbb{P}} = \mathcal{O}_{X,P}$.

Definition. If $\mathcal{F}$ and $\mathcal{G}$ are presheaves on X, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ consists of a morphism of abelian groups $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set U such that whenever $V \subseteq U$ is an inclusion, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \big\downarrow \rho^{\mathcal{F}}_{UV} & & \big\downarrow \rho^{\mathcal{G}}_{UV} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(U) \end{array}$$

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, we use the same definition for a morphism of sheaves. An isomorphism is a morphism with a with a two sided inverse.

This last statement amounts to saying that all the $\varphi(U)$'s are just isomorphisms. Well suppose the diagram above commutes and $\varphi(U)$ and $\varphi(V)$ are isomorphisms. Does the following diagram commute:

$$\begin{array}{ccc} \mathcal{F}(U) & \xleftarrow[\varphi^{-1}(U)]{} & \mathcal{G}(U) \\ \big\downarrow \rho^{\mathcal{F}}_{UV} & & \big\downarrow \rho^{\mathcal{G}}_{UV} \\ \mathcal{F}(V) & \xleftarrow[\varphi^{-1}(V)]{} & \mathcal{G}(U) \end{array}$$

Take $g \in \mathcal{G}(U)$, $\rho^{\mathcal{F}}_{UV}\varphi^{-1}(U)(g) =?\varphi^{-1}(V)\rho^{\mathcal{G}}_{UV}(g)$. Taking $\phi(V)$ of both sides (note this is an isomorphism), we obtain $\rho^{\mathcal{G}}_{UV}\varphi(U)\varphi^{-1}(g) = \rho^{\mathcal{G}}_{UV}(g)$ and this works.

Note that a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves induces a morphism $\varphi_P : \mathcal{F}_P \to \mathcal{G}_P$ on the stalks, for any $P \in X$. $\varphi_P(U, s) = (U, \varphi_U(s))$, $s \in \mathcal{F}(U)$. The following proposition would be false for presheaves illustrates the local nature of sheaves.

Proposition 2.13. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. Then φ is an isomorphism if and only if the induced map on the stalk $\varphi_P : \mathcal{F}_P \to \mathcal{G}_P$ is an isomorphism for every $P \in X$.*

Two examples:

(i) Show Hartshorne's statement that this proposition is false for presheaves is true.

(ii) The following similar sounding proposition is false for sheaves. FALSE PROPOSITION. Let $\mathcal{F}, \mathcal{G}$ be sheaves on X. Then $\mathcal{F} \cong \mathcal{G}$ if and only if $\mathcal{F}_P \cong \mathcal{G}_P$ for all $P \in X$.

(i) $X = \mathbb{P}^1$. Let $\mathcal{F}$ be the constant presheaf associated to a group G with at least two elements. Clearly, $\mathcal{F}_P \cong G$ for all $P \in X$. Now let $\mathcal{G}$ be the same as $\mathcal{F}$ except take $\mathcal{G}(\mathbb{P}^1) = 0$ for all other $\mathcal{G}(U) = \mathcal{F}(U)$. $\rho_{\mathbb{P}^1,U} = 0$ all other ρ's the same. Stalks are still G. $\varphi : \mathcal{G} \to \mathcal{F}$, $\varphi(U) = \mathrm{id}$ except when $U = \mathbb{P}^1$, $\varphi_{\mathbb{P}^1}(0) \to G$ induces isomorphism on stalks, $\mathcal{F} \cong \mathcal{G}$, $\mathcal{F}(\mathbb{P}^1) = G$, $\mathcal{G}(\mathbb{P}^1) = 0$.

(ii) $X = \mathbb{P}^1$, $\mathcal{O}$ structure sheaf, fix one point, say $P = [0,1] \in X$. Define the sheaf $\mathcal{I}(P)$, $\mathcal{I}(P)(U) = \{f \in \mathcal{O}(U) \colon f(P) = 0\}$, use restriction maps from $\mathcal{O}$, $O_P = \mathcal{O}_{P,X}$ local rings, $\mathcal{I}(P)_Q$, $P \neq Q = \mathcal{O}_{Q,X}$,

$\mathcal{I}(P)_P = \mathfrak{m}_{P,X} \subseteq \mathcal{O}_{P,X}$

$\mathcal{O}_{P,\mathbb{P}^1} \cong k[x]_{(x)} \supset \mathfrak{m}_{P,X}(x)_{(x)}$

t coordinate: $k[x]_{(x)} = k[t]_{(t)}$ $\mathfrak{m}_{P,X}(x)_{(x)} = (t)$

as a module (group)

$(t) \subset k[t]_{(t)} \cong k[t]_{(t)}$

$\mathcal{O}(\mathbb{P}^1) = k$ $\mathcal{I}(P)(\mathbb{P}^1) = 0$

Definition. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. We define the presheaf kernel of φ, presheaf cokernel of φ, and presheaf image of φ to be the presheaves given by $U \mapsto \ker \varphi(U)$, use $\mathcal{F}$ restriction maps. Need to show $V \subseteq U$, $\rho_{UV} \ker \varphi(U) \subseteq \ker \varphi(V)$.
$U \mapsto \operatorname{coker} \varphi(U)$, use $\mathcal{G}$ restriction maps $U \mapsto \operatorname{im} \varphi(U)$, use $\mathcal{G}$ restriction maps

If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the presheaf kernel of φ is a sheaf. But presheaf cokernel and presheaf image might not be sheaves. This leads to the sheaf associated to a presheaf.

Example where presheaf image is not a sheaf
$X = \mathbb{P}^1$ $Y \subseteq \mathbb{P}^1$, $Y = \{[0,1],[1,0]\}$ (actually any two points will do).
Define the sheaf $\mathcal{F}$ by $\mathcal{F}(U)$ regular functions on $U \cap Y$. $\mathcal{F}$ sheaf on $\mathbb{P}^1$
restriction maps are just restriction of functions.
$\mathcal{F}(U) = 0$ if $U \cap Y = \varnothing$ $\mathcal{F}(U) = k$ if $U \cap Y$ is one point $\mathcal{F}(U) = k \oplus k$ if $U \cap Y$ is two points. $\varphi : \mathcal{O}_{\mathbb{P}^1} \to \mathcal{F}$ by restriction of functions
$\mathcal{O}_{\mathbb{P}}(U) \to \mathcal{F}(U)$
Call $\mathcal{G}$ the presheaf image of φ $\mathcal{G}(\mathbb{P}^1) = k$
$\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \to \mathcal{F}(\mathbb{P}^1)$ regular function on Y, $k \oplus k$ $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = k$, $a \mapsto (a,a)$
Consider the following open cover of $\mathbb{P}^1$ $\mathbb{P}^1 \setminus [0,1]$ $\mathbb{P}^1 \setminus [1,0]$
$\mathcal{G}(\mathbb{P}^1 \setminus [0,1]) = k$, $\mathcal{G}(\mathbb{P}^1 \setminus [1,0]) = k$ $\mathcal{G}((\mathbb{P}^1 \setminus [0,1]) \cap (\mathbb{P}^1 \setminus [1,0])) = 0$
Pick two unequal elements of k, say s,t.
$t \in \mathcal{G}(\mathbb{P}^1 \setminus [0,1])$, $s \in \mathbb{P}^1 \setminus [1,0]$
Both restrict to 0 on the intersection. If $\mathcal{G}$ were a sheaf, we could fine $y \in \mathcal{G}(\mathbb{P}^1)$ that restricts to t on $\mathbb{P}^1 \setminus [0,1]$ and s on $\mathbb{P}^1 \setminus [1,0]$, $\mathcal{G}(\mathbb{P}^1) = k$. Anything in $\mathcal{G}(\mathbb{P}^1)$ must restrict

to the same element of k in $\mathcal{G}(\mathbb{P}^1 \setminus [1,0])$ and $\mathcal{G}(\mathbb{P}^1 \setminus [0,1])$, violates the 4th property of a sheaf.

Proposition-Definition 1.2. Given any presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ with the property that for any sheaf $\mathcal{G}$ and any morphism $\varphi : \mathcal{F} \to \mathcal{G}$, there is a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \psi \circ \theta$. Furthermore, the pair $(\mathcal{F}^+, \theta)$ us unique up to isomorphism. $\mathcal{F}^+$ is called the sheaf associated to the presheaf $\mathcal{F}$. This is also called the sheafification of $\mathcal{F}$.

Propery 3 of sheaf wants to reduce the number of sections whereas property 4 wants to increase the number of sections.

Proof. We construct the sheaf $\mathcal{F}^+$ as follows. For any open U, let $\mathcal{F}^+(U)$ be the set f functions s from U to the union $\cup_{P \in U} \mathcal{F}_P$ of the stalks of $\mathcal{F}$ over points of U such that

(i) for each $P \in U$, $s(P) \in \mathcal{F}_P$ and (ii) for each $P \in U$, there is a neighborhood V of P contained in U and an element $t \in \mathcal{F}(V)$ such that for all $Q \in V$, the germ t_Q of t at Q is equal to $s(Q)$.

$\{$ for any $Q \in V, \langle V, t \rangle \in \mathcal{F}_Q \}$

Given $s \in \mathcal{F}(U), \langle U, s \rangle \in \mathcal{F}_P$ for all $P \in U\}$. Now one can verify immediately that $\mathcal{F}^+$ with the natural restriction maps is a sheaf.

Definition. A subsheaf of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that for every open set $U \subseteq X$, $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$, and the restriction maps of the sheaf $\mathcal{F}'$ are induced by those from $\mathcal{F}$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\rho_{UV}"] & \mathcal{F}(V) \\ \mathcal{F}'(U) \arrow[u, hook] \arrow[r, "\rho'_{UV}"] & \mathcal{F}'(V) \arrow[u, hook] \end{tikzcd}$$

where $V \subseteq U$. It follows that for any point P, the stalk $\mathcal{F}'_p$ is a subgroup of $\mathcal{F}'_P$.

If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, we define the kernel of φ, denoted $\ker \varphi$, to be the presheaf kernel of φ (which is a sheaf). Thus, $\ker \varphi$ is a subsheaf of $\mathcal{F}$.

We say that a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is injective if and only if $\ker \varphi = 0$. Thus, φ is injective if and only if the induced map $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every open set of X.

If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, we define the image of φ, denoted $\operatorname{im} \varphi$, to be sheaf associated to the presheaf image of φ, there is a natural map $\operatorname{im} \varphi \to \mathcal{G}$. In fact, this map is injective (see Ex. 1.4) and thus $\operatorname{im} \varphi$ can be identified with a subsheaf of $\mathcal{G}$. We say that a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective if and only if $\operatorname{im} \varphi = \mathcal{G}$. We say that a sequence

$$\cdots \longrightarrow \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \longrightarrow \cdots$$

of sheaves and morphisms is exact if and only if at each stage $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$. Thus, $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}$ is exact if and only if φ is injective, $\mathcal{F} \to \mathcal{G} \to 0$ is exact if and only if φ is surjective. Now let $\mathcal{F}'$ be a subsheaf of $\mathcal{F}$. We define the quotient sheaf $\mathcal{F}/\mathcal{F}'$ to be the sheaf associated to the presheaf $U \to \mathcal{F}(I)/\mathcal{F}'(U)$. The restriction maps come from $\mathcal{F}$. It follows that for any point P, the stalk $(\mathcal{F}/\mathcal{F}')_P$ is the quotient $\mathcal{F}_P/\mathcal{F}'_P$. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. We define the cokernel of φ, denoted coker φ, to be the sheaf associated to the presheaf cokernel of φ.

Caution: We saw that a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves is injective if and only if the map of sections $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for each U. The corresponding statement for surjective morphisms is not true: if $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective the maps $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ need not be surjective.

Example: $X = \mathbb{P}^1$, $Y \subseteq X$, $Y = \{[0,1],[1,]0\}$, $\mathcal{O}_{\mathbb{P}^1}$ sheaf of regular functions on $\mathbb{P}^1$. $\mathcal{F}$ on $\mathbb{P}^1$, $\mathcal{F}(U) =$ regular functions on $U \cap Y$.

$\mathcal{F}(U) = 0$ if $U \cap Y = \varnothing$ $\mathcal{F}(U) = k$ if $U \cap Y$ is one-pont $\mathcal{F}(U) = k \oplus k$ if $U \cap Y$ is two points restriction maps are just restriction of functions $\varphi : \mathcal{O}_{\mathbb{P}^1} \to \mathcal{F}$ by restriction of functions Call $\mathcal{G}$ the presheaf image of φ $\mathcal{G}(\mathbb{P}^1) = k$ because $\mathcal{O}_{\mathbb{P}}(\mathbb{P}^1) = k$ $\varphi(\mathbb{P}^1) : \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \to \mathcal{G}(\mathbb{P}^1)$ not surjective (this is $k \mapsto k$ in $k \oplus k$). For any other U, $\varphi(U)$ will be surjective. As soon as you delete one point from $\mathbb{P}^1$, you get an $\mathbb{A}^1$. $\mathcal{O}_{\mathbb{P}^1}(U)$ is at least all polynomials, $U \neq \mathbb{P}^1$. Can always find a polynomial taking prescribed values at any finite set of points.

The stalks of $\mathcal{G}$ at either $[0,1]$ or $[1,0]$ are k. Because for small enough open sets containing only one of the points $\mathcal{O}_{\mathbb{P}^1}(U) = k$. Sections of $\mathcal{G}^+(\mathbb{P}^1)$ only need to locaclly come from $\mathcal{G}$. By taking the two open sets $\mathbb{P}^1 \setminus [0,1]$, $\mathbb{P}^1 \setminus [0,1]$ you get that $\mathcal{G}^+(\mathbb{P}^1) = k \oplus k$. Thus, $\mathcal{G}^+ = \mathcal{F}$. φ surjective even though $\varphi(\mathbb{P}^1)$ not.

back to caution 1.2.1. However, we can say that φ is surjective iff the maps $\varphi_P : \mathcal{F}_P \to \mathcal{G}_P$ are surjective for each P. More generally, a sequence of sheaves is exact iff it is exact on stalks. (Ex.1.2)

Definition. Let $f : X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on X, we define the direct image sheaf $f_*\mathcal{F}$ on Y by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ for any open set $V \subseteq Y$.

For any $V \subseteq U$ open in Y, then $f^{-1}(V) \subseteq f^{-1}(U)$ so you can use the restriction maps from $\mathcal{F}$. Not too hard to check that this gives a sheaf on Y>

For any sheaf $\mathcal{G}$ on Y, we define the inverse image sheaf $f^{-1}\mathcal{G}$ on X to be the sheaf associated to the presheaf $U \mapsto \lim_{V \supseteq f(U)} \mathcal{G}(U)$.

$U' \subseteq U$ any $V \supseteq f(U)$ contains $f(U')$

$f^{-1}\mathcal{G}$ is different than $f^*\mathcal{G}$, f^* will be defined later for certain sheaves on certain spaces.

f_* functor from sheaves on X to sheaves on Y f^{-1} functor from sheaves on Y to sheaves on X

Definition. If Z is a subset of X regarded as a topological space with the induced topology. Let $i : Z \to X$ be the inclusion map. For a sheaf $\mathcal{F}$ on X, $i^{-1}\mathcal{F}$ is called the restriction of $\mathcal{F}$ to Z. $\mathcal{F}|_Z, P \in Z, (\mathcal{F}|_Z)_P = \mathcal{F}_p$.

Exercise 1.8 very important.
Schemes

Definition. Let A be any commutative ring with identity. We define Spec A to be the set of prime ideals of A. For any ideal I of A, we define $V(I)$ to be the set of all prime ideals that contain I.

Remember: A is not a prime ideal but it is an ideal.
Rem: We could have defined for any subset $S \subseteq A$, $V(S)$ to be the set of all prime ideals containing S. it is clear that if $\langle S \rangle$ is the ideal generated by S, then $V(S) = V(\langle S \rangle)$. That is why they only define V for ideals.

Lemma 2.2. *If I, J are two ideals of A, then $V(IJ) = V(I) \cup V(J)$ If $\{I_i\}$ is any set of ideals of A, then $V(\sum I_i) = \cap V(I_i)$ If I, J are two ideals of A, $V(I) \subseteq V(J)$ if and only if $\sqrt{I} \supseteq \sqrt{J}$ $V(A) = \emptyset$ $V((0)) = \operatorname{Spec} A$*

Then these can be taken as the closed sets of a topology on Spec A called the Zariski topology.

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