Syracuse University

MAT 738: Algebraic Geometry

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0 Introduction

0.1 Course Description

MAT 738 Introduction to Algebraic Geometry: The study of the zeros of polynomials. Classical algebraic varieties in affine and projective space, followed by introduction to modern theory of sheaves, schemes, and cohomology.

0.2 Disclaimer

These notes were taken in Spring 2019 in a course taught by Professor Steven Diaz. In some places, notation/material has been changed or added. Any errors in this text should be attributed to the typist – Caleb McWhorter – and not the instructor or any referenced text.

0.3 Conventions

A ring is a commutative ring with identity—unless otherwise stated. All ring homomorphisms $\phi : R \to S$ are assumed to have the property that $\phi(1_R) = 1_S$. For a ring R, the whole ring is considered to be an ideal, but not a prime ideal.

1 Varieties

1.1 Introduction

One could define Algebraic Geometry as the study of solutions to systems of polynomial equations. The early history of Algebraic Geometry was focused on what we will eventually know as affine varieties, especially the simplest cases plane algebraic curves, e.g. lines, circles, parabolas, ellipses, hyperbolas, and cubic curves. The development of Algebraic Geometry was slow, due to cumbersome language, notation, varying approaches, and especially varying or ineffective definitions. However motived by work in various fields including Complex Analysis, (Algebraic) Topology, Number Theory, and especially Commutative Algebra, Algebraic Geometry developed rapidly. Modern Algebraic Geometry is due largely to the development of the theory of sheaves and schemes by Grothendieck and Serre, but is also due to the contribution of many others such as Zariski, Čech, Leray, Cartan, et al..

1.2 Affine Varieties

Let k be a fixed algebraically closed field.[1] Often, we will focus on the case where $k = \mathbb{C}$.

Definition (Affine n-space)**.** Denote by k^n the set of ordered n-tuples of elements of k. Then define $\mathbb{A}^n_k := k^n$, called affine n-space over k. Note that this is also denoted $\mathbb{A}^n(k)$ or simply $\mathbb{A}^n$ when k is understood.

For ease of notation, we often denote the polynomial ring $k[x_1, \ldots, x_n]$ simply by 'A'. We can think of elements of $A := k[x_1, \ldots, x_n]$ as being functions $f : \mathbb{A}^n \to k$ via $(k_1, \ldots, k_n) \mapsto f(k_1, \ldots, k_n)$, i.e. via evaluation; that is, given $f \in A$ and $P = (k_1, \ldots, k_n)$, we have $f(P) = f(k_1, \ldots, f_n) \in k$. This allows us to consider the vanishing set of the polynomial f.

Definition (Zero Set)**.** For $f \in A := k[x_1, \ldots, x_n]$, define $Z(f) := \{P \in \mathbb{A}^n \mid f(P) = 0\}$, called the zero set of f. For $T \subseteq A$, we define the zeros of T, denoted $Z(T)$, by

$$Z(T) := \bigcap_{f \in T} Z(f) = \{P \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } f \in T\}.$$

If $T = \{f_1, \ldots, f_r\}$, we will often write $Z(T) = Z(f_1, \ldots, f_r)$.

Note the underlying field k does matter here. For example, $Z(x^2 + 1) = \emptyset$ if $k = \mathbb{R}$ but if $k = \mathbb{C}$, then $Z(x^2 + 1) = \{\pm i\}$. As another example in $\mathbb{R}^2$, $Z(y - x^2, x - y^2)$ consists of two points, namely the points $(0, 0), (1, 1)$ of intersection, see Figure 1. However if $k = \mathbb{C}^2$, then $Z(y - x^2, x - y^2)$ consists of four points: $(0, 0), (1, 1), (-\zeta_3, \zeta_3^2), (\zeta_3^2, -\zeta_3)$, where ζ_3 is a primitive cube root of unity.

[1] Note that much of elementary Algebraic Geometry, the requirement that k be algebraically closed is unnecessary, and k being an arbitrary field, or even simply an integral domain would suffice.

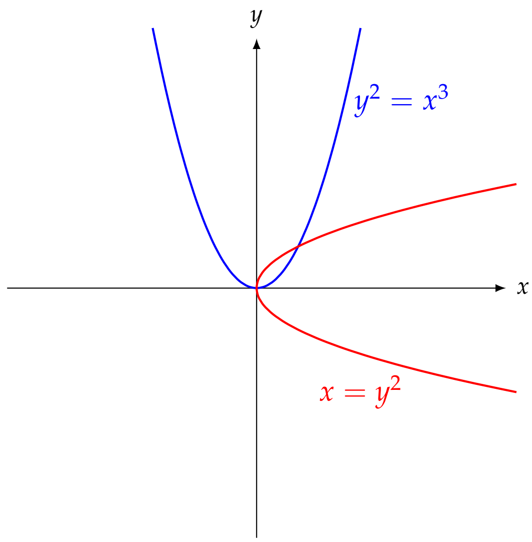

Figure 1: If $k = \mathbb{R}^2$, then the set $Z(y - x^2, x - y^2)$ is the intersection of $y = x^2$ with $x = y^2$.

We now have enough to define the basic building block of Algebraic Geometry.

Definition (Algebraic Set)**.** A subset $Y \subseteq \mathbb{A}^n$ is called an algebraic set if and only if there exists a subset $T \subseteq A$ such that $Y = Z(T)$.

That is, a set of points Y is an algebraic set if Y is a set of common zeros for a collection of polynomials T. This is the start of the bridge between Algebraic Geometry, describing geometric objects using algebraic terms, as some of the following examples shall being to show (though one will have to wait for coordinate rings and much later material to see the deeper connections).

Example 1.1.

(i) The emptyset, $\emptyset$, is algebraic since $\emptyset = Z(1)$, where $f = 1$ is the constant polynomial $f = 1$. Furthermore, $\mathbb{A}^n$ is an algebraic set since $\mathbb{A}^n = Z(0)$, where $f = 0$ is the zero polynomial.

(ii) Any single point $P = (k_1, \ldots, k_n) \in \mathbb{A}^n$ is an algebraic set since $\{P\} = Z(x_1 - k_1, \ldots, x_n - k_n)$. In fact, any *finite* collection of points is algebraic, c.f. Proposition 1.1. [One should check that you can write down an explicit polynomial to confirm this.]

(iii) The set $\{(x, y) : y - f(x) = 0\}$, where $f(x)$ is a polynomial, is trivially algebraic. But this is precisely the graph of the function $y = f(x)$. For example, the cubic $y = x^3$ is algebraic (meaning its graph). This easily generalizes to $y = f(x_1, \ldots, x_n)$, where $f(x_1, \ldots, x_n)$ is a polynomial. Furthermore considering the graph of $Ax^2 + By^2 +$

$Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$, we see that the graph of every quadratic surface an algebraic surface, e.g. the cone, ellipsoid, cylinder, hyperboloid, etc..

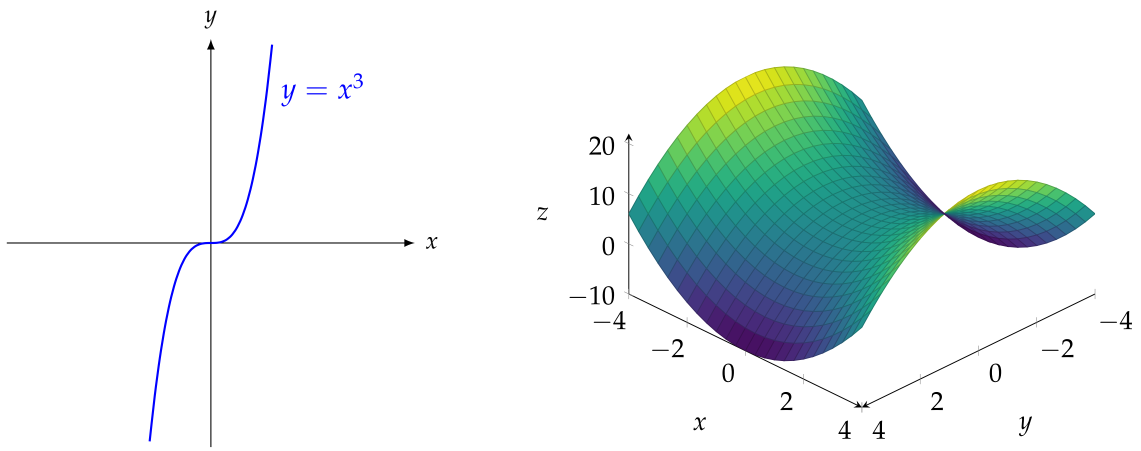

Figure 2: One the left, the algebraic set given by the curve $y = x^3$ (not including axes). On the right, the algebraic set given by $z = x^2 - y^2 + 6$ (not including axes).

(iv) A 'line glued to a circle' is an algebraic set, i.e. the image in Figure 3 (excluding the axes) is an algebraic set since it represents the set $Z((x^2+y^2-1)(y-2)) = \{(x,y) \colon (x^2+y^2-1)(y-2)\}$. Why? If $(x^2+y^2-1)(y-2) = 0$, then either $x^2+y^2 = 1$, in which case the point lies on the circle, or $y - 2 = 0$ in which case the point (x,y) lies on the line $y = 2$. In fact, even the coordinate axes are algebraic sets because together they are $Z(xy)$, by a similar reasoning.

(v) The set of all $n \times n$ matrices can be identified with the set $\mathbb{C}^{n^2}$. This space contains many subsets of interest. For example, the matrices of determinant 1, $\mathrm{SL}_n(\mathbb{C})$, forms an affine algebraic variety in $\mathbb{C}^{n^2}$ because it is the vanishing set of the polynomial given by $\Delta - 1$, where

$$\Delta(x_{ij}) = \det \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

is the determinant. In fact, a determinantal variety is the set of all matrices (considered as a subset of $\mathbb{C}^{n^2}$) of rank at most k. For $k \geq n$, the determinantal variety is the whole space $\mathbb{C}^{n^2}$. But for $k < n$, the rank of a matrix A is at most k if and only if all its $(k+1) \times (k+1)$ subdeterminants vanish. But as these subdeterminants are

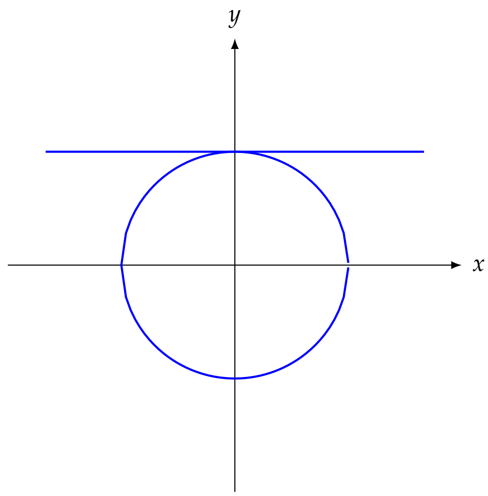

Figure 3: The algebraic set given by a 'line glued to a circle.'

polynomials in the variables x_{ij}, the set of matrices of rank at most k is an affine algebraic variety. $\triangleleft$

Of course, not all sets are algebraic. Not even all curves in $\mathbb{R}^2$ are algebraic, as the following exercise demonstrates.

Exercise.

(a) Prove that in $\mathbb{A}^2_{\mathbb{R}}$, the graph of $y = \sin x$ is not an algebraic set, i.e. prove the set $\{(x, y) : y = \sin x\}$ is not algebraic. [*Hint: The graph intersects the x-axis at infinitely many points, but single variable polynomials have finitely many roots.*]

(b) Prove that in $\mathbb{A}^2_{\mathbb{R}}$, the graph of $y = e^x$ is not an algebraic set, i.e. prove the set $\{(x, y) : y = e^x\}$ is not an algebraic set. [*Hint: e^x grows 'faster' than any polynomial, i.e.* $\lim_{x \to \infty} \frac{e^x}{x} = \infty$.]

(c) Prove that the open ball in the usual Euclidean topology on $\mathbb{C}^n$ is not an algebraic set by showing that every affine algebraic variety in $\mathbb{C}^n$ is closed in the Euclidean topology. [*Hint: Polynomials are continuous functions from $\mathbb{C}^n$ to $\mathbb{C}$, so their zero sets are closed.*] Explain then why the set of invertible matrices, $\mathrm{GL}_n(\mathbb{C})$ is not an affine algebraic variety.

Note that if $N \subseteq M$, then $Z(M) \subseteq Z(N)$, since any polynomial which vanishes on all of M certainly vanishes on all of N. Combining this with the Hilbert Basis Theorem: if R is noetherian, then $R[x_1, \ldots, x_n]$ is noetherian, we obtain the following result.

Proposition 1.1.

(a) Let $T \subseteq A$ and J be the ideal generated by T, then $Z(T) = Z(J)$. In particular, every algebraic set in $\mathbb{A}^n$ is of the form $Z(J)$ for some ideal J of A.

(b) Every algebraic set in $\mathbb{A}^n$ is of the form $Z(T)$ for some finite set $T \subseteq A$.

Proof.

(a) Since T generates J, we know that $T \subseteq J$ which immediately implies $Z(J) \subseteq Z(T)$. For the reverse inclusion, let $P \in Z(T)$ so that for all $f \in T$, $f(P) = 0$. If $g \in J$, we have $g = \sum_{i=1}^{m} a_i f_i$, where $a_i \in A$, $f_i \in T$. But then we have

$$g(P) = \sum_{i=1}^{m} a_i(P) f_i(P) = \sum_{i=1}^{m} a_i(P) \cdot 0 = 0.$$

Therefore, $Z(T) = Z(J)$. By definition, $Y \subseteq \mathbb{A}^n$ is algebraic if and only if there is $T \subseteq A$ with $Y = Z(T)$. Taking $J := \langle T \rangle$, the second claim follows.

(b) Since k is a field, we know that $A = k[x_1, \ldots, x_n]$ is noetherian by the Hilbert Basis Theorem. Recall that every ideal in a noetherian ring is finitely generated (in fact this is equivalent to the definition). By (a), we know that $Y \subseteq \mathbb{A}^n$ is algebraic if and only if $Y = Z(J)$ for some ideal J of A. But as A is noetherian, we can find a finite number of generators, say $T := \{f_1, \ldots, f_r\}$, for J. Therefore, $Y = Z(J) = Z(\langle T \rangle) = Z(T) = Z(f_1, \ldots, f_r)$, as desired. $\square$

Our goal is to build an underlying topological structure with which to work. This will form the basis for much of what is to come later. But first, we will need a proposition.

Proposition 1.2.

(a) The empty set and whole space are algebraic sets.

(b) The finite union of algebraic sets is algebraic.

(c) The intersection of any algebraic sets is algebraic.

Proof.

(a) Recall from Example 1.1 that $\emptyset = Z(1)$ and $\mathbb{A}^n = Z(0)$ are algebraic sets.

(b) By induction, it suffices to prove that the union of two algebraic sets is algebraic. Suppose that S_1, S_2 are algebraic sets. Then by Proposition 1.1, we know that $S_1 = Z(T_1)$ and $S_2 = Z(T_2)$ for some $T_1, T_2 \subseteq A$. We need show that $S_1 \cup S_2 = Z(T_1) \cup Z(T_2) = Z(T)$ for some set T. We shall show that $Z(T_1) \cup Z(T_2) = Z(T_1 T_2)$, i.e.

$T = T_1T_2 = \{t_1t_2 \colon t_1 \in T_1, t_2 \in T_2\}$ works, so that again by Proposition 1.1, $S_1 \cup S_2$ is algebraic.

To see that $Z(T_1) \cup Z(T_2) \subseteq Z(T_1T_2)$, let $P \in Z(T_1) \cup Z(T_2)$. But then $P \in Z(T_1)$ or $Z(T_2)$. Without loss of generality, suppose $P \in Z(T_1)$. Any element of T_1T_2 is of form fg, where $f \in T_1, g \in T_2$. But then $P \in Z(T_1T_2)$.

To see that $Z(T_1T_2) \subseteq Z(T_1) \cup Z(T_2)$, choose $P \in Z(T_1T_2)$. We will show that if $P \notin Z(T_1)$ then $P \in Z(T_2)$. So suppose $P \notin Z(T_1)$. Then there is $f \in T_1$ with $f(P) \neq 0$. Choose any $g \in T_2$. Now $fg \in T_1T_2$ so $(fg)(P) = 0$. But $0 = (fg)(P) = f(P)g(P)$. We know $f(P) \neq 0$ whenever$k$ is an integral domain, so that we must have $g(P) = 0$. As $g \in T_2$ was arbitrary, then $P \in Z(T_2)$.

(c) Suppose $\{S_i\}_{i \in \mathcal{I}}$ is a collection of algebraic sets. Then for each $i \in \mathcal{I}$, we know that $S_i = Z(T_i)$ for some $T_i \subseteq A$. But then we have

$$\bigcap_{i \in \mathcal{I}} S_i = \bigcap_{i \in \mathcal{I}} Z(T_i) = Z\left(\bigcup_{i \in \mathcal{I}} T_i\right),$$

where the last equality follows since $P \in Z(\bigcup_{i \in \mathcal{I}} T_i)$ if and only if every polynomial in every T_i vanishes at P, i.e. $P \in \bigcap_{i \in \mathcal{I}} Z(T_i)$. □

Notice that Proposition 1.2 shows that the algebraic sets are precisely the closed sets in some topology, called the Zariski topology on $\mathbb{A}^n$.

Definition (Zariski Topology)**.** The Zariski topology on $\mathbb{A}^n$ is defined by taking the open sets to be the complements of algebraic sets.

Example 1.2. What is the Zariski topology on $\mathbb{A}^1$? Since k is a field, $A = k[x]$ is a PID, so every ideal in A is principal. But by Propositon 1.1, every algebraic set Y is of the form $Y = Z(J)$ for some ideal J. But then $Y = Z(f)$ for some polynomial. Since k is algebraically closed, we can factor $f(x)$ as $f(x) = c(x - a_1) \cdots (x - a_n)$, where $c, a_1, \ldots, a_n \in k$. Therefore, $Z(f) = \{a_1, \ldots, a_n\}$. We know also from Example 1.1 that any finite set in $\mathbb{A}^n$ is algebraic. The closed sets in the Zariski topology are then $\emptyset$, $\mathbb{A}^1$, and all finite subsets. Therefore, the Zariski topology on $\mathbb{A}^1$ consists of $\emptyset$, $\mathbb{A}^1$, and the complements of finite subsets. But then every set which is not the whole set or $\emptyset$ contain all but finitely many elements of k. In particular, every two open sets intersect so that the Zarisiki topology on $\mathbb{A}^1$ is (extremely) not Hausdorff. Note that one did not need the fact that $k[x]$ a PID. Any algebraic set is if the form $\bigcap_{i \in \mathcal{I}} Z(f_i)$, where $f_i \in k[x]$. However, f_i can only be 0, in which case $Z(f_i) = \mathbb{A}^1$, a nonzero constant, in which case $Z(f_i) = \emptyset$, or a nonconstant polynomial in which case $Z(f_i)$ consists on finitely many points. In particular, the intersection consists of finitely many points. ◁

When studying objects geometrically, one breaks them into their constituent pieces and studies them individually, as well as how they fit together. So when studying geometric objects from an algebraic or topological structure, we will also want to break pieces into smaller structures.

Definition (Irreducible)**.** A nonempty subset Y of a topological space X is irreducible if and only if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets each of which is closed in Y (in the induced topology). The empty set is not considered to be irreducible. A set which is not irreducible is reducible.

Example 1.3. Whenever k is infinite, e.g. when k is algebraically closed, $\mathbb{A}^1$ is irreducible as the only proper closed subsets are finite, so $\mathbb{A}^1$ could not be written as a finite union of closed sets.

Example 1.4. In $\mathbb{A}^2$, the set $Z(xy)$ is reducible as we can write $Z(xy) = Z(x) \cup Z(y)$, i.e. for xy to vanish either x must vanish or y must vanish. This should be expected since the coordinate axes can be broken up into the x-axis and y-axis individually, c.f. Example 1.1(iv).

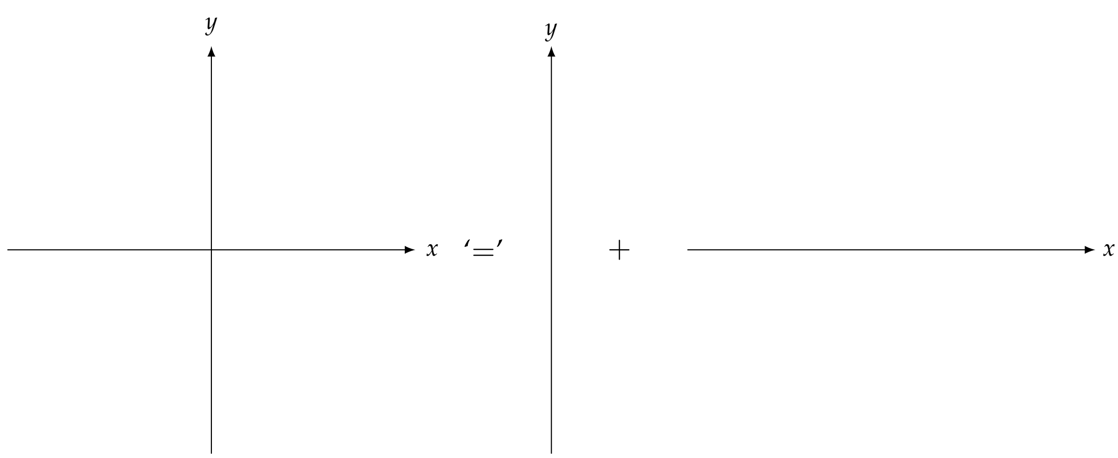

Figure 4: The decomposition $Z(xy) = Z(x) \cup Z(y)$.

Definition (Affine Algebraic Variety)**.** An affine algebraic variety is an irreducible closed subset of $\mathbb{A}^n$ (with the induced topology). This is also sometimes referred to as simply an affine variety. An open subset of an affine variety is a quasi-affine variety.

Notice all the proceeding definitions and examples connected subsets of $\mathbb{A}^n$ and ideals in A. Before moving forward, we first need to examine this correspondence more closely. Notice before one began with a collection of polynomials and then obtained a collection of points. One can go in the other direction as well, which gives the following definition.

Definition ($I(Y)$)**.** Let $Y \subseteq \mathbb{A}^n$ be a subset. Define the ideal of Y, denoted $I(Y)$, by $I(Y) := \{f \in A \colon f(P) = 0 \text{ for all } P \in Y\}$.

Of course, one should actually check that this is an ideal.

Proposition 1.3. *Let $Y \subseteq \mathbb{A}^n$ be any subset. Then $I(Y)$ is an ideal.*

Proof. We know that $0 \in I(Y)$ since 0 vanishes everywhere. Now suppose that $f, g \in I(Y)$ and $h \in A$. For any $P \in Y$,

$$\begin{aligned}(f+g)(P) &= f(P) + g(P) = 0 + 0 = 0 \\ (hf)(P) &= h(P)f(P) = h(P)(0) = 0\end{aligned}$$

so that $f + g, hf \in I(Y)$. □

We then have two functions between two different important sets, namely

$$\left\{ \text{ Subsets of } \mathbb{A}^n \ \right\} \underset{Z(-)}{\overset{I(-)}{\rightleftarrows}} \left\{ \begin{array}{c} \text{subsets of} \\ k[x_1, \ldots, x_n] \end{array} \right\}$$

To find the relationship between this connection, we will need the following result.

Proposition 1.4.

(i) If $T_1 \subseteq T_2$ are subsets of A, then $Z(T_1) \supseteq Z(T_2)$.

(ii) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{A}^n$, then $I(Y_1) \supseteq I(Y_2)$.

(iii) For any two subsets Y_1, Y_2 of $\mathbb{A}^2_1$, we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(iv) For any ideal $J \subseteq A$, $I(Z(J)) = \sqrt{J}$, where $\sqrt{J} := \{f \in A \colon f^r \in J \text{ for some } r > 0\}$ is the radical of J.

(v) For any subset $Y \subseteq \mathbb{A}^n$, $Z(I(Y)) = \overline{Y}$, the closure of Y in the Zariski topology.

Proof.

(i) If $P \in Z(T_2)$, then $f(P) = 0$ for all $f \in T_2$. But then $f(P) = 0$ for all $f \in T_1$, showing that $P \in Z(T_1)$.

(ii) If $f \in I(Y_2)$, then $f(P) = 0$ for all $P \in Y_2$. But then $f(P) = 0$ for all $P \in Y_1$, showing that $f \in I(Y_1)$.

(iii) We know that $f \in I(Y_1 \cup Y_2)$ if and only if $f(P) = 0$ for all $P \in Y_1 \cup Y_2$ if and only if $f(P) = 0$ for all $P ß n Y_1$ and $f(P) = 0$ for all $P \in Y_2$ if and only if $f \in I(Y_1)$ and $f \in I(Y_2)$ if and only if $f \in I(Y_1) \cap I(Y_2)$.

(iv) This follows immediately from Hilbert's Nullstellensatz, c.f. Theorem 1.1: $f \in \sqrt{J}$ if and only if $f^r \in J$ for some $r > 0$ if and only if $f^r(P) = 0$ for all $P \in Z(J)$, where the last "if and only if" follows from the definition of $Z(-)$ and the Nullstellensatz.

(v) We know that $Z(I(Y))$ is closed because it is algebraic. We know also that $Z(I(Y)) \supseteq Y$ because if $P \in Y$, then $f(P) = 0$ for all $f \in I(Y)$, showing $P \in Z(I(Y))$. Now if W is any closed set, say $W = Z(J)$, then $I(W) \supseteq J$ because if $f \in J$, then $f(P) = 0$ for all $P \in Z(J) = W$, showing $f \in I(W)$. Thus, $Z(I(W)) \subseteq Z(J) = W$. We know that $Y = W$ and $Z(I(W)) \supseteq W$. But then $W = Z(I(W))$.

Now suppose W is any closed subset containing Y. From (ii), we have $I(W) \subseteq I(Y)$. But then from (i), we know that $Z(I(W)) \supseteq Z(I(Y))$. Then from (iii), we know $W = Z(I(W))$ so that $W \supseteq Z(I(Y))$. Therefore, $Z(I(Y))$ is a closed set containing Y and contained in any closed set containing Y. Therefore, $Z(I(Y)) = \overline{Y}$. $\square$

As a reminder for the reader,

Theorem 1.1 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field, J be an ideal in $A = k[x_1, \ldots, x_n]$, and $f \in A$ be a polynomial vanishing at all points of $Z(J)$. Then $f^r \in J$ for some $r > 0$.*

Proof. See any 'standard' introductory text on Commutative Algebra. $\square$

Note for Hilbert's Nullstellensatz, one does need k to be algebraically closed, as the following examples show:

Example 1.5.

(i) If $k = \mathbb{R}$ and $J = (x^2 + x + 1) \subseteq \mathbb{R}[x]$, then $Z(J) = \emptyset$. Now 1 vanishes at all points of $\emptyset = Z(J)$, but no power of 1 is in $(x^2 + x + 1)$.

(ii) If $k = \mathbb{R}$ and $J = (x^2 + y^2) \subseteq \mathbb{R}[x, y]$, then $Z(J) = \{(0, 0)\}$. Now x vanishes at $(0, 0)$ but no power of x is in $(x^2 + y^2)$. $\triangleleft$

As a result, we can now see the connection between the the subsets of $\mathbb{A}^n$ and the subsets of $k[x_1, \ldots, x_n]$ that we had discussed earlier.

$$
\begin{array}{ccc}
\left\{ \text{ Subsets of } \mathbb{A}^n \right\} & \underset{Z(-)}{\overset{I(-)}{\rightleftarrows}} & \left\{ \begin{array}{c} \text{subsets of} \\ k[x_1, \ldots, x_n] \end{array} \right\} \\
\text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\
\left\{ \begin{array}{c} \text{Closed subsets} \\ \text{of } \mathbb{A}^n \end{array} \right\} & \underset{Z(-)}{\overset{I(-)}{\underset{\sim}{\rightleftarrows}}} & \left\{ \begin{array}{c} \text{Radical ideals} \\ \text{of } k[x_1, \ldots, x_n] \end{array} \right\}
\end{array}
$$

Corollary 1.1. *There is a one-to-one (inclusion-reversing) correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals in A given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. Furthermore, an algebraic set is irreducible if and only if its ideal is a prime ideal.*

Proof. All of this was shown in Proposition 1.4 with the exception of the final statement. Now suppose that Y is irreducible. We need show $I(Y)$ is prime. Suppose that $fg \in I(Y)$. Then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$. But then $Y = Y \cap Z(fg) = (Y \cap Z(f)) \cup (Y \cap Z(g))$, both of which are closed subsets of Y (in the subspace topology). Since Y is irreducible, Y must be one of the sets in this union. Without loss of generality, assume $Y = Y \cap Z(f)$. But then $f \in I(Y)$, as desired. Now suppose that $\mathfrak{p}$ is a prime ideal, and $Z(\mathfrak{p}) = Y_1 \cup Y_2$. Then we must have $\mathfrak{p} = I(Y_1) \cap I(Y_2)$. But then $\mathfrak{p} = I(Y_1)$ or $\mathfrak{p} = I(Y_2)$. Without loss of generality, assume $\mathfrak{p} = I(Y_1)$. Then we have $Z(\mathfrak{p}) = Z(I(Y_1)) = Y_1$. Therefore, $Z(\mathfrak{p})$ is irreducible. $\square$

This gives the most complete diagram as

$$
\begin{array}{ccc}
\left\{ \text{ Subsets of } \mathbb{A}^n \right\} & \underset{Z(-)}{\overset{I(-)}{\rightleftarrows}} & \left\{ \begin{array}{c} \text{subsets of} \\ k[x_1, \ldots, x_n] \end{array} \right\} \\
\text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\
\left\{ \begin{array}{c} \text{Closed subsets} \\ \text{of } \mathbb{A}^n \end{array} \right\} & \underset{Z(-)}{\overset{I(-)}{\underset{\sim}{\rightleftarrows}}} & \left\{ \begin{array}{c} \text{Radical ideals} \\ \text{of } k[x_1, \ldots, x_n] \end{array} \right\} \\
\text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\
\left\{ \begin{array}{c} \text{Subvarieties} \\ \text{of } \mathbb{A}^n \end{array} \right\} & \underset{Z(-)}{\overset{I(-)}{\underset{\sim}{\rightleftarrows}}} & \left\{ \begin{array}{c} \text{Prime ideals} \\ \text{of } k[x_1, \ldots, x_n] \end{array} \right\}
\end{array}
$$

Example 1.6.

(i) $\mathbb{A}^n$ is irreducible because it corresponds to the zero ideal in A, which is prime.

(ii) If f is an irreducible polynomial in $A = k[x, y]$ (a UFD), then f generates a prime ideal (irreducibles are primes in a UFD). Then it must be that $Y = Z(f)$ is irreducible. This is called the affine curve defined by the equation $f(x, y) = 0$. If the degree of f is d, we say that Y is a curve of degree d. Generally, if f is an irreducible polynomial in $A = k[x_1, \ldots, x_n]$, the affine variety $Y = Z(f)$ is called a hypersurface for n>3 (or simply a surface if $n = 3$).

(iii) A maximal ideal $\mathfrak{m}$ of $A = k[x_1, \ldots, x_n]$ corresponds to a minimal irreducible set (by 'inclusion-reversingness') of $\mathbb{A}^n$, which must be a point, say $P = (a_1, \ldots, a_n)$ is the corresponding point. Then every maximal ideal of A is of the form $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$. Note that we can express $\{P\} = Z(\langle x_1 - a_1, \ldots, x_n - a_n \rangle)$, $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$ is maximal as $k[x_1, \ldots, x_n] / \langle x_1 - a_1, \ldots, x_n - a_n \rangle \cong k$ is a field, and that this result requires that k be algebraically closed. $\triangleleft$

Note that polynomials in more than one variable can have infinitely many roots. For example, $f(x,y) = xy$ has zeros whenever $x = 0$ or $y = 0$. Clearly, $Z(xy) = \{(x,0)\ x \in \mathbb{R}\} \cup \{(0,y) \mid y \in \mathbb{R}\}$. But in the case where k is infinite, there will always exist a point where the polynomial does not have a root.

Proposition 1.5. *Let k be an infinite field and $0 \neq f \in k[x_1, \ldots, x_n]$. Then there exists a point $P \in \mathbb{A}^n$ such that $f(P) \neq 0$.*

Proof. We proceed by induction on n. The case where $n = 1$ is routine since any polynomial in a single variable has finitely many roots by degree considerations. Assume then that the result is true for n, and let $f \in k[x_1, \ldots, x_{n+1}]$. Write f as $a_m x_{n+1}^m + a_{m-1} x_{n+1}^{n-1} + \cdots + a_1 x_{n+1} + a_0$, where $a_i \in k[x_1, \ldots, x_n]$. Since f is nonzero, some $a_i \neq 0$. Recalling $k[x_1, \ldots, x_{n+1}] = k[x_1, \ldots, x_n][x_{n+1}]$ and using the induction hypothesis, there exists $Q = (b_1, \ldots, b_n)$, where $b_i \in k$, such that $a_i(Q) \neq 0$. Now $f(b_1, \ldots, b_n, x_{n+1})$ is a nonzero polynomial in a single variable, which can have only finitely many roots. Because k is infinite, there must then be $b_{n+1} \in k$ so that $f(b_1, \ldots, b_n, b_{n+1}) \neq 0$, as desired. □

Of course, this is not true when k is finite. For example, if $k = \{k_1, \ldots, k_{p^n}\}$, then $f(x) = \prod_{i=1}^{p^n} (x - k_i)$ is a nonzero polynomial with every element of k as a root.

Definition ((Affine) Coordinate Ring)**.** If $Y \subseteq \mathbb{A}^n$ is an algebraic set, we define the affine coordinate ring, denoted $A(Y)$, to be $A/I(Y)$.

One can think of elements of $A(Y)$ to be functions from $Y \to k$: if $Y \subseteq \mathbb{A}^n, f \in A = k[x_1, \ldots, x_n[$, then f certainly gives a function $f : \mathbb{A}^n \to k$ via restriction. Note that $f, g \in k[x_1, \ldots, x_n]$ induce the same function from $Y \to k$ if and only if $f - g \in I(Y)$. Hence, $A(Y)$ is the ring of all functions $f : Y \to k$ that are restrictions of polynomials.

Example 1.7. Let $Y \subseteq \mathbb{A}^2$ be the set given by $xy = 1$, i.e. $Z(xy - 1)$. It is routine to verify that $\sqrt{\langle xy - 1 \rangle} = \langle xy - 1 \rangle$. But then we have

$$k[x,y]/(xy - 1) = k[x, x^{-1}],$$

the ring of Laurent polynomials. ◁

Remark. We know that $A(Y)$ is an integral domain if and only if $I(Y)$ is prime if and only if Y is irreducible. Now $A(Y)$ is finitely generated as a k-algebra because it is a quotient of $k[x_1, \ldots, x_n]$, which is finitely generated as a k-algebra (one can take the image of the generators to be a generating set). On the other hand, any finitely generated k-algebra, say B, which is a domain is the affine coordinate ring of some variety. Why? Consider a map $\psi : k[x_1, \ldots, x_n] \to B$ is a surjection with kernel I. Then taking $Y = Z(I)$, we have $A(Y) = B$.

To study the topology of the affine varieties we have defined, we will need to introduce an important class of topological spaces, which is just the topological variant of its algebraic cousin.

Definition (Noetherian)**.** A topological space X is called noetherian if it satisfies the ascending chain condition on open subsets (every chain of open subsets stabilizes), i.e. for any chain of open subsets $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_r \subseteq \cdots$, there exists an n such that $U_n = U_{n+1} = \cdots$. Equivalently, a topological space X is noetherian if it satisfies the descending chain condition on closed subsets.

Example 1.8. The topological space $\mathbb{A}^n$, with the Zariski topology, is noetherian as given a set of closed subsets $Y_1 \supseteq Y_2 \supseteq \cdots$, applying $Y(-)$ (noting it is inclusion-reversing), we have $I(Y_1) \subseteq I(Y_2) \subseteq \cdots$. But this is an ascending chain of ideals in $A = k[x_1, \ldots, x_n]$. As A is noetherian (by the Hilbert Basis Theorem), the chain of ideals must stabilize, say at n. Finally as $Y_i = Z(I(Y_i))$, so too must the chain $Y_1 \supseteq Y_2 \supseteq \cdots$ stabilize at n. $\triangleleft$

Example 1.9. The topological space $\mathbb{R}^n$, with the usual Euclidean topology, is not noetherian. Let $\{a_n\}$ be a strictly monotone decreasing set with $a_n \to 0$. Letting Y_i be the closed sphere centered at the origin with radius a_i, we see that $\{Y_i\}$ is a chain of closed subsets that does not stabilize. $\triangleleft$

Proposition 1.6. *In a noetherian topological space, every nonempty set of closed subsets has a minimal element.*

Proof. Let S be a nonempty collection of closed subsets. Choose $Y_1 \in S_1$. If Y_1 is minimal, we are done. If not, choose $Y_2 \subseteq Y_1$. If Y_2 is minimal, we are done. If not, choose Y_3 such that $Y_3 \subseteq Y_2$. Continuing in this process, we construct a chain $Y_1 \supseteq Y_2 \subseteq Y_3 \supseteq \cdots$. Since the space is noetherian, this chain must stabilize, say at n. But then Y_n must be a minimal element of S. $\square$

This is equivalent to the algebraic property that every collection of submodules (or ideals) must contain a maximal element.

Proposition 1.7. *In a noetherian topological space X, every nonempty closed subset Y can be expressed a finite union $Y = Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets of Y. If requires $Y_i \supsetneq Y_j$ for $i \neq j$ (which is possible), the Y_i are uniquely determined up to reordering. In this case, the Y_i are called the irreducible components of Y.*