Syracuse University

MAT 830:
The McKay Correspondence

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0 Introduction

The M^{c}Kay Correspondence is an umbrella for a family of correspondences linking finite groups, resolutions of singularities of algebraic varieties, Lie Algebras, Character Theory, Invariant Theory, Representations of Quivers, and Cohen-Macaulay modules. It will not be our goal to see any particular connection in depth, but rather a surface level introduction to these correspondences generally, with a strong emphasis on examples.

The organizational scheme for the M^{c}Kay Correspondence is the Coxeter-Dynkin diagrams.

> *"The problem is to find a common origin of the A-D-E Classification Theorems, and to substitute a priori proofs to a posteriori verifications of the parallels of the classification.*
> – V.I. Arnold ,1976

The Coxeter-Dynkin ADE diagrams classify objects in each of the areas above, plus subadditive functions, root systems, Weyl groups, String Theory, Cluster Algebras, etc.. An example theorem from October demonstrating the M^{c}Kay Correspondence is the following

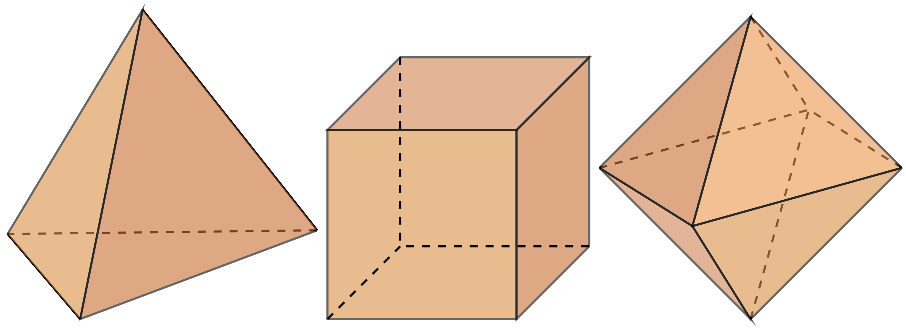
Theorem 0.1. *Let G be a small finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, acting linearly on $S = \mathbb{C}[x, y]$. Denote by $R = S^G$ the ring of invariants. Then there is a one-to-one correspondence between the following:*

- *Irreducible representations of G*
- *Indecomposable reflexive R-modules*
- *Irreducible Components of the exceptional fiber of minimal resolution of singularities of* $\operatorname{Spec} R$.

These correspondences extend to isomorphisms between

- *the M^{c}Kay Correspondence of G*
- *the Auslander-Reiten quiver of R*
- *the dual desingularization graph of* $\operatorname{Spec} R$.

1 Platonic Solids and Finite Groups of Matrices



Platonic solids are defined by the property that the faces are each convex and pairwise congruent. These five solids above are the only possible five solids satisfying these properties.

Proposition 1.1. *The solids above are the only possible Platonic solids.*

Proof. Suppose a solid has faces with p sides, and q meet at each vertex. We write this as a pair $\{p, q\}$, called the Schäfli symbol. The external angles of each face add to 2π radians. The internal angles are then $\pi - \frac{2\pi}{p}$. So around each vertex, we have

$$q\left(\pi - \frac{2\pi}{p}\right) < 2\pi.$$

If this were 2π, the solid would be flat, i.e. a tiling of the plane. If the angle were $> 2\pi$, the angle would be concave if and only if $\dfrac{2}{p} + \dfrac{2}{q} > 1$. The integer solutions are $\{p, 2\}$, $\{2, q\}$ or $\{3, 3\}$, $\{3, 4\}$, or $\{3, 5\}$. We can also use Euler's Formula $V - E + F = 1$ to work out V, E, F.

$\{p,q\}$	Name	F	E	V
$\{p,2\}$	dihedron	2	q	q
$\{2,q\}$	hosohedron	p	p	2
$\{3,3\}$	tetrahedron	4	6	4
$\{3,4\}$	octahedron	8	12	6
$\{4,3\}$	cube	6	12	8
$\{3,5\}$	icosahedron	20	30	12
$\{5,3\}$	dodecahedron	12	30	20

$\square$

Notice that the above proof is merely a 'uniqueness proof' but does *not* show existence of these solids. We shall prove existence by classifying their (rotational) symmetry groups. Observe that dual pairs[1] of polyhedra have the same rotational symmetry groups. We shall find

- the rotational symmetries of a dihedron of hosohedron is D_{2k}.
- tetrahedral group $\mathbf{T}$ of 12 rotational symmetries of a tetrahedron.
- the octahedral group, $\mathbb{O}$, has 24 rotational symmetries of the octahedron.
- the icoahedral group $\mathcal{I}$ of 60 rotational symmetries of the icosahedron/dodecahedron

Furthermore, these will all be familiar groups.

Theorem 1.1. *Along with the degenerate case of the cyclic group C_k for any $k \geq 1$ corresponding to rotation of $\mathbb{R}^3$ by $\dfrac{2\pi}{k}$, these are all of the finite groups of rotations of $\mathbb{R}^3$.*

1.1 Matrix Groups

To classify the finite groups of rotational symmetries, recall the following definitions

Definition (Orthogonal Group)**.** The orthogonal group, $O(n)$, is the set of all invertible orthogonal matrices, i.e. $O(n) := \{A \in \mathrm{GL}_n(\mathbb{R})\colon AA^T = I_n\}$.

A routine exercise verifies that the orthogonal group is equivalent to $\{A\colon |Ax| = |x| \text{ for all } x \in \mathbb{R}^n\} = \{A\colon Ax \cdot Ay = x \cdot y, \text{ for all } x, y \in \mathbb{R}^n\} = \{A\colon \text{ rows of } A \text{ form orthonormal basis for } \mathbb{R}^n\} = \{A\colon \text{ columns of } A \text{ form orthonormal basis for } \mathbb{R}^n\} = \{\text{ set of linear isometries of } \mathbb{R}^n\}$. Generally, if $\langle \cdot\ ,\ \cdot \rangle$ is a symmetric bilinear form, then you can define the orthogonal group of the form $\langle \cdot\ ,\ \cdot \rangle$ to be $\{A \in \mathrm{GL}(\mathbb{R})\colon \langle Ax\ ,\ Ay \rangle = \langle x\ ,\ y \rangle, \text{ for all } x, y \in \mathbb{R}\}$.

Recall also that $AA^T = I_n$ so that $\det(AA^T) = 1$, $\det(A)^2 = 1$ and then $\det A = \pm 1$.

Definition (Special Orthogonal Group)**.** The special orthogonal group, $\mathrm{SO}(n)$, is the subgroup of $O(n)$ of matrices having determinant 1, i.e. $\mathrm{SO}(n) := \{A \in \mathrm{GL}_n(\mathbb{R})\colon AA^T = I_n, \det A = 1\}$.

Example 1.1. The case of $n = 1$ is dull. The case of $n = 2$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(n) \Rightarrow \text{ columns ar orthogonal}$$

$$\begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = 0$$

[1]The dual of a polyhedron P has a vertex at the center of each face of P, and two vertices joined by an edge if the faces abut each other.

But we know that $\begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} c \\ -a \end{pmatrix} = 0$ so $\begin{pmatrix} b \\ d \end{pmatrix}$ must be a multiple of $\begin{pmatrix} c \\ -a \end{pmatrix}$. But these are also unit vectors so the multiplier is ± 1. So either

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ or } \begin{pmatrix} a & c \\ c & -a \end{pmatrix}$$

Also, $a^2 + c^2 = 1$, so we can find an angle $\theta \in [0, 2\pi)$ so that $a = \cos\theta$ and $c = \sin\theta$. But then

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

However, the left is in $\mathrm{SO}(2)$ and the other has determinant -1. Note that A is rotation by θ counterclockwise and the second matrix is reflection across the line at angle $\theta/2$.

Theorem 1.2. *The finite subgroups of linear isometries of the plane are cyclic or dihedral.*

Proof. (Sketch) If $G \subsetneq \mathrm{SO}(2)$ is a finite group, then G consists only of rotations by the work in the example. One can easily check that G is generated by the rotation with smallest positive angle. If $G \subseteq O(2) \setminus \mathrm{SO}(2)$, then G must contain a reflection B, and $G \cap \mathrm{SO}(2) = \langle A \rangle$ will be cyclic. One easily verifies that $G = \{I, A, \ldots, A^{n-1}, B, BA, \ldots, BA^{n-1}\}$. $\square$

Corollary 1.1. $\mathrm{SO}(2)$ *consists of the rotations of* $\mathbb{R}^2$.

Definition (Rotation of $\mathbb{R}^n$)**.** A rotation of $\mathbb{R}^n$ is a linear map ϕ on $\mathbb{R}^n$ so that ϕ fixes a line through the origin, and $\phi\big|_{\ell^\perp}$ is a rotation of the subspace orthogonal to ℓ.

Note that $\mathrm{SO}(3) = \{$ rotations of $\mathbb{R}^3\}$ but $\mathrm{SO}(n)$ strictly contains the rotations of $\mathbb{R}^4$ for $n \geq 4$.

Definition (Rotation)**.** Recall a linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ is called a rotation if

- T fixes a unit vector p, called a *pole*.
- The restriction of T to $(\mathrm{span}(p))^\perp \cong \mathbb{R}^2$ is a rotation of $\mathbb{R}^2$.

Theorem 1.3 (Euler's Theorem)**.**

$$\mathrm{SO}(3) = \{\textit{rotation of } \mathbb{R}^3\}$$

In particular, the composition of two rotations of $\mathbb{R}^3$ *is another rotation.*

Proof. Suppose T is a rotation, we can find a basis for $\mathbb{R}^3$ of the form $\{p, x_1, x_2\}$, where p a pole for T and x_1, x_2 is a basis for $\mathbb{R}^2 = (\mathrm{span}(p))^\perp$. With respect to this basis,

$$[T] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(3).$$

Now let $A \in \mathrm{SO}(3)$. We need find a pole for A, i.e. A fixes a nonzero vector. But then A must have 1 as an eigenvector. Using the fact that $\det A = 1$, we have

$$\begin{aligned}
\det(A - I) &= \det(A)\det(A - I) \\
&= \det(A^T)\det(A - I) \\
&= \det(A^T A - A^T) \\
&= \det(I - A^T) \\
&= \det(I - A) \\
&= \det(-(A - I)) \\
&= (-1)^3 \det(A - I) \\
&= -\det(A - I).
\end{aligned}$$

Therefore, $\det(A - I) = 0$. But then A has a unit eigenvector, say p. The restriction of A to $(\mathrm{span}(p))^{\perp}$ still preserves the dot product, and so is a rotation by the $\mathrm{SO}(2)$ case. □

Theorem 1.4. *The finite subgroups of* $\mathrm{SO}(3)$ *are cyclic, dihedral, or the group of rotational symmetries of a tetrahedron, an octahedron, or an icosahedron.*

Proof. Let $G \subseteq \mathrm{SO}(3)$ be finite with $|G| = N > 1$. Set $P = \{p \in \mathbb{R}^3 \colon p \text{ pole of some } 1 \neq g \in G\} = \{p \in \mathbb{R}^3 \colon |p| = 1, gp = g \text{ for some } g \neq 1\}$. We claim that G acts on P, i.e. if $p \in P$, $g \in G$, then $gp \in P$. If p is a pole of $h \in G$, then $(ghg^{-1})(gp) = gp$ since h fixes p. But then gp is a pole of ghg^{-1}. Observe each $1 \neq g \in G$ has two poles, so $|P| < \infty$. For $p \in P$, let $G_p := \mathrm{stab}_p = \{g \in G \colon p \text{ pole of } g\} \cup \{1_G\}$.

Now G_p is the set of all rotations with pole p, and G_p is cyclic by the $n = 2$ case. Furthermore, G_p is generated by the smallest nonzero rotation. Let $r_p := |G_p|$, and $n_p := |O_p|$, where O_p is the orbit of p. So $r_p n_p = |G|$ by the Orbit-Stabilizer Theorem. We count pairs (p, g), where p is a pole of $g \neq 1$. Now each g has two poles so

$$|\{(p, g) \colon p \in P, gp = p, g \neq 1\}| = 2(N - 1) = \sum_{p \in P} r_p - 1$$

where the last equality follows since G_p is the set of g's with pole p. Replacing r_p with N/n_p, we obtain

$$2N - 2 = \sum_{p \in P} \frac{N}{n_p} - 1 = \sum_{\text{orbits } O_p} n_p \left(\frac{N}{n_p} - 1\right) = \sum_{\text{orbits } O_p} N - \frac{N}{r_p}.$$

But then we have

$$2 - \frac{2}{N} = \sum_{i=1}^{k} \left(1 - \frac{1}{r_i}\right),$$

where we have labeled the orbits $O_1, \ldots, O_k$ and $r_i = |G_{p_i}|$. This equation is known as Lüroth's Equation.

Lüroth's equation implies that $k \leq 3$, as each term on the right hand side is at least $1/2$ and the left hand side is less than 2. If $k = 1$, then there is a unique orbit or poles and thus $2 - \frac{2}{N} = 1 - \frac{1}{r}$. But the left hand side is at least 1, while the right hand side is less than 1, a contradiction. Now if $k = 2$, then there are two orbits of poles so that

$$\left(1 - \frac{1}{r_1}\right) + \left(1 - \frac{1}{r_2}\right) = 2 - \frac{2}{N} \iff \frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{N} \iff n_1 + n_2 = 2,$$

where the last equivalence follows since $r_i n_i = N$, where $n_i = |O_i|$. But then each orbit is a singleton set, implying there are two poles. Furthermore, $r_i = N$ for $i = 1, 2$ so that every group element fixes both poles. Then $G \cong C_N$ by the $n = 2$ case. In the case of $k = 3$, we have

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1 + \frac{2}{N} > 1.$$

The number of algebraic possibilities are limited. We can assume $r_1 \leq r_2 \leq r_3$ and so $r_1 < 3$. The solutions are $(2, 2, k)$ for any $k \geq 2$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$. We can construct the polyhedron in each case.

- $(2, 2, k)$: We have $\frac{1}{2} + \frac{1}{2} + \frac{1}{k} = 1 + \frac{2}{N}$, so $N = 2k$. But then there are two orbits of size k and one of size 2, say $O_3 = \{p, p'\}$. Half the elements of G, i.e. k elements, fix p and p', while the remaining elements swap the elements. But then $G \cong D_k$.

- $(2, 3, 5)$: We have $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} = 1 + \frac{2}{N}$ so that $N = 60$. The orbits have sizes 30, 20, and 12. Let $V = O_3$ be the orbit of size 12. Choose $p \in V$ to be the north pole, and let $H = G_p$ be the stabilizer of p. We have $|H| = \frac{60}{12} = 5$. In particular, H is cyclic with order 5. Now H (with order 5) acts on V (of order 12), fixing p and $-p$. But then the orbits have size 1, 1, 5, and 5. Now V is the set of vertices of the icosahedron.

- The cases of $(2, 3, 3)$ and $(2, 3, 4)$ are handled the same as the case above.

$\square$

Corollary 1.2. *The finite subgroups of* $\mathrm{SO}(3)$ *have presentations*

$$\begin{aligned}
\mathbb{C}_n &= \langle x \mid x^n = 1 \rangle \\
\mathbb{D}_n &= \langle x, y \mid x^2 = y^n = (xy)^2 = 1 \rangle \\
\mathbb{T} &= \langle x, y \mid x^2 = y^3 = (xy)^3 = 1 \rangle \\
\mathbb{O} &= \langle x, y \mid x^2 = y^3 = (xy)^4 = 1 \rangle \\
\mathbb{I} &= \langle x, y \mid x^2 = y^3 = (xy)^5 = 1 \rangle
\end{aligned}$$

Proof. (Sketch) Suppose we have the Schäfli symbol $\{p, q\}$, i.e. each face has p sides, and q meet at each vertex. Fix a vertex, and let τ be rotation by $2\pi/q$ around this vertex. But then $|\tau| = q$. Also, fix an edge incident to our vertex, and let σ be the rotation swapping the ends of this edge. Then $|\sigma| = 2$.

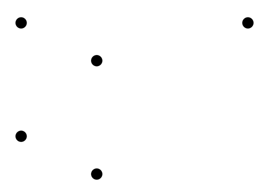
Focus on the face fo the right of our edge, and consider $\sigma\tau$. This rotates the face by $2\pi//p$. So we must have $|\sigma\tau| = p$, and we have elements σ, τ satisfying $\sigma^2 = \tau^q = (\sigma\tau)^p = 1$. One needs to check that σ, τ generate G and that

$$|\langle x, y \mid x^2 = y^q = (xy)^p = 1\rangle| = |G|.$$

$\square$

Corollary 1.3. *We have isomorphisms* $\mathbb{T} \cong A_4$, $\mathbb{O} \cong S_4$, $\mathbb{I} \cong A_5$.

Note that the group $\langle x, y \mid x^r = y^s = (xy)^t = 1\rangle$ is *only* finite in the cases above. Associate to this the graph $T_{r,s,t}$, shown below.



with total $+s + t - 2$ vertices.

$C_n : (n, 1, n) givehorizontallinewithdots, thedynkin$ A_{2n-1} $D_n : (2, n, 2)$ D_{n+1} $T : (2, 3, 3)$ E_6 $O(2, 3, 4)$ E_7 $I(2, 3, 5)$ E_8

These are the ADE Coxeter-Dynkin diagrams.

Our next goal is to classify the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. We travel from $\mathrm{SO}(3)$, to $\mathrm{SU}(2)$, onto $\mathrm{SL}_2(\mathbb{C})$.

Definition (Unitary Group). $U(n) := \{A \in \mathrm{GL}_n(\mathbb{C})\colon A^*A = I_n\}$, where $A^* = \overline{A^T}$. But this is also $\{A\colon |Ax| = |x|\}$ Euclidean norm $= \{A\colon (Ax)^*(Ay) = x^*y\}$ i.e. A preserves the Hermitian inner product $\langle x, y\rangle = x^*y$. That is the $\{A\colon$ rows/col of A are orthogonal basis for $\mathbb{C}^n\}$.

As before $U(n)$ can be described as the set of matrices preserving an arbitrary Hermitian inner product.

Definition (Special Unitary Group). $\mathrm{SU}(n) := \{A \in U(n)\colon \det A = 1\}$.

Lemma 1.1. *Every finite subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *is conjugate to a subgroup of* $U(n)$*. In particular, every subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *is conjugate to a subgroup of* $\mathrm{SL}_n(\mathbb{C})$ *and* $\mathrm{SU}(n)$*.*

Proof. We construct a new Hermitian inner product on $\mathbb{C}^n$ so that a given finite group G preserves the product. Define $\langle u, v\rangle = \frac{1}{|G|}\sum_{g\in G}(gu)^*(gv)$. Then for any $h \in G$, $\langle hu, hv\rangle = \frac{1}{|G|}\sum_{g\in G}(ghu)^*(ghv) = \frac{1}{|G|}\sum_{k\in G}(ku)^*(kv) = \langle u, v\rangle$.