Syracuse University

MAT 830:
The McKay Correspondence

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0 Introduction

The M^{c}Kay Correspondence (pronounced mc-eye) is an umbrella for a family of correspondences linking finite groups, resolutions of singularities of algebraic varieties, Lie Algebras, Character Theory, Invariant Theory, Representations of Quivers, and Cohen-Macaulay modules. It will not be our goal to see any particular connection in depth, but rather a surface level introduction to these correspondences generally, with a strong emphasis on examples.

> *"The problem is to find a common origin of the ADE Classification Theorems, and to substitute a priori proofs to a posteriori verifications of the parallels of the classification.*
> – V.I. Arnold ,1976

The organizational scheme for the M^{c}Kay Correspondence is the Coxeter-Dynkin diagrams. The Coxeter-Dynkin ADE diagrams classify objects in each of the areas above, plus subadditive functions, root systems, Weyl groups, String Theory, Cluster Algebras, etc.. An example theorem demonstrating the M^{c}Kay Correspondence is the following, due to M^{c}Kay , Auslander, Reiten, Artin, Verdier, Gonzalzez-Springber, Herzog, et al.,

Theorem 0.1. *Let G be a small finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, acting linearly on $S = \mathbb{C}[x, y]$. Denote by $R = S^G$ the ring of invariants. Then there is a one-to-one correspondence between the following:*

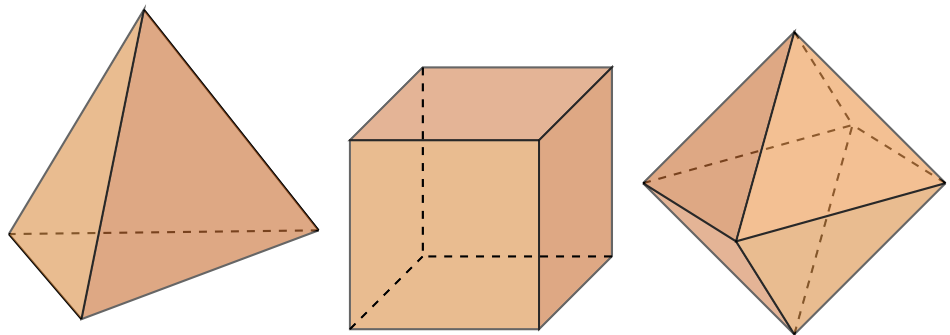
- *Irreducible representations of G*
- *Indecomposable reflexive R-modules*
- *Irreducible Components of the exceptional fiber of minimal resolution of singularities of* Spec *R.*

These correspondences extend to isomorphisms between

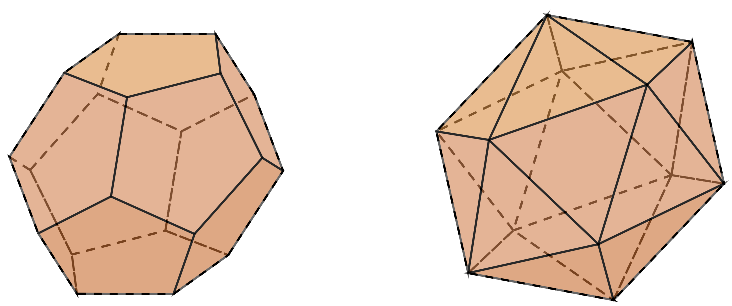
- *the M^{c}Kay Correspondence of G*
- *the Auslander-Reiten quiver of R*
- *the dual desingularization graph of* Spec *R.*

1 Platonic Solids and Finite Groups of Matrices

A Platonic solid is a regular, convex polyhedron, constructed by congruent regular polygonal faces with the same number of faces meeting at each vertex; that is, the platonic solids are defined by the property that the faces are each convex and pairwise congruent. The solids shown below are the only five solids satisfying these properties.



Tetrahedron Cube Octahedron



Dodecahedron Icosahedron

Proposition 1.1. *The solids above are the only possible Platonic solids.*

Proof. Suppose a solid has faces with p sides and q faces meeting at each vertex. We write this as a pair $\{p, q\}$, called the Schläfli symbol. The external angles of each face add to 2π radians, as is the case with any convex polygon. Each exterior angle is then $\frac{2\pi}{p}$ radians. The internal angles are then $\pi - \frac{2\pi}{p}$. So around each vertex, the sum of the angles is $q(\pi - \frac{2\pi}{p})$. This angle cannot be larger than 2π as the faces are concave if and only if $\dfrac{2}{p} + \dfrac{2}{q} > 1$, and we require convex faces. Furthermore, this sum cannot be 2π for then the solid would be flat, i.e. a tiling of the plane. Therefore, we have the relation

$$q\left(\pi - \frac{2\pi}{p}\right) < 2\pi.$$

$\{p,q\}$	Name	F	E	V
$\{p,2\}$	dihedron	2	q	q
$\{2,q\}$	hosohedron	p	p	2
$\{3,3\}$	tetrahedron	4	6	4
$\{3,4\}$	octahedron	8	12	6
$\{4,3\}$	cube	6	12	8
$\{3,5\}$	icosahedron	20	30	12
$\{5,3\}$	dodecahedron	12	30	20

The integer solutions are $\{p,2\}$, $\{2,q\}$ or $\{3,3\}$, $\{3,4\}$, or $\{3,5\}$. Then Euler's Formula $V - E + F = 1$ allows one to compute V, E, F, as found in the table. $\square$

Notice that the above proof is merely a uniqueness proof and does *not* show the existence of these solids. We shall prove existence by classifying the rotational symmetry groups of these solids. We shall find

- the dihedral group, D_{2k}, of the symmetries of the dihedron/hosohedron.
- the tetrahedral group, $\mathbb{T}$, of the 12 rotational symmetries of a tetrahedron.
- the octahedral group, $\mathbb{O}$, of the 24 rotational symmetries of the octahedron.
- the icoahedral/dodecahedral group, $\mathbb{I}$, of 60 the rotational symmetries of the icosahedron/dodecahedron

Note that dual pairs[1] of polyhedra have the same rotational symmetry groups. Furthermore, these will all be familiar groups. For example, we shall find $\mathbb{T} \cong A_4$ and $\mathbb{O} \cong S_4$. This follows from the fact that symmetries of the faces of one polyhedron correspond to symmetries of the centers of their faces, and vice versa. For instance, the cube and the octahedron both have the same symmetry group, $\mathbb{O} \cong S_4$, because they are dual. From the table in Proposition 1.1, we can see that the dihedron and hosohedron are dual, the octahedron and cube are dual, the icosahedron and dodecahedron are dual, and the tetrahedron is self-dual. Finally, the groups D_{2k}, $\mathbb{T}$, $\mathbb{O}$, and $\mathbb{I}$, along with the cyclic groups C_n for $n \in \mathbb{N}$, are the *only* finite groups of rotations of $\mathbb{R}^3$.

Theorem 1.1. *Along with the degenerate case of the cyclic group C_k for any $k \geq 1$ corresponding to rotation of $\mathbb{R}^3$ by $\dfrac{2\pi}{k}$, the groups D_{2k}, $\mathbb{T}$, $\mathbb{O}$, and $\mathbb{I}$ are all of the finite groups of rotations of $\mathbb{R}^3$.*

[1] The dual of a polyhedron P has a vertex at the center of each face of P, and two vertices joined by an edge if the faces abut each other.

1.1 Matrix Groups

To classify the finite groups of rotational symmetries, we begin by recalling a few definitions.

Definition (Orthogonal Group)**.** The orthogonal group, $O(n)$, is the set of all invertible orthogonal matrices, i.e. $O(n) := \{A \in \mathrm{GL}_n(\mathbb{R})\colon AA^T = I_n\}$.

Example 1.1. The following are all orthogonal matrices:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad B = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

◁

A routine exercise verifies that the orthogonal group is also equivalent to any of the following:

$$\begin{aligned} O(n) &:= \{A \in \mathrm{GL}_n(\mathbb{R})\colon AA^T = I_n\} \\ &= \{A\colon |Ax| = |x| \text{ for all } x \in \mathbb{R}^n\} \\ &= \{A\colon Ax \cdot Ay = x \cdot y, \text{ for all } x, y \in \mathbb{R}^n\} \\ &= \{A\colon \text{ rows of } A \text{ form orthonormal basis for } \mathbb{R}^n\} \\ &= \{A\colon \text{ columns of } A \text{ form orthonormal basis for } \mathbb{R}^n\} \\ &= \{\text{set of linear isometries of } \mathbb{R}^n\}. \end{aligned}$$

Note that we have defined $O(n)$ in terms of the symmetric bilinear form $\langle A, B\rangle = AB^T$. Generally, if $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form then you can define the orthogonal group of the form $\langle \cdot, \cdot \rangle$ to be $\{A \in \mathrm{GL}(\mathbb{R})\colon \langle Ax, Ay\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbb{R}\}$. Observe also that from the relation $AA^T = I_n$, we obtain $\det(AA^T) = 1$. Recall that $\det(AB) = \det(A)\det(B)$, and $\det(A) = \det(A^T)$. It then follows that $\det(A)^2 = 1$, and then $\det A = \pm 1$. A special subset of these matrices are our next group of interest.

Definition (Special Orthogonal Group)**.** The special orthogonal group, $\mathrm{SO}(n)$, is the subgroup of $O(n)$ of matrices having determinant 1, i.e. $\mathrm{SO}(n) := \{A \in \mathrm{GL}_n(\mathbb{R})\colon AA^T = I_n, \det A = 1\}$.

Example 1.2. The case of $n = 1$ is dull, consisting only of the identity matrix. The case of $n = 2$ is a bit more interesting. Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SO}(2) \subseteq O(2)$. Since $A \in O(2)$, we know that the columns of A are orthogonal, giving

$$\begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix}^T = 0.$$

A simple calculation shows that $\begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} c \\ -a \end{pmatrix}^T = 0$. But then by linear independence, $\begin{pmatrix} b \\ d \end{pmatrix}$ must be a multiple of $\begin{pmatrix} c \\ -a \end{pmatrix}$. But these are also unit vectors, so the multiplier is ± 1. This gives two possible cases for A:

$$A = \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \text{ or } \begin{pmatrix} a & c \\ c & -a \end{pmatrix}.$$

Since $A \in \mathrm{SO}(2)$, we know that $\det A = 1$. This gives $a^2 + c^2 = 1$. Then we can find an angle $\theta \in [0, 2\pi)$ so that $a = \cos\theta$ and $c = \sin\theta$. Using this in our possibilities above, we have

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

While the left matrix is an element of SO(2), the other has determinant -1. Therefore, we must have $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, a rotation by θ counterclockwise about the origin. The second matrix on the right above corresponds to the reflection across the line at angle $\theta/2$ through the origin. Therefore, the group SO(2) is precisely the group of rotations in the plane. Similarly, SO(3) is the group of rotations for three-dimensional space, see Theorem 1.3. $\triangleleft$

Definition (Rotation of $\mathbb{R}^n$). Let $n > 2$. A rotation of $\mathbb{R}^n$ is a linear map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

- ϕ fixes a line ℓ through the origin
- $\phi\big|_{\ell^\perp}$ is a rotation of the subspace orthogonal to ℓ

Sometimes the definition of rotations of $\mathbb{R}^n$ is instead given as follows: a rotation of $\mathbb{R}^n$ is a linear operator if T fixes a unit vector p, called a pole, and the restriction of T to $(\mathrm{span}(p))^\perp \cong \mathbb{R}^2$ is a rotation of $\mathbb{R}^2$. We will make use of this alternate definition later. For now, we shall prove that the finite subgroups of linear isometries of the plane are a cyclic group or a dihedral group.

Theorem 1.2. *The finite subgroups of linear isometries of the plane are a cyclic group or a dihedral group.*

Proof (Sketch). If $G \subseteq \mathrm{SO}(2)$ is a finite group, then G consists only of rotations by Example 1.2. One can check that G is generated by the rotation with smallest positive angle. Now if $G \subseteq O(2) \setminus \mathrm{SO}(2)$, then G must contain a reflection B, and $G \cap \mathrm{SO}(2) = \langle A \rangle$ will be cyclic. One then verifies that $G = \{I, A, \ldots, A^{n-1}, B, BA, \ldots, BA^{n-1}\}$. $\square$

Corollary 1.1. $\mathrm{SO}(2)$ *consists of the rotations of* $\mathbb{R}^2$.

Proof. By Example 1.2, we know that $\mathrm{SO}(2)$ is contained in the group of rotations of $\mathbb{R}^2$. Since every rotation of $\mathbb{R}^2$ can be represented by a matrix in the form of $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, the containment holds in the other direction. $\square$

Note that $\mathrm{SO}(3) = \{\text{rotations of } \mathbb{R}^3\}$ but $\mathrm{SO}(n)$ strictly contains the rotations of $\mathbb{R}^n$ for $n \geq 4$. We now come to yet another theorem of Euler.

Theorem 1.3 (Euler's Theorem)**.**

$$\mathrm{SO}(3) = \{\textit{rotation of } \mathbb{R}^3\}$$

In particular, the composition of two rotations of $\mathbb{R}^3$ *is another rotation.*

Proof. Suppose T is a rotation. We can find a basis for $\mathbb{R}^3$ of the form $\mathcal{B} := \{p, x_1, x_2\}$, where p a pole for T and $\{x_1, x_2\}$ is a basis for $\mathbb{R}^2 = (\mathrm{span}(p))^\perp$. With respect to the basis $\mathcal{B}$,

$$[T]_\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(3).$$

Now let $A \in \mathrm{SO}(3)$. We need find a pole for A, i.e. a nonzero vector fixed by A. If this were the case, then A must have 1 as an eigenvector. Using the fact that $\det A = 1$, we have

$$\begin{aligned}
\det(A - I) &= \det(A)\det(A - I) \\
&= \det(A^T)\det(A - I) \\
&= \det(A^T A - A^T) \\
&= \det(I - A^T) \\
&= \det(I - A) \\
&= \det(-(A - I)) \\
&= (-1)^3 \det(A - I) \\
&= -\det(A - I).
\end{aligned}$$

Therefore, $\det(A - I) = 0$. But then A has a unit eigenvector, say p. The restriction of A to $(\mathrm{span}(p))^{\perp}$ still preserves the dot product, and so A is a rotation of $\mathbb{R}^3$ by the SO(2) case. □

Theorem 1.4. *The finite subgroups of* SO(3) *are cyclic, dihedral, or the group of rotational symmetries of a tetrahedron, an octahedron, or an icosahedron (the symmetry groups of the Platonic solids).*

Proof. Let $G \subseteq \mathrm{SO}(3)$ be finite with $|G| = N > 1$, and define $P = \{\vec{p} \in \mathbb{R}^3 \colon \vec{p} \text{ pole of some } 1 \neq g \in G\} = \{\vec{p} \in \mathbb{R}^3 \colon |\vec{p}| = 1, g\vec{p} = g \text{ for some } g \neq 1\}$. We claim that G acts on P, i.e. if $\vec{p} \in P$, $g \in G$, then $g\vec{p} \in P$. If $\vec{p}$ is a pole of $h \in G$, then $(ghg^{-1})(g\vec{p}) = g\vec{p}$ since h fixes $\vec{p}$. But then $g\vec{p}$ is a pole of ghg^{-1}. Observe each $1 \neq g \in G$ has two poles, so $|P| < \infty$. For $\vec{p} \in P$, let $G_{\vec{p}} := \mathrm{stab}_{\vec{p}} = \{g \in G \colon \vec{p} \text{ pole of } g\} \cup \{1_G\}$.

Now $G_{\vec{p}}$ is the set of all rotations with pole $\vec{p}$, and $G_{\vec{p}}$ is cyclic by the $n = 2$ case. Furthermore, $G_{\vec{p}}$ is generated by the smallest nonzero rotation. Let $r_{\vec{p}} := |G_{\vec{p}}|$, and $n_{\vec{p}} := |O_{\vec{p}}|$, where $O_{\vec{p}}$ is the orbit of $\vec{p}$. So $r_{\vec{p}} n_{\vec{p}} = |G|$ by the Orbit-Stabilizer Theorem. We count pairs $(\vec{p}, g)$, where $\vec{p}$ is a pole of $g \neq 1$. Now each g has two poles so

$$|\{(\vec{p}, g) \colon \vec{p} \in P, g\vec{p} = \vec{p}, g \neq 1\}| = 2(N-1) = \sum_{\vec{p} \in P} r_{\vec{p}} - 1$$

where the last equality follows since $G_{\vec{p}}$ is the set of g's with pole $\vec{p}$. Replacing $r_{\vec{p}}$ with $N/n_{\vec{p}}$, we obtain

$$2N - 2 = \sum_{\vec{p} \in P} \frac{N}{n_{\vec{p}}} - 1 = \sum_{\text{orbits } O_{\vec{p}}} n_{\vec{p}} \left(\frac{N}{n_{\vec{p}}} - 1 \right) = \sum_{\text{orbits } O_{\vec{p}}} N - \frac{N}{r_{\vec{p}}}.$$

But then we have

$$2 - \frac{2}{N} = \sum_{i=1}^{k} \left(1 - \frac{1}{r_i} \right),$$

where we have labeled the orbits $O_1, \ldots, O_k$ and $r_i = |G_{\vec{p}_i}|$. This equation is known as Lüroth's Equation.

Lüroth's equation implies that $k \leq 3$, as each term on the right hand side is at least $1/2$ and the left hand side is less than 2. If $k = 1$, then there is a unique orbit or poles and thus $2 - \frac{2}{N} = 1 - \frac{1}{r}$. But the left hand side is at least 1, while the right hand side is less than 1, a contradiction. Now if $k = 2$, then there are two orbits of poles so that

$$\left(1 - \frac{1}{r_1}\right) + \left(1 - \frac{1}{r_2}\right) = 2 - \frac{2}{N} \iff \frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{N} \iff n_1 + n_2 = 2,$$

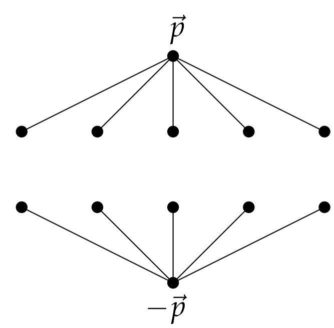
where the last equivalence follows since $r_i n_i = N$, where $n_i = |O_i|$. But then each orbit is a singleton set, implying there are two poles. Furthermore, $r_i = N$ for $i = 1, 2$ so that every group element fixes both poles. Then $G \cong C_N$ by the $n = 2$ case. In the case of $k = 3$, we have

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1 + \frac{2}{N} > 1.$$

The number of algebraic possibilities are limited. We can assume $r_1 \le r_2 \le r_3$ and so $r_1 < 3$. The solutions are $(2, 2, k)$ for any $k \ge 2$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$. We can construct the polyhedron in each case.

- $(2, 2, k)$: We have $\frac{1}{2} + \frac{1}{2} + \frac{1}{k} = 1 + \frac{2}{N}$, so $N = 2k$. But then there are two orbits of size k and one of size 2, say $O_3 = \{\vec{p}, \vec{p}'\}$. Half the elements of G, i.e. k elements, fix $\vec{p}$ and $\vec{p}'$, while the remaining elements swap the elements. But then $G \cong D_k$.

- $(2, 3, 5)$: We have $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} = 1 + \frac{2}{N}$ so that $N = 60$. The orbits have sizes 30, 20, and 12. Let $V = O_3$ be the orbit of size 12. Choose $p \in V$ to be the north pole, and let $H = G_{\vec{p}}$ be the stabilizer of p. We have $|H| = \frac{60}{12} = 5$. In particular, H is cyclic with order 5. Now H (with order 5) acts on V (of order 12), fixing $\vec{p}$ and $-\vec{p}$. But then the orbits have size 1, 1, 5, and 5. Now V is the set of vertices of the icosahedron.



- The cases of $(2, 3, 3)$ and $(2, 3, 4)$ are handled the same as the case above.

□

Corollary 1.2. *The finite subgroups of* SO(3) *have presentations*

$$\begin{aligned}
\mathbb{C}_n &= \langle x \mid x^n = 1 \rangle \\
\mathbb{D}_n &= \langle x, y \mid x^2 = y^n = (xy)^2 = 1 \rangle \\
\mathbb{T} &= \langle x, y \mid x^2 = y^3 = (xy)^3 = 1 \rangle \\
\mathbb{O} &= \langle x, y \mid x^2 = y^3 = (xy)^4 = 1 \rangle \\
\mathbb{I} &= \langle x, y \mid x^2 = y^3 = (xy)^5 = 1 \rangle
\end{aligned}$$

Proof (Sketch). Suppose we have the Schäfli symbol $\{p, q\}$, i.e. each face has p sides, and q meet at each vertex. Fix a vertex, and let τ be rotation by $2\pi/q$ around this vertex. But then $|\tau| = q$. Also, fix an edge incident to our vertex, and let σ be the rotation swapping the ends of this edge. Then $|\sigma| = 2$.

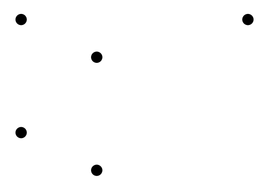
Focus on the face fo the right of our edge, and consider $\sigma\tau$. This rotates the face by $2\pi//p$. So we must have $|\sigma\tau| = p$, and we have elements σ, τ satisfying $\sigma^2 = \tau^q = (\sigma\tau)^p = 1$. One needs to check that σ, τ generate G, and that

$$|\langle x, y \mid x^2 = y^q = (xy)^p = 1\rangle| = |G|.$$

$\square$

Corollary 1.3. *We have isomorphisms* $\mathbb{T} \cong A_4$, $\mathbb{O} \cong S_4$, $\mathbb{I} \cong A_5$.

Note that the group $\langle x, y \mid x^r = y^s = (xy)^t = 1\rangle$ is *only* finite in the cases above. Associate to this the graph $T_{r,s,t}$, shown below.



with total $+s + t - 2$ vertices.

$C_n : (n, 1, n) givehorizontallinewithdots, thedynkin$ A_{2n-1} $D_n : (2, n, 2)$ D_{n+1} $T : (2, 3, 3)$ E_6 $O(2, 3, 4)$ E_7 $I(2, 3, 5)$ E_8

These are the ADE Coxeter-Dynkin diagrams.

Our next goal is to classify the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. We travel from $\mathrm{SO}(3)$, to $\mathrm{SU}(2)$, onto $\mathrm{SL}_2(\mathbb{C})$.

Definition (Unitary Group). $U(n) := \{A \in \mathrm{GL}_n(\mathbb{C}) \colon A^*A = I_n\}$, where $A^* = \overline{A^T}$. But this is also $\{A \colon |Ax| = |x|\}$ Euclidean norm $= \{A \colon (Ax)^*(Ay) = x^*y\}$ i.e. A preserves the Hermitian inner product $\langle x, y\rangle = x^*y$. That is the $\{A \colon$ rows/col of A are orthogonal basis for $\mathbb{C}^n\}$.

As before $U(n)$ can be described as the set of matrices preserving an arbitrary Hermitian inner product.

Definition (Special Unitary Group). $\mathrm{SU}(n) := \{A \in U(n) \colon \det A = 1\}$.

Lemma 1.1. *Every finite subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *is conjugate to a finite subgroup of* $U(n)$. *In particular, every subgroup of* $\mathrm{SL}_n(\mathbb{C})$ *is conjugate to a finite subgroup of* $\mathrm{SL}_n(\mathbb{C})$ *and* $\mathrm{SU}(n)$.

Proof. Let $G \subseteq \mathrm{GL}_n(\mathbb{C})$ be a finite subgroup. We construct a new Hermitian inner product on $\mathbb{C}^n$ so that a given finite group G preserves the product. Define $\langle u, v\rangle := \frac{1}{|G|}\sum_{g\in G}(gu)^*(gv)$. Then for any $h \in G$, $u, v \in \mathbb{C}^n$,

$$\langle hu, hv\rangle = \frac{1}{|G|}\sum_{g\in G}(ghu)^*(ghv) = \frac{1}{|G|}\sum_{k\in G}(ku)^*(kv) = \langle u, v\rangle.$$

Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be an orthonormal basis with respect to the form $\langle\ ,\ \rangle$ for $\mathbb{C}^n$, and let $\rho : \mathbb{C}^n \to \mathbb{C}^n$ be the change of basis taking the standard basis to B. Then

$$\langle \rho e_i, \rho e_j\rangle = \langle b_i, b_j\rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

It follows from linearity that $\langle \rho e_i, \rho e_j\rangle = u^*v$. Then for any $g \in G$, we claim that $\rho^{-1}g\rho \in U(n)$. It is sufficient to show that $\rho^{-1}g\rho$ preserves the usual Hermitian inner product. We have

$$u^*v = \langle \rho u, \rho v\rangle = \langle g\rho u, g\rho v\rangle = (\rho^{-1}g\rho u)^*(\rho^{-1}h\rho v),$$

since ρ^{-1} is the opposite change of basis. Therefore, $\rho^{-1}G\rho \subseteq U(n)$. As conjugation preserves the dot product, if $G \subseteq \mathrm{SL}_n(\mathbb{C})$, then $\rho^{-1}G\rho \subseteq \mathrm{SU}(n)$. □

In order to classify the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$, we need to understand $\mathrm{SU}(2)$. We know

$$\mathrm{SU}(2) = \left\{ A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;\middle|\; A^* = A^{-1}, \det A = 1 \right\} = \left\{ \begin{pmatrix} \alpha & -\beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

To relate $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$, we define a map $\pi : \mathrm{SU}(2) \to \mathrm{SO}(3)$. The group $\mathrm{SO}(3)$ is the group of symmetries of the unit sphere S^2. We define an action of $\mathrm{SU}(2)$ on S^2 by rotations. Since $\mathrm{SU}(2)$ acts naturally on $\mathbb{C}^2$, i.e. 2×2 matrices, hence on $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C}^2/\sim$ (since the determinant is 1).

Topologically, $\P^1_{\mathbb{C}}$ is a real 2-sphere. But this gives a natural map $\pi : \mathrm{SU}(2) \to \mathrm{SO}(3)$, which one routinely verifies is a group homomorphism, and $-I_2$ acts trivially. In fact, one can verify that $\ker \pi = \{\pm I_2\}$. Therefore, π is a two-to-one cover of $\mathrm{SO}(3)$.

Lemma 1.2. *The only element of order 2 in* $\mathrm{SU}(2)$ *is* $-I_2$.

Proof (Sketch). Use the explicit form of the elements of $\mathrm{SU}(2)$. □

Theorem 1.5. *A finite subgroup of* $\mathrm{SU}(2)$ *is either cyclic of odd order or a double cover of a finite subgroup of* $\mathrm{SO}(3)$.

Proof. Let $\Gamma \subseteq \mathrm{SU}(2)$ be a finite subgroup. If Γ has odd order, then by Lagrange's Theorem Γ has no elements of order 2. Then $\Gamma \cap \ker \pi = \{I_2\}$ so that $\pi|_\Gamma : \Gamma \to \mathrm{SO}(3)$ maps Γ bijectively to a finite subgroup of $\mathrm{SO}(3)$. The only such of odd order are the cyclic groups. If Γ has even order, then by Cauchy's Theorem Γ contains an element of order 2. Then $\ker \pi \subseteq \Gamma$ so that $\pi|_\Gamma$ is a two-to-one homomorphism onto a finite subgroup of $\mathrm{SO}(3)$. $\square$

Theorem 1.6. *The finite subgroups of* $\mathrm{SL}_2(\mathbb{C})$*, up to conjugacy, are*

$$\mathbb{C}_n = \left\langle \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \right\rangle, B\mathcal{D}_n := \left\langle \mathbb{C}_{2n}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle$$

$$B\mathcal{T} := \left\langle B\mathcal{D}_2, \frac{1}{\sqrt{2}} \begin{pmatrix} \omega_8 & \omega_8^3 \\ \omega_8 & \omega_8^7 \end{pmatrix} \right\rangle$$

$$B\mathcal{O} = \left\langle B\mathcal{T}, \begin{pmatrix} \omega_8^3 & 0 \\ 0 & \omega_8^5 \end{pmatrix} \right\rangle$$

$$B\mathbb{I} = \langle ???? \rangle$$

where ω is a primitive n^{th} root of unity. The group $B\mathcal{D}_n$ is the binary dihedral group of order $4n$, $B\mathcal{T}$ is the binary tetrahedral group of order 24, $B\mathcal{O}$ the binary octahedral group of order 48, and $B\mathbb{I}$ the binary icosahedral group of order 120.

The explicit generators come from the quaternionic description of π. There is also a classification of finite subgroups of $\mathrm{GL}_2(\mathbb{C})$, coming from the extension of groups

$$1 \longrightarrow \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_2(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^\times \longrightarrow 1.$$

So any $G \subseteq \mathrm{GL}_2(\mathbb{C})$ is an extension of $G \cap \mathrm{SL}_2(\mathbb{C})$ by a finite subgroup of $\mathbb{C}^\times$—which are cyclic. Though it takes a certain amount of work, one can classify the finite subgroups of $\mathrm{SL}_3(\mathbb{C})$ using $A \in \mathrm{SL}_3(\mathbb{C})$

$$A \in \mathrm{SL}_3(\mathbb{C}) \rightsquigarrow \left(\begin{array}{c|c} \det B^{-1} & \\ \hline & B \end{array}\right), B \in \mathrm{GL}_2(\mathbb{C}).$$

2 Group Representations & Characters

Our final goal for this section will be M[c]Kay original observation that the character tables of the binary polyhedral groups 'are' the extended A-D-E diagrams. We begin with an introduction to group representations.

Definition (Representation)**.** Let G be a group. A (complex) representation of G is a group homomorphism

$$\rho : G \to \mathrm{GL}_n(\mathbb{C}),$$

for some $n \geq 1$. We call n the dimension of ρ. We call the representation $G \to \mathbb{C}^* = \mathrm{GL}_1(\mathbb{C})$ given by $g \mapsto 1$ for all $g \in G$ the trivial representation.

We will identify $\mathrm{GL}_n(\mathbb{C})$ as the automorphism group of $\mathbb{C}^n$, i.e. invertible linear maps. In this way, a representation is equivalent to an action of G on $\mathbb{C}^n$. Write ρ_g for the linear operator $\rho(g) : \mathbb{C}^n \to \mathbb{C}^n$. Avoiding a choice of basis, we write $\rho : G \to \mathrm{GL}(V)$ for a vector space V. Often, we will not distinguish between ρ and V, unless doing so would cause confusion.

Remark. Recall the group algebra $\mathbb{C}[G]$ is the $\mathbb{C}$-vector space spanned by the elements of G,

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} \alpha_g g \;\middle|\; \alpha_g \in \mathbb{C} \right\},$$

with addition given componentwise and multiplication given by $(\alpha\beta)(gh)$, extended by linearity. Suppose M is a finitely generated $\mathbb{C}[G]$-module, then it is also a finitely generated $\mathbb{C}$-module, i.e. a $\mathbb{C}$-vector space. Therefore, $M \cong \mathbb{C}^n$, as vector spaces. Multiplication by group elements defines linear operators $(M \xrightarrow{\cdot g} M) \in \mathrm{GL}(M) \cong \mathrm{GL}_n(\mathbb{C})$. Therefore, we obtain a map $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ given by $g \mapsto (M \xrightarrow{g} M)$, i.e. a representation. Conversely, a representation V is equivalent to a $\mathbb{C}[G]$-module. Therefore, the following are equivalent

- a representation of a group G
- a $\mathbb{C}[G]$-module
- an action of G on $\mathbb{C}^n$

The direct sum of two representations $\rho : G \to \mathrm{GL}_n(\mathbb{C})$, $\rho' : G \to \mathrm{GL}_m(\mathbb{C})$ is $\rho \oplus \rho' : G \to \mathrm{GL}_{n+m}(\mathbb{C})$ given by

$$g \mapsto \left(\begin{array}{c|c} \rho(g) & 0 \\ \hline 0 & \rho'(g) \end{array} \right).$$

Definition (Indecomposable)**.** If ρ cannot be written as a direct sum of two representations, then we call the representation indecomposable. Otherwise, we call the representation decomposable.

If ρ is decomposable, there are invariant, i.e. stabilized, subspaces of the vector space $V \oplus V'$.

Definition (Irreducible)**.** We say that a representation $\rho : G \to \mathrm{GL}(V)$ is irreducible if ρ has no invariant subspaces, i.e. no submodules other than $\{0\}$ and V. Otherwise, we say that ρ is reducible.

Clearly, a decomposable representation must be reducible, which immediately gives the following by contrapositive.

Theorem 2.1. *Any irreducible representation is indecomposable.*

Example 2.1. Let $G = S_3$. What are the representations of S_3? There is always the trivial representation $1 : S_3 \to \mathbb{C}^*$ given by $\sigma \mapsto 1$ for all $\sigma \in S_3$. We also have the sign (or alternating) representation $a : S_3 \to \mathbb{C}^\times$ given by $\sigma \mapsto (-1)^{\operatorname{sign}\sigma}$, which restricts to an injection from A_3 to 1, i.e. $S_3 \setminus A_3$ injects to -1 under a. We know that every permutation can be represented by a matrix given by mapping a permutation σ to the result of the permutation σ acting on the rows of I_3. For example,

$$(2\ 3) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This gives a homomorphism $S_3 \to \mathrm{GL}_3(\mathbb{C})$, called the natural representation. This defines an action of S_3 on $\mathbb{C}^3$ by permutation of basis, i.e. $\sigma(\{z_1, z_2, z_3\}) = \{z_{\sigma^{-1}(1)}, z_{\sigma^{-1}(2)}, z_{\sigma^{-1}(3)}\}$.

Clearly, the trivial representation is both indecomposable and irreducible. The sign representation has dimension one, so it is both indecomposable and irreducible. The natural representation has stable subspaces, namely the one spanned by $(1, 1, 1)$, so that it cannot be indecomposable, i.e. the natural representation is decomposable. But then the natural representation is also reducible. We can also create submodule/subrepresentations by 'modding out.' For example, define $V = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon z_1 + z_2 + z_3 = 0\}$—the natural representation modulo the trivial representation, with the permutation action. This is called the standard representation. This space has dimension two and one can check the permutation representation is isomorphic to $1 \oplus V$. $\triangleleft$

Theorem 2.2 (Maschke's Theorem)**.** *Every indecomposable representation over $\mathbb{C}$ of a finite group is irreducible. Therefore, a representation over $\mathbb{C}$ of a finite group is indecomposable if and only if it is irreducible.*

Proof. Suppose that V is a representation of G and $W \subseteq V$ is a subrepresentation, i.e. a G-stable subspace. Fix a linear projection $\pi : V \twoheadrightarrow W$, and G-linearize it:

$$\widetilde{\pi}(v) = \frac{1}{|G|} \sum_{g \in G} (g \pi g^{-1})(v).$$

Now notice we have

$$\begin{aligned} h\widetilde{\pi}(v) &= \frac{1}{|G|} h \sum_g (g\pi g^{-1})(v) \\ &= \frac{1}{|G|} \sum_g hg\pi g^{-1} h^{-1} hv \\ &= \frac{1}{|G|} \sum_{hg} (hg)\pi(hg^{-1})(hv) \\ &= \widetilde{\pi}(hv). \end{aligned}$$

Therefore, $h\widetilde{\pi}(v) = \widetilde{\pi}(hv)$ so $\widetilde{\pi}$ is G-linear. It is routine to verify that $\widetilde{\pi}$ fixes W, and we know $\widetilde{\pi}$ projects V onto W. Hence, $V \cong W \oplus \ker \widetilde{\pi}$. But then V is reducible. □

Remark. This works over any field with $|G| \neq 0$. Another way to say this is that the group algebra $\mathbb{C}[G]$ is semisimple, i.e. short exact sequence of $\mathbb{C}[G]$-modules splits.

2.1 Characters

Definition (Character)**.** Let $\rho ?LG \to \mathrm{GL}_n(\mathbb{C})$ be a representation of G. The character of ρ is $\chi_\rho := \mathrm{tr} \circ \rho$, i.e. the composition When the representation is apparent, we denote this simply as χ.

Observe that χ_ρ is *not* generally a homomorphism as the trace is not generally multiplicative, i.e. $\mathrm{tr}(AB) \neq \mathrm{tr}(A)\mathrm{tr}(B)$. If $n = 1$, then clearly χ_ρ is a homomorphism. Now while the trace map is not generally multiplicative, we do have that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. More generally, $\mathrm{tr}(\cdot)$ is invariant under cyclic permutation of products. Therefore, χ_ρ is a *class function*, i.e. χ_ρ is constant on conjugacy classes:

$$\chi_\rho(g^{-1}hg) = \mathrm{tr}(\rho(g^{-1}hg)) = \mathrm{tr}(\rho(g)^{-1}\rho(h)\rho(g)) = \mathrm{tr}(\rho(h)) = \chi_\rho(h).$$

Example 2.2. Take $G = S_3 = \{(1), (1\ 2), (2\ 3), (1\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$. The conjugacy classes of S_n are classified by cycle type—corresponding to integer partitions of n. These are

$$\{(1)\}, \{(1\,2), (2\,3), (1\,3)\}, \{(1\,2\,3), (1\,3\,2)\}.$$

We shall create a table for the characters of S_n. We only need one column per conjugacy class, and one row per representation. The trivial representation takes every σ to the identity, so $\chi_{\mathrm{triv}}(\sigma) = 1$ for all σ. We know for the alternating representation that

$$a(\sigma) = \begin{cases} 1, & \sigma \text{ even} \\ -1, & \sigma \text{ odd} \end{cases}$$

Therefore, χ_{a} is the same as $\chi_{\text{triv}}(\sigma) = 1$. For the permutation representation, we have

$$1 \mapsto \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad (1\ 2) \mapsto \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \quad (1\ 2\ 3) \mapsto \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$$

We knew also that the permutation representation was isomorphic to the standard representation summed with the trivial representation. Since the trace of a block matrix is the sum of the traces, we know that $\chi_{\text{perm}} = \chi_{\text{std}} + \chi_{\text{triv}}$. We can then subtract to find χ_{std}, making its row on the table below redundant. If we remove the redundant χ_{perm} row, we

	(1)	(1 2)	(1 2 3)
χ_{triv}	1	1	1
χ_{a}	1	−1	1
χ_{perm}	3	1	0
χ_{std}	2	0	−1

obtain the following table: We make the following observations:

	(1)	(1 2)	(1 2 3)
χ_{triv}	1	1	1
χ_{a}	1	−1	1
χ_{perm}	3	1	0
χ_{std}	2	0	−1

1. The table is square, and the number of characters is the number of conjugacy classes.

2. The columns are orthogonal.

3. The rows are orthogonal if one weights each column by the number of elements in that class, e.g.
$$\langle \chi_{\text{triv}}, \chi_{\text{std}} \rangle = 1(1 \cdot 2) + 3(1 \cdot 0) + 2(1 \cdot -1) = 0.$$

4. The first column yields the dimension of ρ. In general, $\chi_p(1) = \operatorname{tr}(\rho(1)) = \operatorname{tr}(I_n) = n$.

5. The sum of the squares of the 1^{st} column is $6 = |S_3|$.

Proposition 2.1. *Let G be a finite group, ρ a finite dimensional representation of G and χ its corresponding character. Then*

(i) *χ is a class function.*

(ii) *$\chi(1) = n$.*

(iii) The characters of a direct sum of representations is the sum of the characters.

(iv) The character of a tensor product of representations is the product of the characters.

(v) $\chi(g^{-1}) = \overline{\chi(g)}$

(vi) If $|g| = k$*, then the eigenvalues of the matrix* ρ_g *are powers of the* k^{th} *roots of unity, and* $\chi(g)$ *is a sum of such things.*

Recall that if V and W are vector spaces with basis $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$, respectively, then $V \otimes W$ is the vector space with basis $\{e_i \otimes f_j\}_{i=1,\ldots,n; j=1,\ldots,m}$ and scalar multiplication $\alpha(e_i \otimes f_j) = \alpha e_i \otimes f_j = e_i \otimes f_j$. If V and W carry actions of G, then so does $V \otimes W$ by $g(v \otimes w) = g(v) \otimes g(w)$. If $g^k = 1$, then $(\rho_g)^k = I_n$ so the minimal polynomial of ρ_g divides $x^k - 1$. Therefore, its roots are roots of unity. The trace of a matrix is the sum of its eigenvalues.

2.2 Orthogonality Relations

Let $\mathcal{H}$ denote the set of all class functions $G \to \mathbb{C}$. This contains the characters of G. Define a Hermitian inner product on $\mathcal{H}$

$$\langle \phi, \psi \rangle := \frac{1}{|G} \sum_{g \in G} \overline{\phi(g)} \psi(g).$$

Theorem 2.3. *The irreducible characters, i..e the characters of irreducible representations, are an orthonormal basis for* $\mathcal{H}$ *with respect to this inner product. In particular,*

$$\langle \chi_\rho, \chi_{\rho'} \rangle = \begin{cases} 1, & \rho \cong \rho' \\ 0, & \rho \neq \rho' \end{cases}$$

Proof. The proof will proceed in eight steps.

(i) For any representation V, the fixed subspace $V^G := \{v \in V : gv = v \text{ for all } g \in G\}$ is a subrepresentation of V. There is a natural projection

$$\pi : V \longrightarrow V^G \subseteq V$$

$$v \longmapsto \frac{1}{|G|} \sum_{g \in G} gv.$$

(ii) Compute the trace of π. First, extend a basis for V^G to a basis for V. Then and so $\mathrm{tr}(\pi) = \dim V^G$. Now the trace of a sum is the sum of the traces, so

$$\mathrm{tr}(\pi) = \mathrm{tr}\left(\frac{1}{|G|}\sum_{g\in G} g(\cdot)\right) = \frac{1}{|G|}\sum_{g\in G}\mathrm{tr}(\rho_g) = \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g).$$

In other words, $\dim V^G$ is the average value of χ_ρ.

(iii) For representations V and W, we have

$$\mathrm{Hom}_{\mathbb{C}}(V, W) = \{\text{linear maps } V \to W\}$$

$$\mathrm{Hom}_G(V, W) = \{G\text{-linear maps, i.e. } gf(v) = f(gv)\}$$

Now $\dim \mathrm{Hom}_{\mathbb{C}}(V, W) = \dim V \cdot \dim W$. However, what is $\dim \mathrm{Hom}_G(V, W)$?

(iv) We know that $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is again a representation of G: for $g \in G$, $f : V \to W$ a linear map, define $(gf)(v) := g(f(g^{-1}v))$.

(v) Now $\mathrm{Hom}_{\mathbb{C}}(V, W)^G = \{f \in \mathrm{Hom}_{\mathbb{C}}(V, W) \colon gf = f \text{ for all } g \in G\}$. That is, $\{f \colon (gf)(v) = f(v) \text{ for all } g \in G, v \in V\}$. which is $\{f \colon g(f(g^{-1}(v))) = f(v) \text{ for all } g, v\}$, rearranging is $f(g^{-1}(v)) = g^{-1}(f(v))$ for all g, which is $\mathrm{Hom}_G(V, W)$.

So if V and W are irreducible,

$$\dim \mathrm{Hom}_G(V, W) = \begin{cases} 1, & V \cong W \\ 0, & V \ncong W \end{cases}$$

and on the other hand, $\dim \mathrm{Hom}_G(V, W) = \dim(\mathrm{Hom}_{\mathbb{C}}(V, W)^G)$ by 5, which is $= \dim((V^* \otimes_{\mathbb{C}} W)^G)$, which is the average value of $\chi_{V^*\otimes W}$ which is the average value of $\overline{\chi_V}\chi_W$ which is $\langle \chi_V, \chi_W \rangle$.

Consequently, the characters determine the representations; that is, $\rho \cong \rho'$ if and only if $\chi_\rho = \chi_{\rho'}$.

The number of irreducible representations of G is equal to the number of conjugacy classes. [Because $\mathcal{H}$ has a basis given by the characteristic functions of the conjugacy classes.]

A representation ρ is irreducible if and only if $\langle \chi_\rho, \chi_\rho \rangle = 1$. [For any ρ, Maschke's Theorem allows one to write $\rho \cong \rho_1^{a_1} \oplus \cdots \oplus \rho_r^{a_r}$, where the ρ_i are distinct, irreducible, and a_i is its multiplicity. But then $\chi_\rho = a_1 \chi_{\rho_1} + \cdots + a_r \chi_{\rho_r}$. But then

$$\langle \chi_\rho, \chi_\rho \rangle = \sum_{i,j} a_i a_j \langle \chi_{\rho_i}, \chi_{\rho_j} \rangle = \sum_{i=1}^{r} a_i^2.$$

Therefore, $\rho = \rho_i$ must be irreducible. The other direction follows straight from the theorem.

The multiplicity of an irreducible representation ρ_i in a given representation is $\langle \chi_{\rho_i}, \chi_\rho \rangle$.

Definition (Regular Representation)**.** The regular representation of G is a $\mathbb{C}$-vector space $\mathbb{C}[G]$. Equivalently, $\mathbb{C}[G]$, as a module over itself, or $G \to \mathrm{GL}(\mathbb{C}[G])$.

Recall $\mathbb{C}[G]$ has a basis $\{g \in G\}$. The action of $h \in G$ is given by $g \mapsto hg$, i.e. h permutes basis elements.

Proposition 2.2. *Every irreducible representation V appears as a direct summand in the regular representation, with multiplicity equal to its dimension, i.e.*

$$\mathbb{C}[G] \cong \bigoplus_{i=1}^{r} V_i^{\dim V_i},$$

where $V_1, \ldots, V_r$ are the irreducible representations. In particular,

$$|G| = \sum_{i=1}^{r} (\dim V_i)^2 = \sum_{i=1}^{r} (\chi_i(1))^2.$$

Proof. L.T.R. □

Corollary 2.1. *G is abelian if and only if every representation is 1-dimensional.*

Proof. G is abelian if and only if every conjugacy class is a singleton if and only if there are $|G|$ classes if and only if there are $|G|$ irreducibles if and only if all the representations have dimension 1. □

Example 2.3. (i) $G = S_3$. We found three irreducible representations: $\chi_{\text{triv}}, \chi_{\text{akt}}, \chi_{\text{std}}$. Since $1^2 + 1^2 + 2^2 = 6 = |S_3|$, this must be all the irreducible representations for S_3.

(ii) Let $G = C_n = \langle x \colon x^n = 1 \rangle$. Every irreducible is a map $G \to \mathbb{C}^\times$ completely determined by the image of x. Since the map is a morphism, $1 \mapsto 1$, which implies that the image of x must be an n^{th} root of unity. But then we obtain

$$\begin{aligned} \rho_k : \; x &\mapsto \omega_n^j \\ x^r &\mapsto \omega_n^{jr} \end{aligned}$$

for $j = 0, \dots, n-1$, where ρ_0 is the trivial representation. Then we have character table

	$\{1\}$	$\{x\}$	$\{x^2\}$	$\cdots$	$\{x^{n-1}\}$
ρ_0	1	1	$\cdots$	1	
ρ_1	ω	ω^2	$\cdots$	ω^{n-1}	
ρ_2	1	ω^2	ω^4	$\cdots$	$\omega^{2(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ρ_{n-1}	1	ω^{n-1}	$\omega^{2(n-1)}$	$\cdots$	$\omega^{(n-1)^2}$

(iii) Let $G = S_4$. We have cycle types $1, (1\,2), (1\,2\,3), (1\,2\,3\,4), (1\,2)(3\,4)$, with multiplicity, $1, 6, 2\binom{4}{3} = 8$, $3! = 6$, 3, respectively.

χ_{triv}	1	1	1	1	1
χ_{alt}	1	-1	1	-1	1
χ_{std}	3	1	0	-1	-1
χ_{perm}	4	2	1	0	0
$\chi_{\text{std}} \otimes \chi_{\text{alt}}$	3	-1	0	1	-1
R	2	0	-1	0	2

Is the standard representation irreducible> We have $\langle \chi_{\text{std}}, \chi_{\text{std}} \rangle = \dfrac{1}{|G|}(1 \cdot 3^2 + 6 \cdot 1^2 + 6 \cdot 0^2 + 6(-1)^2 + 3(-1)^3) = \dfrac{1}{24} \cdot 24 = 1$, so yes. Are we done? Well, we have $1^2 + 1^2 + 3^2 = 11 < 24$, so no. A sneaky trick is to tensor with the known representations. Tensoring with the trivial one does nothing so we proceed with the others. The tensor of the standard with the alternating representation is 3-dimensional and irreducible by the same calculation. So now $1^2 + 1^2 + 3^2 + 3^2 = 20$, missing 4. So missing one two dimensional or 4 1-dimensional. In either case, there is (at least one) 2-dimensional representation, say R. So the first row of the entry for R must be 2. Call the other entries a, b, c, and d respectively. Using the orthogonality relations, one finds a system of four equations and four unknowns only to find $a = c = 0$, $b = -1$, $d = 2$. Finally, $\langle \chi_R, \chi_R \rangle = \cdots = 1$, so R is irreducible, as expected. This is then the complete character table.

(iv) $G = A_4 \subseteq S_4$. The Class Equation tells that the number of things in each conjugacy class must divide the order of the group. So unlike the situation in S_4, $(1\ 2\ 3)$ is not a conjugacy class (size 8, $|A_4| = 12$). ?????????

So $(1\ 2\ 3) \not\sim (1\ 3\ 2)$.

So it must split into at least to conjugacy classes. These are 1, $(1\ 2\ 3)$, $(1\ 3\ 2)$, $(1\ 2)(3\ 4)$, of sizes 1, 4, 4, and 3, respectively.

	1	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(1\ 2)(3\ 4)$
χ_{triv}	1	1	1	1
χ_{std}	3	0	0	-1
R	2	-1	-1	2

We always have the trivial representation. We can always restrict any representation of S_4 to A_4. Doing so with the alternating representation gives the trivial representation. The standard tensor alternating restricted to A_4 is the standard representation on A_4. Note one can restrict an irreducible representation and no longer be irreducible. We have $\langle \chi_{\text{std}}, \chi_{\text{std}} \rangle = 1$, $\langle R, R \rangle = \frac{1}{12}(1 \cdot 2^2 + 4(-1)^2 + 4(-1)^2 + 3 \cdot 2^2) = 2$, not irreducible. So the restriction of R to A_4 splits into two 1-dimensional representations. So we must split the R row into two, say U and U'.

	1	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(1\ 2)(3\ 4)$
χ_{triv}	1	1	1	1
χ_{std}	3	0	0	-1
U	1	a	b	c
U'	1	$-1-a$	$-1-b$	$2-c$

Once we have a, b, c, we know that the rows of U, U' add to the rows of R, hence the last row must be what is given above. Linear Algebra gives $a = \omega_3$, $b = \omega_3^2$, and $c = 1$. Could we have seen that without Linear Algebra? If we have a quotient of A_4, where characters we know, we can restrict along the quotient map. Normal subgroup in A_4: $\{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. The quotient is, having order 3, C_3, which has two nontrivial irreducible representations. Let's say $C_3 = \langle (1\ 2\ 3) \rangle$. Then these representations are $(1\ 2\ 3) \mapsto \omega_3$, $(1\ 3\ 2) \mapsto \omega_3^2$ and $(1\ 2\ 3) \mapsto \omega_3^2$, $(1\ 3\ 2) \mapsto \omega_3$.

	1	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(1\ 2)(3\ 4)$
χ_{triv}	1	1	1	1
χ_{std}	3	0	0	-1
U	1	ω	ω^2	1
U'	1	ω^2	ω	1

(v) Take $G = B\mathcal{T} \subseteq \mathrm{SL}_2(\mathbb{C})$, which has a 2-to-1 map $B\mathcal{T} \to \mathcal{T}$, where $\mathcal{T}$ is the tetrahedral group of order 12. We know that $\mathcal{T} \cong A_4$. Then we know that we can restrict the

4 irreducibles of A_4 to $B\mathcal{T}$. Then the preimage of a conjugacy class in $\mathcal{T}$ is either a single conjugacy class in $B\mathcal{T}$ of twice the size, or 2 classes, each of the same size as the original. So to start, just lump the classes together. The classes are 1, (1 2 3), (1 3 2), (1 2)(3 4), of sizes 2, 8, 8, 6.

χ_{triv}	1	1	1	1
χ_{std}	3	0	0	-1
U	1	ω	ω^2	1
U'	1	ω^2	ω	1
$\rho \otimes U$	$\rho \otimes U'$			

There is also the "given rep" $B\mathcal{T} \hookrightarrow \mathrm{GL}_2(\mathbb{C})$, which is two-dimensional. Also, $\rho \otimes U$, $\rho \otimes U'$, there are two more 2-dimensional. So we'll have different values in the (1 2 3) column, a, ωa, $\omega^2 a$ for some a. So they are pairwise nonisomorphic. Fact, ρ is irreducible (we had explicit matrix generators for $B\mathcal{T}$). So $\rho \otimes U$, $\rho \otimes U'$ are too. Then $1^2 + 3^2 + 1^2 + 1^2 + 2^2 + 2^2 + 2^2 = 24$, and that is all. There are then seven conjugacy classes in $B\mathcal{T}$. The preimage of $\{1\}$ is $\{\pm 1\}$, the identity matrix. So that class splits in two. Fact: the class $\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ in $\mathcal{T}$ lifts to a single class of size 6. Then the other two split into two.

	1	1	4	4	4	4	6
χ_{triv}	1	1	1	1	1	1	1
χ_{std}	3	3	0	0	0	0	-1
U	1	1	ω	ω	ω^2	ω^2	1
U'	1	1	ω^2	ω^2	ω	ω	1
ρ	2	-2	1	-1	1	-1	0
$\rho \otimes U$	2	-2	ω	$-\omega$	ω^2	$-\omega^2$	0
$\rho \otimes U'$	2	-2	ω^2	$-\omega^2$	ω	$-\omega$	0

Note that the character table does not determine the group. For example D_4 and Q_8 have the same character table. [Lose a lot passing to conugacy classes.] However, the character table carries a lot of information about the group. For example, if $\rho_1, \ldots, \rho_r$ are the irreducible representations, then for every i, j,

$$\rho_i \otimes \rho_j \cong \bigoplus_{k=1}^{r} \rho_k^{c_{i,j}^k}$$

for some *structure constants* of the group, $c_{i,j}^k$. When G is given to us as a subgroup of GL, it's already interesting to look at

$$\rho \otimes \rho_j = \bigoplus_{i=1}^{r} \rho_i^{c_{i,j}},$$

where ρ is the given representation. Then

$$\chi\chi = \sum c_{i,j}\chi_i$$

and we can read the $c_{i,j}$'s from the character table. Back to $B\mathcal{T}$. We are given ρ, the 5th row of the table. Let's decompose $\rho \otimes \text{std}$. We have $\rho \otimes \text{std} : 6, -6, 0, 0, 0, 0, 0$. Checking carefully and using the properties of ω, this is the sum of ρ, $|\rho \otimes U$, and $\rho \otimes U'$.

Definition (M$^{\text{c}}$Kay Quiver)**.** Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. The M$^{\text{c}}$Kay quiver of G has vertices $\rho_1, \ldots, \rho_r$, the irreducible representations of G, arrows m_{ij} for $\rho_i \to \rho_j$ if ρ_i appears with multiplicity m_{ij} in $\rho \otimes \rho_j$.

Recall $m_{ij} = \dim \mathrm{Hom}_G(V_i, V \otimes V_j) = \langle \chi_i, \chi\chi_j \rangle$ (abstract stuff on boses). We have $G = C_2$ embedded in $\mathrm{GL}_3(\mathbb{C})$ as $\left\langle \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \right\rangle$. We know the irreducible representations of $C_2 = \langle \sigma \colon \sigma^2 = 1 \rangle$. The two representations must be $\sigma \mapsto 1$, $\sigma \mapsto -1$, call the first ρ_1 and the second ρ_{-1}. Notice $\rho(\text{given}) \cong \rho_{-1}^{(3)}$ and

	ρ_1	ρ_{-1}
ρ_1	ρ_1	ρ_{-1}
ρ_{-1}	ρ_{-1}	ρ_1

So $\rho \otimes \rho_1 = \rho = \rho_{-1}^{(3)}$, $\rho \otimes \rho_{-1} = \rho_{-1}^{(3)} \otimes \rho_{-1} = \rho_1^{(3)}$. Then we have

Now consider $C_n = \left\langle \begin{pmatrix} \omega_n & \\ & \omega_n^{-1} \end{pmatrix} \right\rangle \subseteq \mathrm{SL}_2(\mathbb{C})$. We know the irreducible representations of $C_n : \rho_0, \rho_1, \ldots, \rho_{n-1}$, where ρ_j takes the generator of C_n to ω_n^j. Our given representation is $\rho \cong \rho_1 \otimes \rho_{n-1}$. What is $\rho_j \otimes \rho_k$? It's ρ_{j+k}, with jk taken mod n. So $\rho \otimes \rho_j = (\rho_1 \oplus \rho_{n-1}) \otimes \rho_j = \rho_{j+1} \oplus \rho_{j-1}$, where again indices are taken mod n.

We get the above diagram for every j.

Example

D_4	$\{1\}$	$\{x^2\}$	$\{x, x^3\}$	$\{y, x^2y\}$	$\{y, x^3y\}$
β_{++}	1	1	1	1	1
β_{+-}	1	1	1	-1	-1
β_{-+}	1	1	-1	-1	-1
β_{--}	1	1	-1	-1	1
ρ	2	-2	0	0	0

where $\rho_{\pm\pm}(x) = \pm 1$, $\beta_{\pm\pm}(y) = \pm 1$, and ρ is the "geometric" representation as symmetries of an n-gon in $\mathbb{C}^2$.

Now let's compute the M$^{\text{c}}$Kay quiver of D_4 with respect to $\rho : D_4 \hookrightarrow \mathrm{GL}_2(\mathbb{C})$.

$$\begin{aligned} \rho \otimes \beta_{++} &\cong \rho \\ \rho \otimes \beta_{+-} &\cong \rho \\ \rho \otimes \beta_{-+} &\cong \rho \\ \rho \otimes \beta_{--} &\cong \rho \\ \rho \otimes \rho &\cong \beta_{++} \oplus \beta_{+-} \oplus \beta_{-+} \oplus \beta_{--} \end{aligned}$$

Then picture

M^{c}Kay observed that the arrows in the M^{c}Kay quiver of the binary tetrahedral groups come in opposing pairs, no more than one between any two vertices, and if you remove the trivial representation, one obtains an ADE Dynkin diagram.

The first two parts are relatively simple to prove without knowing the classification. For example, $m_{ji} = \langle \chi_j, \chi\chi_i \rangle$. Now χ is the character of $G \hookrightarrow \mathrm{SL}_2(\mathbb{C})$ is self-adjoint since it is in SL_2. But then $m_{ji} = \langle \chi_j, \chi\chi_i \rangle = \langle \chi_j\chi, \chi_i \rangle = \langle \chi_i, \chi, \chi_j \rangle = m_{ij}$.

Our next goal is to give a uniform proof (meaning without classification) of M^{c}Kay 's observation about ADE diagrams.

2.3 ADE and Extended ADE Diagrams

A list:

The extended ADE diagrams have one extra vertex, circled. Then have $n+1$ vertices. They have the weird property that ...

Lemma 2.1. *Let T be a connected finite graph (possibly with multiple edges). Then either T is an ADE diagram or T contains an extended ADE, and not both.*

Proof. If T does not contain an extended ADE, $\tilde{A}_n$ then no cycles, so a tree. not contain $\tilde{D}_n$ so then at most one branch point of valence $= 3$. So its a T_{pqr},

Assume that $p \leq q \leq r$. Not contain $\tilde{E}_6$ so that $p \leq 2$. Not contain $\tilde{E}_7$, so then $q \leq 3$. Not contain $\tilde{E}_8$ so that $r \leq 5$. But then $T_{1,1,n}$, $T_{2,2,n}$, or $T_{2,3,3}$, $T_{2,3,4}$, $T_{2,3,5}$, and these are A_n, D_{n+2}, E_6, E_7, or E_8, respectively. □

There are two 'birthplaces' for extended ADE diagrams. The first is additive functions on graphs, the second is Tits quadratic forms of graphs.

Definition. Let T be a finite connected graph on a vertex set $\{1, \ldots, n\}$. Then an additive function on T is a function $a : \{1, \ldots, n\} \to \mathbb{N}_{>0}$ such that for every i

$$\sum_{\text{there is edge i - j}} a_j = 2a_i$$

whee $a_i = a(i)$. It is subadditive if less than or equal to. Strictly subadditive if strict inequalty.

Example is a subadditive function since $2 \geq 1$. Could there be an additive function? We would need $2a_1 = a_2$ and $2a_2 = a_1$, impossible.

Example Now $a_1 = 1 = a_2$ is an additive function because we count each edge separately. $2a_1 = 2a_2$.

Ecample We would need $2a_1 \leq 3a_2$, $2a_2 \leq 3a_1$, impossible.

Example $\tilde{D}_5$

So carries additive function.

The crucial observation is that if T is the M^{c}Kay graph of a subgroup of $\mathrm{SL}_2(\mathbb{C})$ (replace each left/right arrow with dash), then labeling each vertex with the dimension of the corresponding representation is an additive function! This is because we tensor with the given 2-dimensional representation ρ, and connect ρ_i to all the ρ_j appearing in $\rho \otimes \rho_i$. So

$$2 \dim \rho_i = \dim(\rho \otimes \rho_i) = \sum_{\rho_i - \rho_j} \dim \rho_j.$$

Theorem 2.4. *A graph T carries an additive function if and only if it is extended ADE. It carries a strictly subadditive function if and only if it is ADE.*

Lemma 2.2. *The extended ADE graphs carry additive functions, and the ADE graphs carry strictly subadditive functions.*

Proof. Write them down. $\square$

We need to reinterpret additive functions. Write a function $a : \{1, \ldots, n\} \to \mathbb{N}_{>0}$ as a column vector $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. Then a is subadditive if and only if $2 \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \geq \begin{pmatrix} \sum a_j \\ \vdots \\ \sum a_j \end{pmatrix}$

$= \text{indicdence matrix} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, i.e. $2I - A$ has nonnegative entries, where I is the identity matrix and A is the indicence matrix and $[A]_{ij}$ is the number of edges between i and j.

Lemma 2.3. *If T admits an additive function, then every subadditive function on T is additive.*

Proof. Set $C = 2I_n - A$, where A is the incidence matrix of T.. Assume a is an additive function and b is a subadditive function. We need show that b is additive. Consider $b^T Ca$. Since a is additive, $Ca = 0$. But we also know $b^T Ca = b^T C^T a$ since C is symmetric, which is $(Cb) \cdot a$, which is a positive linear combination of entries of Cb (since entries of a are positive). But then $Cb = 0$, which implies that b is additive. $\square$

Corollary 2.2. *Every subadditive function on an extended ADE diagram is additive.*

Lemma 2.4. *Suppose that $T \subsetneq T'$ are finite connected graphs. If T' carries a subadditive function a, the restriction of a to T is strictly subadditive.*

Proof. We know

$$2a_i \geq \sum_{i-j \in T'} a_j \geq \sum_{i-j \in T}$$

where second inequality follows since every edge in T is an edge in T'. Since $T' \neq T$, there is at least one edge in T' not in T, so the inequality must be strict. □

We can now prove a previously stated theorem.

Theorem 2.5. *A graph T carries an additive function if and only if it is extended ADE. It carries a strictly subadditive function if and only if it is ADE.*

Proof. First, Lemma 2.2 does $\leftarrow$ for both. For $\rightarrow$, assume carries an additive function a and is not an extended ADE. Then by Lemma 2.1, either T is ADE or T *strictly* contains an extended ADE.

If T is ADE, then T carries a strictly subadditive function, contradicting Lemma 2.3. If T strictly contains an extended ADE, then this ADE carries a strictly subadditive function by Lemma 2.4, contradicting Lemma 2.3.

Finally if T carries a strictly subadditive function and its not ADE, then T contains an extended ADE, which must carry a strictly subadditive function by Lemma 2.4, contradicting Corollary 2.2. □

Corollary 2.3. *The M^c^Kay graph of a binary polyhedral group is extended ADE, with additive function given by the dimensions of the irreducible representations. In particular, the 'extra vertex' has value 1.*

2.4 Another Aside: The Quadratic Form of a Graph

Definition. Let T be a finite connected graph, possibly with multiple edges. A quadratic form of T, also known as a Tits form, is the polynomial $q_T(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2 - \sum_{i-j} x_i x_j$, where as usual we count edges with multiplicity.

Example 2.4. $q_T(x_1, x_2) = x_1^2 + x_2^2 - x_1 x_2$.
$q_T(x_1, x_2) = x_1^2 + x_2^2 - 3x_1 x_2$

Observe that $q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T C \mathbf{x}$, where C is the Coxeter matrix of T. So we wonder if q_T is related to (sub)additive functions.

Theorem 2.6. *The quadratic form q_T is positive definite, i.e. $q(x) \geq 0$ for all x and only zero for $x = 0$, if and only if T is an ADE diagram, and positive semidefinite, i.e. $q(x) \geq 0$, if and only if T is extended ADE.*

The theorem can be proved directly. Show that if q_T is positive definite, then T does not contain any cycles, or more than one branch point, or a vertex of degree > 3. So T is a T_{pqr} tree. Show that q_T is positive definite if and only if $1/p + 1/q + 1/r > 1.$, so then ADE. $\square$

One can also prove the theorem by translating q_T into the additive function notation:

Proposition 2.3. *If a is an additive function on T, then q_T is positive semidefinite. (also strictly subadditive then positive definite).*

Proof. Assume a is an additive function. For each edge $e : i - j$, define $q_e(x_1, \ldots, x_n) = \dfrac{1}{2a_i a_j}(a_i x_j - a_j x_i)^2$. The coefficient of x_i^2 is $\dfrac{1}{2a_i a_j} a_j^2 = \dfrac{a_j}{2a_i}$. The coefficient of $x_i x_j$ is $\dfrac{1}{a_i a_j}(-2a_i a_j) = -1$. Consider the sum $\sum_e q_e(x_1, \ldots, x_n)$. Coefficient of $x_i x_j$ is number of edges $i - j$. The coefficient of x_i^2 is $\sum_{\text{Edges e containing i}} \dfrac{a_j}{2a_i} = \dfrac{1}{2a_i} \sum a_j = 1$. So $q_T = \sum q_e$'s is a sum of squares, so positive semidefinite.

3 Invariant Theory

3.1 A Motivating Example

We now make a transition from groups and graphs to Commutative Algebra and Algebraic Geometry. We begin with a motivating example.

Example 3.1. Consider $C_2 \subseteq \mathrm{SL}_2(\mathbb{C})$, with generator $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then C_2 acts on $\mathbb{C}^2$ by $\sigma(p) = -p$, i.e. σ is a rotation of Arg p by π. The quotient space of this group action has for points the orbit of the action: for every nonzero point $\{p, -p\}$, along with $\{0\}$. A fundamental domain for this action, i.e. a subset of $\mathbb{C}^2$ containing exactly one point from each orbit.

More precisely, let $\mathcal{C}$ denote a cone. We can define a continuous surjective map $\pi : \mathbb{C}^2 \to \mathcal{C}$ such that the fibers of π are precisely the orbits of the action. Choose coordinates so that the cone is defined by $y^2 = xz$, then $\pi(u, v) = (u^2, uv, v^2)$ is such a map. To be more systematic, instead consider the ring of polynomials $\mathbb{C}[u, v]$, thought of as the set of polynomials on $\mathbb{C}^2$. [The function u picks out the first coordinate of a point $p \in \mathbb{C}^2$ and so forth.]

The polynomial functions on the quotient space $\mathbb{C}^2/C_2$ are exactly the polynomials on $\mathbb{C}^2$ that are constant on orbits; that is, the polynomial functions are the set $\{f \in \mathbb{C}[u, v] \colon f(p) = f(-p) \text{ for all } p \in \mathbb{C}^2\} = \mathbb{C}[u^2.uv, v^2] \subseteq \mathbb{C}[u, v]$, note $\mathbb{C}[u^2, uv, v^2] \cong \mathbb{C}[x, y, z]/(y^2 - xz)$. Note that the ring $R := \mathbb{C}[u^2, uv, v^2]$ is an integral domain of dimension two, graded, integrally closed, Cohen-Maccaulay (in fact gorenstein), reflexive, and the polynomial ring $\mathbb{C}[u, v]$ is a finitely generated module over it. In fact, $\mathbb{C}[u, v] \cong R \oplus (Ru + Rv)$. Finally, every indecomposable reflexive R-module appears as a direct summand of the polynomial ring $\mathbb{C}[u, v]$. ◁

The question we shall explore is how many properties of the ring R in Example 3.1 are specific to this example, and how many are properties hold more generally.

3.2 Classical Invariant Theory of Finite Groups

Invariant Theory has connections to many fields, including tori and Lie groups. However, we shall only consider finite groups, so these shall not make an appearance. For this material, we follow Kraft-Procesi. In particular, we take a coordinate free approach whenever possible. Now let k be an infinite field and W a finite dimensional vector space. We say that a function $f : W \to k$ is regular if it is a polynomial in the elements of some basis of W—this is independent of basis. Let $k[W]$ be the ring of regular functions on W. If $\{x_1, \ldots, x_n\}$ were a basis for $W^* = \mathrm{Hom}(W, k)$, then $k[W] \cong k[x_1, \ldots, x_n]$ is a polynomial ring. This holds because the field is infinite, and this is not true for finite fields (the obstruction is nonzero vanishing functions).

Definition (Homogenous Function)**.** A regular function $f \in k[W]$ is homogeneous of degree d if $f(\lambda w) = \lambda^d w$ for all $\lambda \in k$ and $w \in W$.

Concretely in terms of a basis for W^*, this means that f is a linear combination of monomials of degree d, i.e. $x_1^{d_1} \cdots x_n^{d_n}$ with $d_1 + \cdots + d_n = d$. Since every polynomial is a sum of such monomials, every polynomial is a sum of homogeneous polynomials; that is, $f \in k[W]$ is uniquely a sum of homogeneous polynomials, so $k[W] \cong \bigoplus_{d \geq 0} k[W]_d$, where $k[W]_d =$ homogeneous regular functions of degree d. In particular, $k[W]$ is a graded ring, i.e. $A = \bigoplus A_i$ as abelian groups such that $A_i A_j \subseteq A_{i+j}$. As a final remark, we know that $k[W] \cong k[x_1, \ldots, x_n]$. Fixing a basis $\{e_1, \ldots, e_n\}$ for W, then $x_1, \ldots, x_n$ is a dual basis for W^*, i.e. $x_i(e_j) = \delta_{ij}$.

Now suppose we have a subgroup $G \subseteq \mathrm{GL}(W)$, or more generally a representation $\rho : G \to \mathrm{GL}(W)$. This gives an action of G on W: $gw := \rho(g)w$. In turn, this gives an action of G on $k[W]$, $(gf)(w) := f(g^{-1}w)$ (the (-1)-power is needed to get a left action). Moreover, this action is compatible (in fact the same as) the action of G on the dual space W^*. Keep in mind that $k[W]_1$ is the set of linear maps from $W \to k$, i.e. W^*. In fact, $k[W]_d = \mathrm{Sym}_d(W^*)$, the d$^{\text{th}}$ symmetric power of W^*, as such it inherits the action of G on W^*.

Definition (Invariant Function)**.** A function $f \in k[W]$ is invariant (G-invariant) if $gf = f$ for all $g \in G$. Equivalently, $f(w) = f(gw)$ for all $g^{-1} \in G$, i.e. $g \in G$. We write $k[W]^G$ for the set $\{f \in k[W] \colon f \text{ invariant}\}$.

One can check that $k[W]^G$ is a ring: each $g \in G$ acts as an automorphism of $k[W]$.

Example 3.2. Let S_n be the symmetric group on n letters. Now S_n has an action on $W = k^n$ via $\sigma(e_i) = e_{\sigma(i)}$. Equivalently, $\sigma(a_1, \ldots, a_n) = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)})$. Then S_n also acts on $k[W] \cong k[x_1, \ldots, x_n]$, where $\{x_i\}_{i=1}^n$ is the dual basis. What is $\sigma(x_i)$? We know $x_i(e_j) = \delta_{ij}$, so

$$(\sigma x_i)(e_j) = x_i(\sigma^{-1}(e_j)) = x_i e_{\sigma^{-1}(j)} = \delta_{i\sigma^{-1}(j)}.$$

But this means $(\sigma x_i)(e_j) = \delta_{i\sigma^{-1}(j)} = 1$ if and only if $\sigma^{-1}(j) = i$ if and only if $\sigma(i) = j$. Therefore, $\sigma x_i = x_{\sigma(i)}$. Generally for any $f \in k[x_1, \ldots, x_n]$, $(\sigma f)(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Now the question is which functions are invariant; that is, which functions of $k[x_1, \ldots, x_n]$ are independent of the order of the x_i? These are the symmetric polynomials, e.g. $x_1 + \cdots + x_n$, $x_1 \cdots x_n$, $x_1^7 + \cdots + x_n^7$. $\triangleleft$

The symmetric polynomials in n variables are all composed of the elementary symmetric polynomials, given as follows:

$$\begin{aligned}
s_0(x_1, \dots, x_n) &:= 1 \\
s_1(x_1, \dots, x_n) &:= \sum_{1 \le i \le n} x_i = x_1 + \cdots + x_n \\
s_2(x_1, \dots, x_n) &:= \sum_{1 \le i < j \le n} x_i x_j \\
s_3(x_1, \dots, x_n) &:= \sum_{1 \le i < j < k \le n} x_i x_j x_k \\
&\vdots \\
s_n(x_1, \dots, x_n) &:= x_1 \cdots x_n
\end{aligned}$$

Theorem 3.1 (Fundamental Theorem of Symmetric Functions/Newton's Theorem)**.** *Any symmetric polynomial in $x_1, \dots, x_n$ is uniquely expressible as a linear combination of elementary symmetric polynomials.*

In particular, the s_i's are algebraically independent of each other, i.e. there are no nontrivial polynomial relations among them. Therefore, it must be that $k[x_1, \dots, x_n]^{S_n} = k[s_1, \dots, s_n]$. This is indeed a polynomial ring since the s_i have no relations between them. There are algorithms to write any symmetric polynomial in the elementary symmetric polynomials.

Example 3.3.

(i) $x^2 + y^2 = (x+y)^2 - 2xy = s_1^2 - 2s_2$

(ii) $x_1^3 + x_2^3 + x_3^3 = s_1^3 - 3s_1s_2 + 3s_3$

(iii) $x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 = s_1s_2 - 3s_3$

$\triangleleft$

Remark. The power sums, $p_1(x_1, \dots, x_n) = x_1 + \cdots + x_n$, $p_2(x_1, \dots, x_n) = x_1^2 + \cdots + x_n^2$, …, $p_n(x_1, \dots, x_n) = x_1^n + \cdots + x_n^n$, also generate the ring of symmetric polynomials. The complete symmetric polynomials, the Schur polynomials, etc. all also generate the ring of symmetric polynomials. Hence, there are procedures from going from one set of these polynomials to another. The transition functions between them are crucial in the representation of S_n and GL_n (Schur-Weyl Theory).

As another aside, the discriminant of S_n acting on $x_1, \dots, x_n$ is[2]

$$\Delta = \prod_{i<j} (x_i - x_j)^2$$

The discriminant is symmetric, so it must be a polynomial in $s_1, \dots, s_n$.

[2] To some, this is the square of what they would define as the discriminant.

Example 3.4. If $n = 2$, then $\Delta = (x-y)^2 = x^2 - 2xy + y^2 = s_1^2 - 4s_2$. If $n = 3$, $\Delta = (x-y)^2(x-z)^2(y-z)^2 = s_1^2s_2^2 - 4s_2^3 - 4s_1^3s_3 - 27s_3^2 + 18s_1s_2s_3$.

Given a polynomial $g(t) \in \mathbb{C}[t]$ with roots $a_1, \ldots, a_n$ (with multiplicity), the discriminant of g is

$$\Delta(g) = \Delta(a_1, \ldots, a_n) = \prod_{i<j}(a_i - a_j)^2$$

Observe $\Delta(g) = 0$ if and only if g has a repeated root. For example, $g(t) = t^2 + bt + c$, then $\Delta(g) = b^2 - 4c$. An exercise for the reader is to show $\Delta(g) = (\det V)^2$, where V is the Vandermonde matrix:

$$V = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix}$$

3.3 Restricting the Action of S_n

We want to restrict the action of S_n on $k[x_1, \ldots, x_n]$ to the subgroup $A_n \subseteq S_n$. All the symmetric functions are still invariant. Is anything else invariant? Notice that $(i\ j)\sqrt{\Delta} = -\sqrt{\Delta}$. So if σ is an even permutation, $\sigma(\sqrt{\Delta}) = \sqrt{\Delta}$.

FACT: $k[x_1, \ldots, x_n]^{A_n} = k[s_1, \ldots, s_n, \sqrt{\Delta}]$.

Indeed, we can think of $k[x_1, \ldots, x_n]^{A_n}$ as consisting of the symmetric polynomials and the sign-symmetric polynomials: $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (-1)^{\mathrm{sgn}(\sigma)} f(x_1, \ldots, x_n)$.

Moreover, $\Delta = (\sqrt{\Delta})^2$ is a polynomial in the 'variables' $s_1, \ldots, s_n$. But then $k[x_1, \ldots, x_n]^{A_n}$ is isomorphic to a hypersurface ring: $k[y_1, \ldots, y_n, z]/(z^2 - f(y_1, \ldots, y_n))$.

First, a few basic questions about $k[W]^G$:

1. (Generators and Relations): Given a finite group $G \subseteq \mathrm{GL}(W)$, is the ring of invariants $k[W]^G$ a finitely generated k-algebra?

2. If so, describe them explicitly and also is the ideal of relations among the generators finitely generated?If so, describe them explicitly.

Following theorem due to Hilbert and Noether:

Theorem 3.2 (First Fundamental Theorem of Invariant Theory for Finite Groups)**.** *Let $k = \mathbb{C}$. The invariant ring $\mathbb{C}[W]^G$ is generated as a $\mathbb{C}$-algebra by at most $\binom{|G|+\dim W}{\dim W}$ homogeneous polynomials of degree at most $|G|$.*

Note $\binom{n+d}{d}$ is the vector space dimension of homogenous polynomials of degree d in n variables. Hilbert proved finiteness as an application of the Hilbert-Basis Theorem [1890]. The proof given was nonconstructive. There is a story that Gordan (rep. theory of binary forms, was constructive), is said to have a said thats not math thats theology. Mostly believed to be a story. Hilbert later gave a constructive proof. [1890s]. Noether gave the bound in the theorem which is tight, by showing $k[W]^G$ is generated by

$$\left\{ \frac{1}{|G|} \sum_{g \in G} gm \colon m \text{ runs over monomials of degree } \leq |G| \right\}$$

Sketch of Hilbert's (nonconstructive proof)

Theorem 3.3 (Hilbert Basis Theorem)**.** *The polynomial ring $k[x_1, \ldots, x_n]$ is noetherian, i.e. every ideal of $k[x_1, \ldots, x_n]$ is finitely generated, where k is a field.*

Let $S = k[x_1, \ldots, x_n]$, $R = k[x_1, \ldots, x_n]^G \subseteq S$. Let I be the ideal of R generated by all invariants of positive degree.

Exercise: If I is a finitely generated ideal of $R = k[f_1, \ldots, f_t]$, say $I = Rf_1 + \cdots + Rf_t$, then $\{f_i\}$ generate the ring of invariants as a k-algebra. The proof follows by induction on the degree.

We know that IS is a finitely generated ideal of S by the Hilbert Basis Theorem. Define the Reynold's operator

$$\begin{aligned} \rho : S &\to R \\ f &\mapsto \frac{1}{|G|} \sum_{g \in G} gf. \end{aligned}$$

Observe that

1. $\rho(S) \subseteq R$. We have seen this before in a different form. 2. ρ fixes R elementwise. 3. ρ is a ring homomorphism, and is R-linear: if $h \in S^G$, $f \in S$, then $\rho(hf) = h\rho(f)$ 4. For any ideal J of R, $JS = \{\sum as \colon a \in J, s \in S\}$. So $\rho(JS) = \{\rho(\sum as) \colon a \in J, s \in S\} = \{\sum \rho(as) \colon a \in J, s \in S\} = \{\sum \rho(s)a \colon a \in J, s \in S\} = \{\sum ra \colon r \in R, a \in J\} = J$. But then $\rho(JS) = J$. (3rd $=$ sign follows from 3) and last from ρ maps S onto R.

Then the ideal I generated by the invariants of positive degree is the same as $\rho(IS)$, and IS is finitely generated so I is as well.

Theorem 3.4 (The Second Fundamental Theorem of Invariant Theory for Finite Groups)**.** *The invariant ring is finitely generated. [Hilbert's Syzygy Theorem]*

There are versions of the 1st and 2nd Fundamental Theorem of Invariant Theory for many classes of groups.