

Name: Caleb McWhorter — Solutions

MATH 307

Spring 2023

HW 1: Due 02/13 (14)

*"In learning you will teach, and in
teaching you will learn."*

—Phil Collins

Problem 1. (10pt) Let $\mathcal{U} = \{-10, -9, \dots, 9, 10\}$. Define the following subsets of \mathcal{U} :

$$A = \{-2, 0, 5, 10\}$$

$$B = \text{even numbers in } \mathcal{U}$$

$$C = \{-9, -7, -5, -3, -1, 1, 3, 5, 7, 9\}$$

$$D = \text{positive prime numbers in } \mathcal{U}$$

$$E = \{-5, -4, \dots, 4, 5\}$$

Using the sets defined above, answer the following:

(a) $A \cap B$

(b) $B \cup E$

(c) $E - A$

(d) B^c

(e) $|D|$

Solution.

- (a) The set $A \cap B$ is the collection of objects that are elements of *both* A and B . But then we have...

$$A \cap B = \{-2, 0, 10\}$$

- (b) The set $B \cup E$ is the collection of objects that are elements of either B or E . But then we have...

$$B \cup E = \{-10, -8, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 8, 10\}$$

- (c) The set $E - A$ is the collection of objects that are elements of E but *not* elements of A . But then we have...

$$E - A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$$

- (d) The set B^c is the collection of objects that are elements (of \mathcal{U}) that are *not* elements of B . But then we have...

$$B^c = \{-9, -7, -5, -3, -1, 1, 3, 5, 7, 9\}$$

- (e) The number $|D|$ is the cardinality or 'size' of D ; that is, $|D|$ is the number of distinct (unique) elements of D . But then as $D = \{2, 3, 5, 7\}$, we have $|D| = 4$.

Problem 2. (10pt) Define the following sets:

A = set of multiples of 3

B = set of divisors of 30

C = set of even numbers less than 10

Using the sets defined above, answer the following:

- (a) List the elements of B .
- (b) Give the largest element of A less than 50 and the largest negative element of A .
- (c) What are the elements of $B - A$?
- (d) What are the elements of $A \cap B$?
- (e) Are B and C disjoint? Explain.

Solution.

- (a) The divisors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30. Therefore,

$$B = \{1, 2, 3, 5, 6, 10, 15, 30\}$$

- (b) The largest element of A less than 50 will be the largest multiple of 3 that is less than 50. We know that 50 is not divisible by 3, nor is 49. However, as $48 = 16(3)$, we know that 48 is divisible by 3. Therefore, 48 is the largest element of A (an integer divisible by 3) that is less than 50. The negative elements of A , i.e. the negative integers that are a multiple of 3, are $-3, -6, -9, -12, -15, \dots$. The largest of these negative multiples of 3 is -3 .

- (c) The objects of $B - A$ are the elements of B that are *not* elements of A . The elements of B are the divisors of 30 and the elements of A are the multiples of 3. Then the elements of $B - A$ are the integers that are divisors of 30 that are not multiples of 3. But then we have...

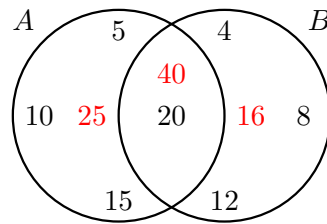
$$B - A = \{1, 2, 5, 10\}$$

- (d) The objects of $A \cap B$ are objects that are elements of *both* A and B . The elements of A are the multiples of 3 and the elements of B are the divisors of 30. Therefore, the elements of $A \cap B$ are the integers that are multiples of 3 that are also divisors of 30. But then we have...

$$A \cap B = \{3, 6, 15, 30\}$$

- (e) The sets B and C are disjoint if $B \cap C = \emptyset$; that is, if there is no integer that is an element of both B and C . However, 2 is an even number less than 10 so that $2 \in C$. But 2 is also a divisor of 30 so that $2 \in B$. This shows $2 \in B \cap C$ so that $B \cap C \neq \emptyset$. Therefore, B and C are not disjoint.

Problem 3. (10pt) Look at the Venn diagram given below:



Use this diagram to answer the following:

- Assuming only a few of the elements of A and B are given in the diagram above, describe what the sets A and B likely represent.
- Place the numbers 25, 16, and 40 in appropriate places in the given Venn diagram.
- Using words, explain what numbers go in the same region of the Venn diagram in which 20 is found.
- Using words, explain what numbers go in the same region of the Venn diagram in which 4, 8, and 12 are found.
- What numbers would be placed outside of both the regions A and B ? Give an example.

Solution.

- The elements of A are the integers 5, 10, 15, and 20. We may then possibly describe A as the set of positive multiples of 5. The elements of B are the integers 4, 8, 12, 20. We may then possibly describe B as the set of positive multiples of 4.
- Because 25 is a multiple of 5 but not 4, 25 belongs only in the set A . Because 16 is a multiple of 4 but not a multiple of 5, 16 belongs only in the set B . Because 40 is a multiple of both 5 and 4, 40 belongs in both the set A and B . We then place 25, 16, and 40 in appropriate places in the diagram above.
- The number 20 is in the overlap of the sets A and B . But then this region represents the collection of elements that are in A and B , i.e. $A \cap B$. This is the collection of integers that are both in A (hence a positive multiple of 5) and in B (hence a positive multiple of 4). To be a multiple of 4 and 5 implies that you are a multiple of $\text{lcm}(4, 5) = 20$. Then the region with 20 is the set of positive integers that are a multiple of 20.
- The numbers 4, 8, 12 are in B , i.e. are a positive multiple of 4, but are not in A , i.e. they are not a positive multiple of 5. But then region containing 4, 8, and 12 are the positive integers which are a multiple of 4 but not a multiple of 5, i.e. an element of the set $B - A$.
- The region outside the circles A and B consist of the elements that are not in A , i.e. not a positive multiple of 5, and not in B , i.e. not a positive multiple of 4. But then this region consists of the positive integers which are not multiples of 4 and not multiples of 5. For example, 1, 2, 3, 6, 7, 9, 11, etc. are all elements of the region outside the circles A and B .

Problem 4. (10pt) You are working with a student named Lucy. You give her the following sets: $A = \{a, b, c, d, a\}$ and $B = \{c, d, e, f\}$.

- (a) Lucy states that the cardinality of A is 5. Explain why Lucy is wrong. How might you correct her?
- (b) You ask Lucy to find $A \cup B$ and she states that this is $\{c, d\}$. What has Lucy done wrong?
- (c) Cameron overhears Lucy's answer in (b) and shouts that the answer is $\{a, b, e, f\}$. How has Cameron misunderstood the mathematical word *or* in this context?
- (d) Both Lucy and Cameron state that you cannot find $A - B$ because they are filled with letters and you cannot subtract letters. Explain what they have misunderstood about sets.

Solution.

- (a) The cardinality, or size, of a set is the number of distinct (unique) elements of a set. Repetition or order of elements of a set do not matter. But then we know that $A = \{a, b, c, d, a\} = \{a, a, b, c, d\} = \{a, b, c, d\}$. But then A only has 4 distinct elements so that $|A| = 4$. Lucy is misunderstanding that it is not the number of 'objects' of A that matters; it is the number of *distinct* objects of A . You should explain to her that she should eliminate duplicates before counting the elements of a set.
- (b) The set $A \cup B$ should consist of the elements that are in either A or B . The set $A \cap B$ is the set of elements that are in both A and B . Notice that c is in A and B and d is in A and B . There are no other elements that are in both A and B . But then $A \cap B = \{c, d\}$. It is then likely that Lucy has confused the symbols \cup and \cap .
- (c) The set $A \cup B$ should consist of the elements that are in A or B . However, 'or' in Mathematics always refers to one or the other or both; that is, the elements of $A \cup B$ are the elements in either A or B . The elements 'a' and 'b' are only in A and the elements 'e' and 'f' are only in B . Cameron then seems to be treating the word 'or' as 'exclusive-or' and is not using it in the mathematical sense.
- (d) They are treating '-' as if it refers to subtraction. However, in this context, we should treat '-' as 'removing.' The set $A - B$ is then the collection of elements of A after having 'removed' the elements that are in B ; that is, the set $A - B$ is the collection of elements of A that are not elements of B .