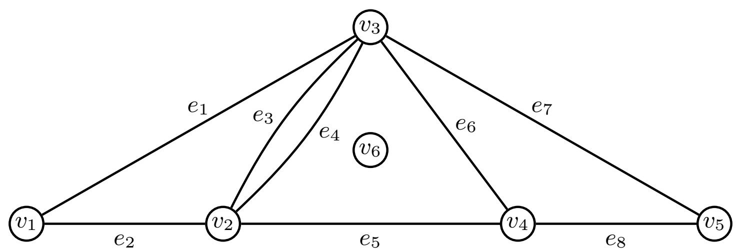
Name: ______________________________
MATH 308
Fall 2021
HW 19: Due 12/15

"The origins of graph theory are humble, even frivolous."
–Norman Biggs

Problem 1. (10pt) Consider the following graph:



(a) Is the graph directed or undirected?

(b) Give the vertex set and the edge set for the graph.

(c) Give the adjacency matrix of the graph.

(d) Is the graph simple?

(e) Are there any isolated vertices?

(f) List all pairs of parallel edges.

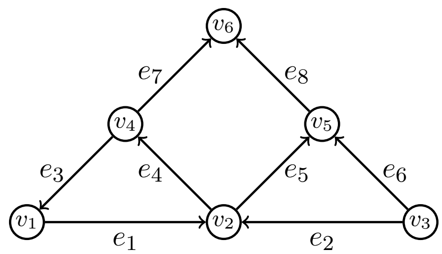
(g) Compute the degree of the graph.

(h) Are the vertices v_1 and v_6 connected? What about the vertices v_1 and v_4?

(i) Does the graph $G \setminus \{v_6\}$ have an Eulerian circuit? Find one or explain why none exists.

(j) Does the graph $G \setminus \{v_6\}$ have an Hamiltonian circuit? Find one or explain why none exists.

Problem 2. (10pt) Give vertex and edge set for a graph (directed) and tell if....



(a) Is the graph directed or undirected?

(b) Give the vertex set and the edge set for the graph.

(c) Give the adjacency matrix of the graph.

(d) Is the graph simple?

(e) Are there any isolated vertices?

(f) Compute the in- and out-degree of each vertex.

(g) Compute the degree of the graph.

(h) Is the vertex v_1 connected to v_6? Is the vertex v_6 connected to v_1?

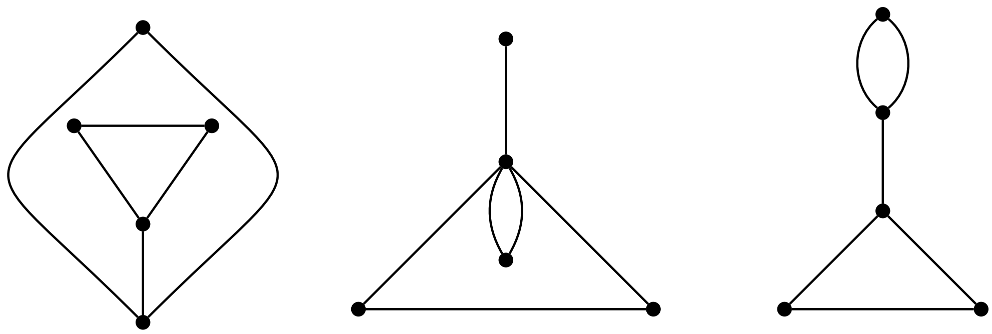
(i) Does the graph have an Eulerian circuit? Find one or explain why none exists.

(j) Does the graph have an Hamiltonian circuit? Find one or explain why none exists.

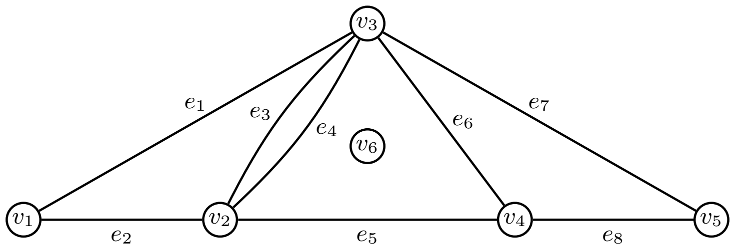
Problem 3. (10pt) Draw a graph that has the adjacency matrix given below. How can you tell from this matrix if the graph is undirected or directed?

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Problem 4. (10pt) Determine which of the following graphs are isomorphic. In each case, explain the isomorphism or explains why an isomorphism cannot exist.



Problem 5. (10pt) Let G be an undirected graph. The *degree sequence* of G is a monotonic non-increasing sequence of the degrees for vertices of G. For instance, consider the graph from Problem i:



This graph has degree sequence $(5, 4, 3, 2, 2, 0)$. For each of the monotonic non-increasing sequences below, determine whether they are the degree sequence for some simple graph G. If there is such a simple graph G, give an example. If no such simple graph is possible, explain why.

(a) $(4, 3, 2, 2, 2)$

(b) $(2, 2, 2, 0)$

(c) $(5, 4, 3, 2, 1, 0)$

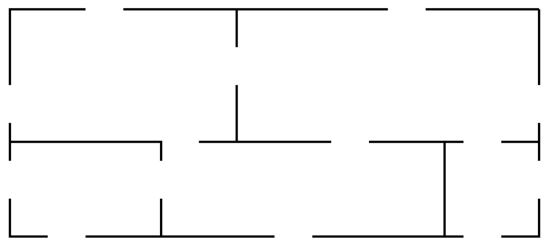
(d) $(4, 3, 3, 3, 3)$

(e) $(4, 3, 2, 1)$

Problem 6. (10pt) Prove that every simple graph with $n \geq 2$ vertices has at least two distinct vertices of the same degree. [Hint: Pigeonhole Principle]

Problem 7. (10pt) Prove that in a tree, T, every pair of vertices u and v has a unique path connecting them.

Problem 8. (10pt) Suppose the ground floor plan of a building is given below. Is it possible to walk through every door on the first floor exactly once, ending up in your starting room? Explain. Is it possible to visit every room exactly once, ending up in your starting room? Explain.



Problem 9. (10pt) Define the following matrices:

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \\ 0 & 3 \\ -2 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

For each of the following operations, either compute the given expression or explain why it is undefined.

(a) $2A - B$

(b) AB

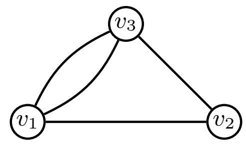
(c) $A + C$

(d) Av

(e) AC

(f) CA

Problem 10. (10pt) Consider the graph G given below:



(a) Compute the adjacency matrix, A.

(b) Using a computer system, compute A, A^2, A^3, and A^4.

(c) Compute the number of walks from v_1 to v_3 of lengths one, two, three, and four, respectively.

(d) Compare your answers from (b) and (c). Make a conjecture on what the a_{ij} entry of A^k represents.