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MATH 101
Summer 2022
HW 11: Due 06/15

"It is strange that only extraordinary men make the discoveries, which later appear so easy and simple."
–Georg C. Lichtenberg

Problem 1. (10pt) Write the following exponential functions in the form $y = Ab^x$:

(a) $y = -3(2^{3x})$

(b) $f(x) = 4\left(\frac{5}{7}\right)^{-x}$

(c) $g(x) = -6(5^{1-3x})$

(d) $h(x) = 9\left(\frac{3}{2}\right)^{2x-1}$

Solution.

(a)
$$y = -3(2^{3x}) = -3(2^3)^x = -3(8^x)$$

(b)
$$f(x) = 4\left(\frac{5}{7}\right)^{-x} = 4\left(\left(\frac{5}{7}\right)^{-1}\right)^x = 4\left(\frac{7}{5}\right)^x$$

(c)
$$g(x) = -6(5^{1-3x}) = -6 \cdot 5^1 \cdot 5^{-3x} = -30(5^{-3})^x = -30\left(\frac{1}{125}\right)^x$$

(d)
$$h(x) = 9\left(\frac{3}{2}\right)^{2x-1} = 9 \cdot \left(\frac{3}{2}\right)^{2x} \cdot \left(\frac{3}{2}\right)^{-1} = 9 \cdot \left(\left(\frac{3}{2}\right)^2\right)^x \cdot \frac{2}{3} = 6 \cdot \left(\frac{9}{4}\right)^x$$

Problem 2. (10pt) Write the following exponential functions in the form $y = Ab^{-x}$:

(a) $y = 6(2^x)$

(b) $f(x) = -7\left(\frac{1}{3}\right)^x$

(c) $g(x) = 5\left(\frac{1}{6}\right)^{2x}$

(d) $h(x) = 3^{3x+1}$

Solution.

(a)

$$y = 6(2^x) = 6(2^{-1})^{-x} = 6\left(\frac{1}{2}\right)^{-x}$$

(b)

$$f(x) = -7\left(\frac{1}{3}\right)^x = -7\left(\left(\frac{1}{3}\right)^{-1}\right)^{-x} = -7(3^{-x})$$

(c)

$$g(x) = 5\left(\frac{1}{6}\right)^{2x} = 5\left(\left(\frac{1}{6}\right)^2\right)^x = 5\left(\frac{1}{36}\right)^x = 5\left(\left(\frac{1}{36}\right)^{-1}\right)^{-x} = 5(36^{-x})$$

(d)

$$h(x) = 3^{3x+1} = 3^{3x} \cdot 3^1 = 3(3^3)^x = 3(27^x) = 3(27^{-1})^{-x} = 3\left(\frac{1}{27}\right)^{-x}$$

Problem 3. (10pt) Find an integer n so that each of the following logarithms are between n and $n+1$, i.e. estimate the logarithm without the use of a calculator. Be sure to show all your work.

(a) $\log_2(11)$

(b) $\log_3(187)$

(c) $\log_{1/2}(5)$

(d) $\log_5(\frac{1}{20})$

Solution.

(a) Because $2^3 = 8 < 11$ and $2^4 = 16 > 11$, we know that $3 < \log_2(11) < 4$.

(b) Because $3^4 = 81 < 187$ and $3^5 = 243 > 187$, we know that $4 < \log_3(187) < 5$.

(c) Because $(\frac{1}{2})^{-2} = (2^{-1})^{-2} = 2^2 = 4 < 5$ and $(\frac{1}{2})^{-3} = (2^{-1})^{-3} = 2^3 = 8 > 5$, we know that $-3 < \log_{1/2}(5) < -2$.

(d) Because $5^{-2} = \frac{1}{25} < \frac{1}{20}$ and $5^{-1} = \frac{1}{5} > \frac{1}{20}$, we know that $-2 < \log_5(\frac{1}{20}) < -1$.

Problem 4. (10pt) For each of the following, either express the given exponential equation in terms of logarithms or express the given logarithmic equation in terms of exponentials:

(a) $5^x = 9$

(b) $\log_3(x) = 4$

(c) $2^3 = x$

(d) $\log_7(2) = x$

Solution.

(a) $5^x = 9 \iff \log_5(9) = x$

(b) $\log_3(x) = 4 \iff 3^4 = x$

(c) $2^3 = x \iff \log_2(x) = 3$

(d) $\log_7(2) = x \iff 7^x = 2$

Problem 5. (10pt) Showing all your work, compute the following exactly:

(a) $\log_2(64)$

(b) $\log_3\left(\frac{1}{27}\right)$

(c) $\ln(1)$

(d) $\log_{2/3}\left(\frac{3}{2}\right)$

(e) $\log_8(8)$

Solution.

(a) $\log_2(64) = \log_2(2^6) = 6$

(b) $\log_3\left(\frac{1}{27}\right) = \log_3\left(\frac{1}{3^3}\right) = \log_3(3^{-3}) = -3$

(c) $\ln(1) = 0$

(d) $\log_{2/3}\left(\frac{3}{2}\right) = \log_{2/3}\left(\left(\frac{2}{3}\right)^{-1}\right) = -1$

(e) $\log_8(8) = 1$

Problem 6. (10pt) For each of the following, express the given logarithm in terms of $\log_b$ for the given base b:

(a) $\log_5(25)$, $b = 2$

(b) $\log_7(64)$, $b = 8$

(c) $\log_3(10)$, $b = e$

(d) $\log_{20}(6)$, $b = 6$

Solution.

(a)

$$\log_5(25) = \frac{\log_2(25)}{\log_2(5)}$$

(b)

$$\log_7(64) = \frac{\log_8(64)}{\log_8(7)} = \frac{2}{\log_8(7)}$$

(c)

$$\log_3(10) = \frac{\log_e(10)}{\log_e(3)} = \frac{\ln(10)}{\ln(3)}$$

(d)

$$\log_{20}(6) = \frac{\log_6(6)}{\log_6(20)} = \frac{1}{\log_6(20)}$$

Problem 7. (10pt) Express each of the following logarithms in terms of $\log x$, $\log y$, $\log z$, and a constant term:

(a) $\log_2(x^2 y)$

(b) $\log_7\left(\dfrac{xy^2}{z^3}\right)$

(c) $\ln\left(\dfrac{xz^{-1}}{\sqrt[3]{y}}\right)$

(d) $\log_5(25x\sqrt{y})$

Solution.

(a)

$$\log_2(x^2 y) = \log_2(x^2) + \log_2(y) = 2\log_2(x) + \log_2(y)$$

(b)

$$\log_7\left(\frac{xy^2}{z^3}\right) = \log_7(xy^2) - \log_7(z^3) = \log_7(x) + \log_7(y^2) - \log_7(z^3) = \log_7(x) + 2\log_7(y) - 3\log_7(z)$$

(c)

$$\ln\left(\frac{xz^{-1}}{\sqrt[3]{y}}\right) = \ln(xz^{-1}) - \ln(\sqrt[3]{y}) = \ln(x) + \ln(z^{-1}) - \ln(y^{1/3}) = \ln(x) - \ln(z) - \frac{1}{3}\ln(y)$$

(d)

$$\log_5(25x\sqrt{y}) = \log_5(25) + \log_5(x) + \log_5(\sqrt{y}) = 2 + \log_5(x) + \frac{1}{2}\log_5(y)$$

Problem 8. (10pt) Express each of the following logarithms in terms of a single logarithm involving no negative powers:

(a) $\log_2(x) - 5\log_2(y)$

(b) $-\frac{1}{2}\big(6\log_3(x) - \log_3(y)\big)$

(c) $5\ln(x^2) - 2\ln\left(\frac{1}{y}\right)$

(d) $\log_6(x) - 5\log_6(y) + 2$

Solution.

(a)

$$\log_2(x) - 5\log_2(y) = \log_2(x) - \log_2(y^5) = \log_2\left(\frac{x}{y^5}\right)$$

(b)

$$-\frac{1}{2}\big(6\log_3(x) - \log_3(y)\big) = -3\log_3(x) + \frac{1}{2}\log_3(y) = \log_3(x^{-3}) + \log_3(\sqrt{y}) = \log_3(x^{-3}\sqrt{y}) = \log_3\left(\frac{\sqrt{y}}{x^3}\right)$$

(c)

$$5\ln(x^2) - 2\ln\left(\frac{1}{y}\right) = \ln\big((x^2)^5\big) - \ln\left(\left(\frac{1}{y}\right)^{1/2}\right) = \ln(x^{10}) - \ln\left(\frac{1}{\sqrt{y}}\right) = \ln\left(\frac{x^{10}}{1/\sqrt{y}}\right) = \ln(x^{10}\sqrt{y})$$

(d)

$$\log_6(x) - 5\log_6(y) + 2 = \log_6(x) - 5\log_6(y) + \log_6(6^2) = \log_6(x) - \log_6(y^5) + \log_6(36) = \log_6\left(\frac{36x}{y^5}\right)$$

Problem 9. (10pt) Showing all your work, solve the following equation:

$$15^x + 10 = 20$$

Solution.

$$15^x + 10 = 20$$

$$15^x = 10$$

$$\log_1 5(15^x) = \log_{15}(10)$$

$$x = \log_{15}(10)$$

Problem 10. (10pt) Showing all your work, solve the following equation:

$$6(2^{3x}) - 2 = 34$$

Solution.

$$6(2^{3x}) = 36$$

$$2^{3x} = 6$$

$$\log_2(2^{3x}) = \log_2(6)$$

$$3x = \log_2(6)$$

$$x = \frac{\log_2(6)}{3}$$