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MATH 101
Fall 2023
HW 15: Due 12/06

"There should be no such thing as boring mathematics."
–Edsger Dijkstra

Problem 1. (10pt) Complete the square in $12x - 31 - x^2$ to find the vertex form of this quadratic function.

Solution. We have...

$$12x - 31 - x^2$$

$$-x^2 + 12x - 31$$

$$-(x^2 - 12x + 31)$$

$$-\left(x^2 - 12x + \left(\frac{-12}{2}\right)^2 - \left(\frac{-12}{2}\right)^2 + 31\right)$$

$$-\left(x^2 - 12x + 36 - 36 + 31\right)$$

$$-\left((x^2 - 12x + 36) + (-36 + 31)\right)$$

$$-\left((x-6)^2 - 5\right)$$

$$-(x-6)^2 + 5$$

$$5 - (x-6)^2$$

Therefore, the vertex form is $5 - (x-6)^2$, which implies the vertex is $(6, 5)$. Because $a = -1 < 0$, this quadratic function opens downwards.

Problem 2. (10pt) Use the 'evaluation method' to find the vertex form of the quadratic function $3x^2 + 6x - 7$.

Solution. A quadratic function $f(x) = ax^2 + bx + c$ has vertex located at $x_0 = -\frac{b}{2a}$. The y-coordinate of the vertex is then $y_0 = f(x_0)$. For the quadratic function $3x^2 + 6x - 7$, we have $a = 3$, $b = 6$, and $c = -7$. Now $x_0 = -\frac{b}{2a} = -\frac{6}{2(3)} = -\frac{6}{6} = -1$. The y-coordinate of the vertex is then $f(-1) = 3(-1)^2 + 6(-1) - 7 = 3(1) - 6 - 7 = 3 - 6 - 7 = -10$. Therefore, the vertex is $(-1, -10)$. We know that for this quadratic function, $a = 3$. The vertex form of a quadratic function is $a(x - P)^2 + Q$, where (P, Q) is the vertex. Therefore, the vertex form of this quadratic function is...

$$3x^2 + 6x - 7 = 3\big(x - (-1)\big)^2 + (-10) = 3(x + 1)^2 - 10$$

Problem 3. (10pt) Use completing the square to solve the following quadratic equation:

$$x(x+2) = 7$$

Solution. We have...

$$x(x+2) = 7$$

$$x^2 + 2x = 7$$

$$x^2 + 2x + \left(\frac{2}{2}\right)^2 = 7 + \left(\frac{2}{2}\right)^2$$

$$x^2 + 2x + 1 = 7 + 1$$

$$(x+1)^2 = 8$$

$$\sqrt{(x+1)^2} = \sqrt{8}$$

$$x + 1 = \pm\sqrt{8}$$

$$x = -1 \pm \sqrt{8}$$

$$x = -1 \pm \sqrt{4 \cdot 2}$$

$$x = -1 \pm 2\sqrt{2}$$

Problem 4. (10pt) Use the quadratic formula to solve the following quadratic equation:

$$x^2 = 2(5x - 11)$$

Solution. The quadratic formula solves quadratic equations of the form $f(x) = 0$, where $f(x)$ is a quadratic function. We first need find a $f(x)$. We have...

$$x^2 = 2(5x - 11)$$

$$x^2 = 10x - 22$$

$$x^2 - 10x + 22 = 0$$

Let $f(x) = x^2 - 10x + 22$. This is a quadratic function, i.e. a function of the form $ax^2 + bx + c$, with $a = 1$, $b = -10$, and $c = 22$. But then the quadratic function gives the solutions are...

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)22}}{2(1)} \\ &= \frac{10 \pm \sqrt{100 - 88}}{2} \\ &= \frac{10 \pm \sqrt{12}}{2} \\ &= \frac{10 \pm \sqrt{4 \cdot 3}}{2} \\ &= \frac{10 \pm 2\sqrt{3}}{2} \\ &= 5 \pm \sqrt{3} \end{aligned}$$