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MATH 308
Fall 2022
HW 18: Due 12/06

"Combinatorialists use recurrence, generating functions, and such transformations as the Vandermonde convolution; others, to my horror, use contour integrals, differential equations, and other resources of mathematical analysis."

–John Riordan

Problem 1. (10pt) By counting functions ('ordinary' functions, injections, or surjections), showing all your work and fully explaining your reasoning, answer the following:

(a) How many ways 5 people can be assigned to 8 tasks, where each person can only be assigned to a single task but a task may have more than one person assigned to it. [Ans: 32,768]

(b) How many ways 5 people can be assigned to 8 tasks, where each person can only be assigned to a single task and each task may only have one person assigned to it. [Ans: 6,720]

(c) How many ways can 5 people be assigned to 3 tasks, where each task must have at least one person assigned to it? [Ans: 150]

Solution.

(a) This is the number of functions from the 5 people to the 8 tasks. This is $8^5 = 32768$.

(b) This is the number of injections from the set of 5 people to the set of 8 tasks. This is ${}_8P_5 = 6,720$.

(c) This is the number of surjections from the set of 5 people to the set of 3 tasks. This is…

$$\sum_{k=0}^{3-1} (-1)^k \binom{3}{k} (3-k)^5 = \binom{3}{0}3^5 - \binom{3}{1}2^5 + \binom{3}{2}1^5 - \binom{3}{3}0^5 = 243 - 96 + 3 - 0 = 150$$

Problem 2. (10pt) Using the principle of inclusion-exclusion, how many integers between 1 and 1000, inclusive, are...

(a) Divisible by at least one of 2, 3, 5? [Ans: 734]

(b) Divisible by 2 and 3 but not by 5? [Ans: 133]

(c) Divisible by 5 but not 2 nor 3? [Ans: 67]

(d) Divisible by 2, 3, and 5? [Ans: 33]

Solution. Let A be the set of multiples of 2, B be the set of multiples of 3, and C be the set of multiples of 5.
We then have...

$$\begin{aligned}
|A| &= \lfloor\frac{1000}{2}\rfloor = 500 \\
|B| &= \lfloor\frac{1000}{3}\rfloor = 333 \\
|C| &= \lfloor\frac{1000}{5}\rfloor = 200 \\
|A \cap B| &= \lfloor\frac{1000}{6}\rfloor = 166 \\
|A \cap C| &= \lfloor\frac{1000}{10}\rfloor = 100 \\
|B \cap C| &= \lfloor\frac{1000}{15}\rfloor = 66 \\
|A \cap B \cap C| &= \lfloor\frac{1000}{30}\rfloor = 33
\end{aligned}$$

(a) This is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 500 + 333 + 200 - 166 - 100 - 66 + 33 = 734$$

(b) This is

$$|(A \cap B) \setminus C| = |A \cap B| - |A \cap B \cap C| = 166 - 33 = 133$$

(c) This is

$$|C \setminus (A \cup B)| = |C| - |C \cap (A \cup B)| = |C| - |C \cap A| - |C \cap B| + |C \cap A \cap B| = 200 - 100 - 66 + 33 = 67$$

(d) This is

$$|A \cap B \cap C| = 33$$

Problem 3. (10pt) Showing all your work and fully explaining your reasoning, use the (general) binomial theorem to answer the following:

(a) What is the coefficient of x^4y^{10} in $(x+y)^{14}$? [Ans: 1001]

(b) What is the coefficient of x^6y^5 in $(2x-3y)^{11}$? [Ans: −7,185,024]

(c) What is the coefficient of $x^{17}yz^2$ in $(x+y+z)^{20}$? [Ans: 3,420]

Solution.

(a) By the binomial theorem, the coefficient is $\binom{14}{4} = 1001$.

(b) Writing $X = 2x$ and $Y = -3y$, this is the coefficient of X^6Y^5 in $(X+Y)^{11}$, which is $\binom{11}{6}$. But because $X^6 = (2x)^6 = 2^6x^6$ and $Y^5 = (-3y)^5 = (-3)^5y^5$, the coefficient is $2^6 \cdot (-3)^5 \cdot \binom{11}{6} = -7185024$.

(c) The coefficient is given by the generalized binomial theorem and is $\binom{20}{17,1,2} = \dfrac{20!}{17!1!2!} = 3420$.

Problem 4. (10pt) Using the theory of dearrangements, showing all your work, and fully explaining your reasoning, answer the following:

(a) Find all the dearrangements of the set $S = \{1, 2, 3\}$.

(b) How many dearrangements are there for a set with four elements? [Ans: 9]

(c) Approximate how many dearrangements there are for a set with 10 elements. [Ans: 1,334,961]

Solution.

(a) The dearrangements are 231 and 312.

(b) The number of dearrangements are...

$$4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 4! - \frac{4!}{4!} + \frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!} = 0 + 12 - 4 + 1 = 9$$

total dearrangements.

(c) There are approximately

$$D_{10} \approx \frac{n!}{e} = \frac{10!}{e} \approx 1334960.91612293 \approx 1334961$$

total dearrangements.