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MATH 108
Spring 2023
HW 14: Due 05/01

"Of the many forms of false culture, a premature converse with abstractions is perhaps the most likely to prove fatal to the growth of a masculine vigour of intellect."

–George Boole

Problem 1. (10pt) Consider the following system of equations:

$$\begin{aligned} 3x - 2y &= -8 \\ -x + 3y &= 5 \end{aligned}$$

(a) Find the coefficient matrix, A.

(b) Show that A has an inverse.

(c) Use your answer from (b) to find the solution to the system of equations.

Solution.

(a) The coefficient matrix is the matrix of column-by-column coefficients for the variables—properly aligned. Because the variables are already aligned, we have...

$$A = \begin{pmatrix} 3 & -2 \\ -1 & 3 \end{pmatrix}$$

(b) We know that A has an inverse, i.e. that A^{-1} exists, if and only if $\det A \neq 0$. We have...

$$\det A = \det \begin{pmatrix} 3 & -2 \\ -1 & 3 \end{pmatrix} = 3(3) - (-2)(-1) = 9 - 2 = 7 \neq 0$$

Because $\det A \neq 0$, we know that A^{-1} exists.

(c) Recall that that when written in vector form, i.e. $A\mathbf{x} = \mathbf{b}$, the matrix A is the coefficient matrix (written column-by-column in the same order as the variable vector), $\mathbf{x}$ is the variable vector, and $\mathbf{b}$ is the constant vector. If A^{-1} exists, multiplying both sides of $A\mathbf{x} = \mathbf{b}$ on the left by A^{-1}, we have...

$$A\mathbf{x} = \mathbf{b}$$

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

From (b), we know that A^{-1} exists. We need to find A^{-1}. But we know how to find the inverse of a 2×2 matrix:

$$\text{If } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \det A \neq 0, \text{ then } A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

But then we have...

$$A^{-1} = \frac{1}{7}\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}$$

Therefore, we know...

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{7}\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -8 \\ 5 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3(-8)+2(5) \\ 1(-8)+3(5) \end{pmatrix} \frac{1}{7}\begin{pmatrix} -24+10 \\ -8+15 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -14 \\ 7 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Then the solution is...

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

That is, the solution is $x = -2$ and $y = 1$.

Problem 2. (10pt) The RREF form of a matrix coming from a system of equations is shown below. Determine if there is a solution. If so, find the solution(s). If not, explain why the system does not have a solution.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

Problem 3. (10pt) The RREF form of a matrix coming from a system of equations is shown below. Determine if there is a solution. If so, find the solution(s). If not, explain why the system does not have a solution.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$