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MATH 100
Fall 2023
HW 7: Due 10/02

"This is the worst kind of discrimination — the kind against me!"
–Bender Bending Rodríguez, Futurama

Problem 1. (10pt) Consider the function given by $W(t) = 568.1 - 13.4t$.

(a) Is $W(t)$ a linear function? Explain.

(b) Find the slope of $W(t)$.

(c) Find the y-intercept of $W(t)$.

(d) Find the x-intercept of $W(t)$.

(e) Find a value of t for which $W(t) = 100$.

Solution.

(a) The function $W(t)$ is a linear function. We can see that $W(t)$ has the form $\ell(x) = mx + b$ with $m = -13.4$ and $b = 568.1$. Therefore, $W(t)$ is linear.

(b) From (a), we can see that the slope of $W(t)$ is $m = -13.4$.

(c) From (a), we can see that the y-intercept of $W(t)$ is $b = 568.1$.

(d) The x-intercept of $W(t)$ is the t-value for which $W(t) = 0$. But then, we have...

$$\begin{aligned} 568.1 - 13.4t &= 0 \\ 13.4t &= 568.1 \\ t &\approx 42.3955 \end{aligned}$$

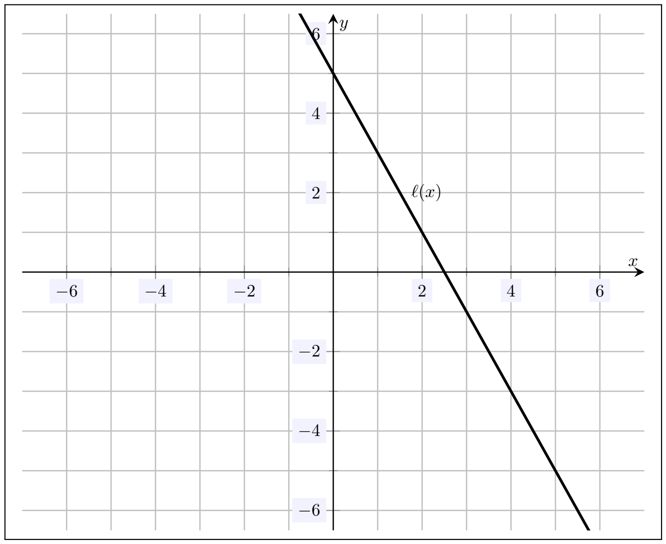
Therefore, the x-intercept of $W(t)$ is $t = 42.3955$, i.e. the point $(42.3955, 0)$.

(e) If t_0 is a value for which $W(t_0) = 100$, then we have...

$$\begin{aligned} W(t_0) &= 100 \\ 568.1 - 13.4t &= 100 \\ -13.4t &= -468.1 \\ t &\approx 34.9328 \end{aligned}$$

Because the steps above are reversible, we know that $W(34.9328) \approx 100$.

Problem 2. (10pt) Consider the relation plotted below.



(a) Is $\ell(x)$ a linear function? Explain.

(b) Find the equation for $\ell(x)$.

(c) Find the x and y-intercepts for $\ell(x)$.

(d) Find a value of x for which $\ell(x) = -3$.

Solution.

(a) The relation $\ell(x)$ is linear because its graph is a line.

(b) From (a), we know that $\ell(x)$ is linear. Therefore, $\ell(x) = mx + b$ for some m, b. Examining the plot above, we can see that $\ell(x)$ contains the points $(0, 5)$, $(1, 3)$, $(2, 1)$, $(3, -1)$, $(4, -3)$, and $(5, 5)$. Using the first two points, we have...

$$m = \frac{\Delta y}{\Delta x} = \frac{5 - 3}{0 - 1} = \frac{2}{-1} = -2$$

But then $\ell(x) = mx + b = -2x + b$. But because the line contains the point $(0, 5)$, we have...

$$\begin{aligned} \ell(0) &= -2(0) + b \\ 5 &= 0 + b \\ 5 &= b \end{aligned}$$

Therefore, $\ell(x) = -2x + 5$.

(c) The x-intercept for $\ell(x)$ is a value, say x_0, for which $f(x_0) = 0$. But then, we have...

$$\begin{aligned} \ell(x_0) &= 0 \\ -2x_0 + 5 &= 0 \\ -2x &= -5 \\ x &= 2.5 \end{aligned}$$

Because the steps above are reversible, we know $f(2.5) = 0$. Therefore, the x-intercept for $\ell(x)$ is $x = 2.5$, i.e. the points $(2.5, 0)$. We can also see and estimate this in the plot of $\ell(x)$ given above.

The y-intercept for $\ell(x)$ is the point where $\ell(x)$ intersects the y-axis. But this is where $x = 0$. Then we have $\ell(0) = -2(0) + 5 = 0 + 5 = 5$. Therefore, the y-intercept for $\ell(x)$ is $y = 5$, i.e. the point $(0, 5)$.

Alternatively, from (b), we know that $\ell(x) = -2x + 5$ has the form $y = mx + b$ with $m = -2$ and $b = 5$. But from (a), we know that $\ell(x)$ is linear so that $b = 5$ must represent the y-intercept, i.e. the point $(0, 5)$.

(d) Suppose that there is an x, say x_0, such that $\ell(x_0) = -3$. But then, we have...

$$\begin{aligned} \ell(x_0) &= -3 \\ -2x_0 + 5 &= -3 \\ -2x_0 &= -8 \\ x_0 &= 4 \end{aligned}$$

As all the steps above are reversible, we know that $\ell(4) = -3$.

Problem 3. (10pt) Consider the linear function that goes through the points $(-4, 5)$ and $(6, 0)$.

(a) Find the slope of this linear function.

(b) Find the equation of this linear function.

Solution.

(a) We know that $m = \frac{\Delta y}{\Delta x}$. But then...

$$m = \frac{\Delta y}{\Delta x} = \frac{0-5}{6-(-4)} = \frac{0-5}{6+4} = \frac{-5}{10} = -\frac{1}{2}$$

(b) Because $\ell(x)$ is a linear function, we know that $\ell(x) = mx + b$ for some m, b. From (a), we know that $m = -\frac{1}{2}$. Then $\ell(x) = -\frac{1}{2}\,x + b$. Because the line contains the point $(-4, 5)$, we know that it satisfies the equation for $\ell(x)$. But then...

$$\begin{aligned} \ell(-4) &= -\frac{1}{2} \cdot -4 + b \\ 5 &= \frac{4}{2} + b \\ 5 &= 2 + b \\ b &= 3 \end{aligned}$$

Therefore, $\ell(x) = -\frac{1}{2}\,x + 3$.

Problem 4. (10pt) A certain product requires \$800 of upfront costs to produce—the *fixed costs*. After this investment, it costs \$8.50 produce every item.

(a) Explain why the cost to produce q items, $C(q)$, is a linear function.

(b) Find the equation for $C(q)$.

(c) What does the y-intercept for $C(q)$ represent?

(d) How much does it cost to produce 10,000 items?

(e) What is the maximum number of items you could produce with \$6,000?

Solution.

(a) We know that $C(q)$ is a function because given any number of items produced, there is only one total cost of production associated with this production level. We know that $C(q)$ is a linear function because the rate of change of $C(q)$, i.e. the cost of producing additional items, is constant.

(b) From (a), we know that $C(q)$ is linear. Therefore, $C(q) = mq + b$ for some m, b. Because the rate of change of $C(q)$, i.e. the cost of producing additional items, is $\$8.50$, we know that $m = 8.50$. But then $C(q) = 8.50q + b$. We know that there are \$800 of upfront costs, i.e. costs before any production of items. We then know that $C(0) = \$800$. This shows...

$$\begin{aligned} C(0) &= 8.50q + b \\ 800 &= 8.50(0) + b \\ 800 &= 0 + b \\ b &= 800 \end{aligned}$$

Therefore, $C(q) = 8.50q + 800$.

(c) The y-intercept for $C(q)$ is the point where $C(q)$ intersects the y-axis. But the y-axis is where $q = 0$. Therefore, the y-intercept of $C(q)$ is $C(0) = 8.50(0) + 800 = \$800$, i.e. the point $(0, \$800)$. Equivalently, because $C(q)$ is linear with $m = 8.50$ and $b = 800$, we know that the y-intercept is $b = 800$, i.e. the point $(0, 800)$. This represents the upfront cost of \$800 to produce the items, i.e. the *fixed costs*.

(d) This is precisely $C(10000)$, which is...

$$C(10000) = 8.50(10000) + 800 = 85000 + 800 = \$85{,}800$$

Therefore, it costs \$85,800 in total to produce 10,000 items.

(e) One could only produce an amount of items q such that $C(q) \geq \$6{,}000$ with \$6,000. Then we know...

$$C(q) \leq 6000$$
$$8.50q + 800 \leq 6000$$
$$8.50q \leq 5200$$
$$q \leq 611.765$$

We assume you can not produce a partial item. So either $q = 611$ or $q = 612$. Clearly, one cannot afford to produce $q = 612$ items. Therefore, $q = 611$; that is, one could only afford to produce 611 items.