Name: ______________________________

MATH 308
Fall 2023
HW 9: Due 10/12

"We will always have STEM with us. Some things will drop out of the public eye and will go away, but there will always be science, engineering, and technology. And there will always, always be mathematics."

–Katherine Johnson

Problem 1. (10pt) For each of the following functions, determine whether the function is injective, surjective, or bijective. Be sure to fully justify your answer.

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 7 - 3x$

(b) $g : \mathbb{R} \to \mathbb{R}^{\geq 0}$, $g(x) = x^2 + 1$

(c) $h : \mathbb{R}^2 \to \mathbb{R}^2$, $h(x, y) = (x - 2y, -2x + 4y)$

(d) $k : [0, \infty) \to \mathbb{R}$, $k(x) = 6 - x^2$

Problem 2. (10pt) Let $A = B = \mathbb{R}$. Consider the function $f : A \to B$ given by $f(x) = x^2 - 4x + 7$.

(a) Sketch a graph of $f(x)$. Be sure your graph includes an interval around the vertex of $f(x)$.

(b) Is $f(x)$ injective? Explain. [Hint: $f(x) = (x-2)^2 + 3$.]

(c) Is $f(x)$ surjective? Explain. [Hint: $f(x) = (x-2)^2 + 3$.]

(d) Do your responses in (b) and (c) change if $A = [2, \infty)$? Explain.

(e) Do your responses in (b) and (c) change if $B = [3, \infty)$? Explain.

Problem 3. (10pt) Let $X = \{1, 2, 3, 4, 5\}$, $Y = \{1, 2, 3, 4\}$, and $A = \{1, 2, 3\}$. If S is a set and $\phi : S \to S$ is a function, we say that $s \in S$ is a *fixed point* for ϕ if $\phi(s) = s$. Recall that a function $\psi : \mathbb{R} \to \mathbb{R}$ is *strictly increasing* if $\psi(x) < \psi(y)$ for all $x, y \in \mathbb{R}$ with $x < y$.

(a) Determine a function $F : A \to Y$ that is nondecreasing with no fixed point. Be sure to fully specify the function and justify that F has the required properties.

(b) Determine a function $G : X \to Y$ such that $G\big|_A = F$ and G is neither surjective nor injective but so that G does have a fixed point. Be sure to fully specify the function and justify that G has the required properties.

(c) Is G a strictly increasing function? Explain.

Problem 4. (10pt) Below is a partial proof of the fact that if $f : X \to Y$ is a function and $A, B \subseteq Y$, then $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. By filling in the missing portions, complete the partial proof below so that it is a correct, logically sound proof with 'no gaps.'

Proposition. If $f : X \to Y$ is a function and $A, B \subseteq Y$, then $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

Proof. To prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, we need to show ______________________________

and ______________________________.

Clearly, if $f^{-1}(A \cup B) = \varnothing$, then $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$. Similarly, if $f^{-1}(A) \cup f^{-1}(B) = \varnothing$, then $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$. Assume neither $f^{-1}(A \cup B)$ nor $f^{-1}(A) \cup f^{-1}(B)$ are empty.

$f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$: Let $x \in$ __________________. But then $f(x) \in A \cup B$. This implies

that either $f(x) \in$ __________________ or $f(x) \in$ __________________.

Case 1, $f(x) \in$ __________________: If $f(x) \in A$, then __________________ $\in f^{-1}(A)$. But then

$x \in$ ________________________.

Case 2, $f(x) \in B$: If $x \in B$, then $x \in$ __________________. But then $x \in f^{-1}(A) \cup f^{-1}(B)$.

Therefore, if $x \in f^{-1}(A \cup B)$, we know that $x \in$ ______________________________. This shows

that $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$.

$f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$: Suppose that ______________________________. This implies

that $x \in f^{-1}(A)$ or $f^{-1}(B)$.

Case 1, __________________: If $x \in f^{-1}(A)$, then $f(x) \in A$. But then $x \in$ __________________.

This shows that $x \in f^{-1}(A \cup B)$.

Case 2, $x \in f^{-1}(B)$: If $x \in f^{-1}(B)$, then $f(x) \in$ __________________. But then $f(x) \in f(A \cup B)$.

This shows that $x \in$ __________________.

Therefore, if $x \in f^{-1}(A) \cup f^{-1}(B)$, we know that $x \in f^{-1}(A \cup B)$. This shows that ________________________.

But then we have shown that ______________________ and ______________________. Therefore, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.