Name: *Caleb McWhorter — Solutions*
MATH 308
Fall 2023
HW 11: Due 11/10

"Mathematics is the art of giving the same name to different things."
–Henri Poincaré

Problem 1. (10pt) Consider the relation $\sim$ on $\mathbb{Z}\times\mathbb{Z}$ given by $(a,b) \sim (c,d)$ if and only if $a+c = b+d$.

(a) What assumptions does this relation need to satisfy to be an equivalence relation?

(b) Is $(1,0) \sim (3,4)$? Explain.

(c) Is $(-2,1) \sim (1,1)$? Explain.

(d) Is this relation symmetric? Explain.

(e) Is this relation reflective? Explain.

Solution.

(a) A relation $\sim$ on a set $\mathcal{R}$ is an equivalence relation, if for all $x, y \in \mathcal{R}$,

  (i) *Reflexive*: $x \sim x$
  (ii) *Symmetric*: if $x \sim y$, then $y \sim x$.
  (iii) *Transitive*: if $x \sim y$ and $y \sim z$, then $x \sim z$.

(b) We know that $(a,b) \sim (c,d)$ if and only if $a+c = b+d$. Observe that $4 = 1+3 = 0+4 = 4$. Therefore, $(1,0) \sim (3,4)$.

(c) We know that $(a,b) \sim (c,d)$ if and only if $a+c = b+d$. Observe that $-1 = -2+1 \neq 1+1 = 2$. Therefore, $(-2,1) \not\sim (1,1)$.

(d) A relation on a set $\mathcal{R}$, $(\mathcal{R}, \sim)$, is symmetric if and only if for all $x, y \in \mathcal{R}$, if $x \sim y$, then $y \sim x$. Assume $(a,b) \sim (c,d)$. We then know that $a+c = b+d$. But $a+c = c+a$ and $b+d = d+b$. Then we know that $c+a = d+b$, which implies that $(c,d) \sim (a,b)$.

(e) This reflection is *not* reflective. Recall that a relation on a set $\mathcal{R}$, $(\mathcal{R}, \sim)$, is reflexive if $x \sim x$ for all $x \in \mathcal{R}$. Consider $(0,1) \in \mathbb{Z}\times\mathbb{Z}$. We do not have $(0,1) \sim (0,1)$, so that $(\sim, \mathbb{Z}\times\mathbb{Z})$ is not reflective and thus not an equivalence relation. We know $(0,1) \not\sim (0,1)$ because $0 = 0+0 \neq 1+1 = 2$.

Generally, let $(a,b) \in \mathbb{Z}\times\mathbb{Z}$. If $(a,b) \sim (a,b)$, then $2a = a+a = b+b = 2b$. This implies that $2a = 2b$, so that $a = b$. Therefore, if $(a,b) \sim (a,b)$, then $a = b$. Conversely, consider $(a,a) \in \mathbb{Z}\times\mathbb{Z}$. We know that $(a,a) \sim (a,a)$ because $2a = a+a = a+a = 2a$. Therefore, $(a,b) \sim (a,b)$ if and only if $a = b$.

Problem 2. (10pt) Showing all your work, compute the following:

(a) $\displaystyle\sum_{k=-3}^{100} 5$

(b) $\displaystyle\sum_{k=0}^{200} k^2$

(c) $\displaystyle\sum_{k=100}^{200} k$

(d) $\displaystyle\sum_{k=0}^{150} (2k-3)$

Solution. Let $\{a_k\}, \{b_k\}$ be sequences and $c \in \mathcal{R}$. Recall the following formulas:

$$\sum_{k=a}^{b}(a_k+b_k) = \sum_{k=a}^{b} a_k + \sum_{k=a}^{b} b_k, \quad \sum_{k=a}^{b} c a_k = c\sum_{k=a}^{b} a_k, \quad \sum_{k=a}^{b} c = (b-a+1)c, \quad \sum_{k=0}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) Using the third formula from above, we have...

$$\sum_{k=-3}^{100} 5 = 5\big(100-(-3)+1\big) = 5 \cdot 104 = 520$$

(b) Using the last formula above, we have...

$$\sum_{k=0}^{200} k^2 = \left.\frac{n(n+1)(2n+1)}{6}\right|_{n=200} = \frac{200(200+1)(2\cdot 200+1)}{6} = \frac{200 \cdot 201 \cdot 401}{6} = 2{,}686{,}700$$

(c) Index shifting the summation and applying the first, fourth, and third formulae above, we have...

$$\sum_{k=100}^{200} k = \sum_{k=0}^{100}(k+100) = \sum_{k=0}^{100} k + \sum_{k=0}^{100} 100 = \left.\frac{n(n+1)}{2}\right|_{n=100} + 100(100-0+1) = \frac{100(101)}{2} + 100(101) = 5{,}050 + 10{,}100 = 15{,}150$$

Alternatively, we can use the fact that for $a, b \in \mathbb{N}$ with $a < b$, we have $\sum_{k=0}^{b} a_k = \sum_{k=0}^{a} a_k + \sum_{k=a}^{b} a_k$, i.e. $\sum_{k=a}^{b} a_k = \sum_{k=0}^{b} a_k - \sum_{k=0}^{a} a_k$. Using this observation and the fourth formula above, we have...

$$\sum_{k=100}^{200} k = \sum_{k=0}^{200} k - \sum_{k=0}^{99} k = \left.\frac{n(n+1)}{2}\right|_{n=200} - \left.\frac{n(n+1)}{2}\right|_{n=99} = \frac{200(201)}{2} - \frac{99(100)}{2} = 20{,}100 - 4{,}950 = 15{,}150$$

(d) Applying the first, second, fourth, and third formulae from above, we have...

$$\sum_{k=0}^{150}(2k-3) = \sum_{k=0}^{150} 2k - \sum_{k=0}^{150} 3 = 2\sum_{k=0}^{150} k - \sum_{k=0}^{150} 3 = 2 \cdot \left.\frac{n(n+1)}{2}\right|_{n=150} - 3(150-0+1) = 2 \cdot 11{,}325 - 3 \cdot 151 = 22{,}650 - 453 = 22{,}197$$

Problem 3. (10pt) Showing all your work, find a closed-form expression for the following sum:

$$\sum_{k=2}^{n}(2k^2 - k + 4n)$$

Solution. Let $\{a_k\}, \{b_k\}$ be sequences and $c \in \mathcal{R}$. Recall the following formulas:

$$\sum_{k=a}^{b}(a_k+b_k) = \sum_{k=a}^{b} a_k+\sum_{k=a}^{b} b_k, \quad \sum_{k=a}^{b} ca_k = c\sum_{k=a}^{b} a_k, \quad \sum_{k=a}^{b} c = (b-a+1)c, \quad \sum_{k=0}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Index shifting the summation and using the formulae above, we have...

$$\begin{aligned}
\sum_{k=2}^{n}(2k^2 - k + 4n) &= \sum_{k=0}^{n-2}\big(2(k+2)^2 - (k+2) + 4n\big) \\
&= \sum_{k=0}^{n-2}\big(2(k^2+4k+4) - (k+2) + 4n\big) \\
&= \sum_{k=0}^{n-2}\big(2k^2+8k+8-k-2+4n\big) \\
&= \sum_{k=0}^{n-2}\big(2k^2+7k+(6+4n)\big) \\
&= \sum_{k=0}^{n-2} 2k^2 + \sum_{k=0}^{n-2} 7k + \sum_{k=0}^{n-2}(6+4n) \\
&= 2\sum_{k=0}^{n-2} k^2 + 7\sum_{k=0}^{n-2} k + \sum_{k=0}^{n-2}(4n+6) \\
&= 2\cdot \left.\frac{N(N+1)(2N+1)}{6}\right|_{N=n-2} + 7\cdot \left.\frac{N(N+1)}{2}\right|_{N=n-2} + (4n+6)\cdot\big((n-2)-0+1\big) \\
&= 2\cdot\frac{(n-2)(n-1)\big(2(n-2)+1\big)}{6} + 7\cdot\frac{(n-2)(n-1)}{2} + (4n+6)(n-1) \\
&= \frac{(n-2)(n-1)(2n-4+1)}{3} + \frac{7(n-2)(n-1)}{2} + (4n+6)(n-1) \\
&= \frac{(n-2)(n-1)(2n-3)}{3} + \frac{7(n-2)(n-1)}{2} + (4n+6)(n-1) \\
&= \frac{4n^3+27n^2-25n-6}{6}
\end{aligned}$$