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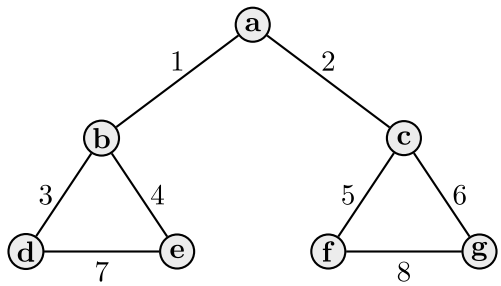
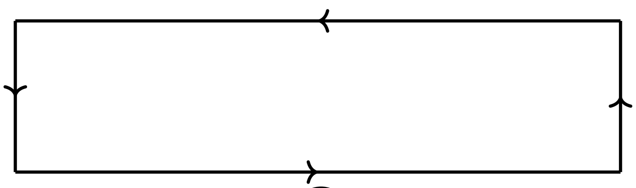
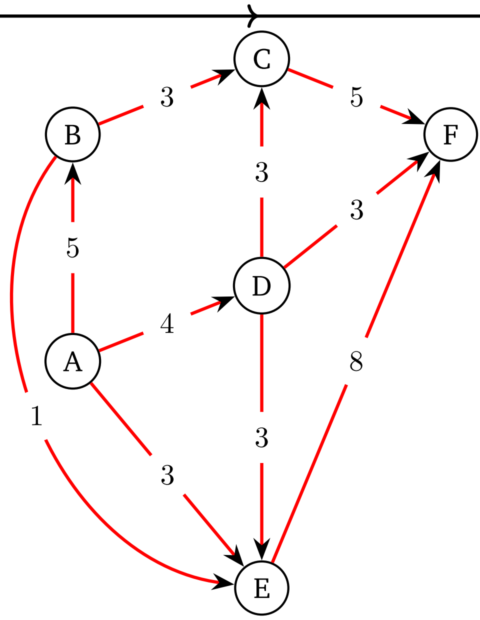
MATH 308
Fall 2023
HW 18: Due 12/12

> *"It has been said that geometry is the art of applying good reasoning to bad diagrams."*
>
> –*Richard J. Trudeau*

Problem 1. (10pt) Consider the graph G shown below.







(a) Is the graph G connected? Explain.

(b) Is $d7e4b1a2c5f8g$ a trail? Explain. Is it a path? Explain.

(c) Is $c5f8g6c2a$ a path? Explain. Is this walk closed? Explain.

(d) Does this graph have a circuit? Explain.

(e) Let A_G denote the adjacency matrix of G. Given the following:

$$A_G^{10} = \begin{pmatrix} 860 & 746 & 746 & 681 & 681 & 681 & 681 \\ 746 & 1282 & 940 & 884 & 884 & 543 & 543 \\ 746 & 940 & 1282 & 543 & 543 & 884 & 884 \\ 681 & 884 & 543 & 743 & 742 & 401 & 401 \\ 681 & 884 & 543 & 742 & 743 & 401 & 401 \\ 681 & 543 & 884 & 401 & 401 & 743 & 742 \\ 681 & 543 & 884 & 401 & 401 & 742 & 743 \end{pmatrix}$$

How many closed walks of length 10 are there starting at a? Explain.

Solution.

(a) The graph G is connected because between any distinct vertices v, w, there is a walk from v to w.

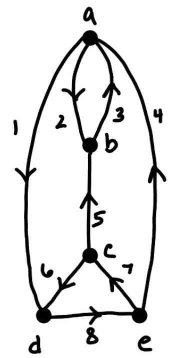
(b) This walk does not repeat an edge. Therefore, this walk is a trail. Because this walk does not repeat an edge or a vertex, this walk is also a path.

(c) This walk is not a path because the vertex c is repeated. This walk is also open and not closed because the walk does not start and end at the same point.

(d) This graph has six circuits. For instance, the walk $b3d7e4b$ is a circuit.

(e) A closed walk is a walk that starts and ends at the same point. So a closed walk starting at a must also end at a. Therefore, we need find the number of walks of length 10 from a to itself. We know the number of walks from v_i to v_j of length k is the a_{ij} entry of A_G^k. Therefore, the number of walks from a to a is the a_{11} entry of $A_G^{!0}$. We can see there are 860 walks of length 10 from a to a.

Problem 2. (10pt) Consider the graph G shown below.



(a) Does there exist a Hamiltonian circuit for this graph? Explain.

(b) Does there exist an Euler circuit for this graph? Explain.

(c) Find the adjacency matrix for this graph.

(d) Find the number of walks from a to b of length 4. Be sure to justify your answer.

Solution.

(a) A Hamiltonian circuit is a simple circuit (a closed walk with at least one edge and no repeated edge nor repeated vertex—except the first and last) that includes every vertex of G. The walk $b3a1d8e7c5b$ includes every vertex of G, starts and stops at the same vertex (a closed walk), and has no repeated edge or vertex (except the first and last). Therefore, G has a Hamiltonian circuit.

(b) An Euler circuit is a circuit (a closed walk with at least one edge that does not contain a repeated edge) that includes every every edge (and hence every vertex) of G. If a graph G has an Euler circuit, then one can begin the circuit at any vertex.[1] Without loss of generality, assume the Euler circuit begins at b. One must then move alone 3 to a. If one then moves along 2 to b, then one must repeat edge 3 to walk to the remaining vertices of G. Therefore, if one begins at b, one must move along 3 to a. One then must move along 1 to d and then along 8 to e. One must then either travel along edge 4 or 7. If one travels along 4, then one must either travel along vertex 2—which again forces a repetition of edge 3—or travel along edge 1, which is a repeated edge. Therefore, one must travel along 7 to vertex c. One can then only travel along vertex 5 or 6. If one travels along edge 5, then one must repeat edge 3. If one travels along edge 6, then one must repeat edge 8. But then one cannot proceed from vertex c—which is not the starting vertex—without repeating an edge. Therefore, G cannot have an Euler circuit. Alternatively, recall that a directed graph has an Euler circuit if and only if the graph is connected and $\deg^+ v = \deg^- v$ for all $v \in V(G)$. Observe that that the graph G is connected. However, observe that $\deg^-(b) = 2 \neq 1 = \deg^+(b)$. Therefore, G does not have an Euler circuit.

[1] If the Euler circuit is $v_0e_0v_1e_1 \cdots e_{i-1}v_ie_iv_{i+1}e_{i+1} \cdots e_{n-1}v_0$ and one wishes to begin at v_i, one can use the Euler circuit $v_ie_iv_{i+1}e_{i+1} \cdots e_{n-1}v_0e_0v_1e_1 \cdots e_{i-1}v_i$.

(c) Recall that the adjacency matrix for a graph G, A_G, is the matrix given by $A_G = (a_{ij})$, where a_{ij} is the number of edges from v_i to v_j. Ordering the vertices as a, b, c, d, e, the adjacency matrix is...

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(d) The number of walks from v_i to v_j of length k is the a_{ij} entry of the matrix A_G^k, where A_G is the adjacency matrix of G. We have...

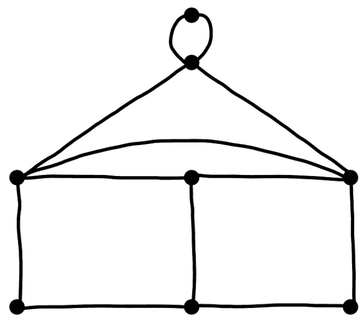
$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A_G^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \end{pmatrix}$$

$$A_G^4 = (A_G^2)^2 = \begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 2 & 1 \\ 2 & 0 & 0 & 0 & 2 \\ 2 & 2 & 2 & 2 & 0 \end{pmatrix}$$

Therefore, the number of walks from a to b of length 4 is the a_{12} entry of A_G^4, which is 2. There are two walks of length four from a to b. We can find all such walks: $a1d8e7c5b$ and $a1d8e4a2b$.

Problem 3. (10pt) Consider the graph G shown below.



(a) Does there exist an Euler trail for this graph? If so, find one. If not, explain why.

(b) Does there exist an Euler circuit for this graph? If so, find one. If not, explain why.

(c) Does there exist a Hamiltonian circuit for this graph? If so, find one. If not, explain why.

Solution.

(a) For an undirected graph G, there exists an Euler trail in G if and only if G is connected and there are exactly two vertices of odd degree. If G has an Euler trail, it must be between the vertices with odd degree. First, observe that G is connected. Finally, observe that the degree of every vertex of G is even except the middle two vertices. Therefore, there exists an Euler trail for G.

(b) For an undirected graph G, there exists an Euler circuit in G if and only if G is connected and every vertex has positive even degree. Observe that the 'middle' two vertices of G have odd degree. Therefore, G does not have an Euler circuit.

Problem 4. (10pt) Showing all your work and fully justifying your reasoning, respond to the following:

(a) Does there exist a tree with 2023 vertices and 2024 edges? Explain.

(b) Does a graph with five vertices and four edges have to be a tree? Explain.

(c) Find two non-isomorphic trees with five vertices. Be sure to explain why they cannot be isomorphic.