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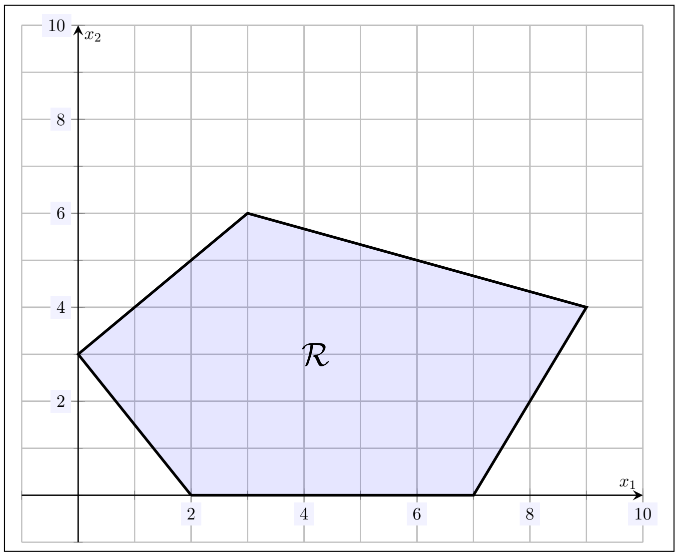
MATH 108
Fall 2023
HW 15: Due 12/12

"The linear programming was—and is—perhaps the single most important real-life problem."
–Keith Devin

Problem 1. (10pt) Consider the function $z = 5x_1 - 6x_2$ on the region $\mathcal{R}$ shown below. Does z have a maximum or minimum value on $\mathcal{R}$? Explain. If the function has a maximum or minimum value on $\mathcal{R}$, find the maximum and minimum value.

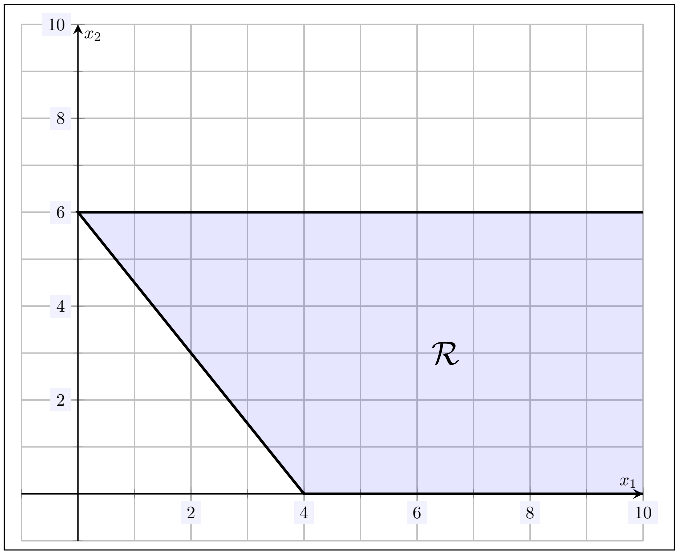


Solution. Observe that the function $z = 5x_1 - 6x_2$ is linear in x_1, x_2. The region $\mathcal{R}$ is nonempty and bounded. Therefore, the Fundamental Theorem of Linear Programming states that z must have a maximum and minimum value on $\mathcal{R}$ and these values occurs at a corner point for $\mathcal{R}$. So we need only evaluate z at the corner points of $\mathcal{R}$.

Corner Point	$z(x_1, x_2)$
$(2, 0)$	$z(2, 0) = 5(2) - 6(0) = 10 - 0 = 10$
$(7, 0)$	$z(7, 0) = 5(7) - 6(0) = 35 - 0 = 35$
$(9, 4)$	$z(9, 4) = 5(9) - 6(4) = 45 - 24 = 21$
$(3, 6)$	$z(3, 6) = 5(3) - 6(6) = 15 - 36 = -21$
$(0, 3)$	$z(0, 3) = 5(0) - 6(3) = 0 - 18 = -18$

Therefore, the maximum value for z is 35 and occurs at $(7, 0)$ and the minimum value for z is -21 and occurs at $(3, 6)$.

Problem 2. (10pt) Consider the function $z = -3x_1 + 8x_2$ on the region $\mathcal{R}$ shown below. Does z have a maximum or minimum value on $\mathcal{R}$? Explain. If the function has a maximum or minimum value on $\mathcal{R}$, find the maximum and minimum value.

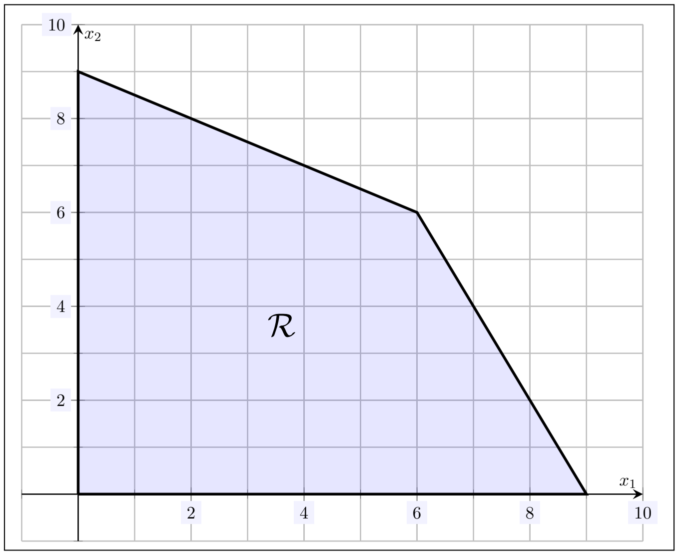


Solution. The function $z = -3x_1 + 8x_2$ is linear in x_1, x_2. The region $\mathcal{R}$ is nonempty. However, the region $\mathcal{R}$ is unbounded. Therefore, the Fundamental Theorem of Linear Programming does not apply. There may or may not be a maximum or minimum value for z on $\mathcal{R}$.

Observe that the larger the value of x_1, the smaller the value of z. If (x_1, x_2) is a point in the region $\mathcal{R}$, increasing x_1 moves the point to the right. But there is no limit on how far one can move to the point (x_1, x_2) in $\mathcal{R}$ to the right and remain in $\mathcal{R}$. Therefore, there is no limit to how much one can decrease z. This shows that z has no minimum value.

Observe that increasing x_1, i.e. decreasing x_1 increases z, and increasing x_2 increases z. If (x_1, x_2) is a point in the region $\mathcal{R}$, decreasing x_1 moves the point to the left and increasing x_2 moves the point upwards. However, there is a limit to how much one can move a point upwards and to the left and stay in the region. Therefore, z has a maximum value and it must occur at the point $(0, 6)$. The maximum value of z is then $z(0, 6) = -3(0) + 8(3) = 0 + 24 = 24$.

Problem 3. (10pt) Consider the function $z = x_1 - 9x_2$ on the region $\mathcal{R}$ shown below. Does z have a maximum or minimum value on $\mathcal{R}$? Explain. If the function has a maximum or minimum value on $\mathcal{R}$, find the maximum and minimum value.



Solution. The function $z = x_1 - 9x_2$ is linear in x_1, x_2. The region $\mathcal{R}$ is nonempty and bounded. By the Fundamental Theorem of Linear Programming, the function z has a maximum and minimum value on $\mathcal{R}$ and occurs at a corner point of $\mathcal{R}$. So we need only evaluate z at the corner points of $\mathcal{R}$.

Corner Point	$z(x_1, x_2)$
$(0, 0)$	$z(0, 0) = 0 - 9(0) = 0 - 0 = 0$
$(9, 0)$	$z(9, 0) = 9 - 9(0) = 9 - 0 = 9$
$(6, 6)$	$z(6, 6) = 6 - 9(6) = 6 - 54 = -48$
$(0, 9)$	$z(0, 9) = 0 - 9(9) = 0 - 81 = -81$

Therefore, the maximum value for z is 9 and occurs at $(9, 0)$ and the minimum value for z is -81 and occurs at $(0, 9)$.

Problem 4. (10pt) Find the dual problem for the minimization problem shown below.

$$\min w = 4y_1 + 6y_2 - 9y_3$$
$$\begin{cases} 7y_1 + 3y_2 + 8y_3 \geq 37 \\ 4y_1 - y_2 + 5y_3 \geq 55 \\ y_1 - y_2 + 3y_3 \leq 18 \\ y_1, y_2, y_3 \geq 0 \end{cases}$$

Solution. First, we need every inequality to be of the form '$\geq$' an integer. We multiply both sides of the third inequality by -1 to place this inequality in this form. This gives us the following inequalities (ignoring the non-negativity inequalities):

$$\begin{cases} 7y_1 + 3y_2 + 8y_3 \geq 37 \\ 4y_1 - y_2 + 5y_3 \geq 55 \\ -y_1 + y_2 - 3y_3 \geq -18 \end{cases}$$

We then form a matrix M from these inequalities with the function $w = 4y_1 + 6y_2 - 9y_3$ as the bottom row. This gives us the following matrix:

$$M = \begin{pmatrix} 7 & 3 & 8 & 37 \\ 4 & -1 & 5 & 55 \\ -1 & 1 & -3 & -18 \\ 4 & 6 & -9 & 0 \end{pmatrix}$$

We then compute the transpose of this matrix:

$$M^T = \begin{pmatrix} 7 & 4 & -1 & 4 \\ 3 & -1 & 1 & 6 \\ 8 & 5 & -3 & -9 \\ 37 & 55 & -18 & 0 \end{pmatrix}$$

This is the 'matrix of coefficients' for the inequalities for the corresponding dual maximization problem—the bottom row representing the function. The dual problem is a maximization problem so that the inequalities are '$\leq$.' Because there are 4 columns, there are $4 - 1 = 3$ variables in this system. [The last column corresponds to the 'opposite' side of the inequalities.] Therefore, the dual maximization problem is...

$$\max z = 37x_1 + 55x_2 - 18x_3$$
$$\begin{cases} 7x_1 + 4x_2 - x_3 \leq 4 \\ 3x_1 - x_2 + x_3 \leq 6 \\ 8x_1 + 5x_2 - 3x_3 \leq -9 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$