Name: ____________________________________

MATH 361
Spring 2024
HW 2: Due 02/01

"You say impossible, but all I hear is, 'I'm possible.'"

— Ted Lasso, Ted Lasso

Problem 1. (10pts) Showing all your work and fully justifying your reasoning, compute the following:

(a) $\displaystyle \lim_{x \to 0} \frac{\sin(3x)}{\cos(5x)}$

(b) $\displaystyle \frac{d}{dx}\left(\frac{xe^x}{x^2+1}\right)$

(c) $\displaystyle \frac{d^2}{dx^2}(x^2+5)^{10}$

(d) $\displaystyle \int xe^x \, dx$

Problem 2. (10pts) One of the first 'non-trivial' approximation techniques one learns is the process of linearization. Recall that if $f(x)$ is differentiable at c, the linearization of $f(x)$ at c, denoted $L(x)$, is the tangent line of $f(x)$ at $x = c$. But then for $x \approx c$, we have $f(x) \approx L(x)$. Consider the function $f(x) = \sqrt{x}$.

(a) Find the linearization of $f(x)$ at $x = 144$.

(b) Use (a) to approximation $\sqrt{150}$. What is the error for your approximation?

(c) Is this generally a useful method for computing $f(x) = \sqrt{x}$? Explain.

Problem 3. (10pts) Another of the first 'non-trivial' approximation techniques one learns is Taylor series. The Taylor series of a function can be used to approximate values of the function. In fact, the (infinite) Taylor series can be exactly equal to the function. Consider the polynomial $f(x) = x^3 - 5x^2 + 7$.

(a) Find the Taylor Series for $f(x)$ at $x = 1$.

(b) Show your Taylor Series in (a) is exactly $f(x)$.

(c) Assuming that $(x-1)^n$ is 'negligible' whenever $n > 1$ and $x \approx 1$, use (b) to approximate $f(1.01)$. What is the error for this approximation?

Problem 4. (10pts) Taylor series can also be used to approximate integrals that are not exactly computable. For instance, to find the percentage of values within one standard deviation of the mean for a normal distribution one would need to compute...

$$\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-x^2/2}\, dx$$

However, the integral $\displaystyle\int e^{-x^2/2}\, dx$ has no elementary antiderivative. Therefore, approximation must be used. Recall the Maclaurin series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and that this series has an infinite radius of convergence.

(a) Find the Maclaurin series for $e^{-x^2/2}$. Show that this series converges to $e^{-x^2/2}$ everywhere.

(b) Let $T_3(x)$ denote the first three nonzero terms from your series in (a). Approximate the integral above by using the fact that $e^{-x^2/2} \approx T_3(x)$ on $[-1, 1]$.

(c) It is a well-known fact in Statistics that approximately 68% of values in a normal distribution are within one standard deviation of the mean. Does your answer in (b) agree with this fact?