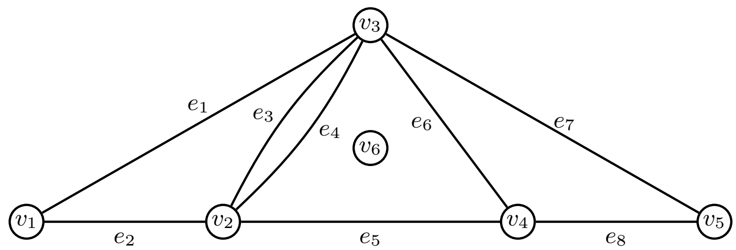
Name: *Caleb McWhorter — Solutions*
MATH 308
Fall 2021
HW 19: Due 12/15

"The origins of graph theory are humble, even frivolous."
–Norman Biggs

Problem 1. (10pt) Consider the following graph:



(a) Is the graph directed or undirected?

(b) Give the vertex set and the edge set for the graph.

(c) Give the adjacency matrix of the graph.

(d) Is the graph simple?

(e) Are there any isolated vertices?

(f) List all pairs of parallel edges.

(g) Compute the degree of the graph.

(h) Are the vertices v_1 and v_6 connected? What about the vertices v_1 and v_4?

(i) Does the graph $G \setminus \{v_6\}$ have an Eulerian circuit? Find one or explain why none exists.

(j) Does the graph $G \setminus \{v_6\}$ have an Hamiltonian circuit? Find one or explain why none exists.

Solution.

(a) This graph is undirected.

(b) We have. . .

$$V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

$$\begin{aligned} E(G) &= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\} \\ &= \{\{v_1, v_3\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}\} \end{aligned}$$

(c)

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(d) There are multiple edges as there are two edges between v_2 and v_3. Therefore, the graph is not simple.

(e) The vertex v_6 is isolated.

(f) The only pair of parallel edges are e_3 and e_4.

(g) For each i, we compute the degree of vertex v_i, denoted $\deg_i$.

$$\begin{aligned} v_1 &= 2 & v_4 &= 3 \\ v_2 &= 4 & v_5 &= 2 \\ v_3 &= 5 & v_6 &= 0 \end{aligned}$$

Then the degree of G is...

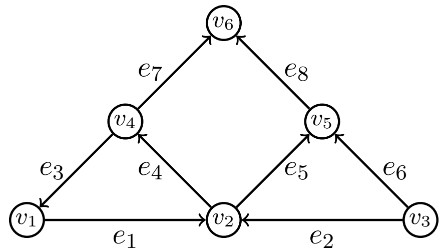
$$\deg G = \sum \deg_i = 2 + 4 + 5 + 3 + 2 + 0 = 16$$

(h) There is no walk from v_1 to v_6; therefore, v_1 and v_6 are not connected. However, there is a walk from v_1 to v_4, e.g. e_2, e_5 or $e_1, e_3, e_2, e_1, e_7, e_8$; therefore, v_1 and v_4 are connected.

(i) Recall that if a graph G is connected and the degree of every vertex of G is a positive even integer, then G has an Euler circuit. Therefore, if G does not have an Eulerian circuit then either G is not connected or not every vertex of G has positive even degree. The graph $G \setminus \{v_6\}$ is connected but not every vertex has positive even degree. For instance, the vertex v_4 has degree 3. Therefore, G does not have an Eulerian circuit.

(j) The graph $G \setminus \{v_6\}$ has a Hamiltonian circuit. For instance, beginning at v_1, the walk e_1, e_7, e_8, e_5, e_2 forms a Hamiltonian circuit.

Problem 2. (10pt) Give vertex and edge set for a graph (directed) and tell if....



(a) Is the graph directed or undirected?

(b) Give the vertex set and the edge set for the graph.

(c) Give the adjacency matrix of the graph.

(d) Is the graph simple?

(e) Are there any isolated vertices?

(f) Compute the in and out degree of each vertex.

(g) Compute the degree of the graph.

(h) Are the vertices v_1 and v_6 connected? What about the vertices v_6 and v_1?

(i) Does the graph have an Eulerian circuit? Find one or explain why none exists.

(j) Does the graph have an Hamiltonian circuit? Find one or explain why none exists.

Solution.

(a) This graph is directed.

(b) We have...

$$V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

$$\begin{aligned} E(G) &= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\} \\ &= \{(v_1, v_2), (v_3, v_2), (v_4, v_1), (v_2, v_4), (v_2, v_5), (v_3, v_5), (v_4, v_6), (v_5, v_8)\} \end{aligned}$$

(c)

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(d)

(e)

(f)

(g)

(h)

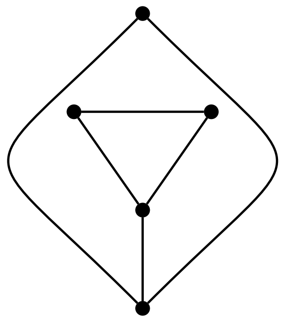
(i)

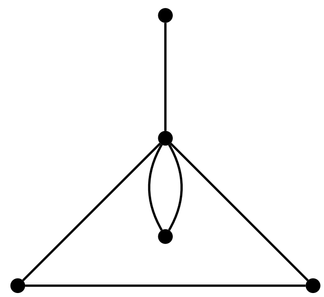
(j)

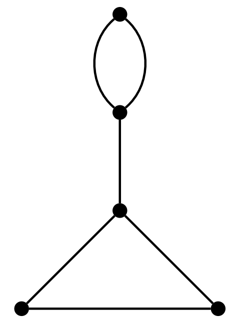
Problem 3. (10pt) Draw a graph that has the adjacency matrix given below. How can you tell from this matrix if the graph is undirected or directed?

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Problem 4. (10pt) Determine which of the following graphs are isomorphic. In each case, explain the isomorphism or explains why an isomorphism cannot exist.







Problem 5. (10pt) Given an undirected graph G, the *degree sequence* of G is a monotonic non-increasing sequence of the vertex degrees of G. For each of the following degree sequences for a simple graph G, either give an example of a graph G with the given degree sequence or prove that one cannot exist.

(a) $(4, 3, 2, 2, 2)$

(b) $(2, 2, 2, 0)$

(c) $(5, 4, 3, 2, 1, 0)$

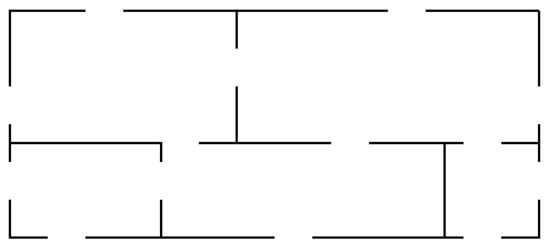
(d) $(4, 3, 3, 3, 3)$

(e) $(4, 3, 2, 1)$

Problem 6. (10pt) Prove that every nontrivial simple graph has two vertices of the same degree. [Hint: Pigeonhole Principle]

Problem 7. (10pt) Prove that in a tree, T, every distinct pair of vertices u and v has a unique path connecting them.

Problem 8. (10pt) Suppose the ground floor plan of a building is given below. Is it possible to walk through every door on the first floor exactly once, ending up in your starting room? Explain. Is it possible to visit every room exactly once, ending up in your starting room? Explain.



Problem 9. (10pt) Define the following matrices:

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \\ 0 & 3 \\ -2 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

For each of the following operations, either compute the given expression or explain why it is undefined.

(a) $2A - B$

(b) AB

(c) $A + C$

(d) Av

(e) AC

(f) CA

Solution.

(a)

(b)

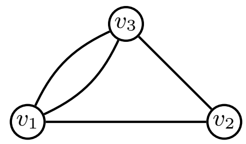
(c)

(d)

(e)

(f)

Problem 10. (10pt) Consider the graph G given below:



(a) Compute the adjacency matrix, A.

(b) Using a computer system, compute A, A^2, A^3, and A^4.

(c) Compute the number of walks from v_1 to v_3 of lengths one, two, three, and four, respectively.

(d) Compare your answers from (b) and (c). Make a conjecture on what the a_{ij} entry of A^k represents.

Solution.

(a)

(b)

(c)

(d)