**MAT 308: Exam 1** **Name:** ____________________

**Fall – 2023**

**10/12/2023**

**'$\infty$' Minutes**

Write your name on the appropriate line on the exam cover sheet. This exam contains 13 pages (including this cover page) and 10 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| 10 | 10 | |
| Total: | 100 | |

1. (10 points) Consider the logical expressions $\neg(P \wedge Q)$ and $(P \to \neg Q) \vee (\neg P \wedge \neg Q)$.

   (a) Show that these two expressions are logically equivalent by constructing the truth table for both expressions.

   (b) Show that $\neg(P \wedge Q) \equiv (P \to \neg Q) \vee (\neg P \wedge \neg Q)$ by 'simplifying' the expression $(P \to \neg Q) \vee (\neg P \wedge \neg Q)$.

   (c) Without computing the truth table for $P \wedge Q \Leftrightarrow (P \to \neg Q) \vee (\neg P \wedge \neg Q)$, determine whether this expression is a tautology, contradiction, or neither. Explain how you came to your conclusion.

**Solution.**

(a) Constructing the truth tables for $\neg(P \wedge Q)$ and $(P \to \neg Q) \vee (\neg P \wedge \neg Q)$, we have...

| $P$ | $Q$ | $P \wedge Q$ | $\neg(P \wedge Q)$ | $\neg P$ | $\neg Q$ | $P \to \neg Q$ | $\neg P \wedge \neg Q$ | $(P \to \neg Q) \vee (\neg P \wedge \neg Q)$ |
|---|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | **F** | $F$ | $F$ | $F$ | $F$ | **F** |
| $T$ | $F$ | $F$ | **T** | $F$ | $T$ | $T$ | $F$ | **T** |
| $F$ | $T$ | $F$ | **T** | $T$ | $F$ | $T$ | $F$ | **T** |
| $F$ | $F$ | $F$ | **T** | $T$ | $T$ | $T$ | $T$ | **T** |

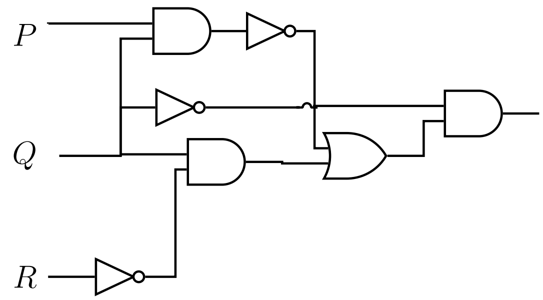
Because the fourth and last column (the bolded columns) have the same output for each input of $P, Q$, we know that $\neg(P \wedge Q) \equiv (P \to \neg Q) \vee (\neg P \wedge \neg Q)$.

(b) We use the fact that $A \to B \equiv \neg A \vee B$, the associativity of $\vee$, the absorption law $A \vee (A \wedge B) \equiv A$, and finally De Morgan's Law $\neg(A \wedge B) \equiv \neg A \vee \neg B$:

$$\begin{aligned}
(P \to \neg Q) \vee (\neg P \wedge \neg Q) &\equiv (\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q) \\
&\equiv \neg P \vee \neg Q \vee (\neg P \wedge \neg Q) \\
&\equiv \neg P \vee \big(\neg Q \vee (\neg P \wedge \neg Q)\big) \\
&\equiv \neg P \vee \neg Q \\
&\equiv \neg(P \wedge Q)
\end{aligned}$$

(c) We know from (a) that $\neg(P \wedge Q) \equiv (P \to \neg Q) \vee (\neg P \wedge \neg Q)$. This shows that $\neg(P \wedge Q)$ and $(P \to \neg Q) \vee (\neg P \wedge \neg Q)$ have the same logical outputs for each input of $P, Q$. Now $\neg(P \wedge Q)$ and $\neg\big(\neg(P \wedge Q)\big) \equiv P \wedge Q$ must have opposite outputs for each input of $P, Q$. But then $P \wedge Q$ and $(P \to \neg Q) \vee (\neg P \wedge \neg Q)$ will have opposite logical outputs for each input $P, Q$. Therefore, the biconditional $P \wedge Q \Leftrightarrow (P \to \neg Q) \vee (\neg P \wedge \neg Q)$ will be false for each input of $P, Q$. This shows that $P \wedge Q \Leftrightarrow (P \to \neg Q) \vee (\neg P \wedge \neg Q)$ is a contradiction.

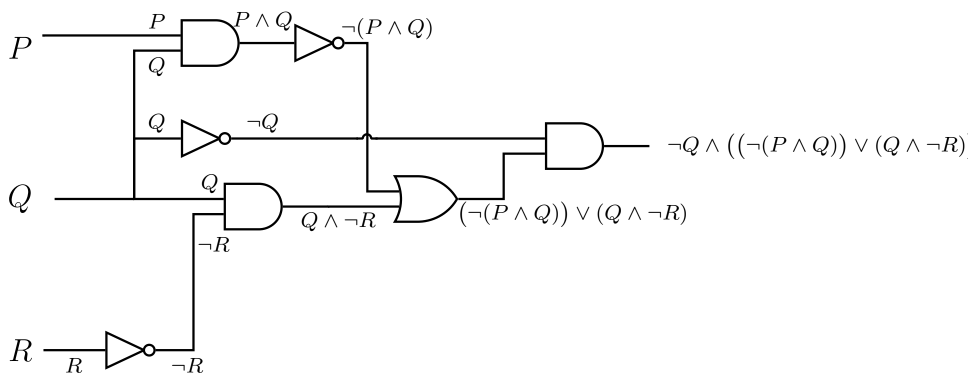
2. (10 points) Consider the digital logic circuit given below. We shall find a simpler circuit that has the same 'behavior', i.e. input-output table, as the given circuit.




(a) Find the logic expression corresponding to this circuit.
(b) Simplify the logical expression in (a) as much as possible.
(c) Construct the digital logic circuit corresponding to your answer in (b).

**Solution.**

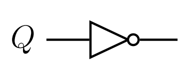
(a) We label each input/output on the circuit diagram.




(b) We have...

$$\begin{aligned}
\neg Q \wedge \big((\neg(P \wedge Q)) \vee (Q \wedge \neg R)\big) &\equiv \neg Q \wedge ((\neg P \vee \neg Q) \vee (Q \wedge \neg R)) \\
&\equiv \neg Q \wedge (\neg P \vee \neg Q \vee (Q \wedge \neg R)) \\
&\equiv \neg Q \wedge (\neg P \vee ((\neg Q \vee Q) \wedge (\neg Q \vee \neg R))) \\
&\equiv \neg Q \wedge (\neg P \vee (T_0 \wedge (\neg Q \vee \neg R))) \\
&\equiv \neg Q \wedge (\neg P \vee (\neg Q \vee \neg R)) \\
&\equiv (\neg Q \wedge \neg P) \vee (\neg Q \wedge (\neg Q \vee \neg R)) \\
&\equiv (\neg Q \wedge \neg P) \vee \neg Q \\
&\equiv \neg Q
\end{aligned}$$

(c)



3. (10 points) Let $z, q$ be free variables with universe $\mathbb{Z}$ and $\mathbb{Q}$, respectfully. Define the predicate $P(z, q)\colon z \neq 0 \to qz = 1$. Consider the quantified statement...

$$(\forall z)(\exists q)P(z, q)$$

(a) Write the quantified statement above as a complete English sentence.

(b) Determine whether the given quantified statement is true or false. Be sure to justify your answer.

(c) Write the negation, converse, and contrapositive of the given quantified statement as complete English sentences.

**Solution.**

(a) For all integers $z$, there exists a rational number $q$ such that if $z \neq 0$, then $qz = 1$.

(b) The statement is true. If $z = 0$, take $q$ to be any rational number. Because $z = 0$, $z \neq 0$ is false, which implies that $z \neq 0 \to qz = 1$ is true. If $z \neq 0$, define $q := \frac{1}{z}$, which is possible because $z \neq 0$. But then $z \neq 0$ and $qz = \frac{1}{z} \cdot z = 1$. Therefore, $z \neq 0 \to qz = 1$ is true. Every $z \in \mathbb{Z}$ is either zero or nonzero. For each such $z$, we have found at least one rational number $q$ such that $z \neq 0 \to qz = 1$ is true. Therefore, the quantified statement is true. The statement given is the statement that every nonzero integer has a multiplicative inverse which is rational.

(c) First, recall that $\neg(P \to Q) \equiv P \wedge \neg Q$. But then...

$$\neg P(z, q) \equiv \neg(z \neq 0 \to qz = 1) \equiv \neg(z \neq 0) \wedge (qz = 1) \equiv (z = 0) \wedge (qz = 1)$$

*Negation:* The negation of the quantified statement is...

$$\neg(\forall z)(\exists q)P(z, q) \equiv (\exists z)\neg(\exists q)P(z, q) \equiv (\exists z)(\forall q)\neg P(z, q) \equiv (\exists z)(\forall q)\left[(z = 0) \wedge (qz = 1)\right]$$

As an English sentence, this is the statement, "There exists an integer $z$ such that for all integers $q$, $z = 0$ and $qz = 1$." Because the original quantified statement was true, this negation clearly must be false.

4. (10 points) Define sets $A, B, C, D$ as given below with universe $\mathcal{U} = (-\infty, \infty)$.

$$A = (-10, 10) \qquad C = [5, 20)$$
$$B = (-15, 0] \qquad D = [-6, 6]$$

(a) $B^c$

(b) $A \Delta C$

(c) $D \setminus B$

(d) $(A \cup B)^c$

(e) $A \cap (B \cup C)$

5. (10 points) Let $X = \{a, b, c\}$ and $Y = \{1, 2\}$.

   (a) Compute $\mathcal{P}(X)$.

   (b) Compute $\mathcal{P}(X) \times Y$.

   (c) Compute $|\mathcal{P}(X) \times Y|$.

   (d) Is $\big(\{b, a, a\}, 1\big) \in \mathcal{P}(X) \times Y$?

6. (10 points) For natural numbers $n$, let $A_n = \left[-\frac{1}{n}, \frac{n+1}{n}\right)$, and for real numbers $x$, let $B_x = (x-1, x+1)$. Compute the following:

   (a) $\displaystyle\bigcup_{n\in\mathbb{N}} A_n$

   (b) $\displaystyle\bigcap_{n\in\mathbb{N}} A_n$

   (c) $\displaystyle\bigcup_{x\in\mathbb{R}} B_x$

   (d) $\displaystyle\bigcap_{x\in\mathbb{R}} B_x$

7. (10 points) Let $\Phi\colon \mathbb{R} \to \mathbb{R}$ be given by $x \mapsto 2x^2 + 11x - 6$.

   (a) Is $7 \in \operatorname{im}\Phi$? Explain.

   (b) Is $-11 \in \Phi^{-1}(-5)$? Explain.

   (c) Compute $\operatorname{im}\Phi$.

   (d) Compute $\Phi^{-1}(0)$.

   (e) Is $(3, 11)$ on the graph of $\Phi$? Explain.

   (f) Is $\Phi$ a decreasing function? Explain.

   (g) Is $\Phi$ a negative function? Explain.

**Solution.**

(a) If $7 \in \operatorname{im}\Phi$, then there exists $x_0 \in \mathbb{R}$ such that $\Phi(x_0) = 7$. But then...

$$\begin{aligned} \Phi(x) &= 7 \\ 2x_0^2 + 11x_0 - 6 &= 7 \end{aligned}$$

*— Continued Space for Problem 7 —*

8. (10 points) Complete the following parts, being sure to fully justify your reasoning:

   (a) Is the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + y^3 - 7$ injective?

   (b) Is the function $g : \mathbb{R} \to (-\infty, 10]$ given by $x \mapsto 10 - (x - 3)^2$ surjective?

   (c) Is the function $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = \pi x - \sqrt{2}$ bijective?

**Solution.**

(a)

(b)

(c) The function $h(x) = \pi x - \sqrt{2}$ is a linear function, i.e. a function of the form $mx+b$ with $m = \pi$ and $b = -\sqrt{2}$. Because $m = \pi \neq 0$, $h(x)$ is not constant. All non-constant linear functions are injective and surjective—hence bijective. Therefore, $h(x)$ is a bijective function.

9. (10 points) Below is a partial proof of the fact that if $A, B, C$ are sets, then $A \times (B \setminus C) = (A \times B) - (A \times C)$. By filling in the missing portions, complete the partial proof below so that it is a correct, logically sound proof with 'no gaps.'

**P**roposition. If $A, B, C$ are sets, then $A \times (B \setminus C) = (A \times B) - (A \times C)$.

*Proof.* To show that $A \times (B \setminus C) = (A \times B) - (A \times C)$, we need to show that $A \times (B \setminus C) \subseteq (A \times B) - (A \times C)$ and $(A \times B) - (A \times C) \subseteq A \times (B \setminus C)$.

If $A \times (B \setminus C) = \varnothing$, we clearly have $A \times (B \setminus C) \subseteq (A \times B) - (A \times C)$. Similarly, if $(A \times B) - (A \times C) = \varnothing$, then we have $(A \times B) - (A \times C) \subseteq A \times (B \setminus C)$. Now assume that $A \times (B \setminus C)$ and $(A \times B) - (A \times C)$ are nonempty.

$A \times (B \setminus C) \subseteq (A \times B) - (A \times C)$: Let $(x, y) \in A \times (B \setminus C)$. We want to show that $(x, y) \in (A \times B) - (A \times C)$. By definition, $(x, y) \in A \times (B \setminus C)$ implies that _______________ and _______________. Because $y \in B \setminus C$, we know that ____________________ and ____________________. Now $x \in A$ and $y \in B$, so that $(x, y) \in A \times B$. Because $y \notin C$, we know that $(x, y) \notin A \times C$. Because ____________________ and ____________________, we know that $(x, y) \in (A \times B) - (A \times C)$. But then if $(x, y) \in A \times (B \setminus C)$, we know that $(x, y) \in (A \times B) - (A \times C)$. Therefore, ______________________________.

$(A \times B) - (A \times C) \subseteq A \times (B \setminus C)$: Let ______________________________. We want to show that $(x, y) \in A \times (B \setminus C)$. Because $(x, y) \in (A \times B) - (A \times C)$, we know that ______________________ and ______________________. By definition, because $(x, y) \in A \times B$, $x \in A$ and $y \in B$. We also know that $(x, y) \notin A \times C$. So either ____________________ or ____________________. But we know that $x \in A$, so that it must be that $y \notin C$. But then $y \in B$ and ________________, which implies that $y \in B \backslash C$. Because ________________ and $y \in B \setminus C$, we know that $(x, y) \in A \times (B \setminus C)$. Then if $(x, y) \in (A \times B) - (A \times C)$, we know ________________________________. Therefore, $(A \times B) - (A \times C) \subseteq A \times (B \setminus C)$.

Because _____________________________ and _____________________________, we know that $A \times (B \setminus C) = (A \times B) - (A \times C)$. $\square$

10. (10 points) Below is a partial proof of results about the surjectivity of functions. By filling in the missing portions, complete the partial proof below so that it is a correct, logically sound proof with 'no gaps.'

**P**roposition. Let $A, B, C$ be sets and $f : A \to B, g : B \to C$ be functions.

(i) If $f, g$ are surjective, then $g \circ f$ is surjective.

(ii) If $g \circ f$ is surjective, then $g$ is surjective.

*Proof.*

(i) Suppose that $f, g$ are surjective. We want to show that ________________________.

We need to show that if ______________, there exists ______________ such that $(g \circ f)(a) = c$. Because $g$ is surjective, there exists $b \in B$ such that ______________.

Because ______________________, there exists $a \in A$ such that $f(a) = b$. But then ________________________________. Therefore, $g \circ f$ is surjective.

(ii) Assume that __________________________. We want to show that $g$ is surjective.

We need to show that if ______________, there exists ______________ such that ____________________. Let $c \in C$. Because $g \circ f : A \to C$ is surjective, there exists $a \in A$ such that $(g \circ f)(a) = c$. We know that $(g \circ f)(a) = g\big(f(a)\big)$. Because $f : A \to B$, we know that $f(a) \in$ ______________. Then there is ______________ such that $b = f(a)$. Then $g(b) =$ ___________________. Therefore, $g$ is surjective.

$\square$