MAT 104: Exam 1
Spring – 2023
03/03/2023
85 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 16 pages (including this cover page) and 15 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
15	10	
Total:	150	

1. (10 points) Suppose that $f(x)$ is a function with domain $[-10, 10)$ and $g(x)$ is a function with domain $(-5, 15]$. Several outputs for the functions $f(x)$ and $g(x)$ are given below.

x	0	1	2	3	5
$f(x)$	3	0	2	4	4
$g(x)$	3	5	1	2	0

(a) What is the domain of $f - g$?

(b) Given the information above, what is the largest possible domain for $\frac{f}{g}$?

(c) Find $(f + g)(2)$.

(d) Find $(fg)(0)$.

(e) Find $(g \circ f)(0)$.

(a) For $f - g$ to be defined, both f, g need to be defined because $(f - g)(x) = f(x) - g(x)$, so $f(x)$ and $g(x)$ need to be defined. Therefore, the domain of $f - g$ is the 'overlap' of the domains for $f(x)$ and $g(x)$. Therefore, the domain of $f - g$ is $(-5, 10)$.

$$\textit{Domain } f - g\colon (-5, 10)$$

(b) For $\frac{f}{g}$ to be defined, both f, g need to be defined because $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, so $f(x)$ and $g(x)$ need to be defined. Therefore, the domain of $f - g$ is the 'overlap' of the domains for $f(x)$ and $g(x)$. The overlap of these domains is $(-5, 10)$. However, for $\frac{f(x)}{g(x)}$ to be defined, we also need $g(x) \neq 0$. From the table, we see that if $x = 5$, then $g(x) = g(5) = 0$. Therefore, this also cannot be in the domain of $\frac{f}{g}$. Excluding this value from the interval $(-5, 10)$ yields $(-5, 5) \cup (5, 10)$. Therefore, the largest the domain of $\frac{f}{g}$ can be is $(-5, 5) \cup (5, 10)$.

$$\textit{Domain } \frac{f}{g}\colon (-5, 5) \cup (5, 10)$$

(c) We have...

$$(f + g)(2) = f(2) + g(2) = 2 + 1 = 3$$

(d) We have...

$$(fg)(0) = f(0)g(0) = 3 \cdot 3 = 9$$

(e) We have...

$$(g \circ f)(0) = g\big(f(0)\big) = g(3) = 2$$

2. (10 points) Let $f(x)$ be an exponential function with $f(-2) = 12$ and $f(3) = \frac{3}{8}$. Showing all your work, find $f(x)$.

Solution. *Because $f(x)$ is an exponential function, we know that $f(x) = ab^x$ for some a, b. But then we know…*

$$\begin{aligned} f(-2) &= ab^{-2} \\ f(3) &= ab^3 \end{aligned}$$

But using the fact that $f(-2) = 12$ and $f(3) = \frac{3}{8}$, we know…

$$\begin{aligned} 12 &= ab^{-2} \\ \frac{3}{8} &= ab^3 \end{aligned}$$

Dividing each of these, we have…

$$\frac{12}{3/8} = \frac{ab^{-2}}{ab^3} = \frac{\cancel{a}b^{-2}}{\cancel{a}b^3} = \frac{b^{-2}}{b^3} = \frac{1}{b^5}$$

Then using the fact that $\frac{12}{3/8} = 12 \cdot \frac{8}{3} = 4 \cdot 8 = 32$. Therefore, we have…

$$\begin{aligned} \frac{1}{b^5} &= 32 \\ b^5 &= \frac{1}{32} \\ b &= \sqrt[5]{\frac{1}{32}} \\ b &= \frac{1}{2} \end{aligned}$$

But then using the fact that $12 = ab^{-2}$, we have…

$$12 = ab^{-2} = a\left(\frac{1}{2}\right)^{-2} = a \cdot 2^2 = 4a$$

But then $a = \frac{12}{4} = 3$. Therefore, we know that…

$$f(x) = 3\left(\frac{1}{2}\right)^x$$

3. (10 points) Showing all your work, find the inverse of the function $f(x) = \log_3(x^2) - 4$.

Solution. *We have...*

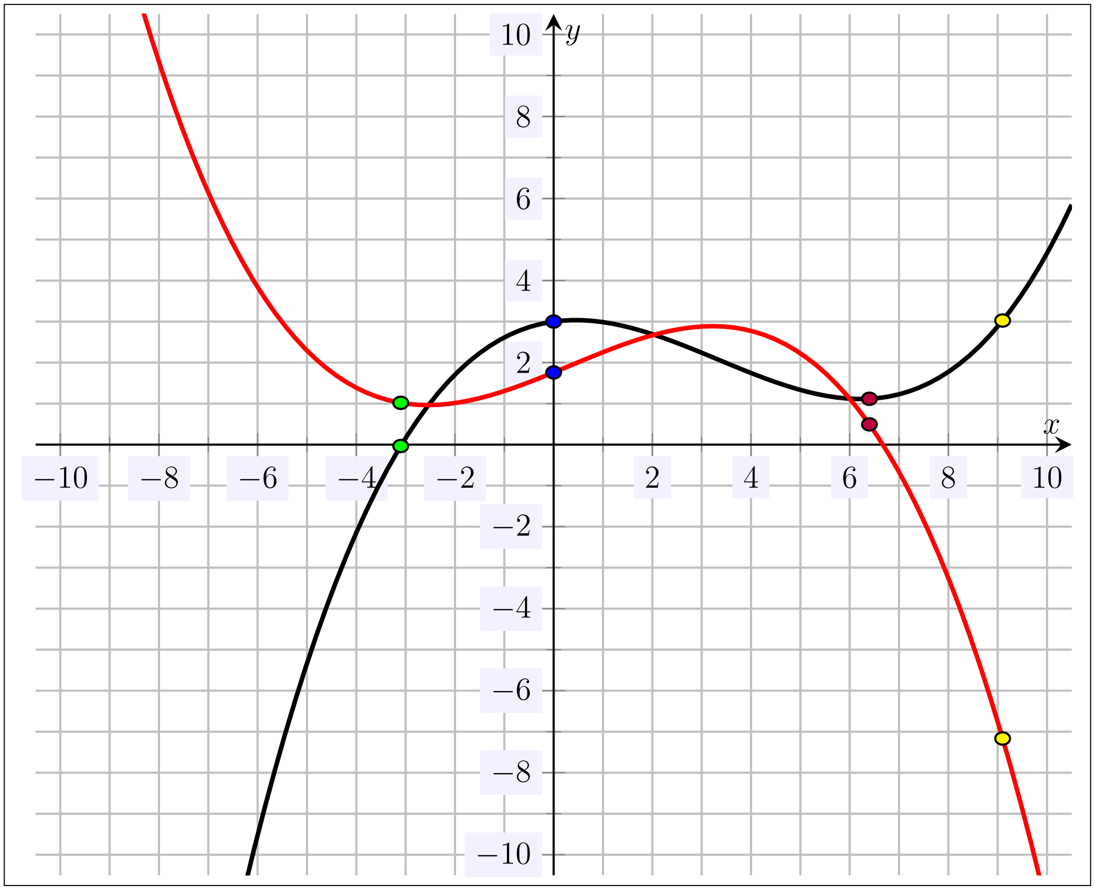
$$f(x) = \log_3(x^2) - 4$$
$$y = \log_3(x^2) - 4$$
$$\Downarrow$$
$$x = \log_3(y^2) - 4$$
$$x = 2\log_3(y) - 4$$
$$x + 4 = 2\log_3(y)$$
$$\frac{x+4}{2} = \log_3(y)$$
$$3^{\frac{x+4}{2}} = 3^{\log_3(y)}$$
$$y = 3^{\frac{x+4}{2}}$$
$$f^{-1}(x) = 3^{\frac{x+4}{2}}$$

We can also verify that this is the inverse:

$$(f \circ f^{-1}(x) = f\big(f^{-1}(x)\big) = f\left(3^{\frac{x+4}{2}}\right) = \log_3\left(\left(3^{\frac{x+4}{2}}\right)^2\right) - 4 = \log_3\left(3^{x+4}\right) - 4 = (x+4) - 4 = x$$

$$(f^{-1} \circ f)(x) = f^{-1}\big(f(x)\big) = f^{-1}\left(\log_3(x^2) - 4\right) = 3^{\frac{\log_3(x^2)-4+4}{2}} = 3^{\frac{\log_3(x^2)}{2}} = 3^{\frac{2\log_3(x)}{2}} = 3^{\log_3(x)} = x$$

4. (10 points) A function $f(x)$ is plotted below. As accurately as possible, sketch the function $4 - f(x+3)$ on the graph below.



Solution. *Compared to $f(x)$, the graph of $f(x+3)$ is shifted 3 to the left. Compared to the graph of $f(x)$, the graph of $-f(x+3)$ is shifted 3 to the left and then reflected across the x-axis. Finally, compared to the graph of $f(x)$, the graph of $f(x)$, the graph of $4-f(x+3)$ is shifted 3 to the left, reflected across the x-axis, and shifted up by 4. We can then sketch the function $4-f(x+3)$ above. We can also transform several points on the original graph as described above to find the resulting point and connect these to sketch the graph $4-f(x+3)$. These are shown by matching colored points on the plot above.*

5. (10 points) Let $f(x) = 3 - 2x$ and $g(x) = x^2 + x - 1$. Showing all your work and simplifying as much as possible, find the following:

   (a) $f\big(g(0)\big)$
   (b) $(2f - g)(0)$
   (c) $(g - f)(x)$
   (d) $(f \circ g)(x)$
   (e) $(g \circ f)(x)$

Solution.

(a) We have...

$$f\big(g(0)\big) = f(0^2 + 0 - 1) = f(0 + 0 - 1) = f(-1) = 3 - 2(-1) = 3 + 2 = 5$$

(b) We have...

$$(2f - g)(0) = 2f(0) - g(0) = 2\big(3 - 2(0)\big) - (0^2 + 0 - 1) = 2(3 - 0) - (0 + 0 - 1) = 2(3) - (-1) = 6 + 1 = 7$$

(c) We have...

$$(g - f)(x) = g(x) - f(x) = (x^2 + x - 1) - (3 - 2x) = x^2 + x - 1 - 3 + 2x = x^2 + 3x - 4 = (x - 1)(x + 4)$$

(d) We have...

$$(f \circ g)(x) = f\big(g(x)\big) = f(x^2 + x - 1) = 3 - 2(x^2 + x - 1) = 3 - 2x^2 - 2x + 2 = -2x^2 - 2x + 5$$

(e) We have...

$$(g \circ f)(x) = g\big(f(x)\big) = g(3 - 2x) = (3 - 2x)^2 + (3 - 2x) - 1 = (4x^2 - 12x + 9) + (3 - 2x) - 1 = 4x^2 - 14x + 11$$

6. (10 points) Showing all your work, find the inverse of the function $f(x) = 3e^{1-2x}$.

Solution. *We have...*

$$\begin{aligned} f(x) &= 3e^{1-2x} \\ y &= 3e^{1-2x} \\ &\Downarrow \\ x &= 3e^{1-2y} \\ \frac{x}{3} &= e^{1-2y} \\ \ln\left(\frac{x}{3}\right) &= \ln\left(e^{1-2y}\right) \\ \ln\left(\frac{x}{3}\right) &= 1-2y \\ 2y &= 1-\ln\left(\frac{x}{3}\right) \\ y &= \frac{1-\ln\left(\frac{x}{3}\right)}{2} \\ f^{-1}(x) &= \frac{1-\ln\left(\frac{x}{3}\right)}{2} \end{aligned}$$

We can also verify that this is the inverse:

$$(f \circ f^{-1}(x) = f\big(f^{-1}(x)\big) = f\left(\frac{1-\ln\left(\frac{x}{3}\right)}{2}\right) = 3e^{1-2\cdot\frac{1-\ln\left(\frac{x}{3}\right)}{2}} = 3e^{1-(1-\ln\left(\frac{x}{3}\right)} = 3e^{1-1+\ln\left(\frac{x}{3}\right)} = 3e^{\ln\left(\frac{x}{3}\right)} = 3\cdot\frac{x}{3} = x$$

$$(f^{-1} \circ f)(x) = f^{-1}\big(f(x)\big) = f^{-1}\left(3e^{1-2x}\right) = \frac{1-\ln\left(\frac{3e^{1-2x}}{3}\right)}{2} = \frac{1-\ln\left(e^{1-2x}\right)}{2} = \frac{1-(1-2x)}{2} = \frac{1-1+2x}{2} = \frac{2x}{2} = x$$

7. (10 points) Let $f(x)$ be the function $f(x) = \dfrac{2^{3x+1}}{5^{1-x}}$.

   (a) Write $f(x)$ in the form ab^x for some a, b.

   (b) Is $f(x)$ increasing or decreasing?

   (c) Is $f(x)$ concave up or concave down?

Solution.

(a) We have…

$$f(x) = \frac{2^{3x+1}}{5^{1-x}} = \frac{2^{3x} \cdot 2}{5 \cdot 5^{-x}} = \frac{2}{5} \cdot \frac{2^{3x}}{5^{-x}} = \frac{2}{5}\left((2^3)^x \cdot 5^x\right) = \frac{2}{5}\left(8^x \cdot 5^x\right) = \frac{2}{5}(8 \cdot 5)^x = \frac{2}{5} \cdot 40^x$$

Therefore, $f(x)$ has the form ab^x with $a = \frac{2}{5}$ and $b = 40$.

(b) Because $f(x) = \frac{2}{5} \cdot 40^x$ has the form ab^x with $a = \frac{2}{5}$ and $b = 40$, because $b = 40 > 1$ and $a = \frac{2}{5} > 0$, the function $f(x)$ is increasing.

(c) Because $f(x) = \frac{2}{5} \cdot 40^x$ has the form ab^x with $a = \frac{2}{5}$ and $b = 40$, because $b = 40 > 1$ and $a = \frac{2}{5} > 0$, the function $f(x)$ is concave up.

8. (10 points) Showing all your work, find the exact solution to the following:

$$\log_2\left(50 - e^{x+1}\right) + 5 = 10$$

Solution.

$$\log_2\left(50 - e^{x+1}\right) + 5 = 10$$

$$\log_2\left(50 - e^{x+1}\right) = 5$$

$$2^{\log_2\left(50 - e^{x+1}\right)} = 2^5$$

$$50 - e^{x+1} = 32$$

$$e^{x+1} = 18$$

$$\ln\left(e^{x+1}\right) = \ln 18$$

$$x + 1 = \ln 18$$

$$x = \ln(18) - 1$$

We can also verify this solution:

$$\log_2\left(50 - e^{x+1}\right) + 5 = 10$$

$$\log_2\left(50 - e^{\ln(18)-1+1}\right) + 5 \stackrel{?}{=} 10$$

$$\log_2\left(50 - e^{\ln(18)}\right) + 5 \stackrel{?}{=} 10$$

$$\log_2(50 - 18) + 5 \stackrel{?}{=} 10$$

$$\log_2(32) + 5 \stackrel{?}{=} 10$$

$$5 + 5 \stackrel{?}{=} 10$$

$$10 = 10$$

✓

9. (10 points) Showing all your work, write the following as a sum of terms involving only $\log_2(x)$, $\log_2(y)$, and possibly a constant:

$$\log_2\left(\frac{16\sqrt[3]{x}}{y^{-5}}\right)$$

Solution.

$$\begin{aligned}
\log_2\left(\frac{16\sqrt[3]{x}}{y^{-5}}\right) &= \log_2\left(16\sqrt[3]{x}\right) - \log_2(y^{-5}) \\
&= \log_2(16) + \log_2(\sqrt[3]{x}) - (-5)\log_2(y) \\
&= 4 + \log_2(x^{1/3}) + \log_2(y) \\
&= 4 + \frac{1}{3}\log_2(x) + \log_2(y)
\end{aligned}$$

10. (10 points) Let $f(x) = 2 - e^x$ and $g(x) = 6\log_5(2 - x)$.

(a) What is the domain of $f(x)$?

(b) What is the range of $f(x)$?

(c) What is the domain of $g(x)$?

(d) What is the range of $g(x)$?

Solution.

(a) The domain of $f(x) = 2 - e^x$ is the set of all real numbers, $\mathbb{R} = (-\infty, \infty)$.

(b) Because the range of e^x is $(0, \infty)$, the range of $-e^x$ is $(-\infty, 0)$. But then the range of $f(x) = 2 - e^x$ is $(-\infty, 2)$.

(c) For a logarithm to be defined, the input need be positive. But then we need $2 - x > 0$. But then the domain of $g(x) = 6\log_5(2 - x)$ is $2 > x$, i.e. $x < 2$. Equivalently, the domain of $g(x)$ is $(-\infty, 2)$.

(d) The range of $g(x) = 6\log_5(2 - x)$ is all real numbers, i.e. $\mathbb{R} = (-\infty, \infty)$.

11. (10 points) Showing all your work, find the exact solution to the following:

$$3^{2x} = 5 \cdot 2^x$$

Solution.

$$3^{2x} = 5 \cdot 2^x$$

$$\frac{3^{2x}}{2^x} = 5$$

$$\left(\frac{3^2}{2}\right)^x = 5$$

$$\left(\frac{9}{2}\right)^x = 5$$

$$\log_{9/2}\left(\frac{9}{2}\right)^x = \log_{9/2}(5)$$

$$x = \log_{9/2}(5)$$

We can verify the solution:

$$3^{2x} = 5 \cdot 2^x$$

$$3^{2\log_{9/2}(5)} \stackrel{?}{=} 5 \cdot 2^{\log_{9/2}(5)}$$

$$3^{\log_{9/2}(5^2)} \stackrel{?}{=} 5 \cdot 2^{\log_{9/2}(5)}$$

$$3^{\log_{9/2}(25)} \stackrel{?}{=} 5 \cdot 2^{\log_{9/2}(5)}$$

$$3^{\ln(25)/\ln(9/2)} \stackrel{?}{=} 5 \cdot 2^{\ln(5)/\ln(9/2)}$$

$$\left(3^{\ln(25)/\ln(9/2)}\right)^{\ln(9/2)} \stackrel{?}{=} \left(5 \cdot 2^{\ln(5)/\ln(9/2)}\right)^{\ln(9/2)}$$

$$3^{\ln(25)} \stackrel{?}{=} 5^{\ln(9/2)} \cdot 2^{\ln 5}$$

$$3^{2\ln(5)} \stackrel{?}{=} 5^{\ln(9/2)} \cdot 2^{\ln 5}$$

$$\left(3^{2\ln(5)}\right)^{1/\ln(5)} \stackrel{?}{=} \left(5^{\ln(9/2)} \cdot 2^{\ln 5}\right)^{1/\ln(5)}$$

$$3^2 \stackrel{?}{=} 5^{\ln(9/2)/\ln(5)} \cdot 2$$

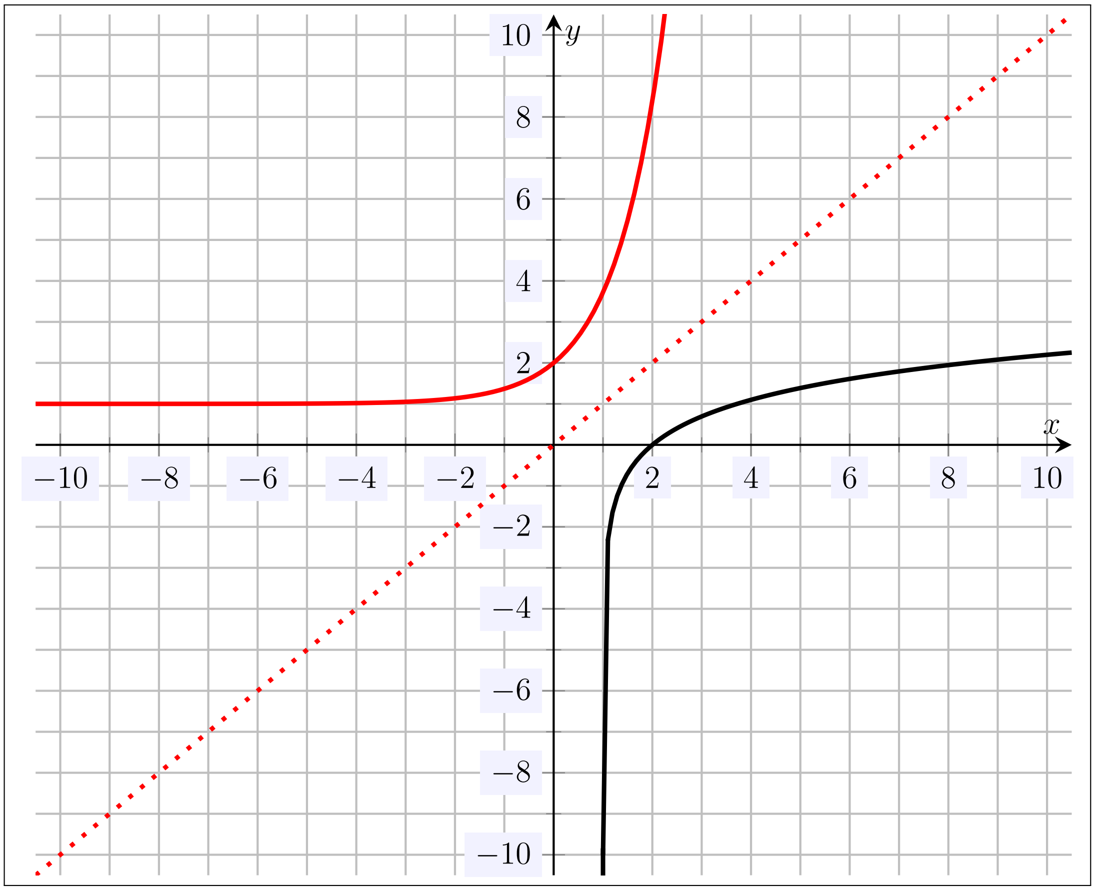
$$9 \stackrel{?}{=} 5^{\log_5(9/2)} \cdot 2$$

$$9 \stackrel{?}{=} \frac{9}{2} \cdot 2$$

$$9 = 9$$

✓

12. (10 points) A relation $f(x)$ is plotted below.



(a) Using the plot above, explain why $f(x)$ is a function.

(b) Using the plot above, explain why $f^{-1}(x)$ exists.

(c) Sketch the function $f^{-1}(x)$ on the plot above.

Solution.

(a) The relation $f(x)$ passes the vertical line test. Therefore, $f(x)$ is a function.

(b) The function $f(x)$ passes the horizontal line tests. Therefore, $f(x)$ is one-to-one and hence has an inverse, i.e. $f^{-1}(x)$ exists.

(c) To sketch $f^{-1}(x)$, we reflect the function $f(x)$ across the line $y = x$ (the dotted red line above). This gives the inverse function $f^{-1}(x)$, which is plotted as a solid red curve above.

13. (10 points) Showing all your work, write each of the following as a single logarithm involving no negative powers:

(a) $\ln(x) + 3\ln(y)$

(b) $\log_5(x) - \log_5(y^{-2})$

(c) $4\log_3(x) - \frac{1}{2}\log_3(y) + 2$

Solution.

(a)

$$\ln(x) + 3\ln(y) = \ln(x) + \ln(y^3) = \ln(xy^3)$$

(b)

$$\log_5(x) - \log_5(y^{-2}) = \log_5\left(\frac{x}{y^{-2}}\right) = \log_5(xy^2)$$

(c)

$$\begin{aligned} 4\log_3(x) - \frac{1}{2}\log_3(y) + 2 &= 4\log_3(x) - \frac{1}{2}\log_3(y) + \log_3(3^2) \\ &= \log_3(x^4) + \log_3(y^{-1/2}) + \log_3(9) \\ &= \log_3(x^4) + \log_3\left(\frac{1}{\sqrt{y}}\right) + \log_3(9) \\ &= \log_3\left(\frac{9x^4}{\sqrt{y}}\right) \end{aligned}$$

14. (10 points) Showing all your work, compute the following "by hand":

(a) $\ln(e^{3/2})$
(b) $\log_5(\sqrt{5})$
(c) $\log_4\left(\frac{1}{64}\right)$
(d) $\log_9(3)$
(e) $\log_8(128)$

Solution.

(a)

$$\ln(e^{3/2}) = \frac{3}{2}$$

(b)

$$\log_5(\sqrt{5}) = \log_5(5^{1/2}) = \frac{1}{2}$$

(c)

$$\log_4\left(\frac{1}{64}\right) = \log_4\left(64^{-1}\right) = \log_4\left((4^3)^{-1}\right) = \log_4\left(4^{-3}\right) = -3$$

(d)

$$\log_9(3) = \log_9(\sqrt{9}) = \log_9(9^{1/2}) = \frac{1}{2}$$

(e)

$$\log_8(128) = \frac{\log_2(128)}{\log_2(8)} = \frac{\log_2(2^7)}{\log_2(2^3)} = \frac{7}{3}$$

15. (10 points) If one invests P dollars at an annual interest rate r (written as a decimal), compounded monthly, then the amount of money in the account after t years, $M(t)$, is given by...
$$M(t) = P\left(1 + \frac{r}{12}\right)^{12t}$$
Suppose that you invest \$8,000 at an annual interest rate of 6.2%, compounded monthly.

(a) Find the value of the investment after 5 years.

(b) Find how long until the investment is worth \$20,000.

Solution. *We have* $P = \$8000$ *and* $r = 0.062$. *But then we have...*

$$M(t) = P\left(1 + \frac{r}{12}\right)^{12t} = \$8000\left(1 + \frac{0.062}{12}\right)^{12t} = \$8000(1.005166667)^{12t}$$

(a) We have...

$$M(5) = \$8000(1.005166667)^{12(5)} = \$8000(1.005166667)^{60} = \$8000(1.36233745) = \$10{,}898.70$$

(b) We have...

$$M(t) = \$8000(1.005166667)^{12t}$$
$$\$20000 = \$8000(1.005166667)^{12t}$$
$$(1.005166667)^{12t} = \frac{\$20000}{\$8000}$$
$$(1.005166667)^{12t} = 2.5$$
$$\ln\left((1.005166667)^{12t}\right) = \ln(2.5)$$
$$12t\ln(1.005166667) = \ln(2.5)$$
$$t = \frac{\ln(2.5)}{12\ln(1.005166667)}$$
$$t \approx \frac{0.916291}{0.0618404}$$
$$t \approx 14.817$$