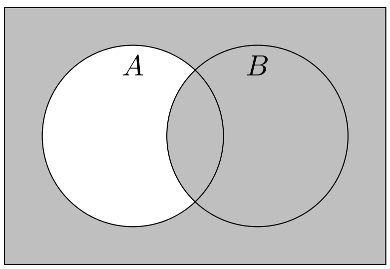
Name: *Caleb McWhorter — Solutions*
MATH 308
Fall 2023
HW 7: Due 10/05

"Since, as is well known, God helps those who help themselves, presumably the Devil helps all those, and only those, who don't help themselves. Does the Devil help himself?"

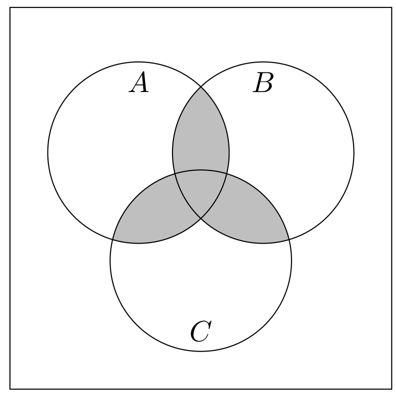
–Douglas Hofstadter

Problem 1. (10pt) For each of the following, if a Venn diagram is given, then express the shaded region as a set or set operation, and if a set operation is given, express the given set with a Venn diagram:

(a)



(c)



(b) $(A \cap B) \cup (A \cup B)^c$

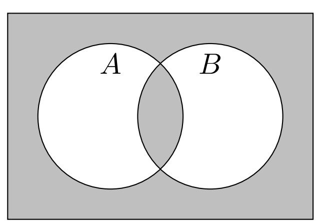
(d) $(A \setminus C) \cap B$

Solution.

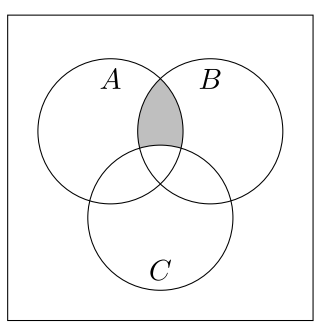
(a) $B \cup A^c = (A - B)^c = (A \cap B) \cup (B - A) \cup (A \cup B)^c$

(b) $(A \cap B) \cup (A \cap C) \cup (B \cap C)$

(c)



(d)



Problem 2. (10pt) We shall create a new mathematical term: let A, B be sets. We say A is a *pseudo-subset* of B, written $A \sqsubset B$, if there is an element of A that is also an element of B and also an element of A that is not an element of B.

(a) We know if S is a set, then $\varnothing \subseteq S$. Is the same true for *pseudo-subsets*? That is, do we have $\varnothing \sqsubset S$ for all sets S? Explain.

(b) If A is a *pseudo-subset* of B, are A and B disjoint? Explain.

(c) Express the definition of being a *pseudo-subset* as a quantified logical statement.

(d) If what it means for $A \not\sqsubset B$ by negating your expression in (c). Write this quantified statement as a complete English sentence.

Solution.

(a) It is *not* true that $\varnothing \sqsubset S$ for all sets S. If $A \sqsubset B$, then there must be an element of A that is an element of B and an element of A that is not in B. However, there are no elements in $\varnothing$. Therefore, $\varnothing \not\sqsubset S$ for all sets S.

(b) If $A \sqsubset B$, then there exists $a_0 \in A$ such that $a_0 \in B$ and there exists and element $a_1 \in A$ such that $a_1 \notin B$. But then $a_0 \in A$ and $a_0 \in B$, which implies that $a_0 \in A \cap B$. But then $A \cap B \neq \varnothing$. Therefore, A and B are not disjoint.

(c) If $A \sqsubset B$, then there exists $a_0 \in A$ such that $a_0 \in B$ and there exists and element $a_1 \in A$ such that $a_1 \notin B$. But then a quantified logical statement that defines A being a pseudo-subset of B is...[1]

$$(\exists a_0)[a_0 \in A \wedge a_0 \in B] \wedge (\exists a_1)[a_1 \in A \wedge a_1 \notin B]$$

(d) Negating our quantified expression in (c), we have...

$$\begin{aligned}
\neg\Big((\exists a_0)[a_0 \in A \wedge a_0 \in B] \wedge (\exists a_1)[a_1 \in A \wedge a_1 \notin B]\Big) &\equiv \neg\big((\exists a_0)[a_0 \in A \wedge a_0 \in B]\big) \vee \neg\big((\exists a_1)[a_1 \in A \wedge a_1 \notin B]\big) \\
&\equiv (\forall a_0)\neg[a_0 \in A \wedge a_0 \in B] \vee (\forall a_1)\neg[a_1 \in A \wedge a_1 \notin B] \\
&\equiv (\forall a_0)[a_0 \notin A \vee a_0 \notin B] \vee (\forall a_1)[a_1 \notin A \vee a_1 \in B]
\end{aligned}$$

Directly 'translated' to English, this quantified expression is, "For all a_0, either $a_0 \notin A$ or $a_0 \notin B$, or for all a_1, either $a_1 \notin A$ or $a_1 \in B$."[2]

[1] There are several quantified statements that can serve as the definition of being a pseudo-subset. One of the simplest involves some work with the definition. If there is an element in A and also an element of B, there is an element in $A \cap B$. If there is an element of A that is also not an element of B, there is an element in $A - B$. But then we can quantify the definition of a pseudo-subset as follows: $(\exists x)(x \in A \cap B) \wedge (\exists y)(y \in A - B)$.

[2] A simpler quantified statement for the definition of *not* being a pseudo-subset negates the alternative quantified statement given in (c). If we negate $(\exists x)(x \in A \cap B) \wedge (\exists y)(y \in A - B)$, we obtain $(\forall x)(x \notin A \cap B) \vee (\forall y)(y \notin A - B)$. Stated as an English sentence, we see that this is, "For all x, $x \notin A \cap B$ or for all x, $x \notin A - B$." Alternatively, we can state this as, "Either A and B are disjoint or there are no elements that are in A but not B."

Problem 3. (10pt) Below is a partial proof of the fact that if A, B, C are sets, then $A \cap (B - C) = (A \cap B) - (A \cap C)$. By filling in the missing portions, complete the partial proof below so that it is a correct, logically sound proof with 'no gaps.'

Proposition. If A, B, C are sets, then $A \cap (B - C) = (A \cap B) - (A \cap C)$.

Proof. To prove that $A \cap (B - C) = (A \cap B) - (A \cap C)$, we need to show $\underline{A \cap (B - C) \subseteq (A \cap B) - (A \cap C)}$ and $\underline{(A \cap B) - (A \cap C) \subseteq A \cap (B - C)}$.

If $A \cap (B - C) = \varnothing$ or $(A \cap B) - (A \cap C) = \varnothing$, then $\varnothing = A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$ and $\varnothing = (A \cap B) - (A \cap C) \subseteq A \cap (B - C)$, respectively. Assume neither $A \cap (B - C)$ nor $(A \cap B) - (A \cap C)$ are empty.

$A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$: Let $x \in \underline{\quad A \cap (B - C) \quad}$. Then $\underline{\quad x \in A \quad}$ and $\underline{\quad x \in (B - C) \quad}$. Because $x \in B - C$, we know that $\underline{\quad x \in B \quad}$ and $\underline{\quad x \notin C \quad}$. But then $x \in A$ and $x \in B$ so that $\underline{\quad x \in A \cap B \quad}$. Now $x \in A$ but $x \notin \underline{\quad C \quad}$ so that $x \notin \underline{\quad A \cap C \quad}$. This shows that $x \in (A \cap B) - (A \cap C)$. Therefore, $\underline{A \cap (B - C) \subseteq (A \cap B) - (A \cap C)}$.

$(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$: Let $x \in (A \cap B) - (A \cap C)$. Then $x \in \underline{\quad A \cap B \quad}$ and $x \notin A \cap C$. Because $x \in A \cap B$, we know that $x \in \underline{\quad A \quad}$ and $x \in \underline{\quad B \quad}$. Because $x \notin A \cap C$, we know that $x \notin \underline{\quad A \quad}$ or $x \notin \underline{\quad C \quad}$. But because $x \in A$, it must be that $x \notin \underline{\quad C \quad}$. Because $x \in B$ and $x \notin C$, we know that $x \in \underline{\quad B - C \quad}$. But then $x \in \underline{\quad A \quad}$ and $x \in \underline{\quad B - C \quad}$, so that $x \in A \cap (B - C)$. Therefore, $\underline{(A \cap B) - (A \cap C) \subseteq A \cap (B - C)}$.

Because $\underline{A \cap (B - C) \subseteq (A \cap B) - (A \cap C)}$ and $\underline{(A \cap B) - (A \cap C) \subseteq A \cap (B - C)}$ we know that $A \cap (B - C) = (A \cap B) - (A \cap C)$. $\square$

Problem 4. (10pt) Let A, B be sets, not necessarily nonempty. Complete the following parts:

(a) Is possible for $A - B = B - A$? Explain.

(b) If $A \not\subseteq B$, does this imply that A is a proper subset of B? Explain.

(c) If A, B are not disjoint, does this imply there is an element $x \in A$ and $x \in B$? Explain.

(d) Is it possible for $A \subseteq A^c$? Explain.

Solution.

(a) Yes, it is possible that $A - B = B - A$. For instance, if $A = \varnothing = B$, then clearly $A - B = \varnothing$ and $B - A = \varnothing$. But then $A - B = B - A$. However, the case where $A = \varnothing = B$ is a special case of the general statement: $A - B = B - A$ if and only if $A = B$. We shall prove this.

Suppose that $A - B = B - A$. Then every element of $A - B$ is also an element of $B - A$. Suppose that $x \in A - B$, then $x \in A$ and $x \notin B$. Because $x \in A - B = B - A$, it must also be that $x \in B - A$. But if $x \in B - A$, then $x \in B$ and $x \notin A$. Thus, we have $x \in A$ and $x \notin A$, and $x \in B$ and $x \notin B$, which are clearly impossible. Suppose that $x \in B - A$. Then $x \in B$ and $x \notin A$. Because $x \in B - A = A - B$, it must also be that $x \in A - B$. But if $x \in A - B$, then $x \in A$ and $x \notin B$. Thus, we have $x \in A$ and $x \notin A$, and $x \in B$ and $x \notin B$, which are clearly impossible. Therefore, if $A - B = B - A$, then $A - B = \varnothing B - A$. If $A - B = \varnothing$, then there is no element x such that $x \in A$ and $x \notin B$. But then every element of A must also be an element of B, i.e. $A \subseteq B$. Similarly, if $B - A = \varnothing$, then there is no element x such that $x \in B$ and $x \notin A$. But then every element of B must also be an element of A, i.e. $B \subseteq A$. Because $A \subseteq B$ and $B \subseteq A$, $A = B$.

For the other direction, suppose that $A = B$. Then $A - B = A - A = \varnothing$ and $B - A = B - B = \varnothing$. Therefore, $A - B = \varnothing = B - A$.

(b) No, $A \not\subseteq B$ does not imply that A is a proper subset of B. If A is a proper subset of B, then for all $a \in A$, we also have $a \in B$.[3] If $A \not\subseteq B$, then there exists $a \in A$ such that $a \notin B$. But then A cannot be a proper subset of B.

(c) Yes, if A and B are not disjoint, there is an element such that $x \in A$ and $x \in B$. If A and B are disjoint, then $A \cap B = \varnothing$. But then if A and B are not disjoint, $A \cap B \neq \varnothing$. Then there exists $x \in A \cap B$, which implies that $x \in A$ and $x \in B$.

(d) Yes, it possible that $A \subseteq A^c$. If $A = \varnothing$, then it must be that A is a subset of *any* set. In particular, $A = \varnothing \subseteq A^c = \varnothing^c = \mathcal{U}$, where $\mathcal{U}$ is the universe. However, if A is nonempty, then it is impossible that $A \subseteq A^c$. Suppose that A is nonempty. Then there exists $x \in A$. But then if $A \subseteq A^c$, then $x \in A$ and $x \in A^c$. Because $x \in A^c$, we know that $x \notin A$, contradicting the fact that $x \in A$. So if $A \subseteq A^c$, it must be that A is empty, i.e. $A = \varnothing$.

[3] There is another requirement. Namely, there exists $b \in B$ such that $b \notin A$, which we will not need to disprove the given statement.