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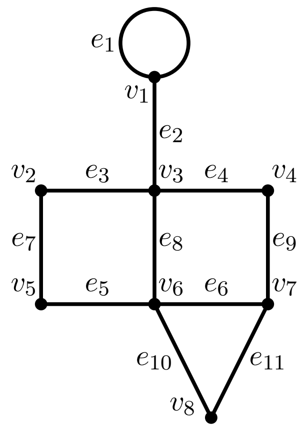
MATH 308
Fall 2022
HW 21: Due 12/15

"All paths are not equal; if they were, they wouldn't be paths but rather the points at each end."

–H.E. Huntley

Problem 1. (10pt) Does the graph G below have an Euler circuit or Euler trail? If it has an Euler circuit or Euler trail, find it. If it does not have an Euler circuit or Euler trail, explain why not.

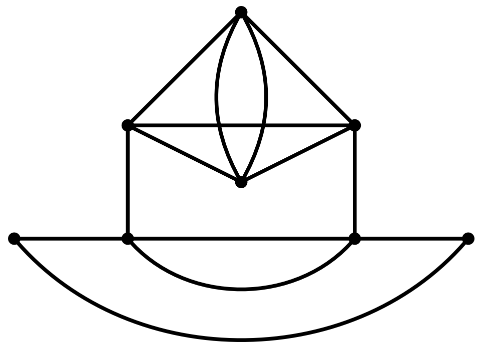


Solution. An undirected graph G has an Euler circuit if and only if G is connected and every vertex has positive even degree. The graph G is connected because given any two vertices there is a walk between them. However, the vertex v_7 has odd degree—we have $\deg v_7 = 3$. Therefore, G does not have an Euler circuit.

An undirected graph G has an Euler trail if and only if G is connected and it has exactly two vertices of odd degree. The graph G is connected because given any two vertices there is a walk between them. Observe that $\deg v_1 = 3$ (the loop contributes 2 to the degree) and $\deg v_7 = 3$. All other vertices of G have even degree. Therefore, G has an Euler trail. For instance, the following is an Euler trail:

$$v_7\, e_{11}\, v_8\, e_{10}\, v_6\, e_6\, v_7\, e_9\, v_4\, e_4\, v_3\, e_3\, v_2\, e_7\, v_5\, e_5\, v_6\, e_8\, v_3\, e_2\, v_1\, e_1$$

Problem 2. (10pt) Let G be the graph given below.



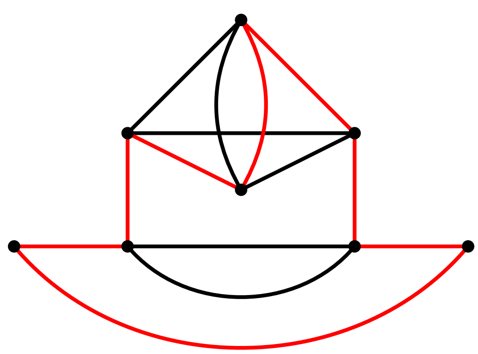
(a) Does G have an Euler trail? If it does, find it. If it does not, explain why.

(b) Does G have a Hamiltonian circuit? If it does, find it. If it does not, explain why.

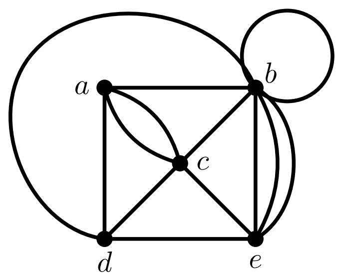
Solution.

(a) A graph G has an Euler trail if and only if it is connected and every vertex has positive even degree. The graph G is connected because every pair of vertices has a walk from one to the other. Furthermore, every vertex of G has degree 4 except the 'leftmost' and 'rightmost' vertices which have degree 2. Therefore, G does not have an Euler trail.

(b) A Hamiltonian circuit is a walk which visits every vertex precisely once and starts/ends at the same vertex. The graph G has a Hamiltonian circuit. For instance, there is a Hamiltonian circuit highlighted in red in the graph below:



Problem 3. (10pt) Suppose G is the graph given below on the left and that H is a directed graph with adjacency matrix A. Showing all your work and fully justifying your responses, answer the questions below.



$$A^8 = \begin{pmatrix} 408628 & 1456983 & 1201872 & 1045608 \\ 217055 & 774044 & 638429 & 555540 \\ 442957 & 1577690 & 1299626 & 1131303 \\ 280444 & 1001303 & 825067 & 716683 \end{pmatrix}$$

(a) How many walks are there of length 1 from b to e? What about from d to b?

(b) How many walks are there of length 2 from c to itself? What about from e to a?

(c) How many walks are there of length 4 from a to c? What about from b to itself?

(d) How many connected components does G have?

(e) How many walks are there of length 8 from v_1 to v_3 in H? What about v_4 to v_2?

Solution.

(a) The number of walks of length k in a graph G from v_i to v_j is the entry a_{ij} in A^k, where A is the adjacency matrix of G. The adjacency matrix of G is...

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 3 \\ 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 1 & 0 \end{pmatrix}$$

The number of walks of length 1 from b to e is $a_{2,5} = 3$. The number of walks of length 1 from d to b is $a_{4,2} = 1$.

(b) The number of walks of length 2 from v_i to v_j is the entry a_{ij} in A^2. We have...

$$A^2 = \begin{pmatrix} 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 3 \\ 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 3 \\ 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 4 & 2 & 3 & 6 \\ 4 & 13 & 7 & 6 & 5 \\ 2 & 7 & 7 & 4 & 4 \\ 3 & 6 & 4 & 4 & 4 \\ 6 & 5 & 4 & 4 & 11 \end{pmatrix}$$

But then the number of walks of length 2 from c to c is $a_{3,3} = 7$. The number of walks of length 2 from e to a is $a_{5,1} = 6$.

(c) The number of walks of length 4 from v_i to v_j is the entry a_{ij} in $A^4 = (A^2)^2$. We have...

$$A^4 = (A^2)^2 = \begin{pmatrix} 6 & 4 & 2 & 3 & 6 \\ 4 & 13 & 7 & 6 & 5 \\ 2 & 7 & 7 & 4 & 4 \\ 3 & 6 & 4 & 4 & 4 \\ 6 & 5 & 4 & 4 & 11 \end{pmatrix} \begin{pmatrix} 6 & 4 & 2 & 3 & 6 \\ 4 & 13 & 7 & 6 & 5 \\ 2 & 7 & 7 & 4 & 4 \\ 3 & 6 & 4 & 4 & 4 \\ 6 & 5 & 4 & 4 & 11 \end{pmatrix} = \begin{pmatrix} 101 & 138 & 90 & 86 & 142 \\ 138 & 295 & 192 & 162 & 196 \\ 90 & 192 & 134 & 108 & 135 \\ 86 & 162 & 108 & 93 & 124 \\ 142 & 196 & 135 & 124 & 214 \end{pmatrix}$$

Therefore, the number of walks of length 4 from a to c is $a_{1,3} = 90$. The number of walks of length 4 from b to itself is $a_{2,2} = 295$.

(d) The graph G is connected because every two distinct vertices in G has a walk connecting them. Therefore, G has one connected component.

(e) The number of walks of length 8 from v_i to v_j is the entry a_{ij} in A^8, where A is the adjacency matrix of the graph. But then the number of walks of length 8 from v_1 to v_3 in H is $a_{1,3} = 1201872$. The number of walks of length 8 from v_4 to v_2 is $a_{4,2} = 1001303$.

Problem 4. (10pt) Suppose that G is an undirected graph with adjacency matrix, A, given below.

$$A = \begin{pmatrix} 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Using only this adjacency matrix, showing all your work, fully justifying your responses, answer the following:

(a) Is G a simple graph?

(b) Is G a multigraph?

(c) How many connected components does G have?

(d) Find the degrees of vertices v_1, v_8, and v_5.

(e) What is the degree of G?

Solution.

(a) A graph G is simple if and only if it does not have loops or multiple edges. A loop is an edge from a vertex to itself. But this occurs if and only if $A_{i,i} > 0$. Because $A_{5,5} > 0$, there is a loop at v_5. Therefore, G is not simple. A graph has multiple edges if and only if there exist two distinct vertices with more than one edge between them. But this occurs if and only if $A_{i,j} > 1$ for some i, j. Because $A_{1,2} = 2 > 1$, there are two edges from v_1 to v_2. But then G is not simple.

(b) A graph G is a multigraph if and only if there exist two distinct vertices with more than one edge between them. But this occurs if and only if $A_{i,j} > 1$ for some i, j. Because $A_{1,2} = 2 > 1$, there are two edges from v_1 to v_2. But then G is a multigraph.

(c) The number of connected components is the number of blocks in its adjacency matrix. Observe that we can block A as follows:

$$A = \left(\begin{array}{ccc|ccc|c|c} 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Therefore, there are 4 connected components.

(d) The degree of a vertex from its adjacency matrix can be found by summing the entries in its row—being sure to count loops, i.e. entries along the diagonal, twice. But then we have $\deg v_1 = 2 + 1 = 3$, $\deg v_8 = 2 \cdot 1 = 2$, and $\deg v_5 = 1 + 2 \cdot 1 + 2 = 5$.

(e) The degree of G is the sum of the degrees of its vertices. Using the procedure outlined in (d), we have...

$$\deg G = \sum_{v_i} \deg v_i = 3 + 3 + 2 + 1 + 5 + 2 + 0 + 2 = 18$$

Alternatively, because G is undirected, we can use the Handshake Theorem. The degree of G is twice the number of edges. The number of edges is the sum of all the entries on or above the diagonal. But then we have $\deg G = 2|E(G)| = 2(1 + 1 + 2 + 1 + 1 + 2 + 1) = 2 \cdot 9 = 18$.