Write your name on the appropriate line on the exam cover sheet. This exam contains 11 pages (including this cover page) and 10 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

| Question | Points | Score |
|:---:|:---:|:---:|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| 10 | 10 | |
| Total: | 100 | |

1. (10 points) Showing all your work and fully justifying your reasoning, compute the following:

(a) $\displaystyle\sum_{k=0}^{5}(2k^2 - 10k + 5)$

(b) $\displaystyle\prod_{\substack{j=-3 \\ j \neq -1}}^{3} \frac{j}{j+1}$

(c) $\displaystyle\sum_{k=1}^{\infty} \left( \frac{1}{k+2} - \frac{1}{k} \right)$

(d) $\displaystyle\sum_{j=1}^{3}\sum_{i=0}^{3}(ij + i - j + 2)$

2. (10 points) Showing all your work and fully justifying your reasoning, find a closed form expression for the following:

$$\sum_{i=-1}^{n} \left(3(i+1)^2 + 5ni - 3n\right)$$

3. (10 points) Define the following vectors and matrices:

$$A = \begin{pmatrix} 1 & -4 & 2 \\ 0 & 3 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & -3 & 4 \\ 1 & 2 & 1 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} -3 \\ 10 \\ 1 \end{pmatrix}$$

Showing all your work and fully justifying your reasoning, answer the following:

(a) Compute $-3\mathbf{v} + \mathbf{u}$.

(b) Compute $\mathbf{u} \cdot \mathbf{v}$.

(c) Compute $2A - B$.

(d) Compute $A\mathbf{u}$.

(e) Only one of $AB$, $B^T\mathbf{u}$, and $A^TB$ can be computed. Explain which cannot and compute the one that can be computed.

4. (10 points) Being sure to show all your work and fully justify your logic, complete the following:

   (a) Use the definition of even and odd integers to show that $-198$ is even and $455$ is odd.

   (b) Is $432 = 18 \cdot 24$ a factorization of $432$? Is this a prime factorization? Find the prime factorization of $432$.

   (c) Are there integers $x, y$ such that $4x + 6y = 5$? Explain.

   (d) Compute $\gcd(2^{10} \cdot 3^{20} \cdot 5^{30} \cdot 11^{80}, \ 2^{50} \cdot 3^{40} \cdot 5^{30} \cdot 7^{80})$.

   (e) Compute $\mathrm{lcm}(2^{10} \cdot 3^{20} \cdot 5^{30} \cdot 11^{80}, \ 2^{50} \cdot 3^{40} \cdot 5^{30} \cdot 7^{80})$.

5. (10 points) Showing all your work, answer the following:

   (a) Express $\dfrac{-1488}{287}$ using the division algorithm.

   (b) Compute $\gcd(287, 1488)$ using the Euclidean algorithm.

   (c) Express $\gcd(287, 1488)$ as a linear combination of $287$ and $1488$.

6. (10 points) Showing all your work, compute the following:

   (a) $(15 \cdot 18 + 43) \mod 8$.

   (b) $(34 \cdot 17) \mod 45$.

   (c) $99^{1234567} \mod 100$.

   (d) The last three digits of $44^{200}$.

   (e) The number of digits in $99^{1234567}$.

7. (10 points) Consider the linear congruence $287x + 584 \equiv 422 \mod 1488$.

   (a) Explain why there exists a unique solution to this equation modulo $1{,}488$.

   (b) How many integers in the range 1, 2, $\ldots$, $1{,}488$ are invertible modulo $1{,}488$?

   (c) Is $287$ invertible modulo $1488$? Explain.

   (d) Solve the given linear congruence.

8. (10 points) Showing all your work, compute the following:

   (a) How many different starting teams of six players can be chosen from a hockey team of 11 players.

   (b) The number of possible top five finishers from a F1 race with 22 drivers.

   (c) The number of distinct arrangements of the letters in 'sassafras.'

   (d) How many arrangements of the digits in the number '12345678' contain the number '72.'

9. (10 points) Showing all your work, compute the following:

   (a) The coefficient of $x^3 y^7$ in $(2x - 5y)^{10}$

   (b) The coefficient of $xy^2 z^3$ in $(2x - y + 5z)^6$

   (c) How many committees with at most two men with a designated chair can be chosen from a collection of five men and 8 women.

   (d) The number of nonnegative integer solutions to $x_1 + x_2 + x_3 = 23$ with $x_2 \geq 2$. [Hint: Consider 'distributing' twenty-three 1's to $x_1$, $x_2$, $x_3$.]

10. (10 points) Let $S$ be the set of nonnegative integers that are at most one million.

   (a) Find the number of elements of $S$ that are not divisible by 7 but are divisible by at least one of 3 or 5.

   (b) Find the number of elements of $S$ that are a multiple of at least one of 4, 6, or 11.

   (c) Find the number of elements of $S$ that are divisible by 2 (but not 4), and at least one of 3, 5, 6, or 9.