Name: *Caleb McWhorter — Solutions*
MATH 307
Spring 2023
HW 1: Due 02/13 (14)

"In learning you will teach, and in teaching you will learn."
–Phil Collins

Problem 1. (10pt) Let $\mathcal{U} = \{-10, -9, \ldots, 9, 10\}$. Define the following subsets of $\mathcal{U}$:

$$
\begin{aligned}
A &= \{-2, 0, 5, 10\} \\
B &= \text{even numbers in } \mathcal{U} \\
C &= \{-9, -7, -5, -3, -1, 1, 3, 5, 7, 9\} \\
D &= \text{positive prime numbers in } \mathcal{U} \\
E &= \{-5, -4, \ldots, 4, 5\}
\end{aligned}
$$

Using the sets defined above, answer the following:

(a) $A \cap B$

(b) $B \cup E$

(c) $E - A$

(d) B^c

(e) $|D|$

Solution.

(a) The set $A \cap B$ is the collection of objects that are elements of *both* A and B. But then we have...

$$A \cap B = \{-2, 0, 10\}$$

(b) The set $B \cup E$ is the collection of objects that are elements of either B or E. But then we have...

$$B \cup E = \{-10, -8, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 8, 10\}$$

(c) The set $E - A$ is the collection of objects that are elements of E but *not* elements of A. But then we have...

$$E - A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$$

(d) The set B^c is the collection of objects that are elements (of $\mathcal{U}$) that are *not* elements of B. But then we have...

$$B^c = \{-9, -7, -5, -3, -1, 1, 3, 5, 7, 9\}$$

(e) The number $|D|$ is the cardinality or 'size' of D; that is, $|D|$ is the number of distinct (unique) elements of D. But then as $D = \{2, 3, 5, 7\}$, we have $|D| = 4$.

Problem 2. (10pt) Define the following sets:

$$\begin{aligned} A &= \text{set of multiples of 3} \\ B &= \text{set of divisors of 30} \\ C &= \text{set of even numbers less than 10} \end{aligned}$$

Using the sets defined above, answer the following:

(a) List the elements of B.

(b) Give the largest element of A less than 50 and the largest negative element of A.

(c) What are the elements of $B - A$?

(d) What are the elements of $A \cap B$?

(e) Are B and C disjoint? Explain.

Solution.

(a) The divisors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30. Therefore,

$$B = \{1, 2, 3, 5, 6, 10, 15, 30\}$$

(b) The largest element of A less than 50 will be the largest multiple of 3 that is less than 50. We know that 50 is not divisible by 3, nor is 49. However, as $48 = 16(3)$, we know that 48 is divisible by 3. Therefore, 48 is the largest element of A (an integer divisible by 3) that is less than 50. The negative elements of A, i.e. the negative integers that are a multiple of 3, are $-3, -6, -9, -12, -15, \ldots$. The largest of these negative multiples of 3 is -3.

(c) The objects of $B - A$ are the elements of B that are *not* elements of A. The elements of B are the divisors of 30 and the elements of A are the multiples of 3. Then the elements of $B - A$ are the integers that are divisors of 30 that are not multiples of 3. But then we have...

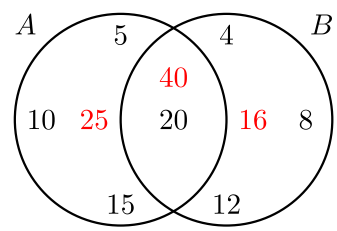
$$B - A = \{1, 2, 5, 10\}$$

(d) The objects of $A \cap B$ are objects that are elements of *both* A and B. The elements of A are the multiples of 3 and the elements of B are the divisors of 30. Therefore, the elements of $A \cap B$ are the integers that are multiples of 3 that are also divisors of 30. But then we have...

$$A \cap B = \{3, 6, 15, 30\}$$

(e) The sets B and C are disjoint if $B \cap C = \emptyset$; that is, if there is no integer that is an element of both B and C. However, 2 is an even number less than 10 so that $2 \in C$. But 2 is also a divisor of 30 so that $2 \in B$. This shows $2 \in B \cap C$ so that $B \cap C \neq \emptyset$. Therefore, B and C are not disjoint.

Problem 3. (10pt) Look at the Venn diagram given below:



Use this diagram to answer the following:

(a) Assuming only a few of the elements of A and B are given in the diagram above, describe what the sets A and B likely represent.

(b) Place the numbers 25, 16, and 40 in appropriate places in the given Venn diagram.

(c) Using words, explain what numbers go in the same region of the Venn diagram in which 20 is found.

(d) Using words, explain what numbers go in the same region of the Venn diagram in which 4, 8, and 12 are found.

(e) What numbers would be placed outside of both the regions A and B? Give an example.

Solution.

(a) The elements of A are the integers 5, 10, 15, and 20. We may then possibly describe A as the set of positive multiples of 5. The elements of B are the integers 4, 8, 12, 20. We may then possibly describe B as the set of positive multiples of 4.

(b) Because 25 is a multiple of 5 but not 4, 25 belongs only in the set A. Because 16 is a multiple of 4 but not a multiple of 5, 16 belongs only in the set B. Because 40 is a multiple of both 5 and 4, 40 belongs in both the set A and B. We then place 25, 16, and 40 in appropriate places in the diagram above.

(c) The number 20 is in the overlap of the sets A and B. But then this region represents the collection of elements that are in A and B, i.e. $A \cap B$. This is the collection of integers that are both in A (hence a positive multiple of 5) and in B (hence a positive multiple of 4). To be a multiple of 4 and 5 implies that you are a multiple of $\text{lcm}(4, 5) = 20$. Then the region with 20 is the set of positive integers that are a multiple of 20.

(d) The numbers 4, 8, 12 are in B, i.e. are a positive multiple of 4, but are not in A, i.e. they are not a positive multiple of 5. But then region containing 4, 8, and 12 are the positive integers which are a multiple of 4 but not a multiple of 5, i.e. an element of the set $B - A$.

(e) The region outside the circles A and B consist of the elements that are not in A, i.e. not a positive multiple of 5, and not in B, i.e. not a positive multiple of 4. But then this region consists of the positive integers which are not multiples of 4 and not multiples of 5. For example, 1, 2, 3, 6, 7, 9, 11, etc. are all elements of the region outside the circles A and B.

Problem 4. (10pt) You are working with a student named Lucy. You give her the following sets: $A = \{a, b, c, d, a\}$ and $B = \{c, d, e, f\}$.

(a) Lucy states that the cardinality of A is 5. Explain why Lucy is wrong. How might you correct her?

(b) You ask Lucy to find $A \cup B$ and she states that this is $\{c, d\}$. What has Lucy done wrong?

(c) Cameron overhears Lucy's answer in (b) and shouts that the answer is $\{a, b, e, f\}$. How has Cameron misunderstood the mathematical word *or* in this context?

(d) Both Lucy and Cameron state that you cannot find $A - B$ because they are filled with letters and you cannot subtract letters. Explain what they have misunderstood about sets.

Solution.

(a) The cardinality, or size, of a set is the number of distinct (unique) elements of a set. Repetition or order of elements of a set do not matter. But then we know that $A = \{a, b, c, d, a\} = \{a, a, b, c, d\} = \{a, b, c, d\}$. But then A only has 4 distinct elements so that $|A| = 4$. Lucy is misunderstanding that it is not the number of 'objects' of A that matters; it is the number of *distinct* objects of A. You should explain to her that she should eliminate duplicates before counting the elements of a set.

(b) The set $A \cup B$ should consist of the elements that are in either A or B. The set $A \cap B$ is the set of elements that are in both A and B. Notice that c is in A and B and d is in A and B. There are no other elements that are in both A and B. But then $A \cap B = \{c, d\}$. It is then likely that Lucy has confused the symbols $\cup$ and $\cap$.

(c) The set $A \cup B$ should consist of the elements that are in A or B. However, 'or' in Mathematics always refers to one or the other or both; that is, the elements of $A \cup B$ are the elements in either A or B. The elements 'a' and 'b' are only in A and the elements 'e' and 'f' are only in B. Cameron then seems to be treating the word 'or' as 'exclusive-or' and is not using it in the mathematical sense.

(d) They are treating '$-$' as if it refers to subtraction. However, in this context, we should treat '$-$' as 'removing.' The set $A - B$ is then the collection of elements of A after having 'removed' the elements that are in B; that is, the set $A - B$ is the collection of elements of A that are not elements of B.