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MATH 308
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HW 6: Due 10/08

"So I was not born with a whole lot of natural talent. But I work hard and I never give up. That is my gift. That is my ninja way!"
–Rock Lee, Naturo

Problem 1. (20pt) Describe all sets (if any) with...

(a) no proper subsets.

(b) one proper subset.

(c) two proper subsets.

Solution.

(a) Because $\varnothing$ is a subset of every set, every nonempty set S has a proper subset. If $S = \varnothing$, then $\varnothing \subseteq S$ but $S = \varnothing$ so that $\varnothing$ is not a proper subset of S. Therefore, there are no sets without at least one proper subset.

(b) We know that $\varnothing$ has no proper subsets. If S is a nonempty set with at least two elements, say $a, b \in S$, then $\{a\} \subseteq S$ and $\{b\} \subseteq S$ so that S has at least two proper subsets. So suppose S is a singleton set, i.e. $S = \{a\}$. Then $\varnothing \subseteq S$ and $S \neq \varnothing$. Therefore, S has exactly one proper subset. But then the only sets with exactly one proper subset are singleton sets.

(c) We know that $\varnothing$ has no proper subsets. From (b), we know that singleton sets have exactly one proper subset. Suppose that S has at least two elements, say $a, b \in S$. But then $\varnothing \subseteq S$, $\{a\} \subseteq S$, and $\{b\} \subseteq S$ are all proper subsets of S so that S has at least three proper subsets. Therefore, there are no sets with exactly two proper subsets.

Remark. If a set S is infinite, then S has an infinite amount of proper subsets: for all $s \in S$, $\{s\} \subseteq S$ is a proper subset. We claim that if S is a finite set with n elements, then S has $2^n - 1$ proper subsets.

Proposition. *If S is a finite set with n elements, then S has $2^n - 1$ proper subsets.*

Proof. Suppose that $|S| = 0$. But then $S = \varnothing$. From (a) above, we know that $S = \varnothing$ has no proper subsets. Furthermore, $2^0 - 1 = 1 - 1 = 0$. Now let S be a singleton set, say $S = \{s\}$. From (b), we know that S has one proper subset—namely, $\varnothing$. Observe that $|S| = 1$ and $2^1 - 1 = 2 - 1 = 1$. Now assume that for any finite set S with $|S| = k$ that S has $2^k - 1$ proper subsets.

Now let S be a set with $|S| = k + 1$. Choose an element $s \in S$. Consider all the proper subsets of S that do not contain S. But each such subset is a subset of $S \setminus \{s\}$. Conversely, every proper subset of $S \setminus \{s\} \subsetneq S$ is a subset of S that does not contain S. Therefore, the number of proper subsets of S not containing s is the number of proper subsets of $S \setminus \{s\}$. We know that $|S \setminus \{s\}| = k$. By the induction hypothesis, the number of proper subsets of $S \setminus \{s\}$ is $2^k - 1$.

Now consider the proper subsets of S containing s. Suppose that $A \subsetneq S$ is a proper subset of S with $s \in A$. Then $A \setminus \{s\} \subseteq A \subsetneq S$ is a proper subset of S not containing s. Conversely, if $B \subsetneq S$ is a proper subset of S not containing s, then $B \cup \{s\} \subseteq S$ is a proper subset of S containing S, unless $B = S \setminus \{s\}$ in which case $B \cup \{s\} = S$ is not proper. Therefore, the proper subsets of S

not containing s, except for $S \setminus \{s\}$, are in one-to-one correspondence with the proper subsets of S containing s. Then there are $(2^k - 1) + (2^k - 1) + 1 = 2 \cdot 2^k - 1 = 2^{k+1} - 1$ proper subsets of S. Therefore, by induction, the number of proper subsets of a set S with $|S| = n$ is $2^n - 1$.

Remark. If one knows that the number of proper subsets of a finite set S with $|S| = n$ is 2^n, then an immediate corollary is the number of proper subsets of S is $2^n - 1$: if S is empty the result is clear, and if $|S| = n > 0$, then the only non-proper subset of S is S itself, making $2^n - 1$ proper subsets.

This makes the problem simple. If S is infinite, it is clear that S cannot have exactly none, one, or two proper subsets. If S is finite with $|S| = n$, then S has $2^n - 1$ proper subsets. But for all $n \in \mathbb{Z}_{\geq 0}$, $2^n - 1 \notin \{0, 2\}$. We only have $2^n - 1 = 1$ if $n = 1$, but then S is a singleton set.

Problem 2. (20pt) The symmetric difference of two sets A and B, denoted $A\Delta B$, is defined by $A\Delta B := (A \setminus B) \cup (B \setminus A)$.

(a) Describe $A\Delta B$ in words.

(b) Show that $A\Delta B = (A \cup B) - (A \cap B)$.

(c) Prove that the symmetric difference is commutative.

(d) Prove that if $A\Delta B = \varnothing$, then $A = B$. Is the converse true?

Solution.

(a) The set $A \setminus B$ is the set of elements that are in A but not in B. The set $B \setminus A$ is the set of elements that are in B but not in A. Therefore, $A\Delta B$ is the set of elements that are only in A or only in B.

(b) Let $x \in A\Delta B := (A \setminus B) \cup (B \setminus A)$. Then $x \in A \setminus B$ or $x \in B \setminus A$. Assume that $x \in A \setminus B$. Then $x \in A$ and $x \notin B$. Because $x \in A$, we know that $x \in A \cup B$. Because $x \in A$ and $x \notin B$, we know that $x \notin A \cap B$. But then $x \in (A \cup B) - (A \cap B)$. Now assume that $x \in B \setminus A$. Then $x \in B$ and $x \notin A$. Because $x \in B$, we know that $x \in A \cup B$. But because $x \in B$ and $x \notin A$, we know $x \notin A \cap B$. Therefore, $x \in (A \cup B) - (A \cap B)$. Therefore, if $x \in A\Delta B$, then $x \in (A \cup B) - (A \cap B)$ so that $A\Delta B \subseteq (A \cup B) - (A \cap B)$.

Now let $x \in (A \cup B) - (A \cap B)$. Then $x \in A$ and $x \notin A \cap B$ or $x \in B$ and $x \notin A \cap B$. Assume that $x \in A$ and $x \notin A \cap B$. But then $x \in A$ and $x \notin B$. Therefore, $x \in A \setminus B$ so that $x \in A\Delta B = (A \setminus B) \cup (B \setminus A)$. Now assume that $x \in B$ and $x \notin A \cap B$. But then $x \in B$ and $x \notin A$. Therefore, $x \in B \setminus A$ so that $x \in A\Delta B = (A \setminus B) \cup (B \setminus A)$. But then if $x \in (A \cup B) - (A \cap B)$, then $x \in A\Delta B$ so that $(A \cup B) - (A \cap B) \subseteq A\Delta B$. Therefore, $A\Delta B = (A \cup B) - (A \cap B)$.

OR

$$
\begin{aligned}
x \in A\Delta B &\iff x \in (A \setminus B) \cup (B \setminus A) \\
&\iff (x \in A \setminus B) \vee (x \in B \setminus A) \\
&\iff (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A) \\
&\iff [(x \in A \wedge x \notin B) \vee x \in B] \wedge [(x \in A \wedge x \notin B) \vee x \notin A] \\
&\iff [(x \in A \vee x \in B) \wedge (x \notin B \vee x \in B)] \wedge [(x \in A \vee x \notin A) \wedge (x \notin B \vee x \notin A)] \\
&\iff [(x \in A \vee x \in B) \wedge T_0] \wedge [T_0 \wedge (x \notin B \vee x \notin A)] \\
&\iff (x \in A \vee x \in B) \wedge (x \notin B \vee x \notin A) \\
&\iff (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B) \\
&\iff (x \in A \cup B) \wedge (x \notin A \cap B) \\
&\iff x \in (A \cup B) - (A \cap B)
\end{aligned}
$$

(c) Using the commutative of unions, observe that. . .

$$A\Delta B := (A \setminus B) \cup (B \setminus A) = (B \setminus A) \cup (A \setminus B) =: B\Delta A$$

(d) Suppose that $A\Delta B = \varnothing$. We then have $(A \setminus B) \cup (B \setminus A) = \varnothing$. Therefore, $A \setminus B = \varnothing$ and $B \setminus A = \varnothing$. Because $A \setminus B = \varnothing$, if $x \in A$, we must have $x \in B$. Because $B \setminus A = \varnothing$, if $x \in B$, then $x \in A$. But then $x \in A$ if and only if $x \in B$. Therefore, $A = B$.

The converse is also true. Suppose that $A = B$. Then $A \setminus B = \varnothing$ and $B \setminus A = \varnothing$ (because $x \in A$ if and only if $x \in B$). Therefore, $A\Delta B = (A \setminus B) \cup (B \setminus A) = \varnothing \cup \varnothing = \varnothing$. Then $A\Delta B = \varnothing$ if and only if $A = B$.

OR

Lemma. If A and B are sets, then $A \cap B^c = \varnothing$ if and only if $A = B$.

Proof. Assume $A \cap B^c = \varnothing$ and suppose that $A \neq B$. Then there exists $a \in A$ such that $a \notin B$. Because $a \notin B$, we know that $a \in B^c$. But then $a \in A$ and $a \in B^c$ so that $a \in A \cap B^c = \varnothing$, a contradiction. Therefore, $A = B$. Now assume that $A = B$. But then $A \cap B^c = A \cap A^c = \varnothing$.

We know use this lemma as follows (in the fourth if and only if):

$$\begin{aligned} A\Delta B = \varnothing &\iff (A \setminus B) \cup (B \setminus A) = \varnothing \\ &\iff (A \setminus B = \varnothing) \wedge (B \setminus A = \varnothing) \\ &\iff (A \cap B^c = \varnothing) \wedge (B \cap A^c = \varnothing) \\ &\iff (A = B) \wedge (B = A) \\ &\iff A = B \end{aligned}$$

Problem 3. (20pt) Let A, B be sets with a common universal set $\mathscr{U}$. Prove the following:

(a) $A - (A - B) = A \cap B$

(b) $A \subseteq B$ if and only if $A^c \supseteq B^c$

Solution.

(a) Let $x \in A - (A - B)$. Then $x \in A$ and $x \notin A - B$. By definition, $A - B = A \cap B^c$. Therefore, $x \notin A - B$ implies that $x \notin A \cap B^c$. But then $x \in (A \cap B^c)^c$. Now $(A \cap B^c)^c = A^c \cup B$ so that $x \in A^c \cup B$. Therefore, $x \in A^c$ or $x \in B$. But $x \in A$ so that $x \notin A^c$. Therefore, $x \in B$. But then $x \in A$ and $x \in B$ so that $x \in A \cap B$. This proves that $A - (A - B) \subseteq A \cap B$.

Now assume that $x \in A \cap B$. This implies that $x \in A$ and $x \in B$. Suppose that $x \notin A - (A - B)$. From the work above, we know that $A - B = A \cap B^c$. But then $A - (A - B) = A - (A \cap B^c)$. By definition, $A - (A \cap B^c)$ is the set $A \cap (A \cap B^c)^c$. But $(A \cap B^c)^c = A^c \cup B$ so that $A \cap (A \cap B^c)^c = A \cap (A^c \cup B)$. Now the set $A \cap (A^c \cup B)$ is $(A \cap A^c) \cup (A \cap B) = \varnothing \cup (A \cap B) = A \cap B$. As $x \notin A - (A - B)$, this implies $x \notin A \cap B$, a contradiction. Therefore, $x \in A - (A - B)$. But then $A \cap B \subseteq A - (A - B)$. Therefore, $A - (A - B) = A \cap B$.

OR

$$\begin{aligned} x \in A \setminus (A \setminus B) &\iff x \in A \setminus (A \cap B^c) \\ &\iff x \in \big(A \cap (A \cap B^c)^c\big) \\ &\iff x \in \big(A \cap (A^c \cup B)\big) \\ &\iff x \in \big((A \cap A^c) \cup (A \cap B)\big) \\ &\iff x \in \big(\varnothing \cup (A \cap B)\big) \\ &\iff x \in A \cap B \end{aligned}$$

(b) Assume that $A \subseteq B$. We want to show that $B^c \subseteq A^c$. Let $x \in B^c$. Now $x \in B^c$ implies that $x \notin B$. Because $A \subseteq B$, it must be that $x \notin A$; otherwise, $x \in A$ and $x \notin B$, contradicting the fact that $A \subseteq B$. But then $x \in B^c$ implies that $x \in A^c$ so that $B^c \subseteq A^c$.

Now assume that $A^c \supseteq B^c$. We want to show that $A \subseteq B$. Let $x \in A$. Because $x \in A$, we know that $x \notin A^c$. But if $x \notin A^c$, we know that $x \notin B^c$; otherwise, $x \notin A^c$ and $x \in B^c$ contradicts the fact that $A^c \supseteq B^c$. Therefore, if $x \in A$, then $x \in B$. But then $A \subseteq B$.

OR

$$\begin{aligned} A \subseteq B &\iff (\forall x)(x \in A \Rightarrow x \in B) \\ &\iff (\forall x)\big(\neg(x \in B) \Rightarrow \neg(x \in A)\big) \\ &\iff (\forall x)(x \notin B \Rightarrow x \notin A) \\ &\iff (\forall x)(x \in B^c \Rightarrow x \in A^c) \\ &\iff B^c \subseteq A^c \end{aligned}$$

Problem 4. (10pt) If $A \subseteq U$ and $B \subseteq V$, is $A \times B \subseteq U \times V$? Justify your answer.

Solution. Yes. Suppose that $A \subseteq U$ and $B \subseteq V$. If either A or B are empty, then $A \times B$ is empty. Clearly, $\varnothing \subseteq U \times V$. So suppose that A and B are nonempty. Let $(x, y) \in A \times B$. Then by definition, $x \in A$ and $y \in B$. Because $A \subseteq U$ and $B \subseteq V$, this implies that $x \in U$ and $y \in V$, respectively. But then $(x, y) \in U \times V$. Therefore, $A \times B \subseteq U \times V$.

Problem 5. (10pt) Suppose that X and Y are sets with a common universal set $\mathscr{U}$. Show that $X = Y$ if and only if $(X \cap Y^c) \cup (X^c \cap Y) = \varnothing$.

Solution. Suppose that $X = Y$. Then $Y^c = X^c$ so that $X \cap Y^c = X \cap X^c = \varnothing$. Similarly, $X^c = Y^c$ so that $X^c \cap Y = Y^c \cap Y = \varnothing$. But then $(X \cap Y^c) \cup (X^c \cap Y) = \varnothing \cup \varnothing = \varnothing$.

Now assume that $(X \cap Y^c) \cup (X^c \cap Y) = \varnothing$. This implies that $X \cap Y^c = \varnothing$ and $X^c \cap Y = \varnothing$. But we already proved in Problem 2 (see the lemma below) that $X \cap Y^c = \varnothing$ implies that $X = Y$. Mutatis mutandis, $X^c \cap Y = \varnothing$ implies that $Y = X$. But then we know that $X = Y$.

Lemma. If A and B are sets, then $A \cap B^c = \varnothing$ if and only if $A = B$.

Proof. Assume $A \cap B^c = \varnothing$ and suppose that $A \neq B$. Then there exists $a \in A$ such that $a \notin B$. Because $a \notin B$, we know that $a \in B^c$. But then $a \in A$ and $a \in B^c$ so that $a \in A \cap B^c = \varnothing$, a contradiction. Therefore, $A = B$. Now assume that $A = B$. But then $A \cap B^c = A \cap A^c = \varnothing$.

Problem 6. (20pt) Prove or disprove:

(a) $(A \cup B) \setminus B = A$

(b) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$

(c) $A \cap (B \setminus C) = (A \cap B) \setminus C$

(d) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Solution.

(a) The statement is *false*. Let $A = \{1, 2, 3\}$ and $B = \{3, 4\}$. Then $A \cup B = \{1, 2, 3, 4\}$ and $(A \cup B) \setminus B = \{1, 2, 3, 4\} \setminus \{3, 4\} = \{1, 2\} \neq \{1, 2, 3\} = A$.

(b) The statement is *true*. Observe that...

$$\begin{aligned}
A \cap (B \setminus C) &= A \cap (B \cap C^c) \\
&= (A \cap B) \cap C^c \\
&= (A \cap B) \cap (\varnothing \cap C^c) \\
&= (A \cap B) \cap \big((A \cap A^c) \cap C^c\big) \\
&= (A \cap B) \cap \big(A \cap (A^c \cap C^c)\big) \\
&= \big((A \cap B) \cap A\big) \cap (A^c \cap C^c) \\
&= \big(A \cap (A \cap B)\big) \cap (A^c \cap C^c) \\
&= \big((A \cap A) \cap B\big) \cap (A^c \cap C^c) \\
&= (A \cap B) \cap (A^c \cup C^c) \\
&= (A \cap B) \cap (A \cap C)^c \\
&= (A \cap B) \setminus (A \cap C)
\end{aligned}$$

(c) The statement is *true*. Observe that...

$$A \cap (B \setminus C) = A \cap (B \cap C^c) = (A \cap B) \cap C^c = (A \cap B) \setminus C$$

(d) The statement is *true*. Observe that...

$$A \setminus (B \cap C) = A \cap (B \cap C)^c = A \cap (B^c \cup C^c) = (A \cap B^c) \cup (A \cap C^c) = (A \setminus B) \cup (A \setminus C)$$

Problem 7. (20pt) Express the following sets as an interval, collection of intervals, or well known set (prove your answer):

(a) $\displaystyle\bigcap_{n\geq 1}\left[0, 1+\frac{1}{n}\right)$

(b) $\displaystyle\bigcup_{n\geq 1}\left[0, 1+\frac{1}{n}\right)$

(c) $\displaystyle\bigcup_{n\in\mathbb{Z}}\bigcap_{m\geq 1}\left(n-\frac{1}{m}, n+\frac{1}{m}\right)$

Solution.

(a) We claim that...

$$\bigcap_{n\geq 1}\left[0, 1+\frac{1}{n}\right) = [0,1]$$

If $x<0$, then $x\notin[0,2)=[0,1+1/1)$ so that $x\notin\bigcap_{n\geq 1}\left[0,1+\frac{1}{n}\right)$. If $x\in[0,1]$, then clearly $x\in[0,1+1/n)$ for all $n\geq 1$, so that $x\in\bigcap_{n\geq 1}\left[0,1+\frac{1}{n}\right)$. Suppose that $x>1$, i.e. $x-1>0$. It is clear that $\frac{1}{x-1}\in\mathbb{R}$ and $\frac{1}{x-1}>0$. Choose $n_0\in\mathbb{N}$ such that $n_0>\frac{1}{x-1}$. But then $\frac{1}{n_0}<x-1$ so that $1+\frac{1}{n_0}<x$. Clearly, this implies that $x\notin[0,1+1/n_0)$. But then $x\notin\bigcap_{n\geq 1}\left[0,1+\frac{1}{n}\right)$. Therefore, $x\in\bigcap_{n\geq 1}\left[0,1+\frac{1}{n}\right)$ if and only if $x\in[0,1]$, as desired.

(b) We claim that...

$$\bigcup_{n\geq 1}\left[0, 1+\frac{1}{n}\right) = [0,2)$$

Clearly, if $x\in[0,2)=[0,1+1/1)$, then $x\in\bigcup_{n\geq 1}\left[0,1+\frac{1}{n}\right)$. But if $x\in\bigcup_{n\geq 1}\left[0,1+\frac{1}{n}\right)$, then $x\in[0,1+1/n_0)$ for some $n_0\in\mathbb{N}$. But $n_0\geq 1$ so that $1/n_0\leq 1$. Then we have $x\in[0,1+1/n_0)\subseteq[0,1+1/1)=[0,2)$. Therefore, $x\in\bigcup_{n\geq 1}\left[0,1+\frac{1}{n}\right)$ if and only if $x\in[0,2)$, as desired.

(c) We claim that...

$$\bigcup_{n\in\mathbb{Z}}\bigcap_{m\geq 1}\left(n-\frac{1}{m}, n+\frac{1}{m}\right) = \mathbb{Z}$$

Fix $N\in\mathbb{Z}$. Then $N\in(N-\frac{1}{m},N+\frac{1}{m})$ for all $m\geq 1$. But then $N\in\bigcup_{n\in\mathbb{Z}}\bigcap_{m\geq 1}\left(n-\frac{1}{m},n+\frac{1}{m}\right)$.

Now suppose that $x\in\bigcup_{n\in\mathbb{Z}}\bigcap_{m\geq 1}\left(n-\frac{1}{m},n+\frac{1}{m}\right)$ and that x is not an integer. Because $x\in\bigcup_{n\in\mathbb{Z}}\bigcap_{m\geq 1}\left(n-\frac{1}{m},n+\frac{1}{m}\right)$, there exists $N_0\in\mathbb{Z}$ such that $x\in\bigcap_{m\geq 1}\left(N_0-\frac{1}{m},N_0+\frac{1}{m}\right)$. We claim that this $N_0\in\mathbb{Z}$ is unique.

Suppose that $x\in\left(N-\frac{1}{m},N+\frac{1}{m}\right)$ for some $N\in\mathbb{Z}, m\in\mathbb{N}$ with $N\neq N_0$. Either $N>N_0$ or $N<N_0$. Suppose that $N>N_0$. Because $N\in\mathbb{Z}$, we know that $N\geq N_0+1$. But as $x\in\bigcap_{m\geq 1}\left(N_0-\frac{1}{m},N_0+\frac{1}{m}\right)$, $x\in(N_0-1/2,N_0+1/2)$. Therefore, $x<N_0+1/2$. But because $x\in\bigcap_{m\geq 1}\left(N-\frac{1}{m},N+\frac{1}{m}\right)$, we know that $x\in(N-1/2,N+1/2)$. Therefore, $x>N-1/2$. But then

$$x>N-\frac{1}{2}\geq N_0+1-\frac{1}{2}=N_0+\frac{1}{2},$$

a contradiction. Suppose then that $N < N_0$. Because $N \in \mathbb{Z}$, we know that $N \leq N_0 - 1$. But as $x \in \bigcap_{m \geq 1} \left(N_0 - \frac{1}{m}, N_0 + \frac{1}{m}\right)$, $x \in (N_0 - 1/2, N_0 + 1/2)$. Therefore, $N_0 - 1/2 < x$. But because $x \in \bigcap_{m \geq 1} \left(N - \frac{1}{m}, N + \frac{1}{m}\right)$, we know that $x \in (N - 1/2, N + 1/2)$. Therefore, $x < N + 1/2$. But then

$$x < N + \frac{1}{2} \leq N_0 - 1 + \frac{1}{2} = N_0 - \frac{1}{2},$$

a contradiction.

Then there is a unique $N_0 \in \mathbb{Z}$ such that $x \in \bigcap_{m \geq 1} \left(N_0 - \frac{1}{m}, N_0 + \frac{1}{m}\right)$. Because $x \in \bigcap_{m \geq 1} \left(N_0 - \frac{1}{m}, N_0 + \frac{1}{m}\right)$, we know that $x \in (N_0 - 1/1, N_0 + 1/1) = (N_0 - 1, N_0 + 1)$. As x is not an integer, we know that $x \neq N_0$. But then $|x - N_0| > 0$. We know also that $|x - N_0| \in \mathbb{R}$. Choose $m_0 \in \mathbb{N}$ such that $m_0 > \frac{1}{|x - N_0|}$. But then $\frac{1}{m_0} < |x - N_0|$. This implies that either $\frac{1}{m_0} < x - N_0$ or $\frac{1}{m_0} < -(x - N_0)$. If $\frac{1}{m_0} < x - N_0$, then $N_0 + \frac{1}{m_0} < x$, contradicting the fact that $x \in \bigcap_{m \geq 1} \left(N_0 - \frac{1}{m}, N_0 + \frac{1}{m}\right)$. If $\frac{1}{m_0} < -(x - N_0)$, then $-\frac{1}{m_0} > x - N_0$, so that $N_0 - \frac{1}{m_0} > x$, contradicting the fact that $x \in \bigcap_{m \geq 1} \left(N_0 - \frac{1}{m}, N_0 + \frac{1}{m}\right)$. Therefore, $x \notin \bigcap_{m \geq 1} \left(N_0 - \frac{1}{m}, N_0 + \frac{1}{m}\right)$. As this was the only $N \in \mathbb{Z}$ such that $x \in \bigcap_{m \geq 1} \left(N - \frac{1}{m}, N + \frac{1}{m}\right)$, it must be that $x \notin \bigcup_{n \in \mathbb{Z}} \bigcap_{m \geq 1} \left(n - \frac{1}{m}, n + \frac{1}{m}\right)$. But then $x \in \bigcup_{n \in \mathbb{Z}} \bigcap_{m \geq 1} \left(n - \frac{1}{m}, n + \frac{1}{m}\right)$ if and only if $x \in \mathbb{Z}$, as desired.