Name: ______________________________
MATH 308
Fall 2022
HW 14: Due 11/10

"Mathematics is the queen of the sciences and number theory is the queen of mathematics."
–Carl Friedrich Gauss

Problem 1. (10pt) For each of the following pairs (a, b), determine the quotient q and remainder r from the division algorithm and express b as $b = qa + r$:

(a) $(a, b) = (4, 17)$

(b) $(a, b) = (3, 117)$

(c) $(a, b) = (-6, 25)$

(d) $(a, b) = (9, -82)$

Problem 2. (10pt) Showing all your work and explaining all your reasoning, answer the following:

(a) Use the Euclidean algorithm to find $\gcd(220, 815)$.

(b) Do there exist integer solutions x, y to the equation $20x - 84y = 25$? Explain.

Problem 3. (10pt) Showing all your work, use the extended Euclidean algorithm to express $\gcd(350, 480)$ as a linear combination of 350 and 480.

Problem 4. (10pt) Recall that a rational number is a real number of the form $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$. A real number which is not rational is called irrational. All integers are rational numbers: if $n \in \mathbb{Z}$, we have $n = \frac{n}{1}$. Some real numbers are rational, e.g. $0.26 = \frac{26}{100} = \frac{13}{50}$ and $0.\overline{3} = \frac{1}{3}$. However, not all real numbers are rational. Write a proof that $\sqrt{2}$ is not rational by completing the following:

(a) We know that $\sqrt{2}$ is either rational or irrational. If $\sqrt{2}$ is not irrational, what do we know about $\sqrt{2}$?

(b) Explain why we can write $\sqrt{2}$ as $\sqrt{2} = \frac{a}{b}$, where $a, b \in \mathbb{Z}$, $b \neq 0$, and $\gcd(a, b) = 1$.

(c) Show that (b) implies that $a^2 = 2b^2$.

(d) Use Euclid's Theorem to show that $2|a$.

(e) Explain why (d) implies that $b^2 = 2k^2$ for some $k \in \mathbb{Z}$.

(f) Explain why (e) implies that $2|b$.

(g) Explain why (f) contradicts (b). What does this imply about $\sqrt{2}$?