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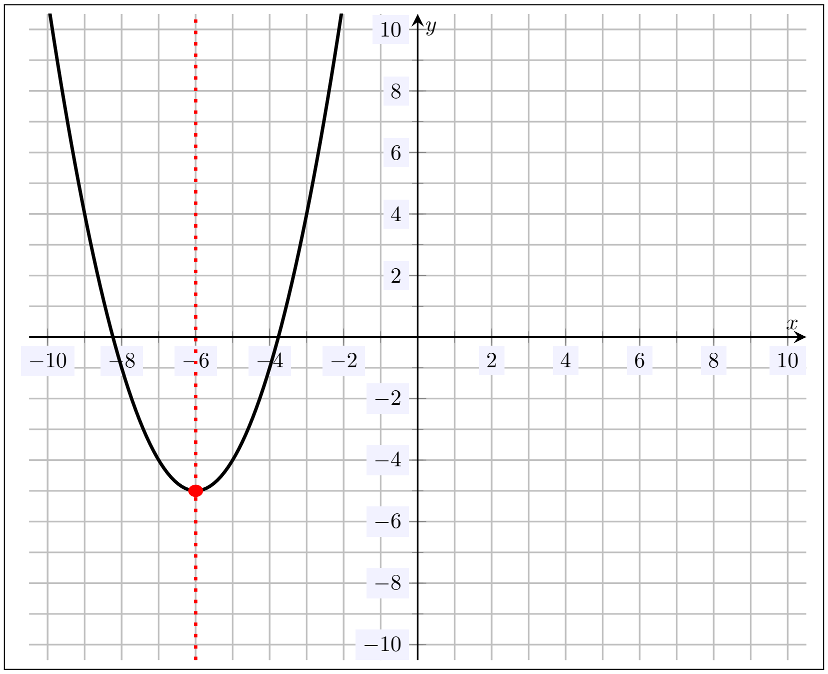
MATH 101
Spring 2024
HW 16: Due 04/10

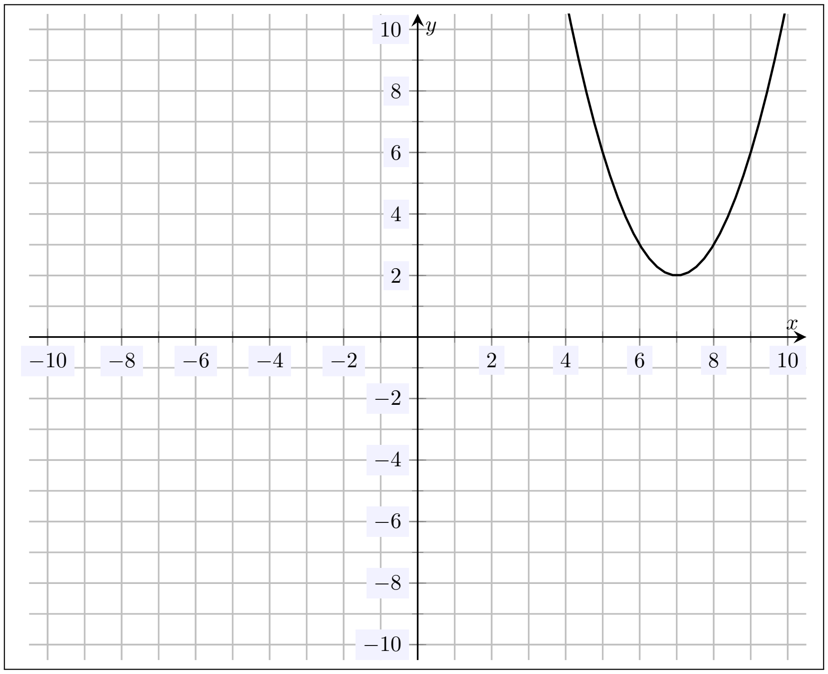
"Mankind was born on Earth... it was never meant to die here."
— Joseph Cooper, Interstellar

Problem 1. (10pts) Sketch the function $f(x) = (x+6)^2 - 5$.



Solution. Recall the vertex form of a quadratic function is $f(x) = a(x-P)^2 + Q$, where (P, Q) is the vertex of the quadratic function and a is the coefficient of x^2 from $f(x) = ax^2 + bx + c$. Observe that $f(x) = (x+6)^2 - 5 = 1\big(x - (-6)\big)^2 + (-5)$. Therefore, $a = 1 > 0$ and $(P, Q) = (-6, -5)$. Therefore, the vertex is $(-6, -5)$ and the parabola opens downwards because $a = 1 > 0$. The axis of symmetry is $x = -6$. Therefore, the plot should be symmetric about this line. This gives the sketch given above.

Problem 2. (10pts) Find the equation of the quadratic function shown below. Be sure to fully justify why your answer is correct.



Solution. Recall the vertex form of a quadratic function is $f(x) = a(x - P)^2 + Q$, where (P, Q) is the vertex of the quadratic function and a is the coefficient of x^2 from $f(x) = ax^2 + bx + c$. We know that if $a > 0$, then the quadratic function opens upwards and if $a < 0$, then it opens downwards. Clearly, because this parabola opens upwards, $a > 0$. Examining the plot, we can see that the vertex is $(P, Q) = (7, 2)$. Then we know that $f(x) = a(x - P)^2 + Q = a(x - 7)^2 + 2$. We can also see that the parabola contains the points $(5, 6)$ and $(9, 6)$. Then we know that when $x = 5$ that $y = 6$. But then...

$$\begin{aligned} f(x) &= a(x-7)^2 + 2 \\ f(5) &= a(5-7)^2 + 2 \\ 6 &= a(-2)^2 + 2 \\ 6 &= 4a + 2 \\ 4 &= 4a \\ a &= 1 \end{aligned}$$

Therefore, $f(x) = (x - 7)^2 + 2 = x^2 - 14x + 51$.

Problem 3. (10pts) Consider the quadratic function $f(x) = -x^2 - 4x + 12$.

(a) Find a, b, c for this quadratic function.

(b) Does $f(x)$ open upwards or downwards? Explain.

(c) Is this quadratic function convex or concave? Explain.

(d) Find the minimum value of $f(x)$, if it exists. If it does not exist, explain why.

(e) Find the maximum value of $f(x)$, if it exists. If it does not exist, explain why.

Solution.

(a) A quadratic function has the form $ax^2 + bx + c$. But then we can see that for $f(x)$, $a = -1$, $b = -4$, and $c = 12$.

(b) Because $a = -1 < 0$, this quadratic function opens downwards.

(c) Because $a = -1 < 0$, this quadratic function is concave.

(d) Because $a = -1 < 0$, this quadratic function has a no minimum value—the outputs of $f(x)$ get arbitrarily small.

(e) Because $a = 1 > 0$, this quadratic function has a maximum value. We know the maximum value occurs at the vertex. So we need to find the vertex of $f(x)$. By completing the square, we have. . .

$$\begin{gathered}
-x^2 - 4x + 12 \\
-(x^2 + 4x - 12) \\
-\left(x^2 + 4x + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2 - 12\right) \\
-\big((x^2 + 4x + 4) + (-4) - 12\big) \\
-\big((x+2)^2 - 16\big) \\
-(x+2)^2 + 16
\end{gathered}$$

Therefore, the vertex is $(-2, 16)$. But then the maximum value for $f(x)$ is 16 and occurs when $x = -2$.

Alternatively, using the 'evaluation method', we know the vertex occurs when $x = -\frac{b}{2a} = -\frac{-4}{2(-1)} = -\dfrac{-4}{-2} = -2$. But then the y-coordinate of the vertex $f(-2) = -(-2)^2 - 4(-2) + 12 = -4 + 8 + 12 = 16$. Therefore, the vertex is $(-2, 16)$ and the maximum output of $f(x)$ is 16.

Problem 4. (10pts) Consider the quadratic function $f(x) = (x+3)^2 - 10$.

(a) Find a, b, c for this quadratic function.

(b) Does $f(x)$ open upwards or downwards? Explain.

(c) Is this quadratic function convex or concave? Explain.

(d) Find the minimum value of $f(x)$, if it exists. If it does not exist, explain why.

(e) Find the maximum value of $f(x)$, if it exists. If it does not exist, explain why.

Solution.

(a) A quadratic function has the form $ax^2 + bx + c$. But because $f(x) = (x+3)^2 - 10 = (x+3)(x+3) - 10 = (x^2 + 6x + 9) - 10 = x^2 + 6x - 1$, we can see that for $f(x)$, $a = 1$, $b = 6$, and $c = -1$.

(b) Because $a = 1 > 0$, this quadratic function opens upwards.

(c) Because $a = 1 > 0$, this quadratic function is convex.

(d) Because $a = 1 > 0$, this quadratic function has a minimum value. We know the minimum value occurs at the vertex. So we need to find the vertex of $f(x)$. Recall the vertex form of a quadratic function is $f(x) = a(x-P)^2 + Q$, where (P, Q) is the vertex of the quadratic function and a is the coefficient of x^2 from $f(x) = ax^2 + bx + c$. Because $f(x) = (x+3)^2 - 10 = 1\big(x - (-3)\big)^2 + (-10)$, we can see that $f(x)$ has vertex $(P, Q) = (-3, -10)$. But then the minimum value for $f(x)$ is -10 and occurs when $x = -3$.

(e) Because $a = 1 > 0$, this quadratic function has no minimum value—the outputs of $f(x)$ get arbitrarily large.