Name: *Caleb McWhorter — Solutions*
MATH 108
Fall 2021
HW 6: Due 11/04

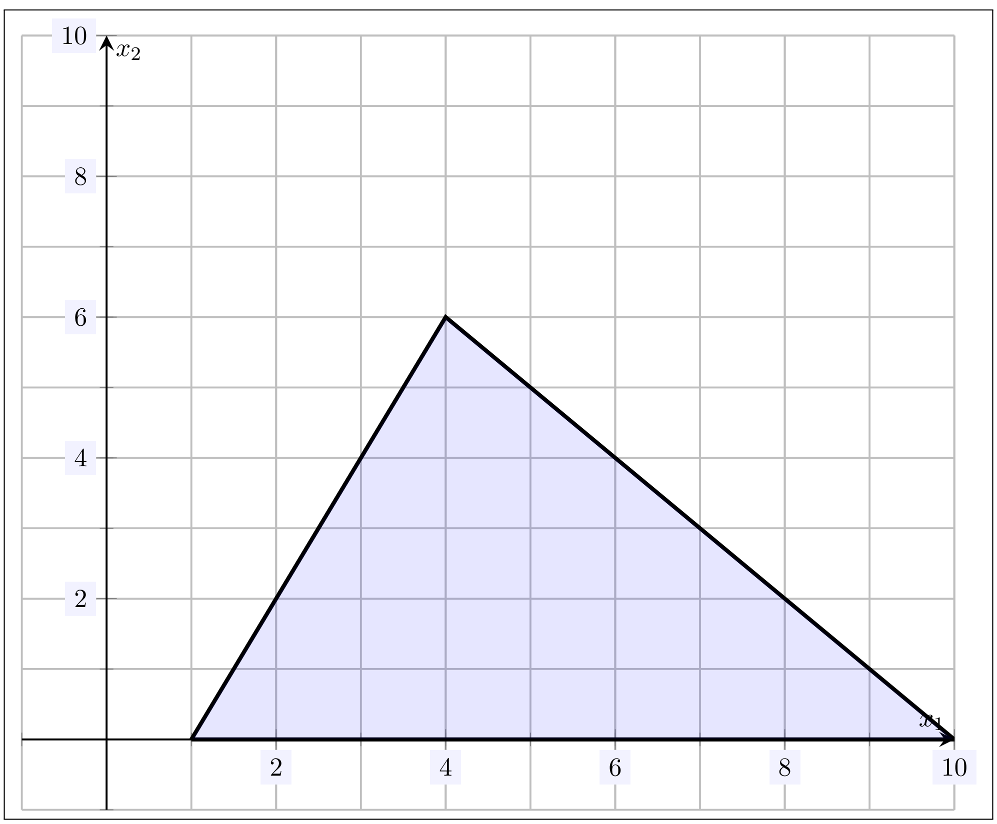
"I used to sell furniture for a living. The trouble was, it was my own."
–Les Dawson

Problem 1. (10pt) As accurately as possible, sketch the feasible region given by the following maximization problem:

$$\begin{aligned} \text{max } z &= 4x_1 + 6x_2 \\ x_1 + x_2 &\leq 10 \\ 2x_1 - x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Is this region bounded or unbounded?

Solution. The first inequality corresponds to the inequality $x + y \leq 10$. Solving for y, we have $y \leq 10 - x$. So we plot the line $y = 10 - x$. On this line, $y = 10 - x$ and because we want $y \leq 10 - x$, we shade below the line. The second equality corresponds to the inequality $2x - y \geq 2$. Solving for y, we have $2x - 2 \geq y$, i.e. $y \leq 2x - 2$. So we plot the line $y = 2x - 2$. On this line, $y = 2x - 2$ and because we want $y \leq 2x - 2$, we shade beneath the line. The final inequalities correspond to the inequalities $x, y \geq 0$. This means we only consider things in Quadrant I. Plotting all these conditions, we obtain the feasible reason shaded below.



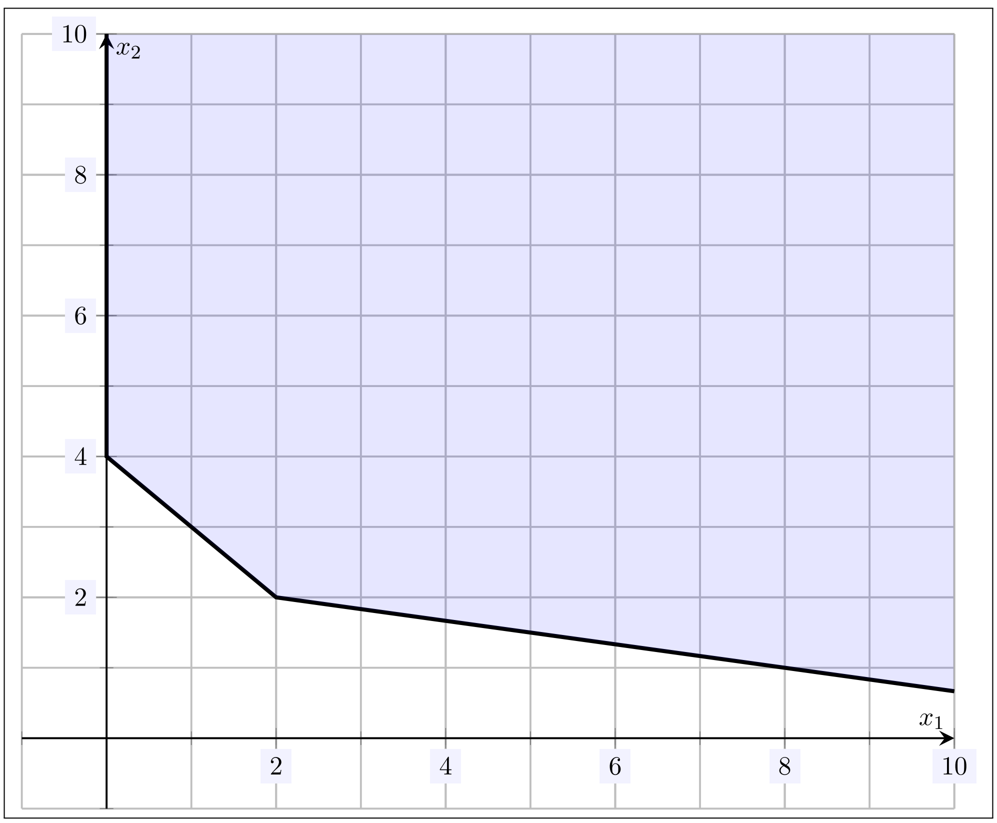
Clearly, this region is bounded.

Problem 2. (10pt) As accurately as possible, sketch the feasible region given by the following minimization problem:

$$\begin{aligned} \min z &= x_1 - 3x_2 \\ x_1 + x_2 &\geq 4 \\ \frac{1}{2}x_1 + 3x_2 &\geq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

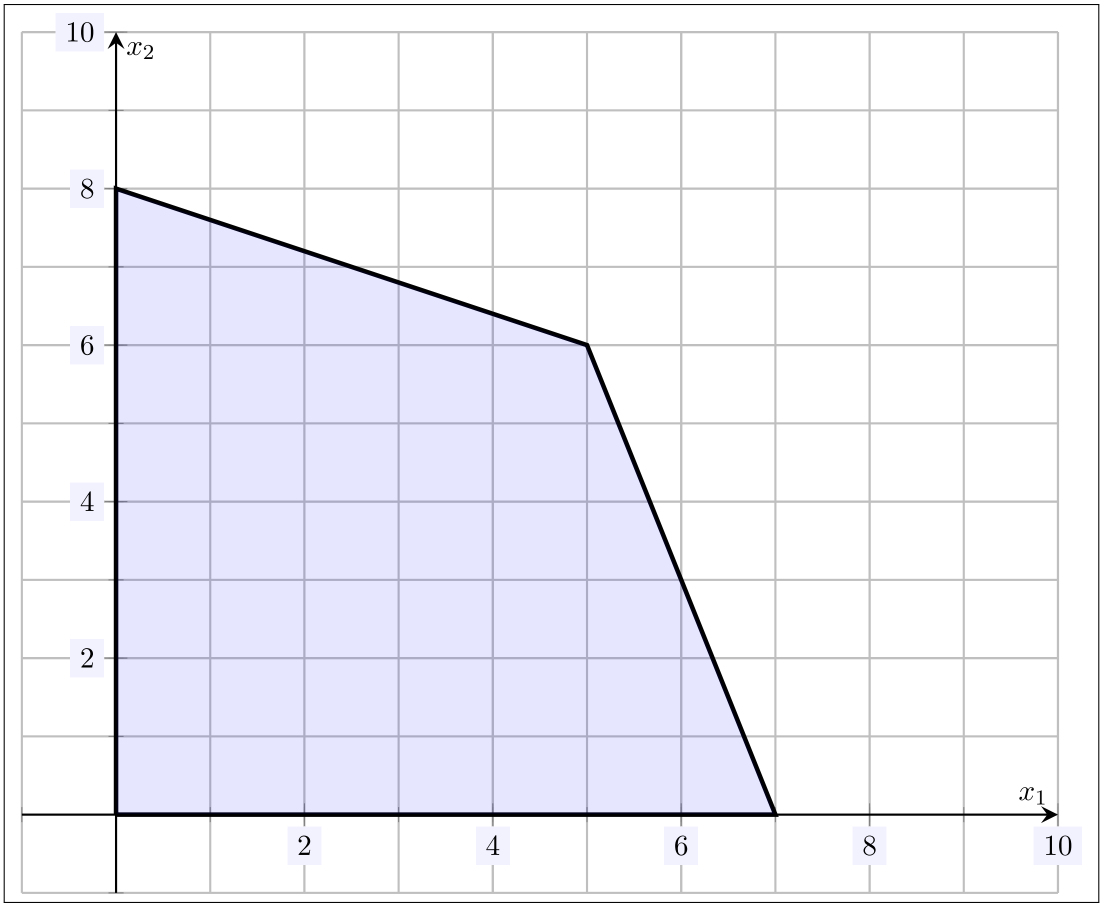
Is this region bounded or unbounded?

Solution. The first inequality correspond to the inequality $x + y \geq 4$. Solving for y, we find $y \geq 4 - x$. So we plot the line $y = 4 - x$. On the line, $y = 4 - x$ and because we want $y \geq 4 - x$, we shade above the line. The second equality corresponds to the inequality $\frac{1}{2}x + 3y \geq 7$. Solving for y, we find $y \geq -\frac{1}{6}x + \frac{7}{3} = \frac{14-x}{6}$. On the line, $y = -\frac{1}{6}x + \frac{7}{3}$ and because we want $y \geq -\frac{1}{6}x + \frac{7}{3}$, we shade above the line. The final inequalities correspond to the inequalities $x, y \geq 0$. This means we only consider things in Quadrant I. Plotting all these conditions, we obtain the feasible reason shaded below.



Clearly, this region is unbounded.

Problem 3. (10pt) Find a system of inequalities that gives the following feasible region:

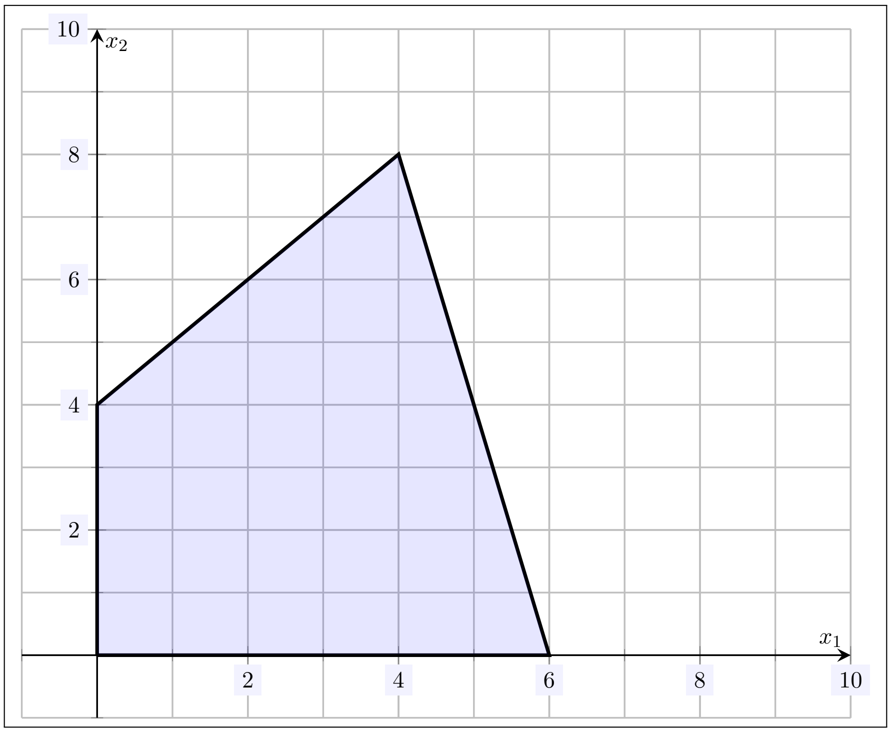


Solution. To force us to be in Quadrant I, we can use the inequalities $x_1, x_2 \geq 0$. Now we find the equations of the two lines, the 'uppermost' containing the points $(0, 8)$ and $(5, 6)$ and the 'rightmost' containing the points $(5, 6)$ and $(7, 0)$. In the former case, the line is $y = -\frac{2}{5}x + 8$, and in the latter case, $y = -3x + 21$. Because we want to shade below these lines, we have $y \leq -\frac{2}{5}x + 8$, i.e. $\frac{2}{5}x + y \leq 8$, and $y \leq -3x + 21$, i.e. $3x + y \leq 21$. Therefore, a system of inequalities (using $x_1 = x$ and $x_2 = y$) representing this feasible region is...

$$\begin{aligned} \frac{2}{5}x_1 + x_2 &\leq 8 \\ 3x_1 + x_2 &\leq 21 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Problem 4. (10pt) Use the Fundamental Theorem of Linear Programming, i.e. the Corner-Point Method, to find the maximum and minimum values for z given the following definition of z and constraints:

$$\begin{aligned} z &= -2x_1 + 5x_2 \\ -x_1 + x_2 &\leq 4 \\ 4x_1 + x_2 &\leq 24 \\ x_1, x_2 &\geq 0 \end{aligned}$$



Solution. The Fundamental Theorem of Linear Programming says that if we have a linear programming problem, i.e. a maximization or minimization of a linear function, over a nonempty bounded region, then the maximum and minimum exist and occur at a corner point. So we plot our feasible region, check that it is nonempty and bounded, find the corner points, and evaluate z at these points. The first inequality corresponds to the line $y = x + 4$, shaded below. The second inequality corresponds to the line $y = 24 - 4x$, shaded below. The last inequalities simply imply that we are in Quadrant I. We then plot these lines. We find the line $y = x + 4$ has x-intercept $(-4, 0)$, y-intercept $(0, 4)$, and intersects the line $y = 24 - 4x$ at the point $(4, 8)$. The line $y = 24 - 4x$ has x-intercept $(6, 0)$ and y-intercept $(0, 24)$. Therefore, the corner points for this feasible region (notice the region is bounded) are $(0, 0)$, $(0, 4)$, $(4, 8)$, and $(6, 0)$. We evaluate $z = -2x_1 + 5x_2 = -2x + 5y$ at these points:

(x, y)	$z = -2x + 5y$
$(0, 0)$	$z = 0 + 0 = 0$
$(0, 4)$	$z = 0 + 20 = 20$
$(4, 8)$	$z = -8 + 40 = 32$
$(6, 0)$	$z = -12 + 0 = -12$

Therefore, the minimum value for z is -12 and occurs at $(x_1, x_2) = (6, 0)$, and the maximum value for z is 32 and occurs at $(x_1, x_2) = (4, 8)$.