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MATH 108
Fall 2022
HW 19: Due 12/08

"Linear programming can be viewed as part of a great revolutionary development which has given mankind the ability to state general goals and to lay out a path of detailed decisions to take in order to 'best' achieve its goals when faced with practical situations of great complexity."

–George Dantzig

Problem 1. (10pt) Write down the initial simplex tableau for the following optimization problem:

$$\begin{aligned} \max z = 4.6x_1 + 3.1x_2 + 7.9x_3 \\ 5.5x_1 - 6x_2 + 1.1x_3 \leq 110.3 \\ -6.7x_1 - 8.3x_3 \leq 220.1 \\ x_1 - 7.7x_2 + 4.5x_3 \leq 662.0 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution. Introducing slack variables into each inequality (except the last non-negativity inequality) to obtain equalities, we have...

$$\begin{array}{rcrcrcccccl} 5.5x_1 & + & -6x_2 & + & 1.1x_3 & + & s_1 & & & = & 110.3 \\ -6.7x_1 & + & 0x_2 & + & -8.3x_3 & + & & s_2 & & = & 220.1 \\ 1x_1 & + & -7.7x_2 & + & 4.5x_3 & + & & & s_3 & = & 662.0 \end{array}$$

Moving things to the 'z'-side of the equality in the function, we have $z - 4.6x_1 - 3.1x_2 - 7.9x_3 = 0$. Adding this to the table yields...

$$\begin{array}{ccrcrcrcccccr} & & 5.5x_1 & + & -6x_2 & + & 1.1x_3 & + & s_1 & & & = & 110.3 \\ & & -6.7x_1 & + & 0x_2 & + & -8.3x_3 & + & & s_2 & & = & 220.1 \\ & & 1x_1 & + & -7.7x_2 & + & 4.5x_3 & + & & & s_3 & = & 662.0 \\ z & + & -4.6x_1 & + & -3.1x_2 & + & -7.9x_3 & & & & & = & 0 \end{array}$$

This yields the following initial simplex tableau:

$$\begin{array}{rrrccc|r} 5.5 & -6.0 & 1.1 & 1 & 0 & 0 & 110.3 \\ -6.7 & 0.0 & -8.3 & 0 & 1 & 0 & 220.1 \\ 1.0 & -7.7 & 4.5 & 0 & 0 & 1 & 662.0 \\ \hline -4.6 & -3.1 & -7.9 & 0 & 0 & 0 & 0 \end{array}$$

Problem 2. (10pt) Suppose that the initial simplex tableau below was associated to a standard maximization problem. Write down the function being maximized and the corresponding system of constraints.

$$\begin{array}{rrrrrr|r} 2 & -1 & 4 & 1 & 0 & 0 & 100 \\ 6 & 0 & 2 & 0 & 1 & 0 & 80 \\ -4 & 8 & 3 & 0 & 0 & 1 & 220 \\ \hline -3 & -1 & -5 & 0 & 0 & 0 & 0 \end{array}$$

Solution. Each row of the tableau 'corresponds' to an inequality with the exception of the last row which 'corresponds to the function.' But then there were $4 - 1 = 3$ inequalities in the original system (ignoring the non-negativity inequality). For each inequality, we introduce a slack variable. Therefore, there were 3 slack variables. Each column of the tableau 'corresponds' to a variable in the system with the exception of the last column which 'corresponds to the solutions.' Therefore, there were $7 - 1 = 6$ variables in the system. Because 3 of the variables are slack variables, we have $6 - 3 = 3$ 'original' variables in the system of inequalities. Labeling these columns in the tableau, we have...

$$\begin{array}{rrrrrr|r} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & \\ 2 & -1 & 4 & 1 & 0 & 0 & 100 \\ 6 & 0 & 2 & 0 & 1 & 0 & 80 \\ -4 & 8 & 3 & 0 & 0 & 1 & 220 \\ \hline -3 & -1 & -5 & 0 & 0 & 0 & 0 \end{array}$$

The last row 'corresponds' to the function. But then we have $z - 3x_1 - x_2 - 5x_3 = 0$ so that $z = 3x_1 + x_2 + 5x_3$. Writing the equalities corresponding to the first 3 rows, we have...

$$\begin{aligned} 2x_1 - x_2 + 4x_3 + s_1 &= 100 \\ 6x_1 + 2x_3 + s_2 &= 80 \\ -4x_1 + 8x_2 + 3x_3 + s_3 &= 220 \end{aligned}$$

Removing the slack variables, we have...

$$\begin{aligned} 2x_1 - x_2 + 4x_3 &\leq 100 \\ 6x_1 + 2x_3 &\leq 80 \\ -4x_1 + 8x_2 + 3x_3 &\leq 220 \end{aligned}$$

Therefore, the original minimization problem was...

$$\begin{aligned} \max z = 3x_1 + x_2 + 5x_3 \\ 2x_1 - x_2 + 4x_3 &\leq 100 \\ 6x_1 + 2x_3 &\leq 80 \\ -4x_1 + 8x_2 + 3x_3 &\leq 220 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Problem 3. (10pt) Suppose that the final simplex tableau associated to a maximization problem was the following:

$$\begin{array}{rrrrrrrrrr}
1 & 1.1 & 2 & 0 & 0 & 0.22 & 0.067 & -0.011 & 0 & 140 \\
0 & 2.1 & 1.5 & 1 & 0 & -0.021 & 0.23 & -0.037 & 0 & 85 \\
0 & -1.1 & -0.59 & 0 & 1 & 0.008 & -0.088 & 0.16 & 0 & 42 \\
0 & -6.4 & -12 & 0 & 0 & -0.55 & -0.45 & 0.54 & 1 & 270 \\
0 & 2.3 & 2.3 & 0 & 0 & 0.2 & 0.59 & 0.72 & 0 & 760
\end{array}$$

(a) How many inequalities were considered?

(b) How many variables were there in the original inequalities?

(c) How many slack/surplus variables were introduced?

(d) What was the solution to this maximization problem?

Solution.

(a) Each row of the tableau 'corresponds' to an inequality with the exception of the last row which 'corresponds to the function.' But then there were $5 - 1 = 4$ inequalities in the original system (ignoring the non-negativity inequality).

(b) Each column of the tableau 'corresponds' to a variable in the system with the exception of the last column which 'corresponds to the solutions.' Therefore, there were $10-1 = 9$ variables in the system. Note by (c), there are 4 slack/surplus variables. Therefore, there were $9 - 4 = 5$ 'original' variables in the system of inequalities.

(c) Because we introduce a slack/surplus variable for each inequality and by (a) there were 4 inequalities in the original system, there were 4 slack/surplus variables.

(d) By (b) and (c), there were 5 'original' variables and 4 slack/surplus variables. Therefore, we need find the maximum value along with the values of the variables—-namely, the values for $(x_1, x_2, x_3, x_4, x_5, s_1, s_2, s_3, s_4)$. Adding 'dividers' to the tableau and 'naming' the columns, we have...

$$\begin{array}{rrrrrrrrr|r}
x_1 & x_2 & x_3 & x_4 & x_5 & s_1 & s_2 & s_3 & s_4 & \\
\boxed{1} & 1.1 & 2 & 0 & 0 & 0.22 & 0.067 & -0.011 & 0 & 140 \\
0 & 2.1 & 1.5 & \boxed{1} & 0 & -0.021 & 0.23 & -0.037 & 0 & 85 \\
0 & -1.1 & -0.59 & 0 & \boxed{1} & 0.008 & -0.088 & 0.16 & 0 & 42 \\
0 & -6.4 & -12 & 0 & 0 & -0.55 & -0.45 & 0.54 & \boxed{1} & 270 \\
\hline
0 & 2.3 & 2.3 & 0 & 0 & 0.2 & 0.59 & 0.72 & 0 & 760
\end{array}$$

We indicate the pivot positions above. This yields $x_1 = 140$, $x_4 = 85$, $x_5 = 42$, and $s_4 = 270$. All remaining variables have value 0. The maximum value is 760. Therefore, the maximum value is 760 and occurs at $(x_1, x_2, x_3, x_4, x_5, s_1, s_2, s_3, s_4) = (140, 0, 0, 85, 42, 0, 0, 0, 270)$.