Quiz 1. *True/False*: If P is the proposition $6 < 5$ and Q is the proposition, "Earth is a planet," then the logical statement $P \to Q$ is false.

Solution. The statement is *false*. Recall that the truth table for $P \to Q$ is as follows:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Here, P is the proposition $P : 6 < 5$ and Q is the proposition Q: "Earth is a planet." It is clear that P is false and Q is true. But then examining the logic table above, we can see that $P \to Q$ is true.

Quiz 2. *True/False*: $\neg(P \to \neg Q) \equiv P \wedge Q$

Solution. The statement is *true*. To determine if two propositions are logically equivalent, one can either examine the truth table or apply logical rules to obtain one logical expression from the other. If we construct a truth table, we have...

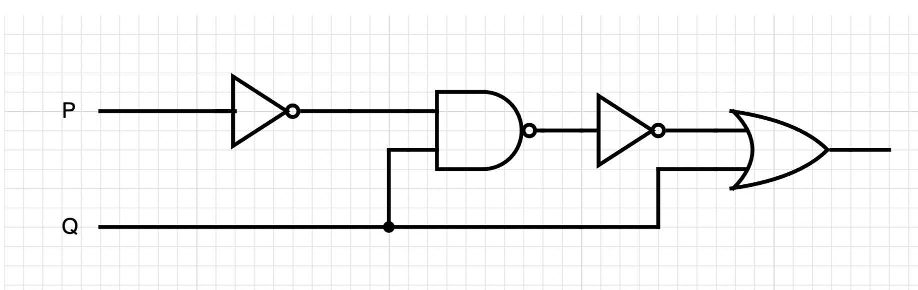
P	Q	$\neg Q$	$P \to \neg Q$	$\neg(P \to \neg Q)$	$P \wedge Q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F

Because for each possible pair of choices for P and Q the outputs for $\neg(P \to \neg Q)$ and $P \wedge Q$ match, $\neg(P \to \neg Q) \equiv P \wedge Q$. Alternatively, we can transform one into the other by applying logical equivalences (recall $P \to Q \equiv \neg P \vee Q$ or $\neg(P \to Q) \equiv P \wedge \neg Q$):

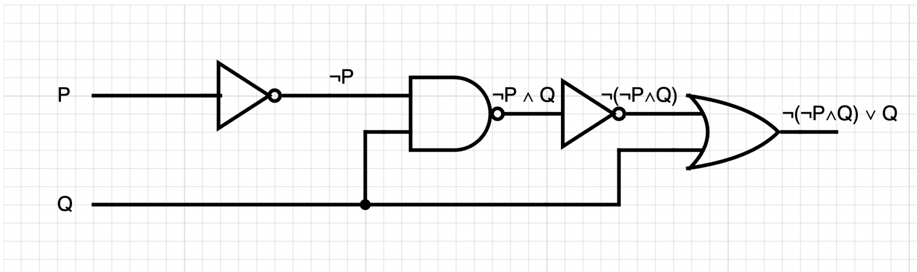
$$\neg(P \to \neg Q) \equiv \neg(\neg P \vee \neg Q) \equiv \neg(\neg P) \wedge \neg(\neg Q) \equiv P \wedge Q.$$

Quiz 3. *True/False*: The logic corresponding to the circuit shown below is the proposition:

$$(\neg P \wedge Q) \vee \neg Q.$$



Solution. The statement is *false*. We can trace through the circuit. We see that the current from P passes through a NOT gate and we obtain $\neg P$. This then feeds into an AND gate along with Q so that we obtain $\neg P \wedge Q$. The resulting current is then passed through a NOT gate, obtaining $\neg(\neg P \wedge Q)$. This finally reaches an OR gate—along with Q—to obtain $\neg(\neg P \wedge Q) \vee Q$. We can see a diagrammatic explanation below.



Quiz 4. *True/False*: Let the universe $\mathcal{U}$ be the set of real numbers and define $P(x)$ to be the predicate $P(x) : x^2 + x - 4 \geq 0$. Then $(\forall x)\big(\neg P(x)\big)$ is true.

Solution. The statement is *false*. If $P(x) : x^2 + x - 4 \geq 0$, then $\neg P(x) : x^2 + x - 4 < 0$. But then $(\forall x)\big(\neg P(x)\big)$ is the statement, "For all x, $x^2+x-4 < 0$." Now if $x = 1$, we have $\neg P(1)\colon 1^2+1-4 < 0$, i.e. $-2 < 0$, which is true. If $x = 0$, we have $\neg P(0)\colon 0^2 + 0 - 4 < 0$, i.e. $-4 < 0$, which is true. However, while $(\forall x)\big(\neg P(x)\big)$ is clearly true for *some* (we found at least two), it is not true *for all* x. As a counterexample, let $x = 10$. Then $\neg P(10)\colon 10^2 + 10 - 4 < 0$, which is $104 < 0$—clearly false. Therefore, $\neg P(x)$ is not true for all x. But then $(\forall x)\big(\neg P(x)\big)$ is false.

Quiz 5. *True/False*: Let the domain of x, y be the integers. Then $(\exists! x)(\forall y)(x + 2y = 5)$.

Solution. The statement is *false*. The logical proposition $(\exists! x)(\forall y)(x + 2y = 5)$ in words states, "There exists a unique x such that for all y, $x + 2y = 5$." Suppose that there were such a x, say x_0. Then we know that $x_0 + 2y = 5$ for all y. In particular, x_0 satisfies this equality when $y = 0$. But then we know that $x_0 = 5$. But also, it must satisfy the equality when $x = 1$. But then $x_0 + 2 = 5$ so that x_0. Then there is not a unique x that works for all y! Therefore, the statement is false. Note that if we reverse the quantifiers, the statement is true: $(\forall y)(\exists! x)(x + 2y = 5)$. In this case, this is the statement, "For all y, there exists a unique x such that $x + 2y = 5$." If you were given any y, define $x_0 := 5 - 2y$. But then $x + 2y = (5 - 2y) + 2y = 5$. So there exists such an x. Is it unique? Well if there were two or more x values that worked for some y, say two of them are x_0 and $\tilde{x}_0$, then we have $x_0 + 2y = 5 = \tilde{x}_0 + 2y$. But then $x_0 + 2y = \tilde{x}_0 + 2y$. Subtracting $2y$, we have $x_0 = \tilde{x}_0$. Therefore, there can only be one such x. Because we have found one, we know that the statement that for all y, there exists a unique y such that $x + 2y = 5$ is true.

Quiz 6. *True/False*: $\{1,2\} \subseteq \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$

Solution. The statement is *false*. We know that $A \subseteq B$ if and only if for all $a \in A$, we have $a \in B$. We test every element of the set $\{1,2\}$. The first element is 1. However, $1 \notin \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$. [Note that $1 \notin \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$ but $\{1\} \in \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$.] However, we do have $\{1,2\} \notin \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$.

Quiz 7. *True/False*: $\displaystyle\bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \varnothing$

Solution. The statement is *false*. For $n = 1$, the set $(-\frac{1}{n}, \frac{1}{n})$ is the interval $(-1, 1)$. For $n = 2$, the set $(-\frac{1}{n}, \frac{1}{n})$ is the interval $(-\frac{1}{2}, \frac{1}{2})$. For $n = 3$, the set $(-\frac{1}{n}, \frac{1}{n})$ is the interval $(-\frac{1}{3}, \frac{1}{3})$. Note that 0 is an element of all these sets. Generally, we have $0 \in (-\frac{1}{n}, \frac{1}{n})$ for all $n \in \mathbb{N}$. But then we know that $0 \in \displaystyle\bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$. This is sufficient to demonstrate that this is not empty. [Note that it is actually true that $\displaystyle\bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$—though this takes more work to prove.]

Quiz 8. *True/False*: Let $E(n)$ denote the relation from $\mathbb{N}$ to $\mathbb{Z}^{\geq 0}$ given by the rule that $E(n)$ is the number of positive even integers less than or equal to n. Then this relation is a function with $E(5) = 2$, i.e. 2 is in the image of 5, and 10 in the preimage of 5.

Solution. The statement is *false*. There are several claims here. First, the claim that $E(n) : \mathbb{N} \to \mathbb{Z}^{\geq 0}$ is a function. Given some $n \in \mathbb{N}$, there is a single number of positive even integers $\leq n$. But then for every input for $E(n)$, there is only one possible output. Therefore, $E(n)$ is a function from $\mathbb{N}$ to $\mathbb{Z}^{\geq 0}$. For 2 to be in the image of 5, we need $E(5) = 2$. There are two positive even integers ≤ 5 (namely, 2 and 4) so that 2 is in the image of 5. For 10 to be in the preimage of 5, we would have to have $E(10) = 5$. Note that there are 5 even integers ≤ 10 (namely 2, 4, 6, 8, 10). Therefore, 10 is in the preimage of 5.

Quiz 9. *True/False*: Let $f : X \to Y$ be a function. Then f^{-1} will be a function if and only if the preimage set satisfies the following: $(\forall y \in \text{im } f)(\exists x \in X)(f^{-1}(y) = x)$.

Solution. The statement is *false*. Take for example the function $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$ given by $f(x) = x^2$. For all $y \in \mathbb{R}^{\geq 0}$, there exists an $x \in \mathbb{R}$ such that $f(x) = y$, namely $\pm\sqrt{y}$. But if $y > 0$, then there are two possibilities: $+\sqrt{y}$ and $-\sqrt{y}$. But this function $f(x)$ has f^{-1} with the property that $(\forall y \in \text{im } f)(\exists x \in X)(f^{-1}(y) = x)$. If we want f^{-1} to be a function, we require $(\forall y \in \text{im } f)(\exists! x \in X)(f^{-1}(y) = x)$.

Quiz 10. *True/False*: Suppose that $f : A \to B$ and $g : B \to C$ are functions and that $g \circ f$ is injective. Then it must be that f is injective.

Solution. The statement is *true*. Observe that $g \circ f : A \to C$. Suppose f were not injective. Then there are two values in A, say a_1, a_2, such that $a_1 \neq a_2$ and $f(a_1) = f(a_2)$. But then we have...

$$\begin{aligned} f(a_1) &= f(a_2) \\ g(f(a_1)) &= g(f(a_2)) \\ (g \circ f)(a_1) &= (g \circ f)(a_2) \end{aligned}$$

But then there are two values in the domain of $g \circ f$, namely a_1, a_2 such that $a_1 \neq a_2$ but $(g \circ f)(a_1) = (g \circ f)(a_2)$. But then $g \circ f$ is not injective, contrary to what we were told. Our assumption that f was not injective must then be wrong. Therefore, it must be that f is injective.