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MATH 108

Spring 2023

HW 16: Due 05/01

*“Everyday life is like programming, I guess. If you love something, you can put beauty into it.”*

*—Donald Knuth*

**Problem 1.** (10pt) Write down the initial simplex tableau for the following optimization problem:

$$\begin{aligned}\max z &= 3.1x_1 - 4.7x_2 + 5.9x_3 \\ 1.1x_1 - 5.7x_2 + 4.0x_3 &\leq 10.4 \\ 6.7x_1 - 0.8x_2 - 8.8x_3 &\geq -8.8 \\ -9.1x_1 + 7.3x_2 - 9.1x_3 &\leq 11.7 \\ x_1, x_2, x_3 &\geq 0\end{aligned}$$

**Solution.** First, observe that this optimization is not in standard form because the third inequality is not ‘ $\leq$ ’ and the constant term is negative. Multiplying both sides of this third inequality by  $-1$ , we have...

$$\begin{aligned}\max z &= 3.1x_1 - 4.7x_2 + 5.9x_3 \\ 1.1x_1 - 5.7x_2 + 4.0x_3 &\leq 10.4 \\ -6.7x_1 + 0.8x_2 + 8.8x_3 &\leq 8.8 \\ -9.1x_1 + 7.3x_2 - 9.1x_3 &\leq 11.7 \\ x_1, x_2, x_3 &\geq 0\end{aligned}$$

Now introducing slack variables into each inequality (except the last non-negativity inequality) to obtain equalities, we have...

$$\begin{array}{ccccccccc} 1.1x_1 & - & 5.7x_2 & + & 4.0x_3 & + & s_1 & & = & 10.4 \\ -6.7x_1 & + & 0.8x_2 & + & 8.8x_3 & + & & s_2 & = & 8.8 \\ -9.1x_1 & + & 7.3x_2 & - & 9.1x_3 & + & & & s_3 & = & 11.7 \end{array}$$

Moving things to the ‘ $z$ ’-side of the equality in the function, we have  $z - 4.6x_1 - 3.1x_2 - 7.9x_3 = 0$ . Adding this to the table yields...

$$\begin{array}{ccccccccc} 1.1x_1 & - & 5.7x_2 & + & 4.0x_3 & + & s_1 & & = & 10.4 \\ -6.7x_1 & + & 0.8x_2 & + & 8.8x_3 & + & & s_2 & = & 8.8 \\ -9.1x_1 & + & 7.3x_2 & - & 9.1x_3 & + & & & s_3 & = & 11.7 \\ z & - & 3.1x_1 & + & 4.7x_2 & - & 5.9x_3 & & & = & 0 \end{array}$$

This yields the following initial simplex tableau:

$$\begin{array}{cccccc|c} 1.1 & -5.7 & 4.0 & 1 & 0 & 0 & 10.4 \\ -6.7 & 0.8 & 8.8 & 0 & 1 & 0 & 8.8 \\ -9.1 & 7.3 & -9.1 & 0 & 0 & 1 & 11.7 \\ \hline -3.1 & 4.7 & -5.9 & 0 & 0 & 0 & 0 \end{array}$$

**Problem 2.** (10pt) Suppose that the final simplex tableau associated to a maximization problem was the following:

1	1.77	0	0	0.74	0	0.26	0.29	208.57
0	0.57	0	1	0.14	0	-0.14	0.29	28.57
0	2.51	0	0	0.83	1	1.17	-0.14	605.71
0	0.09	1	0	-0.03	0	0.03	0.14	34.29
0	3	0	0	2	0	0	2	600

- How many inequalities were considered?
- How many variables were there in the original inequalities?
- How many slack/surplus variables were introduced?
- What was the solution to this maximization problem?

**Solution.**

- Each row of the tableau ‘corresponds’ to an inequality with the exception of the last row which ‘corresponds to the function.’ But then there were  $5 - 1 = 4$  inequalities in the original system (ignoring the non-negativity inequality).
- Each column of the tableau ‘corresponds’ to a variable in the system with the exception of the last column which ‘corresponds to the solutions.’ Therefore, there were  $9 - 1 = 8$  variables in the system. Note by (c), there are 4 slack/surplus variables. Therefore, there were  $8 - 4 = 4$  ‘original’ variables in the system of inequalities.
- Because we introduce a slack/surplus variable for each inequality and by (a) there were 4 inequalities in the original system, there were 4 slack/surplus variables.
- By (b) and (c), there were 5 ‘original’ variables and 4 slack/surplus variables. Therefore, we need find the maximum value along with the values of the variables—namely, the values for  $(x_1, x_2, x_3, x_4, x_5, s_1, s_2, s_3, s_4)$ . Adding ‘dividers’ to the tableau and ‘naming’ the columns, we have...

$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	$s_4$	
<span style="border: 1px solid black;">1</span>	1.77	0	0	0.74	0	0.26	0.29	208.57
0	0.57	0	<span style="border: 1px solid black;">1</span>	0.14	0	-0.14	0.29	28.57
0	2.51	0	0	0.83	<span style="border: 1px solid black;">1</span>	1.17	-0.14	605.71
0	0.09	<span style="border: 1px solid black;">1</span>	0	-0.03	0	0.03	0.14	34.29
0	3	0	0	2	0	0	2	600

We indicate the pivot positions above. This yields  $x_1 = 208.57$ ,  $x_3 = 34.39$ ,  $x_4 = 28.57$ , and  $s_2 = 605.71$ . All remaining variables have value 0. The maximum value is 600. Therefore, the maximum value is 600 and occurs at  $(x_1, x_2, x_3, x_4, s_1, s_2, s_3, s_4) = (208.57, 0, 34.29, 28.57, 0, 605.71, 0, 0)$ .

**Problem 3.** (10pt) Find the dual problem to the minimization problem below.

$$\begin{aligned}\min z &= 2x_1 + 6x_2 \\ 6x_1 + 5x_2 &\leq 10 \\ x_1 + 3x_2 &\leq 9 \\ x_1, x_2 &\geq 0\end{aligned}$$

**Solution.** First, we write the ‘matrix associated’ to this minimization; that is, we create a matrix with rows corresponding to the equality version of the inequalities (with the exception of the non-negativity inequality) with the function being the last row. This yields matrix:

$$\begin{pmatrix} 6 & 5 & 10 \\ 1 & 3 & 9 \\ 2 & 6 & 0 \end{pmatrix}$$

We now find the transpose of this matrix:

$$\begin{pmatrix} 6 & 5 & 10 \\ 1 & 3 & 9 \\ 2 & 6 & 0 \end{pmatrix}^T = \begin{pmatrix} 6 & 1 & 2 \\ 5 & 3 & 6 \\ 10 & 9 & 0 \end{pmatrix}$$

We now find the standard maximization problem corresponding to this matrix:

$$\begin{aligned}\max w &= 10y_1 + 9y_2 \\ 6y_1 + y_2 &\leq 2 \\ 5y_1 + 3y_2 &\leq 6 \\ y_1, y_2 &\geq 0\end{aligned}$$