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MATH 308
Fall 2022
HW 9: Due 10/13

> *"It's fine to work on any problem, so long as it generates interesting mathematics along the way—even if you don't solve it at the end of the day."*
>
> *–Andrew Wiles*

Problem 1. (10pt) Suppose that you have a function $f : \mathbb{R} \to \mathbb{R}$ that is strictly increasing.

(a) Explain why f must be an injective function.

(b) If f is merely increasing, does f have to be an injection? Explain why or give a counterexample.

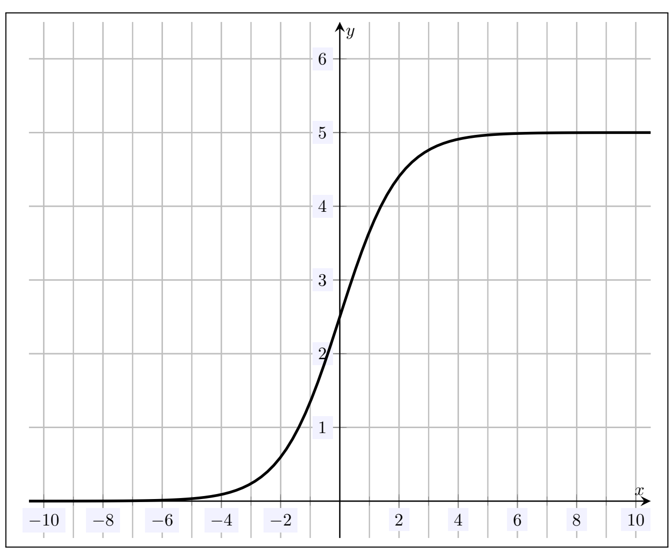
(c) Does f have to be surjective? Explain why or give a counterexample.

Solution.

(a) Because $f(x)$ is strictly increasing, 'subsequent' values must always be larger so that no value could possibly repeat. But then $f(x)$ would be injective. To make this concrete, suppose $f(x)$ is strictly increasing but were not injective, i.e. there exist x_1, x_2 with $x_1 \neq x_2$ but $f(x_1) = f(x_2)$. Without loss of generality, assume $x_1 < x_2$. But then because $f(x)$ is strictly increasing, it must be that $f(x_1) < f(x_2)$, a contradiction. Therefore, it must be that $f(x)$ is injective.

(b) No. For instance, if $c \in \mathbb{R}$, then $f(x) = c$ is increasing because if $x_1 < x_2$, then $c = f(x_1) \leq f(x_2) = c$. However, it is clear that $f(x)$ is not injective because $f(x) = c$ for all $x \in \mathbb{R}$.

(c) Even if $f(x)$ is strictly increasing, it need not be the case that $f(x)$ is surjective. For instance, the logistic function $f(x) = \frac{5}{1+e^{-x}}$ is clearly strictly increasing but not surjective as there is no x such that $f(x) = 6$.



Problem 2. (10pt) Consider the function $f : \mathbb{R}^{\geq 2} \to \mathbb{R}$ given by $f(x) = \sqrt{x-2}$.

(a) Solve the equation $\sqrt{x-2} = \sqrt{y-2}$ for y.

(b) Using your work in (a), explain why this shows that $f(x)$ is injective.

(c) Is $f(x)$ surjective? If $f(x)$ is surjective, explain why. If $f(x)$ is not surjective, find an element of the codomain not in the image of $f(x)$.

Solution.

(a) We have...

$$\sqrt{x-2} = \sqrt{y-2}$$

$$(\sqrt{x-2})^2 = (\sqrt{y-2})^2$$

$$x - 2 = y - 2$$

$$x = y$$

(b) We know a function $f(x)$ is injective if $f(x) = f(y)$ implies that $x = y$. Suppose $f(x) = \sqrt{x-2}$. But then if $f(x) = f(y)$, we know that $\sqrt{x-2} = \sqrt{y-2}$. By (a), we know that this implies that $x = y$. Therefore, $f(x)$ is injective.

(c) The function $f(x) = \sqrt{x-2}$ is not surjective. For instance, $-1 \notin \operatorname{im} f$. Generally, if $c \in \mathbb{R}$ and $c < 0$, then $f(x) \neq c$ for all $x \in \mathbb{R}$ because $f(x) \geq 0$.

Problem 3. (10pt) Let A, B be nonempty sets. Find a bijective function from $A \times B$ to the set $B \times A$. Be sure to explain why your function is bijective. Does this mean that $A \times B$ and $B \times A$ are the same sets? Explain why or why not.

Solution. It should be clear that $A \times B$ and $B \times A$ as they merely differ by a reversal of element order. We can make this rigorous: let $f : A \times B \to B \times A$ be defined by $(a, b) \mapsto (b, a)$, i.e. $f\big((a, b)\big) = (b, a)$. It is clear that f is injective: if $f\big((a, b)\big) = f\big((a', b')\big)$, then we have $(a, b) = f\big((a, b)\big) = f\big((a', b')\big) = (a', b')$. But then $(a, b) = (a', b')$, which forces $a = a'$ and $b = b'$. Therefore, $(a, b) = (a', b')$. Furthermore, given $(b, a) \in B \times A$, we know that $a \in A$, $b \in B$, so that $(a, b) \in A \times B$ and $f\big((a, b)\big) = (b, a)$. Therefore, f is surjective. But as f is injective and surjective, we know that f is bijective.