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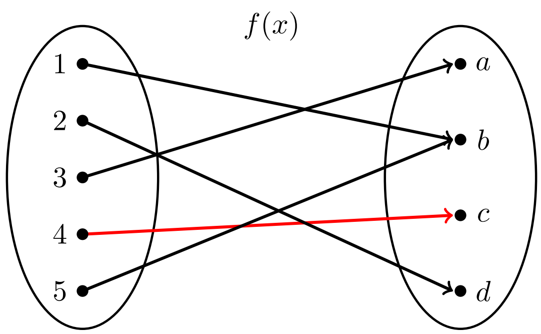
MATH 308
Fall 2022
HW 8: Due 10/13

"The difference between mathematicians and physicists is that after physicists prove a big result they think it is fantastic but after mathematicians prove a big result they think it is trivial."
–Lucien Szpiro

Problem 1. (10pt) Consider the relation $f(x)$ given below.



(a) Explain why $f(x)$ is not a function.

(b) Add an arrow to the diagram so that $f(x)$ is a surjective function.

(c) Identify the domain, codomain, and range for $f(x)$.

(d) Is $f(x)$ an injective function? Explain why or why not.

Solution.

(a) A function must be defined on its entire domain. Because the domain of $f(x)$ is the set $\{1, 2, 3, 4, 5\}$, $f(4)$ need be defined for $f(x)$ to be a function.

(b) For $f(x)$ to be a surjective function, we first need assure that $f(x)$ is a function, i.e. we need to define $f(4)$. We can choose any one of $\{a, b, c, d\}$, i.e. $f(4) \in \{a, b, c, d\}$. For $f(x)$ to be surjective, we need $\operatorname{im} f = \{a, b, c, d\}$. As defined above, $\operatorname{im} f = \{a, b, c, d\}$. But then defining $f(4)$ to be in $\{a, b, c, d\} \setminus \operatorname{im} f = \{c\}$, i.e. $f(4) := c$ (given by the red arrow in the diagram above), $f(x)$ is then a surjective function.

(c) The domain of $f(x)$ is $\{1, 2, 3, 4, 5\}$. The codomain of $f(x)$ is $\{a, b, c, d\}$. The range of the original 'function' was $\{a, b, d\}$, while the range of the $f(x)$ defined is $\{a, b, c, d\}$.

(d) The function $f(x)$ is not injective as $f(1) = b = f(5)$ but $1 \neq 5$.

Problem 2. (10pt) Complete the proof of the proposition stated below by filling in the blanks.

Proposition. Let $f : X \to Y$ be a function and $B \subseteq Y$. Then $X \setminus f^{-1}(B) \subseteq f^{-1}(Y \setminus B)$.

Proof. We know that if $X \setminus f^{-1}(B) = \varnothing$, then $X \setminus f^{-1}(B) \subseteq f^{-1}(Y \setminus B)$. Assume that $X \setminus f^{-1}(B) \neq \varnothing$. To show that $X \setminus f^{-1}(B) \subseteq f^{-1}(Y \setminus B)$, we need to show that if $\underline{\quad x \in X \setminus f^{-1}(B) \quad}$, then $\underline{\quad x \in f^{-1}(Y \setminus B) \quad}$.

Let $x \in X \setminus f^{-1}(B)$. But then we know that $\underline{\qquad x \in X \qquad}$ and $x \notin \underline{\qquad f^{-1}(B) \qquad}$. Because $x \notin \underline{\qquad f^{-1}(B) \qquad}$, we know that $f(x) \notin \underline{\qquad B \qquad}$. It is clear that $f(x) \in Y$. But then $\underline{\qquad f(x) \in Y \qquad}$ and $f(x) \notin \underline{\qquad B \qquad}$. This shows that $f(x) \in \underline{\qquad Y \setminus B \qquad}$.

This shows that $f(x)$ is in the preimage of $Y \setminus B$. But then we know that $x \in \underline{\qquad f^{-1}(Y \setminus B) \qquad}$.

But then if $x \in \underline{\qquad X \setminus f^{-1}(B) \qquad}$, then $x \in \underline{\qquad f^{-1}(Y \setminus B) \qquad}$. Therefore, $X \setminus f^{-1}(B) \subseteq f^{-1}(Y \setminus B)$.

Problem 3. (10pt) Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $x \mapsto x^2 + 3x - 7$.

(a) Without referencing the graph of f, use the definition of decreasing to show that $f(x)$ is not a decreasing function on $\mathbb{R}$ by giving a counterexample.

(b) Determine whether or not $3 \in \operatorname{im} f$. If $3 \in \operatorname{im} f$, find an element in the preimage of 3. If $3 \notin \operatorname{im} f$, explain why.

(c) Is $f^{-1}(x)$ a function? Explain why or why not by referencing the graph of $f(x)$. Give an additional explanation of why or why not using your response in (b).

Solution.

(a) A function is decreasing if $f(x_2) \leq f(x_1)$ whenever $x_1 < x_2$. Observe that $0 < 1$ but $f(1) = -3 \not< -7 = f(0)$. Therefore, $f(x)$ is not decreasing. [Note: We can write $f(x) = x^2 + 3x - 7 = \left(x + \frac{3}{2}\right)^2 - \frac{37}{4}$. Therefore, $f(x)$ is decreasing on $(-\infty, -\frac{3}{2})$ and increasing on $(\frac{3}{2}, \infty)$.]

(b) If $3 \in \operatorname{im} f$, then there exists $x_0 \in \mathbb{R}$ such that $f(x_0) = 3$, i.e. $x \in f^{-1}(3)$. But then we would have...

$$\begin{aligned} f(x_0) &= 3 \\ x_0^2 + 3x_0 - 7 &= 3 \\ x_0^2 + 3x_0 - 10 &= 0 \\ (x_0 + 5)(x_0 - 2) &= 0 \end{aligned}$$

Then $x_0 = -5$ or $x_0 = 2$. One can easily verify that $f(-5) = f(2) = 3$. Therefore, $3 \in \operatorname{im} f$.

(c) If $f^{-1}(x)$ is a function, then given $y \in \operatorname{im} f$, there is a unique x such that $f(x) = y$. From (b), observe that given $3 \in \operatorname{im} f$, $f(-5) = f(2) = 3$. Therefore, $f^{-1}(3) \in \{-5, 2\}$ so that, as a function, $f^{-1}(3)$ is not well defined. Therefore, $f^{-1}(x)$ is not a function.