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MATH 108
Spring 2024
HW 18: Due 04/17

"The problem is, if all you care about in the world is the velvet rope, you will always be unhappy, no matter which side you're on."
— Tahani Al-Jamil, The Good Place

Problem 1. (10pts) The following matrix is the 'RREF' of an augmented matrix coming from a system of equations. Did this system of equations have a solution? If the system of equations had a solution, find all the possible solutions. If the system did not have a solution, explain why.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution. Each of the columns of the matrix corresponds to a variable—except for the last column which corresponds to the 'other' side of the equalities. There are then $5 - 1 = 4$ variables. We mark the pivot columns of the matrix:

$$\begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, x_1, x_2, x_4 will be 'fixed.' We then take x_3 to be a free variable. The first row tells us that $x_1 = 0$. The third row tells us that $x_4 = 1$. The second row tells us that $x_2 + x_3 = 2$, which implies that $x_2 = 2 - x_3$. Therefore, the solutions are all of the form:

$$\begin{cases} x_1 = 1 \\ x_2 = 2 - x_3 \\ x_3 \colon \text{free} \\ x_4 = 1 \end{cases}$$

Problem 2. (10pts) Consider the matrix...

$$M = \begin{pmatrix} 1 & 8 \\ 1 & 4 \end{pmatrix}$$

(a) Compute $\det M$.

(b) Does M^{-1} exist? Explain. If M^{-1} exists, find M^{-1}.

Solution.

(a) Recall that $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$. But then...

$$\det \begin{pmatrix} 1 & 8 \\ 1 & 4 \end{pmatrix} = 1(4) - 8(1) = 4 - 8 = -4$$

(b) Recall that if A is a square matrix, then A^{-1} exists if and only if $\det A \neq 0$. Because $\det M = -4 \neq 0$, we know that M^{-1} exists. If A is a two-by-two matrix whose inverse exists, then...

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

But then...

$$M^{-1} = \frac{1}{-4} \begin{pmatrix} 4 & -8 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$$

Problem 3. (10pts) Use 'the method of inverses' to solve the following system of linear equations:

$$\begin{aligned} 2x - 9y &= -15 \\ -7x + 4y &= 25 \end{aligned}$$

Solution. We write this system in the form $A\mathbf{x} = \mathbf{b}$, where A is the coefficient matrix, $\mathbf{x}$ is the vector of variables, and $\mathbf{b}$ is the constant vector. Then this system of equations is equivalent to the matrix system...

$$\underbrace{\begin{pmatrix} 2 & -9 \\ -7 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} -15 \\ 25 \end{pmatrix}}_{\mathbf{b}}$$

Observe that $\det A = 2(4) - (-7)(-9) = 8 - 63 = -55$. Because $\det A = -55 \neq 0$, we know that A^{-1} exists. We have...

$$A^{-1} = \frac{1}{-55} \begin{pmatrix} 4 & 9 \\ 7 & 2 \end{pmatrix}$$

But then...

$$\begin{pmatrix} 2 & -9 \\ -7 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -15 \\ 25 \end{pmatrix}$$

$$\frac{1}{-55} \begin{pmatrix} 4 & 9 \\ 7 & 2 \end{pmatrix} \begin{pmatrix} 2 & -9 \\ -7 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-55} \begin{pmatrix} 4 & 9 \\ 7 & 2 \end{pmatrix} \begin{pmatrix} -15 \\ 25 \end{pmatrix}$$

$$A^{-1}A \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{55} \begin{pmatrix} 4(-15) + 9(25) \\ 7(-15) + 2(25) \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{55} \begin{pmatrix} -60 + 225 \\ -105 + 50 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{55} \begin{pmatrix} 165 \\ -55 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Therefore, the solution to the system is $(x, y) = (-3, 1)$.