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MATH 308
Fall 2022
HW 10: Due 10/13

"I think that some intuition leaks out in every step of an induction proof."
–Jim Propp

Problem 1. (10pt) Let $\{a_n\}_{n\in\mathbb{N}}$ be the sequence defined by $a_n := 2^n - 5$ and $\{b_m\}_{m\in\mathbb{Z}^\times}$ be defined by $b_m := \dfrac{m+1}{m}$. Showing all your work, compute the following:

(a) $\displaystyle\sum_{k=0}^{5} a_k$

(b) $\displaystyle\sum_{\substack{j=-3\\ j\neq 0}}^{3} b_m$

(c) $\displaystyle\prod_{k=1}^{3} a_n$

(d) $\displaystyle\sum_{p=0}^{0} a_p$

(e) $\displaystyle\sum_{j=2}^{4} (a_j + b_j)$

(f) $\displaystyle\prod_{n=1}^{10^{50}} b_n$

Solution.

(a)

(b)

(c)

(d)

(e)

(f)

Problem 2. (10pt) Let $a \in \mathbb{R}$. Consider the following sum defined for $n > 7$:

$$\sum_{k=7}^{n} (k + a - 7)^2$$

(a) Reindex the sum above so that it begins at $k = 0$.

(b) Using the given summation formulas below, find the sum from (a) in terms of n, a alone.

$$\sum_{k=0}^{n} 1 = n + 1, \qquad \sum_{k=0}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Problem 3. (10pt) Complete the proof of the given proposition below by filling in the corresponding blanks.

Proposition. For $n \geq 2$, $\displaystyle\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right) = \frac{n+1}{2n}$.

Proof. We prove this using ________________________________. First, we establish a base case.

Base Case: Let $n = 2$. Then we have...

$$\rule{3cm}{0.4pt} = \rule{3cm}{0.4pt} = \rule{3cm}{0.4pt} = \frac{3}{4}$$

$$\left.\frac{n+1}{2n}\right|_{n=2} = \frac{2+1}{2(2)} = \frac{3}{4}$$

But then if $n = 2$, we know that $\displaystyle\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right) = \frac{n+1}{2n}$.

We know establish the induction step.

Induction Step: Assume that for $n = N$, $\displaystyle\prod_{k=2}^{N}\left(1-\frac{1}{k^2}\right) = \frac{N+1}{2N}$. We show that the statement of the proposition is then true for $n =$ __________. We have...

$$\begin{aligned}
\prod_{k=2}^{N+1}\left(1-\frac{1}{k^2}\right) &= \rule{4cm}{0.4pt} \cdot \prod_{k=2}^{N}\left(1-\frac{1}{k^2}\right) \\
&= \rule{4cm}{0.4pt} \cdot \rule{4cm}{0.4pt} \\
&= \rule{4cm}{0.4pt} \cdot \rule{4cm}{0.4pt} \\
&= \rule{4cm}{0.4pt} \cdot \rule{4cm}{0.4pt} \\
&= \frac{N+2}{2(N+1)} \\
&= \frac{(N+1)+1}{2(N+1)}
\end{aligned}$$

But then we know that $\displaystyle\prod_{k=2}^{N+1}\left(1-\frac{1}{k^2}\right) = \frac{(N+1)+1}{2(N+1)}$.

Therefore, by ________________________________, we know that for $n \geq 2$, $\displaystyle\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right) = \frac{n+1}{2n}$.

$\square$

Problem 4. (10pt) Let $\{a_n\}_{n\in\mathbb{Z}^{\geq 0}}$ be the recursive sequence given by $a_0 = 1$, $a_1 = 3$, and $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$. A student observe that $a_0 = 1$, $a_1 = 3$, $a_2 = 5$, $a_3 = 7$, and $a_4 = 9$. They then predict that $a_n = 2n+1$ for $n \geq 0$. Below is a proof of this conjecture, with parts of their proof removed. Complete the missing parts.

Proposition. Let $\{a_n\}_{n\in\mathbb{Z}^{\geq 0}}$ be the recursive sequence given by $a_0 = 1$, $a_1 = 3$, and $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$. Then for all $n \geq 0$, $a_n = 2n+1$.

Proof. We prove this using ________________________________. First, we establish a few bases cases.

Base Case: If ________________, we have $a_0 = 1$ and $2n+1 = 2(0)+1 = 1$. Then if $n = 0$, we have

$a_n = 2n+1$. Now if $n =$ ________________, we have _____________________ and _____________________.

But then if $n = 1$, we have ________________.

We now establish the induction case.

Induction Case: Now assume that $a_k = 2k+1$ for all $0 \leq k \leq n$. Now consider the term

________________.

We have...

$$
\begin{aligned}
a_{n+1} &= 2a_n - a_{n-1} \\
&= \underline{\hspace{8cm}} \\
&= \underline{\hspace{8cm}} \\
&= 2n+3 \\
&= 2(n+1)+1
\end{aligned}
$$

But then we know that $a_{n+1} = 2(n+1)+1$.

Therefore, by ________________________________, we know that $a_n = 2n+1$ for all $n \geq 0$. $\square$