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MATH 308
Fall 2021
HW 10: Due 11/05

"There is only one problem with common sense; it's not very common."
–Milt Bryce

Problem 1. (10pt) Determine if the following functions are injective, surjective, and/or bijective. Which of the functions have an inverse function? [No formal proofs required.]

(a) $f : \mathbb{R} \to [0,1]$ defined by $f(x) = \sin^2 x$.

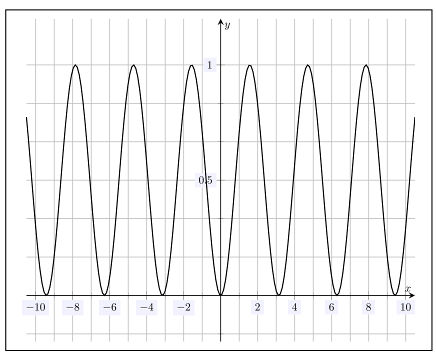
(b) $g : [0, \frac{\pi}{2}] \to [0,1]$ defined by $g(x) = \cos x$.

(c) $h : \mathbb{N} \to \mathbb{Z}$ given by $h(n) = 3^n$.

(d) $j : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $j(n,m) = (n - m + 3)^2$.

Solution.

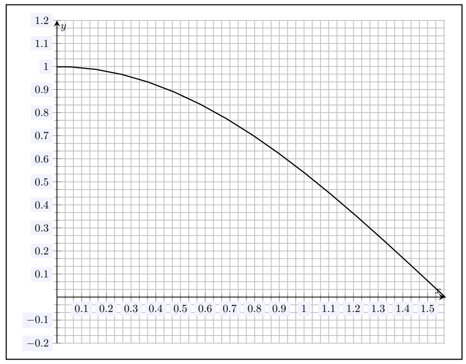
(a) From the plot of $f(x)$, it would seem that $f(x)$ is surjective but not injective. Because $f(x)$ is not both injective and surjective, $f(x)$ is not bijective. Because $f : \mathbb{R} \to [0,1]$ is not bijective, it does not have an inverse function.



To see this directly, observe that $0, \pi \in \mathbb{R}$, $f(0) = \sin^2(0) = 0$, and $f(\pi) = \sin^2(\pi) = 0$. Because $f(0) = f(\pi)$ and $0 \neq \pi$, we know that $f(x)$ is not injective. Alternatively, $f(x)$ is not injective because there is a horizontal line which intersects the graph of $f(x)$ more than once, e.g. the horizontal line $y = 1$. Alternatively, observe that $f(x)$ is surjective because every horizontal line $y = c$, where $c \in [0,1]$, intersects the graph of $f(x)$ at least once. We know that $\arcsin x$ is defined when $-1 \leq x \leq 1$. Let $y \in [0,1]$ and define $x := \arcsin(\sqrt{y})$ (observe that $\sqrt{y}$ is defined because $y \geq 0$ and $\arcsin(\sqrt{y})$ is defined because $-1 \leq \sqrt{y} \leq 1$). Then $f(x) = f(\arcsin(\sqrt{y})) = \big(\sin(\arcsin(\sqrt{y}))\big)^2 = (\sqrt{y})^2 = y$. Therefore, $f(x)$ is surjective.

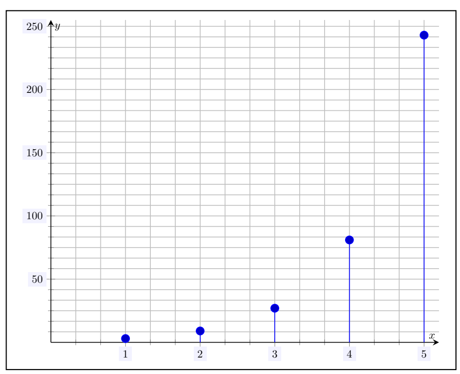
(b) From the plot of $g(x)$, it would seem that $g(x)$ is injective and surjective. Because $g(x)$ is both injective and surjective, it is bijective. Because $g : [0, \frac{\pi}{2}] \to [0,1]$ is bijective, we know that $g(x)$ has an inverse function, i.e. $g^{-1} : [0,1] \to [0, \frac{\pi}{2}]$ exists. In fact, the inverse is

$\arccos x : [0,1] \to [0, \frac{\pi}{2}]$.



We can see these facts directly. To see that $g(x)$ is injective, observe that every horizontal line $y = c \in [0,1]$ intersects the graph of $g(x)$ at most once. Alternatively, recall that $\arccos x$ is defined for $x \in [-1,1]$ and produces values in $[0,\pi]$. Let $x, y \in [0, \frac{\pi}{2}]$ and suppose that $g(x) = g(y)$, i.e. $\cos(x) = \cos(y)$. But then $x = \arccos\big(\cos(x)\big) = \arccos\big(\cos(y)\big) = y$, where we have used the fact that $x, y \in [0, \frac{\pi}{2}]$. Therefore, $g(x)$ is injective. To see that $g(x)$ is surjective, observe that every horizontal line $y = c \in [0,1]$ intersects the graph of $g(x)$ at least once. Alternatively, recall that $g(x) = \cos x$ is continuous. We know that $g(0) = \cos(0) = 1$ and $g(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) = 0$. By the Intermediate Value Theorem, for any $y \in [0,1]$, there exists $x \in [0, \frac{\pi}{2}]$ such that $g(x) = y$. But then $g(x)$ is surjective.

(c) We can plot a few values for this function. From this plot it seems that $h(n)$ is injective but not surjective. Because $h(n)$ is not both injective and surjective, $h(n)$ is not bijective. Because $h : \mathbb{N} \to \mathbb{Z}$ is not bijective, it does not have an inverse function.



We can see these facts directly. To see that $h : \mathbb{N} \to \mathbb{Z}$ is not surjective, observe that not every horizontal line $y = z \in \mathbb{Z}$ intersects the graph of the function. For instance, the lines $y = 2$ and $y = -5$ do not intersect the graph of $h(n)$. Alternatively, consider $0 \in \mathbb{Z}$. Because $h(n) > 0$, there does not exist $n \in \mathbb{N}$ such that $h(n) = 0$. Therefore, $h(n)$ is not surjective. To see that $h(n)$ is injective, observe that every horizontal line $y = z \in \mathbb{Z}$ intersects the graph of $h(n)$ at most once. Alternatively, suppose that $n, m \in \mathbb{N}$ with $n \neq m$. Without loss of generality, assume that $n < m$. Because $\widetilde{h}(x) = 3^x$ has the property that $\widetilde{h}'(x) = 3^x \ln(3) > 0$, we know that $\widetilde{h}(x)$ is increasing. But then $h(n)$ is increasing. But then because $n < m$, $h(n) = 3^n < 3^m = h(m)$. Therefore, if $n \neq m$, we know $g(n) \neq g(m)$. This shows that $h(n)$ is injective.

(d) Though we cannot (usefully) plot $j(n,m)$, we can still easily determine that j is neither injective nor surjective. Because j is not both injective and surjective, j is not bijective. Because $j : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is not bijective, it does not have an inverse function. To see that j is not injective, observe that $(0,0),(1,1) \in \mathbb{Z} \times \mathbb{Z}$, $j(0,0) = (0-0+3)^2 = 3^2 = 9$, and $j(1,1) = (1-1+3)^2 = 3^2 = 9$. Because $(0,0) \neq (1,1)$ but $j(0,0) = j(1,1)$, we know that j is not injective. To see that j is not surjective, observe that $n-m+3 \in \mathbb{Z}$ because $n,m \in \mathbb{Z}$. But then $(n-m+3)^2 \geq 0$. Therefore, there exist no $n,m \in \mathbb{Z}$ such that $(n-m+3)^2 = -1$. But then there exists no $(n,m) \in \mathbb{Z} \times \mathbb{Z}$ such that $j(n,m) = -1$. Therefore, j is not surjective.

Problem 2. (10pt) Show that the function $f : \mathbb{R} \setminus \{-1\} \to \mathbb{R} \setminus \{3\}$ given by $f(x) = \dfrac{3x-5}{x+1}$ is a bijection. Explain why this implies f is invertible and then find the inverse for $f(x)$.

Solution. To show that $f : \mathbb{R} \setminus \{-1\} \to \mathbb{R} \setminus \{3\}$ is bijective, we need to prove that f is both injective and surjective. To see that f is injective, let $x, y \in \mathbb{R} \setminus \{1\}$ and suppose that $f(x) = f(y)$. We need to show $x = y$. But $f(x) = f(y)$ implies...

$$\begin{aligned} f(x) &= f(y) \\ \frac{3x-5}{x+1} &= \frac{3y-5}{y+1} \\ (3x-5)(y+1) &= (3y-5)(x+1) \\ 3xy + 3x - 5y - 5 &= 3xy + 3y - 5x - 5 \\ 3x - 5y &= 3y - 5x \\ 8x &= 8y \\ x &= y \end{aligned}$$

Therefore, f is injective.

To see that f is surjective, suppose that $y \in \mathbb{R} \setminus \{3\}$. We need to show there exists $x \in \mathbb{R} \setminus \{-1\}$ such that $f(x) = y$. Because $y \neq 3$, we can define $x := \frac{y+5}{3-y}$. We need to show that $x \neq -1$. If $x = -1$, then

$$\begin{aligned} x &= -1 \\ \frac{y+5}{3-y} &= -1 \\ y + 5 &= y - 3 \\ 5 &= -3 \end{aligned}$$

which is impossible. Therefore, $x \neq -1$. Observe...

$$f(x) = \frac{3x-5}{x+1} = \frac{3 \cdot \frac{y+5}{3-y} - 5}{\frac{y+5}{3-y} + 1} = \frac{\frac{3y+15}{3-y} - \frac{15-5y}{3-y}}{\frac{y+5}{3-y} + \frac{3-y}{3-y}} = \frac{\frac{8y}{3-y}}{\frac{8}{3-y}} = \frac{8y}{8} = y$$

Therefore, f is surjective.

Because f is both injective and surjective, $f : \mathbb{R} \setminus \{-1\} \to \mathbb{R} \setminus \{3\}$ is a bijection. Therefore, $f^{-1} : \mathbb{R} \setminus \{3\} \to \mathbb{R} \setminus \{-1\}$ exists. We can find it by interchanging the roles of x and y and solving for

y:

$$x = \frac{3y-5}{y+1}$$
$$x(y+1) = 3y-5$$
$$xy + x = 3y-5$$
$$xy - 3y = -x-5$$
$$y(x-3) = -x-5$$
$$y = \frac{-x-5}{x-3}$$
$$y = \frac{x+5}{3-x}$$

Therefore, $f^{-1} : \mathbb{R} \setminus \{3\} \to \mathbb{R} \setminus \{-1\}$[1] is given by $f^{-1}(x) = \dfrac{x+5}{3-x}$.[2]

We can easily verify that this is indeed the inverse:

$$\begin{aligned}(f \circ f^{-1})(x) &= f\big(f^{-1}(x)\big) \\ &= f\left(\frac{x+5}{3-x}\right) \\ &= \frac{3 \cdot \frac{x+5}{3-x} - 5}{\frac{x+5}{3-x} + 1} \\ &= \frac{\frac{3x+15}{3-x} - \frac{15-5x}{3-x}}{\frac{x+5}{3-x} + \frac{3-x}{3-x}} \\ &= \frac{\frac{8x}{3-x}}{\frac{8}{3-x}} \\ &= \frac{8x}{8} \\ &= x\end{aligned} \qquad \begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}\big(f(x)\big) \\ &= f^{-1}\left(\frac{3x-5}{x+1}\right) \\ &= \frac{\frac{3x-5}{x+1} + 5}{3 - \frac{3x-5}{x+1}} \\ &= \frac{\frac{3x-5}{x+1} + \frac{5x+5}{x+1}}{\frac{3x+3}{x+1} - \frac{3x-5}{x+1}} \\ &= \frac{\frac{8x}{x+1}}{\frac{8}{x+1}} \\ &= \frac{8x}{8} \\ &= x\end{aligned}$$

[1] We need prove that f^{-1} has the correct domain and codomain. Because $f : \mathbb{R} \setminus \{-1\} \to \mathbb{R} \setminus \{3\}$, we need to prove $f^{-1} \colon \mathbb{R} \setminus \{3\} \to \mathbb{R} \setminus \{-1\}$. Clearly, f^{-1} is defined on $\mathbb{R} \setminus \{3\}$. We need to prove that $\operatorname{im} f^{-1} \subseteq \mathbb{R} \setminus \{-1\}$. Clearly, $\operatorname{im} f^{-1} \subseteq \mathbb{R}$. So we only need show that $-1 \notin \operatorname{im} f^{-1}$, i.e. $f^{-1}(x) \neq -1$ for all x. If $f^{-1}(x) = -1$ for some x, then $\frac{x+5}{3-x} = -1$. But this implies $x + 5 = x - 3$, which forces $5 = -3$, which is impossible. Therefore, $f^{-1}(x) \neq -1$ for all x. This proves that $-1 \notin \operatorname{im} f$ so that $\operatorname{im} f^{-1} \subseteq \mathbb{R} \setminus \{-1\}$. This proves we can allow the codomain to be $\mathbb{R} \setminus \{-1\}$.

[2] In fact, finding f^{-1} was how we found how to define x given y in the proof of surjectivity.

Problem 3. (10pt) Let $S \subseteq \mathbb{R}$ and $f, g : S \to \mathbb{R}$ be monotone increasing functions.

(a) Prove that $f + g$ is a monotone increasing function.

(b) If f and $f + g$ are increasing on S, then is g necessarily increasing on S? Prove this statement or give a counterexample.

Solution.

(a) Recall that a function $\phi : \mathbb{R} \to \mathbb{R}$ is monotone increasing if for all $x, y \in \mathbb{R}$ with $x < y$, then $\phi(x) \leq \phi(y)$. Assume that $f, g : S \to \mathbb{R}$ are monotone increasing functions. But then if $x, y \in S$ with $x < y$, we know that $f(x) \leq f(y)$ and $g(x) \leq g(y)$.

We need to show that $(f + g)(x)$ is monotone increasing; that is, we need to show that if $x < y$, then $(f + g)(x) \leq (f + g)(y)$. Observe...

$$(f + g)(x) = f(x) + g(x) \leq f(y) + g(x) \leq f(y) + g(y) = (f + g)(y)$$

Therefore, $(f + g)(x) \leq (f + g)(y)$ so that $f + g$ is monotone increasing.

(b) Recall that a function $\phi : \mathbb{R} \to \mathbb{R}$ is increasing if for all $x, y \in \mathbb{R}$ with $x < y$, then $\phi(x) < \phi(y)$. Suppose that $f, f + g : S \to \mathbb{R}$ are increasing functions. It need not be the fact that g is increasing on S.

For example, let $S = [0, \infty)$, $f(x) = x^2 + x$, and $g(x) = -x$. Observe that $(f + g)(x) = f(x) + g(x) = (x^2 + x) + (-x) = x^2$. Because $f'(x) = 2x + 1 \geq 0$ on $[0, \infty)$ and $(f + g)'(x) = 2x \geq 0$ on $[0, \infty)$. Both f and $f + g$ are clearly increasing on $[0, \infty)$. However, $g(x) = -x$ is clearly decreasing on $[0, \infty)$.

Problem 4. (10pt) Let $f : X \to Y$ and let $A, B \in \mathcal{P}(X)$.

(a) Prove that $f(A \cup B) = f(A) \cup f(B)$.

(b) Is it true that $f(A \cap B) = f(A) \cap f(B)$? Prove or give a counterexample.

Solution.

(a) To prove $f(A \cup B) = f(A) \cup f(B)$, we need to prove that $f(A \cup B) \subseteq f(A) \cup f(B)$ and $f(A) \cup f(B) \subseteq f(A \cup B)$.

$f(A \cup B) \subseteq f(A) \cup f(B)$: Let $y \in f(A \cup B)$. We want to show that $y \in f(A) \cup f(B)$. Because $y \in f(A \cup B)$, there exists $x \in A \cup B$ such that $f(x) = y$. Because $x \in A \cup B$, we know that $x \in A$ or $x \in B$. We consider both cases.

Case I, $x \in A$: If $x \in A$, then $x \in A$ and $f(x) = y$ so that $y \in f(A)$. But then $y \in f(A) \cup f(B)$.

Case II, $x \in B$: If $x \in B$, then $x \in B$ and $f(x) = y$ so that $y \in f(B)$. But then $y \in f(A) \cup f(B)$.

Therefore, if $y \in f(A \cup B)$, then $y \in f(A) \cup f(B)$. This proves that $f(A \cup B) \subseteq f(A) \cup f(B)$.

$f(A) \cup f(B) \subseteq f(A \cup B)$: Let $y \in f(A) \cup f(B)$: Let $y \in f(A) \cup f(B)$. We want to show that $y \in f(A \cup B)$. Because $y \in f(A) \cup f(B)$, we know that $y \in f(A)$ or $y \in f(B)$. We consider both cases:

Case I, $y \in f(A)$: If $y \in f(A)$, then there exists $x \in A$ such that $f(x) = y$. Because $x \in A$, we know that $x \in A \cup B$. But then because $x \in A \cup B$ and $f(x) = y$, we know that $y \in f(A \cup B)$.

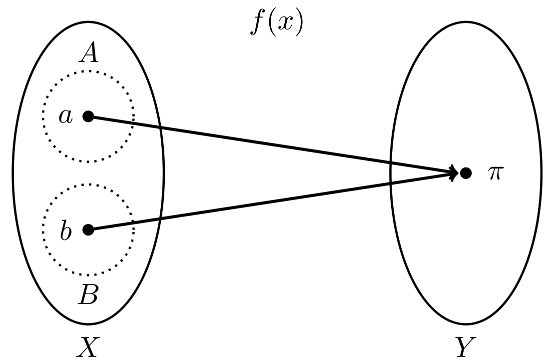
Case II, $y \in f(B)$: If $y \in f(B)$, then there exists $x \in B$ such that $f(x) = y$. Because $x \in B$, we know that $x \in A \cup B$. But then because $x \in A \cup B$ and $f(x) = y$, we know that $y \in f(A \cup B)$.

But then if $y \in f(A) \cup f(B)$, then $y \in f(A \cup B)$. Therefore, $f(A) \cup f(B) \subseteq f(A \cup B)$.

Therefore, because $f(A \cup B) \subseteq f(A) \cup f(B)$ and $f(A) \cup f(B) \subseteq f(A \cup B)$, we know that $f(A \cup B) = f(A) \cup f(B)$.

(b) No, generally, we only know that $f(A \cap B) \subseteq f(A) \cap f(B)$. [Try proving this!] The other inclusion need not hold, i.e. there may be elements of $f(A) \cap f(B)$ that may not be elements of $f(A \cap B)$. This is because there could be elements only in A and B, i.e. $a \in A \setminus B$ and $b \in B \setminus A$ (hence, $a, b \notin A \cap B$), such that $f(a) = f(b)$. For instance, consider the function

$f(x)$ given by the diagram below.



That is, let $f : X \to Y$ be the function given by $f(a) = \pi$ and $f(b) = \pi$. Define $A = \{a\}$ and $B = \{b\}$. Clearly, $A \cap B = \varnothing$. Therefore, $f(A \cap B) = f(\varnothing) = \varnothing$. Moreover, $f(A) = \{f(x)\colon x \in A\} = \{f(a)\} = \{\pi\}$ and $f(B) = \{f(x)\colon x \in B\} = \{f(b)\} = \{\pi\}$. But then $f(A) \cap f(B) = \{\pi\} \cap \{\pi\} = \{\pi\}$. While we have $f(A \cap B) = \varnothing \subseteq \{\pi\} = f(A) \cap f(B)$, we do not have $f(A \cap B) = f(A) \cap f(B)$.

Problem 5. (10pt) Let $f : X \to Y$ and $A, B \subseteq Y$. Prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Solution. To prove $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, we need to prove that $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$.

$f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$: Let $x \in f^{-1}(A \cap B)$. We need to show that $x \in f^{-1}(A) \cap f^{-1}(B)$. Because $x \in f^{-1}(A \cap B)$, we know that $f(x) \in A \cap B$. But then $f(x) \in A$, which implies $x \in f^{-1}(A)$. Similarly, because $f(x) \in A \cap B$, we know that $f(x) \in B$, which implies $x \in f^{-1}(B)$. Because $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$, we know $x \in f^{-1}(A) \cap f^{-1}(B)$. But then if $x \in f^{-1}(A \cap B)$, then $x \in f^{-1}(A) \cap f^{-1}(B)$. Therefore, $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.

$f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$: Let $x \in f^{-1}(A) \cap f^{-1}(B)$. We need to show that $x \in f^{-1}(A \cap B)$. Because $x \in f^{-1}(A) \cap f^{-1}(B)$, we know that $x \in f^{-1}(A)$, which implies that $f(x) \in A$. Similarly, because $x \in f^{-1}(A) \cap f^{-1}(B)$, we know that $x \in f^{-1}(B)$, which implies that $f(x) \in B$. Therefore, because $f(x) \in A$ and $f(x) \in B$, we know that $f(x) \in A \cap B$. This implies that $x \in f^{-1}(A \cap B)$. But then if $x \in f^{-1}(A) \cap f^{-1}(B)$, then $x \in f^{-1}(A \cap B)$. Therefore, $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$.

Therefore, because $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$, we know $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Problem 6. (10pt) For each of the following, find a function $f : \mathbb{N} \to \mathbb{Z}$ with the following properties:

(a) f is injective but not surjective

(b) f is surjective but not injective

(c) f is neither surjective nor injective

(d) f is a bijection

Solution. There are infinitely many possible solutions. Here is a collection of possible solutions:

(a) Let $f : \mathbb{N} \to \mathbb{Z}$ be given by $f(n) = n$. Clearly, f is injective because if $f(n) = f(m)$, then $n = f(n) = f(m) = m$, so that $n = m$. Therefore, f is injective. But f is not surjective. Because $n \in \mathbb{N}$, we know that $n \geq 0$. But then $f(n) \geq 0$ for all n. But then $f(n) \neq -1$ for all $n \in \mathbb{N}$. Therefore, f is not surjective.

(b) Let $g : \mathbb{N} \to \mathbb{Z}$ be the bijection in (d)—no one said parts had to or should be done in order, i.e.

$$g(n) = \begin{cases} 0, & n = 1 \\ \frac{n}{2}, & n \text{ is even} \\ -\frac{n+1}{2}, & n \text{ is odd} \end{cases}$$

From (d), we know that g is a bijection. Suppose we construct a surjection $h : \mathbb{N} \to \mathbb{N}$ that is not injective. We know that $g \circ h : \mathbb{N} \to \mathbb{Z}$ will be a surjection because the composition of surjective functions is surjective. But because h is not injective, there exists $n, m \in \mathbb{N}$ with $n \neq m$ but $h(n) = h(m)$. Define $N := h(n) = h(m)$. Then $(g \circ h)(n) = g\big(h(n)\big) = g(N)$ and $(g \circ h)(m) = g\big(h(m)\big) = g(N)$ so that $(g \circ h)(n) = (g \circ h)(m)$ but $n \neq m$. Thus, $g \circ h$ would not be injective. It remains to construct a surjection $h : \mathbb{N} \to \mathbb{N}$ that is not an injection. Let $h : \mathbb{N} \to \mathbb{N}$ be given by...

$$h(n) = \begin{cases} \frac{n}{2}, & n \text{ even} \\ \frac{n+1}{2}, & n \text{ odd} \end{cases}$$

The function h maps the even positive integers to $1, 2, 3, \ldots$ and maps the odd even integers to $1, 2, 3, \ldots$. To see that h is not injective, observe that $h(1) = \frac{1+1}{2} = \frac{2}{2} = 1$ and $h(2) = \frac{2}{2} = 1$. Obviously $1 \neq 2$ but $h(1) = h(2)$, so that h is not injective. It only remains to show that h is surjective. Let $n \in \mathbb{N}$. We want to show there exists $m \in \mathbb{N}$ such that $h(m) = n$. Observe that $2n \in \mathbb{N}$ is even. Defining $m := 2n$, we have $h(m) = \frac{m}{2} = \frac{2n}{2} = n$. But then for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $h(m) = n$. Therefore, h is surjective.

Now define $f := g \circ h : \mathbb{N} \to \mathbb{Z}$, i.e. $f(n) := g\big(h(n)\big)$. We know that f is surjective (because it is the composition of surjective functions is surjective) and that f is not injective (because h is not injective). But then $f : \mathbb{N} \to \mathbb{Z}$ is a surjection that is not injective. One can even give f

explicitly using the value of n modulo 4[3]

$$f(n) = \begin{cases} \dfrac{n}{4}, & n \equiv 0 \mod 4 \\ -\dfrac{n-1}{4}, & n \equiv 1 \mod 4 \\ -\dfrac{n-2}{4}, & n \equiv 2 \mod 4 \\ \dfrac{n+1}{4}, & n \equiv 3 \mod 4 \end{cases}$$

(c) Let $f : \mathbb{N} \to \mathbb{Z}$ be given by $f(n) = 0$ for all $n \in \mathbb{N}$. Clearly, f is not injective because $f(1) = f(2) = 0$ but $1 \neq 2$. Furthermore, f is not surjective because there is no $n \in \mathbb{N}$ such that $f(n) = 1$.

(d) We will construct a function that will 'count through' $\mathbb{Z}$ in the following way: $0, 1, -1, 2, -2, 3, -3, \ldots$. For now, we shall focus on the sequence $1, -1, 2, -2, 3, -3, \ldots$. We will create a map that assigns values in $\mathbb{N}$ to these values in the order. A table may be helpful:

n	1	2	3	4	5	6	7	8	$\cdots$
$f(n)$	1	-1	2	-2	3	-3	4	-4	$\cdots$

Letting $f(1) = 0$ and shifting each of the outputs one spot to the right in the table above gives a 'valid' description of a bijective function. However, we can be more concrete. Observe that the even integers are assigned to the negation of half their value, i.e. $-n/2$. Using this as a 'hint' for the odd integers, observe that the odd integers are mapped to half of one less than their value, i.e. $(n-1)/2$. If we want to include 0, we can simply 'shift' everything up, insert 0, and use the same idea as above.

n	1	2	3	4	5	6	7	8	$\cdots$
$f(n)$	0	1	-1	2	-2	3	-3	4	$\cdots$

This allows us to explicitly define $f(n)$:[4]

$$f(n) = \begin{cases} 0, & n = 1 \\ \frac{n}{2}, & n \text{ is even} \\ -\frac{n-1}{2}, & n \text{ is odd} \end{cases}$$

Observe that $f(n) = 0$ if and only if $n = 1$, $f(n) > 0$ if and only if n is even, and $f(n) < 0$ if and only if n is odd. We need to prove that f is injective and surjective.

We first prove that f is injective. Suppose that $f(n) = f(m)$ for some $n, m \in \mathbb{N}$. We need to show that $n = m$. There are three cases:

[3]The value of h at n depends upon the parity of n. But then the out $h(n)$ may have different parity than n, which is then input into g whose output depends upon the parity of its input. For instance, if n is even then $h(n) = \frac{n}{2}$, which may be even, e.g. $n = 8$, or odd, e.g. $n = 6$. One would then only use the '$\frac{n}{2}$' definition of h and the $\frac{n}{2}$ definition of g if n were divisible by 2 at least twice, i.e. divisible by four.

[4]One can do this without singling out the case of $n = 1$ if one defines $f(n) = \frac{n}{2}$ when n is even and defining $f(n) = -\lfloor \frac{n}{2} \rfloor$ when n is odd. Of course, because $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$ when n is even, this immediately gives that one could simply define $f(n) = (-1)^n \lfloor \frac{n}{2} \rfloor$ for all $n \in \mathbb{N}$.

(i) $f(n) = f(m) = 0$: Of course, $f(n) = 0$ and $f(m) = 0$ if and only if $n = 0$ and $m = 0$. But then $n = m$.

(ii) $f(n) = f(m) > 0$: Because $f(n) > 0$, we know that n is even. This implies $f(n) = \frac{n}{2}$. Similarly, because $f(m) > 0$, we know that m is even so that $f(m) = \frac{m}{2}$. But then $f(n) = f(m)$ implies that $\frac{n}{2} = \frac{m}{2}$. Therefore, $n = m$.

(iii) $f(n) = f(m) < 0$: Because $f(n) < 0$, we know that n is odd. This implies $f(n) = -\frac{n-1}{2}$. Similarly, because $f(m) < 0$, we know that m is even so that $f(m) = -\frac{m-1}{2}$. But then $f(n) = f(m)$ implies that $-\frac{n-1}{2} = -\frac{m-1}{2}$. This obviously implies $\frac{n-1}{2} = \frac{m-1}{2}$ so that $n - 1 = m - 1$. Therefore, $n = m$.

But then if $f(n) = f(m)$, we know that $n = m$. Therefore, $f(n)$ is injective.

To see that f is surjective, suppose that $z \in \mathbb{Z}$. We need to show that there exists $n \in \mathbb{N}$ such that $f(n) = z$. There are three cases:

(i) $z = 0$: We know that $f(1) = 0 = z$.

(ii) $z > 0$: Let $z > 0$ and define $n := 2z$. Clearly, $2z > 0$ is an even integer. Then $f(n) = f(2z) = \frac{2z}{2} = z$.

(iii) $z < 0$: Let $z < 0$ and define $n := -2z + 1$. Because $z \leq -1$ (because $z < 0$ is an integer), we know that $-2z \geq 2$ so that $-2z + 1 \geq 3$. But then $f(n) = f(-2z + 1) = -\frac{(-2z+1)-1}{2} = -\frac{-2z}{2} = -(-z) = z$.

But then for all $z \in \mathbb{Z}$, there exists $n \in \mathbb{N}$ such that $f(n) = z$. Therefore, f is surjective.

Because f is injective and surjective, f is a bijection.