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MATH 108
Fall 2021
HW 7: Due 11/04

"I like long walks, especially when they are taken by people that annoy me."
–Fred Allen

Problem 1. (10pt) Write down the tableau associated to the following linear programming problem:

$$\begin{aligned} \max\ z &= 3x_1 + x_2 \\ 2x_1 + 3x_2 &\leq 6 \\ 4x_1 + 2x_2 &\leq 8 \\ -3x_1 + 4x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution. Before creating the tableau, we want to be sure the linear programming problem is in standard form (or as close as possible). Because this is a maximization problem, we require all the inequalities to be '$\leq$.' [This is excluding the final inequality $x_1, x_2 \geq 0$, because we require the variables to be nonnegative.] So we multiply both sides of the third inequality by -1 to obtain $3x_1 - 4x_2 \leq -2$. Then our system is

$$\begin{aligned} \max\ z &= 3x_1 + x_2 \\ 2x_1 + 3x_2 &\leq 6 \\ 4x_1 + 2x_2 &\leq 8 \\ 3x_1 - 4x_2 &\leq -2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

We introduce slack variables s_1, s_2, s_3 so that...

$$\begin{aligned} 2x_1 + 3x_2 + s_1 &= 6 \\ 4x_1 + 2x_2 + s_2 &= 8 \\ 3x_1 - 4x_2 + s_3 &= -2 \end{aligned}$$

Moving everything to the left side in $z = 3x_1 + x_2$, we have $z - 3x_1 - x_2 = 0$. Then the associated tableau is...

$$\begin{array}{rrrrr|rl} x_1 & x_2 & s_1 & s_2 & s_3 & & \\ 2 & 3 & 1 & 0 & 0 & 6 & s_1 \\ 4 & 2 & 0 & 1 & 0 & 8 & s_2 \\ 3 & -4 & 0 & 0 & 1 & -2 & s_3 \\ \hline -3 & -2 & 0 & 0 & 0 & 0 & \end{array}$$

Problem 2. (10pt) Assume the following is a tableau associated to a standard maximization problem. Write down the function being maximization and the system of constraints.

$$\begin{array}{rrrrrr|r} 1 & 2 & 1 & 1 & 0 & 0 & 100 \\ 2 & 8 & 2 & 0 & 1 & 0 & 150 \\ 1 & 1 & 1 & 0 & 0 & 1 & 200 \\ \hline -5 & -4 & -5 & 0 & 0 & 0 & 0 \end{array}$$

Solution. It is clear that we had three inequalities. Therefore, we had three slack variables, s_1, s_2, s_3. Because there are seven columns, one of which must correspond to the b's, there must be three variables, x_1, x_2, x_3. Looking at the last row and assuming we are trying to maximize a variable z, we must have $z - 5x_1 - 4x_2 - 5_x3 = 0$. But then $z = 5x_1 + 4x_2 + 5x_3$. Therefore, the standard maximization problem was...

$$\begin{aligned} \text{max } z = 5x_1 &+ 4x_2 + 5x_3 \\ x_1 + 2x_2 + x_3 &\leq 100 \\ 2x_1 + 8x_2 + 2x_3 &\leq 150 \\ x_1 + x_2 + x_3 &\leq 200 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Problem 3. (10pt) Solve the following linear programming problem:

$$\begin{aligned} \text{max } z &= x_1 + 6x_2 + 3x_3 \\ x_1 + x_2 + 2x_3 &\leq 4 \\ x_1 + 2x_2 + x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution. This linear programming problem is already in standard form for a maximization. We introduce slack variables s_1 and s_2 so that...

$$\begin{aligned} x_1 + x_2 + 2x_3 + s_1 \quad\;\; &= 4 \\ x_1 + 2x_2 + x_3 \quad\;\; + s_2 &= 4 \end{aligned}$$

Moving everything to the left side in $z = x_1 + 6x_2 + 3x_3$, we have $z - x_1 - 6x_2 - 3x_3 = 0$. Then the associated tableau is...

$$\begin{array}{rrrrr|rl} x_1 & x_2 & x_3 & s_1 & s_2 & & \\ 1 & 1 & 2 & 1 & 0 & 4 & s_1 \\ 1 & 2 & 1 & 0 & 1 & 4 & s_2 \\ \hline -1 & -6 & -3 & 0 & 0 & 0 & \end{array}$$

We find our first pivot position:

$$\begin{array}{rrrrr|rll} x_1 & x_2 & x_3 & s_1 & s_2 & & & \\ 1 & 1 & 2 & 1 & 0 & 4 & s_1 & 4/1 = 4 \\ 1 & \boxed{2} & 1 & 0 & 1 & 4 & s_2 & 4/2 = 2 \\ \hline -1 & \underline{-6} & -3 & 0 & 0 & 0 & & \end{array}$$

So x_x is the entering variable and s_2 is the exiting variable. We make this pivot entry 1 by $\frac{1}{2}R_2 \to R_2$.

$$\begin{array}{rrrrr|rl} x_1 & x_2 & x_3 & s_1 & s_2 & & \\ 1 & 1 & 2 & 1 & 0 & 4 & s_1 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 2 & x_2 \\ \hline -1 & -6 & -3 & 0 & 0 & 0 & \end{array}$$

Now we perform the first step of the simplex method: $-R_2 + R_1 \to R_1$ and $6R_2 + R_3 \to R_3$.

$$\begin{array}{rrrrr|rl} x_1 & x_2 & x_3 & s_1 & s_2 & & \\ \frac{1}{2} & 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 2 & s_1 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 2 & x_2 \\ \hline 2 & 0 & 0 & 0 & 3 & 12 & \end{array}$$

Because there are no negatives in the bottom row, the simplex method is complete. We see that the maximum value for z is 12 (the bottom-rightmost entry) and occurs at $(x_1, x_2, x_3, s_1, s_2) = (0, 2, 0, 2, 0)$, i.e. $x_1 = 0, x_2 = 2, x_3 = 0, s_1 = 2, s_2 = 0$.

Problem 4. (10pt) Find the dual problem to...

$$\begin{aligned} \text{min } z &= 5x_1 + 4x_2 \\ x_1 + 7x_2 &\geq 7 \\ x_1 + 3x_2 &\geq 9 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution. First, observe that this linear programming problem is a minimization and is already in standard form. Therefore, the associated matrix is...

$$\begin{pmatrix} 1 & 7 & 7 \\ 1 & 3 & 9 \\ 5 & 4 & 0 \end{pmatrix}$$

Taking the transpose, we find...

$$\begin{pmatrix} 1 & 1 & 5 \\ 7 & 3 & 4 \\ 7 & 9 & 0 \end{pmatrix}$$

Therefore, the dual problem is...

$$\begin{aligned} \text{max } z &= 7x_1 + 9x_2 \\ x_1 + x_2 &\leq 5 \\ 7x_1 + 3x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Problem 5. (10pt) Solve the following linear programming problem:

$$\begin{aligned} \text{min } z &= 2x_1 + 3x_2 \\ 2x_1 + x_2 &\geq 1 \\ x_1 + 3x_2 &\geq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution. This is linear programming problem is a minimization and it is already in standard form. Therefore, the associated matrix is...

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 2 & 3 & 0 \end{pmatrix}$$

The transpose is then...

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \\ 1 & 1 & 0 \end{pmatrix}$$

Therefore, the associated maximization problem (the dual problem) is...

$$\begin{aligned} \text{max } z &= x_1 + x_2 \\ 2x_1 + x_2 &\leq 2 \\ x_1 + 3x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Because $z = x_1 + x_2$, we know that $z - x_1 - x_2 = 0$. We introduce slack variables s_1, s_2 so that...

$$\begin{aligned} 2x_1 + x_2 + s_1 &= 2 \\ x_1 + 3x_2 + s_2 &= 3 \end{aligned}$$

Therefore, the associated tableau is...

$$\begin{array}{rrrr|rl} x_1 & x_2 & s_1 & s_2 & & \\ 2 & 1 & 1 & 0 & 2 & s_1 \\ 1 & 3 & 0 & 1 & 3 & s_2 \\ \hline -1 & -1 & 0 & 0 & 0 & \end{array}$$

We find our first pivot position:

$$\begin{array}{rrrr|rll} x_1 & x_2 & s_1 & s_2 & & & \\ \boxed{2} & 1 & 1 & 0 & 2 & s_1 & 2/2 = 1 \\ 1 & 3 & 0 & 1 & 3 & s_2 & 3/1 = 3 \\ \hline \underline{-1} & -1 & 0 & 0 & 0 & & \end{array}$$

So x_1 is the entering variable and s_1 is the exiting variable. We make this pivot entry 1 by $\frac{1}{2}R_1 \to R_1$.

$$\begin{array}{cccc|cl} x_1 & x_2 & s_1 & s_2 & & \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & x_1 \\ 1 & 3 & 0 & 1 & 3 & s_2 \\ \hline -1 & -1 & 0 & 0 & 0 & \end{array}$$

Now we perform the first step of the simplex method: $-R_1 + R_2 \to R_2$ and $R_1 + R_3 \to R_3$.

$$\begin{array}{cccc|cl} x_1 & x_2 & s_1 & s_2 & & \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & x_1 \\ 0 & \frac{5}{2} & -\frac{1}{2} & 1 & 2 & s_2 \\ \hline 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 1 & \end{array}$$

There are still negatives in the last row. So we proceed with the next step of the simplex method. We find our pivot position:

$$\begin{array}{cccc|cll} x_1 & x_2 & s_1 & s_2 & & & \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & x_1 & 1/(1/2) = 2 \\ 0 & \boxed{\frac{5}{2}} & -\frac{1}{2} & 1 & 2 & s_2 & 2/(5/2) = 4/5 \\ \hline 0 & \underline{-\frac{1}{2}} & \frac{1}{2} & 0 & 1 & & \end{array}$$

So x_2 is the entering variable and s_2 is the exiting variable. We make the pivot position 1 by $\frac{2}{5}R_2 \to R_2$.

$$\begin{array}{cccc|cl} x_1 & x_2 & s_1 & s_2 & & \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & x_1 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} & \frac{4}{5} & s_2 \\ \hline 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 1 & \end{array}$$

Now we perform the next step of the simplex method: $-\frac{1}{2}R_2 + R_1 \to R_1$ and $\frac{1}{2}R_2 + R_3 \to R_3$:

$$\begin{array}{cccc|cl} x_1 & x_2 & s_1 & s_2 & & \\ 1 & 0 & \frac{3}{5} & -\frac{1}{5} & \frac{3}{5} & x_1 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} & \frac{4}{5} & x_2 \\ \hline 0 & 0 & \frac{2}{5} & \frac{1}{5} & \frac{7}{5} & \end{array}$$

Because there are no remaining negative entries in the bottom row, the simplex method is complete. We have maximum value $z = \frac{7}{5} = 1.4$ (the bottom-rightmost entry) occurring at $(x_1, x_2, s_1, s_2) = (\frac{3}{5}, \frac{4}{5}, 0, 0) = (0.6, 0.8, 0, 0)$, i.e. $x_1 = \frac{3}{5} = 0.6, x_2 = \frac{4}{5} = 0.8, s_1 = 0, s_2 = 0$.

But this maximum value (and its location) as the minimum value for our original minimization problem (the dual problem). Therefore, the minimum value is $z = \frac{7}{5} = 1.4$ occuring at $(x_1, x_2, s_1, s_2) = (\frac{3}{5}, \frac{4}{5}, 0, 0) = (0.6, 0.8, 0, 0)$, i.e. $x_1 = \frac{3}{5} = 0.6, x_2 = \frac{4}{5} = 0.8, s_1 = 0, s_2 = 0$