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MATH 308

Fall 2022

HW 6: Due 09/27

*“Since, as is well known, god helps those who help themselves,
presumably the devil helps all those, and only those, who don’t help
themselves. Does the devil help himself?”*

–Douglas Hofstadter, Gödel, Escher, Bach: An Eternal Golden Braid

Problem 1. (10pt) Let $S := \{-3, -2, -1, 0, 1, 2, 3\}$ be a universal set and define $X := \{-1, 0, 1\}$. Give an example of...

- (a) a proper subset of S , say A , that is disjoint from X .
- (b) a subset of S , say B , such that $B - X \neq B$.
- (c) a subset of S , say C , such that $X \Delta C = X \cup C$.
- (d) a subset of S , say D , such that D^c contains only nonnegative numbers.
- (e) a subset of S , say E , such that the complement of $X \cup E$ is empty.

Solution. Note: Answers may vary.

- (a) The chosen set, A , needs to be disjoint from X ; that is, the set A needs to contain no elements of X , i.e. $-1, 0, 1$. The set A also needs to be a proper subset of S , i.e. not contain every element of S . Examples of such A are $A = \{-3, -2, 2, 3\}$, $A = \{-2, 2\}$, $A = \{3\}$, \emptyset , etc.
- (b) The set $B - X$ is the set of elements that are in B but *not* in X . For $B - X$ to not contain every element of B , i.e. $B - X \neq B$, B and X cannot be disjoint, i.e. $B \cap X \neq \emptyset$. Then the given set B needs to contain at least one element of X . Examples of such B are $B = \{0\}$, $B = \{-3, -2, -1\}$, $B = \{-1, 0, 1\}$, etc.
- (c) The set $X \Delta C$ is the set of elements that are *only* in X or *only* in C , i.e. the elements in X or C but not in $X \cap C$. The set $X \cup C$ is the set of elements in X or C . For $X \Delta C = X \cup C$, there was nothing from $X \cup C$ ‘excluded’ from $X \Delta C$, i.e. every element of $X \cup C$ is in X or C but not both. Examples of such C are $C = \{-3, -2, 2, 3\}$, $C = \{3\}$, $C = \emptyset$, etc.
- (d) The set D^c is the set of elements of S that are *not* in D . Then for D^c to contain only nonnegative numbers, i.e. real numbers $x \geq 0$, the set D^c must then contain all the negative numbers of S . Therefore, the only such example of D is $D = \{-3, -2, -1\}$.
- (e) The set $X \cup E$ is the set of elements that are in X or in E . The complement of $X \cup E$, i.e. $(X \cup E)^c$, is the set of elements that are not in $X \cup E$. For the set $(X \cup E)^c$ to be empty, there must be no elements in S that are not already in $X \cup E$, i.e. $X \cup E = S$. Examples of such E are $E = \{-3, -2, 2, 3\}$, $E = \{-2, -1, 0, 1, 2\}$, $E = \{-3, -2, -1, 0, 1, 2, 3\}$, etc.

Problem 2. (10pt) Let A and B be sets. By defining $A = B$ by using a quantified open sentence, show that $A \neq B$ is equivalent to the logical statement...

$$(\exists x)(x \in A \wedge x \notin B) \vee (\exists x)(x \in B \wedge x \notin A)$$

Solution. By definition, we know that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$, i.e. every element of A is an element of B and every element of B is an element of A . More precisely, $A = B$ if and only if we have: if $a \in A$, then $a \in B$ and if $b \in B$, then $b \in A$. Writing this as a qualified open statement, we have...

$$(\forall x)(x \in A \rightarrow x \in B) \wedge (\forall x)(x \in B \rightarrow x \in A)$$

Then recalling $\neg(\forall x) \equiv \exists x$, $\neg(P \rightarrow Q) \equiv P \wedge \neg Q$, and $\neg(x \in X) \equiv x \notin X$, as well as the fact that $A \neq B \equiv \neg[A = B]$, we must have...

$$\begin{aligned} A \neq B &\equiv \neg[A = B] \\ &\equiv \neg((\forall x)(x \in A \rightarrow x \in B) \wedge (\forall x)(x \in B \rightarrow x \in A)) \\ &\equiv \neg(\forall x)(x \in A \rightarrow x \in B) \vee \neg(\forall x)(x \in B \rightarrow x \in A) \\ &\equiv ((\exists x)\neg(x \in A \rightarrow x \in B)) \vee (\exists x)(\neg(x \in B \rightarrow x \in A)) \\ &\equiv (\exists x)(x \in A \wedge \neg(x \in B)) \vee (\exists x)(x \in B \wedge \neg(x \in A)) \\ &\equiv (\exists x)(x \in A \wedge x \notin B) \vee (\exists x)(x \in B \wedge x \notin A) \end{aligned}$$

Problem 3. (10pt) Let A and B be sets in a universe \mathcal{U} and consider the set $A\Delta B$.

- (a) Using set-builder notation and logical propositions, define the set $A\Delta B$.
- (b) Construct a Venn diagram for the set $(A\Delta B)^c$.
- (c) Construct a Venn diagram for the set $(A\cup B)^c \cup (A\cap B)$
- (d) What might you conjecture from your answers in (b) and (c)?

Solution.

- (a) We know that the set $A\Delta B$ is the set of elements of \mathcal{U} that are in A or B but *not* in both A and B . From this description of $A\Delta B$, we have...

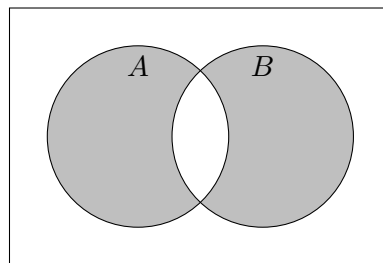
$$A\Delta B = \{x \in \mathcal{U}: (x \in A \vee x \in B) \wedge \neg(x \in A \cap B)\}$$

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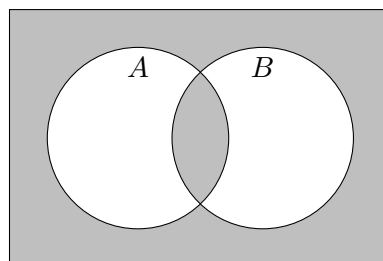
Equivalently, the set $A\Delta B$ is the set of elements of \mathcal{U} that are in A but not B or that are in B but not A . From this description, we have...

$$A\Delta B = \{x \in \mathcal{U}: (x \in A \vee x \in B) \wedge (x \notin A \cap B)\}$$

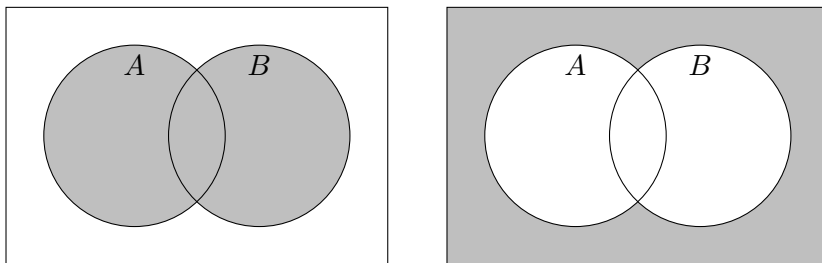
- (b) Using any of the descriptions of $A\Delta B$ given in (a), the Venn diagram for $A\Delta B$ is...



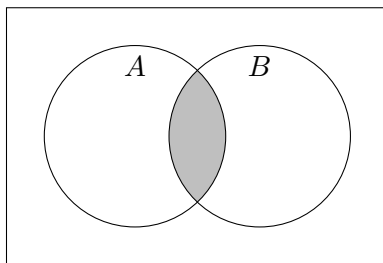
But then the Venn diagram for $(A\Delta B)^c$ is...



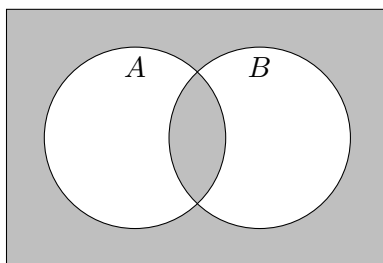
- (c) The Venn diagram for $A \cup B$ is given below on the left, which gives the Venn diagram for $(A \cup B)^c$ below on the right.



The Venn diagram for $A \cap B$ is...



But then the diagram for $(A \cup B)^c \cup (A \cap B)$ is...



- (d) Because the Venn diagram for $(A \Delta B)^c$ in (b) is the same as the Venn diagram for $(A \cup B)^c \cup (A \cap B)$ in (d), we conjecture that $(A \Delta B)^c = (A \cup B)^c \cup (A \cap B)$. In fact, one can prove this:

$$\begin{aligned}
 (A \Delta B)^c &= ((A \cup B) - (A \cap B))^c \\
 &= ((A \cup B) \cap (A \cap B)^c)^c \\
 &= (A \cup B)^c \cup ((A \cap B)^c)^c \\
 &= (A \cup B)^c \cup (A \cap B)
 \end{aligned}$$

Problem 4. (10pt) Let A , B , and C be sets in some universe \mathcal{U} . Find the *complement* of the following sets, showing all your work and ‘simplifying’ as much as possible:

(a) $A \setminus B$

(b) $(A^c \cup C) \cap B$

(c) $\left(\left((A \cup B) \cap C\right)^c \cup B^c\right)^c$

Solution. Recall that if A and B are sets, then by DeMorgan’s Laws, $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$. We also have $(A^c)^c = A$ and the distributive laws $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(a) Recall that $A \setminus B$ is the set of elements that are in A but not in B , i.e. the set $A \cap B^c$. But then we have...

$$\begin{aligned}(A \setminus B)^c &= (A \cap B^c)^c \\ &= A^c \cup (B^c)^c \\ &= A^c \cup B\end{aligned}$$

That is, $(A \setminus B)^c$ are the elements that are either not in A or in B .

(b) We have...

$$\begin{aligned}\left((A^c \cup C) \cap B\right)^c &= (A^c \cup C)^c \cup B^c \\ &= ((A^c)^c \cap C^c) \cup B^c \\ &= (A \cap C^c) \cup B^c\end{aligned}$$

(c) We have...

$$\begin{aligned}\left(\left(\left((A \cup B) \cap C\right)^c \cup B^c\right)^c\right)^c &= ((A \cup B) \cap C)^c \cup B^c \\ &= ((A \cup B)^c \cup C^c) \cup B^c \\ &= ((A^c \cap B^c) \cup C^c) \cup B^c \\ &= ((A^c \cap B^c) \cup B^c) \cup (C^c \cup B^c) \\ &= B^c \cup (C^c \cup B^c) \\ &= C^c \cup B^c \\ &= (C \cap B)^c\end{aligned}$$

Problem 5. (10pt) Define $S := \{1, 2, \{1\}, \{\{2\}\}\}$. Determine whether the following are true or false—no justification is necessary:

- | | |
|------------------------------------|--|
| (a) $\emptyset \in S$ | (g) $\{1\} \subseteq \mathcal{P}(S)$ |
| (b) $\emptyset \subseteq S$ | (h) $\{\{1\}\} \subseteq \mathcal{P}(S)$ |
| (c) $1 \in \mathcal{P}(S)$ | (i) $\emptyset \in \mathcal{P}(S)$ |
| (d) $\{1\} \in \mathcal{P}(S)$ | (j) $\{\emptyset\} \in \mathcal{P}(S)$ |
| (e) $\{\{1\}\} \in \mathcal{P}(S)$ | (k) $\emptyset \subseteq \mathcal{P}(S)$ |
| (f) $1 \subseteq \mathcal{P}(S)$ | (l) $\{\emptyset\} \subseteq \mathcal{P}(S)$ |

Solution. It would be useful to write S and compute $\mathcal{P}(S)$:

$$\mathcal{P}(S) = \left\{ \begin{array}{ccccccc} \emptyset, & & & & & & \\ \{1\}, & \{2\}, & \{\{1\}\}, & \{\{\{2\}\}\}, & & & \\ \{1, 2\}, & \{1, \{1\}\}, & \{1, \{\{2\}\}\}, & \{2, \{1\}\}, & \{2, \{\{2\}\}\}, & \{\{1\}, \{\{2\}\}\} \\ \{1, 2, \{1\}\}, & \{1, 2, \{\{2\}\}\}, & \{1, \{1\}, \{\{2\}\}\}, & \{2, \{1\}, \{\{2\}\}\}, & & & \\ S = \{1, 2, \{1\}, \{\{2\}\}\} & & & & & & \end{array} \right\}$$

- | | |
|---------|---------|
| (a) F | (g) F |
| (b) T | (h) T |
| (c) F | (i) T |
| (d) T | (j) F |
| (e) T | (k) T |
| (f) F | (l) T |

Problem 6. (10pt) Define $A := \{3, 5, 7\}$ and $B := \{\pi, e, \sqrt{2}, \varphi\}$.

- (a) Determine $A \times B$.
- (b) Is $(3, \pi) \in A \times B$? Is $(\pi, 3) \in A \times B$? Explain the relation between your responses.
- (c) Is $A \times B = B \times A$? Explain.

Solution.

- (a) We have...

$$A \times B = \{(a, b) : a \in A, b \in B\} = \left\{ \begin{array}{llll} (3, \pi), & (3, e), & (3, \sqrt{2}), & (3, \varphi) \\ (5, \pi), & (5, e), & (5, \sqrt{2}), & (5, \varphi) \\ (7, \pi), & (7, e), & (7, \sqrt{2}), & (7, \varphi) \end{array} \right\}$$

- (b) From (a), we can see that $(3, \pi) \in A \times B$ but $(\pi, 3) \notin A \times B$. The set $A \times B$ consists of ordered pairs—ordered. The order in an order pair matters. So while $(3, \pi) \in A \times B$ because $3 \in A$ and $\pi \in B$, we know that $(\pi, 3) \notin A \times B$ because $\pi \notin A$ and $3 \notin B$. This is in contrast to sets where order does not matter so that $\{3, \pi\} = \{\pi, 3\}$.

- (c) We have...

$$A \times B = \{(b, a) : a \in A, b \in B\} = \left\{ \begin{array}{llll} (\pi, 3), & (e, 3), & (\sqrt{2}, 3), & (\varphi, 3) \\ (\pi, 5), & (e, 5), & (\sqrt{2}, 5), & (\varphi, 5) \\ (\pi, 7), & (e, 7), & (\sqrt{2}, 7), & (\varphi, 7) \end{array} \right\}$$

We can see that $(3, \pi) \in A \times B$ but $(3, \pi) \notin B \times A$. Because the sets do not contain the same elements, we know that these sets cannot be equal. In fact, $A \times B$ will never be the same as $B \times A$ unless A and B contain all the same elements.

Problem 7. (10pt) Determine $\bigcup_{i \in \mathcal{I}} A_n$ and $\bigcap_{i \in \mathcal{I}} A_n$ for the given A_n and \mathcal{I} below—no justification is necessary. However, if the set is finite, enumerate its elements; otherwise, either give the set in set-builder notation or using set operations with ‘standard’ sets, e.g. \mathbb{Q} , $\mathbb{Z} \setminus \mathbb{N}$, etc.

(a) $A_n := (\frac{1}{n}, 1 + \frac{1}{n})$; $\mathcal{I} := \mathbb{N}$

(b) $A_n := (n, n + 1)$; $\mathcal{I} := \mathbb{Z}$

(c) $A_n := (n - \frac{1}{2}, n + \frac{1}{2})$; $\mathcal{I} := \mathbb{R}$

Solution.

(a) $\bigcup_{i \in \mathcal{I}} A_n = (0, 2]$, $\bigcap_{i \in \mathcal{I}} A_n = \{1\}$

(b) $\bigcup_{i \in \mathcal{I}} A_n = \mathbb{R} \setminus \mathbb{Z}$, $\bigcap_{i \in \mathcal{I}} A_n = \emptyset$

(c) $\bigcup_{i \in \mathcal{I}} A_n = \mathbb{R}$, $\bigcap_{i \in \mathcal{I}} A_n = \emptyset$

Problem 8. (10pt) Below is a partial proof of the fact that $A \setminus B = A \cap B^c$. By filling in the missing portions, complete the partial proof below so that it is a correct, logically sound proof with ‘no gaps’:

Proposition. If A and B are sets, then $A \setminus B = A \cap B^c$.

Proof. If $A \setminus B = \emptyset$, then there is no element in A that is not also in B . But then $A \subseteq B$ so that $A^c \supseteq B^c$. But then $A \cap B^c \subseteq A \cap A^c = \emptyset$ so that $A \cap B^c = \emptyset$. Therefore, if $A \setminus B = \emptyset$, then $A \setminus B = A \cap B^c$. If $A \cap B^c = \emptyset$, then there is no element in both A and B^c . Now if there were an element in $A \setminus B$, there would be an element in A that is not in B , i.e. an element in A that is in B^c , a contradiction to the fact that $A \cap B^c = \emptyset$, i.e. that there is no element in both A and B^c . This shows that $A \setminus B = \emptyset$. Therefore, if $A \cap B^c = \emptyset$, then $A \setminus B = A \cap B^c$. Then we have shown that if either $A \setminus B$ or $A \cap B^c$ are empty then $A \setminus B = A \cap B^c$. Now assume that both $A \setminus B$ and $A \cap B^c$ are nonempty.

To prove that $A \setminus B = A \cap B^c$, we need to show $A \setminus B \subseteq A \cap B^c$ and $A \cap B^c \subseteq A \setminus B$.

$A \setminus B \subseteq A \cap B^c$: We prove that $A \setminus B \subseteq A \cap B^c$. Let $x \in$ $A \setminus B$. Then by definition,

$x \in A$ and $x \notin B$. But then $x \in$ A and $x \in B^c$. This shows that

$x \in$ $A \cap B^c$. Therefore, this shows that $A \setminus B \subseteq A \cap B^c$.

$A \cap B^c \subseteq A \setminus B$: We need to show that $A \cap B^c \subseteq A \setminus B$. Let $x \in$ $A \cap B^c$. Then

$x \in$ A and $x \in$ B^c . But then $x \in$ A and

$x \notin$ B . This shows that $x \in$ $A \setminus B$. Therefore, we know that $A \cap B^c \subseteq A \setminus B$.

Because $A \setminus B \subseteq A \cap B^c$ and $A \cap B^c \subseteq A \setminus B$, we know that $A \setminus B = A \cap B^c$. □