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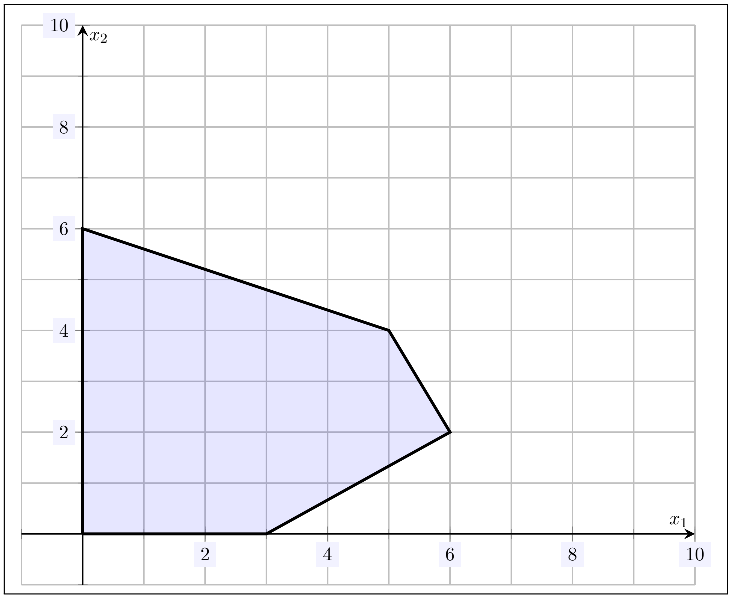
**MATH 108**
**Spring 2023**
**HW 15: Due 05/01**

*"True optimization is the revolutionary contribution of modern research to decision processes."*

*–George Dantzig*

**Problem 1.** (10pt) Find the maximum and minimum values for the function $z = 4x_1 + 5x_2$ on the region shown below. Be sure to fully justify that your answers are correct.



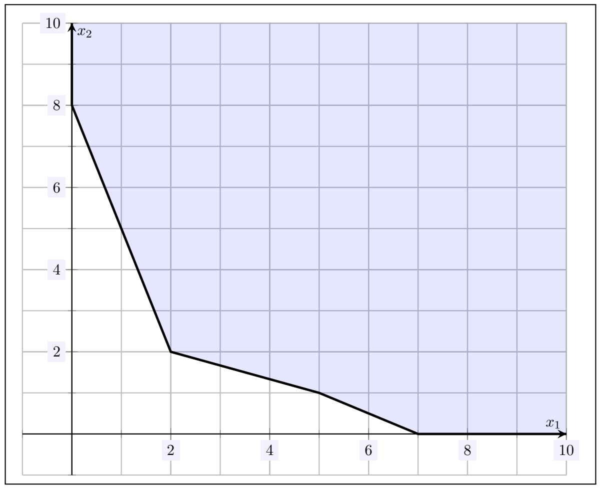

**Solution.** First, observe that the region above is nonempty. For instance, the point $(0,0)$ is in the region so that the region is not empty. The region is closed. Finally, because the region is contained in—for instance—the rectangle $[0,6] \times [0,6]$, the region is bounded. The function $z = 4x_1 + 5x_2$ is linear. Therefore, because the function $z$ is linear and the region is nonempty, closed, and bounded, the Fundamental Theorem of Linear Programming applies. But then we know that $z$ has a maximum and minimum on this region and that they must occur at a corner point. To find the maximum and minimum, we can simply find the value of $z$ at each corner point:

| Corner Point | $z$ |
|---|---|
| $(0,0)$ | $z = 4(0) + 5(0) = 0 + 0 = 0$ |
| $(0,6)$ | $z = 4(0) + 5(6) = 0 + 30 = 30$ |
| $(5,4)$ | $z = 4(5) + 5(4) = 20 + 20 = 40$ |
| $(6,2)$ | $z = 4(6) + 5(2) = 24 + 10 = 34$ |
| $(3,0)$ | $z = 4(3) + 5(0) = 12 + 0 = 12$ |

Therefore, the minimum value is $0$ and occurs at $(0,0)$ and the maximum value is $40$ and occurs at $(5,4)$.

$$\boxed{\begin{array}{c} \min z = 0 \text{ at } (0,0) \\ \max z = 40 \text{ at } (5,4) \end{array}}$$

**Problem 2.** (10pt) Consider the function $z = 5x_1 - x_2$. Does this function has a maximum on the region shown below? If so, explain and find the maximum. If not, explain why. Answer the same question for the minimum of $z$ on the region shown below.




**Solution.** First, observe that the region shown above is nonempty. For instance, the point $(2, 2)$ is in the region so that the region is not empty. The region is closed. The function $z = 5x_1 - x_2$ is linear. However, the region is *not* bounded. For instance, the region contains the point $(x, x)$ for $x \geq 2$. But then the Fundamental Theorem of Linear Programming does not apply because the region is unbounded. Therefore, we will have to reason about the existence of maxima and minima for $z$ directly.

Observe that if $x_1$ is increased and $x_2$ is fixed, the function $z$ increases. If $x_1$ is fixed and $x_2$ is decreased, $z$ increases. Therefore, $z$ increases when we 'move to the right and down.' But then if we choose the point $(x_1, 0)$ for $x_1 \geq 7$, this point is always in the region. But at such a point, we have $z(x_1, 0) = 5x_1 - 0 = 5x_1$. But then choosing $x_1$ arbitrarily large, $z$ is arbitrarily large. Therefore, the function $z$ does not have a maximum on this region.

Observe that if $x_1$ is decreased and $x_2$ is fixed, the function $z$ decreases. If $x_1$ is fixed and $x_2$ is increased, the function $z$ decreases. Therefore, $z$ decreases when we 'move to the left and down.' But there is a limit to how far 'to the left and down' we can move in our region. By the logic used to 'prove' the Fundamental Theorem of Linear Programming in two-variables, the minimum of $z$ must exist and must occur at a corner point. Therefore, to find the minimum, we can simply evaluate $z$ at each corner point for this region. We have...

| Corner Point | $z$ |
|---|---|
| $(0, 8)$ | $z = 5(0) - 8 = 0 - 8 = -8$ |
| $(2, 2)$ | $z = 5(2) - 2 = 10 - 2 = 8$ |
| $(5, 1)$ | $z = 5(5) - 1 = 25 - 1 = 24$ |
| $(7, 0)$ | $z = 5(7) - 0 = 35 - 0 = 35$ |

Therefore, the minimum value is $-8$ and occurs at $(0, 8)$.

$$\boxed{\begin{array}{c} \min z = -8 \text{ at } (0, 8) \\ \max z: \text{ DNE} \end{array}}$$