Name: *Caleb McWhorter — Solutions*
MATH 108
Fall 2022
HW 16: Due 11/22

"Algebra is the offer made by the devil to the mathematician... All you need to do, is give me your soul: give up geometry."
–Michael Atiyah

Problem 1. (10pt) Suppose you have a 3-day drive. You drive at an average speed of 55 mph, 65 mph, and 60 mph each day, respectively. Furthermore, you drive for 10 hours, 8 hours, and 5 hours each day, respectively. Represent your speeds each day as a vector $\mathbf{v}$ and your drive times as a vector $\mathbf{t}$. Compute $\mathbf{v} \cdot \mathbf{t}$ and interpret the result.

Solution. We have...

$$\mathbf{v} = \begin{pmatrix} 55 \\ 65 \\ 60 \end{pmatrix} \qquad \text{and} \qquad \mathbf{t} = \begin{pmatrix} 10 \\ 8 \\ 5 \end{pmatrix}$$

Then we have...

$$\mathbf{v} \cdot \mathbf{t} = \begin{pmatrix} 55 \\ 65 \\ 60 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 8 \\ 5 \end{pmatrix} = 55(10) + 65(8) + 60(5) = 550 + 520 + 300 = 1370$$

Observe that $\mathbf{v} \cdot \mathbf{t} = \sum_i v_i t_i$. As $d = vt$, each $v_i t_i$ is a distance traveled on one of the particular days. But then $\sum v_i t_i$ is a total distance traveled. Therefore, $\mathbf{v} \cdot \mathbf{t}$ computes the 1,370 miles traveled in total.

Problem 2. (10pt) Bill, Bob, and JoBob had a three day work week. The number of hours they worked each day, in the order listed, is represented as a column of the matrix A given below. Their hourly pay, again in the order listed, is represented as a row in the column vector $\mathbf{u}$ given below.

$$A = \begin{pmatrix} 7 & 8 & 6 \\ 8 & 8 & 8 \\ 5 & 12 & 9 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} 15 \\ 12 \\ 20 \end{pmatrix}$$

Compute $A\mathbf{u}$ and interpret the entries of the resulting vector.

Solution. We have...

$$\begin{aligned} A\mathbf{u} &= \begin{pmatrix} 7 & 8 & 6 \\ 8 & 8 & 8 \\ 5 & 12 & 9 \end{pmatrix} \begin{pmatrix} 15 \\ 12 \\ 20 \end{pmatrix} \\ &= \begin{pmatrix} 7(15) + 8(12) + 6(20) \\ 8(15) + 8(12) + 8(20) \\ 5(15) + 12(12) + 9(20) \end{pmatrix} \\ &= \begin{pmatrix} 105 + 96 + 120 \\ 120 + 96 + 160 \\ 75 + 144 + 180 \end{pmatrix} \\ &= \begin{pmatrix} 321 \\ 376 \\ 399 \end{pmatrix} \end{aligned}$$

Observe that each entry in the resulting vector is a sum of the form $\sum_i a_{ij}u_{jk}$. But then each entry is a sum of a number of hours worked times a pay rate, i.e. a amount paid that day. But then each entry is a sum of amounts paid that day. But then each entry is a total daily payroll. Therefore, on the first day the payroll was \$321, the second \$376, and on the third \$399.

Problem 3. (10pt) Assume that each of the following matrices are the reduced-row echelon form from some system of equations. For each, indicate whether there was a solution to the system or not. If there was a solution, either give the solution or give a parametrization of all possible solutions.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution. The matrix A is in RREF and represents a system with a unique solution:

$$\begin{cases} x_1 = -4 \\ x_2 = 3 \\ x_3 = 5 \\ x_4 = 0 \end{cases}$$

The matrix B is in RREF and represents a system without a solution as the last row represents the equation $0 = 1$, which is clearly impossible. Finally, the matrix C is in RREF and the zero row indicates that there is at least one free variable. Because column three does not have a pivot position, we choose x_3 : free. But then the second row indicates $x_2 - 2x_3 = 5$ so that $x_2 = 2x_3 + 5$. The first row indicates $x_1 = 7$. Therefore, the solutions are…

$$\begin{cases} x_1 = 7 \\ x_2 = 2x_3 + 5 \\ x_3 : \text{free} \end{cases}$$