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MATH 308
Fall 2021
HW 5: Due 10/08

"Penny, while I subscribe to the many worlds theory which posits the existence of an infinite number of Sheldons in an infinite number of universes—I assure you that in none of them am I dancing."
–Sheldon Cooper, Big Bang Theory

Problem 1. (10pt) List at least 3 elements from each of the following sets:

(a) $\{n \in \mathbb{N} \colon \exists k \in \mathbb{N}, n = 6k\}$

(b) $\{x \in \mathbb{R} \colon \exists y \in \mathbb{R}, x = y^2\}$

(c) $\{m \in \mathbb{N} \colon \sqrt[3]{m} \in \mathbb{N}\}$

(d) $\{q \in \mathbb{Q} \colon 4q + 1 \in \mathbb{N}\}$

(e) $\{a \in \mathbb{N} \colon \exists b\, \exists c,\, b, c \in \mathbb{N}, a^2 + b^2 = c^2\}$

Solution.

(a) Clearly, if $N \in \{n \in \mathbb{N} \colon \exists k \in \mathbb{N}, n = 6k\}$, then $N = 6k$ for some $k \in \mathbb{N}$, i.e. N is a positive multiple of 6. Furthermore, $6K \in \{n \in \mathbb{N} \colon \exists k \in \mathbb{N}, n = 6k\}$ for all $K \in \mathbb{N}$, i.e. choosing $k = K$. But then $\{n \in \mathbb{N} \colon \exists k \in \mathbb{N}, n = 6k\}$ consists of all the positive multiples of 6. Then, for instance, we have...

$$6, 12, 18, 24, 30, 36, 42, 48 \in \{n \in \mathbb{N} \colon \exists k \in \mathbb{N}, n = 6k\}$$

(b) Clearly, if $w \in \{x \in \mathbb{R} \colon \exists y \in \mathbb{R}, x = y^2\}$, then $w \geq 0$ is a perfect square because $w = y^2$ for some $y \in \mathbb{R}$. Conversely, if $w^2 \in \{x \in \mathbb{R} \colon \exists y \in \mathbb{R}, x = y^2\}$ for all $w \in \mathbb{R}$, i.e. choose $y = w$. Therefore, $\{x \in \mathbb{R} \colon \exists y \in \mathbb{R}, x = y^2\}$ is the set of perfect squares in $\mathbb{R}$. Then, for instance, we have...

$$0, 1, 4, 9, \sqrt{2}, \sqrt{12.9845}, \sqrt{\frac{1}{3}}, \sqrt{\pi} \in \{x \in \mathbb{R} \colon \exists y \in \mathbb{R}, x = y^2\}$$

(c) Clearly, if $N \in \{m \in \mathbb{N} \colon \sqrt[3]{m} \in \mathbb{N}\}$, then there exists $m \in \mathbb{N}$ such that $m = \sqrt[3]{N}$. But then $N = m^3$. Conversely, $N^3 \in \{m \in \mathbb{N} \colon \sqrt[3]{m} \in \mathbb{N}\}$ for all $N \in \mathbb{N}$ because $\sqrt[3]{N^3} = N$. Therefore, $\{m \in \mathbb{N} \colon \sqrt[3]{m} \in \mathbb{N}\}$ is the set of positive perfect cubes. Then, for instance, we have...

$$1, 8, 27, 64, 125, 216, 343 \in \{m \in \mathbb{N} \colon \sqrt[3]{m} \in \mathbb{N}\}$$

(d) Observe $Q \in \{q \in \mathbb{Q} \colon 4q + 1 \in \mathbb{N}\}$ if and only if $4Q + 1 \in \mathbb{N}$ if and only if $4Q \in \mathbb{N} \cup \{0\}$. Therefore, $\{q \in \mathbb{Q} \colon 4q + 1 \in \mathbb{N}\}$ is the set of rational numbers q such that $4q \in \mathbb{Z}_{\geq 0}$. Then, for instance,

$$0, \pm 1, \pm 2, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{5}{4} \in \{q \in \mathbb{Q} \colon 4q + 1 \in \mathbb{N}\}$$

(e) Clearly, $A \in \{a \in \mathbb{N} \colon \exists b\, \exists c,\, b, c \in \mathbb{N}, a^2 + b^2 = c^2\}$ if and only if $A^2 + b^2 = c^2$ for some b, c, where $A, b, c \in \mathbb{N}$, if and only if A and b are legs of a right triangle with integer legs. Then, for instance,

$$3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17, 20, 21, 24, 28, 33, 35 \in \{a \in \mathbb{N} \colon \exists b\, \exists c,\, b, c \in \mathbb{N}, a^2 + b^2 = c^2\}$$

Problem 2. (10pt) Use the set-builder notation to give a set equal to each of the following sets:

(a) $\{1, 4, 9, 16, 25, 36, 49, 64, \ldots\}$

(b) $\{0, \pm 3, \pm 6, \pm 9, \pm 12, \pm 15, \ldots\}$

(c) The set of rational numbers between 0 and 1.

(d) The set of functions passing through the point $(6, 5)$.

(e) The set of differentiable functions with a horizontal tangent line at $x = 1$.

Solution.

(a) There are many possibilities. For instance, ...

$$\begin{aligned}\{1, 4, 9, 16, 25, 36, 49, 64, \ldots\} &= \{n^2 \colon n \in \mathbb{N}\} \\ &= \{n \colon \exists k \in \mathbb{N}, n = k^2\} \\ &= \{n \colon \exists k \in \mathbb{Z}, n = k^2\} \\ &= \{n^2 \colon n \in \mathbb{Z} \setminus \{0\}\}\end{aligned}$$

(b) There are many possibilities. For instance, ...

$$\begin{aligned}\{0, \pm 3, \pm 6, \pm 9, \pm 12, \pm 15, \ldots\} &= \{3k \colon k \in \mathbb{Z}\} \\ &= \{z \in \mathbb{Z} \colon \exists k \in \mathbb{Z}, z = 3k\} \\ &= \{n \in \mathbb{Z} \colon 3 \mid n\} \\ &= \left\{n \in \mathbb{Z} \colon \frac{n}{3} \in \mathbb{Z}\right\}\end{aligned}$$

(c) There are a few possibilities. For instance,

$$\{q \in \mathbb{Q} \colon 0 < q < 1\} = (0, 1) \cap \mathbb{Q}$$

(d) This is the set...

$$\{f : \mathbb{R} \to \mathbb{R} \mid f(6) = 5\}$$

(e) This is the set...

$$\{f : \mathbb{R} \to \mathbb{R} \mid f'(1) = 0\}$$

Problem 3. (10pt) Let $\mathscr{U} = \{1, 2, 3, \{1\}, \{2\}, \{1, 2\}\}$. Let $A = \{2, 1, 2\}$ and $B = \{1\}$.

(a) Is $A \in \mathscr{U}$? Explain.

(b) Is $A \subseteq \mathscr{U}$? Explain.

(c) Is $B \in \mathscr{U}$? Explain.

(d) Is $B \subseteq \mathscr{U}$? Explain.

Solution.

(a) We know $A = \{2, 1, 2\} = \{1, 2\}$. But $\{1, 2\} \in \mathscr{U}$. But then $A \in \mathscr{U}$.

(b) We know $A = \{2, 1, 2\} = \{1, 2\}$. Then the only elements of A are 1 and 2. But $1 \in \mathscr{U}$ and $2 \in \mathscr{U}$. Therefore, $A \subseteq \mathscr{U}$.

(c) We know $B = \{1\}$. But $\{1\} \in \mathscr{U}$. Therefore, $B \in \mathscr{U}$.

(d) We know $B = \{1\}$. Then the only element of B is 1. But $1 \in \mathscr{U}$. Therefore, $B \subseteq \mathscr{U}$.

Problem 4. (20pt) Define the following sets:

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$B = \{1, 3, 5, 7, 9\}$$
$$C = \{2, 4, 6, 8, 10\}$$
$$D = \{2, 3, 5, 7\}$$
$$E = \{4, 8, 9\}$$
$$F = \{1, 2, \{3\}\}$$

Compute the following sets:

(a) $A \cap B$

(b) $C \cup D$

(c) $D \cap E$

(d) $D \setminus B$

(e) $B \setminus A$

(f) $B \times C$

(g) $(D \cap F) \cup (B \cap E)$

In addition, answer the following:

(h) Is $F \subseteq A$? Explain.

(i) Is $B \cap F = \{1, 3\}$? Explain.

(j) Is A a universal set for B, C, D, E, F? If it is, compute D^c. If not, explain why.

Solution.

(a) $A \cap B = \{1, 3, 5, 7, 9\}$

(b) $C \cup D = \{2, 3, 4, 5, 6, 7, 8, 10\}$

(c) $D \cap E = \varnothing$

(d) $D \setminus B = \{2\}$

(e) $B \setminus A = \varnothing$

(f) $B \times C = \{(1, 2), (1, 4), (1, 6), (1, 8), (1, 10), (3, 2), (3, 4), (3, 6), (3, 8), (3, 10), (5, 2), (5, 4), (5, 6), (5, 8), (5, 10), (7, 2), (7, 4), (7, 6), (7, 8), (7, 10), (9, 2), (9, 4), (9, 6), (9, 8), (9, 10)\}$.

(g) $(D \cap F) \cup (B \cap E) = \{2, 9\}$

(h) No, because $\{3\} \in F$ but $\{3\} \notin A$.

(i) No, because $3 \in B$ but $3 \notin F$.

(j) No, because from part (h), we know that $F \not\subseteq A$.

Problem 5. (10pt) Compute each of the following sets:

(a) $\mathscr{P}(\varnothing)$

(b) $\mathscr{P}(\{1, \{1\}\})$

(c) $\mathscr{P}(\{1, e, \pi\})$

(d) $\mathscr{P}(\{1\} \times \{a, b\})$

Solution.

(a) $\mathscr{P}(\varnothing) = \{\varnothing\}$.

(b) $\mathscr{P}(\{1, \{1\}\}) = \{\, \varnothing, \{1\}, \{\{1\}\}, \{1, \{1\}\}\,\}$

(c) $\mathscr{P}(\{1, e, \pi\}) = \{\, \varnothing, \{1\}, \{e\}, \{\pi\}, \{1, e\}, \{1, \pi\}, \{e, \pi\}, \{1, e, \pi\}\,\}$

(d) $\mathscr{P}(\{1\} \times \{a, b\}) = \mathscr{P}(\{(1, a), (1, b)\}) = \{\, \varnothing, \{(1, a)\}, \{(1, b)\}, \{(1, a), (1, b)\}\,\}$

Problem 6. (10pt) Suppose A, B are sets with a common universal set $\mathscr{U}$. Denote each of the following sets with a Venn diagram:

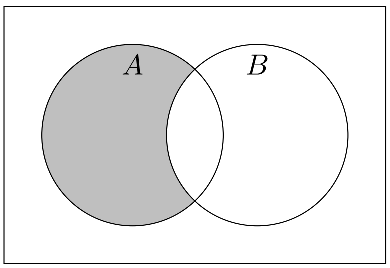
(a) $A \cap B^c$

(b) $(A \cup B)^c$

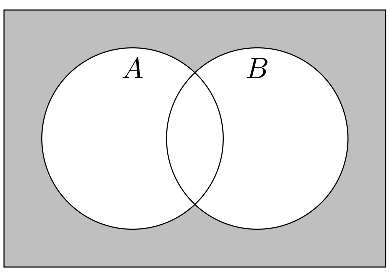
(c) $(A \cup B) \setminus (A \cap B)$

Solution.

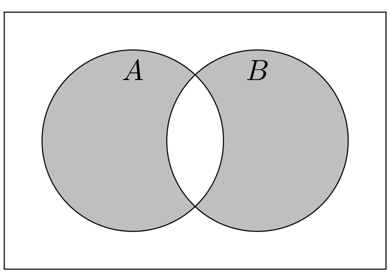
(a)



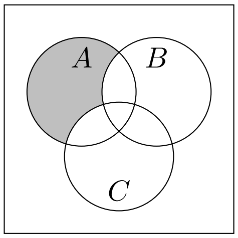
(b)

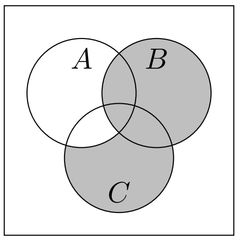


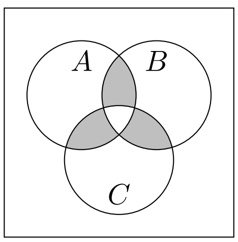
(c)



Problem 7. (10pt) Suppose A, B, C are sets with a common universal set $\mathscr{U}$. For each of the Venn diagrams, write down the shaded sets.

(a) 

(b) 

(c) 

Solution.

(a) There are several possibilities. For instance, $A \cap (B \cup C)^c = A \cap (B^c \cap C^c) = A \setminus (B \cup C)$

(b) There are several possibilities. For instance, $[(B \cup C) \cap A^c] \cup (A \cap B) = (B \cup C) \setminus [(A \cap C) \setminus B] = B \cup (C \cap (A \cap C)^c) = B \cup (C \cap (A^c \cup C^c)) = B \cup (A^c \cap C)$.

(c) There are several possibilities. For instance, $[(A \cap B) \cup (B \cup C) \cup (A \cap C)] \cap (A \cap B \cap C)^c = [(A \cap B) \cup (B \cup C) \cup (A \cap C)] \setminus (A \cap B \cap C)$.

Problem 8. (10pt) Let $A = \{b, c\}$. Suppose that $A \cup B = \{a, b, c, e\}$ and $B \cup C = \{a, c, d, e, f\}$. From this information can we determine the sets A, B, C? Explain. If not, what is the minimal additional information (in terms of unions and intersections of the sets alone) would uniquely determine the three sets?

Solution. Observe that if $A = \{b, c\}, B = \{a, b, c, e\}, C = \{a, c, d, e, f\}$, then $A \cup B = \{a, b, c, e\}$ and $B \cup C = \{a, c, d, e, f\}$. Furthermore, if $A = \{b, c\}, B = \{a, e\}, C = \{c, d, f\}$, then $A \cup B = \{a, b, c, e\}$ and $B \cup C = \{a, c, d, e, f\}$. Therefore, the given information does not uniquely determine B and C.

Observe that $B = (A \cap B) \cup (B \setminus A)$ and $(A \cup B) \setminus A = B \setminus A$. The sets A and $A \cup B$ are known. Therefore, if one knew $A \cap B$, one could uniquely determine B. Similarly, $C = (B \cap C) \cup (C \setminus B)$ and $(B \cup C) \setminus B = C \setminus B$. Given $A \cap B$, the set B is known and by assumption we know the set $B \cup C$. But then if one knew $B \cap C$, one could uniquely determine the set C. The examples above show that knowing any one of $A \cap B, B \cap C, A \cap C, A \cup C, A \cap B \cap C, A \cup B \cup C$ does not uniquely determine A, B, C. Therefore, knowing $A \cap B$ and $B \cap C$ is a minimal set of information (in terms of unions and intersections) to determine A, B, C.