Name: Solutions — Caleb McWhorter
MATH 308
Fall 2021
HW 12: Due 11/12

"People think that computer science is the art of geniuses but the actual reality is the opposite, just many people doing things that build on each other, like a wall of mini stones."

–Donald Knuth

Problem 1. (10pt) Define a relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ via $(x, y) \sim (a, b)$ if and only if $x - y = a - b$.

(a) Is $(3, 1) \sim (2, 5)$? Explain.

(b) Is $(7, 3) \sim (5, 1)$? Explain.

(c) Show that $\sim$ is an equivalence relation on X.

(d) Find at least 3 elements in each of the equivalence classes $[(1, 1)]$ and $[(3, 5)]$.

Solution.

(a) Let $(x, y) = (3, 1)$ and $(a, b) = (2, 5)$. Then $x - y = 3 - 1 = 2$ and $a - b = 2 - 5 = -3$. Because $x - y \neq a - b$, $(3, 1) \not\sim (2, 5)$.

(b) Let $(x, y) = (7, 3)$ and $(a, b) = (5, 1)$. Then $x - y = 7 - 3 = 4$ and $a - b = 5 - 1 = 4$. Because $x - y = a - b$, $(7, 3) \sim (5, 1)$.

(c) To show $\sim$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$, we need to show the relation is reflexive (for all $x \in X$, $x \sim x$), symmetric (if $x, y \in X$ and $x \sim y$, then $y \sim x$), and transitive (if $x, y, z \in X$, $x \sim y$, and $y \sim z$, then $x \sim z$):

- *Reflexive:* Let $(x, y) \in \mathbb{N} \times \mathbb{N}$. We need to show that $(x, y) \sim (x, y)$. Observe that $x - y = 0 = x - y$. Therefore, $(x, y) \sim (x, y)$.
- *Symmetric:* Let $(x, y), (a, b) \in \mathbb{N} \times \mathbb{N}$ and $(x, y) \sim (a, b)$. We need to show that $(a, b) \sim (x, y)$. Because $(x, y) \sim (a, b)$, we know that $x - y = a - b$. But then $a - b = x - y$. Therefore, $(a, b) \sim (x, y)$.
- *Transitive:* Let $(x, y), (a, b), (n, m) \in \mathbb{N} \times \mathbb{N}$ with $(x, y) \sim (a, b)$ and $(a, b) \sim (n, m)$. Because $(x, y) \sim (a, b)$, we know $x - y = a - b$. Similarly, because $(a, b) \sim (n, m)$, we know $a - b = n - m$. But then $x - y = a - b = n - m$, so that $x - y = n - m$. Therefore, $(x, y) \sim (n, m)$.

(d) If $(x, y) \in [(1, 1)]$, then $x - y = 1 - 1 = 0$. But then $x - y = 0$ so that $x = y$. Clearly, if $x = y$, then $x - y = 0 = 1 - 1$ so that $(x, y) \in [(1, 1)]$. Therefore, $(x, y) \in [(1, 1)]$ if and only if $x = y$. This shows the elements of $[(1, 1)]$ are of the form (x, x), where $x \in \mathbb{N}$. This shows that $[(1, 1)] = \{(x, x) \colon x \in \mathbb{N}$. But then, for example, $(1, 1), (2, 2), (15, 15), (23^{23}, 23^{23}) \in [(1, 1)]$.

(e) If $(x, y) \in [(3, 5)]$, then $x - y = 3 - 5 = -2$. But then $x - y = -2$ so that $x = y - 2$. Suppose $x = y - 2$. Because $x \in \mathbb{N}$, $x \geq 1$. As $x = y - 2$, we know that $x = y - 2 \geq 1$. But then $y \geq 3$. Clearly, if $x = y - 2$, then $x - y = (y - 2) - y = -2 = 3 - 5$ so that $(x, y) \in [(3, 5)]$. Therefore, $(x, y) \in [(3, 5)]$ if and only if $x = y - 2$ and $y \geq 3$. This shows the elements of $[(3, 5)]$ are of the form $(x, y) = (y - 2, y)$, where $y \geq 3$. Therefore, $[(3, 5)] = \{(y - 2, y) \colon y \in \mathbb{N}, y \geq 3\}$. But then, for example, $(3, 5), (4, 6), (10, 12), (103, 105) \in [(3, 5)]$.

Problem 2. (10pt) Define a relation on $\mathbb{R}$ via $x \sim y$ if and only if $x \leq y$. Prove or disprove whether $\sim$ is an equivalence relation on $\mathbb{R}$.

Solution. We prove that $\sim$ is *not* an equivalence relation on $\mathbb{R}$. If $\sim$ were an equivalence relation on $\mathbb{R}$, we would need to show the relation is reflexive (for all $x \in X$, $x \sim x$), symmetric (if $x, y \in X$ and $x \sim y$, then $y \sim x$), and transitive (if $x, y, z \in X$, $x \sim y$, and $y \sim z$, then $x \sim z$):

- *Reflexive:* Let $x \in \mathbb{R}$. We need to show that $x \sim x$. But $x \leq x$ so that $x \sim x$. Therefore, $\sim$ is a reflexive relation on $\mathbb{R}$.

- *Symmetric:* Let $x, y \in \mathbb{R}$ with $x \sim y$. We need to show that $y \sim x$. Because $x \sim y$, we know that $x \leq y$. However, it need not be the case that $y \leq x$, implying $y \sim x$. For instance, we know that $1 \sim 3$ because $1 \leq 3$. But $3 \not\leq 1$ so that $3 \not\sim 1$. Observe that if $x \sim y$, then $x \leq y$, and if $y \sim x$, then $y \leq x$. But then $x \sim y$ and $y \sim x$ implies that $x \leq y$ and $y \leq x$ so that $x = y$. Clearly, if $x = y$, then $x \sim y$ and $y \sim x$. Therefore, $x \sim y$ and $y \sim x$ if and only if $x = y$. The relation $\sim$ on $\mathbb{R}$ is only symmetric only for the equal elements in $\mathbb{R}$.

- *Transitive:* Let $x, y, z \in \mathbb{R}$ with $x \sim y$ and $y \sim z$. We need to show that $x \sim z$. Because $x \sim y$, we know $x \leq y$. Furthermore, because $y \sim z$, we know $y \leq z$. But then $x \leq y \leq z$ so that $x \leq z$. This shows that $x \sim z$. Therefore, $\sim$ is a transitive relation on $\mathbb{R}$.

Despite the fact that $\sim$ is a reflexive and transitive relation on $\mathbb{R}$, $\sim$ is *not* an equivalence relation on $\mathbb{R}$ because $\sim$ is not symmetric on $\mathbb{R}$.

Problem 3. (10pt) Define a relation on $\mathbb{R}^2$ via $(x, y) \sim (a, b)$ if and only if (x, y) and (a, b) are the same distance from the origin.

(a) Prove that $\sim$ is an equivalence relation.

(b) Explicitly find the equivalences classes as a set.

(c) Describe the equivalence classes graphically.

Solution.

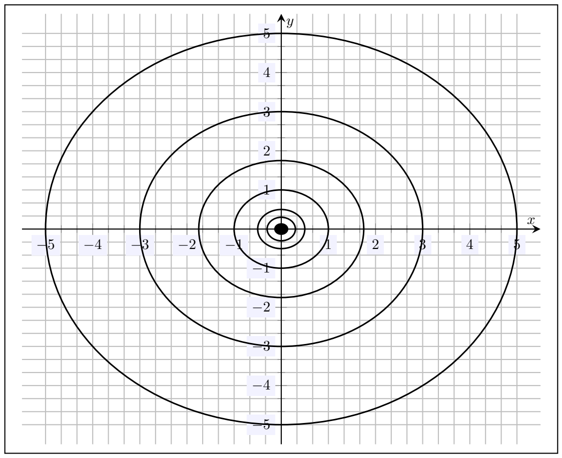
(a) We know the distance, d, between two points, (x, y) and (a, b), in the place is given by $d = \sqrt{(x-a)^2 + (y-b)^2}$. Therefore, the distance from (x, y) to the origin is $d = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$. To show $\sim$ is an equivalence relation on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, we need to show the relation is reflexive (for all $x \in X$, $x \sim x$), symmetric (if $x, y \in X$ and $x \sim y$, then $y \sim x$), and transitive (if $x, y, z \in X$, $x \sim y$, and $y \sim z$, then $x \sim z$):

- *Reflexive:* Let $(x, y) \in \mathbb{R}^2$. We need to show that $(x, y) \sim (x, y)$. But it is clear that (x, y) has the same distance to the origin as itself. Therefore, $(x, y) \sim (x, y)$.
- *Symmetric:* Let $(x, y), (a, b) \in \mathbb{R}^2$ and $(x, y) \sim (a, b)$. Because $(x, y) \sim (a, b)$, we know that (x, y) has the same distance to the origin as (a, b). But then it is immediate that (a, b) has the same distance to the origin as (x, y). Therefore, $(a, b) \sim (x, y)$.
- *Transitive:* Let $(x, y), (a, b), (r, s) \in \mathbb{R}^2$ with $(x, y) \sim (a, b)$ and $(a, b) \sim (r, s)$. Because $(x, y) \sim (a, b)$, we know (x, y) and (a, b) have the same distance to the origin, i.e. $\sqrt{x^2 + y^2} = \sqrt{a^2 + b^2}$. Because $(a, b) \sim (r, s)$, we know (a, b) and (r, s) have the same distance to the origin, i.e. $\sqrt{a^2 + b^2} = \sqrt{r^2 + s^2}$. But then $\sqrt{x^2 + y^2} = \sqrt{a^2 + b^2} = \sqrt{r^2 + s^2}$, which implies $\sqrt{x^2 + y^2} = \sqrt{r^2 + s^2}$. But then (x, y) and (r, s) have the same distance to the origin. Therefore, $(x, y) \sim (r, s)$.

(b) Let $(a, b), (v, w) \in \mathbb{R}^2$ and suppose $(a, b) \in [(v, w)]$. Suppose that the distance from (v, w) to the origin is $d \in \mathbb{R}$, where $d \geq 0$. We know that $(a, b) \in [(v, w)]$ if and only if (a, b) and (v, w) have the same distance to the origin. But (a, b) and (v, w) have the same distance to the origin if and only if $d = \sqrt{a^2 + b^2}$. But $d = \sqrt{a^2 + b^2}$ if and only if $d^2 = a^2 + b^2$. Therefore, $[(v, w)] = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = d^2\}$.

(c) From (b), we know that $(a, b) \in [(v, w)]$ if and only if $d^2 = a^2 + b^2$, where d is the distance from (v, w) to the origin. But $d^2 = a^2 + b^2$ if and only if (a, b) is a point on the circle $d^2 = x^2 + y^2$, i.e. the circle of radius d centered at the origin. That is, $[(v, w)]$ is the set of points on the circle centered at the origin that passes through the point (v, w). This includes the 'trivial circle $\{(0, 0)\}$ (the origin), i.e. the circle with radius 0 centered at the origin.

The work above shows $\sim$ partitions $\mathbb{R}^2$ into circles centered at the origin (including the 'trivial circle' at the origin).



Problem 4. (10pt) Define a relation on $\mathbb{Z}$ via $a \sim b$ if and only if a and b have the same parity, i.e. a and b are either both even or they are both odd.

(a) Show that $\sim$ is an equivalence relation.

(b) Describe all the equivalence classes, i.e. determine the set $\mathbb{Z}/\sim$.

Solution.

(a) To show $\sim$ is an equivalence relation on $\mathbb{Z}$, we need to show the relation is reflexive (for all $x \in X$, $x \sim x$), symmetric (if $x, y \in X$ and $x \sim y$, then $y \sim x$), and transitive (if $x, y, z \in X$, $x \sim y$, and $y \sim z$, then $x \sim z$):

- *Reflexive:* Let $n \in \mathbb{Z}$. We need to show that $n \sim n$. But n clearly has the same parity as n. Therefore, $n \sim n$.
- *Symmetric:* Let $n, m \in \mathbb{Z}$ with $n \sim m$. Because $n \sim m$, n and m have the same parity. But then m has the same parity as n. Therefore, $n \sim m$.
- *Transitive:* Let $n, m, p \in \mathbb{Z}$ with $n \sim m$ and $m \sim p$. Because $n \sim m$, n has the same parity as m. Because $m \sim p$, m and n have the same parity. But an integer cannot be both even an odd. Thus, m has a fixed parity. The parity of m from $n \sim m$ must then be the same parity of m from $m \sim p$. But then because n has the same parity as m, n must have the same parity as p. Therefore, $n \sim p$.

(b) Consider the equivalence class of $0 \in \mathbb{Z}$. We know that 0 is even. But then if $n \in \mathbb{Z}$ is even, we know that $n \sim 0$. Therefore, $[0] = \{\text{even integers}\} = \{2k \mid k \in \mathbb{Z}\}$. Now consider the equivalence class of $1 \in \mathbb{Z}$. We know that 1 is odd. But then if $n \in \mathbb{Z}$ is odd, we know that $n \sim 1$. Therefore, $[1] = \{\text{odd integers}\} = \{2k + 1 \mid k \in \mathbb{Z}\}$. If $n \in \mathbb{Z}$, then n is either even or odd (but not both). But then $n \in [0]$ or $n \in [1]$. Therefore, $\mathbb{Z}/\sim = \{[0], [1]\}$, i.e. there are only two equivalence classes:

$$[0] = \{2k \mid k \in \mathbb{Z}\} = \{\text{even integers}\} = \{0, \pm 2, \pm 4, \pm 6, \ldots\}$$

$$[1] = \{2k + 1 \mid k \in \mathbb{Z}\} = \{\text{odd integers}\} = \{\pm 1, \pm 3, \pm 5, \pm 7, \ldots\}$$

Problem 5. (10pt) Prove that if X is a set and $S \subsetneq X$ is a nonempty subset of X, then $\{S, X \setminus S\}$ is a partition of X.

Solution. Let X be a nonempty set and $\mathcal{A} \subseteq \mathcal{P}(S)$. Recall that $\mathcal{A}$ is a partition of X if...

- $\varnothing \notin \mathcal{A}$
- $X = \bigcup_{A \in \mathcal{A}} A$
- For all $A, B \in \mathcal{A}$, if $A \neq B$, then $A \cap B = \varnothing$.

To prove that $\mathcal{A} = \{S, X \setminus S\}$ is a partition of X, we need to check each of these conditions.

- $\varnothing \notin \mathcal{A}$: There are only two distinct sets in $\bigcup_{A \in \mathcal{A}} A$—$S$ and $X \setminus S$. So we need only check that they are both nonempty. By assumption, S is nonempty. Because $S \subsetneq X$, we know there exists $x \in X$ such that $x \notin S$. But then $x \in X \setminus S$, which implies that $X \setminus S \neq \varnothing$.

- $X = \bigcup_{A \in \mathcal{A}} A$: To show $X = \bigcup_{A \in \mathcal{A}} A$, we need to show that $X \subseteq \bigcup_{A \in \mathcal{A}} A$ and $\bigcup_{A \in \mathcal{A}} A \subseteq X$. We begin by showing $X \subseteq \bigcup_{A \in \mathcal{A}} A$. We need to show that if $x \in X$, then $x \in \bigcup_{A \in \mathcal{A}} A$. Let $x \in X$. Either $x \in S$ or $x \notin S$. But then either $x \in S$ or $x \in X \setminus S$, respectively. But then...

  $$x \in S \cup (X \setminus S) = \bigcup_{A \in \mathcal{A}} A$$

  This proves that $X \subseteq \bigcup_{A \in \mathcal{A}} A$. Now we need to show that $\bigcup_{A \in \mathcal{A}} A \subseteq X$, i.e. we need to show that if $x \in \bigcup_{A \in \mathcal{A}} A$, then $x \in X$. If $x \in \bigcup_{A \in \mathcal{A}} A = S \cup (X \setminus S)$, then either $x \in S$ or $x \in X \setminus S$. But $S \subseteq X$ and $X \setminus S \subseteq X$. But then either $x \in S \subseteq X$ or $x \in X \setminus S \subseteq X$. This implies that $x \in X$, so that $\bigcup_{A \in \mathcal{A}} A \subseteq X$. Because $X \subseteq \bigcup_{A \in \mathcal{A}} A$ and $\bigcup_{A \in \mathcal{A}} A \subseteq X$, we know that $X = \bigcup_{A \in \mathcal{A}} A$.

- For all $A, B \in \mathcal{A}$, if $A \neq B$, then $A \cap B = \varnothing$: There are only two distinct sets in $\bigcup_{A \in \mathcal{A}} A$—$S$ and $X \setminus S$. So we need only check that $S \cap (X \setminus S) = \varnothing$. Suppose that $x \in S \cap (X \setminus S)$. This implies that $x \in S$ and $x \in X \setminus S$. Because $x \in X \setminus S$, we know that $x \in X$ and $x \notin S$. This contradicts the fact that $x \in S$. Therefore, $S \cap (X \setminus S) = \varnothing$.

Problem 6. (10pt) Let X be a nonempty set. Every equivalence relation $\sim$ on X gives rise to a partition on X. Moreover, every partition on X gives rise to an equivalence relation $\sim$ on X. We proved the first statement in class. Suppose that $\{X_i\}_{i\in\mathcal{I}}$ is a partition of X. Show that this partition induces an equivalence relation $X/\sim$ given by $a \sim b$ if and only if $a, b \in X_i$ for some $i \in \mathcal{I}$.

Solution. Assume that $\{X_i\}_{i\in\mathcal{I}}$ is a partition of a set X. Let $\sim$ be the relation on X given by the following: if $a, b \in X$, then $a \sim b$ if and only if $a, b \in X_i$ for some $i \in \mathcal{I}$. To show $\sim$ is an equivalence relation on X, we need to show the relation is reflexive (for all $x \in X$, $x \sim x$), symmetric (if $x, y \in X$ and $x \sim y$, then $y \sim x$), and transitive (if $x, y, z \in X$, $x \sim y$, and $y \sim z$, then $x \sim z$):

- *Reflexive:* Let $x \in X$. We need to show that $x \sim x$. Because $\{X_i\}_{i\in\mathcal{I}}$ is a partition of X, we know that $X = \bigcup_{i\in\mathcal{I}} X_i$. Therefore, $x \in X = \bigcup_{i\in\mathcal{I}} X_i$. Therefore, there exists $i \in \mathcal{I}$ such that $x \in X_i$. Because $x, x \in X_i$, we know that $x \sim x$.[1]

- *Symmetric:* Let $x, y \in X$ with $x \sim y$. We need to show that $y \sim x$. Because $x \sim y$, we know there exists $i \in \mathcal{I}$ such that $x, y \in X_i$. But then $y, x \in X_i$. Therefore, $y \sim x$.[1]

- *Transitive:* Let $x, y, z \in X$ with $x \sim y$ and $y \sim z$. We need to show that $x \sim z$. Because $x \sim y$, there exists $i \in \mathcal{I}$ such that $x, y \in X_i$. Similarly, because $y \sim z$, there exists $j \in \mathcal{I}$ such that $y, z \in X_j$. We know $y \in X_i$ and $y \in X_j$ so that $y \in X_i \cap X_j$. If $i \neq j$, then $X_i \cap X_j \neq \varnothing$ because $y \in X_i \cap X_j$, contradicting the fact that $\{X_i\}_{i\in\mathcal{I}}$ is a partition of X. Therefore, $i = j$. But then $z \in X_i = X_j$. This shows that $x, z \in X_i$. Therefore, $x \sim z$.

For completeness, we will also include the proof that every equivalence relation $X/\sim$ gives a partition of X—namely, the collection $\mathcal{P} := \{[x] \colon x \in X\}$, i.e. the set of equivalence classes form a partition of X. To show this is a partition, we must show the following:

- $\varnothing \notin \mathcal{P}$
- $X = \bigcup_{A\in\mathcal{P}} A$
- For all $A, B \in \mathcal{P}$, if $A \neq B$, then $A \cap B = \varnothing$.

We check each condition individually.

- $\varnothing \notin \mathcal{P}$: Let $A \in \mathcal{P}$. Because $A \in \mathcal{P}$, we know $A = [x]$ for some $x \in X$. But $x \in [x]$ so that $x \in A$. Therefore, $A \neq \varnothing$.

- $X = \bigcup_{A\in\mathcal{P}} A$: To prove $X = \bigcup_{A\in\mathcal{P}} A$, we need to prove that $\bigcup_{A\in\mathcal{P}} A \subseteq X$ and $X \subseteq \bigcup_{A\in\mathcal{P}} A$. First, we prove that $\bigcup_{A\in\mathcal{P}} A \subseteq X$. Observe that for all $A \in \bigcup_{A\in\mathcal{P}} A$, $A = [x]$ for some $x \in X$. But then $A = [x] \subseteq X$. Because $A \subseteq X$ for all $A \in \bigcup_{A\in\mathcal{P}} A$, it must be that $\bigcup_{A\in\mathcal{P}} A \subseteq X$. Now we need to prove that $X \subseteq \bigcup_{A\in\mathcal{P}} A$. Let $x \in X$ and define $A = [x]$. We know that $x \in [x] = A$. But $A \subseteq \bigcup_{A\in\mathcal{P}} A$ so that $x \in A \subseteq \bigcup_{A\in\mathcal{P}} A$. This implies that $x \in \bigcup_{A\in\mathcal{P}} A$. But then $x \in \bigcup_{A\in\mathcal{P}} A$ for all $x \in X$. Therefore, $X \subseteq \bigcup_{A\in\mathcal{P}} A$.

- For all $A, B \in \mathcal{P}$, if $A \neq B$, then $A \cap B = \varnothing$: Let $A, B \in \bigcup_{P\in\mathcal{P}} P$. Then there exist $x, y \in X$ such that $A = [x]$ and $B = [y]$. Because $A = [x]$ and $B = [y]$ are equivalence classes for $X/\sim$, we know that $A \cap B = [x] \cap [y] = \varnothing$ if and only if $[x] = [y]$, which happens if and only if $A = B$. Therefore, if $A \neq B$, we know that $A \cap B = \varnothing$.

[1] This only used the $X = \bigcup_{A\in\mathcal{A}} A$ property of a partition.