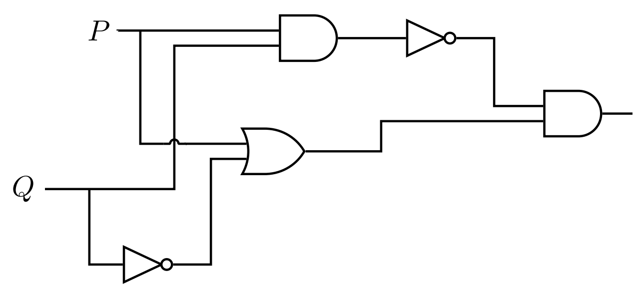
Quiz 1. *True/False*: The expression $P \to Q$ is logically equivalent to $\neg P \vee Q$.

Solution. The statement is *true*. One method of seeing is this is to compute the truth table for $P \to Q$ and $\neg P \vee Q$ and see that the outputs of $P \to Q$ and $\neg P \vee Q$ match, no matter the inputs for P, Q.

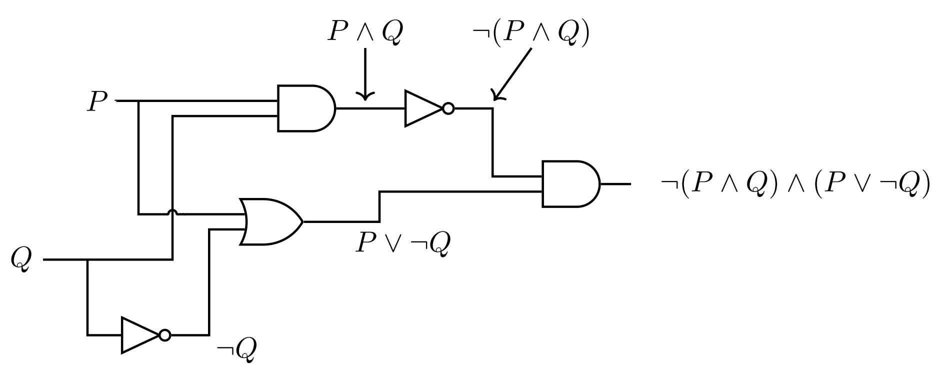
P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$
T	T	**T**	F	**T**
T	F	**F**	F	**F**
F	T	**T**	T	**T**
F	F	**T**	T	**T**

As we can see, the third and fourth columns corresponding to $P \to Q$ and $\neg P \vee Q$, respectively, are the same, $P \to Q \equiv \neg P \vee Q$. Alternatively, $P \to Q$ will be logically equivalent to $\neg P \vee Q$ if they are always simultaneously true. We know for $P \to Q$ to be true, either P must be false or P, Q must both be true. Observe that if P is false, then $\neg P$ is true so that $\neg P \vee Q$ is true. If P, Q are true, then $\neg P \vee Q$ is true. Loosely, $P \to Q$ is true if either P does not occur or if Q occurs. But this is precisely $\neg P \vee Q$. In any case, it is true that $P \to Q \equiv \neg P \vee Q$.

Quiz 2. *True/False*: The logical expression corresponding to the circuit below is $\neg(P \wedge Q) \wedge (P \vee \neg Q)$



Solution. The statement is *true*. To see this, we can follow the circuit, labeling the wires as we go.



Quiz 3. *True/False*: Let $\mathcal{U}$ be the set of integers. Consider the predicate $P(n)\colon n^2 + 5 > 20$. Because $P(5)$ is true, we know that both $\exists n\, P(n)$ and $\forall n\, P(n)$ are true.

Solution. The statement is *false*. Because $P(5)\colon 5^2 + 5 = 25 + 5 = 30 > 20$ is true, we know there exists an integer n—for example $n = 5$—such that $P(n)$ is true. Therefore, $\exists n\, P(n)$ is true. However, the statement $\forall n\, P(n)$ need not be true simply because there is an n such that $P(n)$ is true. For example, $P(1)\colon 1^2 + 5 = 1 + 5 = 6 \not> 20$. But because $P(n)$ is not true when $n = 1$, the predicate $P(n)$ is not true for all n. Therefore, $\forall n\, P(n)$ is false. But then the claim that both $\exists n\, P(n)$ and $\forall n\, P(n)$ are true is false.

Quiz 4. *True/False*: If $P(x)$ is a predicate with nonempty universe $\mathcal{U}$, then there are values of x for which $P(x)$ is true, and there are values for which $P(x)$ is false.

Solution. The statement is *false*. If $P(x)$ is a predicate with universe $\mathcal{U}$, then one of the following must be true: $P(x)$ is true for all $x \in \mathcal{U}$, $P(x)$ is false for all $x \in \mathcal{U}$, or there are values $x, y \in \mathcal{U}$ such that $P(x)$ is true and $P(y)$ is false. Each possibility occurs. For instance, let the universe $\mathcal{U}$ be the set of real numbers. If $P(x)$ is the predicate $P(x)\colon x^2 \geq 0$, then $P(x)$ is true for all $x \in \mathcal{U}$. If $P(x)$ is the predicate $P(x)\colon x^2 < 0$, then $P(x)$ is false for all $x \in \mathcal{U}$. If $P(x)$ is the predicate $P(x)\colon x^2 > 1$, then $P(1)\colon 1^2 = 1 \not> 1$, i.e. $P(1)$ is false, while $P(2)\colon 2^2 = 4 > 1$ is true, i.e. $P(2)$ is true. But then it is not true that for a given predicate $P(x)$ nonempty universe $\mathcal{U}$, there are values of x for which $P(x)$ is true, and there are values for which $P(x)$ is false.

Quiz 5. *True/False*: Let $S = \{x \in \mathbb{Z}\colon (2x-1)(x+6) = 0\}$. The set S has infinitely many elements; in particular, the set S is nonempty.

Solution. The statement is *false*. Suppose that $s \in S$. Then $s \in \mathbb{Z}$ and $(2s-1)(s+6) = 0$. But this implies $2s - 1 = 0$ or $s + 6 = 0$, which in turn implies $s = \frac{1}{2}$ or $s = -6$. Because $s \in \mathbb{Z}$, we know that $s \neq \frac{1}{2}$. It must then be that if $s \in S$, $s = -6$. We can verify that $s \in S$: $-6 \in \mathbb{Z}$ and $(2 \cdot -6 - 1)(-6 + 6) = -5 \cdot 0 = 0$. This shows that $S = \{-6\}$; therefore, S is nonempty. However, clearly S is not infinite. Therefore, the statement of the quiz is false.

Quiz 6. *True/False*: If $S = \varnothing$, then $\mathcal{P}(S) = \varnothing$.

Solution. The statement is *false*. We know that for any set S, $\mathcal{P}(S)$ is the set of subsets of S. For any set S, $\varnothing \subseteq S$ and $S \subseteq S$. Therefore, $\{\varnothing, S\} \subseteq \mathcal{P}(S)$ for all sets S. But then we cannot have $\mathcal{P}(S) = \varnothing$. So the statement of the quiz is false. In fact, $\mathcal{P}(S) = \{\varnothing\}$.

Quiz 7. *True/False*: $\left(\bigcup_{n\in\mathbb{N}}[0,n)\right)^c = (-\infty,0)$

Solution. The statement is *true*. The union of a collection of sets is the set consisting of the elements in any of the sets in the collection. But then the union contains all of the elements of $[0,n)$ for all $n\in\mathbb{N}$. But then given a nonnegative real x, choose $n\in\mathbb{N}$ so that $x<n$. This shows $x\in[0,n)\subseteq\bigcup_{n\in\mathbb{N}}[0,n)$. If x is a negative real, then $x\notin[0,n)\subseteq\left(\bigcup_{n\in\mathbb{N}}[0,n)\right)$ for all $n\in\mathbb{N}$. But this shows that $\left(\bigcup_{n\in\mathbb{N}}[0,n)\right)=[0,\infty)$. Therefore, we have…

$$\left(\bigcup_{n\in\mathbb{N}}[0,n)\right)^c = [0,\infty)^c = (-\infty,0)$$

Alternatively, we have…

$$\left(\bigcup_{n\in\mathbb{N}}[0,n)\right)^c = \bigcap_{n\in\mathbb{N}}[0,n)^c = \bigcap_{n\in\mathbb{N}}\Big((-\infty,0)\cup[n,\infty)\Big) = (-\infty,0)\cup\bigcap_{n\in\mathbb{N}}[n,\infty)$$

Clearly, if x is a negative real, $x\notin[n,\infty)$ for all $n\in\mathbb{N}$, so that $x\notin\bigcap_{n\in\mathbb{N}}[n,\infty)$. If x were a nonnegative real in $\bigcap_{n\in\mathbb{N}}[n,\infty)$, then $x\in[n,\infty)$ for all $n\in\mathbb{N}$. But again, choosing $n\in\mathbb{N}$ with $x<n$ shows that $x\notin[n,\infty)$. But then $x\notin\bigcap_{n\in\mathbb{N}}[n,\infty)$. But this shows…

$$\left(\bigcup_{n\in\mathbb{N}}[0,n)\right)^c = (-\infty,0)\cup\bigcap_{n\in\mathbb{N}}[n,\infty) = (-\infty,0)\cup\varnothing = (-\infty,0)$$