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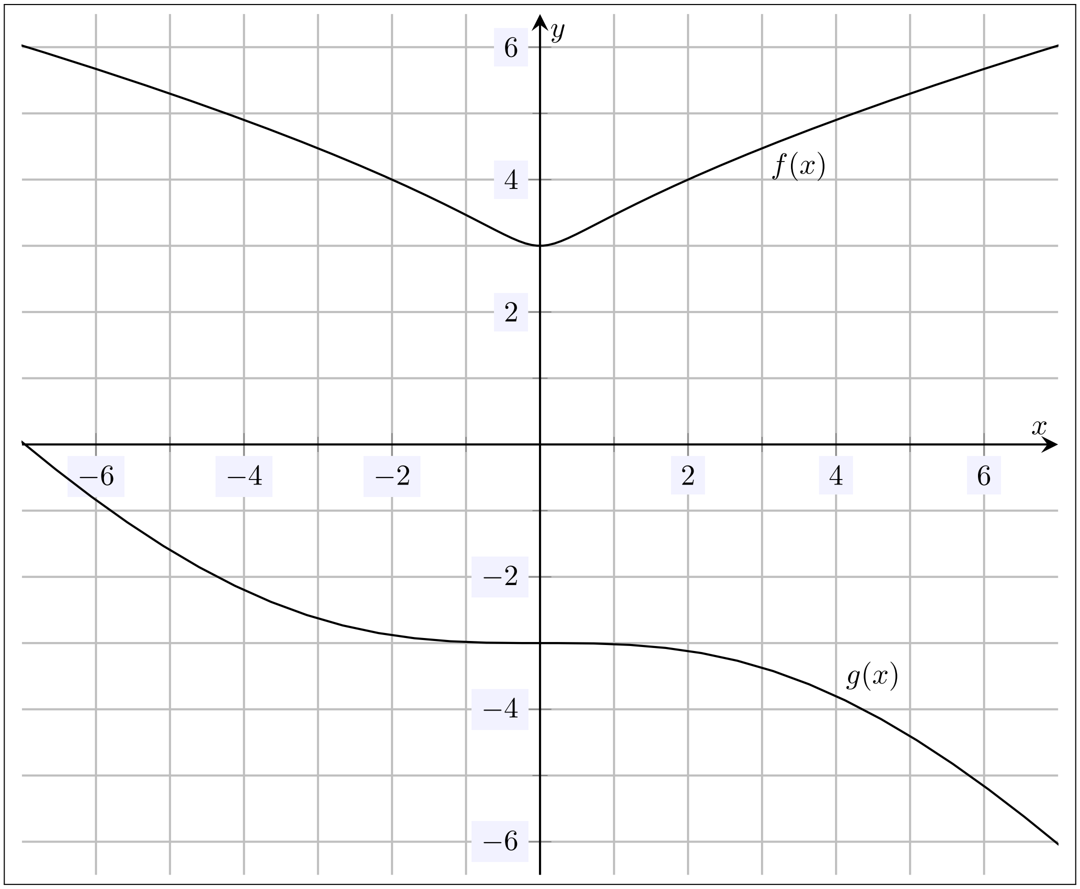
MATH 101
Fall 2021
HW 7: Due 10/08

"I'm pretty but tough, like a diamond or beef jerky in a ball gown."
–Titus Andromedon, Unbreakable Kimmy Schmidt

Problem 1. (10pt) Two functions $f(x)$ and $g(x)$ are plotted below. Are $f(x)$ and $g(x)$ functions? Explain. Do the functions $f(x)$ and $g(x)$ have an inverse? Explain.



Because $f(x)$ and $g(x)$ pass the vertical line test, they are both functions.

Because $f(x)$ fails the horizontal line test and $g(x)$ passes the horizontal line test, $f(x)$ does not have an inverse whereas $g(x)$ does have an inverse.

Problem 2. (10pt) Let $f(x) = 6x - 5$ and $g(x) = 2x^2 + 3x - 5$.

(a) What is $g(2)$?

(b) Assuming g^{-1} exists, what is $g^{-1}(9)$?

(c) Assuming f^{-1} exists, what is $f^{-1}(4)$?

Solution.

(a)

$$g(2) = 2(2^2) + 3(2) - 5 = 2(4) + 6 - 5 = 8 + 6 - 5 = 14 - 5 = 9$$

(b) Because $g(2) = 9$, if g^{-1} exists, then we must have $g^{-1}(9) = 2$ by the work from (a).

(c) Suppose that $f^{-1}(4) = x$, then $f(x) = 4$. But then

$$\begin{aligned} 6x - 5 &= 4 \\ 6x &= 9 \\ x &= \frac{9}{6} \\ x &= \frac{3}{2} \end{aligned}$$

Problem 3. (10pt) Do the points $(1,3)$, $(3,7)$, and $(5,1)$ lie along a line? Justify your answer.

Solution. *Lines have constant slope. Therefore, if we compute the slope between these points, all the slopes computed should be the same. If this is so, then the points lie along a line. If all the slopes computed are not the same, then the points do not lie along a line.*

$$m_1 = \frac{7-3}{3-1} = \frac{4}{2} = 2$$

$$m_2 = \frac{1-7}{5-3} = \frac{-6}{2} = -3$$

Because the slopes are not the same, the points do not lie along a line.

Problem 4. (10pt) Let $\ell(x)$ be the line through the points $(-2, 11)$ and $(3, -4)$.

(a) Find the slope of the line given by $\ell(x)$.

(b) Find the equation for $\ell(x)$.

(c) What is the y-intercept for $\ell(x)$?

(d) What is $\ell(-1)$?

Solution.

(a) We have. . .

$$m = \frac{-4 - 11}{3 - (-2)} = \frac{-4 - 11}{3 + 2} = \frac{-15}{5} = -3$$

(b) The line is not vertical so that we know the line 'looks like' $y = mx + b$. From (a), we know that $m = -3$. But then using the fact that $(-2, 11)$ is on the line, we have

$$\begin{aligned} y &= -3x + b \\ 11 &= -3(-2) + b \\ 11 &= 6 + b \\ b &= 5 \end{aligned}$$

Therefore, the equation of the line is $y = -3x + 5$, i.e. $y = 5 - 3x$. Using the 'ℓ' notation, we have $\ell(x) = -3x + 5$ or $\ell(x) = 5 - 3x$.

(c) The y-intercept occurs when the curve passes through the y-axis, i.e. when $x = 0$. But then $\ell(0) = -3(0) + 5 = 0 + 5 = 5$. Therefore, the y-intercept is $(0, 5)$.

(d) We have. . .

$$\ell(-1) = -3(-1) + 5 = 3 + 5 = 8$$

Problem 5. (10pt) Let $\ell(x)$ be the line through the point $(1,3)$ with slope $\frac{1}{2}$.

(a) Find the equation for $\ell(x)$.

(b) What is $\ell(4)$?

(c) Find the x-intercept for $\ell(x)$.

Solution.

(a) *We know that the line is not vertical so that the line 'looks like' $y = mx + b$. We know the slope is $m = \frac{1}{2}$. Then we know $y = \frac{1}{2}x + b$. But the line contains the point $(1,3)$, so that*

$$\begin{aligned} y &= \frac{1}{2}x + b \\ 3 &= \frac{1}{2} \cdot 1 + b \\ 3 &= \frac{1}{2} + b \\ b &= 3 - \frac{1}{2} \\ b &= \frac{6}{2} - \frac{1}{2} \\ b &= \frac{5}{2} \end{aligned}$$

Therefore, $y = \frac{1}{2}x + \frac{5}{2}$ or equivalently $y = \frac{x+5}{2}$. Using the 'ℓ' notation, we have $\ell(x) = \frac{1}{2}x + \frac{5}{2}$ or equivalently $\ell(x) = \frac{x+5}{2}$.

(b) *We have...*

$$\ell(4) = \frac{1}{2} \cdot 4 + \frac{5}{2} = 2 + \frac{5}{2} = \frac{4}{2} + \frac{5}{2} = \frac{9}{2}$$

(c) *The x-intercept is when the curve passes through the x-axis, i.e. when $y = 0$. But then we have*

$$\begin{aligned} \frac{1}{2}x + \frac{5}{2} &= 0 \\ \frac{1}{2}x &= -\frac{5}{2} \\ x &= -5 \end{aligned}$$

Therefore, the x-intercept is $(-5, 0)$.

Problem 6. (10pt) Determine if the following pairs of lines are parallel, perpendicular, or neither.

(a) $y = 5x, \quad \frac{1}{5}x + y = 8$

(b) $x - 3y = 12, \quad y = x + 7$

(c) $y = 3x - 1, \quad 6x - 2y = 4$

Solution.

(a) The slope of $y = 5x$ is $m = 5$. For the second line, we solve for y: $y = 8 - \frac{1}{5}x$. Then the slope of this line is $m = -\frac{1}{5}$. Because $-\frac{1}{5}$ is the negative reciprocal of 5, the lines are perpendicular.

(b) Solving for y in the first line, we have $y = \frac{1}{3}x - 4$. Then this line has slope $m = \frac{1}{3}$. The slope of the line $y = x + 7$ is $m = 1$. Now $\frac{1}{3} \neq 1$ so that the lines cannot be parallel. But $\frac{1}{3}$ is not the negative reciprocal of 1 so that the lines are not perpendicular. Therefore, the lines are neither parallel nor perpendicular.

(c) The line $y = 3x - 1$ has slope $m = 3$. Solving for y in $6x - 2y = 4$, we have $y = 3x - 2$. This line has slope $m = 3$. Then the lines have the same slope. Therefore, the lines are parallel.

Problem 7. (10pt) Find the equation of the line passing through the point $(1, -1)$ that is perpendicular to the line $y = \frac{1}{3}x - 8$.

Solution. *Because the line is not vertical, we know that the line 'looks like'* $y = mx + b$. *Because our line is perpendicular to the line* $y = \frac{1}{3}x - 8$, *the slope of our line must be the negative reciprocal of the slope for the line* $y = \frac{1}{3}x - 8$. *The slope of the line* $y = \frac{1}{3}x - 8$ *is* $\frac{1}{3}$. *Therefore, the slope of our line is* $-\frac{3}{1} = -3$. *Then we know that* $y = -3x + b$. *But the point* $(1, -1)$ *is on our line so that*

$$\begin{aligned} y &= -3x + b \\ -1 &= -3(1) + b \\ -1 &= -3 + b \\ b &= 2 \end{aligned}$$

Therefore, the equation of the line is $y = -3x + 2$.

Problem 8. (10pt) Let $f(x) = 2x - 1$. Find $f^{-1}(x)$. Show that $f^{-1}(x)$ is the inverse by showing $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Solution. *To find f^{-1}, we interchange the role of x and y in $f(x)$ and then solve for y. So we have...*

$$\begin{aligned} x &= 2y - 1 \\ x + 1 &= 2y \\ y &= \frac{x+1}{2} \end{aligned}$$

Therefore, $f^{-1}(x) = \dfrac{x+1}{2}$.

Now we check that $f^{-1}(x)$ is indeed the inverse for $f(x)$ by checking that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$:

$$f(f^{-1}(x)) = f\left(\frac{x+1}{2}\right) = 2\left(\frac{x+1}{2}\right) - 1 = (x+1) - 1 = x$$

$$f^{-1}(f(x)) = f^{-1}(2x-1) = \frac{(2x-1)+1}{2} = \frac{2x}{2} = x$$

Problem 9. (10pt) A cable internet company offers a high-speed internet package that costs \$62 per month, plus an additional \$85 installation fee.

(a) Find a function that represents the total cost of purchasing internet from this company after n months.

(b) What does the y-intercept for this function represent?

(c) Find the total cost of the internet after 14 months.

(d) How many months of internet can you get for \$500?

Solution.

(a) One must first pay the \$85 installation fee. Then after one month, you pay an additional \$62. After two months, you pay an additional $\$62(2) = \$124$. *Etc. So after* n *months, you pay an additional* $\$62n$. *Therefore, in total, one pays* $C(n) = 62n + 85$.

(b) The y*-intercept is where the function* $C(n) = 62n + 85$ *passes through the* y*-axis, i.e. where* $x = 0$. *But then we have* $C(0) = 62(0) + 85 = 85$. *This is the installation cost. Therefore, the* y*-intercept represents the initial cost of the internet, i.e. the installation cost.*

(c) The total cost is...

$$C(14) = 62(14) + 85 = 868 + 85 = \$953$$

(d) We want to know when $C(n) = \$500$. *But then we have...*

$$\begin{aligned} C(n) &= 500 \\ 62n + 85 &= 500 \\ 62n &= 415 \\ n &= 6.69 \end{aligned}$$

Because the number of months must be an integer, it must be either 6 or 7 months. But because 7 months would cost even more than the \$500, it must be that \$500 can only purchase 6 months of internet service (including the installation).