Quiz 1. *True/False*: The following is a truth table for $P \to Q$:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	F
F	F	F

Solution. The statement is *false*. The correct truth table should be...

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

One way to think about this is as follows: imagine P is a guarantee. Namely, we promise that if P happens, Q must happen. For instance, P could represent the statement, "You do not tamper with your hardware," and Q could be the statement, "I will replace your broken computer." So $P \to Q$ is then the statement, "If you do not tamper with your hardware, then I will replace your broken computer." If both P and Q are true, then this should be true—because I promised to replace the computer if you left it alone. If P is true and Q is false, then the statement should be false because I broke my promise. However, my promise holds true whenever P is false. Why? Because you broke our agreement by tampering with the hardware. So while I may or may not replace the computer, my promise has not been broken in either case, i.e. it remains true. In an implication $P \to Q$, if P is false, then the statement $P \to Q$ is *always* true.

Quiz 2. *True/False*: $\forall x, \exists y,\ x^2 + y = 4$

Solution. The statement is *true*. The statement says that for all x there is a y such that $x^2 + y = 4$. If this is true (which it is), we need to prove it. Fix an x, say x_0. We need to find a y such that $x_0^2 + y = 4$. Define $y_0 := 4 - x_0^2$. But then we have

$$x_0^2 + y_0 = x_0^2 + (4 - x_0^2) = 4,$$

as desired.

Quiz 3. *True/False*: $\neg\left(\forall x, \exists y, P(x,y) \vee \neg Q(x,y)\right) = \exists x, \forall y, \neg P(x,y) \wedge Q(x,y)$

Solution. The statement is *true*. We can simply compute the negation step-by-step:

$$\begin{aligned}
\neg\left(\forall x, \exists y, P(x,y) \vee \neg Q(x,y)\right) &\equiv \exists x, \neg(\exists y, P(x,y) \vee \neg Q(x,y)) \\
&\equiv \exists x, \forall y, \neg(P(x,y) \vee \neg Q(x,y)) \\
&\equiv \exists x, \forall y, \neg P(x,y) \wedge \neg(\neg Q(x,y)) \\
&\equiv \exists x, \forall y, \neg P(x,y) \wedge Q(x,y)
\end{aligned}$$

Quiz 4. *True/False*: To prove $P \Rightarrow Q$, you can prove $Q \Rightarrow P$.

Solution. The statement is *false*. The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$. The converse of a logical statement is not necessarily logically equivalent to the original statement. So proving the converse does not necessarily prove the original statement. However, the contrapositive of $P \Rightarrow Q$, which is $\neg Q \Rightarrow \neg P$, is logically equivalent to $P \Rightarrow Q$. Therefore, to prove $P \Rightarrow Q$, one only need prove $\neg Q \Rightarrow \neg P$. This is called proof by contrapositive.

Quiz 5. *True/False*: Let $A = \{1\}$ and $B = \{3, \{1\}\}$. Then $A \subseteq B$.

Solution. The statement is *false*. Recall that $A \subseteq B$ if every element of A is an element of B. The only element of A is the element 1. However, $1 \notin B$, but rather $\{1\} \in B$, i.e. 1 is not in B but the set consisting of only the element of 1 is in B. However, note that $A \in B$ because $A = \{1\}$ and $\{1\} \in B$.

Quiz 6. *True/False*: Take the universal set to be the integers. Then the following two sets are equal:

$$A = \{n \colon n \text{ odd}\}$$
$$B = \{m \colon m \text{ prime and } m > 2\}$$

Solution. The statement is *false*. We know that $9 \in A$ because 9 is odd. But $9 \notin B$ because $9 = 3 \cdot 3$ is not prime. Therefore, $A \not\subseteq B$ so that $A \neq B$.

Quiz 7. *True/False*: The sets $A \times B \times C$ and $(A \times B) \times C$ are not the same.

Solution. The statement is *true*. Elements in $A \times B \times C$ 'look like' (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. Whereas elements in $(A \times B) \times C$ 'look like' $((a, b), c)$, where $a \in A$, $b \in B$, and $c \in C$. Because elements in these sets are not of the same form, they cannot be the same. As an explicit example, take $A = \{1\}$, $B = \{2, 3\}$, and $C = \{4\}$. Then

$$A \times B \times C = \{(1, 2, 4), (1, 3, 4)\}$$
$$(A \times B) \times C = \{((1, 2), 4), ((1, 3), 4)\}$$

Then $A \times B \times C \neq (A \times B) \times C$.

Quiz 8. *True/False*: There is a set S such that $\mathcal{P}(S)$ has 3 elements.

Solution. The statement is *false*. If S is an infinite set, then clearly there is a subset for each element $s \in S$, i.e. the subset $\{s\}$. Clearly, if there is such a set, it cannot be infinite. Now if S had 3 or more elements—having a subset for each element of S—we know that $\mathcal{P}(S)$ would have more than 3 subsets. Therefore, S must have 0, 1, or 2 elements. If $S = \emptyset$, then $\mathcal{P}(S) = \{\emptyset\}$. If $S = \{s_1\}$, then $\mathcal{P}(S) = \{\emptyset, \{s_1\}\}$. Finally, if $S = \{s_1, s_2\}$, then $\mathcal{P}(S) = \{\emptyset, \{s_1\}, \{s_2\}, S\}$. Therefore, there cannot be such a set S.

Quiz 9. *True/False*: The Principle of Induction is logically equivalent to the Well-Ordering Principle.

Solution. The statement is *true*. We saw in class that the Well-Ordering Principle implied the Principle of Induction. From the homework, we know that the Principle of Induction implies the Well-Ordering Principle.

Quiz 10. *True/False*: If $P(n)$ is a proposition for each $n \in \mathbb{N}$ and $P(1), P(2), P(3), \ldots, P(k)$ are all true, then $P(n)$ is true for all $n \geq 1$.

Solution. The statement is *false*. These are only base cases. For induction to imply that $P(n)$ is true for all $n \in \mathbb{N}$, we need $P(k)$ being true to imply $P(k+1)$ is true. A statement can be true for *many* n and not be true for all n. For instance, the polynomial $p(n) = n^2 - n + 41$ is prime for $n = 1, 2, \ldots, 40$ but not for $n = 41$. In fact, a statement can be true for all but one n!

Quiz 11. *True/False*: If $f : A \to \mathbb{R}$ is positive and $g : A \to \mathbb{R}$ is nonnegative, then $fg : A \to \mathbb{R}$ is positive.

Solution. The statement is *false*. It is possible. For instance, $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := x^2 + 1$ and $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = |x| + 1$ so that $fg = (x^2+1)(|x|+1)$. However, because g is only nonnegative, it can take on the value zero. But then for these values, fg is zero and hence not positive. For instance, let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^2 + 1$ and $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) := |x|$. Then $(fg)(0) = (0^2+1)(|0|) = 0 \not> 0$ so that fg is not positive.

Quiz 12. *True/False*: The sets $\mathbb{Z}$ and $\mathbb{Q}$ have the same cardinality.

Solution. The statement is *true*. We saw this via the diagonalization argument given in class. Alternatively, we know that $\mathbb{Z}$ and $\mathbb{Q}$ are both countably infinite; therefore, there must be a bijection between $\mathbb{Z}$ and $\mathbb{Q}$ so that they must have the same cardinality. We could also use the following approach: the set $\mathbb{Z}$ is countably infinite, so there exists a bijection $f : \mathbb{N} \to \mathbb{Z}$. The set $\mathbb{Q}$ is countably infinite, so there exists a bijection $g : \mathbb{N} \to \mathbb{Q}$. Because f, g are bijections, f^{-1}, g^{-1} and are bijections (because they too have inverses, namely f, g, respectively). But as the composition of bijective functions are bijective, we know that $g \circ f^{-1} : \mathbb{Z} \to \mathbb{Q}$ is a bijection. Therefore, $\mathbb{Z}$ and $\mathbb{Q}$ have the same cardinality.

As another proof, by the Cantor-Schröder-Bernstein Theorem to prove there exists a bijection from $\mathbb{Z}$ to $\mathbb{Q}$, it suffices to prove there are injections $f : \mathbb{Z} \to \mathbb{Q}$ and $g : \mathbb{Q} \to \mathbb{Z}$. Let $f : \mathbb{Z} \to \mathbb{Q}$ be given by $f(x) := x$, i.e. taking advantage of the fact that $\mathbb{Z} \subseteq \mathbb{Q}$. Clearly, f is injective: if $x = f(x) = f(y) = y$, then $x = y$. Now define $g : \mathbb{Q} \to \mathbb{Z}$ be given as follows: if $q \in \mathbb{Q}$, write $q = a/b$ for some $a, b \in \mathbb{Z}$. Without loss of generality, assume that $\gcd(a, b) = 1$ and either $a, b \geq 0$ or $a < 0$ and $b \geq 0$; that is, assume a, b are relatively prime and that if $q \geq 0$, then a_1, b_1 are chosen to be nonnegative and if $q < 0$, then a is chosen to be negative while b is chosen to be nonnegative. Then define $g : \mathbb{Q} \to \mathbb{Z}$ via

$$g(q) = \begin{cases} 2^a 3^b, & q \geq 0 \\ -2^{-a} 3^b, & q < 0 \end{cases}$$

It is clear that if $q \geq 0$, then $g(q) \in \mathbb{Z}$. If $q = a/b < 0$, then $a < 0$ so that $-a > 0$. But then $-2^{-a} 3^b \in \mathbb{Z}$ so that $g(q) \in \mathbb{Z}$. Note that $g(q) \notin \{\pm 1\}$ because this would require $a = b = 0$, but because $q = a/b$, we know $b \neq 0$.

We claim that g is injective. Suppose that $g(q_1) = g(q_2)$, where $q_1, q_2 \in \mathbb{Q}$ with $q_1 = a_1/b_1$, $q_2 = a_2/b_2$ and $a_1, b_1, a_2, b_2 \in \mathbb{Z}$ are chosen as above. Obviously, $g(q_1)$ and $g(q_2)$ must have the same sign. By cancelling negatives, we may assume without loss of generality that $q_1, q_2 \geq 0$. But then $g(q_1) = 2^{a_1} 3^{b_1} = 2^{a_2} 3^{b_2} = g(q_2)$. By the uniqueness of factorization for integers, the number of factors of 2 and 3 on the left and right side of the equality must be the same, respectively. But then $a_1 = a_2$ and $b_1 = b_2$. But then $q_1 = a_1/b_1 = a_2/b_2 = q_2$ so that g is injective.

Quiz 13. *True/False*: The relation on $\mathbb{N}$ given by $x \sim y$ if and only if xy is even is an equivalence relation.

Solution. The statement is *false*. For $\sim$ to be an equivalence relation, $\sim$ must be reflexive, i.e. $n \sim n$ for all $n \in \mathbb{N}$. Take $n = 1$. Then $1(1) = 1$ is odd so that $1 \not\sim 1$. But then $\sim$ is not reflexive.

Quiz 14. *True/False*: Suppose that X is a set of natural numbers and $\sim$ is an equivalence relation on X . If $[2] \cap [5] = \{1, 2, 3, 4, 5, 7\}$, then $[2] = \{1, 2, 3, 4, 5, 7\}$.

Solution. If $(X, \sim)$ is an equivalence relation, then all equivalence classes are either disjoint or equal, i.e. if $[a], [b]$ are equivalence classes, then either $[a] \cap [b] = \emptyset$ or $[a] = [b]$. Observe that $[2] \cap [5] \neq \emptyset$. But then $[2] = [5]$. Therefore,

$$[2] = [2] \cap [2] = [2] \cap [5] = \{1, 2, 3, 4, 5, 7\}$$