"I go. You stay. No following." —Iron Giant, Iron Giant

**Problem 1.** (10pt) Watch the following three videos by 3Blue1Brown (Grant Sanderson):

- (i) Linear transformations and matrices
- (ii) Matrix multiplication as composition
- (iii) The determinant

What did you learn from these videos?

Answers will vary.

**Problem 2.** (10pt) Suppose the reduced-row echelon form for an augmented matrix is the following:

$$\begin{pmatrix}
1 & -4 & 0 & 1 & 0 & 5 \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 3
\end{pmatrix}$$

Using this, find all the solutions to the system of equations.

**Solution.** The last row of the matrix tells us that  $x_5 = 3$ . The second row of the matrix tells us that  $x_3 = -1$ . The first row of the matrix tells us that  $x_1 - 4x_2 + x_4 = 5$ . Fixing any two of  $x_1, x_2, x_4$ , we can solve for the third. Therefore, we have two free variables. For instance, choosing  $x_2$  and  $x_4$  to be free, we have solutions

$$\begin{cases} x_1 = 4x_2 - x_4 + 5 \\ x_2 : \text{free} \\ x_3 = -1 \\ x_4 : \text{free} \\ x_5 = 3 \end{cases}$$

**Problem 3.** (10pt) Can you compute the following product of matrices? If you can, compute the product. If you can not, explain why.

$$\begin{pmatrix} 1 & -1 & 8 \\ 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & -6 \\ 7 & 7 \\ -8 & 0 \end{pmatrix}$$

**Solution.** The first matrix is  $2 \times 3$ . The second matrix is  $4 \times 2$ . For matrix multiplication to be defined, we need the number of columns of the first to equal the number of rows of the second. But  $3 \neq 4$  so that this matrix multiplication is not defined, i.e. we cannot compute this product.

Problem 4. (10pt) Showing all your work, compute the following:

$$\begin{pmatrix} 5 & 0 & 1 \\ -2 & -3 & 4 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Solution.

$$\begin{pmatrix} 5 & 0 & 1 \\ -2 & -3 & 4 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5(2) + 0(-1) + 1(1) \\ -2(2) + (-3)(-1) + 4(1) \\ 1(2) + (-1)(-1) + 1(1) \end{pmatrix}$$
$$= \begin{pmatrix} 10 + 0 + 1 \\ -4 + 3 + 4 \\ 2 + 1 + 1 \end{pmatrix}$$
$$= \begin{pmatrix} 11 \\ 3 \\ 4 \end{pmatrix}$$

**Problem 5.** (10pt) Showing all your work, compute the following:

$$\begin{pmatrix} 1 & -1 & 2 \\ -3 & 6 & 0 \end{pmatrix} \begin{pmatrix} 0 & 4 \\ -2 & 3 \\ 3 & -2 \end{pmatrix}$$

Solution.

$$\begin{pmatrix} 1 & -1 & 2 \\ -3 & 6 & 0 \end{pmatrix} \begin{pmatrix} 0 & 4 \\ -2 & 3 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 1(0) + (-1)(-2) + 2(3) & 1(4) + (-1)3 + 2(-2) \\ -3(0) + 6(-2) + 0(3) & -3(4) + 6(3) + 0(-2) \end{pmatrix}$$
$$= \begin{pmatrix} 0 + 2 + 6 & 4 - 3 - 4 \\ 0 - 12 + 0 & -12 + 18 + 0 \end{pmatrix}$$
$$= \begin{pmatrix} 8 & -3 \\ -12 & 6 \end{pmatrix}$$

**Problem 6.** (10pt) Compute the determinant of the following matrix:

$$\begin{pmatrix} 2 & 1 & 5 \\ -3 & 0 & 3 \\ 7 & 2 & 7 \end{pmatrix}$$

Is this matrix invertible? Explain.

Solution.

$$\det\begin{pmatrix} 2 & 1 & 5 \\ -3 & 0 & 3 \\ 7 & 2 & 7 \end{pmatrix} = -(-3) \begin{vmatrix} 1 & 5 \\ 2 & 7 \end{vmatrix} + 0 \begin{vmatrix} 2 & 5 \\ 7 & 7 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 7 & 2 \end{vmatrix}$$

$$= 3(1(7) - 2(5)) + 0 - 3(2(2) - 7(1))$$

$$= 3(7 - 10) - 3(4 - 7)$$

$$= 3(-3) - 3(-3)$$

$$= -9 - (-9)$$

$$= -9 + 9$$

$$= 0$$

Because the determinant of the matrix is 0, the matrix cannot be invertible.

**Problem 7.** (10pt) Find the inverse of the following matrix:

$$\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$$

Solution.

$$\begin{pmatrix} 2 & -1 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{pmatrix} \qquad 3R_1 + 2R_2 \to R_2$$

$$\begin{pmatrix} 2 & -1 & 1 & 0 \\ 0 & 5 & 3 & 2 \end{pmatrix} \qquad \frac{1}{5}R_2 \to R_2$$

$$\begin{pmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \end{pmatrix} \qquad R_2 + R_1 \to R_1$$

$$\begin{pmatrix} 2 & 0 & \frac{8}{5} & \frac{2}{5} \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \end{pmatrix} \qquad \frac{1}{2}R_1 \to R_1$$

$$\begin{pmatrix} 1 & 0 & \frac{4}{5} & \frac{1}{5} \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \end{pmatrix}$$

Therefore, the inverse is...

$$\begin{pmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$$

Alternatively, because the matrix is  $2 \times 2$ , we can use the shortcut method:

$$\det \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} = 2(4) - (-3)(-1) = 8 - 3 = 5$$

Then we have

$$\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}^{-1} = \frac{1}{5} \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$$