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MATH 101
Fall 2023
HW 13: Due 11/06

"Teachers open the door, but you must enter by yourself."
–Chinese Proverb

Problem 1. (10pt) Find the inverse of the linear function $\ell(x) = \frac{5}{6} - 8x$. Use this inverse function to solve the equation $\ell(x) = 10$.

Solution. We know that $\ell(x) = \frac{5}{6} - 8x$ is a non-constant linear function (because $m = -8 \neq 0$); therefore, $\ell(x)$ has an inverse. To find the inverse of $\ell(x) = \frac{5}{6} - 8x$, we interchange the 'role' of ℓ and x, and then we solve for ℓ. The resulting function will be the inverse of $\ell(x)$:

$$\ell = \frac{5}{6} - 8x \rightsquigarrow x = \frac{5}{6} - 8\ell$$

$$6x = 5 - 48\ell$$

$$6x - 5 = -48\ell$$

$$\ell = \frac{6x-5}{-48}$$

$$\ell = \frac{5-6x}{48}$$

Therefore, $\ell^{-1}(x) = \frac{5-6x}{48}$. We can even verify this:

$$(\ell^{-1} \circ \ell)(x) = \ell^{-1}\big(\ell(x)\big) = \ell^{-1}\left(\frac{5}{6} - 8x\right) = \frac{5 - 6 \cdot \left(\frac{5}{6} - 8x\right)}{48} = \frac{5 - 5 + 48x}{48} = \frac{48x}{48} = x$$

$$(\ell \circ \ell^{-1})(x) = \ell\big(\ell^{-1}(x)\big) = \ell\left(\frac{5-6x}{48}\right) = \frac{5}{6} - 8\left(\frac{5-6x}{48}\right) = \frac{5}{6} - \left(\frac{5-6x}{6}\right) = \frac{5}{6} - \frac{5}{6} + x = x$$

We can now use ℓ^{-1} to solve the equation $\ell(x) = 10$:

$$\ell(x) = 10$$

$$\ell^{-1}\big(\ell(x)\big) = \ell^{-1}(10)$$

$$x = \ell^{-1}(10)$$

$$x = \frac{5 - 6(10)}{48}$$

$$x = \frac{5-60}{48}$$

$$x = \frac{-54}{48}$$

$$x = -\frac{9}{8}$$

Problem 2. (10pt) Explain why the lines $\ell_1(x) = 8x + 3$ and $\ell_2(x) = 9 - 5x$ intersect. Find their point of intersection.

Solution. The slope of the line ℓ_1 is $m_1 = 8$ and the slope of the line ℓ_2 is $m_2 = -5$. Because $m_1 = 8 \neq -5 = m_2$, the lines cannot be parallel. Therefore, the lines must intersect. If the lines intersect at (x_0, y_0), then we know that $\ell_1(x_0) = y_0 = \ell_2(x_0)$. But then...

$$\ell_1(x_0) = \ell_2(x_0)$$

$$8x_3 + 3 = 9 - 5x_0$$

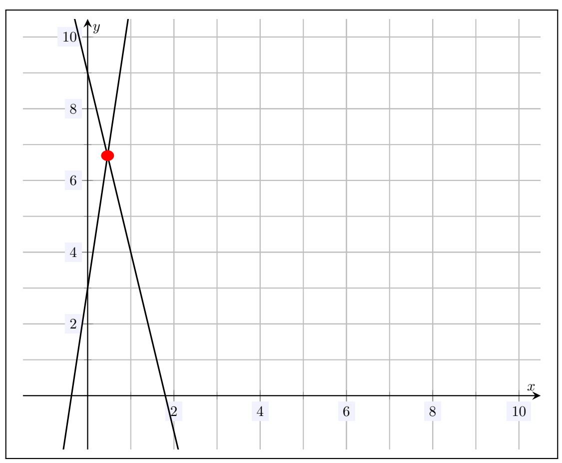
$$13x_0 = 6$$

$$x_0 = \frac{6}{13}$$

But then using this in the first line, we have...

$$\ell_1\left(\frac{6}{13}\right) = 8 \cdot \frac{6}{13} + 3 = \frac{48}{13} + 3 = \frac{48}{13} + \frac{39}{13} = \frac{87}{13}$$

Therefore, the lines intersect at the point $\left(\frac{6}{13}, \frac{87}{13}\right) \approx (0.461, 6.692)$.



Problem 3. (10pt) Find the line perpendicular to the line $y = 7 - \frac{2}{3}x$ that contains the x-intercept of the line $y = 7x + 3$.

Solution. Because the line $y = 7 - \frac{2}{3}x$ is not horizontal (because the slope is $-\frac{2}{3} \neq 0$), the line in question is not vertical; therefore, the line has the form $y = mx + b$ for some m, b. The line is perpendicular to the line $y = 7 - \frac{2}{3}x$. Perpendicular lines have negative reciprocal slopes. The slope of $y = 7 - \frac{2}{3}x$ is $-\frac{2}{3}$. Therefore, our line has slope $m = -\frac{1}{-\frac{2}{3}} = -(-\frac{3}{2}) = \frac{3}{2}$. Then we know $y = \frac{3}{2}x + b$.

The line contains the x-intercept of the line $y = 7x + 3$. The x-intercept is the point(s) where the curve intersects the x-axis, where $y = 0$. But then...

$$\begin{aligned} 0 &= 7x + 3 \\ 7x &= -3 \\ x &= -\frac{3}{7} \end{aligned}$$

Therefore, the x-intercept of $y = 7x+3$ is the point $(-\frac{3}{7}, 0)$. Therefore, the line in question contains the point $(-\frac{3}{7}, 0)$. But then $y = 0$ when $x = -\frac{3}{7}$, so that...

$$\begin{aligned} y &= \frac{3}{2}x + b \\ 0 &= \frac{3}{2} \cdot -\frac{3}{7} + b \\ 0 &= -\frac{9}{14} + b \\ b &= \frac{9}{14} \end{aligned}$$

Therefore, the line is...

$$y = \frac{3}{2}\,x + \frac{9}{14}$$

Problem 4. (10pt) Write down an expression that gives the equation for all linear functions passing through the point $(3, 5)$, then use this to find the line that passes through $(3, 5)$ and has x-intercept -6.

Solution. We know that the graph of a linear function is a line. Given a line with slope m that passes through a point (x_0, y_0), we know that the equation of the line is $y = y_0 + m(x - x_0)$. Because the line contains the point $(3, 5)$, we know that the linear function is $y = 5 + m(x - 3)$. Therefore, every linear function containing the point $(3, 5)$ must have the form $y = m(x - 3) + 5$ for some m.

If the line has x-intercept -6, then the line contains the point $(-6, 0)$—namely, the x-intercept. But then the line contains the point $(3, 5)$ and the point $(-6, 0)$. Therefore, the slope is...

$$m = \frac{\Delta y}{\Delta x} = \frac{5 - 0}{3 - (-6)} = \frac{5}{9}$$

Therefore, the linear function is...

$$y = m(x - 3) + 5$$

$$y = \frac{5}{9}(x - 3) + 5$$

$$y = \frac{5}{9}x - \frac{5}{9} \cdot 3 + 5$$

$$y = \frac{5}{9}x - \frac{5}{3} + 5$$

$$y = \frac{5}{9}x - \frac{5}{3} + \frac{15}{3}$$

$$y = \frac{5}{9}x + \frac{10}{3}$$