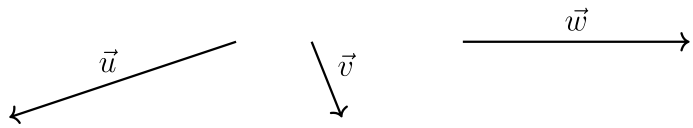
MAT 107: Exam 2
Winter – 2022
01/16/2023
Time Limit: '∞'

Name: *Caleb M^{c}Whorter — Solutions*

Write your name on the appropriate line on the exam cover sheet. This exam contains 11 pages (including this cover page) and 10 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) Define the following vectors:



As accurately as possible, sketch the following:

(a) $\vec{u}+\vec{w}$

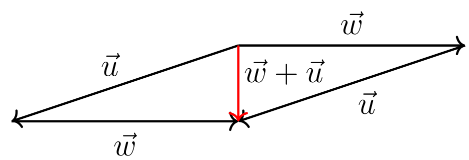
(b) $\vec{v}-\vec{u}$

(c) $-\frac{1}{2}\,\vec{w}$

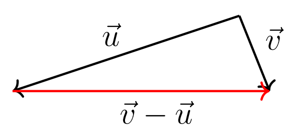
(d) $\vec{u}+2\vec{v}$

Solution.

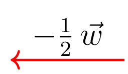
(a)



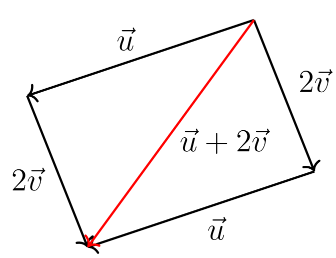
(b)



(c)



(d)



2. (10 points) Let $\vec{u} = \langle 2, -1, 0\rangle$ and $\vec{v} = \hat{\mathbf{i}} + 4\hat{\mathbf{j}} - \hat{\mathbf{k}}$. Find the following:

   (a) $-3\vec{u}$

   (b) $\vec{v} - \vec{u}$

   (c) $3\vec{u} + 2\vec{v}$

   (d) $\vec{u} \cdot \vec{v}$

   (e) The angle between $\vec{u}$ and $\vec{v}$.

Solution. Note that $\vec{v} = \hat{\mathbf{i}} + 4\hat{\mathbf{j}} - \hat{\mathbf{k}} = \langle 1, 4, -1\rangle$.

(a)
$$-3\vec{u} = -3\langle 2, -1, 0\rangle = \langle -3(2), -3(-1), -3(0)\rangle = \langle -6, 3, 0\rangle$$

(b)
$$\vec{v} - \vec{u} = \langle 1, 4, -1\rangle - \langle 2, -1, 0\rangle = \langle 1-2, 4-(-1), -1-0\rangle = \langle -1, 5, -1\rangle$$

(c)
$$3\vec{u}+2\vec{v} = 3\langle 2, -1, 0\rangle + 2\langle 1, 4, -1\rangle = \langle 6, -3, 0\rangle + \langle 2, 8, -2\rangle = \langle 6+2, -3+8, 0+(-2)\rangle = \langle 8, 5, -2\rangle$$

(d)
$$\vec{u} \cdot \vec{v} = \langle 2, -1, 0\rangle \cdot \langle 1, 4, -1\rangle = 2(1) + (-1)4 + 0(-1) = 2 - 4 + 0 = -2$$

(e) Recall that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$. We have...

$$\|\mathbf{u}\| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{4+1+0} = \sqrt{5}$$
$$\|\mathbf{v}\| = \sqrt{1^2 + 4^2 + (-1)^2} = \sqrt{1+16+1} = \sqrt{18}$$

We know from (d) that $\mathbf{u} \cdot \mathbf{v} = -2$. Therefore, we have...

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta \\
-2 &= \sqrt{5} \cdot \sqrt{18} \cos\theta \\
-2 &= 9.4868329805 \cos\theta \\
\cos\theta &= -0.21081851 \\
\theta &= \cos^{-1}(-0.21081851) \\
\theta &\approx 102.17^\circ
\end{aligned}$$

3. (10 points) Suppose you are a sprite in a 2D video game. Currently, you are at $\vec{p} = \langle 2.4, 3.7 \rangle$. You are moving in the direction given by $\langle -2, 1 \rangle$ at speed 1.6. Find your position one game 'tick' from now.

Solution. We have...

$$\|\langle -2, 1 \rangle\| = \sqrt{(-2)^2 + 1^2} = \sqrt{4+1} = \sqrt{5}$$

The velocity vector is then...

$$\vec{v} = 1.6 \cdot \frac{\langle -2, 1 \rangle}{\sqrt{5}} = 0.7155\langle -2, 1 \rangle = \langle -1.431, 0.7155 \rangle$$

Then we have...

$$\begin{aligned} \vec{p}_N &= \vec{p}_0 + \Delta\vec{p} \\ &= \vec{p}_0 + t\vec{v} \\ &= \langle 2.4, 3.7 \rangle + 1 \cdot \langle -1.431, 0.7155 \rangle \\ &= \langle 0.969, 4.4155 \rangle \end{aligned}$$

4. (10 points) Define the following:

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 1 \\ -6 & 1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -1 & 5 \\ 3 & 2 & 1 \\ -2 & 1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 4 & 0 \\ 2 & -1 & 5 \end{pmatrix}$$

Find the following:

(a) $4C$

(b) $A - B$

(c) CA

Solution.

(a)

$$4C = 4\begin{pmatrix} 1 & 4 & 0 \\ 2 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 4(1) & 4(4) & 4(0) \\ 4(2) & 4(-1) & 4(5) \end{pmatrix} = \begin{pmatrix} 4 & 16 & 0 \\ 8 & -4 & 20 \end{pmatrix}$$

(b)

$$A-B = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 1 \\ -6 & 1 & 3 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 5 \\ 3 & 2 & 1 \\ -2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1-0 & 0-(-1) & -2-5 \\ 0-3 & 4-2 & 1-1 \\ -6-(-2) & 1-1 & 3-0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 \\ -3 & 2 & 0 \\ -4 & 0 & 3 \end{pmatrix}$$

(c)

$$\begin{aligned} CA &= \begin{pmatrix} 1 & 4 & 0 \\ 2 & -1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 1 \\ -6 & 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1(1)+4(0)+0(-6) & 1(0)+4(4)+0(1) & 1(-2)+4(1)+0(3) \\ 2(1)+(-1)0+5(-6) & 2(0)+(-1)4+5(1) & 2(-2)+(-1)1+5(3) \end{pmatrix} \\ &= \begin{pmatrix} 1+0+0 & 0+16+0 & -2+4+0 \\ 2+0-30 & 0-4+5 & -4-1+15 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 16 & 2 \\ -28 & 1 & 10 \end{pmatrix} \end{aligned}$$

5. (10 points) Define $A = \begin{pmatrix} -4 & 1 \\ 2 & 6 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

   (a) Compute $A\vec{u}$.

   (b) Explain why you cannot compute $\vec{u}A$.

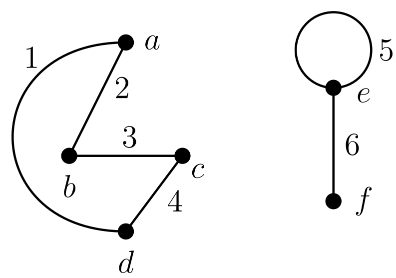
Solution.

(a)

$$\begin{aligned} A\vec{u} &= \begin{pmatrix} -4 & 1 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -4(-1) + 1(3) \\ 2(-1) + 6(3) \end{pmatrix} \\ &= \begin{pmatrix} 4+3 \\ -2+18 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ 16 \end{pmatrix} \end{aligned}$$

(b) To multiply an $m \times n$ matrix with a $r \times s$ matrix, we must have $n = r$. If so, the resulting product has dimension $m \times s$. The matrix A has dimension 2×2 and the matrix (vector) $\vec{u}$ has dimension 2×1. Because $1 \neq 2$, $\vec{u}$ and A have incompatible dimensions to compute $\vec{u}A$.

6. (10 points) Let G be the graph given below:



(a) What is adjacent to a?

(b) What is adjacent to 3?

(c) Are there parallel edges? Explain.

(d) Is the graph connected? Explain.

(e) Is the graph simple? Explain.

Solution.

(a) The vertices b and d are adjacent to a.

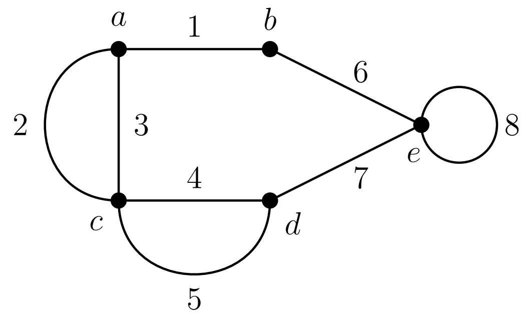
(b) The edges 2 and 4 are adjacent to 3.

(c) There are no parallel edges because there are no two edges that share *both* endpoints.

(d) The graph is not connected. For instance, there is no walk (sequence of edges) to take one from a to e.

(e) The graph is not simple because there is a loop at e—namely edge 5.

7. (10 points) Let G be the graph given below:



(a) Find the degree of the vertex c.

(b) Find the degree of the vertex e.

(c) Find the degree of the graph.

(d) Give an example of a trail from a to c that is not a path.

Solution.

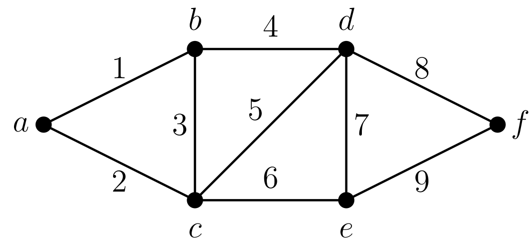
(a) $\deg(c) = 4$

(b) $\deg(e) = 4$

(c) $\deg G = 2 \cdot \#$ edges $= 2 \cdot 8 = 16$

(d) Any path that repeats a vertex but not an edge will work as an example. For instance, $a257613c$.

8. (10 points) Consider the graph below:



(a) Explain why the graph does not have an Euler circuit.

(b) Explain why the graph has an Euler trail.

(c) Find an Euler trial for this graph.

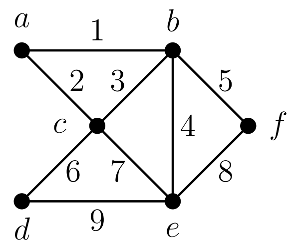
Solution.

(a) A connected graph has an Euler circuit if and only if every vertex has positive even degree. The graph is connected. Vertex b has degree 3, which is odd. So not all vertices have positive even degree. Therefore, the graph does not have an Euler circuit.

(b) A connected graph has an Euler trail if and only if there are exactly vertices with odd degree. Observe that every vertex in the graph has even degree except for vertex b (which has degree 3) and vertex c (which has degree 3).

(c) A possible Euler trail is $b123456789c$.

9. (10 points) Consider the graph below:



(a) Find an Euler circuit for this graph.

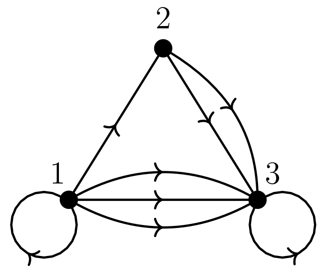
(b) Find a Hamiltonian circuit for this graph.

Solution.

(a) There are many possible Euler circuits for this graph. For instance, $a136945872a$.

(b) There are many possible Hamiltonian circuits for this graph. For instance, $a269851a$.

10. (10 points) Let G be the graph below:



(a) Find the adjacency matrix for G.

(b) Find the number of walks from 1 to 3 of length 2.

Solution.

(a) The adjacency matrix is...

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

(b)

$$\begin{aligned} A^2 &= AA \\ &= \begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1(1)+1(0)+3(0) & 1(1)+1(0)+3(0) & 1(3)+1(2)+3(1) \\ 0(1)+0(0)+2(0) & 0(1)+0(0)+2(0) & 0(3)+0(2)+2(1) \\ 0(1)+0(0)+1(0) & 0(1)+0(0)+1(0) & 0(3)+0(2)+1(1) \end{pmatrix} \\ &= \begin{pmatrix} 1+0+0 & 1+0+0 & 3+2+3 \\ 0+0+0 & 0+0+0 & 0+0+2 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 8 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Because $a_{13} = 8$, there are 8 walks of length 2 from 1 to 3.