MAT 101: Exam 1
Spring – 2022
03/10/2022
85 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 20 pages (including this cover page) and 20 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work.

Question	Points	Score
1	10	
2	8	
3	8	
4	8	
5	8	
6	8	
7	6	
8	8	
9	4	
10	8	
11	8	
12	8	
13	8	
14	8	
15	6	
16	8	
17	6	
18	6	
19	8	
20	8	
Total:	150	

1. (10 points) Mark each of the following statements as True (T) or False (F).

   (a) ___*T*___: $4^{100} + 4^{100} + 4^{100} + 4^{100} = 4^{101}$.

   (b) ___*F*___: The number 1 is a multiple of 12.

   (c) ___*T*___: The number 4 has three divisors.

   (d) ___*F*___: The number 1 is prime.

   (e) ___*T*___: Every integer greater than 1 is a product of prime numbers.

   (f) ___*F*___: For all real numbers x, we know $x^0 = 1$.

   (g) ___*T*___: There is no rational number equal to π.

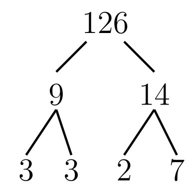
   (h) ___*F*___: The number $31.2 \cdot 10^5$ is in scientific notation.

   (i) ___*T*___: Two lines with positive slopes can never be perpendicular.

   (j) ___*F*___: The line $y = x + 6$ is parallel to the line $y = 7 - x$.

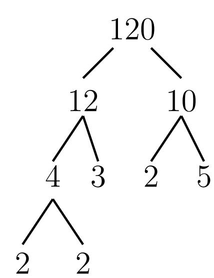
2. (8 points) Find the prime factorization for each of the following integers:

(a) $126 = 2 \cdot 3^2 \cdot 7$

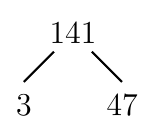


(b) $37 = 37^1$

(c) $120 = 2^3 \cdot 3 \cdot 5$



(d) $141 = 3 \cdot 47$



3. (8 points) Compute the following:

(a) $\gcd(28, 70) = \gcd(2^2 \cdot 7, 2 \cdot 5 \cdot 7) = 2 \cdot 7 = 14$

(b) $\operatorname{lcm}(28, 70) = \operatorname{lcm}(2^2 \cdot 7, 2 \cdot 5 \cdot 7) = 2^2 \cdot 5 \cdot 7 = 140$

(c) $\gcd(2^{500} \cdot 3^{98} \cdot 11^{82} \cdot 53^{17},\ 2^{200} \cdot 3^{50} \cdot 7^{60} \cdot 13^{300}) = 2^{200} \cdot 3^{50}$

(d) $\operatorname{lcm}(2^{500} \cdot 3^{98} \cdot 11^{82} \cdot 53^{17},\ 2^{200} \cdot 3^{50} \cdot 7^{60} \cdot 13^{300}) = 2^{500} \cdot 3^{98} \cdot 7^{60} \cdot 11^{82} \cdot 13^{300} \cdot 53^{17}$

4. (8 points) Showing all your work and simplifying as much as possible, compute the following:

(a) $\dfrac{5}{12} - \dfrac{3}{4} = \dfrac{5}{12} - \dfrac{9}{12} = -\dfrac{4}{12} = -\dfrac{1}{3}$

(b) $\dfrac{11}{6} + \dfrac{4}{15} = \dfrac{55}{30} + \dfrac{8}{30} = \dfrac{63}{30} = \dfrac{21}{10}$

(c) $\dfrac{12}{55} \cdot \dfrac{5}{6} = \dfrac{\cancel{12}^{2}}{\cancel{55}^{11}} \cdot \dfrac{\cancel{5}^{1}}{\cancel{6}^{1}} = \dfrac{2}{11}$

(d) $\dfrac{\frac{20}{21}}{\frac{8}{7}} = \dfrac{20}{21} \cdot \dfrac{7}{8} = \dfrac{\cancel{20}^{5}}{\cancel{21}^{3}} \cdot \dfrac{\cancel{7}^{1}}{\cancel{8}^{2}} = \dfrac{5}{6}$

5. (8 points) Showing all your work and being sure to use no negative powers, simplify the following as much as possible:

(a) $\left(\dfrac{x^3(x^2y^5)^0}{y^7}\right)^{-2} = \left(\dfrac{y^7}{x^3(x^2y^5)^0}\right)^2 = \dfrac{y^{14}}{x^6(x^2y^5)^0} = \dfrac{y^{14}}{x^6}$

(b) $\dfrac{(x^2y^3)^3}{x^{10}y^{-5}} = \dfrac{x^6y^9}{x^{10}y^{-5}} = \dfrac{x^6y^9y^5}{x^{10}} = \dfrac{\cancel{x^6}y^{14}}{x^{\cancel{10}4}} = \dfrac{y^{14}}{x^4}$

(c) $(\sqrt{x^5}\,y^{-3})^4 = (x^{5/2}y^{-3})^4 = x^{20/2}\,y^{-12} = \dfrac{x^{10}}{y^{12}}$

(d) $\left(\dfrac{x^6}{y^5}\right)^{-1/3} = \left(\dfrac{y^5}{x^6}\right)^{1/3} = \dfrac{y^{5/3}}{x^{6/3}} = \dfrac{\sqrt[3]{y^5}}{x^2}$

6. (8 points) Simplify the following radical expressions:

   (a) $\sqrt{36} = \sqrt{6^2} = 6$

   (b) $\sqrt[3]{64} = \sqrt[3]{4^3} = 4$

   (c) $\sqrt{2^5 \cdot 3^2 \cdot 5} = 2 \cdot 2 \cdot 3\sqrt{2 \cdot 5} = 12\sqrt{10}$

   (d) $\sqrt[4]{2^3 \cdot 3^9 \cdot 5^4} = 3 \cdot 3 \cdot 5\sqrt[4]{2^3 \cdot 3} = 45\sqrt[4]{24}$

7. (6 points) Showing all your work and simplifying as much as possible, rationalize the following:

(a) $\dfrac{4}{\sqrt{6}} = \dfrac{4}{\sqrt{6}} \cdot \dfrac{\sqrt{6}}{\sqrt{6}} = \dfrac{4\sqrt{6}}{6} = \dfrac{2\sqrt{6}}{3}$

(b) $\dfrac{1}{\sqrt[3]{7}} = \dfrac{1}{7^{1/3}} = \dfrac{1}{7^{1/3}} \cdot \dfrac{7^{2/3}}{7^{2/3}} = \dfrac{\sqrt[3]{7^2}}{7^{1/3+2/3}} = \dfrac{\sqrt[3]{49}}{7}$

(c) $\dfrac{1}{3-\sqrt{5}} = \dfrac{1}{3-\sqrt{5}} \cdot \dfrac{3+\sqrt{5}}{3+\sqrt{5}} = \dfrac{3+\sqrt{5}}{9+3\sqrt{5}-3\sqrt{5}-5} = \dfrac{3+\sqrt{5}}{9-5} = \dfrac{3+\sqrt{5}}{4}$

8. (8 points) Showing all your work, convert 5 km/s^2 to miles per square minute. Note that 1 km = 1000 m, 1 ft = 0.3048 m, and 60 s = 1 min.

$$\begin{array}{c|c|c|c|c} 5\text{ km} & 1000\text{ m} & 1\text{ ft} & 60\text{ s} & 60\text{ s} \\ \hline 1\text{ s}^2 & 1\text{ km} & 0.3048\text{ m} & 1\text{ min} & 1\text{ min}^2 \end{array} = 59055120.0\text{ mi/min}^2$$

9. (4 points) Complete the following:

   (a) Convert the following number in scientific notation to an ordinary decimal number: $-4.73 \cdot 10^{-6}$

   $$-4.73 \cdot 10^{-6} = -0.00000473$$

   (b) Convert the following ordinary decimal number to scientific notation: 0.054

   $$0.054 = 5.4 \cdot 10^{-2}$$

10. (8 points) Showing all your work, compute the following:

    (a) 97 decreased by 60%

    $$97(1 - 0.60) = 97(0.40) = 38.8$$

    (b) 573 increased by 142%

    $$573(1 + 1.42) = 573(2.42) = 1386.66$$

    (c) 71% of 140

    $$140(0.71) = 99.4$$

11. (8 points) Compute the following, being sure to show all your work and to write your answer in the form $a + bi$:

(a) $(3i)^3 = 3^3 \cdot i^3 = 27(-i) = -27i = 0 - 27i$

(b) $(6 + 4i) - (4 - 4i) = (6 - 4) + \big(4i - (-4i)\big) = 2 + 8i$

(c) $(1 + i)(3 - 2i) = 3 - 2i + 3i - 2i^2 = 3 + i - 2(-1) = 3 + 2 + i = 5 + i$

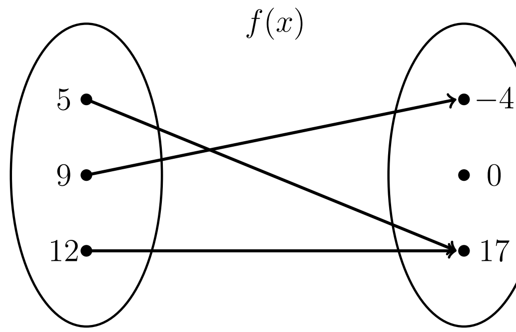
(d) $\dfrac{6+i}{1-i} = \dfrac{6+i}{1-i} \cdot \dfrac{1+i}{1+i} = \dfrac{6 + 6i + i + i^2}{1 + i - i - i^2} = \dfrac{6 + 7i - 1}{1 - (-1)} = \dfrac{5 + 7i}{2}$

12. (8 points) Suppose a student has a 90% participation average, 75% quiz average, 84% homework average, 86% on the midterm, and 66% on the final exam, given that their course grade is computed using the weights below, find their course average:

Participation:	5%
Quizzes:	15%
Homework:	45%
Midterm:	15%
Final:	20%

$$\begin{aligned}\text{Course Average} &= \sum \text{Grade Component} \cdot \text{Average (Decimal)} \\ &= 5(0.90) + 15(0.75) + 45(0.84) + 15(0.86) + 20(0.66) \\ &= 4.5 + 11.25 + 37.8 + 12.9 + 13.2 \\ &= 79.65\end{aligned}$$

13. (8 points) Answer the following:



(a) Explain why the relation $f(x)$ above is a function.

For each input, there is only one possible output, e.g. $f(5) = 17$*,* $f(9) = -4$*, and* $f(12) = 17$*.*

(b) Find the domain, codomain, and range of the function $f(x)$.

$$\text{Domain} = \{5, 9, 12\}$$

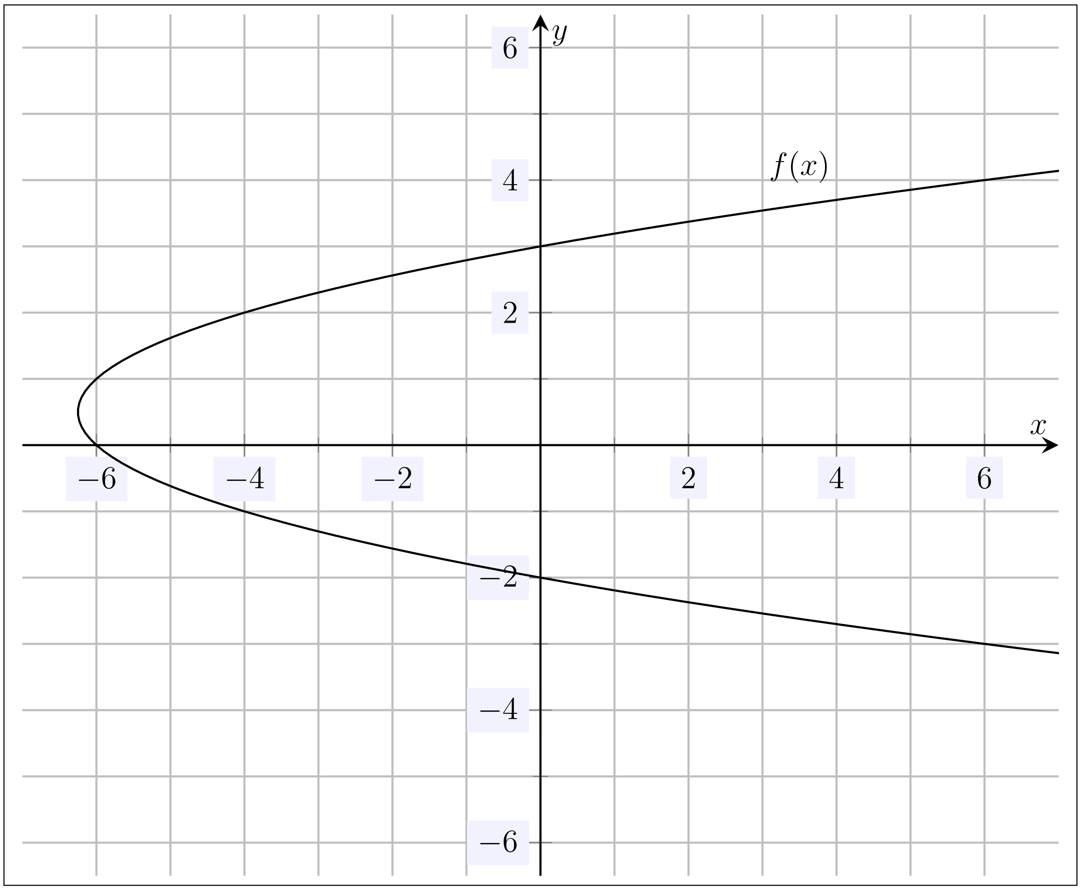
$$\text{Codomain} = \{-4, 0, 17\}$$

$$\text{Range} = \{-4, 17\}$$

(c) Is the relation $g(x) = 17x - x^3$ a function? Explain.

Yes, $g(x)$ *is a function. This is because for each input, there is only one possible output—namely, the one obtained by evaluating* $g(x)$ *at* x *and following order of operations.*

14. (8 points) Consider the relation $f(x)$ plotted below.



(a) Is the relation $f(x)$ plotted above a function? Explain.

No, $f(x)$ fails the vertical line test. For example, the input $x = 0$ has two possible outputs: -2 and 3.

(b) Does the relation above have an inverse function? Explain.

Yes, the relation $f(x)$ has an inverse function because it passes the horizontal line test, i.e. every horizontal line intersects the curve at most once.

(c) Find the y-intercepts of the relation plotted above.

The y-intercepts are $y = -2, 3$, or more precisely the points $(0, -2)$ and $(0, 3)$.

(d) Find the x-intercepts of the relation plotted above.

The only x-intercept is $x = -6$, or more precisely the point $(-6, 0)$.

15. (6 points) Consider the functions given in the table below.

x	-2	-1	0	1	2
$f(x)$	5	-2	-5	-3	2
$g(x)$	6	2	10	7	-5

Compute the following:

(a) $g(2) = -5$

(b) $(f-g)(0) = f(0) - g(0) = -5 - 10 = -15$

(c) $(fg)(1) = f(1)g(1) = -3 \cdot 7 = -21$

(d) $\left(\frac{g}{f}\right)(0) = \dfrac{g(0)}{f(0)} = \dfrac{10}{-5} = -2$

(e) $(f \circ g)(-1) = f(g(-1)) = f(2) = 2$

(f) $(g \circ f)(-1) = g(f(-1)) = g(-2) = 6$

16. (8 points) Consider the function $\ell(x) = 6 - 2x$.

(a) Is $\ell(x)$ linear? Explain.

Yes, $\ell(x)$ is linear because it has the form $y = mx + b$ with $y = \ell(x)$, $x = x$, $m = -2$, and $b = 6$.

(b) Find the slope of $\ell(x)$.

$$m = -2$$

(c) Find the y-intercept of $\ell(x)$.

$y = 6$ (or more precisely, $(0, 6)$)

(d) Is the point $(-1, 4)$ on the graph of $\ell(x)$? Explain.

$$\ell(-1) = 6 - 2(-1) = 6 - (-2) = 6 + 2 = 8$$

Because $\ell(-1) = 8 \neq 4$, the point $(-1, 4)$ is not on the graph of $\ell(x)$.

17. (6 points) Find the equation of the line perpendicular to $y = 6 - 2x$ at its x-intercept.

Because the line $y = 6 - 2x$ *is not horizontal, we know that the line in question is not vertical. Therefore, our line must have the form* $y = mx + b$. *The line* $y = 6 - 2x$ *has slope* $-2 = -\frac{2}{1}$. *Because our line is perpendicular to this line, it must have slope* $m = -\left(-\frac{1}{2}\right) = \frac{1}{2}$. *Then we know that* $y = \frac{1}{2}\,x + b$. *Our line contains the* x*-intercept of the line* $y = 6 - 2x$. *The* x*-intercept occurs when* $y = 0$ *so that we then have...*

$$0 = 6 - 2x$$

$$2x = 6$$

$$x = 3$$

Therefore, the x*-intercept of* $y = 6 - 2x$ *is the point* $(3, 0)$. *But then we know...*

$$y = \frac{1}{2}\,x + b$$

$$0 = \frac{1}{2} \cdot 3 + b$$

$$0 = \frac{3}{2} + b$$

$$b = -\frac{3}{2}$$

Therefore, the equation of the line is...

$$\boxed{y = \frac{1}{2}\,x - \frac{3}{2}}$$

18. (6 points) Solve the following equation and then verify your solution:

$$3x - 14 = 8 - \frac{2}{3}\,x$$

$$3x - 14 = 8 - \frac{2}{3}\,x$$

$$3(3x - 14) = 3(8 - \frac{2}{3}\,x)$$

$$9x - 42 = 24 - 2x$$

$$11x = 66$$

$$x = 6$$

$$3x - 14 = 8 - \frac{2}{3}\,x$$

$$3(6) - 14 \stackrel{?}{=} 8 - \frac{2}{3} \cdot 6$$

$$18 - 14 \stackrel{?}{=} 8 - 4$$

$$4 = 4$$

✓

19. (8 points) You are driving home from university at 55 mph. Your home is 650 miles from your university. Assuming you left the university 2 hours ago and that you drive at a constant speed, find your distance from your home, $D(t)$, as function of time t, in hours.

Let t denote the amount of time (in hours) from the 'current' hour and $D(t)$ be the distance (in miles) from home. Because you are driving at a constant rate of speed, we know that $D(t)$ is a linear function. Therefore, we must have $D(t) = mt + b$ for some m and b. The rate of change is the slope, m. Because you are driving at 55 mph and because the distance between you and your home is decreasing, we must have $m = -55$. Therefore, we know $D(t) = -55t + b$. We also know 2 hours ago that you were 650 miles from home, i.e. that the point $(-2, 650)$ is on the graph of $D(t)$. But then...

$$\begin{aligned} D(t) &= -55t + b \\ 650 &= -55(-2) + b \\ 650 &= 110 + b \\ b &= 540 \end{aligned}$$

Therefore, we have...

$$\boxed{D(t) = 540 - 55t}$$

Note: We could also use the fact that because you have driven two hours, you have traveled $55 \cdot 2 = 110$ miles. So currently, i.e. at time $t = 0$, you are $650 - 110 = 540$ miles from your home. But then $b = 540$.

Note: One could define $D(t)$ using the initial time, $t = 0$, to be the time at the start of the drive. Then because one's home is 650 miles from the school, we have $D(0) = 650$, which is the y-intercept b. We still have the rate of change being $m = -55$. But then $D(t) = 650 - 55t$.

20. (8 points) You rent an apartment in NYC, which you paid a $50 application fee to apply for. The rent is $2500/month. Therefore, the amount you have paid, $R(t)$, to rent the apartment t months after moving in is given by $R(t) = 2500t + 50$.

(a) Without knowing $R(t)$, how do you know that $R(t)$ is linear?

The rate of change of the amount of money you have paid is constant, i.e. the monthly rent is fixed at a rate of $2500/month. Therefore, $R(t)$ is linear.

(b) What is the slope of $R(t)$ and what does it represent in the problem context?

Because $R(t)$ is linear, we know that the slope is $m = 2500$. In the context of the problem, this corresponds to the monthly rent payment of $2,500.

(c) What is the y-intercept of $R(t)$ and what does it represent in the problem context?

Because $R(t)$ is linear, we know that the y-intercept is $b = 50$ (more precisely, (0, 50)). In the context of the problem, this represents that one initially pays $50 for the apartment, i.e. the y-intercept represents the application fee.