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**MATH 108**
**Spring 2023**
**HW 14: Due 05/01**

*"Of the many forms of false culture, a premature converse with abstractions is perhaps the most likely to prove fatal to the growth of a masculine vigour of intellect."*

*–George Boole*

**Problem 1.** (10pt) Consider the following system of equations:

$$\begin{aligned} 3x - 2y &= -8 \\ -x + 3y &= 5 \end{aligned}$$

(a) Find the coefficient matrix, $A$.

(b) Show that $A$ has an inverse.

(c) Use your answer from (b) to find the solution to the system of equations.

**Solution.**

(a) The coefficient matrix is the matrix of column-by-column coefficients for the variables—properly aligned. Because the variables are already aligned, we have. . .

$$A = \begin{pmatrix} 3 & -2 \\ -1 & 3 \end{pmatrix}$$

(b) We know that $A$ has an inverse, i.e. that $A^{-1}$ exists, if and only if $\det A \neq 0$. We have. . .

$$\det A = \det \begin{pmatrix} 3 & -2 \\ -1 & 3 \end{pmatrix} = 3(3) - (-2)(-1) = 9 - 2 = 7 \neq 0$$

Because $\det A \neq 0$, we know that $A^{-1}$ exists.

(c) Recall that that when written in vector form, i.e. $A\mathbf{x} = \mathbf{b}$, the matrix $A$ is the coefficient matrix (written column-by-column in the same order as the variable vector), $\mathbf{x}$ is the variable vector, and $\mathbf{b}$ is the constant vector. If $A^{-1}$ exists, multiplying both sides of $A\mathbf{x} = \mathbf{b}$ on the left by $A^{-1}$, we have. . .

$$A\mathbf{x} = \mathbf{b}$$

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

From (b), we know that $A^{-1}$ exists. We need to find $A^{-1}$. But we know how to find the inverse of a $2 \times 2$ matrix:

$$\text{If } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \det A \neq 0, \text{ then } A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

But then we have...

$$A^{-1} = \frac{1}{7}\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}$$

Therefore, we know...

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{7}\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -8 \\ 5 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3(-8) + 2(5) \\ 1(-8) + 3(5) \end{pmatrix} \frac{1}{7}\begin{pmatrix} -24 + 10 \\ -8 + 15 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -14 \\ 7 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Then the solution is...

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

That is, the solution is $x = -2$ and $y = 1$.

**Problem 2.** (10pt) The RREF form of a matrix coming from a system of equations is shown below. Determine if there is a solution. If so, find the solution(s). If not, explain why the system does not have a solution.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

**Solution.** Adding a dashed line to separate matrix entries representing coefficients and those representing constants in the RREF augmented matrix, we have...

$$\left(\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

We use variables $x_1$, $x_2$, $x_3$, $x_4$ for the four variables (one less than the number of columns because every column corresponds to a variable's coefficients except for the last column). Then we can write out the equation represented by each row in the matrix:

$$x_1 + 0x_2 + 0x_3 + 0x_4 = 0$$

$$0x_1 + 1x_2 + 0x_3 + 0x_4 = -4$$

$$0x_1 + 0x_2 + 1x_3 + 0x_4 = 6$$

$$0x_1 + 0x_2 + 0x_3 + 1x_4 = 4$$

But then we have...

$$\begin{cases} x_1 = 0 \\ x_2 = -4 \\ x_3 = 6 \\ x_4 = 4 \end{cases}$$

Therefore, there is a solution and the solution is unique, namely $(x_1, x_2, x_3, x_4) = (0, -4, 6, 4)$.

**Problem 3.** (10pt) The RREF form of a matrix coming from a system of equations is shown below. Determine if there is a solution. If so, find the solution(s). If not, explain why the system does not have a solution.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**Solution.** Adding a dashed line to separate matrix entries representing coefficients and those representing constants in the RREF augmented matrix, we have...

$$\left(\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We have circled the pivot positions. We use variables $x_1$, $x_2$, $x_3$, $x_4$ for the four variables (one less than the number of columns because every column corresponds to a variable's coefficients except for the last column). We first observe that the columns corresponding to $x_1$, $x_2$, and $x_4$ are pivot columns. The column corresponding to $x_3$ is not a pivot column and thus is naturally chosen as a free variable. We can write out the equation represented by each row in the matrix:

$$\begin{aligned} x_1 + 0x_2 + 0x_3 + 0x_4 &= 4 & &\Longrightarrow & x_1 &= 4 \\ 0x_1 + x_2 + 3x_3 + 0x_4 &= 5 & &\Longrightarrow & x_2 + 3x_3 &= 5 \\ 0x_1 + 0x_2 + 0x_3 + x_4 &= -2 & &\Longrightarrow & x_4 &= -2 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 & &\Longrightarrow & 0 &= 0 \end{aligned}$$

The equation $x_2 + 3x_3 = 5$ implies that $x_2 = 5 - 3x_3$. [We solve for $x_2$ in terms of $x_3$ because $x_3$ is naturally chosen as a free variable.] But then we have...

$$\begin{cases} x_1 = 4 \\ x_2 = 5 - 3x_3 \\ x_3 = \text{free} \\ x_4 = -2 \end{cases}$$

Therefore, there are infinitely many solutions and they are of the form $(x_1, x_2, x_3, x_4) = (4, 5 - 3x_3, x_3, -2)$. For instance, choosing $x_3 = 0$ yields the solution $(x_1, x_2, x_3, x_4) = (4, 5, 0, -2)$, while choosing $x_3 = -8$ yields the solution $(x_1, x_2, x_3, x_4) = (4, 29, -8, -2)$, etc.