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MATH 308
Fall 2022
HW 5: Due 09/22

"To choose one sock from each of infinitely many pairs of socks requires the Axiom of Choice, but for shoes the Axiom is not needed."
–Bertrand Russell

Problem 1. (10pt) For each of the sets described below, either give the set by enumerating all its elements (if possible) or give the set using set-builder notation. Also for each set, give an element and non-element of the set.

(a) The set of integer multiples of 8.

(b) The set of negative solutions to $(x-4)(x+1)(x+6)=0$.

(c) The set of nonnegative rational numbers less than 1.

(d) The set of real numbers with a real-valued square root.

(e) The set of integer cubes with absolute value less than 100.

Solution.

(a) The integer multiples of 8 can be constructed by...

$$\{n\colon (\exists k \in \mathbb{Z})(n = 8k)\} = \{8k\colon k \in \mathbb{Z}\}$$

(b) The set of negative solutions to $(x-4)(x+1)(x+6)=0$ can be constructed by

$$\{x \in \mathbb{R}\colon (x-4)(x+1)(x+6)=0, x<0\}$$

However, we can enumerate this set. If x is a solution to $(x-4)(x+1)(x+6)=0$, then $x-4=0$, $x+1=0$, or $x+6=0$. But this implies that $x=4$, $x=-1$, or $x=-6$, respectively, and one can easily verify that each are a solution. Therefore, the set of negative solutions to $(x-4)(x+1)(x+6)=0$ is...

$$\{-1,-6\}$$

(c) The set of nonnegative rational numbers less than 1 can be constructed by...

$$\{q \in \mathbb{Q}\colon 0 \leq q < 1\} = \{r \in \mathbb{R}\colon (\exists a)(\exists b)(a, b \in \mathbb{Z} \wedge b \neq 0 \wedge r = a/b) \wedge 0 \leq r < 1\}$$

(d) Let $r \in \mathbb{R}$. If $r<0$, then $\sqrt{r}$ is complex but not real, i.e. $\sqrt{r} \in \mathbb{C} \setminus \mathbb{R}$. However, if $r \geq 0$, then $\sqrt{r} \in \mathbb{R}$. Alternatively, $r \in \mathbb{R}$ has a real-valued square root if there is a real number whose square is r. Therefore, the set of real numbers with a real-valued square root can be constructed by...

$$\{r \in \mathbb{R}\colon r \geq 0\} = \{r \in \mathbb{R}\colon (\exists s \in \mathbb{R})(r = s^2)\}$$

(e) We know that $|k^3| < 100$ if and only if $-100 < k^3 < 100$ if and only if $-4.64159 \approx \sqrt[3]{-100} < k < \sqrt[3]{100} \approx 4.64159$. Then set of integer cubes with absolute value less than 100 can be constructed by...

$$\{n \in \mathbb{Z}\colon (\exists k \in \mathbb{Z})(n = k^3 \wedge |n| < 100)\} = \{n \in \mathbb{Z}\colon (\exists k \in \mathbb{Z})(n = k^3 \wedge -100 < n < 100)\} = \{k^3\colon k \in \mathbb{Z}, \sqrt[3]{-100} < k < \sqrt[3]{100}\}$$

However, we can enumerate this set. The only integers with $-4.64159 \approx \sqrt[3]{-100} < k < \sqrt[3]{100} \approx 4.64159$ are $k = -4, -3, -2, -1, 0, 1, 2, 3, 4$. The cube of these numbers are -64, -27, -8, -1, 0, 1, 8, 27, 64. Therefore, the set of integer cubes with absolute value less than 100 is $\{-64, -27, -8, -1, 0, 1, 8, 27, 64\}$.

Problem 2. (10pt) For each of the sets given below, describe the sets in words. Also for each set, give an example of an element and non-element of the set.

(a) $\{2, 3, 5, 7, 11, 13, \ldots\}$

(b) $\{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \ldots\}$

(c) $\{n \in \mathbb{N} \colon n^2 = 30 - n\}$

(d) $\{k \in \mathbb{Z} \colon (3k + 1)/5 \in \mathbb{Z}\}$

(e) $\{n \in \mathbb{N} \colon (\exists k \in \mathbb{N})(n = 3k + 1)\}$

Solution.

(a) The set $P := \{2, 3, 5, 7, 11, 13, \ldots\}$ is the set of prime numbers. Observe that $2 \in P$, $3 \in P$, $17 \in P$, $2\,760\,727\,302\,517 \in P$, and $2^{82\,589\,933} - 1 \in P$ but $1 \notin P$, $4 \notin P$, $6 \notin P$, and $493\,949\,595\,303 \notin P$.

(b) The set $T := \{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \ldots\}$ is the set of integer powers of 2. Observe that $1 \in T$, $2 \in T$, $\frac{1}{2} \in T$, $2^{253\,453} \in T$, and $\frac{1}{2^{642\,443}} \in T$ but $3 \notin T$, $0 \notin T$, $15 \notin T$, $-\frac{1}{2} \notin T$, and $-2 \notin T$.

(c) The set $S := \{n \in \mathbb{N} \colon n^2 = 30 - n\}$ is the set of natural number solutions to $n^2 = 30 - n$. Observe that if $x^2 = 30 - x$ then $x^2 + x - 30 = 0$. But as $(x + 6)(x - 5)$, this implies that $x = -6$ or $x = 5$. But then we know that $\{n \in \mathbb{N} \colon n^2 = 30 - n\} = \{5\}$. Observe that $5 \in S$ and $0 \notin S$, $18 \notin S$, and $-6 \notin S$.

(d) The set $D := \{k \in \mathbb{Z} \colon (3k+1)/5 \in \mathbb{Z}\}$ is the of integers k such that $(3k+1)/5$ is also an integer. Observe that $(3 \cdot -7 + 1)/5 = -4$, $(3 \cdot -2 + 1)/5 = -1$, $(3 \cdot 3 + 1)/5 = 2$, and $(3 \cdot 8 + 1)/5 = 5$, and also $(3 \cdot 0 + 1)/5 = \frac{1}{5}$, $(3 \cdot 7 + 1)/5 = \frac{22}{5}$, and $(3 \cdot -10 + 1)/5 = -\frac{29}{5}$. But then we have $-7 \in D$, $-2 \in D$, $3 \in D$, and $8 \in D$, and also $0 \notin D$, $7 \notin D$, and $-10 \notin D$.

(e) The set $M := \{n \in \mathbb{N} \colon (\exists k \in \mathbb{N})(n = 3k + 1)\}$ is the set of natural numbers that are one more than a multiple of 3. Observe that $3(1) + 1 = 4$, $3(2) + 1 = 7$, and $3(5) + 1 = 16$. But then $4 \in M$, $7 \in M$, and $16 \in M$, but $1 \notin M$, $5 \notin M$, and $18 \notin M$.

Problem 3. (10pt) Define the following sets:

$$\begin{aligned} A &= \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\} \\ B &= \{2,\ 4,\ 6,\ 8,\ 10\} \\ C &= \{1,\ 3,\ 5,\ 7,\ 9\} \\ D &= \{2,\ 3,\ 5,\ 7\} \\ E &= \{1,\ 2,\ 4,\ 8,\ 10\} \\ F &= \{3,\ 5,\ 8,\ 9,\ 10\} \end{aligned}$$

Consider each of the sets above as coming from the universal set $\mathcal{U} := A$. Compute the following:

(a) D^c

(b) $B \cup C$

(c) $C \cup (B \cap D)$

(d) $E \setminus F$

(e) $E \Delta F$

(f) $(B \cup C)^c$

Solution.

(a)

$$D^c = \{1,\ 4,\ 6,\ 8,\ 9,\ 10\}$$

(b)

$$B \cup C = \{2,\ 4,\ 6,\ 8,\ 10\} \cup \{1,\ 3,\ 5,\ 7,\ 9\} = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\} = A$$

(c)

$$\begin{aligned} C \cup (B \cap D) &= \{1,\ 3,\ 5,\ 7,\ 9\} \cup \big(\{2,\ 4,\ 6,\ 8,\ 10\} \cap \{2,\ 3,\ 5,\ 7\}\big) \\ &= \{1,\ 3,\ 5,\ 7,\ 9\} \cup \{2\} \\ &= \{1,\ 2,\ 3,\ 5,\ 7,\ 9\} \end{aligned}$$

(d)

$$E \setminus F = \{1,\ 2,\ 4,\ 8,\ 10\} - \{3,\ 5,\ 8,\ 9,\ 10\} = \{1,\ 2,\ 4\}$$

(e)

$$E \Delta F = \{1,\ 2,\ 4,\ 8,\ 10\} \Delta \{3,\ 5,\ 8,\ 9,\ 10\} = \{1,\ 2,\ 3,\ 4,\ 5,\ 9\}$$

(f)

$$(B \cup C)^c = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\}^c = A^c = \varnothing$$

Problem 4. (10pt) Let the universal set of discourse be the set of integers. Define the following sets:

$$\begin{aligned}
A &= \text{set of even integers} \\
B &= \text{set of odd integers} \\
C &= \text{set of prime integers} \\
D &= \text{set of square integers} \\
E &= \text{set of nonnegative integers} \\
F &= \text{set of positive integers} \\
G &= \text{set of integers strictly between 0 and 20} \\
H &= \text{set of integers that are a multiple of 5}
\end{aligned}$$

Compute the sets below. When giving your solution, either enumerate all the elements of the resulting set (if possible), give the set using set-builder notation, or give the set using some 'standard' notation.

(a) B^c

(b) $A \cup B$

(c) $A \cap C$

(d) $B \cap C$

(e) $G - D$

(f) $E \Delta F$

(g) $C \cap H$

(h) $D \cap E^c$

(i) D^c

Solution.

(a) The elements of B^c are the integers that are not in B, i.e. not odd. Therefore, the elements of B^c are the even integers. We can give this as a set by. . .

$$B^c = \{n \colon (\exists k \in \mathbb{Z})(n = 2k)\} = \{2k \colon k \in \mathbb{Z}\} = A$$

(b) The elements of $A \cup B$ are either even or odd integers. But every integer is either even or odd. Therefore, the union of all even and odd integers is the entire collection of integers, i.e. $A \cup B = \mathbb{Z}$.

(c) The elements of $A \cap C$ are the integers that are both even and prime. However, any even number that is not 2 is divisible by 2 and another integer that is not ± 1. But then the given integer is not prime. Therefore, the only element of $A \cap C$ is 2, i.e. $A \cap C = \{2\}$.

(d) The elements in $B \cap C$ are the integer which are odd and prime. This can be given in *many* ways, e.g.

$$\begin{aligned}
B \cap C &= \{n \in \mathbb{N} \colon \neg(\exists a \in \mathbb{Z})(\exists b \in \mathbb{Z})(a > 1 \wedge b > 1 \wedge n = ab) \wedge \neg(\exists k \in \mathbb{Z})(n = 2k)\} \\
&= \{n \in \mathbb{N} \colon \neg(\exists a \in \mathbb{Z})(\exists b \in \mathbb{Z})(a > 1 \wedge b > 1 \wedge n = ab) \wedge \neg(\exists k \in \mathbb{Z})(n = 2k)\} \\
&= \{n \in \mathbb{Z} \colon (\forall a \in \mathbb{Z})(\forall b \in \mathbb{Z})(n = ab \to a = 1 \vee b = 1) \wedge \neg(\exists k \in \mathbb{Z})(n = 2k)\} \\
&= \{n \in \mathbb{N} \colon \neg(\exists a \in \mathbb{Z})(\exists b \in \mathbb{Z})(a > 1 \wedge b > 1 \wedge n = ab) \wedge (\exists k \in \mathbb{Z})(n = 2k + 1)\} \\
&= \vdots \\
&= \{p \in \mathbb{N} \colon p \text{ prime}, \, p > 2\}
\end{aligned}$$

(e) The elements of $G \setminus D$ are the elements of G that are not in D, i.e. the set of integers strictly between 0 and 20 that are not also square integers. The integers strictly between 0 and 20 are 1, 2, 3, ..., 19. The squares are 0, 1, 4, 9, 16, 25, ... But then the set of integers strictly between 0 and 20 that are not square integers is...

$$G - D = \{2,\ 3,\ 5,\ 6,\ 7,\ 8,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15,\ 17,\ 18,\ 19\}$$

(f) The elements of $E\Delta F$ are the elements that are only in $E = \{0, 1, 2, 3, \ldots\}$ or $F = \{1, 2, 3, 4, \ldots\}$ but not both, i.e. the integers that are either nonnegative or positive but not both. However, every positive integer is nonnegative so that $E \cap F = \{1, 2, 3, \ldots\}$. Therefore, the only element of the set $E\Delta F$ is the only nonpositive, nonnegative integer:

$$E\Delta F = \{0\}$$

(g) The elements of $C \cap H$ are the elements that are in both C and H, i.e. integers that are both prime and a multiple of 5. The primes are 2, 3, 5, 7, 11, 13, 17, 19, ... and the multiples of 5 are $\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots$. But then it is clear that any multiple of 5—other than 5 itself—cannot be prime. Therefore, the only integer that is both prime and a multiple of 5 is 5 itself. Then we know that $C \cap H = \{5\}$.

(h) The elements of $D \cap E^c$ are the elements that are in D and also not in E, i.e. the integers that are square but not nonnegative. If an integer is not nonnegative, i.e. $\neg(n \geq 0) \equiv n < 0$, then the integer is negative. However, if an integer is a square, then it is equal to the square of another integer. In particular, the square numbers are nonnegative. Therefore, a number cannot both be a square and be negative. This shows that...

$$D \cap E^c = \varnothing$$

(i) The elements of D^c are the elements that are not in D, i.e. the integers that are not squares. But we can give this set by...

$$\{n \in \mathbb{Z} \colon \neg(\exists k \in \mathbb{Z})(n = k^2)\} = \{n \in \mathbb{Z} \colon (\forall k \in \mathbb{Z})(n \neq k^2)\}$$