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MATH 108
Fall 2022
HW 20: Due 12/13

"We should forget about small efficiencies, say about 97% of the time: premature optimization is the root of all evil."
–Donald Knuth

Problem 1. (10pt) Write down the initial simplex tableau for the following optimization problem:

$$\begin{gathered} \min z = 5.3x_1 - 3.4x_2 + 6.8x_3 + 8.1x_4 \\ 1.1x_1 - 2.2x_2 + 3.3x_3 - 4.4x_4 \geq 15.6 \\ 8.4x_1 + 5.9x_2 + 17.8x_4 \geq 78.4 \\ 9.9x_1 - x_2 + 6.7x_3 \geq 100.5 \\ x_1, x_2, x_3, x_4 \geq 0 \end{gathered}$$

Solution. Introducing surplus variables into each inequality (except the last non-negativity inequality) to obtain equalities, we have...

$$\begin{array}{rcrcrcrcrrrrcr} 1.1x_1 & + & -2.2x_2 & + & 3.3x_3 & + & -4.4x_4 & + & -s_1 & & & = & 15.6 \\ 8.4x_1 & + & -5.9x_2 & + & 0.0x_3 & + & 17.8x_4 & + & & -s_2 & & = & 78.4 \\ 9.9x_1 & + & -1.0x_2 & + & 6.7x_3 & + & 0.0x_4 & + & & & -s_3 & = & 100.5 \end{array}$$

Moving things to the 'z'-side of the equality in the function, we have $z - 5.3x_1 + 3.4x_2 - 6.8x_3 - 8.1x_4 = 0$. Adding this to the table yields...

$$\begin{array}{rcrcrcrcrcrrrrcr} & & 1.1x_1 & + & -2.2x_2 & + & 3.3x_3 & + & -4.4x_4 & + & -s_1 & & & = & 15.6 \\ & & 8.4x_1 & + & -5.9x_2 & + & 0.0x_3 & + & 17.8x_4 & + & & -s_2 & & = & 78.4 \\ & & 9.9x_1 & + & -1.0x_2 & + & 6.7x_3 & + & 0.0x_4 & + & & & -s_3 & = & 100.5 \\ z & + & -5.3x_1 & + & 3.4x_2 & + & -6.8x_3 & + & 8.1x_4 & + & & & & = & 0 \end{array}$$

This yields the following initial simplex tableau:

$$\begin{array}{rrrrrrr|r} 1.1 & -2.2 & 3.3 & -4.4 & -1 & 0 & 0 & 15.6 \\ 8.4 & -5.9 & 0.0 & 17.8 & 0 & -1 & 0 & 78.4 \\ 9.9 & -1.0 & 6.7 & 0.0 & 0 & 0 & -1 & 100.5 \\ \hline -5.3 & 3.4 & -6.8 & 8.1 & 0 & 0 & 0 & 0 \end{array}$$

Problem 2. (10pt) Find the dual problem for the following minimization problem:

$$\begin{aligned}\min z &= 5x_1 + 4x_2\\ x_1 + x_2 &\geq 4\\ x_1 + 7x_2 &\geq 8\\ x_1 + 5x_2 &\geq 9\\ x_1, x_2 &\geq 0\end{aligned}$$

Solution. First, we write the 'matrix associated' to this minimization; that is, we create a matrix with rows corresponding to the equality version of the inequalities (with the exception of the non-negativity inequality) with the function being the last row. This yields matrix:

$$\begin{pmatrix}1 & 1 & 4\\ 1 & 7 & 8\\ 1 & 5 & 9\\ 5 & 4 & 0\end{pmatrix}$$

We now find the transpose of this matrix:

$$\begin{pmatrix}1 & 1 & 4\\ 1 & 7 & 8\\ 1 & 5 & 9\\ 5 & 4 & 0\end{pmatrix}^T = \begin{pmatrix}1 & 1 & 1 & 5\\ 1 & 7 & 5 & 4\\ 4 & 8 & 9 & 0\end{pmatrix}$$

We now find the standard maximization problem corresponding to this matrix:

$$\begin{aligned}\max w &= 4y_1 + 8y_2 + 9y_3\\ y_1 + y_2 + y_3 &\leq 5\\ y_1 + 7y_2 + 5y_3 &\leq 4\\ y_1, y_2, y_3 &\geq 0\end{aligned}$$

Problem 3. (10pt) Write down the initial simplex tableau for the following optimization problem:

$$\begin{aligned} \max z = 2x_1 + x_2 - 3x_3 \\ x_1 + 2x_2 + 3x_3 \leq 90 \\ x_1 + x_2 \geq 10 \\ x_1 - x_2 - x_3 \leq -20 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution. First, observe that this maximization problem is not in standard form. However, this maximization problem cannot be placed in standard form: the inequality $x_1 + x_2 \geq 10$ is a '$\geq$' inequality and not a '$\leq$' inequality. However, multiplying both sides by -1, we obtain $-x_1 - x_2 \leq -10$, which has a negative number on the right-side of the inequality. Therefore, this is clearly a problem with mixed constraints. If an inequality can be placed in standard form, we are sure that it is. If not, we need be sure that each right-side of the inequalities is a non-negative number. Doing this, we obtain...

$$\begin{aligned} x_1 + 2x_2 + 3x_3 \leq 90 \\ x_1 + x_2 \geq 10 \\ -x_1 + x_2 + x_3 \geq 20 \end{aligned}$$

Now we introduce slack and surplus variables to obtain equalities:

$$\begin{array}{rcrcrcrrrcr} x_1 & + & 2x_2 & + & 3x_3 & + & s_1 & & & = & 90 \\ x_1 & + & x_2 & + & 0x_3 & + & & -s_2 & & = & 10 \\ -x_1 & + & x_2 & + & x_3 & + & & & -s_3 & = & 20 \end{array}$$

Moving everything to the right side of the function in $z = 2x_1 + x_2 - 3x_3$, we obtain $z - 2x_1 - x_2 + 3x_3 = 0$. Adding this to the tableau, we obtain...

$$\begin{array}{rcrcrcrcrrrcr} & & x_1 & + & 2x_2 & + & 3x_3 & + & s_1 & & & = & 90 \\ & & x_1 & + & x_2 & + & 0x_3 & + & & -s_2 & & = & 10 \\ & & -x_1 & + & x_2 & + & x_3 & + & & & -s_3 & = & 20 \\ z & + & -2x_1 & + & -x_2 & + & 3x_3 & + & & & & = & 0 \end{array}$$

This gives an initial simplex tableau:

$$\begin{array}{rrrrrr|r} 1 & 2 & 3 & 1 & 0 & 0 & 90 \\ 1 & 1 & 0 & 0 & -1 & 0 & 10 \\ -1 & 1 & 1 & 0 & 0 & -1 & 20 \\ \hline -2 & -1 & 3 & 0 & 0 & 0 & 0 \end{array}$$