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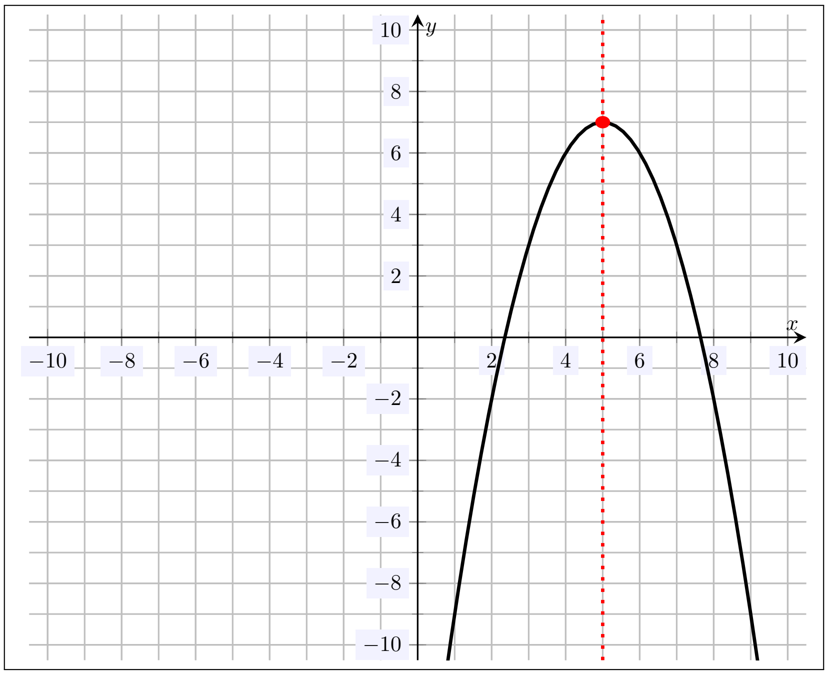
MATH 101
Fall 2023
HW 14: Due 12/06

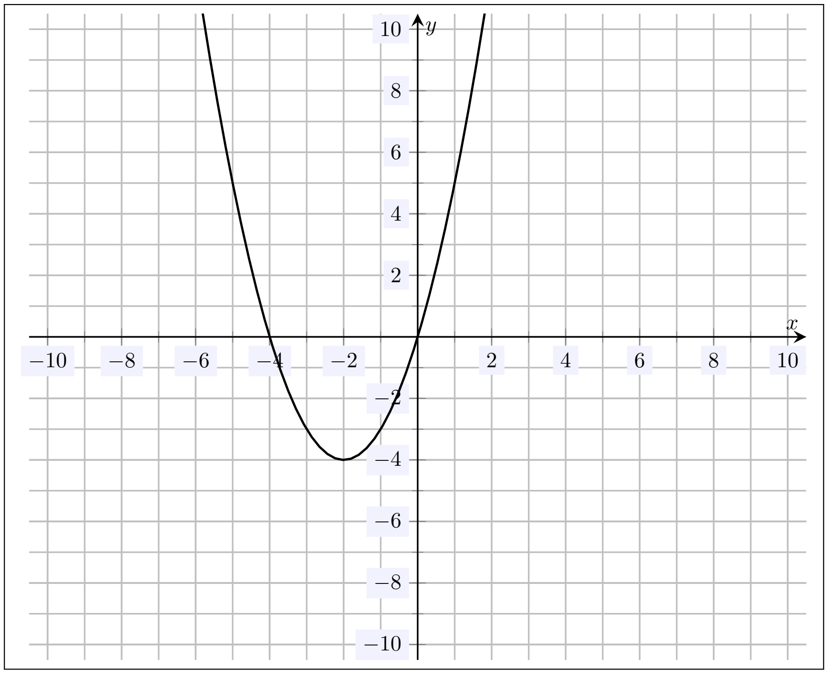
"Nature is written in mathematical language."
–Galileo Galilei

Problem 1. (10pt) Sketch the function $f(x) = 7 - (x-5)^2$.



Solution. Recall the vertex form of a quadratic function is $f(x) = a(x-P)^2 + Q$, where (P, Q) is the vertex of the quadratic function and a is the coefficient of x^2 from $f(x) = ax^2 + bx + c$. Observe that $f(x) = 7 - (x-5)^2 = -1(x-5)^2 + 7$. Therefore, $a = -1 < 0$ and $(P, Q) = (5, 7)$. Therefore, the vertex is $(5, 7)$ and the parabola opens downwards because $a = -1 < 0$. The axis of symmetry is $x = 5$. Therefore, the plot should be symmetric about this line. This gives the sketch given above.

Problem 2. (10pt) Find the equation of the quadratic function shown below. Be sure to fully justify why your answer is correct.



Solution. We see that the quadratic function has zeros $x = -4$ and $x = 0$. We know that every quadratic function can be written in the form $f(x) = a(x - r_0)(x - r_1)$. But then $f(x) = a\big(x - (-4)\big)(x - 0) = a(x + 4)x = ax(x + 4)$. We see also that the point $(-2, -4)$ is on the graph of $f(x)$. But then $f(-2) = -4$. Therefore,

$$\begin{aligned} f(x) &= ax(x+4) \\ f(-2) &= a \cdot -2 \cdot (-2+4) \\ -4 &= -2a \cdot 2 \\ -4 &= -4a \\ a &= 1 \end{aligned}$$

Therefore, $f(x) = x(x + 4) = x^2 + 4x$.

OR

We know the vertex form of a quadratic function is $f(x) = a(x-P)^2+Q$, where (P, Q) is the vertex. We can see from the graph of $f(x)$ that $(P, Q) = (-2, -4)$. Then $f(x) = a\big(x - (-2)\big)^2 + (-4) = a(x + 2)^2 - 4$. We can see also that $(0, 0)$ is on the graph of $f(x)$, i.e. $f(0) = 0$. But then...

$$\begin{aligned} f(x) &= a(x+2)^2 - 4 \\ f(0) &= a(0+2)^2 - 4 \\ 0 &= a \cdot 2^2 - 4 \\ 4a &= 4 \\ a &= 1 \end{aligned}$$

Therefore, $f(x) = (x + 2)^2 - 4 = (x^2 + 4x + 4) - 4 = x^2 + 4x = x(x + 4)$.

Problem 3. (10pt) Consider the quadratic function $f(x) = x^2 - 6x + 14$.

(a) Find a, b, c for this quadratic function.

(b) Does $f(x)$ open upwards or downwards? Explain.

(c) Is this quadratic function convex or concave? Explain.

(d) Find the minimum value of $f(x)$, if it exists. If it does not exist, explain why.

(e) Find the maximum value of $f(x)$, if it exists. If it does not exist, explain why.

Solution.

(a) A quadratic function has the form $ax^2 + bx + c$. But then we can see that for $f(x)$, $a = 1$, $b = -6$, and $c = 14$.

(b) Because $a = 1 > 0$, this quadratic function opens upwards.

(c) Because $a = 1 > 0$, this quadratic function is convex.

(d) Because $a = 1 > 0$, this quadratic function has a minimum value. We know the minimum value occurs at the vertex. So we need to find the vertex of $f(x)$. By completing the square, we have...

$$x^2 - 6x + 14$$
$$x^2 - 6x + \left(\frac{-6}{2}\right)^2 - \left(\frac{-6}{2}\right)^2 + 14$$
$$x^2 - 6x + 9 - 9 + 14$$
$$(x^2 - 6x + 9) + (-9) + 14$$
$$(x-3)^2 + 5$$

Therefore, the vertex is $(3, 5)$. But then the minimum value for $f(x)$ is 5 and occurs when $x = 3$.

Alternatively, using the 'evaluation method', we know the vertex occurs when $x = -\frac{b}{2a} = -\frac{-6}{2(1)} = \frac{6}{2} = 3$. But then the y-coordinate of the vertex $f(3) = 3^2 - 6(3) + 14 = 9 - 18 + 14 = 5$. Therefore, the vertex is $(3, 5)$ and the minimum output of $f(x)$ is 5.

(e) Because $a = 1 > 0$, this quadratic function has no minimum value—the outputs of $f(x)$ get arbitrarily large.

Problem 4. (10pt) Consider the quadratic function $f(x) = 4 - 2(x-2)^2$.

(a) Find a, b, c for this quadratic function.

(b) Does $f(x)$ open upwards or downwards? Explain.

(c) Is this quadratic function convex or concave? Explain.

(d) Find the minimum value of $f(x)$, if it exists. If it does not exist, explain why.

(e) Find the maximum value of $f(x)$, if it exists. If it does not exist, explain why.

Solution.

(a) A quadratic function has the form $ax^2 + bx + c$. We expand $f(x)$:

$$f(x) = 4 - 2(x-2)^2 = 4 - 2(x^2 - 4x + 4) = 4 - 2x^2 + 8x - 8 = -2x^2 + 8x - 4$$

But then we can see that for $f(x)$, $a = -2$, $b = 8$, and $c = -4$.

(b) Because $a = -2 < 0$, this quadratic function opens downwards.

(c) Because $a = -2 < 0$, this quadratic function is concave.

(d) Because $a = -2 < 0$, this quadratic function has a maximum value. We know the maximum value occurs at the vertex. So we need to find the vertex of $f(x)$. But $f(x)$ was given in vertex form: $f(x) = 4 - 2(x-2)^2 = -2(x-2)^2 + 4$. Therefore, the vertex is $(2, 4)$. Then the maximum value for $f(x)$ is 4 and occurs when $x = 2$.

(e) Because $a = -2 < 0$, this quadratic function has no minimum value—the outputs of $f(x)$ get arbitrarily small.