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MATH 108
Spring 2024
HW 20: Due 04/24

"Okay. No hard feelings, but I hate you. Not joking. Bye."
— *Gina Linetti, Brooklyn 99*

Problem 1. (10pts) Find the initial simplex tableau corresponding to the linear programming problem shown below:

$$\max z = 6x_1 + 9x_2$$

$$\begin{cases} x_1 + x_2 \leq 100 \\ -x_1 + 7x_2 \geq 10 \\ -6x_1 + x_2 \leq -70 \\ x_1 + 7x_2 \leq 80 \\ x_1, x_2 \geq 0 \end{cases}$$

Solution. We need all inequalities to have a nonnegative number on the 'right side' of the inequality. So we must multiply both sides of the third inequality by -1, so that we obtain the following inequalities:

$$\begin{cases} x_1 + x_2 \leq 100 \\ -x_1 + 7x_2 \geq 10 \\ 6x_1 - x_2 \geq 70 \\ x_1 + 7x_2 \leq 80 \\ x_1, x_2 \geq 0 \end{cases}$$

We now introduce slack or surplus variables to obtain equalities. We also move everything to 'one side' in the function to obtain $z - 6x_1 - 9x_2 = 0$. Writing all these equalities together, we obtain…

$$\begin{array}{rcrcrcrcrcrcrcr} & & x_1 & + & x_2 & + & s_1 & & & & & & & = & 100 \\ & & -x_1 & + & 7x_2 & & & - & s_2 & & & & & = & 10 \\ & & 6x_1 & + & -1x_2 & & & & & - & s_3 & & & = & 70 \\ & & x_1 & + & 7x_2 & + & & & & & & & s_4 & = & 80 \\ z & + & -6x_1 & + & -9x_2 & & & & & & & & & = & 0 \end{array}$$

Therefore, the initial simplex tableau is…

$$\left[\begin{array}{rrrrrr|r} 1 & 1 & 1 & 0 & 0 & 0 & 100 \\ -1 & 7 & 0 & -1 & 0 & 0 & 10 \\ 6 & -1 & 0 & 0 & -1 & 0 & 70 \\ 1 & 7 & 0 & 0 & 0 & 1 & 80 \\ \hline -6 & -9 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Problem 2. (10pts) Below is the initial simplex tableau corresponding to a linear programming maximization problem. Find the initial maximization problem.

$$\begin{array}{rrrrrrr} 4 & -1 & 2 & 1 & 0 & 0 & 82 \\ -1 & 5 & 9 & 0 & 1 & 0 & 55 \\ 7 & -1 & 4 & 0 & 0 & 1 & 68 \\ -2 & -3 & -1 & 0 & 0 & 0 & 0 \end{array}$$

Solution. We first add the appropriate horizontal line to separate the function from the inequalities and a vertical line to separate the sides of the equalities.

$$\begin{array}{rrrrrr|r} 4 & -1 & 2 & 1 & 0 & 0 & 82 \\ -1 & 5 & 9 & 0 & 1 & 0 & 55 \\ 7 & -1 & 4 & 0 & 0 & 1 & 68 \\ \hline -2 & -3 & -1 & 0 & 0 & 0 & 0 \end{array}$$

The last row corresponds to the function, while the other rows correspond to the inequalities. Therefore, there were three inequalities in the original problem (not including the non-negativity conditions). For each inequality, we introduce a slack or surplus variable. Therefore, three of the variables are slack or surplus variables. Each column—except the last—corresponds to a variable in the system. Therefore, there are 6 total variables. With 3 slack variables, there must then be $6 - 3 = 3$ original variables in the system. We can then label the variables in our system.

$$\begin{array}{rrrrrr|r} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & \\ 4 & -1 & 2 & 1 & 0 & 0 & 82 \\ -1 & 5 & 9 & 0 & 1 & 0 & 55 \\ 7 & -1 & 4 & 0 & 0 & 1 & 68 \\ \hline -2 & -3 & -1 & 0 & 0 & 0 & 0 \end{array}$$

We can see that we had to add s_1, s_2, s_3 to obtain equalities. Therefore, these are slack variables and the corresponding inequalities must have been '$\leq$'. We know that $z - 2x_1 - 3x_2 - x_3$, which implies $z = 2x_1 + 3x_2 + x_3$. Introducing the condition that the variables are nonnegative, the original optimization problem must have been...

$$\max z = 2x_1 + 3x_2 + x_3$$
$$\begin{cases} 4x_1 - x_2 + 2x_3 \leq 82 \\ -x_1 + 5x_2 + 9x_3 \leq 55 \\ 7x_1 - x_2 + 4x_3 \leq 68 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Problem 3. (10pts) Below is the final simplex tableau for a linear programming maximization problem.

$$\begin{array}{cccccccccc}
0 & 1 & 0 & 0 & 0.03 & 0.12 & -0.1 & 0.04 & 8.89 \\
0 & 0 & 1 & 0 & -0.05 & 0.03 & 0.01 & 0.05 & 2.19 \\
0 & 0 & 0 & 1 & 0.05 & -0.01 & 0.05 & 0.02 & 7.31 \\
1 & 0 & 0 & 0 & 0.01 & -0.06 & -0.03 & 0.1 & 1.27 \\
0 & 0 & 0 & 0 & 0.25 & 0.86 & 0.18 & 0.6 & 123.14
\end{array}$$

(a) How many inequalities were considered?

(b) How many variables were there in the original inequalities?

(c) How many slack/surplus variables were introduced?

(d) What was the solution to this maximization problem?

Solution.

(a) Every row in the tableau corresponds to an inequality—except for the last row which corresponds to the function. Because there are 5 rows, there must have been $5 - 1 = 4$ inequalities in the original system (neglecting the non-negativity inequalities).

(b) Every column in the tableau corresponds to a variable—except the last column which corresponds to the 'other' side of an equality. Because there are 9 columns, there are $9 - 1 = 8$ variables in the system. Because we introduce a slack or surplus variable to each inequality and by (a) there are 4 inequalities, 4 of the variables are slack/surplus variables. Therefore, there were $8 - 4 = 4$ 'original' variables in the system.

(c) By (b), we know that there were 4 slack or surplus variables introduced.

(d) Introducing labels for the variables, adding horizontal and vertical lines, and boxing the 'pivot positions', we obtain the following tableau:

$$\begin{array}{cccccccc|c}
x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & s_4 & \\
0 & \boxed{1} & 0 & 0 & 0.03 & 0.12 & -0.1 & 0.04 & 8.89 \\
0 & 0 & \boxed{1} & 0 & -0.05 & 0.03 & 0.01 & 0.05 & 2.19 \\
0 & 0 & 0 & \boxed{1} & 0.05 & -0.01 & 0.05 & 0.02 & 7.31 \\
\boxed{1} & 0 & 0 & 0 & 0.01 & -0.06 & -0.03 & 0.1 & 1.27 \\
\hline
0 & 0 & 0 & 0 & 0.25 & 0.86 & 0.18 & 0.6 & 123.14
\end{array}$$

This gives $x_1 = 1.27$, $x_2 = 8.89$, $x_3 = 2.19$, and $x_4 = 7.31$. All the remaining variables have value 0. From the bottom-rightmost entry, we see that $\max z = 123.14$. Therefore, the maximum values is 123.14 and occurs at $(x_1, x_2, x_3, x_4, s_1, s_2, s_3, s_4) = (1.27, 8.89, 2.19, 7.31, 0, 0, 0, 0)$.

Problem 4. (10pts) Below is the final simplex tableau for a linear programming minimization problem.

$$\begin{array}{rrrrrrr} 3 & 9 & 0 & 1 & 0 & 2 & 5 \\ 0 & -5 & 0 & 0 & 1 & -1 & 2 \\ 1 & 4 & 1 & 0 & 0 & 1 & 2 \\ 5 & 48 & 0 & 0 & 0 & 15 & 30 \end{array}$$

(a) How many inequalities were considered?

(b) How many variables were there in the original inequalities?

(c) How many slack/surplus variables were introduced?

(d) What was the solution to this minimization problem?

Solution.

(a) Each row of the tableau corresponds to an inequality—except for the last row which corresponds to the function. Because there are 4 rows, the original system had $4 - 1 = 3$ inequalities.

(b) Each column of the tableau corresponds to a variable—except for the last column which corresponds to the 'other side' of the equalities. Because there are 7 columns, there are $7 - 1 = 6$ variables in the system. A slack or surplus variable is introduced for each inequality. We know there are three inequalities from (a). Therefore, there are three slack or surplus variables. For a minimization problem, the variables in the original system correspond to the slack or surplus variables in the dual maximization problem. Therefore, there were 3 variables in the original minimization problem. For the dual maximization problem, we know there were 3 variables involved and that there are 3 slack or surplus variables. Therefore, the dual maximization problem has $6 - 3 = 3$ 'original' variables.

(c) From (b), we know that 3 slack or surplus variables were introduced into the dual maximization problem.

(d) We add a horizontal line to separate the function from the equalities and a vertical line to separate the variables from the 'other side' of the equalities. We also label the 3 'original' variables and the 3 slack/surplus variables.

$$\begin{array}{rrrrrr|r} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & \\ 3 & 9 & 0 & 1 & 0 & 2 & 5 \\ 0 & -5 & 0 & 0 & 1 & -1 & 2 \\ 1 & 4 & 1 & 0 & 0 & 1 & 2 \\ \hline 5 & 48 & 0 & 0 & 0 & 15 & 30 \end{array}$$

The minimum value to the original minimization problem is the maximum value for the dual problem. From the table above, we can see that the maximum value is 30. The original minimization problem had two variables, say y_1, y_2, y_3. The value of the original minimization variables are the 'values' of the slack/surplus variables in the dual minimization problem. Therefore, we know $y_1 = 0$, $y_2 = 0$, and $y_3 = 15$. Therefore, the minimum value is 30 and occurs at the point $(y_1, y_2, y_3) = (0, 0, 15)$.