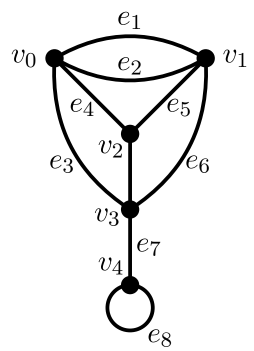
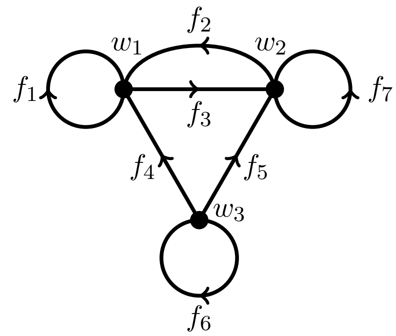
Name: *Caleb McWhorter — Solutions*
MATH 308
Fall 2022
HW 20: Due 12/15

> *"The origins of graph theory are humble, even frivolous."*
> –*Norman Biggs*

Problem 1. (10pt) Consider the graphs G and H given below.



G



H

(a) Find $\deg v_2$, $\deg v_4$, and $\deg G$.

(b) Is G connected? Explain. Is G simple? Explain.

(c) Find $\deg^+ w_1$ and $\deg^- w_1$ as well as $\deg^+ w_3$ and $\deg^- w_3$.

(d) Find a trail in H that is not a path.

(e) Does H have any sources or sinks? Explain.

Solution.

(a) The degree of a vertex in an indirected graph is the number of edges incident to the vertex with loops counted twice. Therefore, $\deg v_2 = 3$ and $\deg v_4 = 3$. The degree of an undirected graph is the sum of the degrees of its vertices. Therefore, we have. . .

$$\deg G = \sum_{v_i} \deg v_i = \deg v_0 + \deg v_1 + \deg v_2 + \deg v_3 + \deg v_4 = 4 + 4 + 3 + 4 + 3 = 18$$

Alternatively, by the Handshake Theorem, the degree of an undirected graph G is twice the number of edges. Therefore, $\deg G = 2|E(G)| = 2 \cdot 9 = 18$.

(b) The graph G is connected because given any two vertices, there is a walk from one vertex to the other. However, the graph G is not simple because there is a loop. Alternatively, the graph G is not simple because there are multiple edges, e.g. there are distinct vertices with more than one edge between them (for instance, there are two edges between v_0 and v_1).

(c) In a directed graph, the in-degree of a vertex v, $\deg^- v$, is the number of edges 'coming in' to the vertex. The out-degree of a vertex v, $\deg^+ v$, is the number of edges 'going out' of the vertex. A loop is counted once in the in and out degree of a vertex. Therefore, $\deg^+ w_1 = 2$, $\deg^- w_1 = 2$ and $\deg^+ w_3 = 3$, $\deg^- w_3 = 1$.

(d) A trail is a walk with no repeated edges whereas a path is a walk where no vertex (and hence no edges) is repeated. Therefore, a trail which is not a path is a walk which does not repeat an edge but does repeat a vertex. For example, $w_1 f_3 w_2 f_2$ is a walk that is a trail but not a path.

(e) In a directed graph, a source is a vertex v with $\deg^- v = 0$ while a sink is a vertex v with $\deg^+ v = 0$. A vertex with neither of these properties is called interval. Because $\deg^+ v \neq 0$ and $\deg^- v \neq 0$ for all $v \in V(H)$, there are no sources or sinks in H; that is, every vertex in H is internal.

Problem 2. (10pt) Suppose you have two graphs, G and H, where G is undirected and H may be undirected or directed. The adjacency matrices of G and H are given below.

$$\underbrace{\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}}_{G} \qquad \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}}_{H}$$

(a) Draw the graph of G.

(b) Does G have a loop? Justify your answer using *only* the adjacency matrix of G.

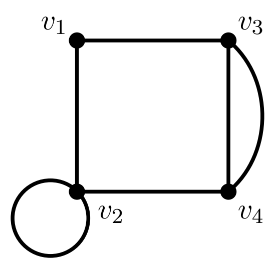
(c) Draw the graph of H.

(d) Is H undirected or directed? Justify your answer using *only* the adjacency matrix of H.

(e) Do G or H have multiple edges? Explain your answer using only the adjacency matrix of G and H, respectively.

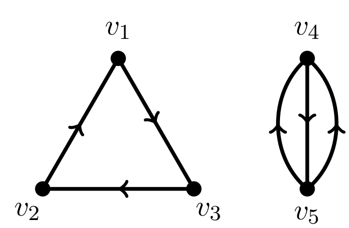
Solution.

(a) We have...



(b) A loop is an edge from a vertex v_i to itself. But then $G_{i,i} > 0$. Because $G_{2,2} > 0$, there is a loop incident to vertex v_2.

(c) We have...



(d) If a graph is undirected, then its adjacency matrix is symmetric, i.e. $A^T = A$. But then $a_{ij} = a_{ji}$ for all i, j. By contrapositive, if the adjacency matrix for a graph is not symmetric, then the graph cannot eb directed. Observe that the adjacency matrix for H is not symmetric. For instance, $H_{2,1} = 1 \neq 0 = H_{1,2}$. Therefore, H is a directed graph.

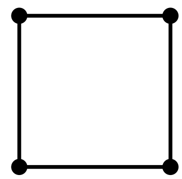
(e) A graph has multiple edges if there exist two distinct vertices with more than one edge between them. But then if A is the adjacency matrix for the graph, then $a_{ij} > 1$ for some i, j. Because $G_{3,4} = 2 > 1$ and $H_{5,4} = 2 > 1$, both G and H have multiple edges.

Problem 3. (10pt) Showing all your work and fully justifying your responses, complete the following:

(a) Draw the graph of K_5. How many vertices and edges does K_5 have? For $n \geq 1$, what are $|V(K_n)|$ and $|E(K_n)|$?

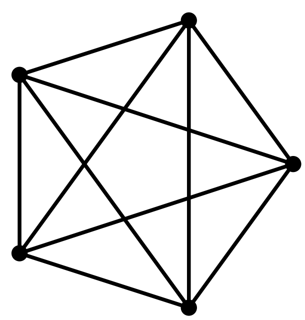
(b) Draw the graph of $K_{3,5}$. How many vertices and edges does $K_{3,5}$ have? For $m, n \geq 1$, what are $|V(K_{m,n})|$ and $|E(K_{m,n})|$?

(c) If G is a simple graph, the complement of G, denoted $\widetilde{G}$, is a graph with $V(G) = V(\widetilde{G})$ and two vertices are adjacent in $\widetilde{G}$ if and only if they are not adjacent in G. Find the complement of the graph given below. Is G connected? Is $\widetilde{G}$ connected?



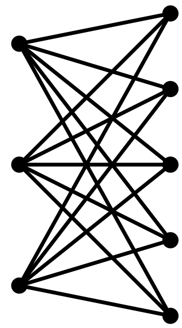
Solution.

(a) We have...



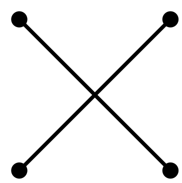
This graph has 5 vertices and 10 edges. In general, to draw K_n, we first place n vertices. Therefore, $|V(K_n)| = n$. We then have an edge between every distinct pair of vertices. But then the number of edges is the number of ways of selecting any two distinct pair of vertices. There are $\binom{n}{2} = \dfrac{n(n-1)}{2}$ ways of choosing any distinct pair of vertices. Therefore, we have $|E(K_n)| = \binom{n}{2} = \dfrac{n(n-1)}{2}$. Alternatively, we can draw an edge between every (not necessarily distinct) vertex. There are $n \cdot n = n^2$ such edges. But then we have also drawn n loops (one at each vertex)—which are not in K_n. Removing these loops, we then have $n^2 - n$ edges. However, once one has drawn a edge from one vertex to another, one need not draw it 'the other way.' We have then doubles the amount of edges in K_n. Therefore, the number of edges is $\frac{1}{2} \cdot (n^2 - n) = \dfrac{n^2 - n}{2} = \dfrac{n(n-1)}{2}$.

(b) We have...

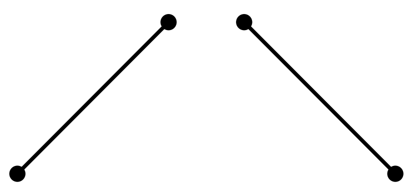


This graph has 8 vertices and 15 edges. In general, to draw $K_{m,n}$, we must first draw a collection of m vertices and a collection of n vertices. But then there are $m+n$ vertices, i.e. $|V(K_{m,n})| = m+n$. For each of the m vertices in the 'first' collection, we need to draw an edge to each of the n vertices in the 'other' collection of vertices. But then there are $m \cdot n$ total edges, i.e. $|E(K_{m,n})| = mn$.

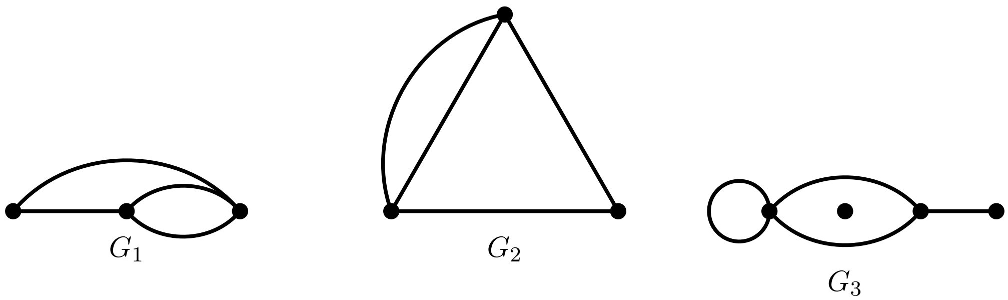
(c) The complement of the graph is...



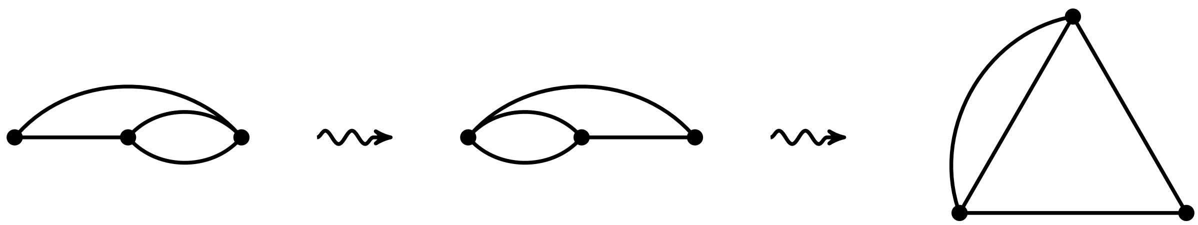
Observe that original graph G is connected because there is a walk between every pair of vertices. However, $\widetilde{G}$ is not connected because there is not a walk between each pair of vertices. Although, it may look like this is the case with how the graph is drawn above. There is no vertex at the intersection of edges above. We could have drawn the complement as below to emphasize that the graph is disconnected:



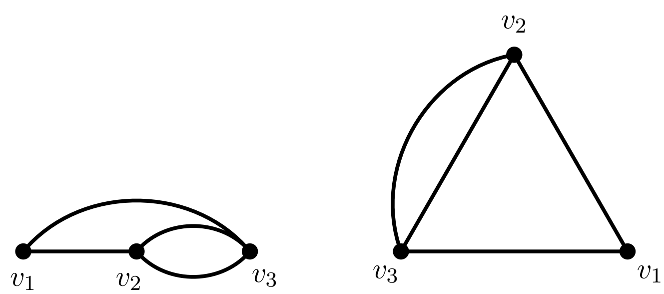
Problem 4. (10pt) Below are three graphs: G_1, G_2, and G_3. Determine which, if any, of the graphs are isomorphic. If two given graphs are isomorphic, show that they are isomorphic. If two graphs are not isomorphic, give at least two reasons why they are not isomorphic.



Solution. First, observe that G_1 is isomorphic to G_2 because we can slowly transform one graph into the other:



Alternatively, label the graphs G_1 and G_2 as below:



Observe that both graphs then have the following adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

Two graphs are isomorphic if and only if they have the same adjacency matrix for some labeling of their vertices. But then the work above shows that G_1 is isomorphic to G_2.

Now G_3 cannot be isomorphic to G_1 or G_2. The number of vertices a graph has is invariant under isomorphism. Because G_3 has four vertices, i.e. $|V(G_3)| = 4$, and G_1, G_2 have three vertices, i.e. $|V(G_1)| = |V(G_2)| = 3$, G_3 cannot be isomorphic to G_1 or G_2. Furthermore, the number of loops a graph has is invariant under isomorphism. But because G_3 has a loop while G_1 and G_2 do not, G_3 cannot be isomorphic to G_1 or G_2. Furthermore, the degrees of vertices is invariant under isomorphism. Because G_3 has a vertex of degree 4 (the 'leftmost' vertex) while G_1 and G_2 do not have a vertex of degree 4, G_3 cannot be isomorphic to G_1 or G_2. Furthermore, the number of isolated vertices is invariant under isomorphism. Because G_3 has one isolated vertex while G_1 and G_3 have none, G_3 cannot be isomorphic to G_1 or G_2.