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MATH 308
Fall 2022
HW 6: Due 09/27

"Since, as is well known, god helps those who help themselves, presumably the devil helps all those, and only those, who don't help themselves. Does the devil help himself?"
–Douglas Hofstadter, Gödel, Escher, Bach: An Eternal Golden Braid

Problem 1. (10pt) Let $S := \{-3, -2, -1, 0, 1, 2, 3\}$ be a universal set and define $X := \{-1, 0, 1\}$. Give an example of. . .

(a) a proper subset of S, say A, that is disjoint from X.

(b) a subset of S, say B, such that $B - X \neq B$.

(c) a subset of S, say C, such that $X \Delta C = X \cup C$.

(d) a subset of S, say D, such that D^c contains only nonnegative numbers.

(e) a subset of S, say E, such that the complement of $X \cup E$ is empty.

Solution. *Note: Answers may vary.*

(a) The chosen set, A, needs to be disjoint from X; that is, the set A needs to contain no elements of X, i.e. -1, 0, 1. The set A also needs to be a proper subset of S, i.e. not contain every element of S. Examples of such A are $A = \{-3, -2, 2, 3\}$, $A = \{-2, 2\}$, $A = \{3\}$, $\varnothing$, etc.

(b) The set $B - X$ is the set of elements that are in B but *not* in X. For $B - X$ to not contain every element of B, i.e. $B - X \neq B$, B and X cannot be disjoint, i.e. $B \cap X \neq \varnothing$. Then the given set B needs to contain at least one element of X. Examples of such B are $B = \{0\}$, $B = \{-3, -2, -1\}$, $B = \{-1, 0, 1\}$, etc.

(c) The set $X \Delta C$ is the set of elements that are *only* in X or *only* in C, i.e. the elements in X or C but not in $X \cap C$. The set $X \cup C$ is the set of elements in X or C. For $X \Delta C = X \cup C$, there was nothing from $X \cup C$ 'excluded' from $X \Delta C$, i.e. every element of $X \cup C$ is in X or C but not both. Examples of such C are $C = \{-3, -2, 2, 3\}$, $C = \{3\}$, $C = \varnothing$, etc.

(d) The set D^c is the set of elements of S that are *not* in D. Then for D^c to contain only nonnegative numbers, i.e. real numbers $x \geq 0$, the set D^c must then contain all the negative numbers of S. Therefore, the only such example of D is $D = \{-3, -2, -1\}$.

(e) The set $X \cup E$ is the set of elements that are in X or in E. The complement of $X \cup E$, i.e. $(X \cup E)^c$, is the set of elements that are not in $X \cup E$. For the set $(X \cup E)^c$ to be empty, there must be no elements in S that are not already in $X \cup E$, i.e. $X \cup E = S$. Examples of such E are $E = \{-3, -2, 2, 3\}$, $E = \{-2, -1, 0, 1, 2\}$, $E = \{-3, -2, -1, 0, 1, 2, 3\}$, etc.

Problem 2. (10pt) Let A and B be sets. By defining $A = B$ by using a quantified open sentence, show that $A \neq B$ is equivalent to the logical statement. . .

$$(\exists x)\big(x \in A \wedge x \notin B\big) \vee (\exists x)\big(x \in B \wedge x \notin A\big)$$

Solution. By definition, we know that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$, i.e. every element of A is an element of B and every element of B is an element of A. More precisely, $A = B$ if and only if we have: if $a \in A$, then $a \in B$ and if $b \in B$, then $b \in A$. Writing this as a qualified open statement, we have. . .

$$(\forall x)(x \in A \to x \in B) \wedge (\forall x)(x \in B \to x \in A)$$

Then recalling $\neg(\forall x) \equiv \exists x$, $\neg(P \to Q) \equiv P \wedge \neg Q$, and $\neg(x \in X) \equiv x \notin X$, as well as the fact that $A \neq B = \neg[A = B]$, we must have. . .

$$\begin{aligned}
A \neq B &\equiv \neg[A = B] \\
&\equiv \neg\big((\forall x)(x \in A \to x \in B) \wedge (\forall x)(x \in B \to x \in A)\big) \\
&\equiv \neg(\forall x)(x \in A \to x \in B) \vee \neg(\forall x)(x \in B \to x \in A) \\
&\equiv \big((\exists x)\neg(x \in A \to x \in B)\big) \vee (\exists x)\big(\neg(x \in B \to x \in A)\big) \\
&\equiv (\exists x)\big(x \in A \wedge \neg(x \in B)\big) \vee (\exists x)\big(x \in B \wedge \neg(x \in A)\big) \\
&\equiv (\exists x)\big(x \in A \wedge x \notin B\big) \vee (\exists x)\big(x \in B \wedge x \notin A\big)
\end{aligned}$$

Problem 3. (10pt) Let A and B be sets in a universe $\mathcal{U}$ and consider the set $A\Delta B$.

(a) Using set-builder notation and logical propositions, define the set $A\Delta B$.

(b) Construct a Venn diagram for the set $(A\Delta B)^c$.

(c) Construct a Venn diagram for the set $(A\cup B)^c \cup (A\cap B)$

(d) What might you conjecture from your answers in (b) and (c)?

Solution.

(a) We know that the set $A\Delta B$ is the set of elements of $\mathcal{U}$ that are in A or B but *not* in both A and B. From this description of $A\Delta B$, we have...

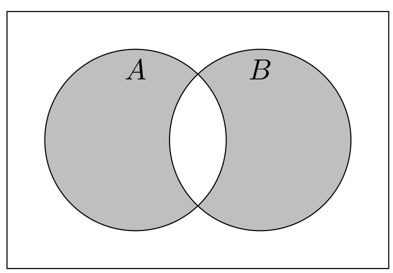
$$A\Delta B = \{x \in \mathcal{U} \colon (x \in A \vee x \in B) \wedge \neg(x \in A \cap B)\}$$

$$A\Delta B = \{x \in \mathcal{U} \colon (x \in A \vee x \in B) \wedge \neg(x \in A \cap B)\}$$

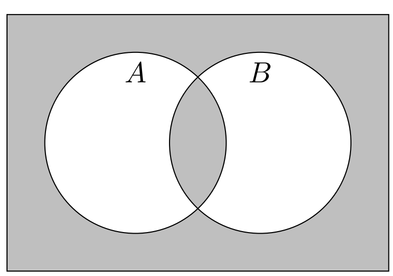
Equivalently, the set $A\Delta B$ is the set of elements of $\mathcal{U}$ that are in A but not B or that are in B but not A. From this description, we have...

$$A\Delta B = \{x \in \mathcal{U} \colon (x \in A \vee x \in B) \wedge (x \notin A \cap B)\}$$

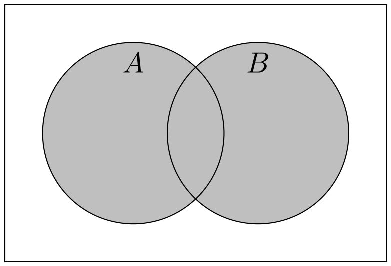
(b) Using any of the descriptions of $A\Delta B$ given in (a), the Venn diagram for $A\Delta B$ is...

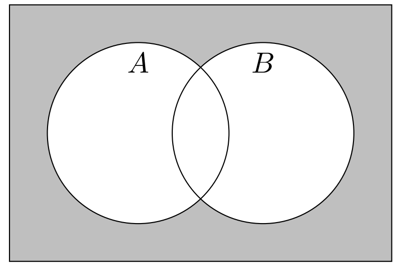


But then the Venn diagram for $(A\Delta B)^c$ is...

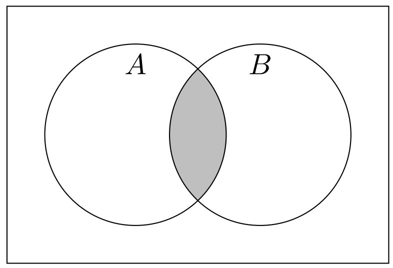


(c) The Venn diagram for $A \cup B$ is given below on the left, which gives the Venn diagram for $(A \cup B)^c$ below on the right.

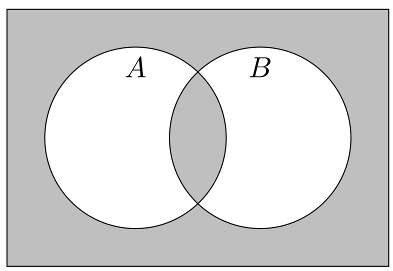




The Venn diagram for $A \cap B$ is...



But then the diagram for $(A \cup B)^c \cup (A \cap B)$ is...



(d) Because the Venn diagram for $(A \Delta B)^c$ in (b) is the same as the Venn diagram for $(A \cup B)^c \cup (A \cap B)$ in (d), we conjecture that $(A \Delta B)^c = (A \cup B)^c \cup (A \cap B)$. In fact, one can prove this:

$$\begin{aligned}
(A \Delta B)^c &= \big((A \cup B) - (A \cap B)\big)^c \\
&= \big((A \cup B) \cap (A \cap B)^c\big)^c \\
&= (A \cup B)^c \cup \big((A \cap B)^c\big)^c \\
&= (A \cup B)^c \cup (A \cap B)
\end{aligned}$$

Problem 4. (10pt) Let A, B, and C be sets in some universe $\mathcal{U}$. Find the *complement* of the following sets, showing all your work and 'simplifying' as much as possible:

(a) $A \setminus B$

(b) $(A^c \cup C) \cap B$

(c) $\big(((A \cup B) \cap C)\big)^c \cup B^c\big)^c$

Solution. Recall that if A and B are sets, then by DeMorgan's Laws, $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$. We also have $(A^c)^c = A$ and the distributive laws $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(a) Recall that $A \setminus B$ is the set of elements that are in A but not in B, i.e. the set $A \cap B^c$. But then we have...

$$\begin{aligned}(A \setminus B)^c &= (A \cap B^c)^c \\ &= A^c \cup (B^c)^c \\ &= A^c \cup B\end{aligned}$$

That is, $(A \setminus B)^c$ are the elements that are either not in A or in B.

(b) We have...

$$\begin{aligned}\big((A^c \cup C) \cap B\big)^c &= (A^c \cup C)^c \cup B^c \\ &= \big((A^c)^c \cap C^c\big) \cup B^c \\ &= (A \cap C^c) \cup B^c\end{aligned}$$

(c) We have...

$$\begin{aligned}\Big(\big(((A \cup B) \cap C)\big)^c \cup B^c\big)^c\Big)^c &= \big((A \cup B) \cap C)\big)^c \cup B^c \\ &= \big((A \cup B)^c \cup C^c\big) \cup B^c \\ &= \big((A^c \cap B^c) \cup C^c\big) \cup B^c \\ &= \big((A^c \cap B^c) \cup B^c\big) \cup (C^c \cup B^c) \\ &= B^c \cup (C^c \cup B^c) \\ &= C^c \cup B^c \\ &= (C \cap B)^c\end{aligned}$$

Problem 5. (10pt) Define $S := \{1, 2, \{1\}, \{\{2\}\}\}$. Determine whether the following are true or false—no justification is necessary:

(a) $\varnothing \in S$

(b) $\varnothing \subseteq S$

(c) $1 \in \mathcal{P}(S)$

(d) $\{1\} \in \mathcal{P}(S)$

(e) $\{\{1\}\} \in \mathcal{P}(S)$

(f) $1 \subseteq \mathcal{P}(S)$

(g) $\{1\} \subseteq \mathcal{P}(S)$

(h) $\{\{1\}\} \subseteq \mathcal{P}(S)$

(i) $\varnothing \in \mathcal{P}(S)$

(j) $\{\varnothing\} \in \mathcal{P}(S)$

(k) $\varnothing \subseteq \mathcal{P}(S)$

(l) $\{\varnothing\} \subseteq \mathcal{P}(S)$

Solution. It would be useful to write S and compute $\mathcal{P}(S)$:

$$\mathcal{P}(S) = \left\{ \begin{array}{cccccc} \varnothing, & & & & & \\ \{1\}, & \{2\}, & \{\{1\}\}, & \{\{\{2\}\}\}, & & \\ \{1,2\}, & \{1,\{1\}\}, & \{1,\{\{2\}\}\}, & \{2,\{1\}\}, & \{2,\{\{2\}\}\}, & \{\{1\},\{\{2\}\}\} \\ \{1,2,\{1\}\}, & \{1,2,\{\{2\}\}\}, & \{1,\{1\},\{\{2\}\}\}, & \{2,\{1\},\{\{2\}\}\}, & & \\ S = \{1,2,\{1\},\{\{2\}\}\} & & & & & \end{array} \right\}$$

(a) F

(b) T

(c) F

(d) T

(e) T

(f) F

(g) F

(h) T

(i) T

(j) F

(k) T

(l) T

Problem 6. (10pt) Define $A := \{3, 5, 7\}$ and $B := \{\pi, e, \sqrt{2}, \varphi\}$.

(a) Determine $A \times B$.

(b) Is $(3, \pi) \in A \times B$? Is $(\pi, 3) \in A \times B$? Explain the relation between your responses.

(c) Is $A \times B = B \times A$? Explain.

Solution.

(a) We have...

$$A \times B = \{(a, b) \colon a \in A, b \in B\} = \begin{Bmatrix} (3,\pi), & (3,e), & (3,\sqrt{2}), & (3,\varphi) \\ (5,\pi), & (5,e), & (5,\sqrt{2}), & (5,\varphi) \\ (7,\pi), & (7,e), & (7,\sqrt{2}), & (7,\varphi) \end{Bmatrix}$$

(b) From (a), we can see that $(3, \pi) \in A \times B$ but $(\pi, 3) \notin A \times B$. The set $A \times B$ consists of ordered pairs—ordered. The order in an order pair matters. So while $(3, \pi) \in A \times B$ because $3 \in A$ and $\pi \in B$, we know that $(\pi, 3) \notin A \times B$ because $\pi \notin A$ and $3 \notin B$. This is in contrast to sets where order does not matter so that $\{3, \pi\} = \{\pi, 3\}$.

(c) We have...

$$A \times B = \{(b, a) \colon a \in A, b \in B\} = \begin{Bmatrix} (\pi,3), & (e,3), & (\sqrt{2},3), & (\varphi,3) \\ (\pi,5), & (e,5), & (\sqrt{2},5), & (\varphi,5) \\ (\pi,7), & (e,7), & (\sqrt{2},7), & (\varphi,7) \end{Bmatrix}$$

We can see that $(3, \pi) \in A \times B$ but $(3, \pi) \notin B \times A$. Because the sets do not contain the same elements, we know that these sets cannot be equal. In fact, $A \times B$ will never be the same as $B \times A$ unless A and B contain all the same elements.

Problem 7. (10pt) Determine $\bigcup_{i \in \mathcal{I}} A_n$ and $\bigcap_{i \in \mathcal{I}} A_n$ for the given A_n and $\mathcal{I}$ below—no justification is necessary. However, if the set is finite, enumerate its elements; otherwise, either give the set in set-builder notation or using set operations with 'standard' sets, e.g. $\mathbb{Q}$, $\mathbb{Z} \setminus \mathbb{N}$, etc.

(a) $A_n := \left(\frac{1}{n}, 1 + \frac{1}{n}\right)$; $\mathcal{I} := \mathbb{N}$

(b) $A_n := (n, n+1)$; $\mathcal{I} := \mathbb{Z}$

(c) $A_n := \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$; $\mathcal{I} := \mathbb{R}$

Solution.

(a) $\bigcup_{i \in \mathcal{I}} A_n = (0, 2)$, $\quad \bigcap_{i \in \mathcal{I}} A_n = \{1\}$

(b) $\bigcup_{i \in \mathcal{I}} A_n = \mathbb{R} \setminus \mathbb{Z}$, $\quad \bigcap_{i \in \mathcal{I}} A_n = \varnothing$

(c) $\bigcup_{i \in \mathcal{I}} A_n = \mathbb{R}$, $\quad \bigcup_{i \in \mathcal{I}} A_n = \varnothing$

Problem 8. (10pt) Below is a partial proof of the fact that $A \setminus B = A \cap B^c$. By filling in the missing portions, complete the partial proof below so that it is a correct, logically sound proof with 'no gaps':

Proposition. If A and B are sets, then $A \setminus B = A \cap B^c$.

Proof. If $A \setminus B = \varnothing$, then there is no element in A that is not also in B. But then $A \subseteq B$ so that $A^c \supseteq B^c$. But then $A \cap B^c \subseteq A \cap A^c = \varnothing$ so that $A \cap B^c = \varnothing$. Therefore, if $A \setminus B = \varnothing$, then $A \setminus B = A \cap B^c$. If $A \cap B^c = \varnothing$, then there is no element in both A and B^c. Now if there were an element in $A \setminus B$, there would be an element in A that is not in B, i.e. an element in A that is in B^c, a contradiction to the fact that $A \cap B^c = \varnothing$, i.e. that there is no element in both A and B^c. This shows that $A \setminus B = \varnothing$. Therefore, if $A \cap B^c = \varnothing$, then $A \setminus B = A \cap B^c$. Then we have shown that if either $A \setminus B$ or $A \cap B^c$ are empty then $A \setminus B = A \cap B^c$. Now assume that both $A \setminus B$ and $A \cap B^c$ are nonempty.

To prove that $A \setminus B = A \cap B^c$, we need to show $\underline{\ A \setminus B \subseteq A \cap B^c\ }$ and $\underline{\ A \cap B^c \subseteq A \setminus B\ }$.

$A \setminus B \subseteq A \cap B^c$: We prove that $A \setminus B \subseteq A \cap B^c$. Let $x \in \underline{\quad A \setminus B \quad}$. Then by definition,

$x \in A$ and $\underline{\quad x \notin B \quad}$. But then $x \in \underline{\quad A \quad}$ and $x \in B^c$. This shows that

$x \in \underline{\quad A \cap B^c \quad}$. Therefore, this shows that $\underline{\ A \setminus B \subseteq A \cap B^c\ }$.

$\underline{\ A \cap B^c \subseteq A \setminus B\ }$: We need to show that $A \cap B^c \subseteq A \setminus B$. Let $x \in \underline{\quad A \cap B^c \quad}$. Then

$x \in \underline{\quad A \quad}$ and $x \in \underline{\quad B^c \quad}$. But then $x \in \underline{\quad A \quad}$ and

$x \notin \underline{\quad B \quad}$. This shows that $x \in \underline{\quad A \setminus B \quad}$. Therefore, we know that $\underline{\ A \cap B^c \subseteq A \setminus B\ }$.

Because $\underline{\ A \setminus B \subseteq A \cap B^c\ }$ and $\underline{\ A \cap B^c \subseteq A \setminus B\ }$, we know that $A \setminus B = A \cap B^c$. $\square$