

**Quiz 1.** *True/False:* The following is a truth table for  $P \rightarrow Q$ :

$P$	$Q$	$P \rightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	F

**Solution.** The statement is *false*. The correct truth table should be...

$P$	$Q$	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

One way to think about this is as follows: imagine  $P$  is a guarantee. Namely, we promise that if  $P$  happens,  $Q$  must happen. For instance,  $P$  could represent the statement, “You do not tamper with your hardware,” and  $Q$  could be the statement, “I will replace your broken computer.” So  $P \rightarrow Q$  is then the statement, “If you do not tamper with your hardware, then I will replace your broken computer.” If both  $P$  and  $Q$  are true, then this should be true—because I promised to replace the computer if you left it alone. If  $P$  is true and  $Q$  is false, then the statement should be false because I broke my promise. However, my promise holds true whenever  $P$  is false. Why? Because you broke our agreement by tampering with the hardware. So while I may or may not replace the computer, my promise has not been broken in either case, i.e. it remains true. In an implication  $P \rightarrow Q$ , if  $P$  is false, then the statement  $P \rightarrow Q$  is *always* true.

**Quiz 2.** *True/False:*  $\forall x, \exists y, x^2 + y = 4$

**Solution.** The statement is *true*. The statement says that for all  $x$  there is a  $y$  such that  $x^2 + y = 4$ . If this is true (which it is), we need to prove it. Fix an  $x$ , say  $x_0$ . We need to find a  $y$  such that  $x_0^2 + y = 4$ . Define  $y_0 := 4 - x_0^2$ . But then we have

$$x_0^2 + y_0 = x_0^2 + (4 - x_0^2) = 4,$$

as desired.

**Quiz 3.** *True/False:*  $\neg(\forall x, \exists y, P(x, y) \vee \neg Q(x, y)) = \exists x, \forall y, \neg P(x, y) \wedge Q(x, y)$

**Solution.** The statement is *true*. We can simply compute the negation step-by-step:

$$\begin{aligned} \neg(\forall x, \exists y, P(x, y) \vee \neg Q(x, y)) &\equiv \exists x, \neg(\exists y, P(x, y) \vee \neg Q(x, y)) \\ &\equiv \exists x, \forall y, \neg(P(x, y) \vee \neg Q(x, y)) \\ &\equiv \exists x, \forall y, \neg P(x, y) \wedge \neg(\neg Q(x, y)) \\ &\equiv \exists x, \forall y, \neg P(x, y) \wedge Q(x, y) \end{aligned}$$

**Quiz 4. True/False:** To prove  $P \Rightarrow Q$ , you can prove  $Q \Rightarrow P$ .

**Solution.** The statement is *false*. The converse of  $P \Rightarrow Q$  is  $Q \Rightarrow P$ . The converse of a logical statement is not necessarily logically equivalent to the original statement. So proving the converse does not necessarily prove the original statement. However, the contrapositive of  $P \Rightarrow Q$ , which is  $\neg Q \Rightarrow \neg P$ , is logically equivalent to  $P \Rightarrow Q$ . Therefore, to prove  $P \Rightarrow Q$ , one only need prove  $\neg Q \Rightarrow \neg P$ . This is called proof by contrapositive.

**Quiz 5. True/False:** Let  $A = \{1\}$  and  $B = \{3, \{1\}\}$ . Then  $A \subseteq B$ .

**Solution.** The statement is *false*. Recall that  $A \subseteq B$  if every element of  $A$  is an element of  $B$ . The only element of  $A$  is the element 1. However,  $1 \notin B$ , but rather  $\{1\} \in B$ , i.e. 1 is not in  $B$  but the set consisting of only the element of 1 is in  $B$ . However, note that  $A \in B$  because  $A = \{1\}$  and  $\{1\} \in B$ .

**Quiz 6. True/False:** Take the universal set to be the integers. Then the following two sets are equal:

$$A = \{n: n \text{ odd}\}$$
$$B = \{m: m \text{ prime and } m > 2\}$$

**Solution.** The statement is *false*. We know that  $9 \in A$  because 9 is odd. But  $9 \notin B$  because  $9 = 3 \cdot 3$  is not prime. Therefore,  $A \not\subseteq B$  so that  $A \neq B$ .

**Quiz 7. True/False:** The sets  $A \times B \times C$  and  $(A \times B) \times C$  are not the same.

**Solution.** The statement is *true*. Elements in  $A \times B \times C$  'look like'  $(a, b, c)$ , where  $a \in A$ ,  $b \in B$ , and  $c \in C$ . Whereas elements in  $(A \times B) \times C$  'look like'  $((a, b), c)$ , where  $a \in A$ ,  $b \in B$ , and  $c \in C$ . Because elements in these sets are not of the same form, they cannot be the same. As an explicit example, take  $A = \{1\}$ ,  $B = \{2, 3\}$ , and  $C = \{4\}$ . Then

$$A \times B \times C = \{(1, 2, 4), (1, 3, 4)\}$$
$$(A \times B) \times C = \{((1, 2), 4), ((1, 3), 4)\}$$

Then  $A \times B \times C \neq (A \times B) \times C$ .

**Quiz 8.** *True/False:* There is a set  $S$  such that  $\mathcal{P}(S)$  has 3 elements.

**Solution.** If  $S$  is an infinite set, then clearly there is a subset for each element  $s \in S$ , i.e. the subset  $\{s\}$ . Clearly, if there is such a set, it cannot be infinite. Now if  $S$  had 3 or more elements—having a subset for each element of  $S$ —we know that  $\mathcal{P}(S)$  would have more than 3 subsets. Therefore,  $S$  must have 0, 1, or 2 elements. If  $S = \emptyset$ , then  $\mathcal{P}(S) = \{\emptyset\}$ . If  $S = \{s_1\}$ , then  $\mathcal{P}(S) = \{\emptyset, \{s_1\}\}$ . Finally, if  $S = \{s_1, s_2\}$ , then  $\mathcal{P}(S) = \{\emptyset, \{s_1\}, \{s_2\}, S\}$ . Therefore, there cannot be such a set  $S$ .