MAT 104: Exam 2
Spring – 2023
04/13/2023
85 Minutes

Name: _____ *Caleb McWhorter — Solutions* _____

Write your name on the appropriate line on the exam cover sheet. This exam contains 16 pages (including this cover page) and 15 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work.

Question	Points	Score
1	15	
2	15	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
15	10	
Total:	160	

1. (15 points) Complete the table of exact values for the trigonometric figures given below.

θ	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$
$\sin\theta$	0	$\dfrac{1}{2}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{1}{2}$	0
$\tan\theta$	0	$\dfrac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undef.

2. (15 points) Compute the exact values for the trigonometric functions given below.

(a) $\cos\left(\dfrac{5\pi}{6}\right)$

(b) $\tan\left(\dfrac{4\pi}{3}\right)$

(c) $\sin\left(-\dfrac{7\pi}{4}\right)$

(d) $\tan\left(\dfrac{3\pi}{2}\right)$

(e) $\sec\left(\dfrac{\pi}{4}\right)$

(f) $\cot\left(\pi\right)$

(g) $\csc\left(\dfrac{2\pi}{3}\right)$

Solution.

(a)
$$\cos\left(\frac{5\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

(b)
$$\tan\left(\frac{4\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

(c)
$$\sin\left(-\frac{7\pi}{4}\right) = -\sin\left(\frac{7\pi}{4}\right) = -\left(-\sin\left(\frac{\pi}{4}\right)\right) = \frac{1}{\sqrt{2}}$$

(d)
$$\tan\left(\frac{3\pi}{2}\right) = \textit{undefined}$$

(e)
$$\sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{4}\right)} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

(f)
$$\cot\left(\pi\right) = \frac{1}{\tan(\pi)} = \frac{1}{0} = \textit{undefined}$$

(g)
$$\csc\left(\frac{2\pi}{3}\right) = \frac{1}{\sin\left(\frac{2\pi}{3}\right)} = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

3. (10 points) Compute the exact values (in radians) for the inverse trigonometric functions given below.

(a) $\sin^{-1}\left(-\dfrac{1}{2}\right)$

(b) $\arccos\left(\dfrac{1}{2}\right)$

(c) $\tan^{-1}\left(-\sqrt{3}\right)$

(d) $\arcsin\left(\sin\left(\dfrac{7\pi}{6}\right)\right)$

(e) $\arctan(\infty)$

Solution.

(a)
$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

(b)
$$\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

(c)
$$\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

(d)
$$\arcsin\left(\sin\left(\frac{7\pi}{6}\right)\right) = \arcsin\left(-\sin\left(\frac{\pi}{6}\right)\right) = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

(e)
$$\arctan(\infty) = \frac{\pi}{2}$$

4. (10 points) Given the right triangle below, compute the exact values of the indicated trigonometric functions.
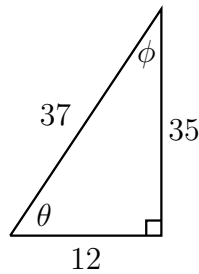
(a) $\sin\theta$

(b) $\cos\phi$

(c) $\tan\theta$

(d) $\sec\theta$

(e) $\csc\phi$

Solution.

(a)
$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{35}{37}$$

(b)
$$\cos\phi = \frac{adjacent}{hypotenuse} = \frac{35}{37}$$

(c)
$$\tan\theta = \frac{opposite}{adjacent} = \frac{35}{12}$$

(d)
$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\dfrac{adjacent}{hypotenuse}} = \frac{hypotenuse}{adjacent} = \frac{37}{12}$$

(e)
$$\csc\phi = \frac{1}{\sin\phi} = \frac{1}{\dfrac{opposite}{hypotenuse}} = \frac{hypotenuse}{opposite} = \frac{37}{12}$$

5. (10 points) Showing all your work, compute the exact value of the expressions given below.
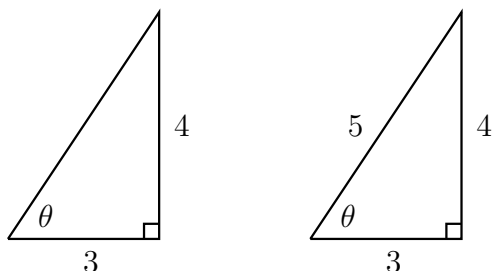
(a) $\sin(2\theta)$, if $\tan(\theta) = \dfrac{4}{3}$ and $0 < \theta < \dfrac{\pi}{2}$

(b) $\tan\left(\cos^{-1}\left(-\dfrac{3}{4}\right)\right)$
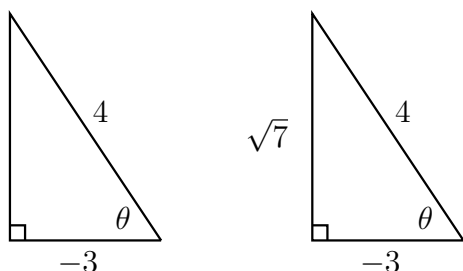
Solution.

(a)

Because $0 < \theta < \frac{\pi}{2}$ is acute and using the fact that $\tan(\theta) = opposite/adjacent$, because $\tan(\theta) = \dfrac{4}{3}$, we can draw the following triangle on the left below:

Because the triangle is a right triangle, using the Pythagorean Theorem, we know that $a^2 + b^2 = c^2$.

But then...

$$a^2 + b^2 = c^2$$
$$3^2 + 4^2 = c^2$$
$$9 + 16 = c^2$$
$$25 = c^2$$
$$c = \sqrt{25}$$
$$c = 5$$

This gives the triangle on the right above. Finally, using the fact that $\sin\theta = opposite/hypotenuse$, $\cos\theta = adjacent/hypotenuse$, and $\sin(2\theta) = 2\sin\theta\cos\theta$, we have...

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2 \cdot \frac{4}{5} \cdot \frac{3}{5} = \frac{24}{25}$$

(b)

Suppose that $\theta = \cos^{-1}(-3/4)$. Applying cosine to both sides we have $\cos\theta = \cos\left(\cos^{-1}(-3/4)\right) = -\frac{3}{4}$. Replacing θ with its reference angle, using the fact that $0 \le \cos^{-1}(x) \le \pi$, $\cos\theta < 0$, and the fact that $\cos\theta = adjacent/hypotenuse$, because $\cos\theta = -\frac{3}{4}$, we can draw the triangle below.

Applying the Pythagorean Theorem, we find $(-3)^2 + b^2 = 4^2$ so that $9 + b^2 = 16$. But then $b^2 = 7$, which implies $b = \pm\sqrt{7}$. Because $\cos\theta = -\frac{3}{4} < 0$, we know that $\frac{\pi}{2} < \cos^{-1}(-3/4) < \pi$. But then we know $b = \sqrt{7}$. But then using the fact that $\tan\theta = opposite/adjacent$, we have...

$$\tan\left(\cos^{-1}\left(-\frac{3}{4}\right)\right) = \tan(\theta) = \frac{\sqrt{7}}{-3} = -\frac{\sqrt{7}}{3}$$

6. (10 points) Complete the parts given below.

    (a) Find the value of $\sin^2(1-x) + \cos^2(1-x)$.

    (b) Express $\sin(\theta \pm \phi)$ in terms of $\sin\theta$, $\sin\phi$, $\cos\theta$, and $\cos\phi$.

    (c) Express $\cos(2\theta)$ in terms of $\cos\theta$ and $\sin\theta$.

    (d) Express $\sin^2\theta$ in terms of $\cos(2\theta)$.

    (e) Express $\cos^2\theta$ in terms of $\cos(2\theta)$.

Solution.

    (a)
$$\sin^2(1-x) + \cos^2(1-x) = 1$$

    (b)
$$\sin(\theta \pm \phi) = \sin(\theta)\cos(\phi) \pm \sin(\phi)\cos(\theta)$$

    (c)
$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

    (d)
$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

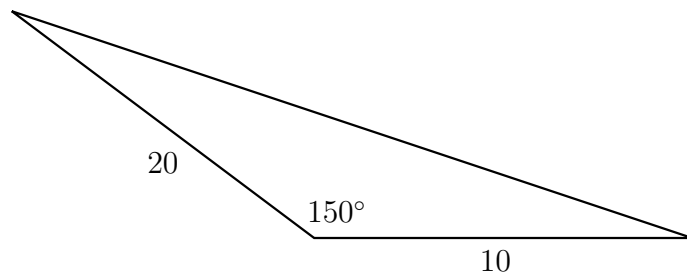   *(e)*
$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

7. (10 points) Given that $\dfrac{11\pi}{12} = \dfrac{5\pi}{4} - \dfrac{\pi}{3}$, compute the exact value of $\cos\left(\dfrac{11\pi}{12}\right)$.
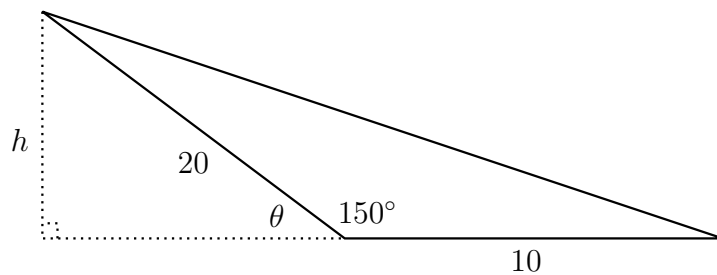
Solution.

$$\cos\left(\frac{11\pi}{12}\right) = \cos\left(\frac{5\pi}{4} - \frac{\pi}{3}\right)$$

$$= \cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)$$

$$= -\cos\left(\frac{\pi}{4}\right) \cdot \cos\left(\frac{\pi}{3}\right) + \left(-\sin\left(\frac{\pi}{4}\right)\right) \cdot \sin\left(\frac{\pi}{3}\right)$$

$$= -\frac{1}{\sqrt{2}} \cdot \frac{1}{2} + \left(-\frac{1}{\sqrt{2}}\right) \cdot \frac{\sqrt{3}}{2}$$

$$= -\frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{-1 - \sqrt{3}}{2\sqrt{2}}$$

$$= -\frac{1 + \sqrt{3}}{2\sqrt{2}}$$

8. (10 points) Compute the exact area of the triangle shown below.

Solution. *Dropping a perpendicular from the upper-left most vertex, we can construct the right triangle shown below with angle θ.*

Using the fact that $\theta + 150° = 180°$, we know that $\theta = 30°$. We know that $\sin \theta = opposite/hypotenuse$, we have...

$$\sin \theta = \frac{opposite}{hypotenuse}$$

$$\sin (30°) = \frac{h}{20}$$

$$\frac{1}{2} = \frac{h}{20}$$

$$20 \cdot \frac{1}{2} = \frac{h}{20} \cdot 20$$

$$h = 10$$

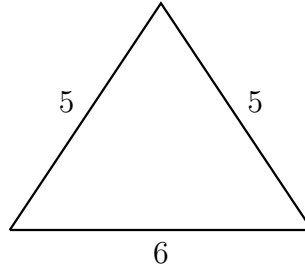*But then we have...*
$$A = \frac{1}{2}\, bh = \frac{1}{2} \cdot 10 \cdot 10 = 5 \cdot 10 = 50$$

Note: One could also use...
$$A = \frac{1}{2}\, ab \sin C = \frac{1}{2} \cdot 20 \cdot 10 \sin(150°) = \frac{1}{2} \cdot 20 \cdot 10 \cdot \frac{1}{2} = 10 \cdot 5 = 50$$

9. (10 points) Compute the exact area of the triangle shown below.

Solution. *To use Heron's area formula, we first find the semiperimeter...*
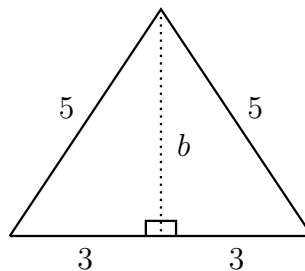
$$s = \frac{a+b+c}{2} = \frac{5+5+6}{2} = \frac{16}{2} = 8$$

But then by Heron's area formula, we have...

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{8(8-5)(8-5)(8-6)} = \sqrt{8 \cdot 3 \cdot 3 \cdot 2} = \sqrt{144} = 12$$

OR

Observe that the triangle is isosceles, dropping a perpendicular from the uppermost vertex bisects the corresponding side. But this gives us the triangle below.
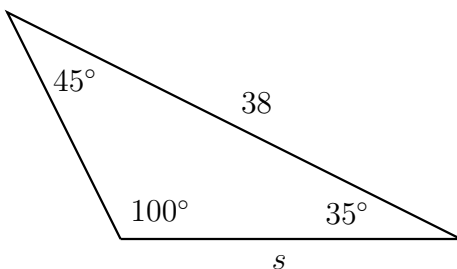

Using the Pythagorean Theorem, we have $a^2 + b^2 = c^2$. But then we have $3^2 + b^2 = 5^2$. This implies that $b^2 = 16$, so that $b = 4$. But then we have...

$$A = \frac{1}{2}bh = \frac{1}{2} \cdot 6 \cdot 4 = 3 \cdot 4 = 12$$

10. (10 points) Find the exact value of the side s in the triangle below.
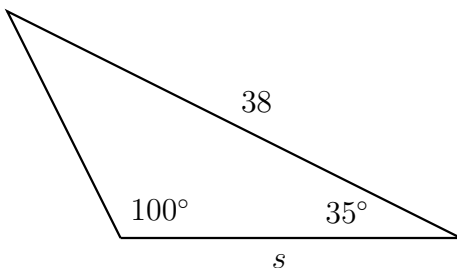
Solution. *Because the sum of the angles of a triangle is* $180°$, *we know the upper-left angle is* $180° - 100° - 35° = 45°$. *This gives the triangle below:*

Using the Law of Sines, we have. . .

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
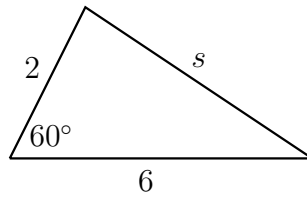
$$\frac{\sin(45°)}{s} = \frac{\sin(100°)}{38}$$

$$s \cdot \sin(100°) = 38 \sin(45°)$$

$$s = \frac{38 \sin(45°)}{\sin(100°)}$$

$$s = \frac{38 \cdot 1/\sqrt{2})}{\sin(100°)}$$

$$s = \frac{38}{\sqrt{2} \sin(100°)}$$

11. (10 points) Find the exact value of the side s in the triangle below.

Solution. *Using the Law of Cosine, we have...*

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$s^2 = 2^2 + 6^2 - 2 \cdot 2 \cdot 6 \cos(60°)$$

$$s^2 = 4 + 36 - 24 \cdot \frac{1}{2}$$

$$s^2 = 40 - 12$$

$$s^2 = 28$$

$$s = \sqrt{28}$$

$$s = 2\sqrt{7}$$

12. **(10 points)** Verify the trigonometric identity given below.
$$\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - \csc^2\theta$$

Solution. *We have...*

$$\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \frac{\tan\theta}{\sin\theta\cos\theta} - \frac{\cot\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta\cos\theta} - \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta\cos\theta}$$

$$= \frac{\sin\theta}{\sin\theta}\cdot\frac{1}{\cos\theta\cdot\cos\theta} - \frac{\cos\theta}{\cos\theta}\cdot\frac{1}{\sin\theta\cdot\sin\theta}$$

$$= \frac{1}{\cos^2\theta} - \frac{1}{\sin^2\theta}$$

$$= \sec^2\theta - \csc^2\theta$$

$$\textbf{OR}$$

We have...

$$\sec^2\theta - \csc^2\theta = \frac{1}{\cos^2\theta} - \frac{1}{\sin^2\theta}$$

$$= \frac{1}{\cos^2\theta}\cdot\frac{\sin^2\theta}{\sin^2\theta} - \frac{1}{\sin^2\theta}\cdot\frac{\cos^2\theta}{\cos^2\theta}$$

$$= \frac{\sin^2\theta}{\sin^2\theta\cos^2\theta} - \frac{\cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$$= \frac{\sin^2\theta - \cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$$= \frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta\cdot\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}\cdot\frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}\left(\frac{\sin^2\theta}{\sin\theta\cos\theta} - \frac{\cos^2\theta}{\sin\theta\cos\theta}\right)$$

$$= \frac{1}{\sin\theta\cos\theta}\left(\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}\right)$$

$$= \frac{1}{\sin\theta\cos\theta}\left(\tan\theta - \cot\theta\right)$$

$$= \frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta}$$

13. (10 points) Verify the trigonometric identity given below.

$$\frac{\sin x}{1 + \cos x} + \frac{\cos x - 1}{\sin x} = 0$$

Solution. *We have...*

$$\frac{\sin x}{1 + \cos x} + \frac{\cos x - 1}{\sin x} = \frac{\sin x}{1 + \cos x} \cdot \frac{\sin x}{\sin x} + \frac{\cos x - 1}{\sin x} \cdot \frac{1 + \cos x}{1 + \cos x}$$

$$= \frac{\sin^2 x}{\sin(x) \cdot (1 + \cos x)} + \frac{(\cos x - 1)(1 + \cos x)}{\sin(x) \cdot (1 + \cos x)}$$

$$= \frac{\sin^2 x + (\cos x - 1)(1 + \cos x)}{\sin(x) \cdot (1 + \cos x)}$$

$$= \frac{\sin^2 x + (\cos x + \cos^2 x - 1 - \cos x)}{\sin(x) \cdot (1 + \cos x)}$$

$$= \frac{\sin^2 x + (\cos^2 x - 1)}{\sin(x) \cdot (1 + \cos x)}$$

$$= \frac{\sin^2 x - (1 - \cos^2 x)}{\sin(x) \cdot (1 + \cos x)}$$

$$= \frac{\sin^2 x - \sin^2 x}{\sin(x) \cdot (1 + \cos x)}$$

$$= 0$$

14. (10 points) Find all the exact solutions to the equation below.

$$3\sin\theta = \sqrt{3}\cos\theta$$

Solution. *Note that* $\cos\theta$ *and* $\sin\theta$ *cannot simultaneously vanish. Therefore, both sides cannot be zero at the same time. Then there can be no solutions when* $\cos\theta = 0$*. We can assume then that* $\cos\theta \neq 0$*. We have...*

$$3\sin\theta = \sqrt{3}\cos\theta$$

$$\frac{3\sin\theta}{\cos\theta} = \frac{\sqrt{3}\cos\theta}{\cos\theta}$$

$$3\tan\theta = \sqrt{3}$$

$$\frac{3\tan\theta}{3} = \frac{\sqrt{3}}{3}$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

But the angles θ *with* $0 \leq \theta < 2\pi$ *such that* $\tan\theta = \frac{1}{\sqrt{3}}$ *are* $\theta = \frac{\pi}{6}$ *and* $\theta = \frac{7\pi}{6}$*. Observe that* $\frac{\pi}{6} + \pi = \frac{7\pi}{6}$*. The period of* $\tan\theta$ *is* π*. Therefore, the solutions are...*

$$\theta = \frac{\pi}{6} \pm k\pi$$

where k *is an integer.*

15. (10 points) Find all the exact solutions to the equation below.

$$2\cos(2x)\sin x + \cos(2x) = 0$$

Solution. *We have. . .*

$$2\cos(2x)\sin x + \cos(2x) = 0$$

$$\cos(2x) \cdot (2\sin x + 1) = 0$$

But then either $\cos(2x) = 0$ or $2\sin x + 1 = 0$. If $\cos(2x) = 0$ and $0 \le 2x < 2\pi$, we know $2x = \frac{\pi}{2}$, so that $x = \frac{\pi}{4}$, or $2x = \frac{3\pi}{2}$, so that $x = \frac{3\pi}{4}$. The period of $\cos(x)$ is 2π so that the period of $\cos(2x)$ is π. But then the solutions to $\cos(2x) = 0$ are $x = \frac{\pi}{4} \pm k\pi$ and $x = \frac{3\pi}{4} \pm k\pi$, where k is an integer.

If $2\sin x + 1 = 0$, then we know that $2\sin x = -1$ so that $\sin x = -\frac{1}{2}$. If $\sin x = -\frac{1}{2}$ and $0 \le x < 2\pi$, then $x = \frac{7\pi}{6}$ or $x = \frac{11\pi}{6}$. Because the period of $\sin x$ is 2π, the solutions to $\sin x = -\frac{1}{2}$ are $x = \frac{7\pi}{6} \pm 2\pi k$ and $x = \frac{11\pi}{6} \pm 2\pi k$, where k is an integer.

Therefore, the solutions are. . .

$$x = \frac{\pi}{4} \pm k\pi$$

$$x = \frac{3\pi}{4} \pm k\pi$$

$$x = \frac{7\pi}{6} \pm 2\pi k$$

$$x = \frac{11\pi}{6} \pm 2\pi k$$