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MATH 361
Spring 2024
HW 1: Due 02/01

"And I knew exactly what to do... but in a much more real sense, I had no idea what to do."
— Michael Scott, The Office

Problem 1. (10pts) Showing all your work and fully justifying your reasoning, compute the following:

(a) $\displaystyle\lim_{x\to -6} \frac{x^2+3x-18}{x^2+5x-6}$

(b) $\displaystyle\lim_{n\to\infty} \frac{2n^2-35n+17}{6n^2+19n-49}$

(c) $\displaystyle\frac{d}{dx}\ln(x\cos x)$

(d) $\displaystyle\int_0^1 \frac{x}{x+1}\,dx$

Solution.

(a)

$$\lim_{x\to -6} \frac{x^2+3x-18}{x^2+5x-6} = \lim_{x\to -6} \frac{(x-3)(x+6)}{(x-1)(x+6)} = \lim_{x\to -6} \frac{(x-3)\cancel{(x+6)}}{(x-1)\cancel{(x+6)}} = \lim_{x\to -6} \frac{x-3}{x-1} = \frac{-6-3}{-6-1} = \frac{-9}{-7} = \frac{9}{7}$$

OR

$$\lim_{x\to -6} \frac{x^2+3x-18}{x^2+5x-6} \overset{\text{L.H.}}{=} \lim_{x\to -6} \frac{2x+3}{2x+5} = \frac{-12+3}{-12+5} = \frac{-9}{-7} = \frac{9}{7}$$

(b)

$$\lim_{n\to\infty} \frac{2n^2-35n+17}{6n^2+19n-49} = \lim_{n\to\infty} \frac{2n^2-35n+17}{6n^2+19n-49}\cdot\frac{1/n^2}{1/n^2} = \lim_{n\to\infty} \frac{2-\frac{35}{n}+\frac{17}{n^2}}{6+\frac{19}{n}-\frac{49}{n^2}} = \frac{2-0+0}{6+0-0} = \frac{2}{6} = \frac{1}{3}$$

OR

$$\lim_{n\to\infty} \frac{2n^2-35n+17}{6n^2+19n-49} \overset{\text{L.H.}}{=} \lim_{n\to\infty} \frac{4n-35}{12n+19} \overset{\text{L.H.}}{=} \lim_{n\to\infty} \frac{4}{12} = \frac{1}{3}$$

(c)

$$\frac{d}{dx}\ln(x\cos x) = \frac{1}{x\cos x}\cdot(\cos x - x\sin x) = \frac{\cos x - x\sin x}{x\cos x} = \frac{\cos x}{x\cos x} - \frac{x\sin x}{x\cos x} = \frac{1}{x} - \tan x$$

(d)

$$\int_0^1 \frac{x}{x+1}\,dx = \int_0^1 \frac{(x+1)-1}{x+1}\,dx = \int_0^1 \left(1-\frac{1}{x+1}\right)\,dx = x-\ln|x+1|\Big|_0^1 = (1-\ln 2)-(0-\ln 1) = 1-\ln 2$$

OR

$$\int_0^1 \frac{x}{x+1}\,dx \quad \begin{array}{c} u=x+1 \\ du=dx \end{array} \quad \int_1^2 \frac{u-1}{u}\,du = \int_1^2 \left(1-\frac{1}{u}\right)\,du = u-\ln|u|\Big|_1^2 = (2-\ln 2)-(1-\ln 1) = 1-\ln 2$$

Problem 2. (10pts) Recall that a sequence $\{a_n\}$ is increasing if $a_{n+1} \geq a_n$ for all n and the sequence is decreasing if $a_{n+1} \leq a_n$ for all n. A sequence $\{a_n\}$ is called bounded above (below) if there exists $M \in \mathbb{R}$ such that $a_n \leq M$ ($a_n \geq M$) for all n. The *Monotone Convergence Theorem* states the following: if $\{a_n\}$ is either increasing or decreasing, i.e. is 'monotone', and bounded above or below, respectively, then $\{a_n\}$ converges. Now consider the sequence with $a_0 = 2$ and given recursively via...

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{5}{a_n}\right)$$

(a) Compute a_1, a_2, a_3.

(b) Compare your values in (a) to $\sqrt{5}$. What might you conjecture?

(c) Explain why the Monotone Convergence Theorem implies that $\{a_n\}$ has a limit.

(d) By (c), we know $L := \lim\limits_{n\to\infty} a_n$ exists. Taking the limit in both sides of the recursive definition for $\{a_n\}$, show that $L = \sqrt{5}$.

Solution.

(a) We use the recurrence relation with $a_0 = 2$.

$$\begin{aligned} a_1 &= \frac{1}{2}\left(a_0 + \frac{5}{a_0}\right) = \frac{1}{2}\left(2 + \frac{5}{2}\right) = \frac{1}{2} \cdot 4.5 = 2.25 \\ a_2 &= \frac{1}{2}\left(a_1 + \frac{5}{a_1}\right) = \frac{1}{2}\left(2.25 + \frac{5}{2.25}\right) = \frac{1}{2} \cdot 4.47222 = 2.23611 \\ a_3 &= \frac{1}{2}\left(a_2 + \frac{5}{a_2}\right) = \frac{1}{2}\left(2.23611 + \frac{5}{2.23611}\right) = \frac{1}{2} \cdot 4.47214 = 2.23607 \end{aligned}$$

(b) We have $\sqrt{5} \approx 2.23607$. Based on the data from (a), we might conjecture (perhaps very foolishly based on so little evidence) that $a_n \to \sqrt{5}$ as $n \to \infty$.

(c) Based on the data above, it seems that the sequence $\{a_n\}$ is decreasing for $n \geq 2$ because $a_3 < a_2$. Furthermore, it seems that the sequence $\{a_n\}$ is bounded (below)—$a_n \geq 0$. If both these are true, then the sequence $\{a_n\}$ is monotone decreasing. But then by the Monotone Convergence Theorem, $\{a_n\}$ would converge to a limit, say L. [We prove this thoroughly at the end of the homework.]

(d) By (c), we that $L := \lim\limits_{n\to\infty} a_n$ exists. But then we also have $L := \lim\limits_{n\to\infty} a_{n+1}$. Moreover, because $a_n > 0$ for all n, we know that $L > 0$. Now...

$$\begin{aligned} a_{n+1} &= \frac{1}{2}\left(a_n + \frac{5}{a_n}\right) \\ \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{5}{a_n}\right) \\ L &= \frac{1}{2}\left(L + \frac{5}{L}\right) \\ 2L^2 &= L^2 + 5 \\ L^2 &= 5 \\ L &= \pm\sqrt{5} \\ L &= \sqrt{5} \end{aligned}$$

Problem 3. (10pts) The Intermediate Value Theorem states the following: if $f(x)$ is continuous on $[a,b]$ and $f(a) < c < f(b)$, then there exists an $x_0 \in (a,b)$ such that $f(x_0) = c$. Consider the function $f(x) = x^2 - 3x + 4$ on the interval $[-1,5]$.

(a) Give a sketch of $f(x)$ on the interval $[-1,6]$.

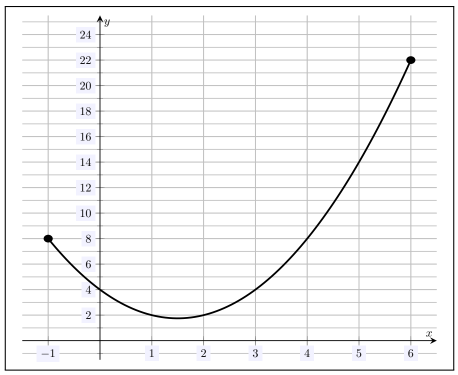
(b) Explain why $f(x)$ is continuous.

(c) Explain why there is a $x_0 \in [-1,6]$ such that $f(x_0) = 14$.

(d) Find the $x_0 \in [-1,6]$ such that $f(x_0) = 14$.

Solution.

(a)



(b) The function $f(x)$ is a polynomial. Hence, $f(x)$ is continuous on $\mathbb{R}$. In particular, it is continuous on $[-1,6]$.

(c) We have $f(-1) = 8$ and $f(6) = 22$. Clearly, $f(-1) = 8 < 14 < 22 = f(6)$. By the Intermediate Value Theorem, there exists $x_0 \in [-1,6]$ such that $f(x_0) = 14$. Alternatively, we can see that a horizontal line at $y = 14$ intersects the graph of $f(x)$. Note that from the graph above, there are also $x_0 \in [-1,6]$ such that $f(x_0) = a$, where $a \in [\frac{7}{4}, 8)$—but the existence of such a x_0 does not follow from the Intermediate Value Theorem applied to $[-1,6]$.

(d) By (c), we know there exists $x_0 \in [-1,6]$ such that $f(x_0) = 14$. But then...

$$\begin{aligned} f(x_0) &= 14 \\ x_0^2 - 3x_0 + 4 &= 14 \\ x_0^2 - 3x_0 - 10 &= 0 \\ (x_0 - 5)(x_0 + 2) &= 0 \end{aligned}$$

But then $x_0 = -2$ or $x_0 = 5$. Clearly, $-2 \notin [-1,6]$. Therefore, we have $x_0 = 5$. We can confirm this: $f(5) = 5^2 - 3(5) + 4 = 25 - 15 + 4 = 14$.

Problem 4. (10pts) The Mean Value Theorem states the following: if $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)\big(b - a\big)$. Consider the function $f(x) = x^3 + x^2 - 4x - 5$. Find the values $c \in [-1, 4]$ that satisfy the Mean Value Theorem.

Solution. We first confirm that the Mean Value Theorem applies to $f(x)$. The function $f(x)$ is a polynomial. Therefore, $f(x)$ is continuous on $\mathbb{R}$ and infinitely differentiable on $\mathbb{R}$. In particular, $f(x)$ is continuous on $[-1, 4]$ and differentiable on $(-1, 4)$. By the Mean Value Theorem, there exists $c \in [-1, 4]$ such that $f(4) - f(-1) = f'(c)\big(4 - (-1)\big)$. We have...

$$f(4) = 4^3 + 4^2 - 4(4) - 5 = 64 + 16 - 16 - 5 = 59$$

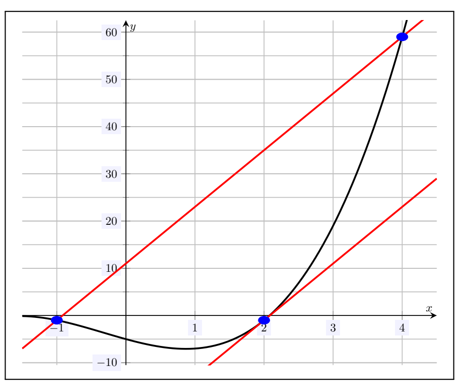
$$f(-1) = (-1)^3 + (-1)^2 - 4(-1) - 5 = -1 + 1 + 4 - 5 = -1$$

$$f'(x) = 3x^2 + 2x - 4 \Rightarrow f'(c) = 3c^2 + 2c - 4$$

But then we have...

$$\begin{gathered} f(4) - f(-1) = f'(c)\big(4 - (-1)\big) \\ 59 - (-1) = (3c^2 + 2c - 4) \cdot 5 \\ 60 = 5(3c^2 + 2c - 4) \\ 12 = 3c^2 + 2c - 4 \\ 0 = 3c^2 + 2c - 16 \\ 0 = (c - 2)(3c + 8) \end{gathered}$$

Therefore, $c = 2$ or $c = -\frac{8}{3}$. As $-\frac{8}{3} \notin (-1, 4)$, it must be that $c = 2$. Indeed, we have $f'(2) = 3(4) + 4 - 4 = 12$, which has the same slope as the line connecting $\big(-1, f(-1)\big)$ and $\big(4, f(4)\big)$.



Problem 2 (Continued). We prove that the Monotone Convergence Theorem can be used to show that $\{a_n\}$ converges to $\sqrt{5}$. We shall show that a_n (or a tail of the sequence) is decreasing and bounded below.

Bounded Below. We need to show that there is an $M \in \mathbb{R}$ such that $M \leq a_n$ for all n. We shall show that $a_n \geq \sqrt{5}$ for all $n \geq 1$. We first show that a_n is positive, i.e. $a_n \geq 0$, for all n by induction. Clearly, $a_0 = 2 > 0$. Assume that a_n is positive. We need show that a_{n+1} is positive. But...

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{5}{a_n}\right)$$

But then a_{n+1} is formed from the quotient, sum, and product of positive real numbers. Therefore, a_{n+1} is positive. By induction, $a_n > 0$ for all n, i.e. a_n is positive for all n. In fact, this shows that $\{a_n\}$ is bounded below (by $M = 0$).

Because $a_n > 0$ for all n, $a_n \geq \sqrt{5}$ if and only if $a_n^2 \geq 5$. Now for $n \geq 1$, we have...

$$a_{n+1}^2 = \left(\frac{1}{2}\left(a_n + \frac{5}{a_n}\right)\right)^2 = \left(\frac{a_n + \frac{5}{a_n}}{2}\right)^2 \geq a_n \cdot \frac{5}{a_n} = 5$$

We have used the fact that for all $x, y \in \mathbb{R}$, $\left(\frac{x+y}{2}\right)^2 \geq xy$. To see this, observe...

$$\left(\frac{x+y}{2}\right)^2 - xy = \frac{x^2+2xy+y^2}{4} - xy = \frac{x^2+2xy+y^2}{4} - \frac{4xy}{4} = \frac{x^2-2xy+y^2}{4} = \frac{(x-y)^2}{4} = \left(\frac{x-y}{2}\right)^2 \geq 0$$

But then $\left(\frac{x+y}{2}\right)^2 \geq xy$. The above result then follows by taking $a = a_n$ and $b = \frac{5}{a_n}$. Because $a_{n+1}^2 \geq 5$ for all n, we know $a_{n+1} \geq 5$ for all n.

Decreasing. We show that $\{a_n\}$ is decreasing for $n \geq 1$. To prove this, we show that $a_{n+1} - a_n \leq 0$, i.e. $a_n - a_{n+1} \geq 0$. For $n \geq 1$, we know that $a_n^2 \geq 5$ so that...

$$\begin{aligned}
a_n - a_{n+1} &= a_n - \frac{1}{2}\left(a_n + \frac{5}{a_n}\right) \\
&= a_n - \frac{a_n}{2} - \frac{5}{2a_n} \\
&= \frac{a_n}{2} - \frac{5}{2a_n} \\
&= \frac{a_n^2}{2a_n} - \frac{5}{2a_n} \\
&= \frac{a_n^2 - 5}{2a_n} \geq \frac{0}{2a_n} = 0
\end{aligned}$$

But then $a_n - a_{n+1} \geq 0$ so that $\{a_n\}$ is decreasing for $n \geq 1$.

Therefore, the sequence $\{a_n\}_{n\geq 1}$ is decreasing and bounded below. Therefore, by the Monotone Convergence Theorem, the sequence $\{a_n\}_{n\geq 1}$ converges. The sequence $\{a_n\}_{n\geq 0}$ has a limit if and only if $\{a_n\}_{n\geq 1}$ has a limit; if so, they have the same limit. Therefore, $\{a_n\}_{n\geq 0}$ has a limit, L.