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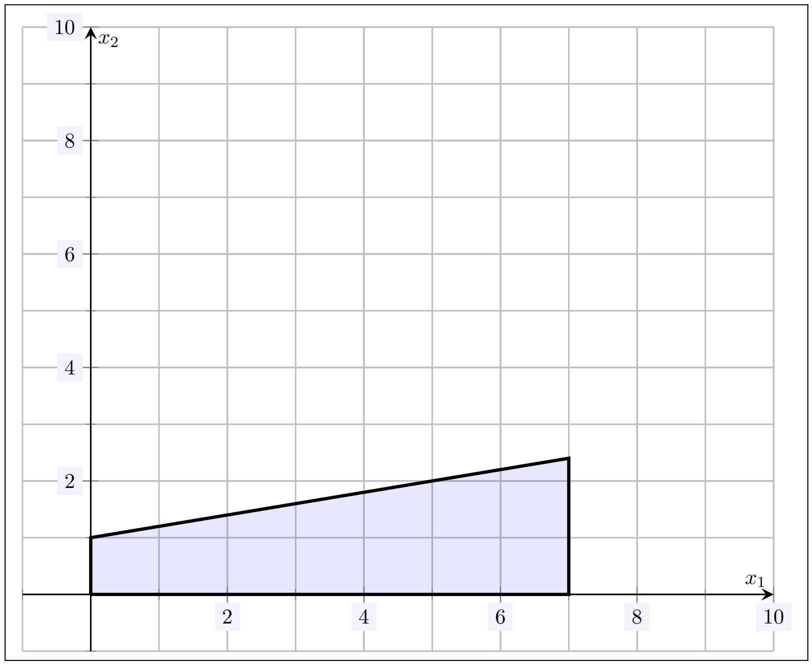
MATH 108
Fall 2022
HW 18: Due 12/06

"True optimization is the revolutionary contribution of modern research to decision processes."
–George Dantzig

Problem 1. (10pt) As accurately as possible, sketch the feasible region given by the following maximization problem on the plot below:

$$\begin{aligned} \max z &= 4x_1 + 6x_2 \\ -x_1 + 5x_2 &\leq 5 \\ x_1 + x_2 &\leq 11 \\ x_1, x_2 &\geq 0 \\ x_1 &\leq 7 \end{aligned}$$



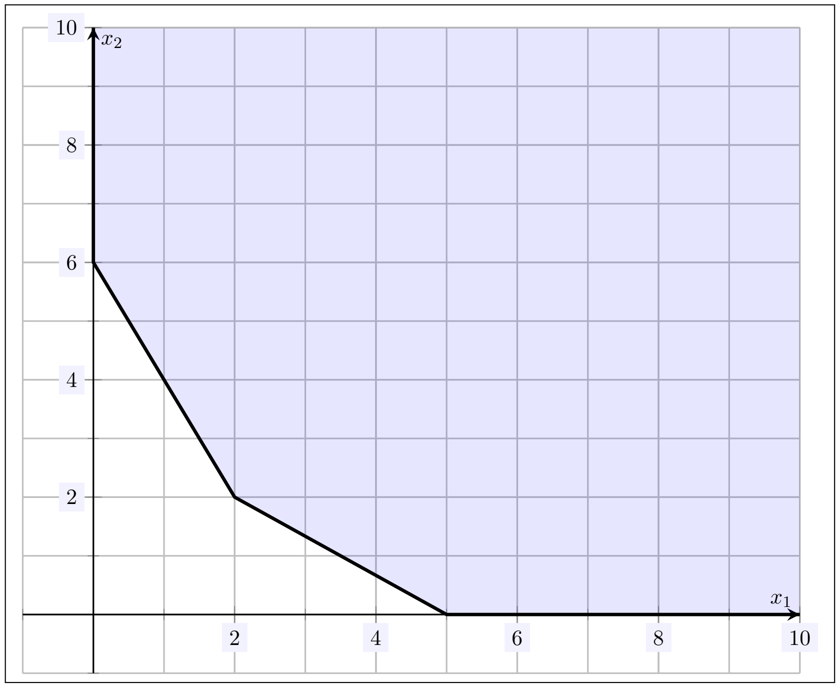
Is this region nonempty? Is this region bounded or unbounded? Solve this maximization problem.

Solution. 'Solving' $-x_1 + 5x_2 \leq 5$ for x_2, we obtain $x_2 \leq \frac{1}{5}\,x_1 + 1$. The line $x_2 = \frac{1}{5}\,x_1 + 1$ has y-intercept 1 and slope $\frac{1}{5}$. Because $x_2 \leq \frac{1}{5}\,x_1 + 1$, we shade below this line. 'Solving' $x_1 + x_2 \leq 11$ for x_2, we obtain $x_2 \leq 11 - x_1$. The line $x_2 = 11 - x_1$ has y-intercept 11 and slope -1. Because $x_2 \leq 11 - x_1$, we shade below this line. The line $x_1 = 7$ is a vertical line with $x_1 = 7$ for all points on the line. Because $x_1 \leq 7$, we shade to the left of this line. The line $x_1 = 0$ is the y-axis. Because $x_1 \geq 0$, we need shade to the right of the y-axis. The line $x_2 = 0$ is the x-axis. Because $x_2 \geq 0$, we need shade above the x-axis. [Note: Together, the inequalities $x_1, x_2 \geq 0$ simply state that the region must be in Quadrant I.] Sketching this above, we see that we need the intersection of $x_2 = \frac{1}{5}\,x_1 + 1$ and $x_1 = 7$. But if $x_1 = 7$, then $x_2 = \frac{1}{5} \cdot 7 + 1 = \frac{12}{5}$. Therefore, the lines intersect at $\left(7, \frac{12}{5}\right)$. Putting all this information together, we obtain the region shaded above.

This region is clearly nonempty. Because we can easily draw a 'ball' around the region, the region is also bounded. The function $z = 4x_1 + 6x_2$ is linear. Therefore, the Fundamental Theorem of Linear Programming states that the function z has a maximum and minimum value on this region and that they occur at a corner point. The corner points are $(0,0)$, $(0,1)$, $\left(7, \frac{12}{5}\right)$, and $(7,0)$. Evaluating z at these points, we find $z(0,0) = 0$, $z(0,1) = 0 + 6 = 6$, $z\left(7, \frac{12}{5}\right) = 28 + \frac{72}{5} = \frac{212}{5}$, and $z(7,0) = 28 + 0 = 28$. Clearly, the maximum value of z on this region is $\frac{212}{5}$ and occurs at $(x_1, x_2) = \left(7, \frac{12}{5}\right)$.

Problem 2. (10pt) As accurately as possible, sketch the feasible region given by the following maximization problem on the plot below:

$$\begin{aligned} \min z &= 4x_1 + 6x_2 \\ 2x_1 + x_2 &\geq 6 \\ 2x_1 + 3x_2 &\geq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$



Is this region nonempty? Is this region bounded or unbounded? Solve this minimization problem.

Solution. 'Solving' for x_2 in $2x_1 + x_2 \geq 6$, we obtain $x_2 \geq 6 - 2x_1$. The line $x_2 = 6 - 2x_1$ has y-intercept 6 and slope -2. Because $x_2 \geq 6 - 2x_1$, we shade above the line. 'Solving' for x_2 in $2x_1 + 3x_2 \geq 10$, we obtain $x_2 \geq \frac{10}{3} - \frac{2}{3}\,x_1$. The line $x_2 = \frac{10}{3} - \frac{2}{3}\,x_1$ has y-intercept $\frac{10}{3}$ and slope $-\frac{2}{3}$. Because $x_2 \geq \frac{10}{3} - \frac{2}{3}\,x_1$, we shade above the line. The line $x_1 = 0$ is the y-axis. Because $x_1 \geq 0$, we need shade to the right of the y-axis. The line $x_2 = 0$ is the x-axis. Because $x_2 \geq 0$, we need shade above the x-axis. [Note: Together, the inequalities $x_1, x_2 \geq 0$ simply state that the region must be in Quadrant I.] Sketching this above, we see we need the intersection of $x_2 = 6 - 2x_1$ and $x_2 = \frac{10}{3} - \frac{2}{3}\,x_1$. This gives $6 - 2x_1 = \frac{10}{3} - \frac{2}{3}\,x_1$. Multiplying both sides by 3, we obtain $18 - 6x_1 = 10 - 2x_1$. But then $4x_1 = 8$ so that $x_1 = 2$. But then $x_2 = 6 - 2(2) = 6 - 2 = 2$. Therefore, the lines intersect at $(x_1, x_2) = (2, 2)$. We also need the x-intercept of $x_2 = \frac{10}{3} - \frac{2}{3}\,x_1$. But then $0 = \frac{10}{3} - \frac{2}{3}\,x_1$ so that $x_1 = 5$, i.e. the x-intercept is $(x_1, x_2) = (5, 0)$. Putting all this information together yields the region shaded in the graph above.

Clearly, the region is nonempty. Because there is no 'ball' which encloses the entire region, the region is unbounded. But then the Fundamental Theorem of Linear Programming does not apply. However, increasing either x_1 or x_2 increases z. Because we can always increase x_1 or x_2 and stay within the region, z does not have a maximum on this region. However, vice versa, decreasing x_1 or x_2 decreases z. Because the region is 'bounded from below', the function has a minimum and must occur at a corner point. The corner points are $(0, 6)$, $(2, 2)$, and $(5, 0)$. Evaluating z at these points, we find $z(0, 6) = 0 + 36 = 36$, $z(2, 2) = 8 + 12 = 20$, and $z(5, 0) = 20 + 0 = 20$. Therefore, the minimum value is 20 and occurs at $(2, 2)$ or $(5, 0)$ (actually, along the line $x_2 = \frac{10}{3} - \frac{2}{3}\,x_1$).