Name: *Caleb McWhorter — Solutions*
MATH 108
Fall 2021
HW 5: Due 10/05

"I go. You stay. No following."
–Iron Giant, Iron Giant

Problem 1. (10pt) Watch the following three videos by 3Blue1Brown (Grant Sanderson):

(i) Linear transformations and matrices

(ii) Matrix multiplication as composition

(iii) The determinant

What did you learn from these videos?

Answers will vary.

Problem 2. (10pt) Suppose the reduced-row echelon form for an augmented matrix is the following:

$$\begin{pmatrix} 1 & -4 & 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

Using this, find all the solutions to the system of equations.

Solution. The last row of the matrix tells us that $x_5 = 3$. The second row of the matrix tells us that $x_3 = -1$. The first row of the matrix tells us that $x_1 - 4x_2 + x_4 = 5$. Fixing any two of x_1, x_2, x_4, we can solve for the third. Therefore, we have two free variables. For instance, choosing x_2 and x_4 to be free, we have solutions

$$\begin{cases} x_1 = 4x_2 - x_4 + 5 \\ x_2 : \text{free} \\ x_3 = -1 \\ x_4 : \text{free} \\ x_5 = 3 \end{cases}$$

Problem 3. (10pt) Can you compute the following product of matrices? If you can, compute the product. If you can not, explain why.

$$\begin{pmatrix} 1 & -1 & 8 \\ 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & -6 \\ 7 & 7 \\ -8 & 0 \end{pmatrix}$$

Solution. The first matrix is 2×3. The second matrix is 4×2. For matrix multiplication to be defined, we need the number of columns of the first to equal the number of rows of the second. But $3 \neq 4$ so that this matrix multiplication is not defined, i.e. we cannot compute this product.

Problem 4. (10pt) Showing all your work, compute the following:

$$\begin{pmatrix} 5 & 0 & 1 \\ -2 & -3 & 4 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Solution.

$$\begin{aligned} \begin{pmatrix} 5 & 0 & 1 \\ -2 & -3 & 4 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 5(2) + 0(-1) + 1(1) \\ -2(2) + (-3)(-1) + 4(1) \\ 1(2) + (-1)(-1) + 1(1) \end{pmatrix} \\ &= \begin{pmatrix} 10 + 0 + 1 \\ -4 + 3 + 4 \\ 2 + 1 + 1 \end{pmatrix} \\ &= \begin{pmatrix} 11 \\ 3 \\ 4 \end{pmatrix} \end{aligned}$$

Problem 5. (10pt) Showing all your work, compute the following:

$$\begin{pmatrix} 1 & -1 & 2 \\ -3 & 6 & 0 \end{pmatrix} \begin{pmatrix} 0 & 4 \\ -2 & 3 \\ 3 & -2 \end{pmatrix}$$

Solution.

$$\begin{aligned}
\begin{pmatrix} 1 & -1 & 2 \\ -3 & 6 & 0 \end{pmatrix} \begin{pmatrix} 0 & 4 \\ -2 & 3 \\ 3 & -2 \end{pmatrix} &= \begin{pmatrix} 1(0) + (-1)(-2) + 2(3) & 1(4) + (-1)3 + 2(-2) \\ -3(0) + 6(-2) + 0(3) & -3(4) + 6(3) + 0(-2) \end{pmatrix} \\
&= \begin{pmatrix} 0 + 2 + 6 & 4 - 3 - 4 \\ 0 - 12 + 0 & -12 + 18 + 0 \end{pmatrix} \\
&= \begin{pmatrix} 8 & -3 \\ -12 & 6 \end{pmatrix}
\end{aligned}$$

Problem 6. (10pt) Compute the determinant of the following matrix:

$$\begin{pmatrix} 2 & 1 & 5 \\ -3 & 0 & 3 \\ 7 & 2 & 7 \end{pmatrix}$$

Is this matrix invertible? Explain.

Solution.

$$\begin{aligned}
\det \begin{pmatrix} 2 & 1 & 5 \\ -3 & 0 & 3 \\ 7 & 2 & 7 \end{pmatrix} &= -(-3) \begin{vmatrix} 1 & 5 \\ 2 & 7 \end{vmatrix} + 0 \begin{vmatrix} 2 & 5 \\ 7 & 7 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 7 & 2 \end{vmatrix} \\
&= 3\big(1(7) - 2(5)\big) + 0 - 3\big(2(2) - 7(1)\big) \\
&= 3\big(7 - 10\big) - 3\big(4 - 7\big) \\
&= 3(-3) - 3(-3) \\
&= -9 - (-9) \\
&= -9 + 9 \\
&= 0
\end{aligned}$$

Because the determinant of the matrix is 0, the matrix cannot be invertible.

Problem 7. (10pt) Find the inverse of the following matrix:

$$\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$$

Solution.

$$\left(\begin{array}{rr|rr} 2 & -1 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array}\right) \qquad 3R_1 + 2R_2 \to R_2$$

$$\left(\begin{array}{rr|rr} 2 & -1 & 1 & 0 \\ 0 & 5 & 3 & 2 \end{array}\right) \qquad \frac{1}{5}R_2 \to R_2$$

$$\left(\begin{array}{rr|rr} 2 & -1 & 1 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \end{array}\right) \qquad R_2 + R_1 \to R_1$$

$$\left(\begin{array}{rr|rr} 2 & 0 & \frac{8}{5} & \frac{2}{5} \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \end{array}\right) \qquad \frac{1}{2}R_1 \to R_1$$

$$\left(\begin{array}{rr|rr} 1 & 0 & \frac{4}{5} & \frac{1}{5} \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \end{array}\right)$$

Therefore, the inverse is...

$$\begin{pmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$$

Alternatively, because the matrix is 2×2, we can use the shortcut method:

$$\det \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} = 2(4) - (-3)(-1) = 8 - 3 = 5$$

Then we have

$$\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}^{-1} = \frac{1}{5}\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$$