MAT 101: Exam 1
Spring — 2024
02/21/2024
85 Minutes

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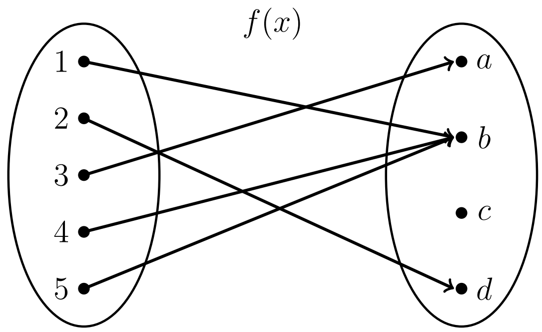
Write your name on the appropriate line on the exam cover sheet. This exam contains 11 pages (including this cover page) and 10 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) Consider the relation given by the diagram below.



(a) Is the relation a function? Explain.

(b) Find the domain of the relation.

(c) Find the codomain of the relation.

(d) Find the range of the relation.

Solution.

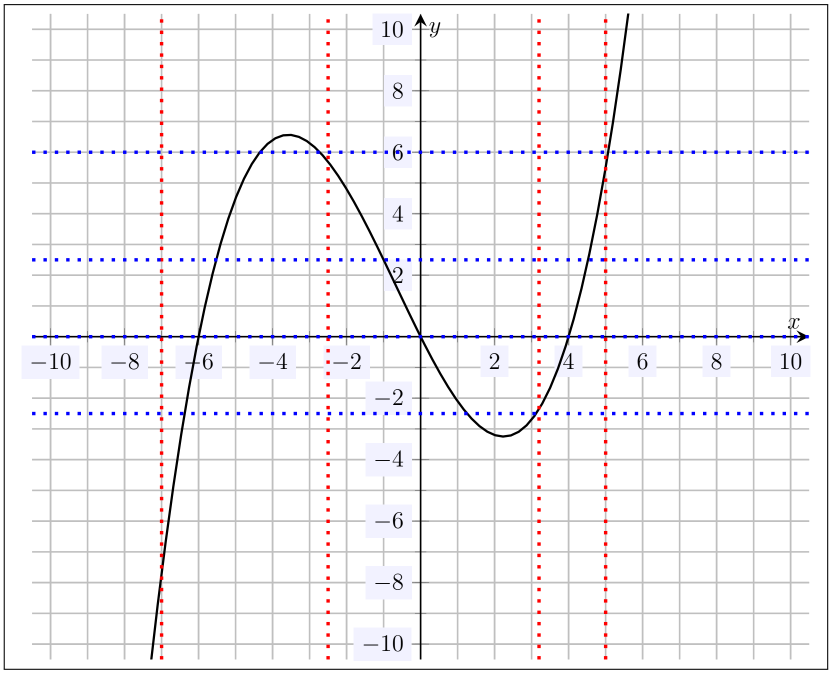
(a) Yes, the relation $f(x)$ is a function. Observe for that each of its inputs, $1, 2, 3, 4, 5$, there is exactly one output, a, b, c, d, respectively.

(b) The domain of this function is $\{1, 2, 3, 4, 5\}$.

(c) The codomain of this function is $\{a, b, c, d\}$.

(d) The range or image of this function is $\{a, b, d\}$.

2. (10 points) Consider the relation plotted below.



(a) Is this relation a function of x? Explain.

(b) Is this relation a function of y? Explain.

(c) If one were to consider this relation as a function of x, would the relation have an inverse? Explain.

Solution.

(a) Yes, this relation is a function of x. This is because the relation passes the vertical line test; that is, every vertical line intersects the graph of the relation at most once. Therefore, for each x, there is at most once y-value associated to x.

(b) No, the relation is not a function of y. This is because the relation fails the horizontal line test; that is, there is a horizontal line that intersects the graph of the relation more than once. Therefore, there is a y-value that is associated to more than one x-value. For instance, the y-value 0 is associated to $x = -6, 0, 4$.

(c) No, the relation would not have an inverse. This is because the relation fails the horizontal line test; that is, there is a horizontal line that intersects the graph of the relation more than once. Therefore, there is a y-value that is associated to more than one x-value. For instance, the y-value 0 is associated to $x = -6, 0, 4$, i.e. if we called this relation $f(x)$ then $f^{-1}(0) = \{-6, 0, 4\}$.

3. (10 points) Let $f(x)$ be the relation given by $f(x) = x(x-1)(x+3)$.

   (a) Is $f(x)$ a function? Explain.

   (b) Find $f(2)$.

   (c) Find the y-intercept(s) of $f(x)$.

   (d) Find the x-intercept(s) of $f(x)$.

Solution.

(a) Yes, the relation is a function. For each input x, there is exactly one output—namely, the one obtained by evaluating $f(x)$ at x and following order of operations.

(b) We have. . .

$$f(2) = 2(2-1)(2+3) = 2(1)5 = 10$$

(c) The y-intercept is the point where the graph of $f(x)$ intersects the y-axis, where $x = 0$. But then. . .

$$f(0) = 0(0-1)(0+3) = 0(-1)3 = 0$$

Therefore, the y-intercept is 0, i.e. the point $(0, 0)$.

(d) The x-intercepts are the point(s) where the graph of $f(x)$ intersects the x-axis, where $y = 0$. But then. . .

$$\begin{aligned} f(x) &= 0 \\ x(x-1)(x+3) &= 0 \end{aligned}$$

This implies that either $x = 0$, or $x - 1 = 0$ so that $x = 1$, or $x + 3 = 0$ so that $x = -3$. Therefore, the x-intercepts are $-3, 0, 1$, i.e. the points $(-3, 0)$, $(0, 0)$, and $(1, 0)$.

4. (10 points) Let $f(x)$ and $g(x)$ be functions for which a table of values is given below.

x	-5	-2	0	1	2	3
$f(x)$	6	-1	3	4	3	-1
$g(x)$	-5	7	-2	4	0	6

Based on the table above, compute the following:

(a) $f(-2) - g(3)$

(b) $(f+g)(2)$

(c) $(fg)(-5)$

(d) $(g \circ f)(0)$

(e) $(f \circ g)(0)$

Solution.

(a)
$$f(-2) - g(3) = -1 - 6 = -7$$

(b)
$$(f+g)(2) = f(2) + g(2) = 3 + 0 = 3$$

(c)
$$(fg)(-5) = f(-5) \cdot g(-5) = 6 \cdot -5 = -30$$

(d)
$$(g \circ f)(0) = g\big(f(0)\big) = g(3) = 6$$

(e)
$$(f \circ g)(0) = f\big(g(0)\big) = f(-2) = -1$$

5. (10 points) Let $g(x) = x^2 + 2x - 3$.

   (a) Find $g(2)$ and $g(-4)$.

   (b) Based on your answer to (a), can $g^{-1}(x)$ exist? Explain.

Solution.

(a) We have...

$$g(2) = 2^2 + 2(2) - 3 = 4 + 4 - 3 = 5$$
$$g(-4) = (-4)^2 + 2(-4) - 3 = 16 - 8 - 3 = 5$$

(b) No, $g^{-1}(x)$ cannot exist. We know that $g(2) = 5 = g(-4)$. Therefore, $g^{-1}(5)$ contains the set $\{-4, 2\}$, i.e. the input 5 would have more than one output. Therefore, $g^{-1}(x)$ cannot be a function.

6. (10 points) Consider the function $\ell(x) = \dfrac{4-3x}{5}$.

   (a) Explain why $\ell(x)$ is linear.

   (b) Find the slope of this function.

   (c) Find the y-intercept of this function.

   (d) Find the x-intercept of this function.

   (e) Does the graph of this function contain the point $(3, -1)$? Explain.

Solution.

(a) Observe that…

$$\ell(x) = \frac{4-3x}{5} = \frac{4}{5} - \frac{3}{5}\,x$$

Therefore, $\ell(x)$ has the form $y = mx + b$ with $y = \ell$, $x = x$, $m = -\frac{3}{5}$, and $b = \frac{4}{5}$.

(b) From (a), we know $\ell(x) = -\frac{3}{5}\,x + \frac{4}{5}$. Therefore, the slope is $m = -\frac{3}{5}$.

(c) From (a), we know $\ell(x) = -\frac{3}{5}\,x + \frac{4}{5}$. Therefore, the y-intercept is $b = \frac{4}{5}$, i.e. the point $(0, \frac{4}{5})$.

(d) The x-intercept is the point(s) where the function intersects the x-axis, where $y = 0$. But then…

$$\begin{aligned}
\ell(x) &= 0 \\
\frac{4-3x}{5} &= 0 \\
4 - 3x &= 0 \\
3x &= 4 \\
x &= \frac{4}{3}
\end{aligned}$$

Therefore, the x-intercept is $\frac{4}{3}$, i.e. the point $(\frac{4}{3}, 0)$.

(e) If the graph of $\ell(x)$ contained the point $(3, -1)$, then $\ell(3) = -1$. We have…

$$\ell(3) = \frac{4-3(3)}{5} = \frac{4-9}{5} = \frac{-5}{5} = -1$$

Therefore, the graph of $\ell(x)$ contains the point $(3, -1)$.

7. (10 points) Explain why the function $f(x) = 3(5-2x)$ has an inverse. Furthermore, find the inverse. Be sure to show all your work. [You do not need to verify that your inverse is indeed the inverse.]

Solution. We have $f(x) = 3(5-2x) = 15-6x$. This is a linear function, i.e. it has the form $y = mx + b$ with $y = f$, $x = x$, $m = -6$, and $b = 15$. Because $m = -6 \neq 0$, we know this is a non-constant linear function. Therefore, we know that $f^{-1}(x)$ exists. Writing $y = 3(5-2x)$, we find the inverse by interchanging the roles of y and x (obtaining $x = 3(5-2y)$) and solving for y:

$$\begin{aligned} x &= 3(5-2y) \\ x &= 15-6y \\ x-15 &= -6y \\ y &= \frac{x-15}{-6} \\ y &= \frac{15-x}{6} \end{aligned}$$

Therefore, $f^{-1}(x) = \dfrac{15-x}{6}$.

Though we were not required to check that this is indeed the inverse, i.e. $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, we shall verify this fact anyway:

$$\begin{aligned} (f \circ f^{-1})(x) &= f\big(f^{-1}(x)\big) \\ &= f\left(\frac{15-x}{6}\right) \\ &= 3\left(5 - 2 \cdot \frac{15-x}{6}\right) \\ &= 3\left(5 - \frac{15-x}{3}\right) \\ &= 3\left(5 - \left(5 - \frac{x}{3}\right)\right) \\ &= 3\left(5 - 5 + \frac{x}{3}\right) \\ &= 3 \cdot \frac{x}{3} \\ &= x \end{aligned}$$

$$\begin{aligned} (f^{-1} \circ f)(x) &= f^{-1}\big(f(x)\big) \\ &= f^{-1}\big(3(5-2x)\big) \\ &= \frac{15 - 3(5-2x)}{6} \\ &= \frac{15 - 15 + 6x}{6} \\ &= \frac{6x}{6} \\ &= x \end{aligned}$$

8. (10 points) Find the exact equation of the line with x-intercept -6 and y-intercept 4. Show all your work.

Solution. Because the line has x-intercept -6, the graph contains the point $(-6, 0)$. Because the line has y-intercept 4, the graph contains the point $(0, 4)$. But then the slope of this line is...

$$m = \frac{\Delta y}{\Delta x} = \frac{0-4}{-6-0} = \frac{-4}{-6} = \frac{2}{3}$$

Clearly, this is not a vertical line so that it has the form $y = mx + b$. We know the y-intercept is 4, which implies $b = 4$. Therefore, the line is...

$$y = \frac{2}{3}x + 4$$

9. (10 points) Find the exact equation of the line parallel to the line $4x - 3y = 6$ whose graph contains the point $(-9, -8)$. Show all your work.

Solution. We solve for y in $4x - 3y = 6$ to identify the slope of this line. We have...

$$\begin{aligned} 4x - 3y &= 6 \\ -3y &= -4x + 6 \\ y &= \frac{4}{3}\,x - 2 \end{aligned}$$

Therefore, this line has slope $\frac{4}{3}$. Because our line is parallel to this line and parallel lines have identical slopes, the slope of the line in question must be $\frac{4}{3}$. Clearly, the line in question is not vertical so that it has the form $y = mx + b$. But then $y = \frac{4}{3}\,x + b$. Because the graph of the line contains the point $(-9, -8)$, we know that $y = -8$ when $x = -9$. But then...

$$\begin{aligned} y &= \frac{4}{3}\,x + b \\ -8 &= \frac{4}{3} \cdot -9 + b \\ -8 &= -12 + b \\ b &= 4 \end{aligned}$$

Therefore, the line is $y = \frac{4}{3}\,x + 4$.

Alternatively, we can use the point-slope form to find y:

$$\begin{aligned} y &= y_0 + m(x - x_0) \\ y &= -8 + \frac{4}{3}\,\big(x - (-9)\big) \\ y &= -8 + \frac{4}{3}\,(x + 9) \\ y &= -8 + \frac{4}{3}\,x + 12 \\ y &= \frac{4}{3}\,x + 4 \end{aligned}$$

10. (10 points) Find the equation of the line perpendicular to $y = \dfrac{5-3x}{6}$ whose graph passes through the x-intercept of the line $-3x + 9y = 15$. Show all your work.

Solution. The line $y = \frac{5-3x}{6} = \frac{5}{6} - \frac{3}{6}\,x = \frac{5}{6} - \frac{1}{2}\,x$ has slope $-\frac{1}{2}$. Because the line in question is perpendicular to this line, it must have a slope which is the negative reciprocal of $-\frac{1}{2}$. Therefore, the line in question has slope $-\frac{1}{-1/2} = 2$. Clearly, the line in question is not vertical so that it has the form $y = mx + b$. But then we know $y = 2x + b$.

The x-intercept of the line $-3x + 9y = 15$ is the point where the graph intersects the x-axis where $y = 0$. But then...

$$\begin{aligned} -3x + 9y &= 15 \\ -3x + 9(0) &= 15 \\ -3x &= 15 \\ x &= -5 \end{aligned}$$

Therefore, the x-intercept of this line is $(-5, 0)$. But then the graph of the line in question contains the point $(-5, 0)$, i.e. $y = 0$ when $x = -5$. But then...

$$\begin{aligned} y &= 2x + b \\ 0 &= 2(-5) + b \\ 0 &= -10 + b \\ b &= 10 \end{aligned}$$

Therefore, the line is $y = 2x + 5$.

Alternatively, we can use the point-slope form to find y:

$$\begin{aligned} y &= y_0 + m(x - x_0) \\ y &= 0 + 2\big(x - (-5)\big) \\ y &= 2(x + 5) \\ y &= 2x + 10 \end{aligned}$$