Quiz 1. *True/False*: If P is the proposition $6 < 5$ and Q is the proposition, "Earth is a planet," then the logical statement $P \to Q$ is false.

Solution. The statement is *false*. Recall that the truth table for $P \to Q$ is as follows:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Here, P is the proposition $P : 6 < 5$ and Q is the proposition Q: "Earth is a planet." It is clear that P is false and Q is true. But then examining the logic table above, we can see that $P \to Q$ is true.

Quiz 2. *True/False*: $\neg(P \to \neg Q) \equiv P \wedge Q$

Solution. The statement is *true*. To determine if two propositions are logically equivalent, one can either examine the truth table or apply logical rules to obtain one logical expression from the other. If we construct a truth table, we have...

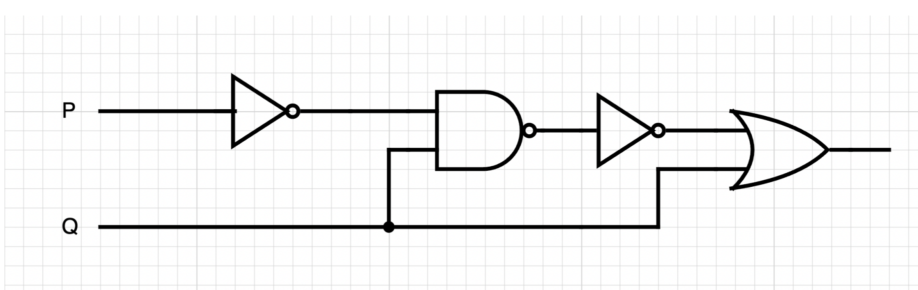
P	Q	$\neg Q$	$P \to \neg Q$	$\neg(P \to \neg Q)$	$P \wedge Q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F

Because for each possible pair of choices for P and Q the outputs for $\neg(P \to \neg Q)$ and $P \wedge Q$ match, $\neg(P \to \neg Q) \equiv P \wedge Q$. Alternatively, we can transform one into the other by applying logical equivalences (recall $P \to Q \equiv \neg P \vee Q$ or $\neg(P \to Q) \equiv P \wedge \neg Q$):

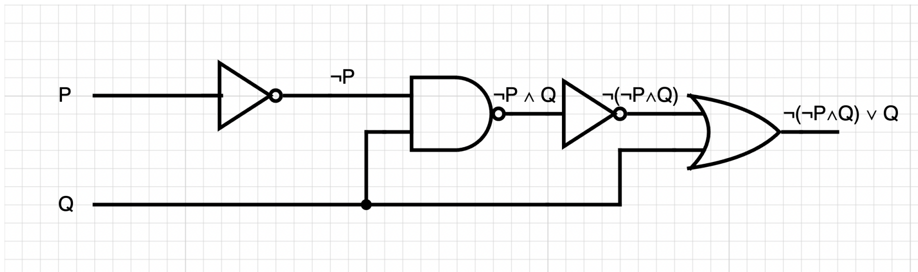
$$\neg(P \to \neg Q) \equiv \neg(\neg P \vee \neg Q) \equiv \neg(\neg P) \wedge \neg(\neg Q) \equiv P \wedge Q.$$

Quiz 3. *True/False*: The logic corresponding to the circuit shown below is the proposition:

$$(\neg P \wedge Q) \vee \neg Q.$$



Solution. The statement is *false*. We can trace through the circuit. We see that the current from P passes through a NOT gate and we obtain $\neg P$. This then feeds into an AND gate along with Q so that we obtain $\neg P \wedge Q$. The resulting current is then passed through a NOT gate, obtaining $\neg(\neg P \wedge Q)$. This finally reaches an OR gate—along with Q—to obtain $\neg(\neg P \wedge Q) \vee Q$. We can see a diagrammatic explanation below.



Quiz 4. *True/False*: Let the universe $\mathcal{U}$ be the set of real numbers and define $P(x)$ to be the predicate $P(x) : x^2 + x - 4 \geq 0$. Then $(\forall x)\big(\neg P(x)\big)$ is true.

Solution. The statement is *false*. If $P(x) : x^2 + x - 4 \geq 0$, then $\neg P(x) : x^2 + x - 4 < 0$. But then $(\forall x)\big(\neg P(x)\big)$ is the statement, "For all x, $x^2+x-4 < 0$." Now if $x = 1$, we have $\neg P(1)\colon 1^2+1-4 < 0$, i.e. $-2 < 0$, which is true. If $x = 0$, we have $\neg P(0)\colon 0^2 + 0 - 4 < 0$, i.e. $-4 < 0$, which is true. However, while $(\forall x)\big(\neg P(x)\big)$ is clearly true for *some* (we found at least two), it is not true *for all* x. As a counterexample, let $x = 10$. Then $\neg P(10)\colon 10^2 + 10 - 4 < 0$, which is $104 < 0$—clearly false. Therefore, $\neg P(x)$ is not true for all x. But then $(\forall x)\big(\neg P(x)\big)$ is false.

Quiz 5. *True/False*: Let the domain of x, y be the integers. Then $(\exists! x)(\forall y)(x + 2y = 5)$.

Solution. The statement is *false*. The logical proposition $(\exists! x)(\forall y)(x + 2y = 5)$ in words states, "There exists a unique x such that for all y, $x + 2y = 5$." Suppose that there were such a x, say x_0. Then we know that $x_0 + 2y = 5$ for all y. In particular, x_0 satisfies this equality when $y = 0$. But then we know that $x_0 = 5$. But also, it must satisfy the equality when $x = 1$. But then $x_0 + 2 = 5$ so that x_0. Then there is not a unique x that works for all y! Therefore, the statement is false. Note that if we reverse the quantifiers, the statement is true: $(\forall y)(\exists! x)(x + 2y = 5)$. In this case, this is the statement, "For all y, there exists a unique x such that $x + 2y = 5$." If you were given any y, define $x_0 := 5 - 2y$. But then $x + 2y = (5 - 2y) + 2y = 5$. So there exists such an x. Is it unique? Well if there were two or more x values that worked for some y, say two of them are x_0 and $\tilde{x}_0$, then we have $x_0 + 2y = 5 = \tilde{x}_0 + 2y$. But then $x_0 + 2y = \tilde{x}_0 + 2y$. Subtracting $2y$, we have $x_0 = \tilde{x}_0$. Therefore, there can only be one such x. Because we have found one, we know that the statement that for all y, there exists a unique y such that $x + 2y = 5$ is true.

Quiz 6. *True/False*: $\{1, 2\} \subseteq \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$

Solution. The statement is *false*. We know that $A \subseteq B$ if and only if for all $a \in A$, we have $a \in B$. We test every element of the set $\{1, 2\}$. The first element is 1. However, $1 \notin \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$. [Note that $1 \notin \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$ but $\{1\} \in \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.] However, we do have $\{1, 2\} \notin \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

Quiz 7. *True/False*: $\displaystyle\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \varnothing$

Solution. The statement is *false*. For $n = 1$, the set $(-\frac{1}{n}, \frac{1}{n})$ is the interval $(-1, 1)$. For $n = 2$, the set $(-\frac{1}{n}, \frac{1}{n})$ is the interval $(-\frac{1}{2}, \frac{1}{2})$. For $n = 3$, the set $(-\frac{1}{n}, \frac{1}{n})$ is the interval $(-\frac{1}{3}, \frac{1}{3})$. Note that 0 is an element of all these sets. Generally, we have $0 \in (-\frac{1}{n}, \frac{1}{n})$ for all $n \in \mathbb{N}$. But then we know that $0 \in \displaystyle\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$. This is sufficient to demonstrate that this is not empty. [Note that it is actually true that $\displaystyle\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$—though this takes more work to prove.]

Quiz 8. *True/False*: Let $E(n)$ denote the relation from $\mathbb{N}$ to $\mathbb{Z}^{\geq 0}$ given by the rule that $E(n)$ is the number of positive even integers less than or equal to n. Then this relation is a function with $E(5) = 2$, i.e. 2 is in the image of 5, and 10 in the preimage of 5.

Solution. The statement is *true*. There are several claims here. First, the claim that $E(n) : \mathbb{N} \to \mathbb{Z}^{\geq 0}$ is a function. Given some $n \in \mathbb{N}$, there is a single number of positive even integers $\leq n$. But then for every input for $E(n)$, there is only one possible output. Therefore, $E(n)$ is a function from $\mathbb{N}$ to $\mathbb{Z}^{\geq 0}$. For 2 to be in the image of 5, we need $E(5) = 2$. There are two positive even integers ≤ 5 (namely, 2 and 4) so that 2 is in the image of 5. For 10 to be in the preimage of 5, we would have to have $E(10) = 5$. Note that there are 5 positive even integers ≤ 10 (namely 2, 4, 6, 8, 10). Therefore, 10 is in the preimage of 5.

Quiz 9. *True/False*: Let $f : X \to Y$ be a function. Then f^{-1} will be a function if and only if the preimage set satisfies the following: $(\forall y \in \text{im } f)(\exists x \in X)(f^{-1}(y) = x)$.

Solution. The statement is *false*. Take for example the function $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$ given by $f(x) = x^2$. For all $y \in \mathbb{R}^{\geq 0}$, there exists an $x \in \mathbb{R}$ such that $f(x) = y$, namely $\pm\sqrt{y}$. But if $y > 0$, then there are two possibilities: $+\sqrt{y}$ and $-\sqrt{y}$. But this function $f(x)$ has f^{-1} with the property that $(\forall y \in \text{im } f)(\exists x \in X)(f^{-1}(y) = x)$. If we want f^{-1} to be a function, we require $(\forall y \in \text{im } f)(\exists! x \in X)(f^{-1}(y) = x)$.

Quiz 10. *True/False*: Suppose that $f : A \to B$ and $g : B \to C$ are functions and that $g \circ f$ is injective. Then it must be that f is injective.

Solution. The statement is *true*. Observe that $g \circ f : A \to C$. Suppose f were not injective. Then there are two values in A, say a_1, a_2, such that $a_1 \neq a_2$ and $f(a_1) = f(a_2)$. But then we have...

$$\begin{aligned} f(a_1) &= f(a_2) \\ g(f(a_1)) &= g(f(a_2)) \\ (g \circ f)(a_1) &= (g \circ f)(a_2) \end{aligned}$$

But then there are two values in the domain of $g \circ f$, namely a_1, a_2 such that $a_1 \neq a_2$ but $(g \circ f)(a_1) = (g \circ f)(a_2)$. But then $g \circ f$ is not injective, contrary to what we were told. Our assumption that f was not injective must then be wrong. Therefore, it must be that f is injective.

Quiz 11. *True/False*: Fix an integer $n > 1$ and let $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ be sequences. Then $\displaystyle\prod_{k=1}^{n} a_n b_n = \prod_{k=1}^{n} a_n \cdot \prod_{k=1}^{n} b_n$ but $\displaystyle\sum_{k=1}^{n} a_n b_n \neq \sum_{k=1}^{n} a_n \cdot \sum_{k=1}^{n} b_n$.

Solution. The statement is *true*. It is true that $\displaystyle\prod_{k=1}^{n} a_n b_n = \prod_{k=1}^{n} a_n \cdot \prod_{k=1}^{n} b_n$. For instance, if $n = 2$, we have...

$$\prod_{k=1}^{2} a_n b_n = a_1 b_1 \cdot a_2 b_2 = (a_1 a_2) \cdot (b_1 b_2) = \prod_{k=1}^{2} a_n \cdot \prod_{k=1}^{2} b_n$$

We can always rearrange the terms in this way for any n. Therefore, the statement is true for products.

However, even in the case of $n = 2$, the statement is untrue for sums. For example, if $n = 2$ in $\displaystyle\sum_{k=1}^{n} a_n b_n \neq \sum_{k=1}^{n} a_n \cdot \sum_{k=1}^{n} b_n$, then we have...

$$\sum_{k=1}^{2} a_n b_n = a_1 b_1 + a_2 b_2$$

$$\sum_{k=1}^{2} a_n \cdot \sum_{k=1}^{n} b_2 = (a_1 + a_2) \cdot (b_1 + b_2) = a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2$$

While this may be true for some sequences $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$, it will not generally be true. The ‘issues’ the distributive property cause for larger n make this even ‘more untrue.’

Quiz 12. *True/False*: Suppose that $a_{n+2} = 6a_n - a_{n+1}$ with $a_0 = 4$ and $a_1 = 3$. Then the characteristic polynomial is given by the equation $x^2 = 6 - x$, i.e. the characteristic polynomial is $x^2 + x - 6$. Because $x^2 + x - 6 = (x+3)(x-2)$ has roots $-3, 2$, the general solution is $a_n = c_1(-3)^n + c_2 2^n$. The specific solution is then $a_n = (-3)^n + 3 \cdot 2^n$.

Solution. The statement is *true*. Because $a_{n+2} = 6a_n - a_{n+1}$ and the 'lowest' term involved is n, we give the nth term power 0 for x. Then we have $x^{0+2} = 6x^0 - x^{0+1}$. This is $x^2 = 6 - x$. We then have $x^2 + x - 6 = 0$. Therefore, the characteristic polynomial for this homogeneous linear recurrence relation is $x^2 + x - 6$. This polynomial has roots -3 and 2 because $x^2 + x - 6 = 0$ is equivalent to $(x+3)(x-2) = 0$, which has solutions $x = -3$ and $x = 2$. Therefore, we know that $a_n = c_1(-3)^n + c_2 \cdot 2^n$. Now we use the fact that when $n = 0$, we have $a_0 = 4$, and when $n = 1$, we have $a_1 = 3$. But then we have…

$$4 = a_0 = c_1(-3)^0 + c_2 \cdot 2^0 = c_1 + c_2$$

$$3 = a_1 = c_1(-3)^1 + c_2 \cdot 2^1 = -3c_1 + 2c_2$$

This is a linear system of two equations in two unknowns. Solving this system yields $c_1 = 1$ and $c_2 = 3$. Therefore, we have $a_n = (-3)^n + 3 \cdot 2^n$.

Quiz 13. *True/False*: $6^{2022} \equiv 1 \text{ mod } 5$

Solution. The statement is *true*. Using the division algorithm, we know that $6 = 1(5) + 1$. But then we know that $6 \equiv 1 \text{ mod } 5$. But then we have…

$$6^{2022} \equiv 1^{2022} \equiv 1 \text{ mod } 5$$

Quiz 14. *True/False*: There is a unique solution to the following system of linear congruences:

$$\begin{aligned} 2x - 1 &\equiv 2 \text{ mod } 3 \\ x &\equiv 0 \text{ mod } 5 \\ 6x &\equiv 5 \text{ mod } 7 \end{aligned}$$

Solution. The statement is *true*. The first congruence is $2x - 1 \equiv 2 \text{ mod } 3$. Adding 1 to both sides, we see that this is equivalent to $2x \equiv 3 \equiv 0 \text{ mod } 3$. Because $\gcd(2, 3) = 1$, we know that 2^{-1} exists mod 3. In fact, because $2 \cdot 2 \equiv 4 \equiv 1 \text{ mod } 3$. Therefore, $2^{-1} \equiv 2 \text{ mod } 3$. Therefore, $2x \equiv 0 \text{ mod } 3$ implies $2^{-1} \cdot 2x \equiv 2^{-1} \cdot 0 \text{ mod } 3$. This is $x \equiv 0 \text{ mod } 3$. In the last congruence, because $\gcd(6, 7) = 1$, we know that 6^{-1} exists mod 7. In fact, because $6 \cdot 6 \equiv 36 \equiv 1 \text{ mod } 7$, we know that $6^{-1} \equiv 6 \text{ mod } 7$. But then $6x \equiv 5 \text{ mod } 7$ implies $6^{-1} \cdot 6x \equiv 6^{-1} \cdot 5 \text{ mod } 7$. But this is $x \equiv 6 \cdot 5 \equiv 30 \equiv 2 \text{ mod } 7$. The original system of congruences is then equivalent to…

$$\begin{aligned} x &\equiv 0 \text{ mod } 3 \\ x &\equiv 0 \text{ mod } 5 \\ x &\equiv 2 \text{ mod } 7 \end{aligned}$$

This system now has the 'proper form' to apply the Chinese Remainder Theorem. Because $\gcd(3, 5, 7) = 1$, we know there exists a unique solution to the system of congruences. In fact, we have solution...

$$x = \sum a_i N_i M_i = 0 \cdot 2 \cdot 35 + 0 \cdot 1 \cdot 21 + 2 \cdot 1 \cdot 15 = 0 + 0 + 30 = 30$$

The solution is then the congruence class of 30 modulo $3 \cdot 5 \cdot 7 = 105$. Therefore, the solution is 30, i.e. $[30] = \{\ldots, -720, -570, -420, -270, -120, 30, 180, 330, 480, 630, 780, \ldots\}$.

Quiz 15. *True/False*: The number 1 is prime.

Solution. The statement is *false*. A prime number is an integer greater than 1 which has no proper divisors. Because 1 is not greater than 1, 1 cannot be prime. Note that 1 is also not composite. To be composite, an integer need have proper divisors. However, the only divisor of 1 is 1. Therefore, 1 also cannot be composite. This shows 1 is neither prime nor composite.

Quiz 16. *True/False*: Using the division algorithm to divide -10 by 3, we have $-10 = 3(3) + 1$.

Solution. The statement is *false*. Recall that given $a, b \in \mathbb{Z}$ with $a \neq 0$, we can write $b = qa + r$ for some $q, r \in \mathbb{Z}$ and $0 \leq r < a$. Clearly, the statement is false because $3(3) + 1 = 9 + 1 = 10 \neq -10$. The multiple of 3 that is less than or equal to -10 is -12. Because $-10 = -12 + 2$, we have $-10 = -4(3) + 2$. Because $-4, 2 \in \mathbb{Z}$ and $0 \leq 2 < 3$, we have expressed -10 divided by 3 using the division algorithm. Note that one cannot use $-10 = -3(3) - 1$ because it is not the case that $0 \leq -1 < 3$.

Quiz 17. *True/False*: The number $2B002B$ in base-10 is 2818091.

Solution. The statement is *true*. The number 2B002B is in hexadecimal. Converting this to base-10 (and recalling $A = 10$, $B = 11$, $C = 12$, $D = 13$, $E = 14$, and $F = 15$), we have...

$$2B002B = 2 \cdot 16^5 + 11 \cdot 16^4 + 0 \cdot 16^3 + 0 \cdot 16^2 + 2 \cdot 16^1 + 11 \cdot 16^0 = 2097152 + 720896 + 0 + 0 + 32 + 11 = 2818091$$

Quiz 18. *True/False*: If A, B are $n \times n$ matrices, then $AB = BA$.

Quiz 19. *True/False*: Let S be a nonempty finite set with $|S| = n$. Then the number of subsets of size at least 2 is...

$$2^n - n - 1$$

Quiz 20. *True/False*: If n is sufficiently large, then the number of dearrangements of n is approximately $n!/e$.

Quiz 21. *True/False*: The number of ways of choosing a committee of 10 people with a president and vice president from 120 people is ${}_{120}C_{10} \cdot {}_{10}C_2$.