MAT 308: Exam 1 **Name:** ______________________________

Fall – 2022

10/21/2022

'∞' Minutes

Write your name on the appropriate line on the exam cover sheet. This exam contains 12 pages (including this cover page) and 10 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) Construct the truth table for the logical expression given below. Is this expression a tautology? Explain.

$$\big[\big(\neg P \wedge (Q \vee R)\big) \to P\big] \iff P$$

2. (10 points) Consider the subsets of $\mathbb{R}$ given by $A = (0, 5]$ and $B = [4, 10)$. Determine the following:

   (a) A^c

   (b) $A \cup B$

   (c) $A \cap B$

   (d) $A - B$

   (e) $A \Delta B$

3. (10 points) Define $P(x)$, $Q(x)$, and $R(x)$ to be the following predicates:

$$
\begin{aligned}
&P(x) : x^2 + x - 20 = 0 \\
&Q(x) : x \text{ is even} \\
&R(x) : x < 0
\end{aligned}
$$

For the universe of integers, write each of the following quantified statements in a sentence (avoiding the word 'not' and stated as 'properly' as possible) and then determine whether the quantified statement is true or false—being sure to justify your answer.

(a) $\forall x \, \big(R(x) \to P(x)\big)$

(b) $\exists x \, \big(P(x) \to \neg Q(x) \wedge R(x)\big)$

(c) $\forall x \, \big(P(x) \to Q(x) \vee R(x)\big)$

(d) $\forall x \, \big(Q(x) \vee R(x) \to P(x)\big)$

(e) $\exists! x \, \big(P(x) \wedge \neg R(x)\big)$

4. (10 points) Recall that DeMorgan's laws, $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$, allows one to compute complements of unions and intersections. Complete the proof below to prove a similar rule to compute complements of relative complements.

   Proposition. Let A and B be sets with common universe $\mathcal{U}$. Then $(A \setminus B)^c = A^c \cup B$.

   Proof. We want to prove $(A \setminus B)^c = A^c \cup B$. So we need to show that ____________________

   and ____________________.

   First, we prove ____________________. We first make an observation. Suppose that

   $y \in A \setminus B$. Then we know that ____________________ and ____________________.

   But then $y \in A \cap B^c$. Now let $x \in (A \setminus B)^c$. By our observation, because $x \in (A \setminus B)^c$,

   it must be that $x \in (A \cap B^c)^c$. But by DeMorgan's Law, we know that

   $(A \cap B^c)^c =$ ____________________. But then because $x \in (A \cap B^c)^c$, we know that

   $x \in A^c \cup B$. So, if $x \in (A \setminus B)^c$, we know $x \in A^c \cup B$. Therefore, ____________________.

   Second, we need to prove that ____________________. Let $x \in A^c \cup B$. Then we know

   that either $x \in$ ____________________ or $x \in$ ____________________. There are two

   cases:

   (i) $x \in A^c$: Observe that if $y \in A \setminus B$, we must have $y \in A$. But then if $y \notin A$, we

   know $y \notin A \setminus B$. This shows that $y \in$ ____________________. Now let

   $x \in A^c$. We need to show that $x \in (A \setminus B)^c$. Because $x \in A^c$, we know

   $x \notin$ ____________________. But then by our observation, we know that

   $x \notin$ ____________________, which implies that $x \in (A \setminus B)^c$.

(ii) $x \in B$: Suppose $x \in B$. If $y \in B$, then we know that $y \notin A \setminus B$. But then

$y \in$ ____________________. But then because $x \in B$, we know that

$x \in$ ____________________.

We have now shown that if $x \in A^c$ or $x \in B$, that $x \in$ ____________________.

Therefore, we know that ____________________.

But then we have shown that ____________________ and ____________________.

Therefore, $(A \setminus B)^c = A^c \cup B$. $\square$

Note: If $(A \setminus B)^c$ or $A^c \cup B$ are empty, then clearly $(A \setminus B)^c \subseteq A^c \cup B$ or $A^c \cup B \subseteq (A \setminus B)^c$, respectfully. We then only needed to prove the result when $(A \setminus B)^c$ and $A^c \cup B$ are nonempty, which was the given proof. We omitted the empty cases for simplicity.

5. (10 points) A certain computer program has n, m as integer variables. Suppose that A is a two-dimensional array of 200 integers values: $A[1,1]$, $A[1,2]$, …, $A[1,20]$, $A[2,1]$, $A[2,2]$, …, $A[2,20]$, …, $A[10,20]$, i.e. A is an array with ten rows and twenty columns. We could represent A visually as a matrix via…

$$\begin{pmatrix} A[1,1] & A[1,2] & \cdots & A[1,20] \\ A[2,1] & A[2,2] & \cdots & A[2,20] \\ \vdots & \ddots & & \vdots \\ A[10,1] & A[10,2] & \cdots & A[10,20] \end{pmatrix}$$

Write each of the following statements as a quantified open statement:

(a) Every element of A is positive.

(b) Some entries of A are larger than 100.

(c) The entries in each row of A are sorted into strictly ascending order.

(d) All of the entries of A are distinct.

(e) All the entries of the first 4 columns of A are distinct.

6. (10 points) Let S_n denote the subset of $\mathbb{R}$ given by $\left[-1+\frac{1}{n}, 2-\frac{1}{n}\right)$. Compute the following:

   (a) S_1

   (b) S_2

   (c) $\bigcup_{n\in\mathbb{N}} S_n$

   (d) $\bigcap_{n\in\mathbb{N}} S_n$

   (e) $\bigcup_{n\in\mathbb{N}} S_n^c$ [Hint: Consider complements and (d).]

7. (10 points) A function $f : \mathbb{R} \to \mathbb{R}$ is called uniformly continuous if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

   (a) Write the definition of a function f being uniformly continuous as a quantified statement.

   (b) Negate your expression in (a).

   (c) State in words what means for a function $f : \mathbb{R} \to \mathbb{R}$ to *not* be uniformly continuous.

8. (10 points) Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. Compute the following:

(a) $\mathcal{P}(\varnothing)$

(b) $A \times B$

(c) $\mathcal{P}(A \setminus B)$

(d) $A \times \varnothing$

(e) $A \cap (A \times B)$

(f) $(A \setminus B) \cap \mathcal{P}(A \setminus B)$

9. (10 points) Negate the logical expression below, simplifying as much as possible. Your answer should not include '¬' symbols.

$$\neg(P \vee Q) \vee \big((\neg P \wedge Q) \vee \neg Q\big)$$

10. (10 points) Mark each of the following as being true (T) or false (F):

(a) ________: $\varnothing \subseteq \varnothing$

(b) ________: $\{A\} \subseteq \mathcal{P}(A)$

(c) ________: $\varnothing \in \{0\}$

(d) ________: $100 < -100 \to 0 = 1$

(e) ________: $A \subseteq \mathcal{P}(A)$

(f) ________: $\forall x \; \forall y \; P(x, y) \equiv \forall y \; \forall x \; P(x, y)$

(g) ________: $\{\varnothing\} \subseteq \{\varnothing\}$

(h) ________: $A \in \mathcal{P}(A)$

(i) ________: $\forall x \; \exists y \; P(x, y) \equiv \exists y \; \forall x \; P(x, y)$

(j) ________: $\varnothing \subsetneq \varnothing$