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MATH 308

Fall 2023

HW 15: Due 11/14

"One reason why Combinatorics has been slow to become accepted as part of mainstream Mathematics is the common belief that it consists of a bag of isolated tricks, a number of areas... with little or no connexion between them. We shall see that they have numerous threads weaving them together into a beautifully patterned tapestry."

−Richard K. Guy

Problem 1. (10pt) Showing all your work, complete the following:

(a) Writing $\left(3x^2 - \frac{1}{x}\right)^{12}$ in descending power of x, find the 'middle' term.

(b) Find the coefficient of $x^3 y^2$ in $(2x - 3y)^5$.

(c) Find the coefficient of $x^2 y z^3$ in $(2x - y + 3z)^6$

Solution.

Recall that the Binomial Theorem states that for $n \in \mathbb{Z}_{\geq 0}$, $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, i.e. the coefficient of $x^k y^{n-k}$ is $\binom{n}{k}$. Furthermore, by the Multinomial Theorem, if $n_1, n_2, \ldots, n_k \in \mathbb{Z}_{\geq 0}$ and $n = n_1 + n_2 + \cdots + n_k$, the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ in $(x_1 + x_2 + \cdots + x_k)^n$ is $\binom{n}{n_1, n_2, \ldots, n_k}$.

(a) The expansion of $\left(3x^2 - \frac{1}{x}\right)^{12}$ has $12 + 1 = 13$ terms. The middle term is then the $\frac{13+1}{2} = \frac{14}{2} = $ 7th term. Writing $\left(3x^2 - \frac{1}{x}\right)$ in descending powers of x, this is the choice of six $3x^2$ and six $-\frac{1}{x}$ terms in the expansion. Using the Binomial Theorem, this term is...

$$\binom{12}{6}(3x^2)^6 \left(-\frac{1}{x}\right)^6 = 924 \cdot 729 x^{12} \cdot \frac{1}{x^6} = 673596 x^6$$

(b) Because there is only one way to obtain $x^3 y^2$, by the Binomial Theorem, the term with $x^3 y^2$ in the expansion of $(2x - 3y)^5$ is...

$$\binom{5}{3}(2x)^3(-3y)^2 = 10 \cdot 8x^3 \cdot 9y^2 = 720 x^3 y^2$$

Therefore, the coefficient of $x^3 y^2$ is 720.

(c) Because there is only one way to obtain the term $x^2 y z^3$, by the Multinomial Theorem, the term with $x^2 y z^3$ in $(2x - y + 3z)^6$ is...

$$\binom{6}{2, 1, 3}(2x)^2(-y)^1(3z)^3 = 60 \cdot 4x^2 \cdot -y \cdot 27z^3 = -6480 x^2 y z^3$$

Therefore, the coefficient of $x^2 y z^3$ is -6480.

Problem 2. (10pt) If you distribute 27 tasks amongst 6 people, what is the maximum number of tasks a person can be assigned? What is the minimum number of tasks someone can be assigned to? Does there have to be a person with at least 5 tasks assigned to them? Explain. Does there have to be a person with at most 4 tasks assigned to them? Explain.

Solution. Clearly, if you distribute 27 tasks amongst 6 people, the maximum number of tasks a person is 27 by assigning all tasks to one person. Furthermore, the minimum number of tasks a person can be assigned is 0 by being sure one person receives no tasks.

By the Pigeonhole Principle, there must be at least one person with at least $\left\lceil \frac{27}{6} \right\rceil = \lceil 4.5 \rceil = 5$ tasks assigned to them. Alternatively, if this were not the case, then every individual would have to be assigned four or less tasks. But then the most number of tasks that could be assigned would be $6 \cdot 4 = 24 < 27$. With $27 - 24 = 3$ tasks left to assign, at least one person must receive at least one more tasks, so that this individual is assigned at least 5 tasks.

Similarly, by the Pigeonhole Principle, there must be a person with at most $\left\lfloor \frac{27}{6} \right\rfloor = \lfloor 4.5 \rfloor = 4$ tasks assigned to them. Alternatively, if this were not the case, then every individual would be assigned at least 5 tasks. But then the minimum number of tasks assigned that would be assigned is $6 \cdot 5 = 30 > 27$, which is impossible. Then there must be an individual assigned less than 5 tasks, i.e. assigned at most 4 tasks.

Problem 3. (10pt) Consider the set of positive integers less than 10,000. How many of these numbers are. . .

(a) . . . divisible by 5 or 7?

(b) . . . divisible by 2 and 5 but not 20?

(c) . . . divisible by 2, 3, or 5?

Solution. Recall the Principle of Inclusion/Exclusion: if $A_1, A_2, \ldots, A_n$ are finite sets, then. . .

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$$

For two sets A, B, this is $|A \cup B| = |A| + |B| - |A \cap B|$. Note that if A, B are disjoint finite sets, this implies that $|A \cup B| = |A| + |B| - |A \cap B| = |A| + |B| - |\varnothing| = |A| + |B| - 0 = |A| + |B|$. For three sets, A, B, C, this is $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. Note that for $n, m \ge 1$, the number of positive multiples of m less than n is $\left\lfloor \frac{n-1}{m} \right\rfloor$.

(a) Let A be the set of multiples of 5 and B be the set of multiples of 7. The set $A \cup B$ is the set of multiples of 5 or 7, i.e. the set of numbers that are divisible by 5 or 7. The set $A \cap B$ is the set of positive integers less than 10,000 that are a multiple of 5 and 7. But any multiples of both 5 and 7 are a multiple of their least common multiple, which is $\mathrm{lcm}(5, 7) = 35$. Therefore, $A \cap B$ is the set of positive integers less than 10,000 that are a multiple of 35. We have. . .

$$|A| = \left\lfloor \frac{10000 - 1}{5} \right\rfloor = \left\lfloor \frac{9999}{5} \right\rfloor = \lfloor 1999.8 \rfloor = 1999$$

$$|B| = \left\lfloor \frac{10000 - 1}{7} \right\rfloor = \lfloor 9999 \rfloor 7 = \lfloor 1428.43 \rfloor = 1428$$

$$|A \cap B| = \left\lfloor \frac{10000 - 1}{35} \right\rfloor = \left\lfloor \frac{9999}{35} \right\rfloor = \lfloor 285.686 \rfloor = 285$$

But then by Inclusion-Exclusion applied to A, B, we have. . .

$$|A \cup B| = |A| + |B| - |A \cap B| = 1999 + 1428 - 285 = 3142$$

(b) Let A be the set of positive integers less than 10,000 that are a multiple of 2, i.e. the set of positive integers less than 10,000 that are divisible by 2. Let B be the set of positive integers less than 10,000 that are a multiple of 5, i.e. the set of positive integers less than 10,000 that are divisible by 5. Finally, let C be the set of positive integers less than 10,000 that are a multiple of 20, i.e. the set of positive integers less than 10,000 that are divisible by 20. The set $A \cap B$ is the set of positive integers less than 10,000 that are a multiple of 2 and 5, i.e. the set of positive integers less than 10,000 that are divisible by 2 and 5. If an integer is divisible by 2 and 5, then it is divisible by $\mathrm{lcm}(2, 5) = 10$. Therefore, $A \cap B$ is the set of positive integers less than 10,000 that are divisible by 10. Because every element of $A \cap B$ is either in C or not, we must have $A \cap B = ((A \cap B) \cap C) \cup ((A \cap B) \cap C^c)$. The sets $(A \cap B) \cap C = A \cap B \cap C$ and $(A \cap B) \cap C^c = A \cap B \cap C^c$ are disjoint because $(A \cap B \cap C) \cap (A \cap B \cap C^c) = (A \cap B) \cap (C \cap C^c) = (A \cap B) \cap \varnothing = \varnothing$. Now the set $(A \cap B) \cap C$ is the set of positive integers less than 10,000 that are divisible by 2 and 5, i.e. divisible by 10,

that are divisible by 20. But an integer divisible by 10 and 20 is divisible by $\operatorname{lcm}(10, 20) = 20$. Therefore, $(A \cap B) \cap C$ is the set of positive integers less than 10,000 that are divisible by 20, i.e. $(A \cap B) \cap C = C$ (which also follows from the fact that $C \subseteq A \cap B$). The set $(A \cap B) \cap C^c$ are the set of positive integers less than 10,000 that are divisible by 2 and 5, i.e. divisible by 10, that are not divisible by 20. We have. . .

$$|A \cap B| = \left\lfloor \frac{10000 - 1}{10} \right\rfloor = \left\lfloor \frac{9999}{10} \right\rfloor = \lfloor 999.9 \rfloor = 999$$

$$|(A \cap B) \cap C| = |C| = \left\lfloor \frac{10000 - 1}{20} \right\rfloor = \left\lfloor \frac{9999}{20} \right\rfloor = \lfloor 499.95 \rfloor = 499$$

Then by the Inclusion-Exclusion Principle and the comments before (a), we have. . .

$$|A \cap B| = |((A \cap B) \cap C)| + |((A \cap B) \cap C^c)|$$
$$|A \cap B| = |C| + |A \cap B \cap C^c|$$
$$999 = 499 + (A \cap B \cap C^c)$$
$$A \cap B \cap C^c = 500$$

(c) Let A, B, C be the set of positive integers less than 10,000 that are a multiple of 2, 3, and 5, respectively, i.e. the set of positive integers less than 10,000 that are divisible by 2, 3, and 5, respectively. The set $A \cup B \cup C$ is the set of positive integers less than 10,000 that are a multiple of 2, 3, or 5, i.e. the set of positive integers less than 10,000 divisible by 2, 3, or 5. The set $A \cap B$ is the set of positive integers that are less than 10,000 that are a multiple of 2 and 3, i.e. the set of positive integers less than 10,000 divisible by 2 and 3. If an integer is divisible by 2 and 3, then it is divisible by $\operatorname{lcm}(2, 3) = 6$. Therefore, $A \cap B$ is the set of positive integers less than 10,000 divisible by 6. Similarly, $A \cap C$ is the set of positive integers less than 10,000 divisible by $\operatorname{lcm}(2, 5) = 10$, $B \cap C$ is the set of positive integers less than 10,000 divisible by $\operatorname{lcm}(3, 5) = 15$, and $A \cap B \cap C$ is the set of positive integers less than 10,000 divisible by $\operatorname{lcm}(2, 3, 5) = 30$. We have. . .

$$|A| = \left\lfloor \frac{10000 - 1}{2} \right\rfloor = \left\lfloor \frac{9999}{2} \right\rfloor = \lfloor 4999.5 \rfloor = 4999$$

$$|B| = \left\lfloor \frac{10000 - 1}{3} \right\rfloor = \left\lfloor \frac{9999}{3} \right\rfloor = \lfloor 3333 \rfloor = 3333$$

$$|C| = \left\lfloor \frac{10000 - 1}{5} \right\rfloor = \left\lfloor \frac{9999}{5} \right\rfloor = \lfloor 1999.8 \rfloor = 1999$$

$$|A \cap B| = \left\lfloor \frac{10000 - 1}{6} \right\rfloor = \left\lfloor \frac{9999}{6} \right\rfloor = \lfloor 1666.5 \rfloor = 1666$$

$$|A \cap C| = \left\lfloor \frac{10000 - 1}{10} \right\rfloor = \left\lfloor \frac{9999}{10} \right\rfloor = \lfloor 999.9 \rfloor = 999$$

$$|B \cap C| = \left\lfloor \frac{10000 - 1}{15} \right\rfloor = \left\lfloor \frac{9999}{15} \right\rfloor = \lfloor 666.6 \rfloor = 666$$

$$|A \cap B \cap C| = \left\lfloor \frac{10000 - 1}{30} \right\rfloor = \left\lfloor \frac{9999}{30} \right\rfloor = \lfloor 333.3 \rfloor = 333$$

Applying the Inclusion-Exclusion Principle to A, B, C, we have. . .

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 4999 + 3333 + 1999 - 1666 - 999 - 666 + 333 = 7333$$