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**MATH 108**
**Spring 2023**
**HW 16: Due 05/01**

*"Everyday life is like programming, I guess. If you love something, you can put beauty into it."*

*–Donald Knuth*

**Problem 1.** (10pt) Write down the initial simplex tableau for the following optimization problem:

$$\begin{gathered}\max z = 3.1x_1 - 4.7x_2 + 5.9x_3 \\ 1.1x_1 - 5.7x_2 + 4.0x_3 \leq 10.4 \\ 6.7x_1 - 0.8x_2 - 8.8x_3 \geq -8.8 \\ -9.1x_1 + 7.3x_2 - 9.1x_3 \leq 11.7 \\ x_1, x_2, x_3 \geq 0\end{gathered}$$

**Solution.** First, observe that this optimization is not in standard form because the third inequality is not '$\leq$' and the constant term is negative. Multiplying both sides of this third inequality by $-1$, we have. . .

$$\begin{gathered}\max z = 3.1x_1 - 4.7x_2 + 5.9x_3 \\ 1.1x_1 - 5.7x_2 + 4.0x_3 \leq 10.4 \\ -6.7x_1 + 0.8x_2 + 8.8x_3 \leq 8.8 \\ -9.1x_1 + 7.3x_2 - 9.1x_3 \leq 11.7 \\ x_1, x_2, x_3 \geq 0\end{gathered}$$

Now introducing slack variables into each inequality (except the last non-negativity inequality) to obtain equalities, we have. . .

$$\begin{array}{rcrcrcccccl} 1.1x_1 & - & 5.7x_2 & + & 4.0x_3 & + & s_1 & & & = & 10.4 \\ -6.7x_1 & + & 0.8x_2 & + & 8.8x_3 & + & & s_2 & & = & 8.8 \\ -9.1x_1 & + & 7.3x_2 & - & 9.1x_3 & + & & & s_3 & = & 11.7 \end{array}$$

Moving things to the '$z$'-side of the equality in the function, we have $z - 4.6x_1 - 3.1x_2 - 7.9x_3 = 0$. Adding this to the table yields. . .

$$\begin{array}{ccrcrcrcccccr} & & 1.1x_1 & - & 5.7x_2 & + & 4.0x_3 & + & s_1 & & & = & 10.4 \\ & & -6.7x_1 & + & 0.8x_2 & + & 8.8x_3 & + & & s_2 & & = & 8.8 \\ & & -9.1x_1 & + & 7.3x_2 & - & 9.1x_3 & + & & & s_3 & = & 11.7 \\ z & - & 3.1x_1 & + & 4.7x_2 & - & 5.9x_3 & & & & & = & 0 \end{array}$$

This yields the following initial simplex tableau:

$$\begin{array}{rrrccc|r} 1.1 & -5.7 & 4.0 & 1 & 0 & 0 & 10.4 \\ -6.7 & 0.8 & 8.8 & 0 & 1 & 0 & 8.8 \\ -9.1 & 7.3 & -9.1 & 0 & 0 & 1 & 11.7 \\ \hline -3.1 & 4.7 & -5.9 & 0 & 0 & 0 & 0 \end{array}$$

**Problem 2.** (10pt) Suppose that the final simplex tableau associated to a maximization problem was the following:

$$\begin{array}{ccccccccc} 1 & 1.77 & 0 & 0 & 0.74 & 0 & 0.26 & 0.29 & 208.57 \\ 0 & 0.57 & 0 & 1 & 0.14 & 0 & -0.14 & 0.29 & 28.57 \\ 0 & 2.51 & 0 & 0 & 0.83 & 1 & 1.17 & -0.14 & 605.71 \\ 0 & 0.09 & 1 & 0 & -0.03 & 0 & 0.03 & 0.14 & 34.29 \\ 0 & 3 & 0 & 0 & 2 & 0 & 0 & 2 & 600 \end{array}$$

(a) How many inequalities were considered?

(b) How many variables were there in the original inequalities?

(c) How many slack/surplus variables were introduced?

(d) What was the solution to this maximization problem?

**Solution.**

(a) Each row of the tableau 'corresponds' to an inequality with the exception of the last row which 'corresponds to the function.' But then there were $5 - 1 = 4$ inequalities in the original system (ignoring the non-negativity inequality).

(b) Each column of the tableau 'corresponds' to a variable in the system with the exception of the last column which 'corresponds to the solutions.' Therefore, there were $9 - 1 = 8$ variables in the system. Note by (c), there are 4 slack/surplus variables. Therefore, there were $8 - 4 = 4$ 'original' variables in the system of inequalities.

(c) Because we introduce a slack/surplus variable for each inequality and by (a) there were 4 inequalities in the original system, there were 4 slack/surplus variables.

(d) By (b) and (c), there were 5 'original' variables and 4 slack/surplus variables. Therefore, we need find the maximum value along with the values of the variables—-namely, the values for $(x_1, x_2, x_3, x_4, x_5, s_1, s_2, s_3, s_4)$. Adding 'dividers' to the tableau and 'naming' the columns, we have...

$$\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & s_4 & \\ \boxed{1} & 1.77 & 0 & 0 & 0.74 & 0 & 0.26 & 0.29 & 208.57 \\ 0 & 0.57 & 0 & \boxed{1} & 0.14 & 0 & -0.14 & 0.29 & 28.57 \\ 0 & 2.51 & 0 & 0 & 0.83 & \boxed{1} & 1.17 & -0.14 & 605.71 \\ 0 & 0.09 & \boxed{1} & 0 & -0.03 & 0 & 0.03 & 0.14 & 34.29 \\ \hline 0 & 3 & 0 & 0 & 2 & 0 & 0 & 2 & 600 \end{array}$$

We indicate the pivot positions above. This yields $x_1 = 208.57$, $x_3 = 34.39$, $x_4 = 28.57$, and $s_2 = 605.71$. All remaining variables have value 0. The maximum value is $600$. Therefore, the maximum value is $600$ and occurs at $(x_1, x_2, x_3, x_4, s_1, s_2, s_3, s_4) = (208.57, 0, 34.29, 28.57, 0, 605.71, 0, 0)$.

**Problem 3.** (10pt) Find the dual problem to the minimization problem below.

$$\min z = 2x_1 + 6x_2$$
$$6x_1 + 5x_2 \geq 10$$
$$x_1 + 3x_2 \geq 9$$
$$x_1, x_2 \geq 0$$

**Solution.** First, observe that the given minimization problem is in standard form; that is, the function is linear, all the inequalities are '$\geq$' a non-negative number, and the variables are non-negative. Now we write the 'matrix associated' to this minimization; that is, we create a matrix with rows corresponding to the equality version of the inequalities (with the exception of the non-negativity inequality) with the function being the last row. This yields matrix:

$$\begin{pmatrix} 6 & 5 & 10 \\ 1 & 3 & 9 \\ 2 & 6 & 0 \end{pmatrix}$$

We now find the transpose of this matrix:

$$\begin{pmatrix} 6 & 5 & 10 \\ 1 & 3 & 9 \\ 2 & 6 & 0 \end{pmatrix}^T = \begin{pmatrix} 6 & 1 & 2 \\ 5 & 3 & 6 \\ 10 & 9 & 0 \end{pmatrix}$$

We now find the standard maximization problem corresponding to this matrix:

$$\max w = 10y_1 + 9y_2$$
$$6y_1 + y_2 \leq 2$$
$$5y_1 + 3y_2 \leq 6$$
$$y_1, y_2 \geq 0$$