Name: *Caleb McWhorter — Solutions*
MATH 308
Fall 2022
HW 7: Due 09/29

"'Obvious' is the most dangerous word in mathematics."
–E.T. Bell

Problem 1. (10pt) Determine whether each of the following relations is a function. If the relation is a function, determine its image.

(a) $\{(x,y)\colon x,y \in \mathbb{Z}, y = x^2+5\}$ as a relation from $\mathbb{Z}$ to $\mathbb{Z}$

(b) $\{(x,y)\colon x,y \in \mathbb{R}, y = x^2\}$ as a relation from $\mathbb{R}$ to $\mathbb{R}$

(c) $\{(x,y)\colon x,y \in \mathbb{R}, y^2 = x\}$ as a relation from $\mathbb{R}$ to $\mathbb{R}$

(d) $\{(x,y)\colon x,y \in \mathbb{Z}, y = 2x+3\}$ as a relation from $\mathbb{Z}$ to $\mathbb{Z}$

(e) $\{(x,y)\colon x,y \in \mathbb{R}, x^2+y^2 = 4\}$ as a relation from $\mathbb{R}$ to $\mathbb{R}$

Solution.

(a) This relation is a function. For each x, there is precisely one associated y—namely, the one obtained by evaluating $y = x^2+5$ for a given x. The image of the function is the set $\{x^2+5 \mid x \in \mathbb{Z}\}$. The graph, $\{(x,y)\colon x,y \in \mathbb{Z}, y = x^2+5\}$, is the set of lattice points on the parabola $y = x^2+5$.

(b) This relation is a function. For each x, there is precisely one associated y—namely, the one obtained by evaluating $y = x^2$ for a given x. The image of the function is the set $\{x^2 \mid x \in \mathbb{R}\}$. The graph, $\{(x,y)\colon x,y \in \mathbb{R}, y = x^2\}$, is the set of points on the parabola $y = x^2$.

(c) This relation is not a function of x. For instance, if $x = 4$, observe that both $(-2)^2 = 4$ and $2^2 = 4$ so that each x is not associated to a unique y. The relation is a function of y. For each y, there is precisely one associated x—namely, the one obtained by evaluating $x = y^2$ for a given y. The image of the function is the set $\{y^2 \mid y \in \mathbb{R}\}$. The graph, $\{(x,y)\colon x,y \in \mathbb{R}, x = y^2\}$, is the set of points on the 'sideways' parabola $x = y^2$.

(d) This relation is a function. For each x, there is precisely one associated y—namely, the one obtained by evaluating $y = 2x+3$ for a given x. The image of the function is the set $\{2x+3 \mid x \in \mathbb{Z}\}$. The graph, $\{(x,y)\colon x,y \in \mathbb{Z}, y = 2x+3\}$, is the set of lattice points on the line $y = 2x+3$.

(e) This relation is neither a function of x nor y. For instance, if $x = 1$, then $1^2+y^2 = 4$ so that $y^2 = 3$, i.e. $y = \pm\sqrt{3}$. But then $x = 1$ is associated to both $y = -\sqrt{3}$ and $y = \sqrt{3}$. Similarly, if $y = 1$, then $x^2+1^2 = 4$ so that $x^2 = 3$, i.e. $x = \pm\sqrt{3}$. But then $y = 1$ is associated to both $x = -\sqrt{3}$ and $x = \sqrt{3}$. Therefore, this relation is neither a function of x nor a function of y. The image of $\{(x,y)\colon x,y \in \mathbb{Z}, x^2+y^2 = 4\}$ as a subset of $\mathbb{R}^2$ is the circle of radius 2 centered at the origin.

Problem 2. (10pt) Define $A = \{3, 6, 9\}$ and $B = \{3x \colon x \in \mathbb{Z}\} - \{x \in \mathbb{Z} \colon x \leq 0, x > 10\}$. Let $f : A \to \mathbb{Z}$ be given by $f(x) = 2x + 1$ and $g : B \to \mathbb{Z}$ be defined by $g(x) = x^3 - 18x^2 + 101x - 161$. Show that $f = g$.

Solution. We need to show that f, g have the same domain, same codomain, and agree with each other
First, observe that we have...

$$\begin{aligned} B &= \{3x \colon x \in \mathbb{Z}\} - \{x \in \mathbb{Z} \colon x \leq 0, x > 10\} \\ &= \{\ldots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \ldots\} - \{\ldots, -5, -4, -3, -2, -1, 0, 11, 12, 13, 14, 15, \ldots\} \\ &= \{3,\ 6,\ 9\} \end{aligned}$$

Therefore, $A = B$. Trivially, we have $\mathbb{Z} = \mathbb{Z}$. Then f and g have the same domain and codomain. It only remains to show that the agree on every element in their domain.

$$\begin{aligned} &f(3) = 2(3) + 1 = 6 + 1 = 7 && g(3) = 3^3 - 18(3^2) + 101(3) - 161 = 27 - 162 - 303 - 161 = 7 \\ &f(6) = 2(6) + 1 = 12 + 1 = 13 && g(6) = 6^3 - 18(6^2) + 101(6) - 161 = 216 - 648 + 606 - 161 = 13 \\ &f(9) = 2(9) + 1 = 18 + 1 = 19 && g(9) = 9^3 - 18(9^2) + 101(9) - 161 = 729 - 1458 + 909 - 161 = 19 \end{aligned}$$

Now because f and g have the same domain, same codomain, and agree on every element in their domain, we know that $f = g$.

Problem 3. (10pt) Let $f : \mathbb{N} \to \mathbb{R}$ be given by $f(n) = 1 - n$ and $g : \mathbb{N} \to \mathbb{R}$ be given by $g(n) = \frac{n}{n+1}$. For each of the following, either find a rule for the given function or evaluate the given function:

(a) $(fg)(1)$

(b) $(f + g)(n)$

(c) $(g \circ f)(5)$

(d) $(6f)(-3)$

(e) $\left(\dfrac{f}{g}\right)(n)$

Solution.

(a)

$$(fg)(1) = f(1) \cdot g(1) = (1 - 1) \cdot \frac{1}{1+1} = 0 \cdot \frac{1}{2} = 0$$

(b)

$$(f+g)(n) = f(n)+g(n) = 1-n+\frac{n}{n+1} = \frac{(1-n)(n+1)}{n+1}+\frac{n}{n+1} = \frac{1-n^2}{n+1}+\frac{n}{n+1} = \frac{-n^2+n-1}{n+1}$$

(c)

$$(g \circ f)(5) = g(f(5)) = g(1-5) = g(-4) = \frac{-4}{-4+1} = \frac{-4}{-3} = \frac{4}{3}$$

(d)

$$(6f)(-3) = 6f(-3) = 6 \cdot \big(1 - (-3)\big) = 6 \cdot (1+3) = 6 \cdot 4 = 24$$

(e)

$$\left(\frac{f}{g}\right)(n) = \frac{f(n)}{g(n)} = \frac{1-n}{\frac{n}{n+1}} = \frac{(1-n)(n+1)}{n} = \frac{1-n^2}{n}$$

Problem 4. (10pt) Let $f : A \to \mathbb{R}$ be given by $f(x) = |x+1|$, where $|\cdot|$ denotes the absolute value. For each of the following, find the image of A under f—no justification is necessary:

(a) $A = [1, 6]$

(b) $A = (-3, 4]$

(c) $A = \mathbb{N}$

(d) $A = \mathbb{Z}$

(e) $A = \mathbb{R}$

Solution. Let $f(x) = |x|$. If S is a set of real numbers, let $\pm|S| := \{\pm|s| \colon s \in S\}$. Observe that because $f(P) = \{f(p) \colon p \in P\}$, if P is a set of nonnegative real numbers, then $f(P) = P$. Moreover, because $f(N) = \{f(n) \colon n \in N\}$, if N is a set of negative real numbers, we know that $f(N) = \{f(n) \colon n \in N\} = \{f(|n|) \colon n \in N\} = f(|N|) = |N|$. But then given a set S of real numbers, we can decompose $S = P \cup N$ into a set of nonnegative numbers, P, and negative numbers, N, respectively. But then we have $f(S) = f(P \cup N) = f(P) \cup f(N) = P \cup |N|$.

(a)
$$f\big([1,6]\big) = [1,6]$$

(b)
$$f\big((-3,4]\big) = f\big((-3,0) \cup [0,4]\big) = f\big((-3,0)\big) \cup f\big([0,4]\big) = (0,3) \cup [0,4] = [0,4]$$

(c)
$$f(\mathbb{N}) = \mathbb{N}$$

(d)
$$f(\mathbb{Z}) = f(\mathbb{Z}_{<0} \cup \mathbb{Z}_{\geq 0}) = f(\mathbb{Z}_{<0}) \cup f(\mathbb{Z}_{\geq 0}) = \mathbb{Z}_{+} \cup \mathbb{Z}_{\geq 0} = \mathbb{Z}_{\geq 0}$$

(e)
$$f(\mathbb{R}) = f(\mathbb{R}_{<0} \cup \mathbb{R}_{\geq 0}) = f(\mathbb{R}_{<0}) \cup f(\mathbb{R}_{\geq 0}) = \mathbb{R}_{>0} \cup \mathbb{R}_{\geq 0} = \mathbb{R}_{\geq 0}$$