MAT 100: Exam 2
Fall – 2021
12/13/2021
85 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 19 pages (including this cover page) and 18 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work.

Question	Points	Score
1	5	
2	10	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	5	
10	5	
11	10	
12	5	
13	5	
14	5	
15	5	
16	5	
17	5	
18	5	
Total:	100	

1. (5 points) Sketch the quadratic function $f(x) = 4 - (x+3)^2$ in the graph below. Your sketch should include the vertex and axis of symmetry for $f(x)$.
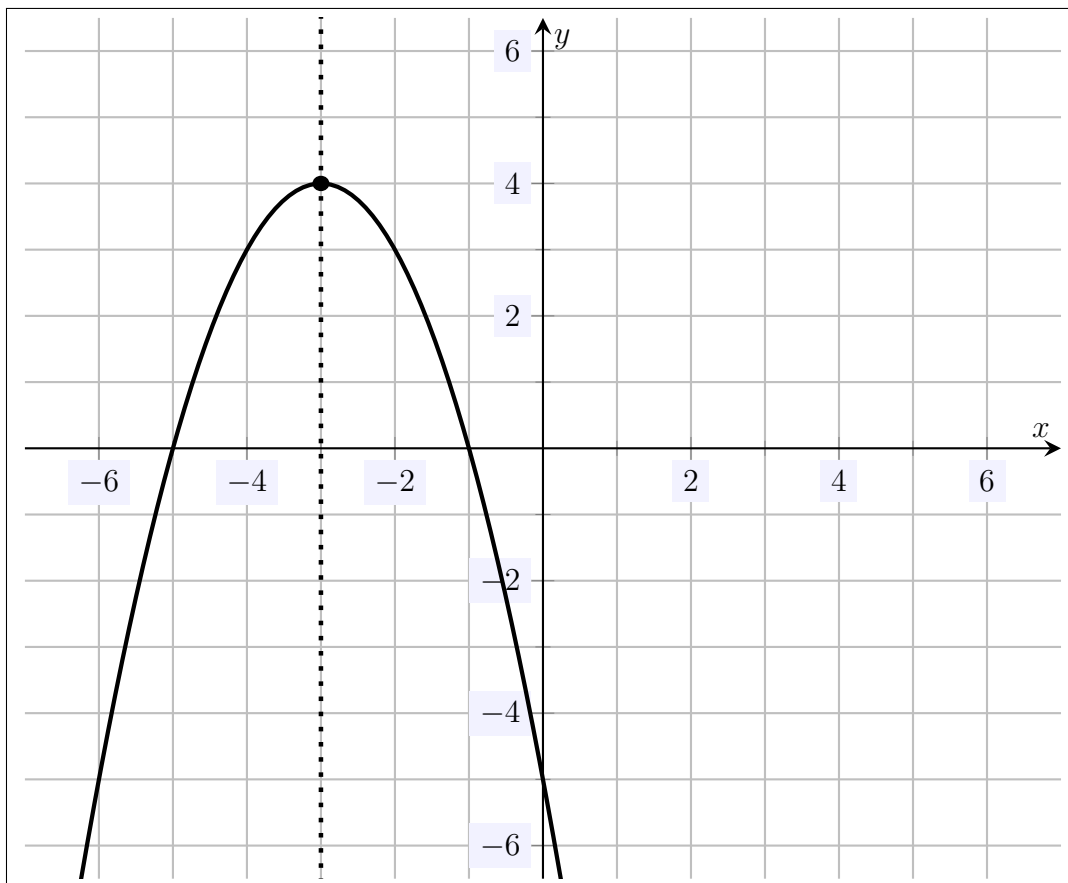
From the form of the function $f(x) = 4 - (x+3)^2$, we can immediately see that the vertex is $(-3, 4)$ and that the parabola opens downwards.

2. (10 points) Let $f(x)$ be the quadratic function $f(x) = 5x^2 + 10x - 7$.

    (a) Find the vertex and axis of symmetry for $f(x)$.

    (b) Does this parabola open upwards or downwards? Explain.

    (c) Is the function convex or concave?

    (d) Does the function have a maximum or minimum value? Explain.

    (e) Find the maximum or minimum value from (d).

Solution.

(a) We know that the x-coordinate of the vertex is...

$$x = \frac{-b}{2a} = \frac{-10}{2(5)} = \frac{-10}{10} = -1$$

The y-coordinate of the vertex is $f(-1) = 5(-1)^2 + 10(-1) - 7 = 5 - 10 - 7 = -12$.
Therefore, the vertex is $(-1, -12)$. It is also immediate that the axis of symmetry is $x = -1$.

> *Vertex: $(-1, -12)$*
> *Axis of Symmetry: $x = -1$*

(b) Because $a = 5 > 0$, the parabola opens upwards.

(c) Because the parabola opens upwards, the function is convex.

(d) Because the parabola opens upwards, the function has a minimum.

(e) We know the minimum value occurs at the vertex. The vertex has coordinate $(-1, -12)$. Therefore, the minimum value is -12.

3. (5 points) Find the vertex form of $y = x^2 - 8x + 21$.

We complete the square. Observe that $\frac{1}{2} \cdot -8 = -4$ and that $(-4)^2 = 16$. Then...

$$y = x^2 - 8x + 21$$

$$y = x^2 - 8x + (16 - 16) + 21$$

$$y = (x^2 - 8x + 16) + (-16 + 21)$$

$$y = (x - 4)^2 + 5$$

$$\boxed{y = (x - 4)^2 + 5}$$

4. (5 points) Factor the polynomial $x^2 - 16x + 55$.

$$\underline{\mathbf{55}}$$

$$
\begin{array}{cc}
1 \cdot 55 & 56 \\
-1 \cdot -55 & -56 \\
5 \cdot 11 & 16 \\
\boxed{-5 \cdot -11 \quad -16} &
\end{array}
$$

$$x^2 - 16x + 55 = (x - 5)(x - 11)$$

$$\boxed{(x - 11)(x - 5)}$$

5. (5 points) Factor the polynomial $x^2 - 2x - 48$.

48

$$1 \cdot -48 \quad -47$$
$$-1 \cdot 48 \quad 47$$
$$2 \cdot -24 \quad -22$$
$$-2 \cdot 24 \quad 22$$
$$3 \cdot -16 \quad -13$$
$$-3 \cdot 16 \quad 13$$
$$4 \cdot -12 \quad -8$$
$$-4 \cdot 12 \quad 8$$
$$\boxed{6 \cdot -8 \quad -2}$$
$$-6 \cdot 8 \quad 2$$

$$x^2 - 2x - 48 = (x + 6)(x - 8)$$

$$\boxed{(x - 8)(x + 6)}$$

6. (5 points) Factor the polynomial $2x^2 + 7x - 30$.

$$1 \cdot -30$$

$1, 2$ $2, 1$

$1, -60$ $2, -30$

$$-1 \cdot 30$$

$1, 2$ $2, 1$

$-1, 60$ $-2, 30$

$$2 \cdot -15$$

$1, 2$ $2, 1$

$2, -30$ $4, -15$

$$-2 \cdot 15$$

$1, 2$ $2, 1$

$-2, 30$ $-4, 15$

$$3 \cdot -10$$

$1, 2$ $2, 1$

$3, -20$ $6, -10$

$$-3 \cdot 10$$

$1, 2$ $2, 1$

$-3, 20$ $-6, 10$

$$5 \cdot -6$$

$1, 2$ $2, 1$

$5, -12$ $10, -6$

$$\boxed{-5 \cdot 6}$$

$1, 2$ $2, 1$

$\boxed{-5, 12}$ $-10, 6$

$$2x^2 + 7x - 30 = (2x - 5)(x + 6)$$

$$\boxed{(2x - 5)(x + 6)}$$

7. (5 points) Consider the function $f(x) = x^2 + 2x - 8$. Find the x and y intercepts for this function.

The y-intercept occurs when $x = 0$. But then $y = f(0) = 0^2 + 2(0) - 8 = 0 + 0 - 8 = -8$. Therefore, the y-intercept is $(0, -8)$.

The x-intercepts occur when $y = 0$. But then $x^2 + 2x - 8 = 0$. Observe...

$$x^2 + 2x - 8 = 0$$
$$(x - 2)(x + 4) = 0$$

But then either $x - 2 = 0$, i.e. $x = 2$, or $x + 4 = 0$, i.e. $x = -4$. Therefore, the x-intercepts are $(-4, 0)$ and $(2, 0)$.

> *y-intercept:* $(0, -8)$
> *x-intercepts:* $(-4, 0), (2, 0)$

8. (5 points) Find the solutions to $6x - x^2 = 9$.

$$6x - x^2 = 9$$
$$x^2 - 6x + 9 = 0$$
$$(x - 3)(x - 3) = 0$$
$$(x - 3)^2 = 0$$
$$x - 3 = 0$$
$$x = 3$$

$$\boxed{x = 3}$$

9. (5 points) Find the solutions to $x^2 = x + 20$.

$$x^2 = x + 20$$

$$x^2 - x - 20 = 0$$

$$(x + 4)(x - 5) = 0$$

$$x + 4 = 0 \quad \text{or} \quad x - 5 = 0$$

$$x = -4 \quad \text{or} \quad x = 5$$

$$\boxed{x = -4,\ 5}$$

10. (5 points) Using the quadratic equation, find the solutions to $x^2 - 2x - 7 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-7)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4 + 28}}{2}$$

$$x = \frac{2 \pm \sqrt{32}}{2}$$

$$x = \frac{2 \pm \sqrt{16 \cdot 2}}{2}$$

$$x = \frac{2 \pm 4\sqrt{2}}{2}$$

$$x = 1 \pm 2\sqrt{2}$$

$$\boxed{x = 1 - 2\sqrt{2}, \ 1 + 2\sqrt{2}}$$

11. (10 points) Consider the rational function $f(x) = \dfrac{x^2 - 9}{x^2 + 2x - 15}$.

    (a) Find the domain for $f(x)$.

    (b) Find the vertical asymptotes for $f(x)$.

    (c) Find the zeros for $f(x)$.

Solution.

$$f(x) = \frac{x^2 - 9}{x^2 + 2x - 15} = \frac{(x-3)(x+3)}{(x+5)(x-3)}$$

(a) *The domain of a rational function consists of the real numbers for which the denominator is not zero. If the denominator were zero, then $(x+5)(x-3) = 0$. But then either $x + 5 = 0$, i.e. $x = -5$, or $x - 3 = 0$, i.e. $x = 3$. Therefore, the domain is the set of real numbers such that $x \neq -5, 3$.*

$$\boxed{x \in \mathbb{R} \text{ such that } x \neq -5, 3}$$

(b) *We see that for $x \neq -5, 3$,*

$$f(x) = \frac{x^2 - 9}{x^2 + 2x - 15} = \frac{(x-3)(x+3)}{(x+5)(x-3)} = \frac{\cancel{(x-3)}(x+3)}{(x+5)\cancel{(x-3)}} = \frac{x+3}{x+5}$$

The vertical asymptotes for $f(x)$ will be where the denominator vanishes in this reduced function. But then $x + 5 = 0$, i.e. $x = -5$. Therefore, the only vertical asymptote is $x = -5$.

$$\boxed{x = -5}$$

(c) *We see that for $x \neq -5, 3$,*

$$f(x) = \frac{x^2 - 9}{x^2 + 2x - 15} = \frac{(x-3)(x+3)}{(x+5)(x-3)} = \frac{\cancel{(x-3)}(x+3)}{(x+5)\cancel{(x-3)}} = \frac{x+3}{x+5}$$

The zeros for $f(x)$ will be where the numerator vanishes in this reduced function. But then $x + 3 = 0$, i.e. $x = -3$. Therefore, the only zero is $x = -3$.

$$\boxed{x = -3}$$

12. (5 points) Compute the following, being sure to simplify as much as possible:

$$\frac{3}{x^2 - 4} - \frac{x - 1}{x^2 - x - 2}$$

$$\frac{3}{x^2 - 4} - \frac{x - 1}{x^2 - x - 2} = \frac{3}{(x - 2)(x + 2)} - \frac{x - 1}{(x - 2)(x + 1)}$$

$$= \frac{3(x + 1)}{(x - 2)(x + 2)(x + 1)} - \frac{(x - 1)(x + 2)}{(x - 2)(x + 1)(x + 2)}$$

$$= \frac{3(x + 1) - (x - 1)(x + 2)}{(x - 2)(x + 1)(x + 2)}$$

$$= \frac{(3x + 3) - (x^2 + 2x - x - 2)}{(x - 2)(x + 1)(x + 2)}$$

$$= \frac{(3x + 3) - (x^2 + x - 2)}{(x - 2)(x + 1)(x + 2)}$$

$$= \frac{3x + 3 - x^2 - x + 2}{(x - 2)(x + 1)(x + 2)}$$

$$= \frac{-x^2 + 2x + 5}{(x - 2)(x + 1)(x + 2)}$$

$$\boxed{\frac{-x^2 + 2x + 5}{(x - 2)(x + 1)(x + 2)}}$$

13. (5 points) Compute the following, being sure to simplify as much as possible:

$$\frac{x^2 - 2x - 3}{x^2 + 6x - 7} \cdot \frac{x^2 + 9x + 14}{x^2 + 5x + 4}$$

$$\frac{x^2 - 2x - 3}{x^2 + 6x - 7} \cdot \frac{x^2 + 9x + 14}{x^2 + 5x + 4} = \frac{(x-3)(x+1)}{(x-1)(x+7)} \cdot \frac{(x+2)(x+7)}{(x+1)(x+4)}$$

$$= \frac{(x-3)\cancel{(x+1)}}{(x-1)\cancel{(x+7)}} \cdot \frac{(x+2)\cancel{(x+7)}}{\cancel{(x+1)}(x+4)}$$

$$= \frac{(x-3)(x+2)}{(x-1)(x+4)}$$

$$\boxed{\frac{(x-3)(x+2)}{(x-1)(x+4)}}$$

14. (5 points) Compute the following, begin sure to simplify as much as possible:

$$\frac{\dfrac{x^2 + 5x}{x^2 - 1}}{\dfrac{x^2 + 3x - 10}{x^2 + 8x - 9}}$$

$$\frac{\dfrac{x^2 + 5x}{x^2 - 1}}{\dfrac{x^2 + 3x - 10}{x^2 + 8x - 9}} = \frac{x^2 + 5x}{x^2 - 1} \cdot \frac{x^2 + 8x - 9}{x^2 + 3x - 10}$$

$$= \frac{x(x + 5)}{(x - 1)(x + 1)} \cdot \frac{(x - 1)(x + 9)}{(x - 2)(x + 5)}$$

$$= \frac{x\cancel{(x + 5)}}{\cancel{(x - 1)}(x + 1)} \cdot \frac{\cancel{(x - 1)}(x + 9)}{(x - 2)\cancel{(x + 5)}}$$

$$= \frac{x(x + 9)}{(x + 1)(x - 2)}$$

$$\boxed{\frac{x(x + 9)}{(x - 2)(x + 1)}}$$

15. (5 points) Solve the following system of equations:

$$x + y = 4$$
$$x - y = 10$$

Using substitution, from the first equation, we have...

$$x + y = 4$$
$$y = 4 - x$$

Using this in the second equation, we have...

$$x - y = 10$$
$$x - (4 - x) = 10$$
$$x - 4 + x = 10$$
$$2x - 4 = 10$$
$$2x = 14$$
$$x = 7$$

But then we have $y = 4 - x = 4 - 7 = -3$. Therefore, the solution is $(7, -3)$.

OR

Using elimination, we add the two equations:

$$x + y = 4$$
$$\underline{x - y = 10}$$
$$2x = 14$$
$$x = 7$$

Using this in the first equation, we have...

$$x + y = 4$$
$$7 + y = 4$$
$$y = -3$$

Therefore, the solution is $(7, -3)$.

$$\boxed{(x, y) = (7, -3)}$$

16. (5 points) Solve the following system of equations:

$$2x - 3y = 4$$
$$6x - 2y = 5$$

Using substitution, from the first equation, we have...

$$2x - 3y = 4$$
$$-3y = -2x + 4$$
$$y = \frac{2}{3}x - \frac{4}{3}$$

Using this in the second equation, we have...

$$6x - 2y = 5$$
$$6x - 2\left(\frac{2}{3}x - \frac{4}{3}\right) = 5$$
$$6x - \frac{4}{3}x + \frac{8}{3} = 5$$
$$18x - 4x + 8 = 15$$
$$14x + 8 = 15$$
$$14x = 7$$
$$x = \frac{1}{2}$$

Then we have $y = \frac{2}{3}x - \frac{4}{3} = \frac{2}{3} \cdot \frac{1}{2} - \frac{4}{3} = \frac{1}{3} - \frac{4}{3} = -\frac{3}{3} = -1$. Therefore, the solution is $\left(\frac{1}{2}, -1\right)$.

OR

Using elimination, we multiply the first equation by -3. This yields...

$$-6x + 9y = -12$$
$$6x - 2y = 5$$

Adding these equations, we have...

$$-6x + 9y = -12$$
$$\underline{6x - 2y = 5}$$
$$7y = -7$$
$$y = -1$$

Using this in the first equation, we have $2x - 3(-1) = 4$ so that $2x + 3 = 4$. But then $2x = 1$. Therefore, $x = \frac{1}{2}$. The solution is then $\left(\frac{1}{2}, -1\right)$.

$$\boxed{(x, y) = \left(\frac{1}{2}, -1\right)}$$

17. (5 points) Determine whether the point $(1, -5)$ is a solution to the following system of equations:

$$5x - y = 10$$
$$x + y = 0$$

We test to see if the point $(x, y) = (1, -5)$ satisfies both equations:

$$5x - y \stackrel{?}{=} 10$$

$$5(1) - (-5) \stackrel{?}{=} 10$$

$$5 + 5 \stackrel{?}{=} 10$$

$$10 = 10 \checkmark$$

$$x + y \stackrel{?}{=} 0$$

$$1 + (-5) \stackrel{?}{=} 0$$

$$1 - 5 \stackrel{?}{=} 0$$

$$-4 \neq 0 \; \text{✗}$$

Because $(x, y) = (1, -5)$ does not satisfy both equations, $(1, -5)$ is not a solution to the system of equations.

18. (5 points) Explain why the following system of equations does not have a solution. Be sure to justify your answer.

$$-6x + y = 3$$
$$12x - 2y = -4$$

A system of two linear equations has a solution if and only if the lines intersect. But then the lines are not parallel. We can find the slopes of each of the lines:

$$-6x + y = 3$$
$$y = 6x + 3 \rightsquigarrow m_1 = -6$$

$$12x - 2y = -4$$
$$-2y = -12x - 4$$
$$y = 6x + 2 \rightsquigarrow m_2 = 6$$

Because $m_1 = m_2$, the lines are parallel. But then the system of equations does have a solution.