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MATH 108

Fall 2021

HW 7: Due 11/04

*"I like long walks, especially when they  
are taken by people that annoy me."*

*—Fred Allen*

**Problem 1.** (10pt) Write down the tableau associated to the following linear programming problem:

$$\begin{aligned}\max z &= 3x_1 + x_2 \\ 2x_1 + 3x_2 &\leq 6 \\ 4x_1 + 2x_2 &\leq 8 \\ -3x_1 + 4x_2 &\geq 2 \\ x_1, x_2 &\geq 0\end{aligned}$$

**Solution.** Before creating the tableau, we want to be sure the linear programming problem is in standard form (or as close as possible). Because this is a maximization problem, we require all the inequalities to be ' $\leq$ .' [This is excluding the final inequality  $x_1, x_2 \geq 0$ , because we require the variables to be nonnegative.] So we multiply both sides of the third inequality by  $-1$  to obtain  $3x_1 - 4x_2 \leq -2$ . Then our system is

$$\begin{aligned}\max z &= 3x_1 + x_2 \\ 2x_1 + 3x_2 &\leq 6 \\ 4x_1 + 2x_2 &\leq 8 \\ 3x_1 - 4x_2 &\leq -2 \\ x_1, x_2 &\geq 0\end{aligned}$$

We introduce slack variables  $s_1, s_2, s_3$  so that...

$$\begin{aligned}2x_1 + 3x_2 + s_1 &= 6 \\ 4x_1 + 2x_2 + s_2 &= 8 \\ 3x_1 - 4x_2 + s_3 &= -2\end{aligned}$$

Moving everything to the left side in  $z = 3x_1 + x_2$ , we have  $z - 3x_1 - x_2 = 0$ . Then the associated tableau is...

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$		
2	3	1	0	0	6	$s_1$
4	2	0	1	0	8	$s_2$
3	-4	0	0	1	-2	$s_3$
-3	-1	0	0	0	0	

**Problem 2.** (10pt) Assume the following is a tableau associated to a standard maximization problem. Write down the function being maximization and the system of constraints.

$$\begin{array}{cccccc|c}
 1 & 2 & 1 & 1 & 0 & 0 & 100 \\
 2 & 8 & 2 & 0 & 1 & 0 & 150 \\
 1 & 1 & 1 & 0 & 0 & 1 & 200 \\
 \hline
 -5 & -4 & -5 & 0 & 0 & 0 & 0
 \end{array}$$

**Solution.** It is clear that we had three inequalities. Therefore, we had three slack variables,  $s_1, s_2, s_3$ . Because there are seven columns, one of which must correspond to the  $b$ 's, there must be three variables,  $x_1, x_2, x_3$ . Looking at the last row and assuming we are trying to maximize a variable  $z$ , we must have  $z - 5x_1 - 4x_2 - 5x_3 = 0$ . But then  $z = 5x_1 + 4x_2 + 5x_3$ . Therefore, the standard maximization problem was...

$$\begin{aligned}
 \max z &= 5x_1 + 4x_2 + 5x_3 \\
 x_1 + 2x_2 + x_3 &\leq 100 \\
 2x_1 + 8x_2 + 2x_3 &\leq 150 \\
 x_1 + x_2 + x_3 &\leq 200 \\
 x_1, x_2, x_3 &\geq 0
 \end{aligned}$$

**Problem 3.** (10pt) Solve the following linear programming problem:

$$\begin{aligned}\max z &= x_1 + 6x_2 + 3x_3 \\ x_1 + x_2 + 2x_3 &\leq 4 \\ x_1 + 2x_2 + x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0\end{aligned}$$

**Solution.** This linear programming problem is already in standard form for a maximization. We introduce slack variables  $s_1$  and  $s_2$  so that...

$$\begin{aligned}x_1 + x_2 + 2x_3 + s_1 &= 4 \\ x_1 + 2x_2 + x_3 + s_2 &= 4\end{aligned}$$

Moving everything to the left side in  $z = x_1 + 6x_2 + 3x_3$ , we have  $z - x_1 - 6x_2 - 3x_3 = 0$ . Then the associated tableau is...

$$\begin{array}{ccccc|cc}x_1 & x_2 & x_3 & s_1 & s_2 & & \\1 & 1 & 2 & 1 & 0 & 4 & s_1 \\1 & 2 & 1 & 0 & 1 & 4 & s_2 \\\hline-1 & -6 & -3 & 0 & 0 & 0 & \end{array}$$

We find our first pivot position:

$$\begin{array}{ccccc|cc}x_1 & x_2 & x_3 & s_1 & s_2 & & \\1 & 1 & 2 & 1 & 0 & 4 & s_1 \quad 4/1 = 4 \\1 & \boxed{2} & 1 & 0 & 1 & 4 & s_2 \quad 4/2 = 2 \\\hline-1 & -6 & -3 & 0 & 0 & 0 & \end{array}$$

So  $x_2$  is the entering variable and  $s_2$  is the exiting variable. We make this pivot entry 1 by  $\frac{1}{2}R_2 \rightarrow R_2$ .

$$\begin{array}{ccccc|cc}x_1 & x_2 & x_3 & s_1 & s_2 & & \\1 & 1 & 2 & 1 & 0 & 4 & s_1 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 2 & x_2 \\\hline-1 & -6 & -3 & 0 & 0 & 0 & \end{array}$$

Now we perform the first step of the simplex method:  $-R_2 + R_1 \rightarrow R_1$  and  $6R_2 + R_3 \rightarrow R_3$ .

$$\begin{array}{ccccc|cc}x_1 & x_2 & x_3 & s_1 & s_2 & & \\ \frac{1}{2} & 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 2 & s_1 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 2 & x_2 \\\hline2 & 0 & 0 & 0 & 3 & 12 & \end{array}$$

Because there are no negatives in the bottom row, the simplex method is complete. We see that the maximum value for  $z$  is 12 (the bottom-rightmost entry) and occurs at  $(x_1, x_2, x_3, s_1, s_2) = (0, 2, 0, 2, 0)$ , i.e.  $x_1 = 0, x_2 = 2, x_3 = 0, s_1 = 2, s_2 = 0$ .

**Problem 4.** (10pt) Find the dual problem to...

$$\begin{aligned}\min z &= 5x_1 + 4x_2 \\ x_1 + 7x_2 &\geq 7 \\ x_1 + 3x_2 &\geq 9 \\ x_1, x_2 &\geq 0\end{aligned}$$

**Solution.** First, observe that this linear programming problem is a minimization and is already in standard form. Therefore, the associated matrix is...

$$\begin{pmatrix} 1 & 7 & 7 \\ 1 & 3 & 9 \\ 5 & 4 & 0 \end{pmatrix}$$

Taking the transpose, we find...

$$\begin{pmatrix} 1 & 1 & 5 \\ 7 & 3 & 4 \\ 7 & 9 & 0 \end{pmatrix}$$

Therefore, the dual problem is...

$$\begin{aligned}\max z &= 7x_1 + 9x_2 \\ x_1 + x_2 &\leq 5 \\ 7x_1 + 3x_2 &\leq 4 \\ x_1, x_2 &\geq 0\end{aligned}$$

**Problem 5.** (10pt) Solve the following linear programming problem:

$$\begin{aligned}\min z &= 2x_1 + 3x_2 \\ 2x_1 + x_2 &\geq 1 \\ x_1 + 3x_2 &\geq 1 \\ x_1, x_2 &\geq 0\end{aligned}$$

**Solution.** This is linear programming problem is a minimization and it is already in standard form. Therefore, the associated matrix is...

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 2 & 3 & 0 \end{pmatrix}$$

The transpose is then...

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \\ 1 & 1 & 0 \end{pmatrix}$$

Therefore, the associated maximization problem (the dual problem) is...

$$\begin{aligned}\max z &= x_1 + x_2 \\ 2x_1 + x_2 &\leq 2 \\ x_1 + 3x_2 &\leq 3 \\ x_1, x_2 &\geq 0\end{aligned}$$

Because  $z = x_1 + x_2$ , we know that  $z - x_1 - x_2 = 0$ . We introduce slack variables  $s_1, s_2$  so that...

$$\begin{aligned}2x_1 + x_2 + s_1 &= 2 \\ x_1 + 3x_2 + s_2 &= 3\end{aligned}$$

Therefore, the associated tableau is...

$x_1$	$x_2$	$s_1$	$s_2$		
2	1	1	0	2	$s_1$
1	3	0	1	3	$s_2$
-1	-1	0	0	0	

We find our first pivot position:

$x_1$	$x_2$	$s_1$	$s_2$			
<span style="border: 1px solid black; padding: 2px;">2</span>	1	1	0	2	$s_1$	$2/2 = 1$
1	3	0	1	3	$s_2$	$3/1 = 3$
-1	-1	0	0	0		

So  $x_1$  is the entering variable and  $s_1$  is the exiting variable. We make this pivot entry 1 by  $\frac{1}{2}R_1 \rightarrow R_1$ .

$$\begin{array}{cccc|c}
x_1 & x_2 & s_1 & s_2 & \\
1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 \quad x_1 \\
1 & 3 & 0 & 1 & 3 \quad s_2 \\
\hline
-1 & -1 & 0 & 0 & 0
\end{array}$$

Now we perform the first step of the simplex method:  $-R_1 + R_2 \rightarrow R_2$  and  $R_1 + R_3 \rightarrow R_3$ .

$$\begin{array}{cccc|c}
x_1 & x_2 & s_1 & s_2 & \\
1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 \quad x_1 \\
0 & \frac{5}{2} & -\frac{1}{2} & 1 & 2 \quad s_2 \\
\hline
0 & -\frac{1}{2} & \frac{1}{2} & 0 & 1
\end{array}$$

There are still negatives in the last row. So we proceed with the next step of the simplex method. We find our pivot position:

$$\begin{array}{cccc|c}
x_1 & x_2 & s_1 & s_2 & \\
1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 \quad x_1 \quad 1/(1/2) = 2 \\
0 & \boxed{\frac{5}{2}} & -\frac{1}{2} & 1 & 2 \quad s_2 \quad 2/(5/2) = 4/5 \\
\hline
0 & -\frac{1}{2} & \frac{1}{2} & 0 & 1
\end{array}$$

So  $x_2$  is the entering variable and  $s_2$  is the exiting variable. We make the pivot position 1 by  $\frac{2}{5}R_2 \rightarrow R_2$ .

$$\begin{array}{cccc|c}
x_1 & x_2 & s_1 & s_2 & \\
1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 \quad x_1 \\
0 & 1 & -\frac{1}{5} & \frac{2}{5} & \frac{4}{5} \quad s_2 \\
\hline
0 & -\frac{1}{2} & \frac{1}{2} & 0 & 1
\end{array}$$

Now we perform the next step of the simplex method:  $-\frac{1}{2}R_2 + R_1 \rightarrow R_1$  and  $\frac{1}{2}R_2 + R_3 \rightarrow R_3$ :

$$\begin{array}{cccc|c}
x_1 & x_2 & s_1 & s_2 & \\
1 & 0 & \frac{3}{5} & -\frac{1}{5} & \frac{3}{5} \quad x_1 \\
0 & 1 & -\frac{1}{5} & \frac{2}{5} & \frac{4}{5} \quad x_2 \\
\hline
0 & 0 & \frac{2}{5} & \frac{1}{5} & \frac{7}{5}
\end{array}$$

Because there are no remaining negative entries in the bottom row, the simplex method is complete. We have maximum value  $z = \frac{7}{5} = 1.4$  (the bottom-rightmost entry) occurring at  $(x_1, x_2, s_1, s_2) = (\frac{3}{5}, \frac{4}{5}, 0, 0) = (0.6, 0.8, 0, 0)$ , i.e.  $x_1 = \frac{3}{5} = 0.6, x_2 = \frac{4}{5} = 0.8, s_1 = 0, s_2 = 0$ .

But this maximum value (and its location) as the minimum value for our original minimization problem (the dual problem). Therefore, the minimum value is  $z = \frac{7}{5} = 1.4$  occurring at  $(x_1, x_2, s_1, s_2) = (\frac{3}{5}, \frac{4}{5}, 0, 0) = (0.6, 0.8, 0, 0)$ , i.e.  $x_1 = \frac{3}{5} = 0.6, x_2 = \frac{4}{5} = 0.8, s_1 = 0, s_2 = 0$