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MATH 308
Fall 2023
HW 6: Due 10/05

"I know that the great Hilbert said, 'We will not be driven out of the paradise Cantor has created for us,' and I reply, 'I see no reason for walking in!' "

–Richard Hamming

Problem 1. (10pt) Let A and B be sets. For each of the following sets, compute the *complement* of the given set. Be sure to show all your work and simplify your set expression as much as possible.

(a) $(A\Delta B) \cup B^c$

(b) $(A \cup B^c) \cap (A \cap B)^c$

(c) $A - (A - B)$

Solution.

(a) We use the fact that $A\Delta B = (A \cup B) - (A \cap B) = (A \cap B^c) \cup (B \cap A^c)$.

$$\begin{aligned}
\big((A\Delta B) \cup B^c\big)^c &= (A\Delta B)^c \cap (B^c)^c \\
&= (A\Delta B)^c \cap B \\
&= \big((A \cap B^c) \cup (B \cap A^c)\big)^c \cap B \\
&= (A \cap B^c)^c \cap (B \cap A^c)^c \cap B \\
&= (A^c \cup B) \cap (B^c \cup A) \cap B \\
&= [(A^c \cup B) \cap B] \cap (B^c \cup A) \\
&= B \cap (B^c \cup A) \\
&= (B \cap B^c) \cup (B \cap A) \\
&= \varnothing \cup (A \cap B) \\
&= A \cap B
\end{aligned}$$

(b)
$$\begin{aligned}
\big((A \cup B^c) \cap (A \cap B)^c\big)^c &= (A \cup B^c)^c \cup (A \cap B) \\
&= (A^c \cap B) \cup (A \cap B) \\
&= \big((A^c \cap B) \cup A\big) \cap \big((A^c \cap B) \cup B\big) \\
&= \big((A^c \cap B) \cup A\big) \cap B \\
&= \big((A^c \cup A) \cap (B \cup A)\big) \cap B \\
&= \big(\mathcal{U} \cap (B \cup A)\big) \cap B \\
&= (B \cup A) \cap B \\
&= B
\end{aligned}$$

(c) We use the fact that $A - B = A \cap B^c$.

$$\begin{aligned}
\big(A - (A - B)\big)^c &= \big(A \cap (A - B)^c\big)^c \\
&= A^c \cup (A - B) \\
&= A^c \cup (A \cap B^c) \\
&= (A^c \cup A) \cap (A^c \cup B^c) \\
&= \mathcal{U} \cap (A^c \cup B^c) \\
&= A^c \cup B^c \\
&= (A \cap B)^c
\end{aligned}$$

Problem 2. (10pt) Let $X = \{a, \{b\}, \{a, b\}\}$.

(a) Compute $\mathcal{P}(X)$. What is the cardinality of this set?

(b) Determine whether the following are true or false—no justification is necessary:

(i) $\varnothing \in X$

(ii) $\varnothing \subseteq X$

(iii) $a \in X$

(iv) $\{a\} \in X$

(v) $\{a\} \subseteq X$

(vi) $\varnothing \in \mathcal{P}(X)$

(vii) $\mathcal{P}(X) \subseteq \mathcal{P}(X)$

(viii) $\{a, b\} \in \mathcal{P}(X)$

(ix) $\{a, b\} \subseteq \mathcal{P}(X)$

(x) $\{\{a, b\}\} \subseteq \mathcal{P}(X)$

Solution.

(a) It would be useful to write S and compute $\mathcal{P}(S)$:

$$\mathcal{P}(X) = \left\{ \begin{array}{ccc} \varnothing, & & \\ \{a\}, & \{\{b\}\}, & \{\{a,b\}\}, \\ \{a,\{b\}\}, & \{a,\{a,b\}\}, & \{\{b\},\{a,b\}\}, \\ X = \{a,\{b\},\{a,b\}\} & & \end{array} \right\}$$

(b) (i) *F*

(ii) *T*

(iii) *T*

(iv) *F*

(v) *T*

(vi) *T*

(vii) *T*

(viii) *F*

(ix) *F*

(x) *F*

Problem 3. (10pt) For integers n, let $X_n = (n, n+1)$, and for natural numbers m, let $Y_m = \left[\frac{1}{m}, m\right)$. Compute the following:

(a) $\bigcup_{i=-1}^{2} X_i$

(b) $\bigcap_{k=2}^{5} Y_k$

(c) $\bigcup_{n\in\mathbb{Z}} X_n$

(d) $\bigcup_{m\in\mathbb{N}} Y_m$

(e) $\left(\bigcup_{m\in\mathbb{N}} Y_m\right)^c$

Solution.

(a) $\bigcup_{i=-1}^{2} X_i = (-1,0) \cup (0,1) \cup (1,2) \cup (2,3)$

(b) $\bigcap_{k=2}^{5} Y_k = [\frac{1}{2}, 2)$

(c) $\bigcup_{n\in\mathbb{Z}} X_n = \mathbb{R} - \mathbb{Z}$

(d) $\bigcup_{m\in\mathbb{N}} Y_m = (0, \infty)$

(e) $\left(\bigcup_{m\in\mathbb{N}} Y_m\right)^c = (0,\infty)^c = (-\infty, 0]$

Problem 4. (10pt) Let $A = \{-1, 0, 1\}$, $B = \{a, b\}$, and $C = \{\sqrt{2}, \pi\}$.

(a) Compute $A \times B$.

(b) Is $A \times B = B \times A$? Explain.

(c) Compute $\mathcal{P}(B \times C)$.

(d) If $X = \varnothing$, what is $X \times Y$ for any set Y?

(e) If X, Y are sets and $X \times Y = Y \times X$, is it necessarily true that $X = Y$? Explain. [Hint: Use part (d).]

Solution.

(a)

$$A \times B = \begin{Bmatrix} (-1, a), & (-1, b), \\ (0, a), & (0, b), \\ (1, a), & (1, b) \end{Bmatrix}$$

(b) No, $A \times B \neq B \times A$. For instance, $(-1, a) \in A \times B$ because $-1 \in A$ and $a \in B$; however, $(-1, a) \notin B \times A$ because $-1 \notin B$, $a \notin A$. In fact, we have...

$$B \times A = \begin{Bmatrix} (a, -1), & (a, 0), & (a, 1), \\ (b, -1), & (b, 0), & (b, 1) \end{Bmatrix}$$

(c) First, observe that we have $B \times C = \{(a, \sqrt{2}), (a, \pi), (b, \sqrt{2}), (b, \pi)\}$. But then we have...

$$\mathcal{P}(B \times C) = \left\{ \begin{array}{cccc} \varnothing, & & & \\ \{(a,\sqrt{2})\}, & \{(a,\pi)\}, & \{(b,\sqrt{2})\}, & \{(b,\pi)\}, \\ \{(a,\sqrt{2}),(a,\pi)\}, & \{(a,\sqrt{2}),(b,\sqrt{2})\}, & \{(a,\sqrt{2}),(b,\pi)\}, & \\ \{(a,\pi),(b,\sqrt{2})\}, & \{(a,\pi),(b,\pi)\}, & & \\ \{(b,\sqrt{2}),(b,\pi)\}, & & & \\ \{(a,\sqrt{2}),(a,\pi),(b,\sqrt{2})\}, & \{(a,\sqrt{2}),(a,\pi),(b,\pi)\}, & & \\ \{(a,\sqrt{2}),(b,\sqrt{2}),(b,\pi)\}, & & & \\ \{(a,\pi),(b,\sqrt{2}),(b,\pi)\}, & & & \\ B \times A = \{(a,\sqrt{2}),(a,\pi),(b,\sqrt{2}),(b,\pi)\} & & & \end{array} \right\}$$

(d) If $X = \varnothing$, then $X \times Y = \varnothing$. We know that $X \times Y = \{(x, y) \colon x \in X, y \in Y\}$. But if $X = \varnothing$, then there is no $x \in X$. Then there can be no $(x, y) \in X \times Y$, which proves that $X \times Y = \varnothing$. This holds mutatis mutandis for $Y = \varnothing$. Therefore, for all sets X, $X \times \varnothing = \varnothing$ and $\varnothing \times X = \varnothing$.

(e) We know from (d) that if $X = \varnothing$, then $X \times Y = \varnothing = Y \times X$ for all sets Y. But then taking Y to be any nonempty set, we see that if $X = \varnothing$, then $X \times Y = \varnothing = Y \times X$ but $X = \varnothing \neq Y$. Therefore, it is not true that if $X \times Y = Y \times X$ that $X = Y$.

However, if X, Y are nonempty sets and $X \times Y = Y \times X$ does imply that $X = Y$. To see this, suppose that X, Y are nonempty sets with $X \times Y = Y \times X$. Choose any $x \in X$ and $y \in Y$. By definition, $(x, y) \in X \times Y$. But $X \times Y = Y \times X$ so that $(x, y) \in Y \times X$. This implies that $x \in Y$ and $y \in X$. Therefore, for all $x \in X, y \in Y$, if $x \in X$, then $x \in Y$, which implies that $X \subseteq Y$, and if $y \in Y$, then $y \in X$, which implies that $Y \subseteq X$. Because $X \subseteq Y$ and $Y \subseteq X$, we know that $X = Y$. Obviously, $X \times Y = Y \times X$, if $X = Y = \varnothing$.