Name: ______________________________

MATH 361

Spring 2022

Written HW 1: Due 02/15

"Well, that's no reason to cry. One cries because one is sad. For example, I cry because others are stupid and it makes me sad."
–Sheldon Cooper, The Big Bang Theory

Problem 1. (Big $\mathcal{O}$) The following problem will explore $\mathcal{O}$, o, and their relationship. It will also explore the relationship between $\mathcal{O}$ and Taylor series.

(a) For each of the following sequences $\{x_n\}$, determine if $x_n = \mathcal{O}(f_n)$ and $x_n = o(f_n)$ as $x \to \infty$:

   (i) $x_n = 5n + 7$, $f_n = n$

   (ii) $x_n = 2n^2 + 3n + 1$, $f_n = n^3$

   (iii) $x_n = \sqrt{n+5}$, $f_n = n$

(b) Show that if $x_n = \mathcal{O}(y_n)$, then $cx_n = \mathcal{O}(cy_n)$, where $c \in \mathbb{R}$. Is the same true for o?

(c) Determine the 'best' integer k such that $e^{2x} = 1 + 2x + \mathcal{O}(x^k)$ as $x \to 0$.

(d) Explain why every smooth function can be approximated on an interval of length h by a polynomial of degree n with an error that is $\mathcal{O}(h^{n+1})$ as $h \to 0$. [Hint: Appeal to Taylor's Theorem.]

Problem 2. (IEEE Floating Point Numbers) This problem will work through an explicit example of how IEEE floating point numbers work with a 'nice' real number. Define $n := 2^5 + 2^{-17} + 2^{-21}$.

(a) Write n in binary.

(b) Express n as a IEEE single precision number.

(c) Determine $\mathrm{fl}(n)$.

(d) Determine the absolute error $|n - \mathrm{fl}(n)|$ and relative error $\dfrac{|n - \mathrm{fl}(n)|}{|n|}$.

(e) What must the relative error be bounded by? Does your answer in (d) agree with this bound?

Problem 3. (Numerical Stability) The following problem will help develop 'hands-on' experience recognizing and avoiding loss of significance. For each of the following functions, describe when a loss of significance may occur and suggest—with justification—ways to avoid the loss.

(a) $\sqrt{x+4} - 2$

(b) $\ln x - 1$ [Do *not* use Taylor Series]

(c) $e^x - e^{-2x}$ [Hint: Taylor Series]

(d) $2\cos^2 x - 1$

Problem 4. (Condition Number) The following problem will help develop 'hands-on' computing condition numbers and recognizing where a computation might be 'unstable.' For the following functions, compute the condition number and describe the x values where the condition number is large:

(a) x^5

(b) $(x+2)^3$

(c) $\ln x$

(d) $\sin x$

Problem 5. (Algorithm Analysis) The following problem will help develop 'hands-on' experience reading and improving algorithms. Consider the following pseudocode that takes as an input an integer n:

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for i = 1 to n:
  sum = 0;
    for j = 1 to i:
    sum += j
  print(sum)
```

(a) Describe what this algorithm is computing.

(b) How many flops are used in this algorithm? Find x_n such that this algorithm is $\mathcal{O}(x_n)$.

(c) Explain why this algorithm is computationally inefficient.

(d) Rewrite this pseudocode so that it is computationally less expensive. For your algorithm, count the number of flops required.

(e) Is there an even shorter way avoiding loops entirely?

Evaluation.

Complete the following survey by rating each problem. Each area will be rated on a scale of 1 to 5. For interest, 1 is “mind-numbing” while a 5 is “mind-blowing.” For difficulty, 1 is “trivial/routine” while 5 is “brutal.” For learning, 1 means “nothing new” while 5 means ”profound awakening.” Then you to estimate the amount of time you spent on each problem (in minutes).

	Interest	Difficulty	Learning	Time Spent
Problem 1				
Problem 2				
Problem 3				
Problem 4				
Problem 5				

Finally, indicate whether you believe lectures were useful in completing this assignment and whether you believe the problems were useful enough/interesting enough to assign again to future students by checking the appropriate space.

	Lectures		Assign Again	
	Yes	No	Yes	No
Problem 1				
Problem 2				
Problem 3				
Problem 4				
Problem 5				