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MATH 101
Spring 2024
HW 18: Due 04/17

"If you don't learn from your mistakes, there's no sense making them."
— Herbert V. Prochnow

Problem 1. (10pts) Without explicitly solving the quadratic equation, determine whether how many distinct solutions the equation has and whether the solutions are rational, real, or complex. Be sure to justify your answer.

$$x^2 = 36 - 5x$$

Solution. We can determine the nature of solutions for a quadratic equation of the form $f(x) = 0$, where $f(x)$ is a quadratic function, using the discriminant of $f(x)$. We have...

$$\begin{gathered} x^2 = 36 - 5x \\ x^2 + 5x - 36 = 0 \end{gathered}$$

Let $f(x) = x^2 + 5x - 36$. This is a quadratic function, i.e. a function of the form $ax^2 + bx + c$, with $a = 1$, $b = 5$, and $c = -36$. The discriminant of $f(x)$ is...

$$\operatorname{disc} f(x) = b^2 - 4ac = 5^2 - 4(1)(-36) = 25 + 144 = 169$$

Because $\operatorname{disc} f(x) = 169 > 0$, this equation has two distinct, real solutions. Moreover, because $169 = 13^2$ is a perfect square, the solutions are rational (in fact, they are integers). One can show that the solutions are $x = -9, 4$.

Problem 2. (10pts) Without explicitly factoring the function $f(x) = x^2 - 8x + 5$ factors 'nicely' over the integers, reals, or complex numbers. Be sure to justify your answer.

Solution. We can determine the nature of the factorization of a quadratic function $ax^2 + bx + c$ using the discriminant of the function. For the quadratic function $f(x) = x^2 - 8x + 5$, we have $a = 1$, $b = -8$, and $c = 5$. But then...

$$\operatorname{disc} f(x) = b^2 - 4ac = (-8)^2 - 4(1)5 = 64 - 20 = 44$$

Because $\operatorname{disc} f(x) = 44 > 0$, $f(x)$ factors over the real numbers. However, because $\operatorname{disc} f(x) = 44$ is not a perfect square, $f(x)$ does not factor 'nicely' over the real numbers. In fact,

$$x^2 - 8x + 5 = \big(x - (4 - \sqrt{11})\big)\big(x - (4 + \sqrt{11})\big)$$

Problem 3. (10pts) Find the roots for the function $f(x) = 2x^2 - 7x + 1$. Be sure to fully justify your answer and show all your work.

Solution. To find the roots of $f(x)$, we need to solve the equation $f(x) = 0$. Using the fact that for $f(x)$, we have $a = 2$, $b = -7$, and $c = 1$, one can show that $\operatorname{disc} f(x) = 41$. Because this is not a perfect square over the real or complex numbers, $f(x)$ does not factor 'nicely.' We then need either complete the square or use the quadratic formula. Using the quadratic formula, we have...

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)1}}{2(2)} \\ &= \frac{7 \pm \sqrt{49 - 8}}{4} \\ &= \frac{7 \pm \sqrt{41}}{4} \end{aligned}$$

Therefore, the roots of $f(x)$ are $x = \frac{7-\sqrt{41}}{4} \approx 0.149219$ and $x = \frac{7+\sqrt{41}}{4} \approx 3.35078$.

Alternatively, we can complete the square to solve the equation $f(x) = 0$. Using this approach, we have...

$$\begin{gathered} 2x^2 - 7x + 1 = 0 \\ 2x^2 - 7x = -1 \\ x^2 - \frac{7}{2}x = -\frac{1}{2} \\ x^2 - \frac{7}{2}x + \left(\frac{1}{2} \cdot -\frac{7}{2}\right)^2 = -\frac{1}{2} + \left(\frac{1}{2} \cdot -\frac{7}{2}\right)^2 \\ x^2 - \frac{7}{2}x + \left(-\frac{7}{4}\right)^2 = -\frac{1}{2} + \left(-\frac{7}{4}\right)^2 \\ x^2 - \frac{7}{2}x + \frac{49}{16} = -\frac{1}{2} + \frac{49}{16} \\ \left(x - \frac{7}{4}\right)^2 = \frac{41}{16} \\ \sqrt{\left(x - \frac{7}{4}\right)^2} = \sqrt{\frac{41}{16}} \\ x - \frac{7}{4} = \pm\frac{\sqrt{41}}{\sqrt{16}} \\ x = \frac{7}{4} \pm \frac{\sqrt{41}}{4} \\ x = \frac{7 \pm \sqrt{41}}{4} \end{gathered}$$

Problem 4. (10pts) Solve the following equation. Be sure to fully justify your answer and show all your work.

$$x(x+1) = -3$$

Solution. By completing the square, we have...

$$\begin{gathered}
x(x+1) = -3 \\
x^2 + x = -3 \\
x^2 + x + \left(\frac{1}{2}\right)^2 = -3 + \left(\frac{1}{2}\right)^2 \\
x^2 + x + \frac{1}{4} = -3 + \frac{1}{4} \\
\left(x + \frac{1}{2}\right)^2 = -\frac{11}{4} \\
\sqrt{\left(x + \frac{1}{2}\right)^2} = \sqrt{-\frac{11}{4}} \\
x + \frac{1}{2} = \pm i\sqrt{\frac{11}{4}} \\
x + \frac{1}{2} = \pm i\,\frac{\sqrt{11}}{\sqrt{4}} \\
x = -\frac{1}{2} \pm i\,\frac{\sqrt{11}}{2} \\
x = \frac{-1 \pm i\sqrt{11}}{2}
\end{gathered}$$

Therefore, the roots are $x = \frac{-1-i\sqrt{11}}{2}$ and $x = \frac{-1+i\sqrt{11}}{2}$.

Alternatively, this equation is equivalent to...

$$\begin{gathered}
x(x+1) = -3 \\
x^2 + x = -3 \\
x^2 + x + 3 = 0
\end{gathered}$$

This is a quadratic equation of the form $f(x) = 0$, where $f(x)$ is the quadratic function $f(x) = x^2 + x + 3$, i.e. a quadratic function of the form $ax^2 + bx + c$ with $a = 1$, $b = 1$, and $c = 3$. Using the quadratic equation, we have...

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-1 \pm \sqrt{1^2 - 4(1)3}}{2(1)} \\
&= \frac{-1 \pm \sqrt{1 - 12}}{2} \\
&= \frac{-1 \pm \sqrt{-11}}{2} \\
&= \frac{-1 \pm i\sqrt{11}}{2}
\end{aligned}$$