Name: ______________________________

MATH 308

Fall 2022

HW 6: Due 09/27

"Since, as is well known, god helps those who help themselves, presumably the devil helps all those, and only those, who don't help themselves. Does the devil help himself?"

–Douglas Hofstadter, Gödel, Escher, Bach: An Eternal Golden Braid

Problem 1. (10pt) Let $S := \{-3, -2, -1, 0, 1, 2, 3\}$ be a universal set and define $X := \{-1, 0, 1\}$. Give an example of...

(a) a proper subset of S, say A, that is disjoint from X.

(b) a subset of S, say B, such that $B - X \neq B$.

(c) a subset of S, say C, such that $X \Delta C = X \cup C$.

(d) a subset of S, say D, such that D^c contains only nonnegative numbers.

(e) a subset of S, say E, such that the complement of $X \cup E$ is empty.

Problem 2. (10pt) Let A and B be sets. By defining $A = B$ by using a quantified open sentence, show that $A \neq B$ is equivalent to the logical statement...

$$(\exists x)\big(x \in A \wedge x \notin B\big) \vee (\exists x)\big(x \in B \wedge x \notin A\big)$$

Problem 3. (10pt) Let A and B be sets in a universe $\mathcal{U}$ and consider the set $A \Delta B$.

(a) Using set-builder notation and logical propositions, define the set $A \Delta B$.

(b) Construct a Venn diagram for the set $(A \Delta B)^c$.

(c) Construct a Venn diagram for the set $(A \cup B)^c \cup (A \cap B)$

(d) What might you conjecture from your answers in (b) and (c)?

Problem 4. (10pt) Let A, B, and C be sets in some universe $\mathcal{U}$. Find the *complement* of the following sets, showing all your work and 'simplifying' as much as possible:

(a) $A \setminus B$

(b) $(A^c \cup C) \cap B$

(c) $\left(\left((A \cup B) \cap C\right)^c \cup B^c\right)^c$

Problem 5. (10pt) Define $S := \{1, 2, \{1\}, \{\{2\}\}\}$. Determine whether the following are true or false—no justification is necessary:

(a) $\varnothing \in S$

(b) $\varnothing \subseteq S$

(c) $1 \in \mathcal{P}(S)$

(d) $\{1\} \in \mathcal{P}(S)$

(e) $\{\{1\}\} \in \mathcal{P}(S)$

(f) $1 \subseteq \mathcal{P}(S)$

(g) $\{1\} \subseteq \mathcal{P}(S)$

(h) $\{\{1\}\} \subseteq \mathcal{P}(S)$

(i) $\varnothing \in \mathcal{P}(S)$

(j) $\{\varnothing\} \in \mathcal{P}(S)$

(k) $\varnothing \subseteq \mathcal{P}(S)$

(l) $\{\varnothing\} \subseteq \mathcal{P}(S)$

Problem 6. (10pt) Define $A := \{3, 5, 7\}$ and $B := \{\pi, e, \sqrt{2}, \varphi\}$.

(a) Determine $A \times B$.

(b) Is $(3, \pi) \in A \times B$? Is $(\pi, 3) \in A \times B$? Explain the relation between your responses.

(c) Is $A \times B = B \times A$? Explain.

Problem 7. (10pt) Determine $\bigcup_{i\in\mathcal{I}} A_n$ and $\bigcap_{i\in\mathcal{I}} A_n$ for the given A_n and $\mathcal{I}$ below—no justification is necessary. However, if the set is finite, enumerate its elements; otherwise, either give the set in set-builder notation or using set operations with 'standard' sets, e.g. $\mathbb{Q}$, $\mathbb{Z} \setminus \mathbb{N}$, etc.

(a) $A_n := \left(\frac{1}{n}, 1 + \frac{1}{n}\right)$; $\mathcal{I} := \mathbb{N}$

(b) $A_n := (n, n+1)$; $\mathcal{I} := \mathbb{Z}$

(c) $A_n := \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$; $\mathcal{I} := \mathbb{R}$

Problem 8. (10pt) Below is a partial proof of the fact that $A \setminus B = A \cap B^c$. By filling in the missing portions, complete the partial proof below so that it is a correct, logically sound proof with 'no gaps':

Proposition. If A and B are sets, then $A \setminus B = A \cap B^c$.

Proof. If $A \setminus B = \varnothing$, then there is no element in A that is not also in B. But then $A \subseteq B$ so that $A^c \supseteq B^c$. But then $A \cap B^c \subseteq A \cap A^c = \varnothing$ so that $A \cap B^c = \varnothing$. Therefore, if $A \setminus B = \varnothing$, then $A \setminus B = A \cap B^c$. If $A \cap B^c = \varnothing$, then there is no element in both A and B^c. Now if there were an element in $A \setminus B$, there would be an element in A that is not in B, i.e. an element in A that is in B^c, a contradiction to the fact that $A \cap B^c = \varnothing$, i.e. that there is no element in both A and B^c. This shows that $A \setminus B = \varnothing$. Therefore, if $A \cap B^c = \varnothing$, then $A \setminus B = A \cap B^c$. Then we have shown that if either $A \setminus B$ or $A \cap B^c$ are empty then $A \setminus B = A \cap B^c$. Now assume that both $A \setminus B$ and $A \cap B^c$ are nonempty.

To prove that $A \setminus B = A \cap B^c$, we need to show ________________ and ________________.

$A \setminus B \subseteq A \cap B^c$: We prove that $A \setminus B \subseteq A \cap B^c$. Let $x \in$ ________________. Then by definition,

$x \in A$ and ________________. But then $x \in$ ________________ and $x \in B^c$. This shows that

$x \in$ ________________. Therefore, this shows that ________________.

________________: We need to show that $A \cap B^c \subseteq A \setminus B$. Let $x \in$ ________________. Then

$x \in$ ________________ and $x \in$ ________________. But then $x \in$ ________________ and

$x \notin$ ________________. This shows that $x \in$ ________________. Therefore, we know that ________________.

Because ________________ and ________________, we know that $A \setminus B = A \cap B^c$. $\square$