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MATH 308
Fall 2023
HW 8: Due 10/12

"The study of Mathematics, like the Nile, begins in minuteness but ends in magnificence."

–Charles Caleb Colton

Problem 1. (10pt) Let $A = \{2, 6, 8, 10\}$, B be the set of nonnegative even numbers that are at most 10, and C be the set of perfect squares less than 10. Define $f : A \to \mathbb{Z}$ and $g : B \setminus C \to \mathbb{Z}$ via $x \to \frac{15(x+8)}{x}$ and $x \mapsto \frac{5(x^2-16x+88)}{4}$, respectfully. Fully justifying your answer, determine whether $f \equiv g$.

Solution. To show that two functions f, g are equal, i.e. $f = g$ or $f \equiv g$, we need to show that they have the same domain, the same codomain, and their outputs are the same everywhere on their 'common domain.'[1]

Equal Domains, $A = B$: We need to show $A = B$ that is, we need to show that A and B have all the same elements. We know that $A = \{2, 6, 8, 10\}$. Now B is the set of nonnegative even numbers less than 10, i.e. $B = \{0, 2, 4, 6, 8, 10\}$. Furthermore, C is the set of perfect squares less than 10, i.e. $C = \{0, 4, 9\}$. But then $B \setminus C = \{2, 6, 8, 10\}$. Therefore, $A = B \setminus C$.

Equal Codomains, $\mathbb{Z} = \mathbb{Z}$: It is immediately clear that f and g have the same codomain—namely, $\mathbb{Z}$.

Equivalent on their Common Domain: To check whether f and g have the same outputs for every element of their 'common domain', we can simply compute f, g for the values in $\{2, 6, 8, 10\}$:

$$\begin{aligned}
f(2) &= \frac{15(2+8)}{2} = \frac{150}{2} = 75 & \qquad g(2) &= \frac{5(2^2 - 16(2) + 88)}{4} = \frac{300}{4} = 75 \\
f(6) &= \frac{15(6+8)}{6} = \frac{210}{6} = 35 & g(6) &= \frac{5(6^2 - 16(6) + 88)}{4} = \frac{140}{4} = 35 \\
f(8) &= \frac{15(8+8)}{8} = \frac{240}{8} = 30 & g(8) &= \frac{5(8^2 - 16(8) + 88)}{4} = \frac{120}{4} = 30 \\
f(10) &= \frac{15(10+8)}{10} = \frac{270}{10} = 27 & g(10) &= \frac{5(10^2 - 16(10) + 88)}{4} = \frac{140}{4} = 35
\end{aligned}$$

Observe that $f(2) = g(2) = 75$, $f(6) = g(6) = 35$, and $f(8) = g(8) = 30$. However, $f(10) = 27 \neq 35 = g(10)$. Therefore, f and g do not agree on their 'common domain.'

Because f and g do not agree on their 'common domain', f and g are not equal, i.e. $f \not\equiv g$.

[1]Note: This is not the same as the two functions having the same image. For example, take $A = \{1, 2\}$ and $B = \{a, b\}$. Define $f, g : A \to B$ via $f(1) = a$, $f(2) = b$, and $g(1) = b$ and $g(2) = a$. Clearly, f, g have the same domain and codomains. The image of both f and g are the same—namely, the set $\{a, b\}$, but observe $a = f(1) \neq g(1) = b$ and $b = f(2) \neq g(2) = a$.

Problem 2. (10pt) Define the following real-valued functions:

$$\begin{aligned} f(x) &= 2x - 1 & \qquad j(x) &= \frac{x-1}{x+2} \\ g(x) &= x^2 + x + 1 & \qquad k(x) &= \sin(\pi x) \\ h(x) &= x2^x & \qquad \ell(x) &= 1 - x^2 \end{aligned}$$

Showing all your work, for each of the following, either compute the function at the specified value or find a general rule for the given function operation:

(a) $(f+g)(0)$

(b) $(j-\ell)(2)$

(c) $(gk)(5)$

(d) $\left(\frac{f}{j}\right)(3)$

(e) $(h \circ k)(1)$

(f) $(2f + \ell)(x)$

(g) $(fg)(x)$

(h) $\left(\frac{h}{f}\right)(x)$

(i) $(k \circ \ell)(x)$

(j) $(\ell \circ g \circ f)(x)$

Solution.

(a) $(f+g)(0) = f(0) + g(0) = (2 \cdot 0 - 1) + (0^2 + 0 + 1) = -1 + 1 = 0$

(b) $(j-\ell)(2) = j(2) - \ell(2) = \frac{2-1}{2+2} - (1 - 2^2) = \frac{1}{4} - (-3) = \frac{13}{4}$

(c) $(gk)(5) = g(5)k(5) = (5^2 + 5 + 1) \cdot \sin(5\pi) = 31 \cdot 0 = 0$

(d) $\left(\frac{f}{j}\right)(3) = \frac{f(3)}{j(3)} = \frac{2 \cdot 3 - 1}{(3-1)/(3+2)} = \frac{5}{2/5} = \frac{25}{2}$

(e) $(h \circ k)(1) = h\big(k(1)\big) = h\big(\sin(\pi)\big) = h(0) = 0 \cdot 2^0 = 0$

(f) $(2f + \ell)(x) = 2f(x) + \ell(x) = 2(2x-1) + (1 - x^2) = -x^2 + 4x - 1$

(g) $(fg)(x) = f(x)g(x) = (2x-1)(x^2 + x + 1) = 2x^3 + x^2 + x - 1$

(h) $\left(\frac{h}{f}\right)(x) = \frac{h(x)}{f(x)} = \frac{x2^x}{2x-1}$

(i) $(k \circ \ell)(x) = k\big(\ell(x)\big) = k(1 - x^2) = \sin\big(\pi(1 - x^2)\big) = \sin(\pi - \pi x^2) = \sin(\pi)\cos(\pi x^2) - \cos(\pi)\sin(\pi x^2) = \sin(\pi x^2)$

(j) $(\ell \circ g \circ f)(x) = \ell\big(g(f(x))\big) = \ell\big(g(2x-1)\big) = \ell\big((2x-1)^2 + (2x-1) + 1\big) = \ell(4x^2 - 2x + 1) = 1 - (4x^2 - 2x + 1)^2 = -16x^4 + 16x^3 - 12x^2 + 4x$

Problem 3. (10pt) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $x \mapsto x^2 + 4x - 5$.

(a) Determine $f(-5)$.

(b) Compute $f([0,1])$.

(c) Is $16 \in \operatorname{im} f$? Explain.

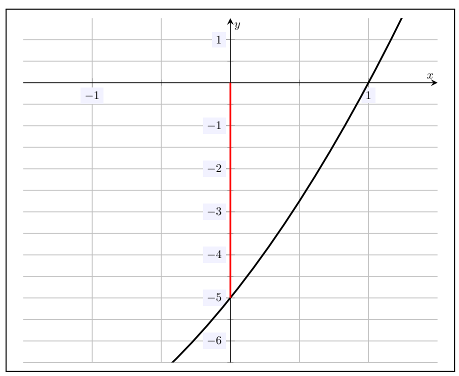
(d) Determine $f^{-1}(0)$.

(e) Find the domain, codomain, and range for $f(x)$.

Solution.

(a) $f(-5) = (-5)^2 + 4(-5) - 5 = 25 - 20 - 5 = 0$.

(b) We know that $f([0,1]) = \{f(x) \colon x \in [0,1]\}$. Observe that if f is strictly increasing, then if $x < y$, then $f(x) < f(y)$. But observe that f is differentiable and $f'(x) = 2x + 4$. Observe that f' is positive on the interval $[0,1]$. But because $f'(x) > 0$ for all $x \in [0,1]$, we know that f is strictly increasing on $[0,1]$. Finally, observe that because f is differentiable, f is continuous. Using the Intermediate Value Theorem, we see that f takes on every value between $f(0)$ and $f(1)$. Because $f(0) = -5$ and $f(1) = 0$, we have $f([0,1]) = [-5,0]$. We can also see this from the graph of $f(x) = x^2 + 4x - 5 = (x+2)^2 - 9$:



(c) If $16 \in \operatorname{im} f$, then there is an $x \in \mathbb{R}$ such that $f(x) = 16$. But then...

$$\begin{aligned} f(x) &= 16 \\ x^2 + 4x - 5 &= 16 \\ x^2 + 4x - 21 &= 0 \\ (x-3)(x+7) &= 0 \end{aligned}$$

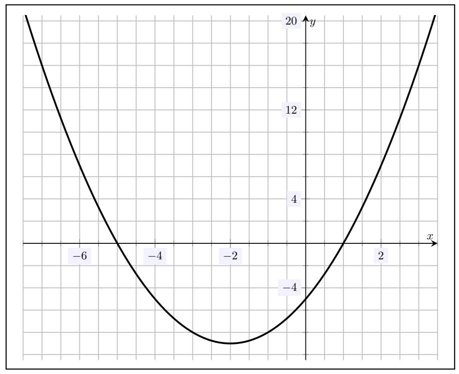
But then $x = -7$ or $x = 3$. Observe that $f(-7) = 16$ or $f(3) = 16$. Therefore, $16 \in \operatorname{im} f$.

(d) If $x \in f^{-1}(0)$, then $f(x) = 0$. But then...

$$\begin{aligned} f(x) &= 0 \\ x^2 + 4x - 5 &= 0 \\ (x-1)(x+5) &= 0 \end{aligned}$$

But then $x = -5$ or $x = 1$. Observe that $f(-5) = 0$ and $f(1) = 0$. Therefore, $f^{-1}(0) = \{-5, 1\}$.

(e) Clearly, the domain and codomain are $\mathbb{R}$, as given in the problem statement. We know the range of f is the image of f, i.e. $\operatorname{im} f = \{f(x) \colon x \in \mathbb{R}\}$. Examining the graph of f, we can see that $\operatorname{im} f = f(\mathbb{R}) = [-9, \infty)$. We can also see this from the fact that $f(x) = (x+2)^2 - 9$.



To prove this, suppose that $y \in [-9, \infty)$, choose $x := \sqrt{9+y} - 2$, which is well-defined because $y \geq -9$. But then...

$$f(x) = (\sqrt{9+y} - 2)^2 + 4(\sqrt{9+y} - 2) - 5 = (y + 13 - 4\sqrt{9+y}) + 4(\sqrt{9+y} - 2) + 4 = y$$

So if $y \in [-9, \infty)$, $y \in \operatorname{im} f$. Clearly, if $y < -9$, then $y \notin \operatorname{im} f$: if there were an $x \in \mathbb{R}$ such that $f(x) = y$, then $(x+2)^2 - 9 = y < -9$. This shows that $(x+2)^2 < 0$, which is impossible. Therefore, $\operatorname{im} f = [-9, \infty)$. Alternatively, we know that $f(-2) = -9$, $\lim\limits_{x \to -\infty} f(x) = \infty$, and $\lim\limits_{x \to \infty} f(x) = \infty$ (because f is an even degree polynomial). We can see that $f(x) = (x+2)^2 - 9 \geq 0 - 9 = -9$. Finally, because f is continuous, by the Intermediate Value Theorem, it must be that f takes on any value in $[-9, c]$ for all $c \in (-9, \infty)$. This again shows that $\operatorname{im} f = [-9, \infty)$.

Problem 4. (10pt) Being sure to justify your answer, complete the following:

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 5 - x^2$. Is f an increasing function? Explain. Is f a decreasing function? Explain.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = 5x - 8$. Is g a positive function? Explain. Is g a negative function? Explain.

(c) Let g be as in (b) and define $A = [2, \infty)$ and $B = (-\infty, 0)$. Is $g\big|_A$ a positive function? Explain. Is $g\big|_B$ a negative function? Explain.

(d) Let $h : \mathbb{R} \to \mathbb{R}$ be given by...

$$h(x) = \begin{cases} 1 - x, & x < 2 \\ 3x + 5, & x \geq 2 \end{cases}$$

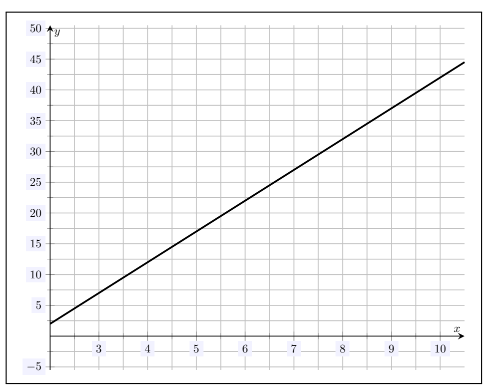
Find the largest possible interval $S \subseteq \mathbb{R}$ such that $h\big|_S$ is a nondecreasing function. Is h monotone on S? Is h strictly monotone on S?

Solution.

(a) If f were increasing, then for all $x, y \in \mathbb{R}$ with $x < y$, we would have $f(y) > f(x)$. However, observe that $0 < 1$ but $4 = f(1) \not> f(0) = 5$. Therefore, f is not an increasing function. If f is a decreasing function, then for all $x, y \in \mathbb{R}$ with $x < y$, we would have $f(y) < f(x)$. However, observe that $-1 < 0$ but $5 = f(0) \not< f(-1) = 4$. Therefore, f is not decreasing.

(b) If g is a positive function, then $g(x) > 0$ for all $x \in \mathbb{R}$. However, observe that $g(0) = -8 \not> 0$. Therefore, g is not a positive function. If g is a negative function, then $g(x) < 0$ for all $x \in \mathbb{R}$. However, observe that $g(2) = 2 \not< 0$. Therefore, g is not a negative function.

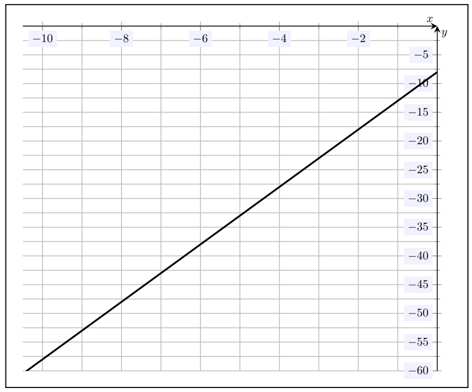
(c) From the graph of $g\big|_A$, we can see that g is not a negative function and g is a positive function.



To see that g is not negative, observe that $2 \in A$ and $g(2) = 2 \not< 0$. To prove that g is positive, observe that if $x \geq 2$, then...

$$\begin{aligned} x &\geq 2 \\ 5x &\geq 10 \\ 5x - 8 &\geq 2 \\ g(x) &\geq 2 \end{aligned}$$

But then for $x \geq 2$, $g(x) \geq 2 > 0$. Therefore, g is positive on $A = [2, \infty)$. Now from the graph of $g\big|_B$, we can see that g is negative but not positive.

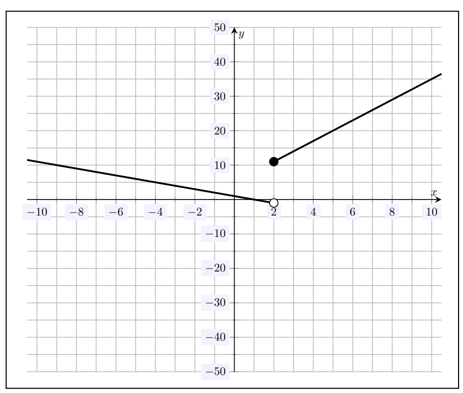


To see that g is not positive, observe that $g(0) = -8 \not> 0$. Therefore, g is not positive. To see that g is negative, observe that if $x < 0$, then...

$$\begin{aligned} x &< 0 \\ 5x &< 0 \\ 5x - 8 &< -8 \\ g(x) &< -8 \end{aligned}$$

But then $g(x) < -8 < 0$. Therefore, g is negative on $B = (-\infty, 0)$.

(d) We first plot h, which is shown below.



Clearly, h is linear on $[2, \infty)$ and $(-\infty, 2)$. On $(-\infty, 2)$, $h \equiv 1 - x$. Because this linear function has negative slope, it is decreasing. In particular, h is not nondecreasing. On $[2, \infty)$, $h \equiv 3x + 5$. Because this linear function has positive slope, it is increasing. In particular, h is then nondecreasing. But then the largest interval on which h is nondecreasing is $S := [2, \infty)$. Because h is nondecreasing or nonincreasing on $[2, \infty)$, h is monotone on $[2, \infty)$. However, because h is not (strictly) increasing or (strictly) decreasing on $S := [2, \infty)$, h is not strictly monotone on $S := [2, \infty)$.