Name: *Caleb McWhorter — Solutions*
MATH 308
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HW 4: Due 09/20

"Pure Mathematics is the world's best game. It is more absorbing than chess, more of a gamble than poker, and lasts longer than Monopoly. It's free. It can be played anywhere—Archimedes did it in a bathtub."
–Richard J. Trudeau

Problem 1. (10pt) Suppose that $P(x)$ is a predicate. Being sure to justify your answer, explain whether the following statements are true or false.

(a) There are choices of x for which $P(x)$ is true and choices of x for which $P(x)$ is false.

(b) Once one quantifies $P(x)$ using $\forall x$ or $\exists x$, the resulting statement is always true or always false—but not both.

(c) If $\exists! x\, P(x)$ is true, then $\exists x\, P(x)$ is true.

(d) The converse of (c) is also true.

Solution.

(a) The statement is false. Recall that a predicate is a 'statement' that depends on finitely many variables such that when the variables are substituted, the resulting statement is a proposition, i.e. has a well-defined truth value of true or false. Assuming that $P(x)$ is a predicate, we then know that for any given x, $P(x)$ is either true or false. Because we required our domains be nonempty, there must be a value x for which $P(x)$ is true or false. However, there need not be an x where $P(x)$ is true and where $P(x)$ is false. For instance, if $P(x) : x^2 < 0$, then for all $x \in \mathbb{R}$, $P(x)$ is false. Alternatively, if $P(x) : x^2 \geq 0$, then for all $x \in \mathbb{R}$, $P(x)$ is true. So the truth set and false set for $P(x)$ need not both be nonempty, though this can be the case. For example, $P(x) : 1 - x^2 \geq 0$ is true for $-1 \leq x \leq 1$ but false for $x \in (-\infty, -1) \cup (1, \infty)$.

(b) The statement is true. From the discussion in (a), we know that at least one of the truth set or the false set of $P(x)$ is nonempty. Once one quantifies a predicate with $\exists$ or $\forall$, the resulting statement is either true or false. For instance, $\forall x\, P(x)$ is true if and only if the truth set is nonempty but is precisely $\mathcal{U}$ and is false otherwise. We know $\exists x\, P(x)$ is true if and only if the truth set of $P(x)$ is nonempty. Finally, we know that the statement cannot be *both* true and false.

(c) The statement is true. For $\exists x\, P(x)$ to be true, there need only be at least one x for which $P(x)$ is true. If $\exists! x\, P(x)$ is true, there is one and only one x such that $P(x)$ is true. But then there is still at least one such x so that $\exists x\, P(x)$ is true.

(d) This statement is false. For $\exists x\, P(x)$ to be true, there need only be at least one x (but may be more) for which $P(x)$ is true. For $\exists! x\, P(x)$ to be true, there is one and only one x such that $P(x)$ is true. There then can be many values of x for such $P(x)$ is true, so that $\exists x\, P(x)$ is true, but then $\exists! x\, P(x)$ is false. As a counterexample to the statement, consider $P(x) : x > 0$. Because $x = 1 > 0$, we know that $\exists x\, P(x)$ is true. However, because $x = 2 > 0$ also gives us $P(x) \equiv T_0$, we know that $\exists! x\, P(x)$ is false.

Problem 2. (10pt) Let the universe for x be the set of real numbers. Let $P(x)$ be the predicate $P(x)\colon 0 < x^2 \leq 50$ and $Q(x)$ be the predicate $Q(x)\colon x^2 = 50$.

(a) Find at least two values for which $P(x)$ is true and two values for which $P(x)$ is false. Do the same for $Q(x)$.

(b) Find the truth set for $P(x)$, and also for $Q(x)$.

(c) Is it true that there is a unique x in the domain such that $P(x) \wedge Q(x)$ is true? Explain.

(d) How would your answer in (b) change if the universe were instead the set of integers? Explain.

Solution.

(a) We know that $P(-2) : 0 < 4 \leq 50$, $P(1) : 0 < 1 \leq 50$, $P(\sqrt{2}) : 0 < 2 \leq 50$, $P(\pi) : 0 < \pi^2 \leq 50$, $P(5.56) : 0 < 30.9136 \leq 50$, and $P(7) : 0 < 49 \leq 50$ are all true, while $P(0) : 0 < 0 \leq 50$, $P(\sqrt{51}) : 0 < 51 \leq 50$, and $P(10) : 0 < 100 \leq 50$ are all false. For $Q(x)$, we know that $Q(-\sqrt{50}) : 50 = 50$ and $Q(\sqrt{50}) : 50 = 50$ are true, while $Q(0) : 0 = 50$, $Q(-1) : 1 = 50$, $Q(20) : 400 = 50$, and $Q(e) : e^2 = 50$ are all false.

(b) We know $P(x)$ is either true or false. It suffices to find when $P(x)$ is true. We know $P(x)$ is true when $0 < x^2 \leq 50$. We know that $x^2 > 0$ so long as $x \neq 0$ because $x^2 \geq 0$ for all $x \in \mathbb{R}$ and that $x^2 = 0$ if and only if $x = 0$. Now if we have $x^2 \leq 50$, then we know that $\sqrt{x^2} \leq \sqrt{50}$. Because $\sqrt{x} = |x|$, this implies that $|x| \leq \sqrt{50}$. But then $-\sqrt{50} \leq x \leq \sqrt{50}$, i.e. $x \in [-\sqrt{50}, \sqrt{50}]$. Combining this with the fact that $x \neq 0$, we know the truth set is $x \in [-\sqrt{50}, \sqrt{50}]$ with $x \neq 0$. Therefore, $P(x)$ is false whenever $x \in (-\infty, -\sqrt{50})$ or $x = 0$ or $x \in (\sqrt{50}, \infty)$.

We know that $Q(x)$ is either true or false. For $Q(x)$ to be true, we need $x^2 = 50$. But then $x^2 - 50 = 0$. This implies that $(x - \sqrt{50})(x + \sqrt{50}) = 0$ so that $x = -\sqrt{50}$, $x = \sqrt{50}$. One can confirm that $Q(-\sqrt{50})$ and $Q(\sqrt{50})$ are true. But then the truth set for $Q(x)$ is $\{-\sqrt{50}, \sqrt{50}\}$, which implies the false set for $Q(x)$ is $x \in (-\infty, -\sqrt{50})$ or $x \in (-\sqrt{50}, \sqrt{50})$ or $x \in (\sqrt{50}, \sqrt{50})$.

(c) This is asking whether $\exists! x \, \big(P(x) \wedge Q(x)\big)$ is true. Because $P(x) \wedge Q(x)$ is true if and only if $P(x)$ and $Q(x)$ are both true, for $\exists! x \, \big(P(x) \wedge Q(x)\big)$ to be true, there would only be a single x that is in both the truth set of $P(x)$ and $Q(x)$. But notice from (b) both $P(x)$ and $Q(x)$ are true when $x = -\sqrt{50}$ or $x = \sqrt{50}$. Therefore, $\exists! x \, \big(P(x) \wedge Q(x)\big)$ is false.

(d) If the universe were the integer, only integers would be 'allowed.' We would then have to restrict the truth sets of $P(x)$ to $Q(x)$ to the integer values (if any) they contained. Taking only the integer values in the truth set of $P(x)$, we know the truth set of $P(x)$ when the universe is the integers is $\{-7, -6, -5, -4, -3, -2, -1, 1, 2, 3, 4, 5, 6, 7\}$ and $P(x)$ is false for all integers not in this set. Taking only the integer values in the truth set of $Q(x)$, we see that the truth set for $Q(x)$ when the universe is the integers is $\varnothing$, meaning $Q(x)$ is always false.

Problem 3. (10pt) Students in their first algebra course may believe that the following rule is true for real numbers: $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$. Write this 'rule' as a quantified open statement in English, being as clear and specific as possible. Then prove or disprove the resulting statement.

Solution. Let $P(x, y)$ be the predicate given by $P(x, y) : \sqrt{x+y} = \sqrt{x} + \sqrt{y}$. The 'rule' that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ can be stated more precisely by saying, "For all real numbers x, y, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$." Written as a quantified statement, this is $\forall x \, \forall y \, P(x, y)$. Certainly, this 'rule', i.e. quantified statement is false. As a counterexample, take $x = 1$ and $y = 1$, then we have...

$$\sqrt{x+y} = \sqrt{x} + \sqrt{y}$$

$$\sqrt{1+1} \stackrel{?}{=} \sqrt{1} + \sqrt{1}$$

$$\sqrt{2} \stackrel{?}{=} 2$$

$$1 \stackrel{?}{=} \frac{2}{\sqrt{2}}$$

$$1 \neq \sqrt{2}$$

Therefore, the statement $\forall x \, \forall y \, P(x, y)$ is false. Note that there are values where $P(x, y)$ is true. For instance, if either $x = 0$ and $y = 1$ or $x = 1$ and $y = 0$, then we have (up to rearrangement) $\sqrt{1+0} = \sqrt{1} + \sqrt{0}$, which is true. Generally, $P(x, y)$ also works for any combination of x, y where one of x, y is zero and the other is a perfect square. However, $\forall x \, \forall y \, P(x, y)$ is still *false* because $P(x, y)$ is not true for *all* x, y.

Problem 4. (10pt) A certain computer program has n as an integer variable. Suppose that A is an array of 20 integers values, i.e. A is a 'list' of the integer values $A[1], A[2], \ldots, A[20]$. Write the following as quantified open statements using $A[k]$:

(a) Every entry in the array is nonnegative.

(b) The value $A[1]$ is the smallest value in the array.

(c) The array is sorted in ascending order.

(d) All the values in the array are distinct.

Solution.

(a) The first entry, $A[1]$ is nonnegative if $A[1] \geq 0$. Similarly, the second entry, $A[2]$, is nonnegative if $A[2] \geq 0$. But then every entry of A is nonnegative if $A[k] \geq 0$ for all k, i.e. $k = 1, 2, \ldots, 20$. We can write this as...

$$\forall k \, (A[k] \geq 0)$$

(b) If the first value of the array, $A[1]$, is smallest, then we know that it is smaller than the second entry, i.e. $A[1] \leq A[2]$. We know also that $A[1]$ must be smaller than the third entry, i.e. $A[1] \leq A[3]$. Generally, we need this true for any entry, i.e. $A[1] \leq A[k]$ for all k. We can write this as...

$$\forall k \, (A[1] \leq A[k])$$

(c) If the array is sorted in ascending order, each subsequent element is larger than the previous. For instance, the second element is at least as large as the previous, i.e. $A[1] \leq A[2]$, and the third is at least as large as the second, i.e. $A[2] \leq A[3]$, etc. Then we want $A[k] \leq A[k+1]$ for all k. We can write this as...

$$\forall k \, (A[k] \leq A$$

Note that if the order is sorted in ascending order, then we also know that any 'previous' element is at most the size of 'later' elements, i.e. if $j < k$, then because $A[j]$ occurs 'before' $A[k]$, we must have $A[j] \leq A[k]$. Then another way of expressing the condition in (c) is...

$$\forall j \, \forall k \, \big(j \leq k \to A[j] \leq A[k]\big)$$

(d) We know that $A[1]$ is distinct from $A[2]$ if $A[1] \neq A[2]$. We know also that $A[1]$ is distinct from $A[15]$ if $A[1] \neq A[15]$. Finally, we see that $A[14]$ is distinct from $A[6]$ if $A[14] \neq A[6]$. Then generally, we know $A[j]$ is distinct from $A[k]$ if $A[j] \neq A[k]$ (obviously, $j \neq k$ or otherwise they would be the same element of the array). We can write this as...

$$\forall j \, \forall k \, \big(j \neq k \to A[j] \neq A[k]\big)$$

Note that one can do this more 'algorithmically' as follows:

$$\forall j \, \forall k \, \big(j < k \to A[j] \neq A[k]\big)$$

Problem 5. (10pt) Showing all your work and simplifying your logical expression as much as possible, negate the following quantified open statements:

(a) $\forall x \big(P(x) \rightarrow \neg Q(x)\big)$

(b) $\exists x \big(P(x) \iff Q(x) \wedge R(x)\big)$

(c) $\forall x \, \exists y \, \big(P(x, y) \vee Q(x, y)\big)$

(d) $\forall x \big(P(x) \rightarrow 1 < x < 3\big)$

Solution.

(a) We have. . .

$$\begin{aligned} \neg \left(\forall x \big(P(x) \rightarrow \neg Q(x)\big)\right) &\equiv \exists x \, \neg \big(P(x) \rightarrow \neg Q(x)\big) \\ &\equiv \exists x \, \left(P(x) \wedge \neg \big(\neg Q(x)\big)\right) \\ &\equiv \exists x \, \big(P(x) \wedge Q(x)\big) \end{aligned}$$

(b) We have. . .

$$\begin{aligned} \neg \left(\exists x \big(P(x) \iff Q(x) \wedge R(x)\big)\right) &\equiv \forall x \, \neg \big(P(x) \iff Q(x) \wedge R(x)\big) \\ &\equiv \forall x \, \big(\neg P(x) \iff Q(x) \wedge R(x)\big) \end{aligned}$$

(c) We have. . .

$$\begin{aligned} \neg \left(\forall x \, \exists y \big(P(x, y) \vee Q(x, y)\big)\right) &\equiv \exists x \, \neg \left(\exists y \big(P(x, y) \vee Q(x, y)\big)\right) \\ &\equiv \exists x \, \forall y \, \neg \big(P(x, y) \vee Q(x, y)\big) \\ &\equiv \exists x \, \forall y \, \big(\neg P(x, y) \wedge \neg Q(x, y)\big) \end{aligned}$$

(d) We have. . .

$$\begin{aligned} \neg \left(\forall x \big(1 < x < 3 \rightarrow P(x)\big)\right) &\equiv \exists x \, \neg \big(1 < x < 3 \rightarrow P(x)\big) \\ &\equiv \exists x \, \big(P(x) \wedge \neg(1 < x < 3)\big) \\ &\equiv \exists x \, \big(P(x) \wedge (x \leq 1 \vee x \geq 3)\big) \end{aligned}$$

Problem 6. (10pt) Recall that the definition of a function, $f(x)$, having a limit as x approaches a was as follows: we say that the limit of $f(x)$ as x approaches a is L, denoted $\lim_{x\to a} f(x) = L$, if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all x, if $|x-a| < \delta$, then $|f(x) - L| < \epsilon$.

(a) Write the definition above using logical symbols and quantifiers.

(b) Find the definition of *not* having a limit by negating the logical expression from (a).

(c) Explain why $\lim_{x\to 0} \dfrac{1}{x}$ does not exist using your response from (b) and considering what happens when $x = 1/n$ and $n \to \infty$.

Solution.

(a) We have...

$$(\forall \epsilon > 0)(\exists \delta > 0)\big(|x-a| < \delta \to |f(x) - L| < \epsilon\big)$$

(b) The definition of $f(x)$ having a limit at $x = a$ is $(\forall \epsilon > 0)(\exists \delta > 0)\big(|x-a| < \delta \to |f(x)-L| < \epsilon\big)$. Then what it means for $f(x)$ to *not* to have a limit is $\neg(\forall \epsilon > 0)(\exists \delta > 0)\big(|x-a| < \delta \to |f(x) - L| < \epsilon\big)$. But this is...

$$\begin{aligned}
\neg(\forall \epsilon > 0)(\exists \delta > 0)\big(|x-a| < \delta \to |f(x) - L| < \epsilon\big) &\equiv (\exists \epsilon > 0)\neg(\exists \delta > 0)\big(|x-a| < \delta \to |f(x) - L| < \epsilon\big) \\
&\equiv (\exists \epsilon > 0)(\forall \delta > 0)\neg\big(|x-a| < \delta \to |f(x) - L| < \epsilon\big) \\
&\equiv (\exists \epsilon > 0)(\forall \delta > 0)\big(|x-a| < \delta \wedge \neg(|f(x) - L| < \epsilon)\big) \\
&\equiv (\exists \epsilon > 0)(\forall \delta > 0)\big(|x-a| < \delta \wedge |f(x) - L| \geq \epsilon\big)
\end{aligned}$$

(c) Let $f(x) = \dfrac{1}{x}$. Observe that $f(1/n) = \dfrac{1}{1/n} = n$. Then $|f(x)-L| = |n-L|$, so that if $|n-L| \geq \epsilon$, then $n - L \leq -\epsilon$ or $n - L \geq \epsilon$, i.e. $n \geq L + \epsilon$. Suppose ϵ were given. If you had any $\delta > 0$, consider the $x = \frac{1}{n}$. If n is sufficiently large, then certainly $|\frac{1}{n}| = |x| = |x-0| = |x-a| < \delta$. But then because n is large, $f(1/n) = n$ is large. We can always choose n even larger so that we also have $n \geq L + \epsilon$ because L and ϵ are fixed. But then we know that the given $f(x)$ with $a = 0$ meets the criterion for *not* having a limit. Therefore, $\lim_{x\to 0} \dfrac{1}{x}$ does not exist.