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MATH 101
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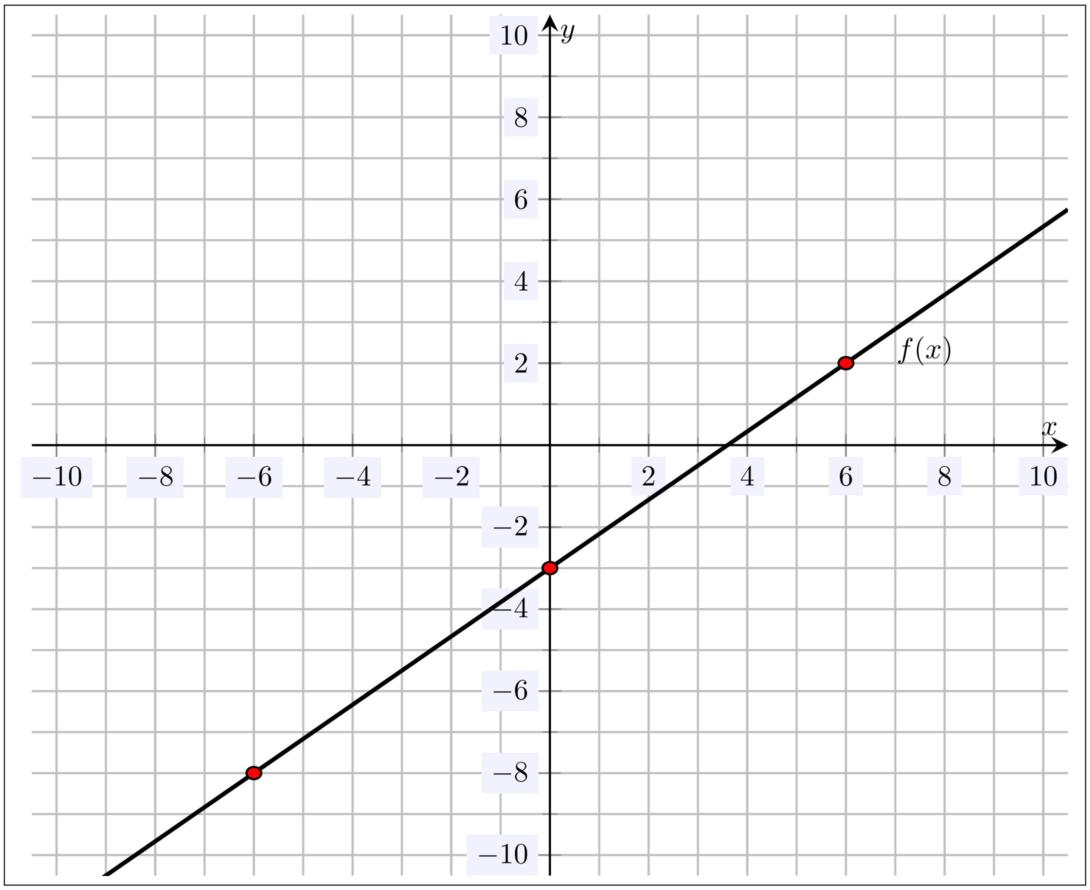
HW 6: Due 01/12

"I'm always thinking one step ahead, like a carpenter that makes stairs."
–Andy Bernard, The Office

Problem 1. (10pt) A linear function is plotted below. Find the equation of this linear function.



Solution. Because the line is not vertical, it has the form $y = mx + b$. We can see from the plot that the the points $(-6, -8)$, $(0, -3)$, and $(6, 2)$ are on the line. Using any two of them, we find the slope of the line:

$$m = \frac{\Delta y}{\Delta x} = \frac{-3 - (-8)}{0 - (-6)} = \frac{-3 + 8}{0 + 6} = \frac{5}{6}$$

Then we know $y = \frac{5}{6}x + b$. Because $(0, -3)$ is the y-intercept, we know that $b = -3$. Therefore, $y = \frac{5}{6}x - 3$. Alternatively, for example, because the line contains the point $(6, 2)$, the point $(6, 2)$ satisfies the equation of the line. But then $y = \frac{5}{6}x + b$ implies $2 = \frac{5}{6} \cdot 6 + b = 5 + b$ so that $b = -3$.

Alternatively, we can see from the plot that the points $(0, -3)$ and $(6, 2)$ are on the line. Therefore, when x increases by 6, y increases by 5. Therefore, the slope is $m = \frac{\Delta y}{\Delta x} = \frac{5}{6}$. Because we know $y = mx + b$, it must be that $y = \frac{5}{6}x + b$. We can then find b as above.

Problem 2. (10pt) Determine if the following pairs of lines are the same, perpendicular, parallel, or none of these.

$$\begin{aligned} \ell_1 : &\quad y = \frac{3}{2}x + 9 \\ \ell_2 : &\quad 9x - 6y = 12 \end{aligned}$$

Solution. Solving for y in the second equation, we find...

$$\begin{aligned} 9x - 6y &= 12 \\ -6y &= -9x + 12 \\ y &= \frac{3}{2}x - 2 \end{aligned}$$

Therefore, the lines are...

$$\begin{aligned} \ell_1 : &\quad y = \frac{3}{2}x + 9 \\ \ell_2 : &\quad y = \frac{3}{2}x - 2 \end{aligned}$$

The slope of the first line is $m_1 = \frac{3}{2}$ and the slope of the second line is $m_2 = \frac{3}{2}$. Because $m_1 = m_2$, either the lines are parallel are the same. But the y-intercept of the first line is $(0, 9)$ while the y-intercept of the second line is $(0, -2)$. Therefore, the lines are parallel, i.e. $\ell_1 \parallel \ell_2$.

Problem 3. (10pt) Determine if the following pairs of lines are the same, perpendicular, parallel, or none of these.

$$\begin{aligned} \ell_1 &: \quad 2x - 3y = 5 \\ \ell_2 &: \quad 6x + 5y = -3 \end{aligned}$$

Solution. Solving for y in the first equation, we find...

$$\begin{aligned} 2x - 3y &= 5 \\ -3y &= -2x + 5 \\ y &= \frac{2}{3}\,x - \frac{5}{3} \end{aligned}$$

Solving for y in the second equation, we find...

$$\begin{aligned} 6x + 5y &= -3 \\ 5y &= -6x - 3 \\ y &= -\frac{6}{5}\,x - \frac{3}{5} \end{aligned}$$

Therefore, the lines are...

$$\begin{aligned} \ell_1 &: \quad y = \frac{2}{3}\,x - \frac{5}{3} \\ \ell_2 &: \quad y = -\frac{6}{5}\,x - \frac{3}{5} \end{aligned}$$

The slope of the first line is $m_1 = \frac{2}{3}$ and the slope of the second line is $m_2 = -\frac{6}{5}$. Because $m_1 \neq m_2$, the lines are not the same or parallel, so that they must intersect. Because the negative reciprocal of $m_1 = \frac{2}{3}$ is $\frac{3}{2} \neq -\frac{6}{5}$. Therefore, the lines are not perpendicular. But then ℓ_1 and ℓ_2 are distinct lines that are not parallel and hence intersect (but not perpendicularly).

Problem 4. (10pt) Determine if the following pairs of lines are the same, perpendicular, parallel, or none of these.

$$\begin{aligned} \ell_1 :& \quad y = -2x + 7 \\ \ell_2 :& \quad -3x + 6y = 15 \end{aligned}$$

Solution. Solving for y in the second equation, we find...

$$\begin{aligned} -3x + 6y &= 15 \\ 6y &= 3x + 15 \\ y &= \frac{1}{2}x + \frac{5}{2} \end{aligned}$$

Therefore, the lines are...

$$\begin{aligned} \ell_1 :& \quad y = -2x + 7 \\ \ell_2 :& \quad y = \frac{1}{2}x + \frac{5}{2} \end{aligned}$$

The slope of the first line is $m_1 = -2$ and the slope of the second line is $m_2 = \frac{1}{2}$. Because $m_1 \neq m_2$, the lines cannot be the same or parallel, so that they must intersect. Because the negative reciprocal of $m_1 = -2 = -\frac{2}{1}$ is $-\frac{1}{2} = m_2$. Therefore, the lines are perpendicular, i.e. $\ell_1 \perp \ell_2$.

Problem 5. (10pt) Find the equation of the line passing through the points $(6, 21)$ and $(-9, -19)$.

Solution. Because the line is not vertical, we know that the line has the form $y = mx + b$. We first find the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{21 - (-19)}{6 - (-9)} = \frac{21 + 19}{6 + 9} = \frac{40}{15} = \frac{8}{3}$$

Therefore, $y = \frac{8}{3}x + b$. Because the point $(6, 21)$ is on the line, it satisfies the equation of the line. Therefore, we know...

$$\begin{aligned} y &= \frac{8}{3}x + b \\ 21 &= \frac{8}{3} \cdot 6 + b \\ 21 &= 8(2) + b \\ 21 &= 16 + b \\ b &= 5 \end{aligned}$$

Therefore, the equation of the line is $y = \frac{8}{3}x + 5$.

Problem 6. (10pt) Find the equation of the line perpendicular to $y = 4 - 5x$ that passes through the point $(3, -1)$.

Solution. Because the line is not vertical, we see that it has the form $y = mx+b$. The line $y = 4-5x$ has slope -5. Because our line is perpendicular to the line $y = 4 - 5x$, our line must have slope equal to the negative reciprocal of -5, which is $-(\frac{1}{-5}) = \frac{1}{5}$. Therefore, $m = \frac{1}{5}$ so that $y = \frac{1}{5}x + b$. Because the line contains the point $(3, -1)$, the point satisfies the equation of the line. Therefore, we know...

$$y = \frac{1}{5}x + b$$

$$-1 = \frac{1}{5} \cdot 3 + b$$

$$-1 = \frac{3}{5} + b$$

$$b = -1 - \frac{3}{5}$$

$$b = -\frac{5}{5} - \frac{3}{5}$$

$$b = -\frac{8}{5}$$

Therefore, the equation of the line $y = \frac{1}{5}x - \frac{8}{5} = \frac{x-8}{5}$.

Problem 7. (10pt) Find the equation of the line parallel to the line $x = -5$ containing the point $(4, 19)$.

Solution. The line $x = -5$ is a vertical line. Because our line is parallel to this line, it must also be vertical, i.e. our line has the form $x = M$ for some M. Because our line passes through the point $(4, 19)$, we know that we must have $x = 4$.

Problem 8. (10pt) Sunita works at an advertising firm. Upon hire, she was paid a $5,000 signing bonus. The company pays her a yearly salary of $63,000.

(a) Write a function which gives the amount of money Sunita has been paid by the company in t years.

(b) What is the slope and y-intercept for the function in (a)? Interpret both of these in the problem context.

(c) Find the amount of money Sunita has been paid in 5 years.

(d) How many years until Sunita has been paid a total of $200,000.

Solution.

(a) Let $P(t)$ be the amount Sunita has been paid after t years. Because she is paid $63,000 each year, after t years, she has been paid a total salary of $63000t$. But she was also paid a signing bonus of 5000. Therefore, $P(t) = 63000t + 5000$.

(b) We know that $P(t)$ is linear because Sunita receives a constant salary of $63000/year. Moreover, $P(t) = 63000t + 5000$ is a linear function because it has the form $y = mx + b$. We then know that $m = 63000$ and $b = 5000$. Interpreting $m = 63000 = \frac{63000}{1}$ as $\frac{\Delta y}{\Delta x}$, we have $\Delta x = 1$ and $\Delta y = 63000$ and using the fact that x has units of years and y has units of dollars, we see that every year Sunita is paid $63,000, i.e. the slope represents her yearly salary of $63,000. The y-intercept is $(0, 5000)$, i.e. when $t = 0$ we know that $P(0) = 5000$. We know that $t = 0$ is the time of hire. Therefore, Sunita must have been paid $5,000 upon hire, i.e. the y-intercept represents Sunita's signing bonus.

(c) Although, Sunita will only receive a salary of $63,000 in 5 years, the total amount of money she has been paid will be $P(5)$, which is...

$$P(5) = 63000(5) + 5000 = 315000 + 5000 = \$320,000$$

(d) When Sunita has been paid a total of $200,000, this is a time t such that $P(t) = 200000$. But then...

$$\begin{aligned} P(t) &= 200000 \\ 63000t + 5000 &= 200000 \\ 63000t &= 195000 \\ t &= 3.09524 \text{ years} \end{aligned}$$

Therefore, she would have been paid a total of $200,000 between her third and fourth year working at the firm. One might say that she will have been paid at least $200,000 after her third year at the firm (or by the start of her fourth year at the firm).

Problem 9. (10pt) A tour bus company charged a group of 30 people a total of \$180 for a tour. The following week, they charged a group of 50 people \$220.

(a) Find a linear function, $C(p)$, for the total cost for a tour for a group of size p.

(b) What is the slope of $C(p)$? Interpret the slope in context.

(c) What is the y-intercept? Does it have meaning in this context? Explain.

(d) Estimate much would the company charge for a group of 60 people.

(e) If you only had \$570, what would you estimate the largest group you could take on the tour?

Solution.

(a) We know $C(30) = 180$ and $C(50) = 220$, i.e. that the graph of $C(p)$ contains the points $(30, 180)$ and $(50, 220)$. Because $C(p)$ is not a vertical line, we know that $C(p) = mp + b$. The slope of $C(p)$ must be...

$$m = \frac{180 - 220}{30 - 50} = \frac{-40}{-20} = 2$$

But then $C(p) = 2p + b$. Because $(30, 180)$ is on the graph of $C(p)$, it satisfies the equation for $C(p)$. Then we know...

$$\begin{aligned} C(p) &= 2p + b \\ C(30) &= 2(30) + b \\ 180 &= 60 + b \\ b &= 120 \end{aligned}$$

Therefore, $C(p) = 2p + 120$.

(b) Because $C(p) = 2p + 120$ has the form $y = mx + b$, we know that $m = 2 = \frac{2}{1}$. Interpreting this as $\frac{\Delta y}{\Delta x}$, we have $\Delta x = 1$ and $\Delta y = 2$. Using the fact that x has units of people and y has units of dollars, we interpret the slope as the fact that the company charges \$2 per person.

(c) Because $C(0) = 2(0) + 120 = 120$, we know that the y-intercept for $C(p)$ is $(0, 120)$, i.e. when $p = 0$, we know that $C(0) = \$120$. This is a charge of \$120 for a tour with no people—which is nonsensical. However, we can interpret this as a fee charged to give a tour at all, i.e. a service charge or booking charge of \$120.

(d) This is $C(60)$, which is $C(60) = 2(60) + 120 = 120 + 120 = \$240$.

(e) We want a number of people, p, so that $C(p) = 570$. But then $2p + 120 = 570$, which implies that $2p = 450$. Therefore, we know that $p = 225$. But then the largest number of people you could bring on the tour would be 225 people.

Problem 10. (10pt) A used car was purchased for \$7,500. Each year, the car loses \$1,200 in value.

(a) Find a function, $V(t)$, which gives the value, V, for the car after t years.

(b) What does the slope of $V(t)$ represent?

(c) What does the y-intercept of $V(t)$ represent?

(d) What is the car worth in 3 years?

(e) How long until the car is essentially worthless?

Solution.

(a) Because the rate of depreciation of the vehicle is constant, we know that $V(t)$ is linear. After t years, the car has lost $1200t$ dollars in value. But because the car was initially worth 7500, we know that $V(t) = 7500 - 1200t$.

(b) Because $V(t) = 7500 - 1200t$ has the form $y = mx + b$, we know that $m = -1200 = \frac{-1200}{1}$. Interpreting this as $\frac{\Delta y}{\Delta x}$, we have $\Delta x = 1$ and $\Delta y = -1200$. Using the fact that x has units of years and y has units of dollars, we interpret the slope as the fact that the car loses \$1,200 in value each year.

(c) We know that the y-intercept occurs when $t = 0$ and that $V(0) = 7500 - 1200(0) = 7500 - 0 = \$7,500$. Because $V(0)$ represents the value of the car at $t = 0$, i.e. at the time of purchase, the y-intercept must represent the initial value of the car.

(d) This is $V(3)$, which is $V(3) = 7500 - 1200(3) = 7500 - 3600 = \$3,900$.

(e) The car is worthless when it has value \$0. But then this is a time when $V(t) = 0$. But then we have...

$$\begin{aligned} V(t) &= 0 \\ 7500 - 1200t &= 0 \\ 7500 &= 1200t \\ t &= 6.25 \text{ years} \end{aligned}$$

Therefore, the car has no value after 6 years and 3 months.