Name: ________________________________

MATH 308
Fall 2022
HW 4: Due 09/20

"Pure Mathematics is the world's best game. It is more absorbing than chess, more of a gamble than poker, and lasts longer than Monopoly. It's free. It can be played anywhere—Archimedes did it in a bathtub."
–Richard J. Trudeau

Problem 1. (10pt) Suppose that $P(x)$ is a predicate. Being sure to justify your answer, explain whether the following statements are true or false.

(a) There are choices of x for which $P(x)$ is true and choices of x for which $P(x)$ is false.

(b) Once one quantifies $P(x)$ using $\forall x$ or $\exists x$, the resulting statement is always true or always false—but not both.

(c) If $\exists! P(x)$ is true, then $\exists P(x)$ is true.

(d) The converse of (c) is also true.

Problem 2. (10pt) Let the universe for x be the set of real numbers. Let $P(x)$ be the predicate $P(x)\colon 0 < x^2 \leq 50$ and $Q(x)$ be the predicate $Q(x)\colon x^2 = 50$.

1. Find at least two values for which $P(x)$ is true and two values for which $P(x)$ is false. Do the same for $Q(x)$.
2. Find the truth set for $P(x)$, and also for $Q(x)$.
3. Is it true that there is a unique x in the domain such that $P(x) \wedge Q(x)$ is true? Explain.
4. How would your answer in (b) change if the universe were instead the set of integers? Explain.

Problem 3. (10pt) Students in their first algebra course may believe that the following rule is true for real numbers: $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$. Write this 'rule' as a quantified open statement in English, being as clear and specific as possible. Then prove or disprove the resulting statement.

Problem 4. (10pt) A certain computer program has n as an integer variable. Suppose that A is an array of 20 integers values, i.e. A is a 'list' of the integer values $A[1], A[2], \ldots, A[20]$. Write the following as quantified open statements using $A[k]$:

(a) Every entry in the array is nonnegative.

(b) The value $A[1]$ is the smallest value in the array.

(c) The array is sorted in ascending order.

(d) All the values in the array are distinct.

Problem 5. (10pt) Showing all your work and simplifying your logical expression as much as possible, negate the following quantified open statements:

(a) $\forall x \big(P(x) \to \neg Q(x) \big)$

(b) $\exists x \big(P(x) \iff Q(x) \wedge R(x) \big)$

(c) $\forall x \exists y \big(P(x, y) \vee Q(x, y) \big)$

(d) $\forall x \big(1 < x < 3 \to P(x) \big)$

Problem 6. (10pt) Recall that the definition of a function, $f(x)$, having a limit as x approaches a was as follows: we say that the limit of $f(x)$ as x approaches a is L, denoted $\lim_{x\to a} f(x) = L$, if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all x, if $|x-a| < \delta$, then $|f(x) - L| < \epsilon$.

(a) Write the definition above using logical symbols and quantifiers.

(b) Find the definition of *not* having a limit by negating the logical expression from (a).

(c) Explain why $\lim_{x\to 0} \frac{1}{x}$ does not exist using your response from (b) and considering what happens when $x = 1/n$ and $n \to \infty$.