MAT 101: Exam 2
Spring – 2022
04/14/2022
85 Minutes

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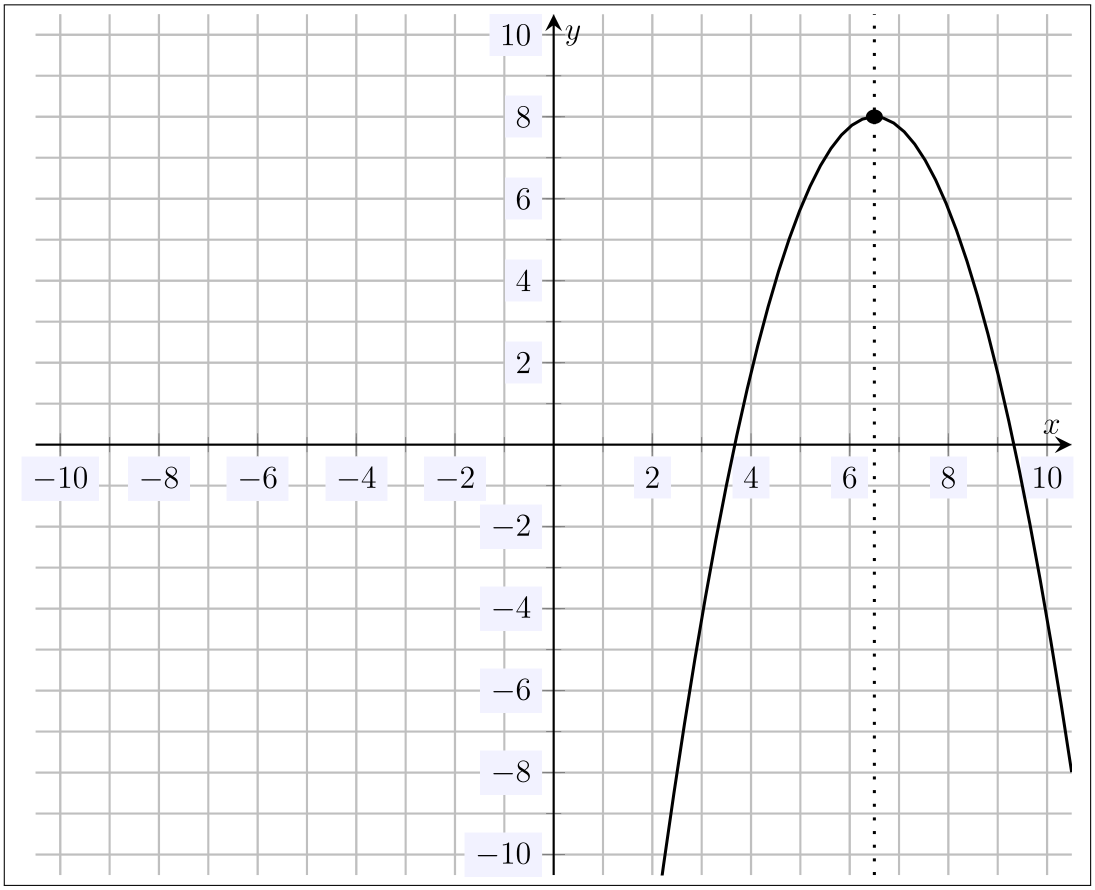
Write your name on the appropriate line on the exam cover sheet. This exam contains 21 pages (including this cover page) and 20 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
15	10	
16	10	
17	10	
18	10	
19	10	
20	20	
Total:	210	

1. (10 points) Sketch the quadratic function $f(x) = 8 - \left(x - \frac{13}{2}\right)^2$ on the plot below. Your sketch should include the vertex and axis of symmetry—being placed as accurately as possible.



We have $f(x) = 8 - \left(x - \frac{13}{2}\right)^2 = -\left(x - \frac{13}{2}\right)^2 + 8$. *Therefore,* $f(x)$ *is in vertex form, i.e. the form* $f(x) = a(x-p)^2 + q$, *where* (p, q) *is the vertex. We then know that the vertex of* $f(x)$ *is* $(-\frac{13}{2}, 8)$ *so that the axis of symmetry is* $x = \frac{13}{2}$. *Because* $a = -1 < 0$, *the parabola opens downwards. This gives the sketch above.*

2. Consider the quadratic function $y = (x+5)^2 + 6$.

   (a) (2 points) Identify a, b, and c for this quadratic function.

   $$y = (x+5)^2 + 6 = (x^2 + 10x + 25) + 6 = x^2 + 10x + 31$$

   Therefore, $a = 1$, $b = 10$, $c = 31$.

   (b) (2 points) Does this quadratic function open upwards or downwards?

   Because $a = 1 > 0$, *the quadratic function opens upwards.*

   (c) (2 points) Is this quadratic function convex or concave?

   Because $a = 1 > 0$, *the quadratic function is convex*

   (d) (2 points) What is the vertex of this quadratic function? What is the axis of symmetry?

   The quadratic function $y = (x+5)^2 + 6$ *is in vertex form, i.e.* $y = a(x-p)^2 + q$, *where* (p, q) *is the vertex. Therefore, the vertex is* $(-5, 6)$ *and the axis of symmetry is* $x = -5$.

   (e) (2 points) Find the maximum and minimum values for y.

   Because the parabola opens upwards, there is no maximum value, i.e. it does not exist. The parabola has a minimum. Because the vertex is $(-5, 6)$, *we know the minimum value is 6—the* y*-coordinate of the vertex.*

3. (10 points) Find the vertex form of the quadratic function $f(x) = -3x^2 + 12x - 7$.

Completing the square, we have...

$$\begin{aligned} f(x) &= -3x^2 + 12x - 7 \\ &= -3\left(x^2 - 4x + \frac{7}{3}\right) \\ &= -3\left(x^2 - 4x + 4 - 4 + \frac{7}{3}\right) \\ &= -3\left((x^2 - 4x + 4) - \frac{12}{3} + \frac{7}{3}\right) \\ &= -3\left((x-2)^2 - \frac{5}{3}\right) \\ &= -3(x-2)^2 + 5 \end{aligned}$$

Alternatively, we can use the 'evaluation method':

$$\begin{aligned} a &= -3 \\ x &= -\frac{b}{2a} = -\frac{12}{2(-3)} = -\frac{12}{-6} = -(-2) = 2 \\ f(2) &= -3(2^2) + 12(2) - 7 = -3(4) + 12(2) - 7 = -12 + 24 - 7 = 5 \end{aligned}$$

This shows that the vertex, (p, q)*, is* $(2, 5)$*. Therefore, the vertex form is...*

$$f(x) = a(x-p)^2 + q = -3(x-2)^2 + 5$$

4. (10 points) Factor $x^2 + 16x - 80$ completely.

80

$1 \cdot -80$	-79
$-1 \cdot 80$	79
$2 \cdot -40$	-38
$-2 \cdot 40$	38
$4 \cdot -20$	-16
$\boxed{-4 \cdot 20}$	$\boxed{16}$
$5 \cdot -16$	-11
$-5 \cdot 16$	11
$8 \cdot -10$	-2
$-8 \cdot 10$	2

Therefore,

$$x^2 + 16x - 80 = (x - 4)(x + 20)$$

5. (10 points) Factor $3x^2 - 9x - 120$ completely.

First, observe that $3x^2 - 9x - 120 = 3(x^2 - 3x - 40)$. *Then...*

$$\begin{array}{rr} \multicolumn{2}{c}{\underline{\mathbf{40}}} \\ 1 \cdot -40 & -39 \\ -1 \cdot 40 & 39 \\ 2 \cdot -20 & -18 \\ -2 \cdot 20 & 18 \\ 4 \cdot -10 & -6 \\ -4 \cdot 10 & 6 \\ \hline \multicolumn{1}{|r}{5 \cdot -8} & \multicolumn{1}{r|}{-3} \\ \hline -5 \cdot 8 & 3 \end{array}$$

Therefore,

$$3x^2 - 9x - 120 = 3(x^2 - 3x - 40) = 3(x+5)(x-8)$$

6. Factor the following completely:

   (a) (5 points) $16x - 20x^2$

   Observe, we can factor out $4x$*:*

$$16x - 20x^2 = 4x(4 - 5x)$$

   (b) (5 points) $49 - x^2$

   Observe that this is a difference of perfect squares:

$$49 - x^2 = (7 - x)(7 + x)$$

7. (10 points) Factor $10x^2 + 43x - 35$ completely.

<u>35</u>

$1 \cdot -35$
$-1 \cdot 35$
$5 \cdot -7$
$-5 \cdot 7$

Then as $10 = 1 \cdot 10$ *or* $10 = 2 \cdot 5$*, we have...*

$$\begin{array}{ccccc} 1,-350 & & & & 10,-35 \\ & 1,10\ \diagdown & & \diagup\ 10,1 & \\ & & 1 \cdot -35 & & \\ & 2,5\ \diagup & & \diagdown\ 5,2 & \\ 2,-175 & & & & 5,-70 \end{array} \qquad \begin{array}{ccccc} -1,350 & & & & -10,35 \\ & 1,10\ \diagdown & & \diagup\ 10,1 & \\ & & -1 \cdot 35 & & \\ & 2,5\ \diagup & & \diagdown\ 5,2 & \\ -2,175 & & & & -5,70 \end{array}$$

$$\begin{array}{ccccc} 5,-70 & & & & \boxed{50,-7} \\ & 1,10\ \diagdown & & \diagup\ 10,1 & \\ & & \boxed{5 \cdot -7} & & \\ & 2,5\ \diagup & & \diagdown\ 5,2 & \\ 10,-35 & & & & 25,-14 \end{array} \qquad \begin{array}{ccccc} -5,70 & & & & -50,7 \\ & 1,10\ \diagdown & & \diagup\ 10,1 & \\ & & -5 \cdot 7 & & \\ & 2,5\ \diagup & & \diagdown\ 5,2 & \\ -10,35 & & & & -25,14 \end{array}$$

Therefore,

$$10x^2 + 43x - 35 = (x+5)(10x-7)$$

Alternatively, we can use the quadratic formula to solve $10x^2 + 43x - 35 = 0$*:*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-43 \pm \sqrt{43^2 - 4(10)(-35)}}{2(10)} = \frac{-43 \pm \sqrt{3249}}{20} = \frac{-43 \pm 57}{20}$$

This gives us $x = \frac{-43-57}{20} = -5$ *and* $x = \frac{-43+57}{20} = \frac{7}{10}$*. Therefore, we know that* $10x^2 + 43x - 35$ *factors as...*

$$10x^2 + 43x - 35 = a(x - r_1)(x - r_2) = 10(x+5)\left(x - \frac{7}{10}\right)$$

We can see this is equivalent to the answer above via...

$$10x^2 + 43x - 35 = 10(x+5)\left(x - \frac{7}{10}\right) = (x+5) \cdot 10\left(x - \frac{7}{10}\right) = (x+5)(10x-7)$$

8. (10 points) Solve the following:

$$5(6-x) = \frac{4}{3}\,x + 30$$

$$\begin{aligned}
5(6-x) &= \frac{4}{3}\,x + 30 \\
30 - 5x &= \frac{4}{3}\,x + 30 \\
-5x &= \frac{4}{3}\,x \\
-15x &= 4x \\
-19x &= 0 \\
x &= 0
\end{aligned}$$

9. (10 points) Solve the following:

$$9 = x(10 - x)$$

$$\begin{aligned} 9 &= x(10 - x) \\ 9 &= 10x - x^2 \\ x^2 - 10x + 9 &= 0 \\ (x - 1)(x - 9) &= 0 \end{aligned}$$

But then either $x - 1 = 0$, so that $x = 1$, or $x - 9 = 0$, so that $x = 9$.

10. (10 points) Solve the following:

$$6 - x = 12x + 7$$

$$6 - x = 12x + 7$$

$$-1 = 13x$$

$$x = -\frac{1}{13}$$

11. (10 points) Solve the following:

$$x(3x-1) = x(x+5)$$

$$\begin{aligned} x(3x-1) &= x(x+5) \\ 3x^2 - x &= x^2 + 5x \\ 2x^2 - 6x &= 0 \\ 2x(x-3) &= 0 \end{aligned}$$

But then either $2x = 0$*, so that* $x = 0$*, or* $x - 3 = 0$*, so that* $x = 3$*.*

12. (10 points) Solve the following:

$$5(x+5) = 4x^2 + 5x$$

$$\begin{aligned} 5(x+5) =& 4x^2 + 5x \\ 5x + 25 =& 4x^2 + 5x \\ 25 &= 4x^2 \\ 0 &= 4x^2 - 25 \\ 0 &= (2x-5)(2x+5) \end{aligned}$$

But then either $2x - 5 = 0$, which implies $x = \dfrac{5}{2}$, or $2x + 5 = 0$, which implies that $x = -\dfrac{5}{2}$.

Alternatively, we have

$$\begin{aligned} 5(x+5) =& 4x^2 + 5x \\ 5x + 25 =& 4x^2 + 5x \\ 25 &= 4x^2 \\ x^2 &= \frac{25}{4} \\ x &= \pm\sqrt{\frac{25}{4}} \\ x &= \pm\frac{5}{2} \end{aligned}$$

Therefore, $x = -\frac{5}{2}$ or $x = \frac{5}{2}$.

13. (10 points) Use the quadratic formula to solve the following:

$$6 - 5x^2 = 4x(1 - x)$$

We have...

$$\begin{aligned} 6 - 5x^2 &= 4x(1 - x) \\ 6 - 5x^2 &= 4x - 4x^2 \\ x^2 + 4x - 6 &= 0 \end{aligned}$$

But then we have...

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-4 \pm \sqrt{4^2 - 4(1)(-6)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{16 + 24}}{2} \\ &= \frac{-4 \pm \sqrt{40}}{2} \\ &= \frac{-4 \pm \sqrt{4 \cdot 10}}{2} \\ &= \frac{-4 \pm 2\sqrt{10}}{2} \\ &= -2 \pm \sqrt{10} \end{aligned}$$

Therefore, the solutions are $x = -2 - \sqrt{10}$ *and* $x = -2 + \sqrt{10}$.

14. (10 points) Use the quadratic formula to factor $x^2 - 10x + 23$.

We find the roots of $x^2 - 10x + 23$ *using the quadratic formula:*

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(23)}}{2(1)} \\ &= \frac{10 \pm \sqrt{100 - 92}}{2} \\ &= \frac{10 \pm \sqrt{8}}{2} \\ &= \frac{10 \pm \sqrt{4 \cdot 2}}{2} \\ &= \frac{10 \pm 2\sqrt{2}}{2} \\ &= 5 \pm \sqrt{2} \end{aligned}$$

Therefore, the roots are $r_1 = 5 - \sqrt{2}$ *and* $r_2 = 5 + \sqrt{2}$. *Using the fact that* $a = 1$*, we then know that* $x^2 - 10x + 23$ *factors as. . .*

$$x^2 - 10x + 23 = a(x - r_1)(x - r_2) = 1\big(x - (5 - \sqrt{2})\big)\big(x - (5 + \sqrt{2})\big) = \big(x - (5 - \sqrt{2})\big)\big(x - (5 + \sqrt{2})\big)$$

15. (10 points) Use the discriminant to determine whether the function $384x^2+232x-175$ factors 'nicely', i.e. over the integers. If not, determine whether it even factors over the real numbers or requires complex numbers to factor.

$$\begin{aligned} D &= b^2 - 4ac \\ &= 232^2 - 4(384)(-175) \\ &= 53824 + 268800 \\ &= 322624 \\ &= 568^2 \end{aligned}$$

Because the discriminant is a perfect square, the function $384x^2 + 232x - 175$ *factors 'nicely', i.e. over the integers. Therefore,* $384x^2 + 232x - 175$ *also factors 'nicely' over the rational numbers, real numbers, and complex numbers.*

16. (10 points) Showing all your work, determine if the point $(5, -4)$ is a solution to the following system of equations:

$$2x + y = 6$$

$$-5x - 6y = -49$$

The point $(x, y) = (5, -4)$ *is a solution to the system of equations if and only if it satisfies both of the equations. We check this:*

$$\begin{aligned} 2x + y &= 6 \\ 2(5) + (-4) &\stackrel{?}{=} 6 \\ 10 - 4 &\stackrel{?}{=} 6 \\ 6 &= 6 \\ &\checkmark \end{aligned}$$

and

$$\begin{aligned} -5x - 6y &= -49 \\ -5(5) - 6(-4) &\stackrel{?}{=} -49 \\ -25 + 24 &\stackrel{?}{=} -49 \\ -1 &\neq -49 \\ &✗ \end{aligned}$$

Because $(5, -4)$ *does not satisfy both of the equations,* $(x, y) = (5, -4)$ *is not a solution to the system of equations. [Indeed, there is no solution to this system of equations because the given lines are parallel.]*

17. (10 points) Showing all your work, determine whether the following system of equations has a solution. If it has a solution, you do not need to find the solution.

$$10x - 4y = -24$$

$$-5x + 2y = -2$$

Solving for y in both equations, we have...

$$\begin{aligned} 10x - 4y &= -24 \\ -4y &= -10x - 24 \\ y &= \frac{5}{2}\,x + 6 \end{aligned}$$

and

$$\begin{aligned} -5x + 2y &= -2 \\ 2y &= 5x - 2 \\ y &= \frac{5}{2}\,x - 1 \end{aligned}$$

Both lines have slope $\frac{5}{2}$. But then the lines are parallel so that they do not interest. Because there cannot be a point on both the lines at once, there are no solutions to the given system of equations.

18. (10 points) Solve the following system of equations:

$$3x + 5y = 16$$

$$x + 6y = 1$$

Using substitution, from the second equation, we have...

$$\begin{aligned} x + 6y &= 1 \\ x &= 1 - 6y \end{aligned}$$

Using this in the first equation, we have...

$$\begin{aligned} 3x + 5y &= 16 \\ 3(1 - 6y) + 5y &= 16 \\ 3 - 18y + 5y &= 16 \\ 3 - 13y &= 16 \\ -13y &= 13 \\ y &= -1 \end{aligned}$$

Then we have $x = 1 - 6y = 1 - 6(-1) = 1 + 6 = 7$. *Therefore, the solution is* $(7, -1)$.

OR

Using elimination, we multiply the second equation by -3. *This yields...*

$$3x + 5y = 16$$

$$-3x - 18y = -3$$

Adding these equations, we have...

$$-13y = 13$$

$$y = -1$$

Using this in the second equation, we have $x + 6y = 1$ *so that* $x + 6(-1) = 1$. *But then* $x - 6 = 1$. *Therefore,* $x = 7$. *The solution is then* $(7, -1)$.

19. (10 points) Solve the following system of equations:

$$4x + 7y = -8$$

$$2x - 5y = -4$$

Using substitution, from the first equation, we have...

$$4x + 7y = -8$$

$$7y = -4x - 8$$

$$y = -\frac{4}{7}x - \frac{8}{7}$$

Using this in the second equation, we have...

$$2x - 5y = -4$$

$$2x - 5\left(-\frac{4}{7}x - \frac{8}{7}\right) = -4$$

$$2x + \frac{20}{7}x + \frac{40}{7} = -4$$

$$7\left(2x + \frac{20}{7}x + \frac{40}{7}\right) = -4 \cdot 7$$

$$14x + 20x + 40 = -28$$

$$34x + 40 = -28$$

$$34x = -68$$

$$x = -2$$

Then we have $y = -\frac{4}{7} \cdot -2 - \frac{8}{7} = \frac{8}{7} - \frac{8}{7} = 0$. *Therefore, the solution is* $(-2, 0)$.

OR

Using elimination, we multiply the second equation by -2. *This yields...*

$$4x + 7y = -8$$

$$-4x + 10y = 8$$

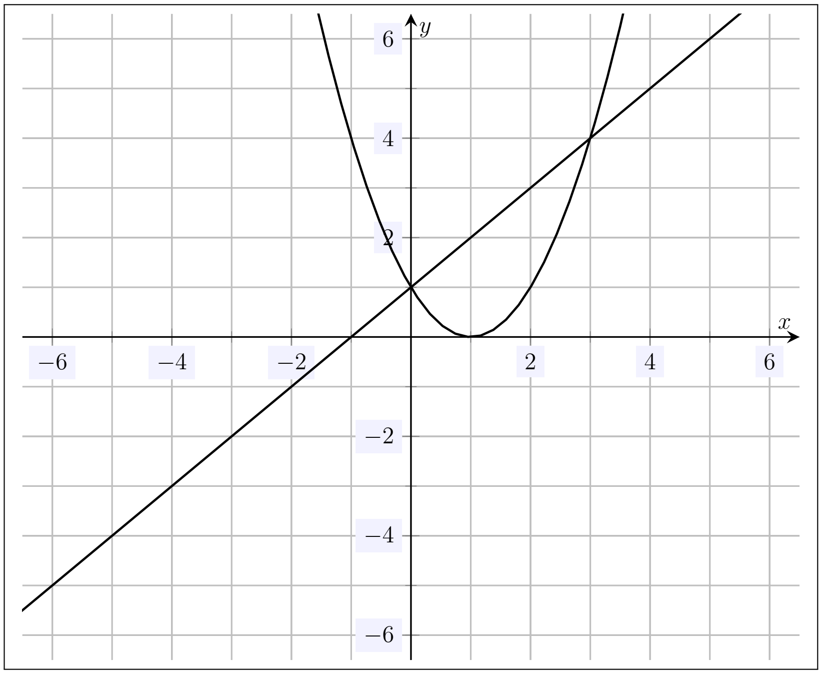
Adding these equations, we have...

$$17y = 0$$

$$y = 0$$

Using this in the first equation, we have $4x + 7y = -8$ *so that* $4x + 7(0) = -8$. *But then* $4x = -8$. *Therefore,* $x = -2$. *The solution is then* $(-2, 0)$.

20. (10 points) A quadratic function $y = x^2 - 2x + 1$ and a linear function $y = x + 1$ are plotted below.



(a) (5 points) Using the plot above, solve the following system of equations:

$$\begin{aligned} -x + y &= 1 \\ 2x + y &= x^2 + 1 \end{aligned}$$

The first equation is equivalent to $y = x + 1$ *and the second is equivalent to* $y = x^2 - 2x + 1$. *Solving the system of equations is then equivalent to finding the intersection points between these curves, i.e. the parabola and the line. From the plot above, we see the solutions are* $(x, y) = (0, 1)$ *and* $(x, y) = (3, 4)$.

(b) (5 points) Setting the functions equal, verify the solution(s) to the system of equations from (a).

We have. . .

$$\begin{aligned} x^2 - 2x + 1 &= x + 1 \\ x^2 - 3x &= 0 \\ x(x - 3) &= 0 \end{aligned}$$

But then either $x = 0$ *or* $x - 3 = 0$, *so that* $x = 3$. *Using the fact that* $y = x + 1$, *we see that if* $x = 0$, $y = 0 + 1 = 1$. *If* $x = 3$, $y = 3 + 1 = 4$. *Therefore, the solutions are* $(0, 1)$ *and* $(3, 4)$, *which is what we obtained in (a).*