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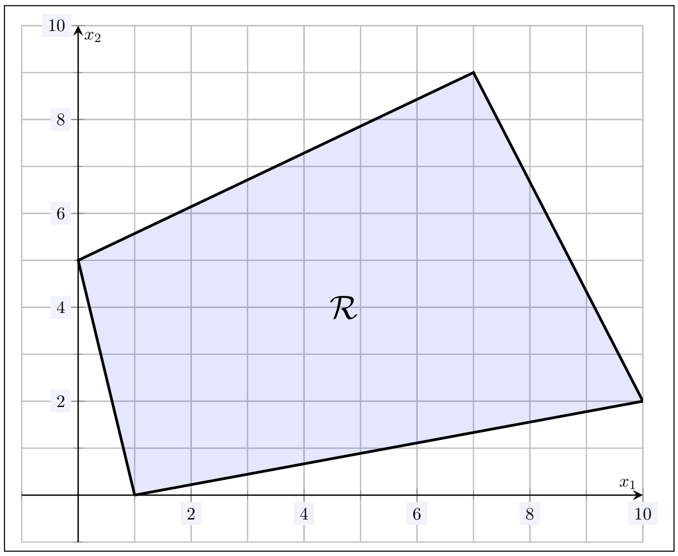
MATH 108
Spring 2024
HW 19: Due 04/22

"I was so unpopular in high school, the crossing guard used to lure me into traffic."
— Annie Edison, Community

Problem 1. (10pts) Consider the function $z = -65x_1 + 5x_2$ on the region $\mathcal{R}$ shown below. Does z have a maximum or minimum value on $\mathcal{R}$? Explain. If the function has a maximum or minimum value on $\mathcal{R}$, find the maximum and minimum value.

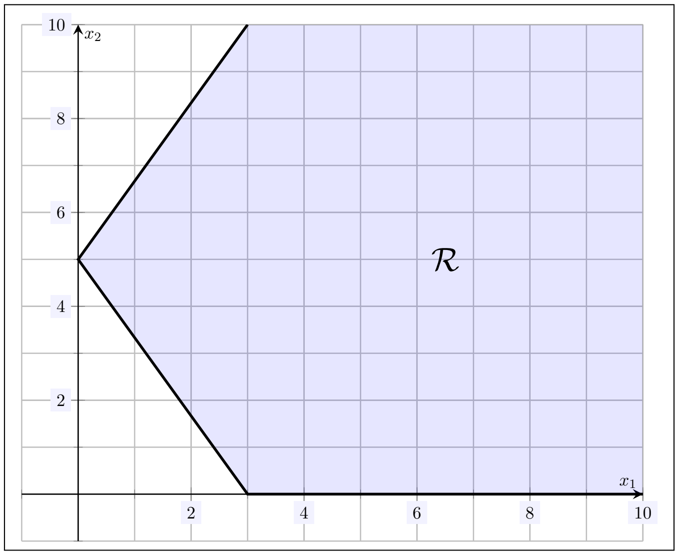


Solution. The function $z = -65x_1 + 5x_2$ is a linear function. The region $\mathcal{R}$ is a nonempty, bounded region. Therefore, by the Fundamental Theorem of Linear Programming, there exists a maximum and minimum for z on $\mathcal{R}$ and they occur at a corner point for $\mathcal{R}$. We need only examine z at these points:

Corner Point	$z(x_1, x_2)$
$(1, 0)$	$z(1, 0) = -65(1) + 5(0) = -65 + 0 = -65$
$(0, 5)$	$z(0, 5) = -65(0) + 5(5) = 0 + 25 = 25$
$(7, 9)$	$z(7, 9) = -65(7) + 5(9) = -455 + 45 = -410$
$(10, 2)$	$z(10, 2) = -65(10) + 5(2) = -650 + 10 = -640$

Therefore, the maximum value for z is 25 and occurs at $(0, 5)$ and the minimum value for z is -640 and occurs at $(10, 2)$.

Problem 2. (10pts) Consider the function $z = 6x_1 + 11x_2$ on the region $\mathcal{R}$ shown below. Does z have a maximum or minimum value on $\mathcal{R}$? Explain. If the function has a maximum or minimum value on $\mathcal{R}$, find the maximum and minimum value.

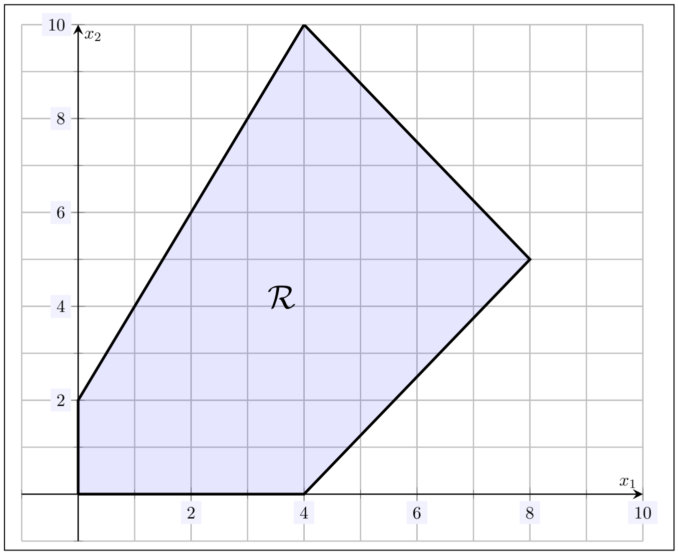


Solution. The function $z = 6x_1 + 11x_2$ is linear. The region $\mathcal{R}$ is nonempty. However, the region $\mathcal{R}$ is not bounded, i.e. it is unbounded. Therefore, the Fundamental Theorem of Linear Programming does not apply. The function z may have a maximum, a minimum, both, or neither. Observe that increasing x_1 increases z. Furthermore, increasing x_2 also increases z. Increasing x_1 and x_2 moves the point (x_1, x_2) to the right and up, respectively. Because there is no limit to how much one can move upwards and to the right in $\mathcal{R}$, there is no limit to how large z can become. Therefore, z has no maximum. Correspondingly, moving downwards and to the left decreases z. One can only move so far in these directions and stay in $\mathcal{R}$. Therefore, there is a minimum value for z and it must occur at a corner point for $\mathcal{R}$. We simply examine z at these points.

Corner Point	$z(x_1, x_2)$
$(3, 0)$	$z(3, 0) = 6(3) + 11(0) = 18 + 0 = 18$
$(0, 5)$	$z(0, 5) = 6(0) + 11(5) = 0 + 55 = 55$

Therefore, the minimum value for z is 18 and occurs at the point $(3, 0)$.

Problem 3. (10pts) Consider the function $z = x_1 + 7x_2$ on the region $\mathcal{R}$ shown below. Does z have a maximum or minimum value on $\mathcal{R}$? Explain. If the function has a maximum or minimum value on $\mathcal{R}$, find the maximum and minimum value.



Solution. The function $z = x_1 + 7x_2$ is a linear function. The region $\mathcal{R}$ is a nonempty, bounded region. Therefore, by the Fundamental Theorem of Linear Programming, there exists a maximum and minimum for z on $\mathcal{R}$ and they occur at a corner point for $\mathcal{R}$. We need only examine z at these points:

Corner Point	$z(x_1, x_2)$
$(0,0)$	$z(0,0) = 0 + 7(0) = 0 + 0 = 0$
$(0,2)$	$z(0,2) = 0 + 7(2) = 0 + 14 = 14$
$(4,10)$	$z(4,10) = 4 + 7(10) = 4 + 70 = 74$
$(8,5)$	$z(8,5) = 8 + 7(5) = 8 + 35 = 43$
$(4,0)$	$z(4,0) = 4 + 7(0) = 4 + 0 = 4$

Therefore, the maximum value for z is 74 and occurs at $(4,10)$ and the minimum value for z is 0 and occurs at $(0,0)$.

Problem 4. (10pts) Find the dual problem for the minimization problem shown below.

$$\min w = y_1 - y_2 + y_3$$
$$\begin{cases} 2y_1 - y_2 + y_3 \leq 9 \\ y_1 + 5y_2 - y_3 \geq 5 \\ 3y_1 + 4y_2 + 6y_3 \geq 10 \\ -y_1 + y_2 + 8y_3 \leq 5 \\ y_1, y_2, y_3 \geq 0 \end{cases}$$

Solution. First, we need every inequality to be of the form '$\geq$' a number. We multiply both sides of the first and fourth inequality by -1 to place this inequality in this form. This gives us the following inequalities (ignoring the non-negativity inequalities):

$$\begin{cases} -2y_1 + y_2 - y_3 \geq -9 \\ y_1 + 5y_2 - y_3 \geq 5 \\ 3y_1 + 4y_2 + 6y_3 \geq 10 \\ y_1 - y_2 - 8y_3 \geq -5 \end{cases}$$

We then form a matrix M from these inequalities with the function $w = y_1 - y_2 + y_3$ as the bottom row. This gives us the following matrix:

$$M = \begin{pmatrix} -2 & 1 & -1 & -9 \\ 1 & 5 & -1 & 5 \\ 3 & 4 & 6 & 10 \\ 1 & -1 & -8 & -5 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

We then compute the transpose of this matrix:

$$M^T = \begin{pmatrix} -2 & 1 & 3 & 1 & 1 \\ 1 & 5 & 4 & -1 & -1 \\ -1 & -1 & 6 & -8 & 1 \\ -9 & 5 & 10 & -5 & 0 \end{pmatrix}$$

This is the 'matrix of coefficients' for the inequalities for the corresponding dual maximization problem—the bottom row representing the function. The dual problem is a maximization problem so that the inequalities are '$\leq$.' Because there are 4 columns, there are $5 - 1 = 4$ variables in this system. [The last column corresponds to the 'opposite' side of the inequalities.] Therefore, the dual maximization problem is...

$$\max z = -9x_1 + 5x_2 + 10x_3 - 5x_4$$
$$\begin{cases} -2x_1 + x_2 + 3x_3 + x_4 \leq 1 \\ x_1 + 5x_2 + 4x_3 - x_4 \leq -1 \\ -x_1 - x_2 + 6x_3 - 8x_4 \leq 1 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$