Name: ____
MATH 361
Spring 2024
HW 1: Due 02/01

"And I knew exactly what to do... but in a much more real sense, I had no idea what to do."

— Michael Scott, The Office

Problem 1. (10pts) Showing all your work and fully justifying your reasoning, compute the following:

(a) $\displaystyle\lim_{x \to -6} \frac{x^2 + 3x - 18}{x^2 + 5x - 6}$

(b) $\displaystyle\lim_{n \to \infty} \frac{2n^2 - 35n + 17}{6n^2 + 19n - 49}$

(c) $\displaystyle\frac{d}{dx} \ln(x \cos x)$

(d) $\displaystyle\int_0^1 \frac{x}{x+1}\, dx$

Problem 2. (10pts) Recall that a sequence $\{a_n\}$ is increasing if $a_{n+1} \geq a_n$ for all n and the sequence is decreasing if $a_{n+1} \leq a_n$ for all n. A sequence $\{a_n\}$ is called bounded above (below) if there exists $M \in \mathbb{R}$ such that $a_n \leq M$ ($a_n \geq M$) for all n. The *Monotone Convergence Theorem* states the following: if $\{a_n\}$ is either increasing or decreasing, i.e. is 'monotone', and bounded above or below, respectively, then $\{a_n\}$ converges. Now consider the sequence with $a_0 = 2$ and given recursively via...

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{5}{a_n}\right)$$

(a) Compute a_1, a_2, a_3.

(b) Compare your values in (a) to $\sqrt{5}$. What might you conjecture?

(c) Explain why the Monotone Convergence Theorem implies that $\{a_n\}$ has a limit.

(d) By (c), we know $L := \lim_{n\to\infty} a_n$ exists. Taking the limit in both sides of the recursive definition for $\{a_n\}$, show that $L = \sqrt{5}$.

Problem 3. (10pts) The Intermediate Value Theorem states the following: if $f(x)$ is continuous on $[a, b]$ and $f(a) < c < f(b)$, then there exists an $x_0 \in (a, b)$ such that $f(x_0) = c$. Consider the function $f(x) = x^2 - 3x + 4$ on the interval $[-1, 5]$.

(a) Give a sketch of $f(x)$ on the interval $[-1, 6]$.

(b) Explain why $f(x)$ is continuous.

(c) Explain why there is a $x_0 \in [-1, 6]$ such that $f(x_0) = 14$.

(d) Find the $x_0 \in [-1, 6]$ such that $f(x_0) = 14$.

Problem 4. (10pts) The Mean Value Theorem states the following: if $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)\big(b - a\big)$. Consider the function $f(x) = x^3 + x^2 - 4x - 5$. Find the values $c \in [-1, 4]$ that satisfy the Mean Value Theorem.