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MATH 308
Fall 2022
HW 3: Due 09/15

"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."

–George Polya

Problem 1. (10pt) Let the universe, $\mathcal{U}$, for m, n, j, k be the set of integers. Define the following predicates:

$$\begin{aligned} P(m)&\colon m \text{ is even} \\ Q(n)&\colon n \text{ is a perfect square} \\ R(j)&\colon j \text{ is divisible by 3} \\ S(k)&\colon k \text{ is divisible by 6} \\ W(m)&\colon 1 < m \leq 8 \end{aligned}$$

Write the open sentences below as complete English sentences as 'simply' as possible and then determine whether the statement is true or false. If the statement is true, explain why. If not, give a counterexample.

(a) $(\exists! n)\big(Q(n) \wedge W(n)\big)$

(b) $(\forall m)\big(P(m) \vee R(m)\big)$

(c) $(\forall m)\big(\neg R(m) \to \neg S(m)\big)$

(d) $(\forall m)(\exists n)\big(S(m) \to [(m = 2n) \wedge P(n)]\big)$

Solution.

(a) Written in words, the proposition $(\exists! n)\big(Q(n) \wedge W(n)\big)$ is the statement, "There exists a unique integer n such that n is a perfect square and $1 < n \leq 8$." The only perfect squares from 1 to 10 are 1, 4, 9. The only one of these greater than 1 and at most 8 is 1. Therefore, there is a unique perfect square greater than 1 and at most 8. Therefore, $(\exists! n)\big(Q(n) \wedge W(n)\big)$ is true.

(b) Written in words, the proposition $(\forall m)\big(P(m) \vee R(m)\big)$ is the statement, "For all integers m, either m is even m is divisible by 3." The statement is false. As a counterexample, if $n = 1$, then n is an integer but n is neither even nor divisible by 3.

(c) Written in words, the proposition $(\forall m)\big(\neg R(m) \to \neg S(m)\big)$ is the statement, "For all integers m, if m is not divisible by 3, then m is not divisible by 6." This statement is true. If m is not divisible by 3, then it does not have 3 as a factor. If m is divisible by 6, then m has a factor of 6. But 6 has 3 as a factor. This would imply that m has 3 as a factor. Therefore, if m is not divisible by 3, it is not divisible by 6. Alternatively, the contrapositive of "if m is not divisible by 3, then m is not divisible by 6" is the statement "if m is divisibly by 6, then m is divisible by 3." Because any integer divisible by 6 must be divisible by 3, the contrapositive is true. But then the original statement is also true.

(d) Written in words, the proposition $(\forall m)(\exists n)\big(S(m) \to [(m = 2n) \wedge P(n)]\big)$ is the statement, "For all integers m, there exists an integer n such that if m is divisible by 6, then $m = 2n$ and n is even." The statement is false. Take $m = 6$. Then m is divisible by 6. If there were an integer n with $m = 2n$, then we know $n = m/6 = 6/6 = 1$. But then n is not even. This shows that the statement $m = 2n$ and n is even is false. Therefore, the statement $S(m) \to [(m = 2n) \wedge P(n)]$ is false so that the statement $(\forall m)(\exists n)\big(S(m) \to [(m = 2n) \wedge P(n)]\big)$.

Problem 2. (10pt) By defining appropriate universes and predicates, quantify the open sentences below. Indicate whether the resulting statement is true or false. No justification is necessary.

(a) For all m, there exists n such that $m = n + 1$.

(b) For all integers n, if n is divisible by 5 then the 1's digit of n is either 0 or 5.

(c) Nonzero real numbers have a unique multiplicative inverse.

(d) Given any pair of distinct integers, there is another integer between them.

(e) Everybody has problems.

Solution.

(a)

(b)

(c)

(d)

(e)

Problem 3. (10pt) Being as clear and detailed as possible, explain why $\exists x P(x) \land \exists x Q(x)$ does not imply $\exists x[P(x) \land Q(x)]$.

Solution.

Problem 4. (10pt) Let $P(x)$ be the predicate $R(x)$: x is a rectangle and let $S(x)$ be the predicate $S(x)$: x is a square.

(a) Write $\forall x\big(R(x) \to S(x)\big)$ as a complete English sentence.

(b) Write the contrapositive, converse, and negation of the open sentence in (a) as complete English sentences.

Solution.

(a)

(b)