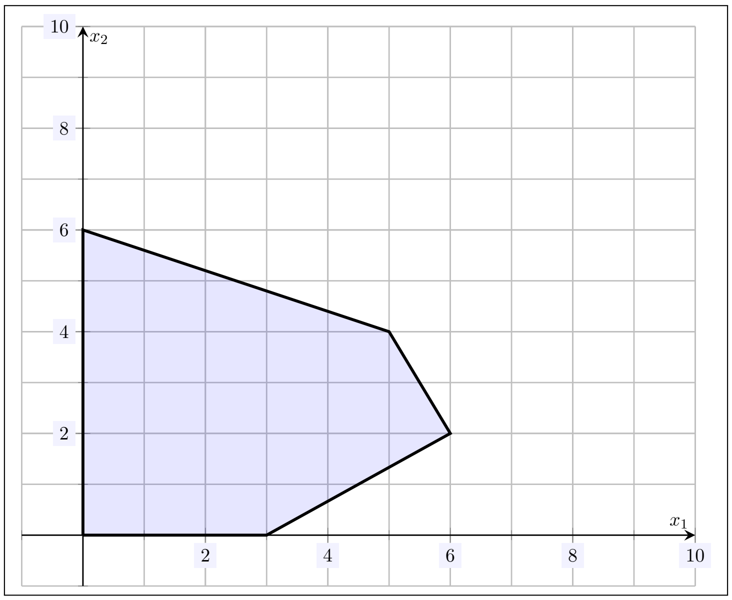
Name: *Caleb McWhorter — Solutions*
MATH 108
Spring 2023
HW 15: Due 05/01

"True optimization is the revolutionary contribution of modern research to decision processes."

–George Dantzig

Problem 1. (10pt) Find the maximum and minimum values for the function $z = 4x_1 + 5x_2$ on the region shown below. Be sure to fully justify that your answers are correct.



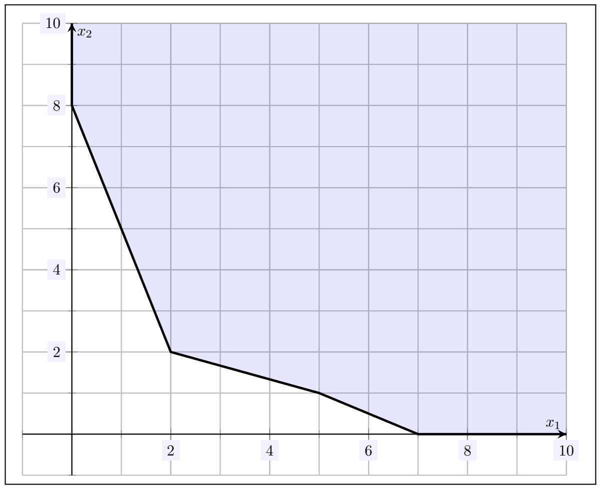
Solution. First, observe that the region above is nonempty. For instance, the point $(0, 0)$ is in the region so that the region is not empty. The region is closed. Finally, because the region is contained in—for instance—the rectangle $[0, 6] \times [0, 6]$, the region is bounded. The function $z = 4x_1 + 5x_2$ is linear. Therefore, because the function z is linear and the region is nonempty, closed, and bounded, the Fundamental Theorem of Linear Programming applies. But then we know that z has a maximum and minimum on this region and that they must occur at a corner point. To find the maximum and minimum, we can simply find the value of z at each corner point:

Corner Point	z
$(0, 0)$	$z = 4(0) + 5(0) = 0 + 0 = 0$
$(0, 6)$	$z = 4(0) + 5(6) = 0 + 30 = 30$
$(5, 4)$	$z = 4(5) + 5(4) = 20 + 20 = 40$
$(6, 2)$	$z = 4(6) + 5(2) = 24 + 10 = 34$
$(3, 0)$	$z = 4(3) + 5(0) = 12 + 0 = 12$

Therefore, the minimum value is 0 and occurs at $(0, 0)$ and the maximum value is 40 and occurs at $(5, 4)$.

$$\boxed{\begin{array}{c} \min z = 0 \text{ at } (0,0) \\ \max z = 40 \text{ at } (5,4) \end{array}}$$

Problem 2. (10pt) Consider the function $z = 5x_1 - x_2$. Does this function has a maximum on the region shown below? If so, explain and find the maximum. If not, explain why. Answer the same question for the minimum of z on the region shown below.



Solution. First, observe that the region shown above is nonempty. For instance, the point $(2, 2)$ is in the region so that the region is not empty. The region is closed. The function $z = 5x_1 - x_2$ is linear. However, the region is *not* bounded. For instance, the region contains the point (x, x) for $x \geq 2$. But then the Fundamental Theorem of Linear Programming does not apply because the region is unbounded. Therefore, we will have to reason about the existence of maxima and minima for z directly.

Observe that if x_1 is increased and x_2 is fixed, the function z increases. If x_1 is fixed and x_2 is decreased, z increases. Therefore, z increases when we 'move to the right and down.' But then if we choose the point $(x_1, 0)$ for $x_1 \geq 7$, this point is always in the region. But at such a point, we have $z(x_1, 0) = 5x_1 - 0 = 5x_1$. But then choosing x_1 arbitrarily large, z is arbitrarily large. Therefore, the function z does not have a maximum on this region.

Observe that if x_1 is decreased and x_2 is fixed, the function z decreases. If x_1 is fixed and x_2 is increased, the function z decreases. Therefore, z decreases when we 'move to the left and up.' But there is no limit to how far 'to the left and up' we can move in our region. For instance, the point $(x_1, x_2) = (0, x_2)$ is always in the region if $x_2 \geq 8$. But $z(0, x_2) = 5(0) - x_2 = -x_2$. Because there is no limit to how large we can make x_2, there is no limit on how negative z can be. Therefore, there is no minimum for the region.

$$\boxed{\begin{array}{c} \min z\colon \text{ DNE} \\ \max z\colon \text{ DNE} \end{array}}$$