Name: ______________________________

MATH 308
Fall 2021
HW 9: Due 11/05

"Computer Science is no more about computers than astronomy is about telescopes."

–Edsger W. Dijkstra

Problem 1. (10pt) Let $f : A \to \mathbb{R}$ be defined by $f(x) := x^3 - 9x^2 + 23x - 12$, where $A = \{1, 3, 6\}$. Let $g : B \to \mathbb{R}$ be defined by $g(x) = x^2 - 4x + 6$, where

$$B = \{x \in \mathbb{N} \mid x \text{ divides } 6\} \setminus \{x \colon x \text{ is an even prime number}\}$$

Prove that $f = g$.

Problem 2. (10pt) Recall the absolute value function, $f(x) = |x|$, is given by

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Considering $f : \mathbb{R} \to \mathbb{R}$, determine the following sets:

(a) $f((-2, 1])$

(b) $f(\mathbb{Z})$

(c) $f^{-1}((-2, 1])$

(d) $f^{-1}(\{-5\})$

(e) $f^{-1}(\mathbb{Z})$

Problem 3. (10pt) Let $f : \mathbb{Z} \to \mathbb{R}$ be given by $f(n) = 2^n$, and let $g : \mathbb{Z} \to \mathbb{R}$ be given by $g(n) = 100 - 3^n$.

(a) Compute $f(1)$.

(b) Compute $g(1)$.

(c) Compute $(fg)(1)$.

(d) Compute $(f \circ g)(1)$.

(e) Find the rule for $(fg)(x)$.

Problem 4. (10pt) Recall that given a function $f : S \to S$, we say that $x \in S$ is a fixed point of f if $f(x) = x$. Let $S = \mathbb{R}$ and let f be the function given by $x \mapsto x^2 + 4x - 10$. Find the fixed points of f. How does the answer change if $S = \mathbb{N}$?

Problem 5. (10pt) Recall that the image of a function $f : S \to S$ (also called the range) is the set $\operatorname{im} f = \{f(s) \colon s \in S\}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{1+x^2}$.

(a) Determine the error in the following 'proof' that $\operatorname{im} f = \mathbb{R}$:

We need prove that $\operatorname{im} f \subseteq \mathbb{R}$ *and* $\mathbb{R} \subseteq \operatorname{im} f$. *Clearly,* $f(x) \in \mathbb{R}$ *so that* $\operatorname{im} f \subseteq \mathbb{R}$. *Now let* $y \in \mathbb{R}$. *Define* $x := \sqrt{\dfrac{1-y}{y}}$. *Then*

$$f(x) = \frac{1}{1+x^2} = \frac{1}{1+\dfrac{1-y}{y}} = \frac{1}{\dfrac{y+1-y}{y}} = \frac{1}{1/y} = y.$$

But then $x \in \mathbb{R}$ *and* $f(x) = y$. *Therefore,* $\mathbb{R} \subseteq \operatorname{im} f$. *Because* $\operatorname{im} f \subseteq \mathbb{R}$ *and* $\mathbb{R} \subseteq \operatorname{im} f$, $\operatorname{im} f = \mathbb{R}$.

(b) Determine $\operatorname{im} f$ and prove that your answer is correct.