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MATH 108
Fall 2023
HW 9: Due 10/24

"I like to play blackjack. I'm not addicted to gambling. I'm addicted to sitting in a semicircle."
–Mitch Hedberg

Problem 1. (10pt) Suppose you play a game where you roll a loaded die. The probabilities for this die are (partially) given below. If you roll an even number, you win \$1. If you roll a one, you lose \$5. If you roll a three, you lose \$2. Finally, if you roll a five, you win/lose nothing.

n	1	2	3	4	5	6
$P(n)$	*1/12*	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{4}{12}$

(a) Find $P(1)$.

(b) Find the probability that if you roll the die three times, you win \$1 each time.

(c) Find the average amount you win per game.

(d) Should you play this game? Explain.

Solution.

(a) We know the sum of the probabilities for all the possibilities must be 1. But then…

$$P(n=1) = 1-P(n=2)-P(n=3)-P(n=4)-P(n=5)-P(n=6) = 1-\frac{2}{12}-\frac{3}{12}-\frac{1}{12}-\frac{1}{12}-\frac{4}{12} = \frac{1}{12} \approx 0.0833$$

(b) The only way one wins \$1 is by rolling an even number. We know the probability of rolling an even number is $P(\text{even}) = P(n=2)+P(n=4)+P(n=6) = \frac{2}{12}+\frac{1}{12}+\frac{4}{12} = \frac{7}{12}$. Because dice rolls are independent, this is…

$$P(\$1\text{three times}) = P(\text{Even three times}) = P(\text{Even})P(\text{Even})P(\text{Even}) = \frac{7}{12}\cdot\frac{7}{12}\cdot\frac{7}{12} = \frac{343}{1728} \approx 0.1985$$

(c) The amount you win on average is the expected value for this game. We construct the random variable, X, given by the win/loss amounts:

n	1	2	3	4	5	6
$P(n)$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{4}{12}$
X	$-\$5$	$\$1$	$-\$2$	$\$1$	$\$0$	$\$1$

We know the expected value for a discrete random variable, X, is $EX = \sum XP(x=X)$. But then we have…

$$EX = \sum XP(x=X) = -\$5\cdot\frac{1}{12}+\$1\cdot\frac{2}{12}+(-\$2)\cdot\frac{3}{12}+\$1\cdot\frac{1}{12}+\$0\cdot\frac{1}{12}+\$1\cdot\frac{4}{12} = -\$\tfrac{1}{3} \approx -\$0.33$$

(d) Because the expected value, $EX \approx -\$0.33 < 0$, is negative, one loses money 'in the long run' playing this game—even if one experiences initial wins. Therefore, one should not play this game for 'long' periods of time.

Problem 2. (10pt) Suppose you are designing a game to 'reallocate' money from your friends to an account that you control... You will have them roll a four-sided dice—each side equally likely to occur. If they roll a four, neither of you wins money. If they roll a two or three, you will pay them \$2 or \$3, respectively. If they roll a one, they will flip a fair coin. If the coin is heads, they win/lose nothing. However, if the coins is tails, they will pay you some amount of money.

(a) Find the amount your friend must pay you if they roll a one and then flip a tails so that you will not lose money at this game 'in the long run.'

(b) If your friend plays this game one-hundred times, are you guaranteed to make money? Explain.

Solution.

(a) Let n be the number one rolls and let the amount of money you pay your friend if they roll a one followed by flipping a tails M. Because each side is equally likely to occur, we know that $P(n = 1) = P(n = 2) = P(n = 3) = P(n = 4) = \frac{1}{4}$. For a coin flip, we know that $P(H) = P(T) = \frac{1}{2}$. Because the dice rolls and coin flips are independent, we know that $P(\text{one and heads}) = P(n = 1)P(H) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$ and $P(\text{one and tails}) = P(n = 1)P(T) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$. But then we can construct a table of the outcomes and their associated payouts—the random variable X.

n	1 & H	1 & T	2	3	4
$P(n)$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{1}{12}$	$\frac{4}{12}$
X	\$0	M	$-\$2$	$-\$3$	\$0

But then the expected value is...

$$EX = \sum XP(x = X) = \$0 \cdot \frac{1}{12} + M \cdot \frac{2}{12} + (-\$2) \cdot \frac{3}{12} + (-\$3) \cdot \frac{1}{12} + \$0 \cdot \frac{4}{12} = \frac{M}{6} - \frac{3}{4} = \frac{2M - 9}{12}$$

If we want to not lose money 'in the long run', we want the expected value to be positive, i.e. $EX > 0$. But then...

$$\begin{aligned}
EX &> 0 \\
\frac{2M - 9}{12} &> 0 \\
2M - 9 &> 0 \\
2M &> 9 \\
M &> \frac{2}{9} \approx 0.22
\end{aligned}$$

Therefore, to win money 'in the long run' playing this game with your friend, you need to make the rule that they pay you any amount more than \$0.22 if they roll a one followed by flipping a tail.

(b) No, you are not guaranteed to win money. It is possible that your friend rolls a 3 one-hundred times in a row! However, choosing the amount M as in (a), we know that if one continues to play this game 'sufficiently many' times, one will make a profit playing this game.