Name: ____________________________________

MATH 308

Fall 2021

HW 11: Due 11/05

"Before software should be reusable, it should be usable."

–Ralph Johnson

Problem 1. (10pt) Show that the following sets have the same cardinality by finding a bijection between them (you need not prove that your function is bijective):

(a) $A = (-2, 2)$, $B = (5, 6)$

(b) $A = \{0, 1\} \times \mathbb{N}$, $B = \mathbb{N}$

(c) $A = [0, 1]$, $B = (0, 1)$

Problem 2. (10pt) Show that $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality using the Schröder-Cantor-Bernstein Theorem.

Problem 3. (10pt) We discussed in class that if S is a set, then the cardinality of $\mathcal{P}(S)$ is strictly larger than the cardinality of S. Therefore, there is no largest cardinality because we can always construct sets with larger cardinality by using power sets. We shall now prove these facts.

(a) If S is a finite set, explain why we already know that $|\mathcal{P}(S)| > |S|$.

(b) Show that $|S| \leq |\mathcal{P}(S)|$ by finding an injection $f : S \to \mathcal{P}(S)$.

(c) Show that $|S| \neq |\mathcal{P}(S)|$ by showing that there is no bijection $\phi : S \to \mathcal{P}(S)$. [Hint: Show there is no such surjection by considering the set $A := \{s \in S \colon s \notin \phi(s)\} \subseteq S$.]

(d) Explain how the previous parts imply that there can be no 'set of all sets.'

Problem 4. (10pt) Determine if the following sets are countable or uncountable (give a brief explanation; however, a formal proof is not necessary):

(a) $A = \{\log n \colon n \in \mathbb{N}\}$.

(b) $B =$ set of perfect squares.

(c) $C = \{(m, n) \in \mathbb{N} \times \mathbb{N} \colon 2 \leq m \leq n^2\}$.

(d) $D =$ set of all irrational numbers.

(e) $E =$ set of linear functions $f : \mathbb{R} \to \mathbb{R}$.

(f) $F =$ set of all finite binary strings.

(g) $G =$ set of *all* binary strings.

(h) $H =$ set of all functions $f : \{0, 1\} \to \mathbb{N}$.

(i) $I =$ set of all functions $f : \mathbb{N} \to \{0, 1\}$.

(j) $J =$ set of all possible dictionary 'words.'

(k) $K =$ set of all subsets of $\mathbb{N}$.

Problem 5. (10pt) Mimic Cantor's proof that the set $\mathbb{R}$ is uncountable to prove that the set of all real numbers without a 7 in their decimal expansion is uncountable.