MAT 101: Exam 3
Spring – 2022
05/12/2022
85 Minutes

Name: *Caleb M^{c}Whorter — Solutions*

Write your name on the appropriate line on the exam cover sheet. This exam contains 21 pages (including this cover page) and 20 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
15	10	
16	10	
17	10	
18	10	
19	10	
20	10	
Total:	200	

1. Compute the functions at the indicated value below. Your answers should be exact.

(a) (2 points) $f(x) = -4(5^x)$

$$f(2) = -4(5^2) = -4(25) = -100$$

(b) (2 points) $g(x) = 5(2^x)$

$$g(0) = 5(2^0) = 5(1) = 5$$

(c) (2 points) $h(x) = -6(4^{1-2x})$

$$h(1) = -6(4^{1-2(1)}) = -6(4^{1-2}) = -6(4^{-1}) = -6 \cdot \frac{1}{4} = -\frac{3}{2}$$

(d) (2 points) $r(x) = 7\left(\frac{3}{5}\right)^x$

$$r(-2) = 7\left(\frac{3}{5}\right)^{-2} = 7\left(\frac{5}{3}\right)^2 = 7 \cdot \frac{25}{9} = \frac{175}{9}$$

(e) (2 points) $s(x) = 8(9^x)$

$$s\left(-\frac{1}{2}\right) = 8(9^{-1/2}) = 8 \cdot \frac{1}{9^{1/2}} = 8 \cdot \frac{1}{\sqrt{9}} = 8 \cdot \frac{1}{3} = \frac{8}{3}$$

2. Using exact values, write the following functions in the form $y = Ab^x$:

(a) (3 points) $f(x) = -5(3^{2x})$

$$f(x) = -5\big((3^2)^x\big) = -5(9^x)$$

(b) (3 points) $g(x) = 3\left(\frac{5}{7}\right)^{-x}$

$$g(x) = 3\left(\frac{5}{7}\right)^{-x} = 3\left(\left(\frac{5}{7}\right)^{-1}\right)^{x} = 3\left(\frac{7}{5}\right)^{x}$$

(c) (4 points) $h(x) = 5(2^{1-3x})$

$$h(x) = 5(2^{1-3x}) = 5(2^1 \cdot 2^{-3x}) = 10(2^{-3x})10\big((2^{-3})^x\big) = 10\left(\frac{1}{2^3}\right)^{x} = 10\left(\frac{1}{8}\right)^{x}$$

3. Determine whether the following functions are increasing or decreasing:

(a) (3 points) $f(x) = 3\left(\frac{10}{11}\right)^x$

Because $f(x)$ *is an exponential function of the form* $y = Ab^{cx}$ *with* $b = \frac{10}{11} < 1$, $c = 1 > 0$, *and* $A = 3 > 0$, $f(x)$ *is a decreasing function.*

(b) (3 points) $g(x) = 17(5^{-2x})$

Because $g(x)$ *is an exponential function of the form* $y = Ab^{cx}$ *with* $b = 5 > 1$, $c = -2 < 0$, *and* $A = 17 > 0$, $f(x)$ *is a decreasing function.*

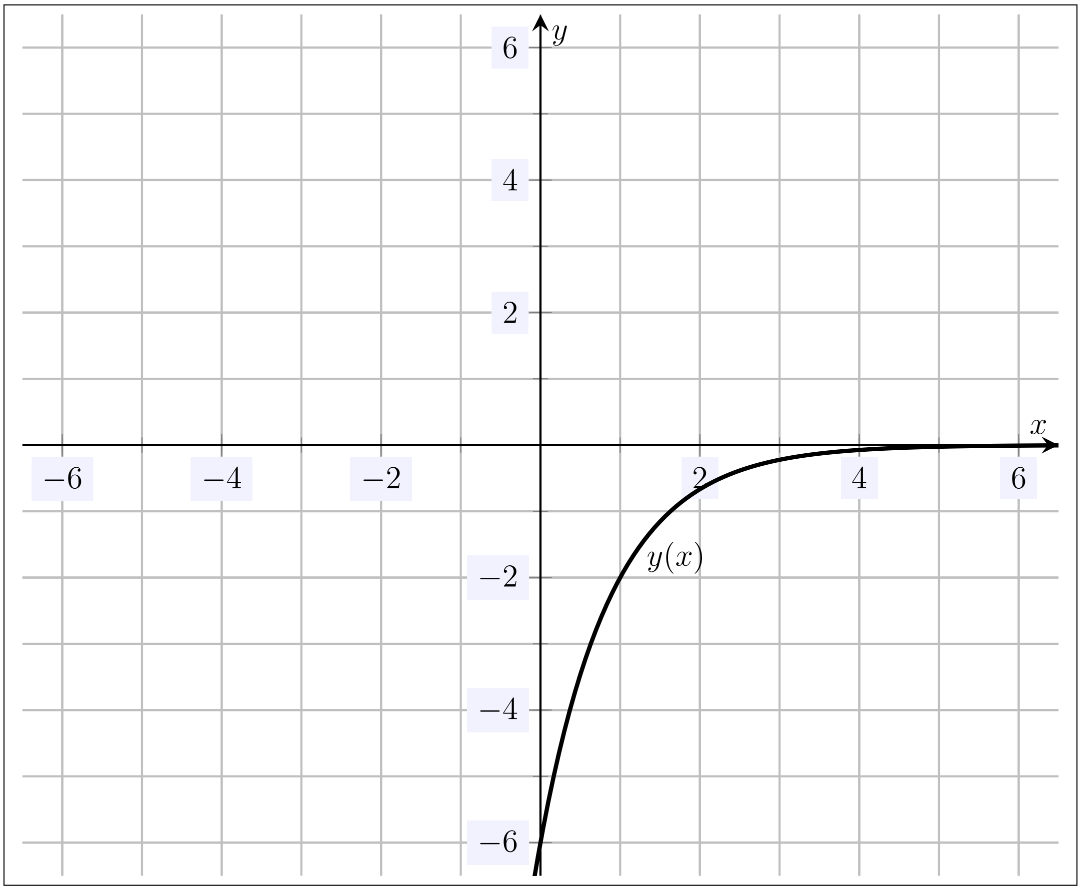
(c) (4 points) $h(x) = -9\left(\frac{7}{5}\right)^{2-x}$

We have...

$$h(x) = -9\left(\frac{7}{5}\right)^{2-x} = -9\left(\frac{7}{5}\right)^2\left(\frac{7}{5}\right)^{-x} = -9\left(\frac{49}{25}\right)\left(\frac{7}{5}\right)^{-x} = -\frac{441}{25}\left(\frac{7}{5}\right)^{-x}$$

Because $h(x)$ *is an exponential function of the form* $y = Ab^{cx}$ *with* $b = \frac{7}{5} > 1$, $c = -1 < 0$, *and* $A = -\frac{441}{25} < 0$, $f(x)$ *is an increasing function.*

4. (10 points) Sketch the function $y = -2\left(\frac{1}{3}\right)^{x-1}$ on the graph below.



$$y = -2\left(\frac{1}{3}\right)^{x-1} = -2\left(\frac{1}{3}\right)^{-1}\left(\frac{1}{3}\right)^{x} = -2\cdot 3\left(\frac{1}{3}\right)^{x} = -6\left(\frac{1}{3}\right)^{x}$$

$$y(0) = -6\left(\frac{1}{3}\right)^{0} = -6\cdot 1 = -6$$

Because y is an exponential function of the form $y = Ab^{cx}$ with $b = \frac{1}{3} < 1$, $c = 1 > 0$, and $A = -6 < 0$, $f(x)$ is an increasing function. By the work above, we know the y-intercept is -6, i.e. $(0, -6)$. Using these two facts about y, we are able to give the sketch above.

5. (10 points) Find the exact values of the following:

(a) $\log_3 3^{2022} = 2022$

(b) $\log_6 \left(\frac{1}{6^{1995}} \right) = \log_6(6^{-1995}) = -1995$

(c) $\log_7 1 = 0$

(d) $\ln(e^{10/11}) = \frac{10}{11}$

(e) $\log_2(64) = \log_2(2^6) = 6$

6. (10 points) Write the following as a single logarithm involving no negative powers:

$$5\log_3(x) - 2\log_3(y) - 6\log_3(z^{-1}) + 4$$

$$\begin{aligned}
5\log_3(x) - 2\log_3(y) - 5\log_3(z^{-1}) + 4 &= 5\log_3(x) - 2\log_3(y) - 5\log_3(z^{-1}) + \log_3(3^4) \\
&= 5\log_3(x) - 2\log_3(y) - 6\log_3(z^{-1}) + \log_3(81) \\
&= \log_3(x^5) + \log_3(y^{-2}) + \log_3(z^6) + \log_3(81) \\
&= \log_3(81x^5y^{-2}z^6) \\
&= \log_3\left(\frac{81x^5z^6}{y^2}\right)
\end{aligned}$$

7. (10 points) Write the following in terms of $\ln x$, $\ln y$, and $\ln z$:

$$\ln\left(\frac{x^{10}}{y^{-5}z^7}\right)$$

$$\begin{aligned}
\ln\left(\frac{x^{10}}{y^{-5}z^7}\right) &= \ln(x^{10}) - \ln(y^{-5}z^7) \\
&= \ln(x^{10}) - \Big(\ln(y^{-5}) + \ln(z^7)\Big) \\
&= 10\ln(x) - \Big(-5\ln(y) + 7\ln(z)\Big) \\
&= 10\ln(x) + 5\ln(y) - 7\ln(z)
\end{aligned}$$

8. (10 points) Use the change of base formula to convert $\log_8(4)$ to base-2 and then find the exact value.

$$
\begin{aligned}
log_8(4) &= \frac{\log_2(4)}{\log_2(8)} \\
&= \frac{\log_2(2^2)}{\log_2(2^3)} \\
&= \frac{2}{3}
\end{aligned}
$$

9. Complete the following:

   (a) (5 points) How many digits does 2021^{2022} have in base-10?

$$\log_{10}(2021^{2022}) = 2022\log_{10}(2021) \approx 2022(3.30557) \approx 6683.86$$

Therefore, 2021^{2022} *has 6,684 digits in base-10.*

   (b) (5 points) How many digits does 2021^{2022} have in base-8?

$$\log_{8}(2021^{2022}) = 2022\log_{8}(2021) \approx 2022(3.66028) \approx 7401.09$$

Therefore, 2021^{2022} *has 7,402 digits in base-8.*

10. (10 points) Is there a whole number k such that 2022^k has 15,000 digits? Explain.

The number of digits in 2022^k *is the smallest whole number that is strictly larger than* $\log_{10}(2022^k)$. *If we want this to be 15,000, then we need…*

$$14999 \leq \log_{10}(2022^k) < 15000$$

$$14999 \leq k\log_{10}(2022) < 15000$$

$$\frac{14999}{\log_{10}(2022)} \leq \frac{k\log_{10}(2022)}{\log_{10}(2022)} < \frac{15000}{\log_{10}(2022)}$$

$$\frac{14999}{3.305781151} \leq k < \frac{15000}{3.305781151}$$

$$4537.2 \leq k < 4537.51$$

However, there is no whole number between 4537.2 *and* 4537.51. *Therefore, there is no whole number* k *such that* 2022^k *has 15,000 digits.*

11. (10 points) Showing all your work, find the exact solution to the following:

$$\frac{1}{25^x} - 4 = 1$$

There are many approaches. For instance,

$$\frac{1}{25^x} - 4 = 1$$

$$\frac{1}{25^x} = 5$$

$$25^{-x} = 5$$

$$(5^2)^{-x} = 5$$

$$5^{-2x} = 5$$

$$-2x = 1$$

$$x = -\frac{1}{2}$$

OR

$$\frac{1}{25^x} - 4 = 1$$

$$\frac{1}{25^x} = 5$$

$$25^{-x} = 5$$

$$\log_{25}(25^{-x}) = \log_{25}(5)$$

$$-x = \frac{1}{2}$$

$$x = -\frac{1}{2}$$

12. (10 points) Showing all your work, find the exact solution to the following:

$$15 - \log_2(4 - x) = 10$$

$$\begin{aligned} 15 - \log_2(4 - x) &= 10 \\ \log_2(4 - x) &= 5 \\ 2^{\log_2(4-x)} &= 2^5 \\ 4 - x &= 32 \\ x &= -28 \end{aligned}$$

13. (10 points) Showing all your work, find the exact solution to the following:

$$e^{x/3} + 12 = 20$$

$$\begin{aligned}
e^{x/3} + 12 &= 20 \\
e^{x/3} &= 8 \\
\ln(e^{x/3}) &= \ln(8) \\
\frac{x}{3} &= \ln(8) \\
x &= 3\ln(8)
\end{aligned}$$

14. (10 points) Showing all your work, find the exact solution to the following:

$$\ln(3x) + 10 = 8$$

$$\begin{aligned}
\ln(3x) + 10 &= 8 \\
\ln(3x) &= -2 \\
e^{\ln(3x)} &= e^{-2} \\
3x &= e^{-2} \\
x &= \frac{e^{-2}}{3} \\
x &= \frac{1}{3e^2}
\end{aligned}$$

15. (10 points) Showing all your work, find the exact solution to the following:

$$3\left(\frac{1}{2}\right)^{5x+3} = 45$$

There are many approaches. For instance,

$$\begin{aligned}
3\left(\frac{1}{2}\right)^{5x+3} &= 45 \\
\left(\frac{1}{2}\right)^{5x+3} &= 15 \\
\ln\left(\frac{1}{2}\right)^{5x+3} &= \ln(15) \\
(5x+3)\ln\left(\frac{1}{2}\right) &= \ln(15) \\
5x+3 &= \frac{\ln(15)}{\ln(1/2)} \\
5x &= \frac{\ln(15)}{\ln(1/2)} - 3 \\
x &= \frac{\frac{\ln(15)}{\ln(1/2)} - 3}{5}
\end{aligned}$$

OR

$$\begin{aligned}
3\left(\frac{1}{2}\right)^{5x+3} &= 45 \\
\left(\frac{1}{2}\right)^{5x+3} &= 15 \\
\log_{1/2}\left(\frac{1}{2}\right)^{5x+3} &= \log_{1/2}(15) \\
5x+3 &= \log_{1/2}(15) \\
5x &= \log_{1/2}(15) - 3 \\
x &= \frac{\log_{1/2}(15) - 3}{5}
\end{aligned}$$

16. (10 points) Showing all your work, find the exact solution to the following:

$$3 - \log_2(x) = \log_2(x+2)$$

Observe that we have...

$$\begin{aligned}
3 - \log_2(x) &= \log_2(x+2) \\
\log_2(x+2) + \log_2(x) &= 3 \\
\log_2\big((x+2)x\big) &= 3 \\
\log_2(x^2+2x) &= 3 \\
2^{\log_2(x^2+2x)} &= 2^3 \\
x^2+2x &= 8 \\
x^2+2x-8 &= 0 \\
(x+4)(x-2) &= 0
\end{aligned}$$

But then either $x+4=0$, i.e. $x=-4$, or $x-2=0$, i.e. $x=2$. However, if $x=-4$, then the original equality would be $3-\log_2(-4)=\log_2(-2)$ and $\log_2 x$ is only defined for $x>0$. Therefore, the only solution is $x=2$.

17. (10 points) Suppose Alice invests \$5,000 into an account which earns 4.5% annual interest, compounded monthly. How much will the investment be worth in 8 years? How much interest has been earned during this time period?

For discrete compounded interest, we know that $F = P\left(1 + \frac{r}{k}\right)^{kt}$*, where* F *is the future value,* P *is the present value,* r *is the annual interest rate,* k *is the number of compounds per year, and* t *is the number of years. But then we have...*

$$F = P\left(1 + \frac{r}{k}\right)^{kt}$$

$$F = 5000\left(1 + \frac{0.045}{12}\right)^{12 \cdot 8}$$

$$F = 5000(1.00375)^{96}$$

$$F \approx 5000(1.432364654)$$

$$F \approx \$7161.82$$

Because only \$5,000 was invested, the interest earned must be $\$7161.82 - \$5000 = \$2161.82$.

18. (10 points) Bob takes out a \$750 loan with an interest rate of 6.4%, compounded continuously. Supposing the loan ends after two and a half years, how much does Bob owe after this time? How much total interest does he pay on this loan?

For continuous compounded interest, we know that $F = Pe^{rt}$, where F is the future value, P is the present value, r is the annual interest rate, and t is the number of years. But then we have…

$$F = Pe^{rt}$$

$$F = 750e^{0.064 \cdot 2.5}$$

$$F \approx 750(1.173511)$$

$$F \approx \$880.13$$

Because only \$750 was taken out on the loan, the interest must be $\$880.13 - \$750 = \$130.13$.

19. (10 points) Suppose Montez takes out a loan to expand his small business. The loan is for $25,000 at a 3.8% annual interest rate, compounded quarterly. How long until the amount of money he owes on the loan has doubled?

For discrete compounded interest, we know that $F = P\left(1+\frac{r}{k}\right)^{kt}$*, where* F *is the future value,* P *is the present value,* r *is the annual interest rate,* k *is the number of compounds per year, and* t *is the number of years. But then we have...*

$$\begin{aligned}
F &= P\left(1+\frac{r}{k}\right)^{kt} \\
50000 &= 25000\left(1+\frac{0.038}{4}\right)^{4t} \\
50000 &= 25000(1.0095)^{4t} \\
(1.0095)^{4t} &= 2 \\
\ln(1.0095)^{4t} &= \ln(2) \\
4t\ln(1.0095) &= \ln(2) \\
t &= \frac{\ln(2)}{4\ln(1.0095)} \\
t &\approx 18.33 \textit{ years}
\end{aligned}$$

20. (10 points) Jillian buys \$900 in a stock which promises annual returns of 1.3%, compounded continuously. How long does Jillian have to wait until this stock is worth \$1,500?

For continuous compounded interest, we know that $F = Pe^{rt}$, where F is the future value, P is the present value, r is the annual interest rate, and t is the number of years. But then we have...

$$\begin{aligned} F &= Pe^{rt} \\ 1500 &= 900e^{0.013t} \\ e^{0.013t} &= 1.66667 \\ \ln(e^{0.013t}) &= \ln(1.66667) \\ 0.013t &= 0.510828 \\ t &\approx 39.29 \textit{ years} \end{aligned}$$