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MATH 101
Fall 2023
HW 5: Due 09/25

"I turned myself into a pickle, Morty! I'm Pickle Rick!"
–Rich Sanchez, Rick & Morty

Problem 1. (10pt) Express each of the following decimal numbers as a rational number in simplest form and express each of the rational numbers as a decimal number:

(a) 0.85

(b) $\frac{5}{12}$

(c) 1.12

(d) $\frac{11}{6}$

Solution.

(a)
$$0.85 = \frac{85}{100} = \frac{5 \cdot 17}{2^2 \cdot 5^2} = \frac{17}{2^2 \cdot 5} = \frac{17}{20}$$

(b)
$$\begin{array}{r} 0.41\overline{6} \\ 12\,\big)\,\overline{5.000} \\ \underline{4.8} \\ 20 \\ \underline{12} \\ 80 \\ \underline{72} \\ 8 \end{array}$$

(c)
$$1.12 = \frac{112}{100} = \frac{2^4 \cdot 7}{2^2 \cdot 5^2} = \frac{2^2 \cdot 7}{5^2} = \frac{28}{25}$$

(d)
$$\begin{array}{r} 1.8\overline{3} \\ 6\,\big)\,\overline{11.00} \\ \underline{6} \\ 5.0 \\ \underline{4.8} \\ 20 \\ \underline{18} \\ 2 \end{array}$$

Problem 2. (10pt) Showing all your work, express the number $0.\overline{2023}$ as a rational number.

Solution. Suppose that $N = 0.\overline{2023} = 0.2023202320232023\overline{2023}$. We have...

$$\begin{array}{rcc} 10000N & = & 2023.2023202320232023\overline{2023} \\ - \quad N & = & 0.2023202320232023\overline{2023} \\ \hline 9999N & = & 2023 \\ N & = & \frac{2023}{9999} \end{array}$$

$$0.\overline{2023} = \frac{2023}{9999}$$

Problem 3. (10pt) Perform the following operations in $\mathbb{C}$:

(a) $(\frac{2}{3} + 5i) + (\frac{1}{2} - \frac{3}{4}i)$

(b) $(15 + 6i) - (9 - 4i)$

(c) $(6 - 3i)(8 + 5i)$

(d) $\dfrac{5 - 7i}{4 + 3i}$

(e) $(1 + 2i)(\overline{1 + 2i})$

Solution.

(a)

$$\left(\frac{2}{3} + 5i\right) + \left(\frac{1}{2} - \frac{3}{4}i\right) = \left(\frac{2}{3} + \frac{1}{2}\right) + \left(5i - \frac{3}{4}\,i\right) = \left(\frac{4}{6} + \frac{3}{6}\right) + \left(\frac{20}{4}\,i - \frac{3}{4}\,i\right) = \frac{7}{6} + \frac{17}{4}\,i$$

(b)

$$(15 + 6i) - (9 - 4i) = 15 + 6i - 9 + 4i = (15 - 9) + (6i + 4i) = 6 + 10i$$

(c)

$$(6-3i)(8+5i) = 48+30i-24i-15i^2 = 48+30i-24i-15(-1) = 48+30i-24i+15 = 63+6i$$

(d)

$$\frac{5 - 7i}{4 + 3i} = \frac{5 - 7i}{4 + 3i}\cdot\frac{4 - 3i}{4 - 3i} = \frac{20 - 15i - 28i + 21i^2}{16 - 12i + 12i - 9i^2} = \frac{(20 - 21) + (-15i - 28i)}{(16 + 9) + (-12i + 12i)} = \frac{-1 - 43i}{25} = -\frac{1}{25} - \frac{43}{25}\,i$$

(e)

$$(1 + 2i)(\overline{1 + 2i}) = (1 + 2i)(1 - 2i) = 1 - 2i + 2i - 4i^2 = 1 - 4(-1) = 1 + 4 = 5$$

Problem 4. (10pt) Every quadratic equation $ax^2 + bx + c = 0$ has exactly two (not necessarily distinct) solutions when the solutions are allowed to be complex numbers. Without explicitly solving the equation, verify that the two solutions to $x^2 - 2x + 5 = 0$ are $x_0 = 1 \pm 2i$; that is, substitute both $x = 1 + 2i$ and $x = 1 - 2i$ into $x^2 - 2x + 5$ and show that one obtains a zero for this function in each case.

Solution. We have...

$$
\begin{aligned}
(x^2 - 2x + 5)\Big|_{x=1+2i} &= (1 + 2i)^2 - 2(1 + 2i) + 5 \\
&= (1 + 2i + 2i + 4i^2) - 2(1 + 2i) + 5 \\
&= (1 + 4i - 4) - 2 - 4i + 5 \\
&= -3 + 4i - 2 - 4i + 5 \\
&= (-3 - 2 + 5) + (4i - 4i) \\
&= 0
\end{aligned}
$$

and also...

$$
\begin{aligned}
(x^2 - 2x + 5)\Big|_{x=1-2i} &= (1 - 2i)^2 - 2(1 - 2i) + 5 \\
&= (1 - 2i - 2i + 4i^2) - 2(1 - 2i) + 5 \\
&= (1 - 4i - 4) - 2 + 4i + 5 \\
&= -3 - 4i - 2 + 4i + 5 \\
&= (-3 - 2 + 5) + (-4i + 4i) \\
&= 0
\end{aligned}
$$