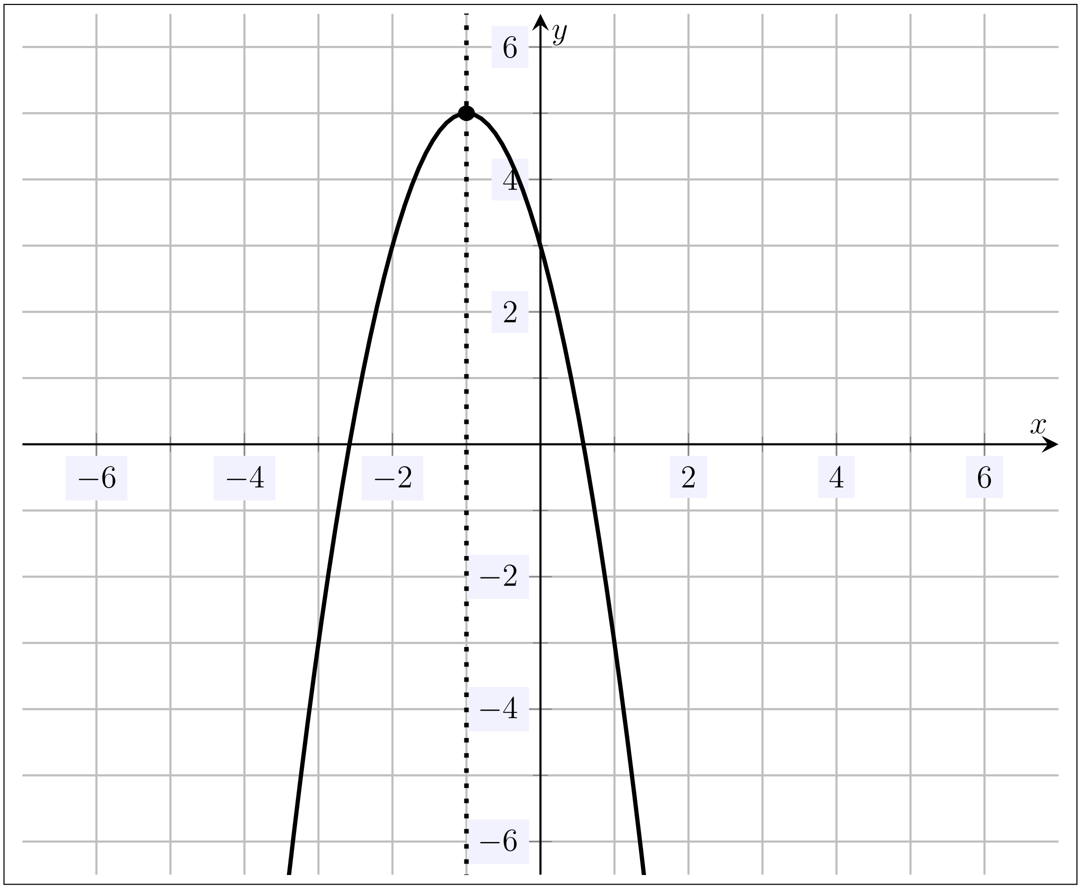
MAT 101: Exam 2
Fall – 2021
12/16/2021
85 Minutes

Name: *Caleb M^{c}Whorter — Solutions*

Write your name on the appropriate line on the exam cover sheet. This exam contains 19 pages (including this cover page) and 18 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work.

Question	Points	Score
1	5	
2	10	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	10	
10	5	
11	5	
12	5	
13	5	
14	5	
15	5	
16	5	
17	5	
18	5	
Total:	100	

1. (5 points) Sketch the quadratic function $f(x) = 5 - 2(x+1)^2$ in the graph below. Your sketch should include the vertex and axis of symmetry for $f(x)$.



From the form of the function $f(x) = 5 - 2(x + 1)^2$, we can immediately see that the vertex is $(-1, 5)$ and that the parabola opens downwards.

2. (10 points) Let $f(x)$ be the quadratic function $f(x) = x^2 + 4x + 9$.

   (a) Find the vertex and axis of symmetry for $f(x)$.

   (b) Does this parabola open upwards or downwards? Explain.

   (c) Is the function convex or concave?

   (d) Does the function have a maximum or minimum value? Explain.

   (e) Find the maximum or minimum value from (d).

Solution.

(a) We know that the x-coordinate of the vertex is...

$$x = \frac{-b}{2a} = \frac{-4}{2(1)} = \frac{-4}{2} = -2$$

The y-coordinate of the vertex is $f(-2) = (-2)^2 + 4(-2) + 9 = 4 - 8 + 9 = 5$. Therefore, the vertex is $(-2, 5)$. It is also immediate that the axis of symmetry is $x = -2$.

$$\boxed{\begin{array}{r} \textit{Vertex: } (-2, 5) \\ \textit{Axis of Symmetry: } x = -2 \end{array}}$$

(b) Because $a = 1 > 0$, the parabola opens upwards.

(c) Because the parabola opens upwards, the function is convex.

(d) Because the parabola opens upwards, the function has a minimum.

(e) We know the minimum value occurs at the vertex. The vertex has coordinate $(-2, 5)$. Therefore, the minimum value is 5.

3. (5 points) Find the vertex form of $y = 3x^2 - 6x + 10$.

We complete the square. First, we factor out a 3 *to obtain* $y = 3(x^2 - 2x + \frac{10}{3})$. *Observe that* $\frac{1}{2} \cdot -2 = -1$ *and that* $(-1)^2 = 1$. *Then...*

$$y = 3x^2 - 6x + 10$$

$$y = 3\left(x^2 - 2x + \frac{10}{3}\right)$$

$$y = 3\left(x^2 - 2x + 1 - 1 + \frac{10}{3}\right)$$

$$y = 3\left((x-1)^2 + \frac{7}{3}\right)$$

$$y = 3(x-1)^2 + 7$$

$$\boxed{y = 3(x-1)^2 + 7}$$

4. (5 points) Factor the polynomial $x^2 - 8x - 33$.

<u>33</u>

$$\begin{array}{rr} 1 \cdot -33 & -32 \\ -1 \cdot 33 & 32 \\ \hline \multicolumn{1}{|r}{3 \cdot -11} & \multicolumn{1}{r|}{-8} \\ \hline -3 \cdot 11 & 8 \end{array}$$

$$x^2 - 8x - 33 = (x - 11)(x + 3)$$

$$\boxed{(x - 11)(x + 3)}$$

5. (5 points) Factor the polynomial $2x^2 + 11x + 15$.

$$\begin{tikzcd}[column sep=small] & 1 \cdot 15 \arrow[dl, dash, "{1,2}"'] \arrow[dr, dash, "{2,1}"] & \\ 1,30 & & 2,15 \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & -1 \cdot -15 \arrow[dl, dash, "{1,2}"'] \arrow[dr, dash, "{2,1}"] & \\ -1,-30 & & -2,-15 \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & 3 \cdot 5 \arrow[dl, dash, "{1,2}"'] \arrow[dr, dash, "{2,1}"] & \\ 3,10 & & 6,5 \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & -3 \cdot -5 \arrow[dl, dash, "{1,2}"'] \arrow[dr, dash, "{2,1}"] & \\ -3,-10 & & -6,-5 \end{tikzcd}$$

$$2x^2 + 11x + 15 = (2x + 5)(x + 3)$$

$$\boxed{(2x + 5)(x + 3)}$$

6. (5 points) Consider the function $f(x) = x^2 + 6x - 40$. Find the x and y intercepts for this function.

The y-intercept occurs when $x = 0$. But then $y = f(0) = 0^2 + 6(0) - 40 = 0 + 0 - 40 = -40$. Therefore, the y-intercept is $(0, -40)$.

The x-intercepts occur when $y = 0$. But then $x^2 + 6x - 40 = 0$. Observe...

$$\begin{aligned} x^2 + 6x - 40 &= 0 \\ (x-4)(x+10) &= 0 \end{aligned}$$

But then either $x - 4 = 0$, i.e. $x = 4$, or $x + 10 = 0$, i.e. $x = -10$. Therefore, the x-intercepts are $(4, 0)$ and $(-10, 0)$.

$$\boxed{\begin{array}{r} y\textit{-intercept: } (0, -40) \\ x\textit{-intercepts: } (-10, 0), (4, 0) \end{array}}$$

7. (5 points) Find the solutions to $x^2 = 8x - 16$.

$$\begin{aligned} x^2 &= 8x - 16 \\ x^2 - 8x + 16 &= 0 \\ (x-4)^2 &= 0 \\ x - 4 &= 0 \\ x &= 4 \end{aligned}$$

$$\boxed{x = 4}$$

8. (5 points) Using the quadratic equation, find the solutions to $3 - 2x^2 = 6x$.

First, we re-arrange the equation:

$$\begin{aligned} 3 - 2x^2 &= 6x \\ 2x^2 + 6x - 3 &= 0 \end{aligned}$$

Now we apply the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{-6 \pm \sqrt{36 + 24}}{4}$$

$$x = \frac{-6 \pm \sqrt{60}}{4}$$

$$x = \frac{-6 \pm \sqrt{4 \cdot 15}}{4}$$

$$x = \frac{-6 \pm 2\sqrt{15}}{4}$$

$$x = \frac{-3 \pm \sqrt{15}}{2}$$

$$\boxed{x = \frac{-3 - \sqrt{15}}{2}, \quad \frac{-3 + \sqrt{15}}{2}}$$

9. (10 points) Consider the rational function $f(x) = \dfrac{x^2 - 25}{x^2 - x - 20}$.

   (a) Find the domain for $f(x)$.

   (b) Find the vertical asymptotes for $f(x)$.

   (c) Find the zeros for $f(x)$.

Solution.

$$f(x) = \frac{x^2 - 25}{x^2 - x - 20} = \frac{(x-5)(x+5)}{(x-5)(x+4)}$$

(a) *The domain of a rational function consists of the real numbers for which the denominator is not zero. If the denominator were zero, then* $(x-5)(x+4) = 0$. *But then either* $x - 5 = 0$, *i.e.* $x = 5$, *or* $x + 4 = 0$, *i.e.* $x = -4$. *Therefore, the domain is the set of real numbers such that* $x \neq 5, -4$.

$$\boxed{x \in \mathbb{R} \textit{ such that } x \neq -4, 5}$$

(b) *We see that for* $x \neq -4, 5$,

$$f(x) = \frac{x^2 - 25}{x^2 - x - 20} = \frac{(x-5)(x+5)}{(x-5)(x+4)} = \frac{\cancel{(x-5)}(x+5)}{\cancel{(x-5)}(x+4)} = \frac{x+5}{x+4}$$

The vertical asymptotes for $f(x)$ *will be where the denominator vanishes in this reduced function. But then* $x + 4 = 0$, *i.e.* $x = -4$. *Therefore, the only vertical asymptote is* $x = -4$.

$$\boxed{x = -4}$$

(c) *We see that for* $x \neq -4, 5$,

$$f(x) = \frac{x^2 - 25}{x^2 - x - 20} = \frac{(x-5)(x+5)}{(x-5)(x+4)} = \frac{\cancel{(x-5)}(x+5)}{\cancel{(x-5)}(x+4)} = \frac{x+5}{x+4}$$

The zeros for $f(x)$ *will be where the numerator vanishes in this reduced function. But then* $x + 5 = 0$, *i.e.* $x = -5$. *Therefore, the only zero is* $x = -5$.

$$\boxed{x = -5}$$

10. (5 points) Compute the following, being sure to simplify as much as possible:

$$\frac{x+2}{x^2-1} - \frac{4}{x^2+4x+3}$$

$$\begin{aligned}
\frac{x+2}{x^2-1} - \frac{4}{x^2+4x+3} &= \frac{x+2}{(x-1)(x+1)} - \frac{4}{(x+1)(x+3)} \\
&= \frac{(x+2)(x+3)}{(x-1)(x+1)(x+3)} - \frac{4(x-1)}{(x+1)(x+3)(x-1)} \\
&= \frac{(x+2)(x+3) - 4(x-1)}{(x-1)(x+1)(x+3)} \\
&= \frac{(x^2+3x+2x+6) - (4x-4)}{(x-1)(x+1)(x+3)} \\
&= \frac{x^2+5x+6-4x+4}{(x-1)(x+1)(x+3)} \\
&= \frac{x^2+x+10}{(x-1)(x+1)(x+3)}
\end{aligned}$$

$$\boxed{\frac{x^2+x+10}{(x-1)(x+1)(x+3)}}$$

11. (5 points) Compute the following, begin sure to simplify as much as possible:

$$\frac{\dfrac{x^2-4x}{x^2-9}}{\dfrac{x^2+2x-24}{x^2+10x+21}}$$

$$\begin{aligned}
\frac{\dfrac{x^2-4x}{x^2-9}}{\dfrac{x^2+2x-24}{x^2+10x+21}} &= \frac{x^2-4x}{x^2-9}\cdot\frac{x^2+10x+21}{x^2+2x-24} \\
&= \frac{x(x-4)}{(x-3)(x+3)}\cdot\frac{(x+3)(x+7)}{(x+6)(x-4)} \\
&= \frac{x\cancel{(x-4)}}{(x-3)\cancel{(x+3)}}\cdot\frac{\cancel{(x+3)}(x+7)}{(x+6)\cancel{(x-4)}} \\
&= \frac{x(x+7)}{(x-3)(x+6)}
\end{aligned}$$

$$\boxed{\frac{x(x+7)}{(x-3)(x+6)}}$$

12. (5 points) Solve the equation $4^{1-x} - 3 = 13$.

$$\begin{aligned} 4^{1-x} - 3 &= 13 \\ 4^{1-x} &= 16 \\ 4^{1-x} &= 4^2 \\ 1 - x &= 2 \\ x &= -1 \end{aligned}$$

OR

$$\begin{aligned} 4^{1-x} - 3 &= 13 \\ 4^{1-x} &= 16 \\ \log_4 4^{1-x} &= \log_4 16 \\ 1 - x &= 2 \\ x &= -1 \end{aligned}$$

$$\boxed{x = -1}$$

13. (5 points) Solve the equation $2e^{-x} + 5 = 17$.

$$\begin{aligned}
2e^{-x} + 5 &= 17 \\
2e^{-x} &= 12 \\
e^{-x} &= 6 \\
\ln e^{-x} &= \ln 6 \\
-x &= \ln 6 \\
x &= -\ln 6
\end{aligned}$$

$$\boxed{x = -\ln 6}$$

14. (5 points) Solve the equation $\log_2(x+5) = 3$.

$$\begin{aligned} \log_2(x+5) &= 3 \\ 2^{\log_2(x+5)} &= 2^3 \\ x+5 &= 8 \\ x &= 3 \end{aligned}$$

$$\boxed{x = 3}$$

15. (5 points) Solve the equation $\log_5(x+7)+\log_5(x+3)=1$.

$$\log_5(x+7)+\log_5(x+3)=1$$

$$\log_5\big((x+7)(x+3)\big)=1$$

$$5^{\log_5\big((x+7)(x+3)\big)}=5^1$$

$$(x+7)(x+3)=5$$

$$x^2+10x+21=5$$

$$x^2+10x+16=0$$

$$(x+2)(x+8)=0$$

$$x+2=0 \textit{ or } x+8=0$$

$$x=-2 \textit{ or } x=-8$$

However, observe that if $x=-8$*, then the left side contains the term* $\log_5(-8+7)=\log_5(-1)$*, which is undefined. Finally, observe that if* $x=-2$*, we have...*

$$\log_5(-2+7)+\log_5(-2+3)=\log_5(5)+\log_5(1)=1+0=1$$

$$\boxed{x=-2}$$

16. (5 points) Suppose you invest \$500 in an account which gains 8% annual interest, compounded semiannually. Find an expression which computes the amount of money in the account after 5 years.

$$F = P\left(1 + \frac{r}{k}\right)^{kt}$$

$$F = 500\left(1 + \frac{0.08}{2}\right)^{2 \cdot 5}$$

$$F = 500\left(1 + 0.04\right)^{10}$$

$$F = 500(1.04)^{10}$$

$$\boxed{500(1.04)^{10} \approx \$740.12}$$

17. (5 points) Solve the following system of equations:

$$\begin{aligned} x - y &= 5 \\ x + y &= 3 \end{aligned}$$

Using substitution, from the first equation, we have...

$$\begin{aligned} x - y &= 5 \\ y &= x - 5 \end{aligned}$$

Using this in the second equation, we have...

$$\begin{aligned} x + y &= 3 \\ x + (x - 5) &= 3 \\ x + x - 5 &= 3 \\ 2x - 5 &= 3 \\ 2x &= 8 \\ x &= 4 \end{aligned}$$

But then we have $y = x - 5 = 4 - 5 = -1$. *Therefore, the solution is* $(4, -1)$.

OR

Using elimination, we add the two equations:

$$\begin{array}{r} x - y = 5 \\ \underline{x + y = 3} \\ 2x = 8 \\ x = 4 \end{array}$$

Using this in the first equation, we have...

$$\begin{aligned} x - y &= 5 \\ 4 - y &= 5 \\ y &= -1 \end{aligned}$$

Therefore, the solution is $(4, -1)$.

$$\boxed{(x, y) = (4, -1)}$$

18. (5 points) Solve the following system of equations:

$$\begin{aligned} -3x + 15y &= 9 \\ 2x + 5y &= -3 \end{aligned}$$

Using substitution, from the first equation, we have...

$$\begin{aligned} -3x + 15y &= 9 \\ 15y &= 3x + 9 \\ y &= \frac{1}{5}\,x + \frac{3}{5} \end{aligned}$$

Using this in the second equation, we have...

$$\begin{aligned} 2x + 5y &= -3 \\ 2x + 5\left(\frac{1}{5}\,x + \frac{3}{5}\right) &= -3 \\ 2x + x + 3 &= -3 \\ 3x + 3 &= -3 \\ 3x &= -6 \\ x &= -2 \end{aligned}$$

Then we have $y = \frac{1}{5}\,x + \frac{3}{5} = \frac{1}{5} \cdot -2 + \frac{3}{5} = -\frac{2}{5} + \frac{3}{5} = \frac{1}{5}$. *Therefore, the solution is* $(-2, \frac{1}{5})$.

OR

Using elimination, we multiply the second equation by -3. *This yields...*

$$\begin{aligned} -3x + 15y &= 9 \\ -6x - 15y &= 9 \end{aligned}$$

Adding these equations, we have...

$$\begin{array}{r} -3x + 15y = 9 \\ \underline{-6x - 15y = 9} \\ -9x = 18 \\ x = -2 \end{array}$$

Using this in the second equation, we have $2(-2) + 5y = -3$ *so that* $5y - 4 = -3$. *But then* $5y = 1$. *Therefore,* $y = \frac{1}{5}$. *The solution is then* $(-2, \frac{1}{5})$.

$$\boxed{(x, y) = \left(-2,\ \frac{1}{5}\right)}$$