Name: *Caleb McWhorter — Solutions*
MATH 308
Fall 2023
HW 9: Due 10/12

"We will always have STEM with us. Some things will drop out of the public eye and will go away, but there will always be science, engineering, and technology. And there will always, always be mathematics."

–Katherine Johnson

Problem 1. (10pt) For each of the following functions, determine whether the function is injective, surjective, or bijective. Be sure to fully justify your answer.

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 7 - 3x$

(b) $g : \mathbb{R} \to \mathbb{R}^{\geq 0}$, $g(x) = x^2 + 1$

(c) $h : \mathbb{R}^2 \to \mathbb{R}^2$, $h(x, y) = (x - 2y, -2x + 4y)$

(d) $k : [0, \infty) \to \mathbb{R}$, $k(x) = 6 - x^2$

Solution.

(a) The function f is linear. We know that any non-constant linear function $\ell : \mathbb{R} \to \mathbb{R}$ is bijective. Because f is not constant (because $m = -3 \neq 0$), we know that f is bijective. By definition, bijective functions are injective and surjective, so that f is both injective and surjective. We can also prove this directly. To show that f is injective, we need to show that if $f(x) = f(y)$, then $x = y$. If $f(x) = f(y)$, then. . .

$$\begin{aligned} f(x) &= f(y) \\ 7 - 3x &= 7 - 3y \\ -3x &= -3y \\ x &= y \end{aligned}$$

Therefore, f is injective. To show that f is surjective, given any $y \in \mathbb{R}$, we need to find an x such that $f(x) = y$. Fix $y \in \mathbb{R}$, and define $x := \frac{7-y}{3}$. But then. . .

$$f(x) = f\left(\frac{7-y}{3}\right) = 7 - 3\left(\frac{7-y}{3}\right) = 7 - (7 - y) = 7 - 7 + y = y$$

Therefore, f is surjective. Because f is injective and surjective, f is bijective. Alternatively, because every horizontal line intersects the graph of f at least once, f is surjective. Because every horizontal line intersects the graph of f at most once, f is injective. Finally, because every horizontal line intersects the graph of f exactly once, f is bijective.

(b) The function g is not surjective or injective. To see that g is not surjective, fix $0 \in \mathbb{R}^{\geq 0}$. If $g(x) = 0$ for some x, then $x^2 + 1 = 0$. This implies that $x^2 = -1$. But $x^2 \geq 0$ for all $x \in \mathbb{R}$. Therefore, there is no x such that $g(x) = 0$. This shows that g is not surjective. To see that g is not injective, observe that $-1, 1 \in \mathbb{R}$ and $-1 \neq 1$, but $g(-1) = 2 = g(1)$. Therefore, g is not injective. Because h is not both surjective and injective, h is not bijective. Alternatively, because every horizontal line $y = c$, where $c \in \mathbb{R}^{\geq 0}$, intersects the graph of g at least once, we know that g is surjective. Because the horizontal line at $y = 2$ intersects the graph of g more than once, we know that g is not injective. Because g is not both surjective and injective, g is not bijective.

(c) The function h is not surjective or injective. To see that h is not surjective, consider $(0, 1) \in \mathbb{R}^2$. If there were $(x, y) \in \mathbb{R}^2$ such that $h(x, y) = (0, 0)$, then $(x - 2y, -2x + 4y) = (0, 1)$. But then $x-2y = 0$ and $-2x+4y = 1$. The first equality implies that $x = 2y$. Using this in $-2x+4y = 1$, we have...

$$\begin{aligned} -2x + 4y &= 1 \\ -2(2y) + 4y &= 1 \\ -4y + 4y &= 1 \\ 0 &= 1 \end{aligned}$$

which is impossible. Therefore, there is no $(x, y) \in \mathbb{R}^2$ such that $h(x, y) = (0, 1)$, which shows that h is not surjective. So see that h is not injective, observe that $(0, 0), (2, 1) \in \mathbb{R}^2$ and $(0, 0) \neq (2, 1)$, but $h(0, 0) = (0, 0) = h(2, 1)$. Therefore, h is not injective. Because h is not both surjective and injective, h is not bijective.

(d) The function k is not surjective or injective. To see that k is not surjective, fix $7 \in \mathbb{R}$. If there $x \in [0, \infty)$ such that $k(x) = 7$, then...

$$\begin{aligned} k(x) &= 7 \\ 6 - x^2 &= 7 \\ -x^2 &= 1 \\ x^2 &= -1 \end{aligned}$$

Because $x^2 \geq 0$ for all $x \in \mathbb{R}$, it is impossible that $x^2 = -1$. Therefore, there is no $x \in \mathbb{R}$ such that $k(x) = 7$, which shows that k is not surjective. To see that k is injective, suppose that $k(x) = k(y)$. Then...

$$\begin{aligned} k(x) &= k(y) \\ 6 - x^2 &= 6 - y^2 \\ -x^2 &= -y^2 \\ x^2 &= y^2 \\ x &= \pm y \end{aligned}$$

Because $x = \pm y$, either $x = y$ or $x = -y$. Because x, y are not both zero, if $x = -y$, then x, y have opposite signs. But because $x, y \in [0, \infty)$, that is not possible. Therefore, it must be that $x = y$, so that k is injective. Because k is not both surjective and injective, k is not bijective. Alternatively, because not every horizontal line intersects the graph of k at least once, k is not surjective. Because every horizontal line intersects the graph of k at most once, k is injective. Because k is not both surjective and injective, k is not bijective.

Problem 2. (10pt) Let $A = B = \mathbb{R}$. Consider the function $f : A \to B$ given by $f(x) = x^2 - 4x + 7$.

(a) Sketch a graph of $f(x)$. Be sure your graph includes an interval around the vertex of $f(x)$.

(b) Is $f(x)$ injective? Explain. [Hint: $f(x) = (x-2)^2 + 3$.]

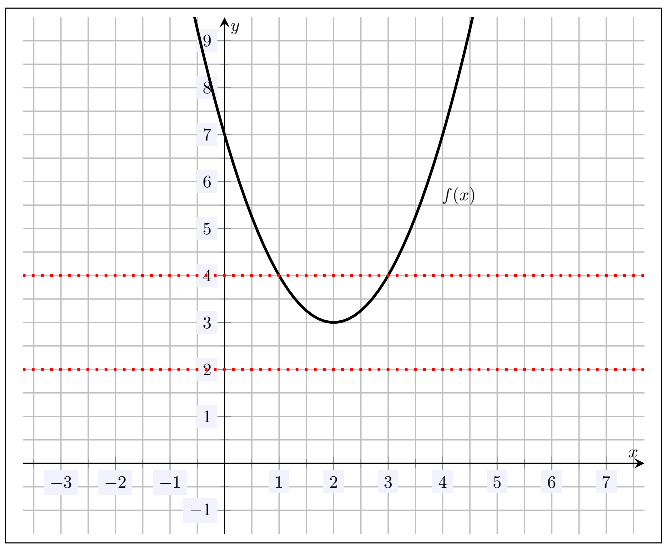
(c) Is $f(x)$ surjective? Explain. [Hint: $f(x) = (x-2)^2 + 3$.]

(d) Do your responses in (b) and (c) change if $A = [2, \infty)$? Explain.

(e) Do your responses in (b) and (c) change if $B = [3, \infty)$? Explain.

Solution.

(a) Because $f(x) = x^2 - 4x + 7$ is a quadratic function of the form $ax^2 + bx + c$ with $a = 1$, $b = -4$, and $c = 7$. Because $a = 1 > 0$, the parabola opens upwards. The vertex form of $f(x)$ is $(x-2)^2 + 3$, so that the vertex is $(2, 3)$. This gives the plot below.



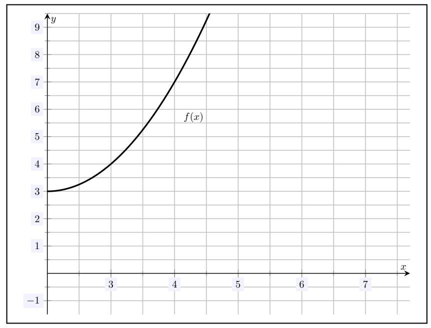
(b) The function f is not injective. We can see that the horizontal line at $y = 4$ intersects the graph of f most than once. Therefore, f is not injective. Alternatively, observe that $1, 3 \in \mathbb{R}$ and $1 \neq 3$, but $f(1) = 4 = f(3)$. Therefore, f is not injective.

(c) The function f is not surjective. We can see that the horizontal line at $y = 2$ does not intersect the graph at least once. Therefore, f is not surjective. Alternatively, observe that if $f(x) = 2$, then...

$$\begin{aligned} f(x) &= 2 \\ (x-2)^2 + 3 &= 2 \\ (x-2)^2 &= -1 \end{aligned}$$

Because $(x-2)^2 \geq 0$ for all $x \in \mathbb{R}$, it is not possible that $(x-2)^2 = -1$. Therefore, there is no x such that $f(x) = 2$, so that f is not surjective.

(d) Observe that if we restrict f to $[2, \infty)$, the graph of f is...

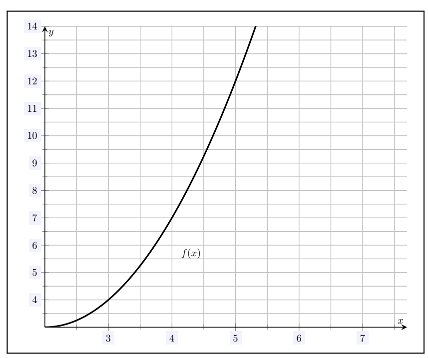


Because the horizontal line at $y = 2$ does not intersect the graph of f at least once, f is not surjective. The proofs work the same as in (c). However, because every horizontal line intersects the graph of f at most once, f is injective. To see this algebraically, assume $f(x) = f(y)$. Then...

$$\begin{aligned} f(x) &= f(y) \\ (x-2)^2 + 3 &= (y-2)^2 + 3 \\ (x-2)^2 &= (y-2)^2 \\ x - 2 &= \pm(y-2) \end{aligned}$$

Because $x, y \in [2, \infty)$, $x - 2, y - 2 \geq 0$. But then we cannot have $x - 2 = -(y - 2)$ unless $x = y = 2$. Therefore, it must be that $x - 2 = y - 2$, so that $x = y$. Therefore, f is injective.

(e) Observe that if we 'restrict' $f : [2, \infty) \to [3, \infty)$, the graph of f is...



Observe that every horizontal line intersects the graph of f at most once. Therefore, f is injective. Again, the same proofs in (d) apply in this case. Observe that every horizontal line intersects the graph of f at least once. Therefore, f is surjective. Alternatively, choose $y \in [3, \infty)$ and define $x := 2 + \sqrt{y - 3}$—which is well-defined because $y \geq 3$. Observe...

$$f(x) = f\left(2 + \sqrt{y-3}\right) = \left((2 + \sqrt{y-3}) - 2\right)^2 + 3 = (\sqrt{y-3})^2 + 3 = (y-3) + 3 = y$$

Therefore, f is surjective. Because f is both injective and surjective, $f : [2, \infty) \to [3, \infty)$ is a bijection.

Problem 3. (10pt) Let $X = \{1,2,3,4,5\}$, $Y = \{1,2,3,4\}$, and $A = \{1,2,3\}$. If S is a set and $\phi : S \to S$ is a function, we say that $s \in S$ is a *fixed point* for ϕ if $\phi(s) = s$. Recall that a function $\psi : \mathbb{R} \to \mathbb{R}$ is *strictly increasing* if $\psi(x) < \psi(y)$ for all $x, y \in \mathbb{R}$ with $x < y$.

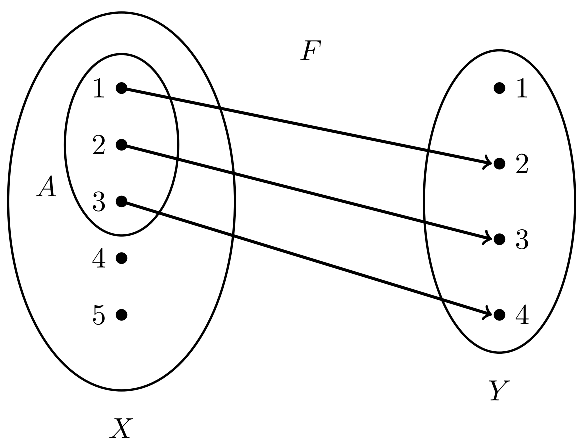
(a) Determine a function $F : A \to Y$ that is nondecreasing with no fixed point. Be sure to fully specify the function and justify that F has the required properties.

(b) Determine a function $G : X \to Y$ such that $G\big|_A = F$ and G is neither surjective nor injective but so that G does have a fixed point. Be sure to fully specify the function and justify that G has the required properties.

(c) Is G a strictly increasing function? Explain.

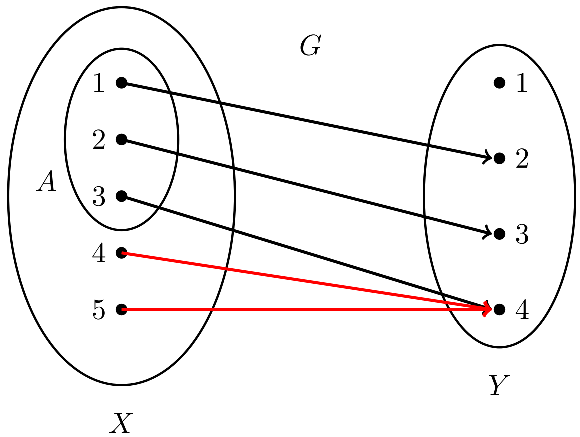
Solution.

(a) Because F need be a function, each value $F(1)$, $F(2)$, and $F(3)$ need be well-defined, i.e. F need be a function. This implies that there can only be one choice for each of $F(1)$, $F(2)$, and $F(3)$. Because $F\colon A \to Y$ is nondecreasing, we know that $F(1) \leq F(2) \leq F(3)$. Because F cannot have a fixed point, we know that $F(1) \neq 1$. But then $F(1) \geq 2$. Again, because F cannot have a fixed point, we know that $F(2) \neq 2$. But then $F(2) \neq 2$ and $F(2) \geq F(1) \geq 2$. Together, this shows that $F(2) > 2$, i.e. $F(2) \geq 3$ given this codomain. Finally, we know that because F does not have a fixed point, $F(3) \neq 3$. Then $F(3) \neq 3$ and $F(3) \geq F(2) \geq 3$. Together, this shows that $F(3) > 3$, i.e. $F(3) \geq 4$ given this codomain. Therefore, the only possible choices for the function F are given by the relations $\{(1,2),(2,3),(3,4)\}$, $\{(1,3),(2,3),(3,4)\}$, and $\{(1,4),(2,4),(3,4)\}$. Define F via the diagram given below (the first relation given):



It is clear that F is a function. Observe that $F(1) \neq 1$, $F(2) \neq 2$, and $F(3) \neq 3$. Therefore, F has no fixed point. Finally, observe that $F(1) \leq F(2) \leq F(3)$ so that F is nondecreasing.

(b) Because G need be a function, each input of G need be well defined. Furthermore, because $G\big|_A = F$, we need choose the same values for G as F on A. So we need only extend F to X by defining F at 4 and 5. Now G cannot be surjective so that $G(4) \neq 1$ and $G(5) \neq 1$; otherwise, for every $y \in Y$, there would be $x \in X$ such that $F(x) = y$ and F would be surjective. But G needs a fixed point. Because $5 \in X$ and $5 \notin Y$, $5 \neq G(5) \in Y$ cannot be a fixed point. Therefore, in order for G to have a fixed point, we must choose $G(4) = 4$. But then because $3 \neq 4$ but $G(3) = 4 = G(4)$, so that G is not injective. We only need define $G(5)$. We can choose any value in $\{2, 3, 4\}$. We simply choose $G(5) = 4$. The function G is given by F (on the set A) and the values at 4 and 5 are given by the lines in red below.



(c) Examining all the possible choices for F in (a), we observe that we must have $F(3) = 4$. But from (b), we know that we must have $G(3) = F(3) = 4$ and $G(4) = 4$. But then $3 < 4$ but $G(3) = 4 \not< G(4)$ so that G is not strictly increasing.

Problem 4. (10pt) Below is a partial proof of the fact that if $f : X \to Y$ is a function and $A, B \subseteq Y$, then $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. By filling in the missing portions, complete the partial proof below so that it is a correct, logically sound proof with 'no gaps.'

Proposition. If $f : X \to Y$ is a function and $A, B \subseteq Y$, then $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

Proof. To prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, we need to show $\underline{f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)}$

and $\underline{f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)}$.

Clearly, if $f^{-1}(A \cup B) = \varnothing$, then $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$. Similarly, if $f^{-1}(A) \cup f^{-1}(B) = \varnothing$, then $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$. Assume neither $f^{-1}(A \cup B)$ nor $f^{-1}(A) \cup f^{-1}(B)$ are empty.

$f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$: Let $x \in \underline{\quad f^{-1}(A \cup B) \quad}$. But then $f(x) \in A \cup B$. This implies

that either $f(x) \in \underline{\qquad A \qquad}$ or $f(x) \in \underline{\qquad B \qquad}$.

Case 1, $f(x) \in \underline{\qquad A \qquad}$: If $f(x) \in A$, then $\underline{\qquad x \qquad} \in f^{-1}(A)$. But then

$x \in \underline{\quad f^{-1}(A) \cup f^{-1}(B) \quad}$.

Case 2, $f(x) \in B$: If $f(x) \in B$, then $x \in \underline{\qquad f^{-1}(B) \qquad}$. But then $x \in f^{-1}(A) \cup f^{-1}(B)$.

Therefore, if $x \in f^{-1}(A \cup B)$, we know that $x \in \underline{\qquad f^{-1}(A) \cup f^{-1}(B) \qquad}$. This shows

that $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$.

$f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$: Suppose that $\underline{\quad x \in f^{-1}(A) \cup f^{-1}(B) \quad}$. This implies

that $x \in f^{-1}(A)$ or $f^{-1}(B)$.

Case 1, $\underline{\quad x \in f^{-1}(A) \quad}$: If $x \in f^{-1}(A)$, then $f(x) \in A$. But then $f(x) \in \underline{\qquad A \cup B \qquad}$.

This shows that $x \in f^{-1}(A \cup B)$.

Case 2, $x \in f^{-1}(B)$: If $x \in f^{-1}(B)$, then $f(x) \in \underline{\qquad B \qquad}$. But then $f(x) \in f(A \cup B)$.

This shows that $x \in \underline{\quad f^{-1}(A \cup B) \quad}$.

Therefore, if $x \in f^{-1}(A) \cup f^{-1}(B)$, we know that $x \in f^{-1}(A \cup B)$. This shows that $\underline{f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)}$.

But then we have shown that $\underline{f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)}$ and $\underline{f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)}$.

Therefore, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.