Name: *Caleb McWhorter — Solutions*
MATH 101
Fall 2022
HW 20: Due 11/30

"The philosophy of the school room in one generation will be the philosophy of government in the next."

–Abraham Lincoln

Problem 1. (10pt) For each of the following quadratic functions, i.e. functions which can be written as $f(x) = ax^2 + bx + c$, identify a, b, c:

(a) $2x^2 - 5x + 7$

(b) $6x + 9 - x^2$

(c) $x^2 - 16$

(d) $(x + 1)^2$

(e) $(x - 2)(x + 3)$

Solution.

(a) We have $a = 2$, $b = -5$, and $c = 7$.

(b) Because $6x + 9 - x^2 = -x^2 + 6x + 9$, we have $a = -1$, $b = 6$, and $c = 9$.

(c) Because $x^2 - 16 = x^2 + 0x - 16$, we have $a = 1$, $b = 0$, and $c = -16$.

(d) Because $(x + 1)^2 = (x + 1)(x + 1) = x^2 + x + x + 1 = x^2 + 2x + 1$, we have $a = 1$, $b = 2$, and $c = 1$.

(e) Becase $(x - 2)(x + 3) = x^2 + 3x - 2x - 6 = x^2 + x - 6$, we have $a = 1$, $b = 1$, and $c = -6$.

Problem 2. (10pt) Consider the quadratic function $f(x) = 4 - (x-2)^2$.

(a) Determine if the given parabola opens upwards or downwards.

(b) Is the parabola convex or concave?

(c) Does the function $f(x)$ have a maximum or a minimum?

(d) Find the vertex and axis of symmetry.

(e) Find the maximum or minimum value of $f(x)$.

(f) Sketch a graph of $f(x)$ on the plot below.

Solution.

(a) Observe $f(x) = 4 - (x-2)^2 = -(x-2)^2 + 4$ is in the form $a(x-p)^2 + q$ with $a = -1$, $p = 2$, and $q = 4$. Because $a = -1 < 0$, the parabola opens downwards.

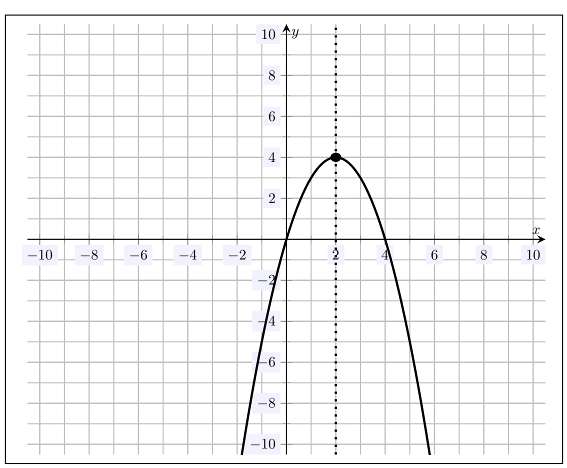
(b) Because the parabola opens downwards, we know that the parabola is concave.

(c) Because the parabola opens downwards, the function $f(x)$ has a maximum.

(d) If $f(x)$ has the form $a(x-p)^2 + q$, then $f(x)$ is a quadratic function with vertex (p, q) and axis of symmetry $x = p$. Observe $f(x) = 4 - (x-2)^2 = -(x-2)^2 + 4$ is in the form $a(x-p)^2 + q$ with $a = -1$, $p = 2$, and $q = 4$. Therefore, the vertex is $(2, 4)$ and the axis of symmetry is $x = 2$.

(e) Because $f(x)$ opens downwards, there is no minimum value for $f(x)$; however, there is a maximum value. The maximum value is the y-coordinate of the vertex. Because the vertex is $(2, 4)$, the maximum value for $f(x)$ is 4.

(f) If $f(x)$ has the form $a(x-p)^2 + q$, then $f(x)$ is a quadratic function with vertex (p, q) and axis of symmetry $x = p$. Furthermore, if $a > 0$ then the parabola opens upwards and if $a < 0$ the parabola opens downwards. Observe $f(x) = 4 - (x-2)^2 = -(x-2)^2 + 4$ is in the form $a(x-p)^2 + q$ with $a = -1$, $p = 2$, and $q = 4$. Therefore, the parabola opens downwards, the vertex is $(2, 4)$, and the axis of symmetry is $x = 2$. This gives the sketch below.



Problem 3. (10pt) Showing all your work, put $f(x) = 2x^2 - 12x - 13$ into vertex form. Also, find the vertex and axis of symmetry for $f(x)$.

Solution. If we complete the square, we have...

$$\begin{aligned} f(x) &= 2x^2 - 12x - 13 \\ &= 2\left(x^2 - 6x - \frac{13}{2}\right) \\ &= 2\left(x^2 - 6x + 3^2 - 3^2 - \frac{13}{2}\right) \\ &= 2\left((x^2 - 6x + 9) - 9 - \frac{13}{2}\right) \\ &= 2\left((x-3)^2 - \frac{18}{2} - \frac{13}{2}\right) \\ &= 2\left((x-3)^2 - \frac{31}{2}\right) \\ &= 2(x-3)^2 - 31 \end{aligned}$$

The vertex form of $f(x)$ is then $f(x) = 2(x-3)^2 - 31$. Therefore, the vertex is $(3, -31)$ and the axis of symmetry is $x = 3$.

If we use the 'evaluation method', we know the vertex occurs at $x = -\frac{b}{2a}$. We find the y-coordinate by evaluation $f(x)$ at this value. Therefore, we have...

$$\begin{aligned} x &= -\frac{b}{2a} = -\frac{-12}{2(2)} = -\frac{-12}{4} = -(-3) = 3 \\ f(3) &= 2(3^2) - 12(3) - 13 = 2(9) - 12(3) - 13 = 18 - 36 - 13 = -31 \end{aligned}$$

Given a quadratic function with leading coefficient a and vertex (p, q), the function is $f(x) = a(x - p)^2 + q$. From the work above, we know that the vertex is $(3, -31)$. Because $f(x) = 2x^2 - 12x - 13$, we know that $a = 2$. Therefore, $f(x) = 2(x-3)^2 - 31$ is the vertex form of $f(x)$. Again, from the work above, we know that the vertex is $(3, -31)$ and that the axis of symmetry is $x = 3$.