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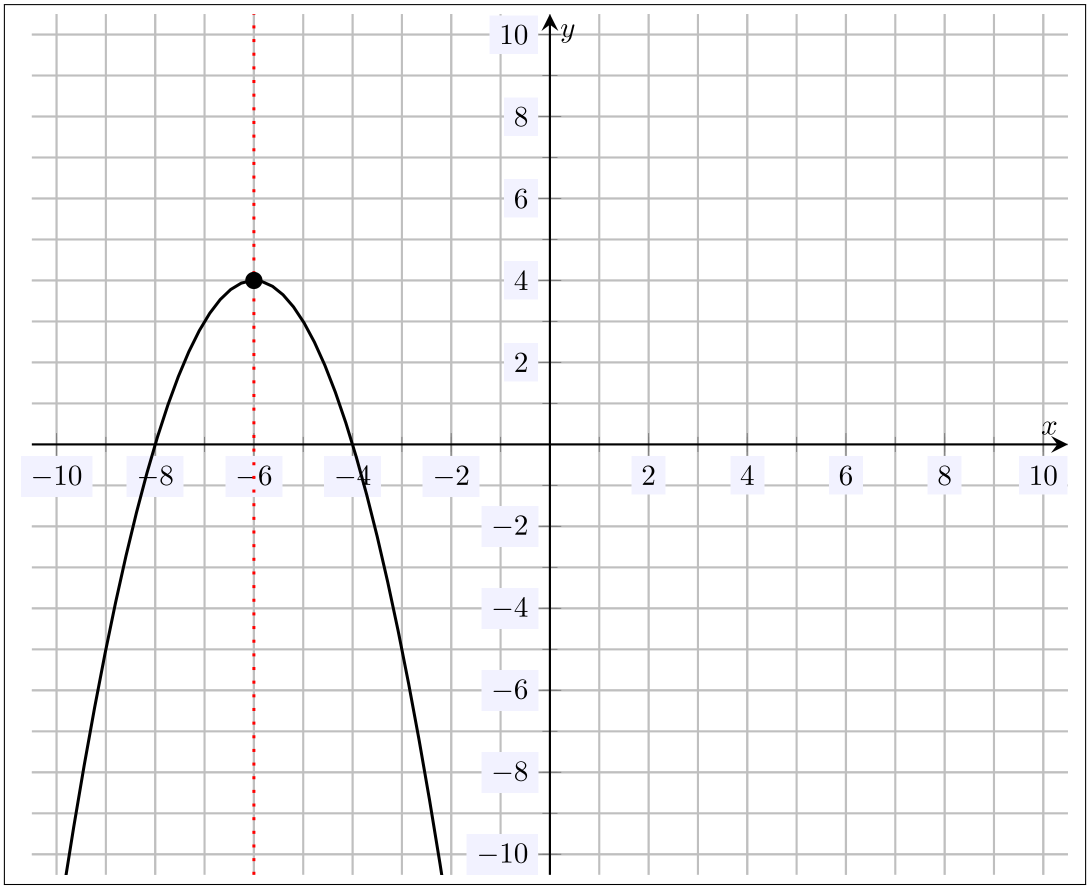
MATH 101
Summer 2022
HW 7: Due 06/07

"Science is simply common sense at its best, that is, rigidly accurate in observation, and merciless to fallacy in logic."

–*Thomas Huxley*

Problem 1. (10pt) Plot the quadratic function $y = 4 - (x+6)^2$ as accurately as possible. Your sketch should include the vertex and axis of symmetry.



Solution. We know that $y = 4 - (x+6)^2 = -(x+6)^2 + 4 = -\big(x - (-6)\big)^2 + 4$ is in vertex form, i.e. the form $y = a(x-P)^2 + Q$ with $a = -1$, $P = -6$, and $Q = 4$. Therefore, the vertex of $y = 4 - (x+6)^2$ is $(-6, 4)$ and the axis of symmetry is $x = -6$. Because $a = -1 < 0$, the parabola opens downwards. This gives the sketch above.

Problem 2. (10pt) Find the vertex form of $f(x) = x^2 - 12x + 41$. Also, find the vertex and axis of symmetry of $f(x)$.

Solution. The vertex form of a function $f(x) = ax^2+bx+c$ is a form $y = a(x-P)^2+Q$, where (P, Q) is the vertex and $x = P$ is the axis of symmetry. We find the vertex form of $f(x) = x^2 - 12x + 41$ by completing the square:

$$\begin{aligned} f(x) &= x^2 - 12x + 41 \\ &= x^2 - 12x + (12/2)^2 - (12/2)^2 + 41 \\ &= x^2 - 12x + 36 - 36 + 41 \\ &= (x^2 - 12x + 36) + (-36 + 41) \\ &= (x-6)^2 + 5 \end{aligned}$$

Therefore, $(6, 5)$ is the vertex and $x = 6$ is the axis of symmetry.

OR

The vertex form of a function $f(x) = ax^2 + bx + c$ is a form $y = a(x - P)^2 + Q$, where (P, Q) is the vertex and $x = P$ is the axis of symmetry. We know the x-coordinate of the vertex is $x = -\frac{b}{2a}$. But we have $x = -\frac{b}{2a} = -\frac{-12}{2(1)} = \frac{12}{2} = 6$. The y-coordinate of the vertex is...

$$f(6) = 6^2 - 12(6) + 41 = 36 - 72 + 41 = 5$$

Therefore, the vertex is $(6, 5)$. We know that $a = 1$. Then vertex form is $f(x) = 1(x - 6)^2 + 5 = (x - 6)^2 + 5$ and the axis of symmetry is $x = 6$.

Problem 3. (10pt) Find the vertex and axis of symmetry of $g(x) = -3x^2 + 24x - 37$.

Solution. The vertex form of a function $f(x) = ax^2+bx+c$ is a form $y = a(x-P)^2+Q$, where (P, Q) is the vertex and $x = P$ is the axis of symmetry. We find the vertex form of $g(x) = -3x^2 + 24x - 37$ by completing the square:

$$\begin{aligned}
g(x) &= -3x^2 + 24x - 37 \\
&= -3\left(x^2 - 8x + \frac{37}{3}\right) \\
&= -3\left(x^2 - 8x + (-8/2)^2 - (-8/2)^2 + \frac{37}{3}\right) \\
&= -3\left(x^2 - 8x + 16 - 16 + \frac{37}{3}\right) \\
&= -3\left((x^2 - 8x + 16) + \left(-16 + \frac{37}{3}\right)\right) \\
&= -3\left((x-4)^2 + \left(-\frac{48}{3} + \frac{37}{3}\right)\right) \\
&= -3\left((x-4)^2 - \frac{11}{3}\right) \\
&= -3(x-4)^2 + 11
\end{aligned}$$

Therefore, $(4, 11)$ is the vertex and $x = 4$ is the axis of symmetry.

OR

The vertex form of a function $f(x) = ax^2 + bx + c$ is a form $y = a(x-P)^2 + Q$, where (P, Q) is the vertex and $x = P$ is the axis of symmetry. We know the x-coordinate of the vertex is $x = -\frac{b}{2a}$. But we have $x = -\frac{b}{2a} = -\frac{24}{2(-3)} = -\frac{24}{-6} = 4$. The y-coordinate of the vertex is...

$$g(4) = -3(4^2) + 24(4) - 37 = -3(16) + 96 - 37 = -48 + 96 - 37 = 11$$

Therefore, the vertex is $(4, 11)$. We know that $a = -3$. Then vertex form is $g(x) = -3(x-4)^2 + 11$ and the axis of symmetry is $x = 6$.

Problem 4. (10pt) Consider the quadratic function $h(x) = 4x^2 - 12x + 6$.

(a) Determine if the parabola opens upwards or downwards.

(b) Is the parabola convex or concave?

(c) Does the parabola have a maximum or minimum?

(d) Find the vertex and axis of symmetry.

(e) Find the maximum/minimum value of $h(x)$.

Solution.

(a) The function $h(x) = 4x^2 - 12x + 6$ is of the form $ax^2 + bx + c$. Because $a = 4 > 0$, we know that the parabola opens upwards.

(b) The function $h(x) = 4x^2 - 12x + 6$ is of the form $ax^2 + bx + c$. Because $a = 4 > 0$, we know that the parabola is convex.

(c) The function $h(x) = 4x^2 - 12x + 6$ is of the form $ax^2 + bx + c$. Because $a = 4 > 0$, we know that the parabola opens upwards or is convex. Therefore, there is a minimum value but no maximum value.

(d) Completing the square, we have...

$$4x^2-12x+6 = 4\left(x^2 - 3x + \frac{3}{2}\right) = 4\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4} + \frac{3}{2}\right) = 4\left(\left(x - \frac{3}{2}\right)^2 - \frac{3}{4}\right) = 4\left(x - \frac{3}{2}\right)^2 - 3$$

Therefore, the vertex is $(\frac{3}{2}, -3)$ and the axis of symmetry is $x = \frac{3}{2}$.

OR

We know that the x-coordinate of the vertex is $x = -\frac{b}{2a} = -\frac{-12}{2(4)} = \frac{12}{8} = \frac{3}{2}$. Then we have...

$$h\left(\frac{3}{2}\right) = 4\left(\frac{3}{2}\right)^2 - 12 \cdot \frac{3}{2} + 6 = 4 \cdot \frac{9}{4} - 6(3) + 6 = 9 - 18 + 6 = -3$$

Therefore, the vertex is $(\frac{3}{2}, -3)$ and the axis of symmetry is $x = \frac{3}{2}$.

(e) We know that $h(x)$ has no maximum value. The minimum value of $h(x)$ is the y–coordinate of the vertex. Therefore, the minimum value of $h(x)$ is -3.

Problem 5. (10pt) Consider the quadratic function $j(x) = -x^2 - 4x + 1$.

(a) Determine if the parabola opens upwards or downwards.

(b) Is the parabola convex or concave?

(c) Does the parabola have a maximum or minimum?

(d) Find the vertex and axis of symmetry.

(e) Find the maximum/minimum value of $j(x)$.

Solution.

(a) The function $j(x) = -x^2 - 4x + 1$ is of the form $ax^2 + bx + c$. Because $a = -1 < 0$, we know that the parabola opens downwards.

(b) The function $j(x) = -x^2 - 4x + 1$ is of the form $ax^2 + bx + c$. Because $a = -1 < 0$, we know that the parabola is concave.

(c) The function $j(x) = -x^2 - 4x + 1$ is of the form $ax^2 + bx + c$. Because $a = -1 < 0$, we know that the parabola opens downwards or is concave. Therefore, there is a maximum value but no minimum value.

(d) Completing the square, we have...

$$-x^2 - 4x + 1 = -(x^2 + 4x - 1) = -(x^2 + 4x + 4 - 4 - 1) = -\big((x+2)^2 - 5\big) = -(x+2)^2 + 5$$

Therefore, the vertex is $(-2, 5)$ and the axis of symmetry is $x = -2$.

OR

We know that the x-coordinate of the vertex is $x = -\frac{b}{2a} = -\frac{-4}{2(-1)} = -\frac{-4}{-2} = -2$. Then we have...

$$j(-2) = -(-2)^2 - 4(-2) + 1 = -4 + 8 + 1 = 5$$

Therefore, the vertex is $(-2, 5)$ and the axis of symmetry is $x = -2$.

(e) We know that $j(x)$ has no minimum value. The maximum value of $j(x)$ is the y–coordinate of the vertex. Therefore, the maximum value of $j(x)$ is 5.

Problem 6. (10pt) Factor the following:

(a) $x^2 - 64$

(b) $4x - 20x^2$

(c) $9x^2 - 25$

(d) $5x^2 + 60x$

Solution.

(a)
$$x^2 - 64 = (x - 8)(x + 8)$$

(b)
$$4x - 20x^2 = 4x(1 - 5x)$$

(c)
$$9x^2 - 25 = (3x - 5)(3x + 5)$$

(d)
$$5x^2 + 60x = 5x(x + 12)$$

Problem 7. (10pt) Factor the following completely: $4x^2 + 20x - 24$

$$4x^2 + 20x - 24 = 4(x^2 + 5x - 6) = 4(x - 1)(x + 6)$$

Problem 8. (10pt) Use completing the square to solve the following equation:

$$4x^2 = 16x - 24$$

Solution.

$$\begin{aligned} 4x^2 &= 16x - 24 \\ x^2 &= 4x - 6 \\ x^2 - 4x &= -6 \\ x^2 - 4x + 4 &= -6 + 4 \\ (x-2)^2 &= -2 \\ x - 2 &= \pm\sqrt{-2} \\ x - 2 &= \pm\sqrt{2}\, i \\ x &= 2 \pm \sqrt{2}\, i \end{aligned}$$

Problem 9. (10pt) Solve the following quadratic equation by factoring:

$$10x = 24 - x^2$$

Solution.

$$10x = 24 - x^2$$

$$x^2 + 10x - 24 = 0$$

$$(x + 12)(x - 2) = 0$$

$$x + 12 = 0 \text{ or } x - 2 = 0$$

$$x = -12 \text{ or } x = 2$$

Problem 10. (10pt) Solve the following quadratic equation:

$$x(10 - x) = 25$$

Solution.

$$\begin{aligned} x(10 - x) &= 25 \\ 10x - x^2 &= 25 \\ x^2 - 10x + 25 &= 0 \\ (x - 5)(x - 5) &= 0 \\ (x - 5)^2 &= 0 \\ x - 5 &= 0 \\ x &= 5 \end{aligned}$$