MAT 308: Exam 2 **Name:** ______________________________

Fall – 2022

11/18/2022

'∞' Minutes

Write your name on the appropriate line on the exam cover sheet. This exam contains 11 pages (including this cover page) and 10 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) Consider the 'rule' $f : \mathbb{R} \to \mathbb{R}^2$ given by $t \mapsto (3\cos t, 3\sin t)$.

   (a) Is $f(t)$ a function? Explain.

   (b) Consider $\operatorname{im} f \subseteq \mathbb{R}^2 = \{(x, y) \colon x, y \in \mathbb{R}\}$. If $(x, y) \in \operatorname{im} f$, show that $x^2 + y^2 = 9$.

   (c) Considered as a subset of $\mathbb{R}^2$, geometrically describe $\operatorname{im} f$.

   (d) Can $\operatorname{im} f$ be given by the image of a function of x? What about a function of y? Explain.

2. (10 points) Define functions $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x + 5|$ and $g(x) = 7 - 3x$.

   (a) Find an element in $\operatorname{im} f$ and also find an element in $\operatorname{im} g$.

   (b) Is $-5 \in \operatorname{im} f$? If not, explain why, and if so, find its preimage.

   (c) Is $12 \in \operatorname{im} g$? If not, explain why, and if so, find its preimage.

   (d) Compute $f\big([-6, 6)\big)$ and $g\big((-6, 6]\big)$.

   (e) Compute $f^{-1}\big([-1, 1]\big)$ and $g^{-1}\big([-1, 1]\big)$.

3. (10 points) Define the function $f : \mathbb{R} \to \mathbb{R}$ via $x \mapsto x^2 + 5$.

   (a) Is $f(x)$ an injective function? If it is injective, explain why; if it is not injective, give a counterexample.

   (b) Is $f(x)$ a surjective function? If it is surjective, explain why; if it is not surjective, give a counterexample.

   (c) Is $f(x)$ a bijective function? Explain.

   (d) Does $f(x)$ have an inverse function? Explain.

4. (10 points) A *fixed point* for a function $f : \mathbb{R} \to \mathbb{R}$ is $x_0 \in \mathbb{R}$ such that $f(x_0) = x_0$.

   (a) Show that -5 is a fixed point for $f(x) = 3x + 10$.

   (b) Show that 4 is not a fixed point for $g(x) = \dfrac{x+4}{2-x}$.

   (c) Find the fixed points for $h(x) = 2x^2 + 6x - 3$.

   (d) Use the quadratic formula to show that $j(x) = x^2 - 3x + 5$ has no fixed points in $\mathbb{R}$ but does have fixed points in $\mathbb{C}$.

5. (10 points) Showing all your work, compute the following:

   (a) $\displaystyle\sum_{k=-2}^{3}(5-k)$

   (b) $\displaystyle\prod_{k=1}^{5}(2k-3)$

   (c) $\displaystyle\sum_{k=0}^{1000}(k-7)$

   (d) $\displaystyle\sum_{k=0}^{1000}\left(\sqrt{k+5}-\sqrt{k}\right)$

   (e) $\displaystyle\prod_{k=1}^{1000}\left(1+\frac{1}{k}\right)$

6. (10 points) Being sure to show all your work and fully justify your logic complete the following:

   (a) Using the definition of odd/even, show that -237 is odd but not even.

   (b) Express $1854/17$ using the division algorithm.

   (c) Find the prime factorization of 2040.

   (d) Compute $\gcd(2^{173} \cdot 3^{187} \cdot 5^{685} \cdot 11^{203},\ 2^{578} \cdot 3^{281} \cdot 7^{323} \cdot 13^{360})$ and find the next largest divisor of the two given numbers.

   (e) Compute $\operatorname{lcm}(2^{173} \cdot 3^{187} \cdot 5^{685} \cdot 11^{203},\ 2^{578} \cdot 3^{281} \cdot 7^{323} \cdot 13^{360})$ and find the next smallest multiple of the two given numbers.

7. (10 points) Showing all your work, compute the following:

   (a) $(2468 \cdot 3579 + 97531)$ mod 2

   (b) $(10 - 18)^{100}$ mod 3

   (c) $(3^{11} + 3^{10})$ mod 4

   (d) $(16 \cdot -7)$ mod 5

   (e) $(-17 \cdot 13 + 145)$ mod 6

8. (10 points) Being sure to show all your work and fully explaining your logic, complete the following:

   (a) What is the remainder when 2022^{2024} is divided by 2023?

   (b) What are the last three digits of 2022^{50}?

   (c) Show that working modulo two that $(x + y)^2 = x^2 + y^2$.

9. (10 points) Let $a = 1561$ and $b = 8525$.

    (a) Use the Euclidean algorithm to find $\gcd(a, b)$.

    (b) Explain why a^{-1} exists $\mod b$.

    (c) Continuing your work in (a), use the extended Euclidean algorithm to compute $a^{-1} \bmod 8525$.

    (d) Prove that your answer in (c) is correct.

10. (10 points) Solve the following system of congruences and show that your solution is correct:

$$\begin{cases} x + 1 \equiv 2 \text{ mod } 3 \\ x \equiv 0 \text{ mod } 5 \\ 3x + 4 \equiv 2 \text{ mod } 7 \\ 1 - x \equiv 4 \text{ mod } 11 \end{cases}$$