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MATH 101
Fall 2023
HW 17: Due 12/11

"Mathematics is a language."
–Josiah Willard Gibbs

Problem 1. (10pt) Showing all your work, factor the following quadratic expression:

$$12x^2 - x - 20$$

Solution. We find factors of $12 \cdot 20 = 240$ that add to -1. Because $-20 < 0$, the factors must have opposite signs. But then we have...

240	
$1 \cdot -240$	-239
$-1 \cdot 240$	239
$2 \cdot -120$	-118
$-2 \cdot 120$	118
$3 \cdot -80$	-77
$-3 \cdot 80$	77
$4 \cdot -60$	-56
$-4 \cdot 60$	56
$5 \cdot -48$	-43
$-5 \cdot 48$	43
$6 \cdot -40$	-34
$-6 \cdot 40$	34
$8 \cdot -30$	-22
$-8 \cdot 30$	22
$10 \cdot -24$	-14
$-10 \cdot 24$	14
$12 \cdot -20$	-8
$-12 \cdot 20$	8
$\boxed{15 \cdot -16}$	$\boxed{-1}$
$-15 \cdot 16$	1

$$12x^2-x-20 = 12x^2+15x-16x-20 = (12x^2+15x)+(-16x-20) = 3x(4x+5)-4(4x+5) = (3x-4)(4x+5)$$

Problem 2. (10pt) Use the quadratic formula to factor the following polynomial:

$$1968x^2 - 18458x + 11495$$

Solution. If the roots of $f(x) = ax^2+bx+c$ are r_0, r_1, then we know that $f(x) = a(x-r_0)(x-r_1)$. So we need to find the roots of the given quadratic function. The polynomial $1968x^2 - 18458x + 11495$ has $a = 1968$, $b = -18458$, and $c = 11495$. But then...

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-18458) \pm \sqrt{(18458)^2 - 4(1968)11495}}{2(1968)} \\ &= \frac{18458 \pm \sqrt{340697764 - 90488640}}{3936} \\ &= \frac{18458 \pm \sqrt{250209124}}{3936} \\ &= \frac{18458 \pm 15818}{3936} \end{aligned}$$

Therefore, the roots are $x = \frac{18458-15818}{3936} = \frac{2640}{3936} = \frac{55}{82}$ and $x = \frac{18458+15818}{3936} = \frac{34276}{3936} = \frac{209}{24}$. Therefore, we have...

$$1968x^2 - 18458x + 11495 = 1968\left(x - \frac{55}{82}\right)\left(x - \frac{209}{24}\right) = 82\left(x - \frac{55}{82}\right) \cdot 24\left(x - \frac{209}{24}\right) = (82x-55)(24x-209)$$

Problem 3. (10pt) Find all the real zeros of the following polynomial:

$$x^6 - 16x^2$$

Solution. The zeros of a function, $f(x)$, are the x-values such that $f(x) = 0$. Recall the difference of perfect squares: $a^2 - b^2 = (a - b)(a + b)$. We have...

$$x^6 - 16x^2 = 0$$

$$x^2(x^4 - 16) = 0$$

$$x^2(x^2 - 4)(x^2 + 4) = 0$$

$$x^2(x - 2)(x + 2)(x^2 + 4) = 0$$

But then either $x^2 = 0$, which implies $x = 0$ or $x - 2 = 0$, which implies $x = 2$ or $x + 2 = 0$, which implies $x = -2$ or $x^2 + 4 = 0$, which implies that $x^2 = -4$. The equation $x^2 = -4$ has no real solutions. However, over the complex numbers, we know that $x^2 = -4$ implies $x = \sqrt{-4} = \pm\sqrt{4}i = \pm 2i$. Therefore, the zeros of the polynomial are...

$$-2, \quad 0, \quad 2, \quad -2i, \quad 2i$$

Equivalently, the zeros are $0, \pm 2, \pm 2i$.

Problem 4. (10pt) Showing all your work, solve the following equation:

$$\frac{x+1}{x-3} = \frac{3x}{x+2}$$

Solution. We have...

$$\begin{aligned}
\frac{x+1}{x-3} &= \frac{3x}{x+2} \\
(x+1)(x+2) &= 3x(x-3) \\
x^2 + 3x + 2 &= 3x^2 - 9x \\
2x^2 - 12x - 2 &= 0 \\
x^2 - 6x - 1 &= 0
\end{aligned}$$

Clearly, this polynomial does not factor. Therefore, we use the quadratic formula with $a = 1$, $b = -6$, and $c = -1$.

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-1)}}{2(1)} \\
&= \frac{6 \pm \sqrt{36 + 4}}{2} \\
&= \frac{6 \pm \sqrt{40}}{2} \\
&= \frac{6 \pm \sqrt{4 \cdot 10}}{2} \\
&= \frac{6 \pm 2\sqrt{10}}{2} \\
&= 3 \pm \sqrt{10}
\end{aligned}$$

Therefore, the solutions are $x = 3 - \sqrt{10}$ and $x = 3 + \sqrt{10}$.