Quiz 1. *True/False*: The following is a truth table for $P \to Q$:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	F
F	F	F

Solution. The statement is *false*. The correct truth table should be...

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

One way to think about this is as follows: imagine P is a guarantee. Namely, we promise that if P happens, Q must happen. For instance, P could represent the statement, "You do not tamper with your hardware," and Q could be the statement, "I will replace your broken computer." So $P \to Q$ is then the statement, "If you do not tamper with your hardware, then I will replace your broken computer." If both P and Q are true, then this should be true—because I promised to replace the computer if you left it alone. If P is true and Q is false, then the statement should be false because I broke my promise. However, my promise holds true whenever P is false. Why? Because you broke our agreement by tampering with the hardware. So while I may or may not replace the computer, my promise has not been broken in either case, i.e. it remains true. In an implication $P \to Q$, if P is false, then the statement $P \to Q$ is *always* true.

Quiz 2. *True/False*: $\forall x, \exists y,\ x^2 + y = 4$

Solution. The statement is *true*. The statement says that for all x there is a y such that $x^2 + y = 4$. If this is true (which it is), we need to prove it. Fix an x, say x_0. We need to find a y such that $x_0^2 + y = 4$. Define $y_0 := 4 - x_0^2$. But then we have

$$x_0^2 + y_0 = x_0^2 + (4 - x_0^2) = 4,$$

as desired.

Quiz 3. *True/False*: $\neg\left(\forall x, \exists y, P(x,y) \vee \neg Q(x,y)\right) = \exists x, \forall y, \neg P(x,y) \wedge Q(x,y)$

Solution. The statement is *true*. We can simply compute the negation step-by-step:

$$\begin{aligned}
\neg\left(\forall x, \exists y, P(x,y) \vee \neg Q(x,y)\right) &\equiv \exists x, \neg(\exists y, P(x,y) \vee \neg Q(x,y)) \\
&\equiv \exists x, \forall y, \neg(P(x,y) \vee \neg Q(x,y)) \\
&\equiv \exists x, \forall y, \neg P(x,y) \wedge \neg(\neg Q(x,y)) \\
&\equiv \exists x, \forall y, \neg P(x,y) \wedge Q(x,y)
\end{aligned}$$

Quiz 4. *True/False*: To prove $P \Rightarrow Q$, you can prove $Q \Rightarrow P$.

Solution. The statement is *false*. The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$. The converse of a logical statement is not necessarily logically equivalent to the original statement. So proving the converse does not necessarily prove the original statement. However, the contrapositive of $P \Rightarrow Q$, which is $\neg Q \Rightarrow \neg P$, is logically equivalent to $P \Rightarrow Q$. Therefore, to prove $P \Rightarrow Q$, one only need prove $\neg Q \Rightarrow \neg P$. This is called proof by contrapositive.

Quiz 5. *True/False*: Let $A = \{1\}$ and $B = \{3, \{1\}\}$. Then $A \subseteq B$.

Solution. The statement is *false*. Recall that $A \subseteq B$ if every element of A is an element of B. The only element of A is the element 1. However, $1 \notin B$, but rather $\{1\} \in B$, i.e. 1 is not in B but the set consisting of only the element of 1 is in B. However, note that $A \in B$ because $A = \{1\}$ and $\{1\} \in B$.

Quiz 6. *True/False*: Take the universal set to be the integers. Then the following two sets are equal:

$$A = \{n \colon n \text{ odd}\}$$
$$B = \{m \colon m \text{ prime and } m > 2\}$$

Solution. The statement is *false*. We know that $9 \in A$ because 9 is odd. But $9 \notin B$ because $9 = 3 \cdot 3$ is not prime. Therefore, $A \not\subseteq B$ so that $A \neq B$.

Quiz 7. *True/False*: The sets $A \times B \times C$ and $(A \times B) \times C$ are not the same.

Solution. The statement is *true*. Elements in $A \times B \times C$ 'look like' (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. Whereas elements in $(A \times B) \times C$ 'look like' $((a, b), c)$, where $a \in A$, $b \in B$, and $c \in C$. Because elements in these sets are not of the same form, they cannot be the same. As an explicit example, take $A = \{1\}$, $B = \{2, 3\}$, and $C = \{4\}$. Then

$$A \times B \times C = \{(1, 2, 4), (1, 3, 4)\}$$
$$(A \times B) \times C = \{((1, 2), 4), ((1, 3), 4)\}$$

Then $A \times B \times C \neq (A \times B) \times C$.

Quiz 8. *True/False*: There is a set S such that $\mathcal{P}(S)$ has 3 elements.

Solution. The statement is *false*. If S is an infinite set, then clearly there is a subset for each element $s \in S$, i.e. the subset $\{s\}$. Clearly, if there is such a set, it cannot be infinite. Now if S had 3 or more elements—having a subset for each element of S—we know that $\mathcal{P}(S)$ would have more than 3 subsets. Therefore, S must have 0, 1, or 2 elements. If $S = \emptyset$, then $\mathcal{P}(S) = \{\emptyset\}$. If $S = \{s_1\}$, then $\mathcal{P}(S) = \{\emptyset, \{s_1\}\}$. Finally, if $S = \{s_1, s_2\}$, then $\mathcal{P}(S) = \{\emptyset, \{s_1\}, \{s_2\}, S\}$. Therefore, there cannot be such a set S.

Quiz 9. *True/False*: The Principle of Induction is logically equivalent to the Well-Ordering Principle.

Solution. The statement is *true*. We saw in class that the Well-Ordering Principle implied the Principle of Induction. From the homework, we know that the Principle of Induction implies the Well-Ordering Principle.

Quiz 10. *True/False*: If $P(n)$ is a proposition for each $n \in \mathbb{N}$ and $P(1), P(2), P(3), \ldots, P(k)$ are all true, then $P(n)$ is true for all $n \geq 1$.

Solution. The statement is *false*. These are only base cases. For induction to imply that $P(n)$ is true for all $n \in \mathbb{N}$, we need $P(k)$ being true to imply $P(k+1)$ is true. A statement can be true for *many* n and not be true for all n. For instance, the polynomial $p(n) = n^2 - n + 41$ is prime for $n = 1, 2, \ldots, 40$ but not for $n = 41$. In fact, a statement can be true for all but one n!

Quiz 11. *True/False*: If $f : A \to \mathbb{R}$ is positive and $g : A \to \mathbb{R}$ is nonnegative, then $fg : A \to \mathbb{R}$ is positive.

Solution. The statement is *false*. It is possible. For instance, $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := x^2 + 1$ and $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = |x| + 1$ so that $fg = (x^2+1)(|x|+1)$. However, because g is only nonnegative, it can take on the value zero. But then for these values, fg is zero and hence not positive. For instance, let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^2 + 1$ and $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) := |x|$. Then $(fg)(0) = (0^2+1)(|0|) = 0 \not> 0$ so that fg is not positive.

Quiz 12. *True/False*: The sets $\mathbb{Z}$ and $\mathbb{Q}$ have the same cardinality.

Solution. The statement is *true*. We saw this via the diagonalization argument given in class. Alternatively, we know that $\mathbb{Z}$ and $\mathbb{Q}$ are both countably infinite; therefore, there must be a bijection between $\mathbb{Z}$ and $\mathbb{Q}$ so that they must have the same cardinality. We could also use the following approach: the set $\mathbb{Z}$ is countably infinite, so there exists a bijection $f : \mathbb{N} \to \mathbb{Z}$. The set $\mathbb{Q}$ is countably infinite, so there exists a bijection $g : \mathbb{N} \to \mathbb{Q}$. Because f, g are bijections, f^{-1}, g^{-1} and are bijections (because they too have inverses, namely f, g, respectively). But as the composition of bijective functions are bijective, we know that $g \circ f^{-1} : \mathbb{Z} \to \mathbb{Q}$ is a bijection. Therefore, $\mathbb{Z}$ and $\mathbb{Q}$ have the same cardinality.

As another proof, by the Cantor-Schröder-Bernstein Theorem to prove there exists a bijection from $\mathbb{Z}$ to $\mathbb{Q}$, it suffices to prove there are injections $f : \mathbb{Z} \to \mathbb{Q}$ and $g : \mathbb{Q} \to \mathbb{Z}$. Let $f : \mathbb{Z} \to \mathbb{Q}$ be given by $f(x) := x$, i.e. taking advantage of the fact that $\mathbb{Z} \subseteq \mathbb{Q}$. Clearly, f is injective: if $x = f(x) = f(y) = y$, then $x = y$. Now define $g : \mathbb{Q} \to \mathbb{Z}$ be given as follows: if $q \in \mathbb{Q}$, write $q = a/b$ for some $a, b \in \mathbb{Z}$. Without loss of generality, assume that $\gcd(a, b) = 1$ and either $a, b \geq 0$ or $a < 0$ and $b \geq 0$; that is, assume a, b are relatively prime and that if $q \geq 0$, then a_1, b_1 are chosen to be nonnegative and if $q < 0$, then a is chosen to be negative while b is chosen to be nonnegative. Then define $g : \mathbb{Q} \to \mathbb{Z}$ via

$$g(q) = \begin{cases} 2^a 3^b, & q \geq 0 \\ -2^{-a} 3^b, & q < 0 \end{cases}$$

It is clear that if $q \geq 0$, then $g(q) \in \mathbb{Z}$. If $q = a/b < 0$, then $a < 0$ so that $-a > 0$. But then $-2^{-a}3^b \in \mathbb{Z}$ so that $g(q) \in \mathbb{Z}$. Note that $g(q) \notin \{\pm 1\}$ because this would require $a = b = 0$, but because $q = a/b$, we know $b \neq 0$.

We claim that g is injective. Suppose that $g(q_1) = g(q_2)$, where $q_1, q_2 \in \mathbb{Q}$ with $q_1 = a_1/b_1$, $q_2 = a_2/b_2$ and $a_1, b_1, a_2, b_2 \in \mathbb{Z}$ are chosen as above. Obviously, $g(q_1)$ and $g(q_2)$ must have the same sign. By cancelling negatives, we may assume without loss of generality that $q_1, q_2 \geq 0$. But then $g(q_1) = 2^{a_1} 3^{b_1} = 2^{a_2} 3^{b_2} = g(q_2)$. By the uniqueness of factorization for integers, the number of factors of 2 and 3 on the left and right side of the equality must be the same, respectively. But then $a_1 = a_2$ and $b_1 = b_2$. But then $q_1 = a_1/b_1 = a_2/b_2 = q_2$ so that g is injective.

Quiz 13. *True/False*: The relation on $\mathbb{N}$ given by $x \sim y$ if and only if xy is even is an equivalence relation.

Solution. The statement is *false*. For $\sim$ to be an equivalence relation, $\sim$ must be reflexive, i.e. $n \sim n$ for all $n \in \mathbb{N}$. Take $n = 1$. Then $1(1) = 1$ is odd so that $1 \not\sim 1$. But then $\sim$ is not reflexive.

Quiz 14. *True/False*: Suppose that X is a set of natural numbers and $\sim$ is an equivalence relation on X . If $[2] \cap [5] = \{1, 2, 3, 4, 5, 7\}$, then $[2] = \{1, 2, 3, 4, 5, 7\}$.

Solution. If $(X, \sim)$ is an equivalence relation, then all equivalence classes are either disjoint or equal, i.e. if $[a], [b]$ are equivalence classes, then either $[a] \cap [b] = \emptyset$ or $[a] = [b]$. Observe that $[2] \cap [5] \neq \emptyset$. But then $[2] = [5]$. Therefore,

$$[2] = [2] \cap [2] = [2] \cap [5] = \{1, 2, 3, 4, 5, 7\}$$

Quiz 15. *True/False*: There exists integers x, y such that $6x + 8y = 5$.

Solution. The statement is *false*. Recall that given integers A and B, there exist integers x, y such that $Ax + By = D$ if and only if $\gcd(A, B)$ divides D. But we know $\gcd(6, 8) = 2$, which does not divide 5.

Quiz 16. *True/False*: The last digit of 11^{2021} is 1.

Solution. The statement is *true*. To find the last digit of 11^{2021}, we compute the remainder upon division by 10, i.e. we compute 11^{2021} modulo 10. Observe that $11 \equiv 1 \mod 10$. But then we have $11^{2021} \equiv 1^{2021} = 1 \mod 10$. Therefore, the last digit of 11^{2021} is 1.

Quiz 17. *True/False*: 7 is invertible modulo 63.

Solution. The statement is *false*. Recall that a is invertible modulo n if and only if $\gcd(a, n) = 1$. To see this, if a is invertible modulo n, then there exists an integer x such that $ax \equiv 1 \mod n$. But then $ax - 1$ is divisible by n, i.e. there exists an integer y such that $ax - 1 = ny$. But then $ax + ny = 1$. But this is possible if and only if $\gcd(a, n)$ divides 1, i.e. $\gcd(a, n) = 1$. Conversely, if $\gcd(a, n) = 1$, then there exists integers x, y such that $ax + ny = 1$. Taking this equation modulo n, we immediately see that $ax \equiv 1 \mod n$. But then x is the inverse of a modulo n, i.e. a is invertible. Now observe $\gcd(7, 63) = 7$. Therefore, 7 is not invertible modulo 63.

Quiz 18. *True/False*: If $P(A \cap B) = 0$, then A and B are independent.

Solution. The statement is *false*. If $P(A \cap B) = 0$, then A and B are disjoint events. But disjoint events can never be independent. If events are disjoint, then the events cannot happen at the same time. But then if one of the events occurs, the other cannot. If the other event is not impossible, then the occurrence or non-occurrence of one event affects the probability of the other occurring, which cannot for independent events.