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MATH 361
Spring 2024
HW 2: Due 02/01

"You say impossible, but all I hear is, 'I'm possible.'"
— Ted Lasso, Ted Lasso

Problem 1. (10pts) Showing all your work and fully justifying your reasoning, compute the following:

(a) $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{\cos(5x)}$

(b) $\displaystyle\frac{d}{dx}\left(\frac{xe^x}{x^2+1}\right)$

(c) $\displaystyle\frac{d^2}{dx^2}(x^2+5)^{10}$

(d) $\displaystyle\int xe^x\,dx$

Solution.

(a)

$$\lim_{x\to 0}\frac{\sin(3x)}{\cos(5x)} = \frac{\sin(0)}{\cos(0)} = \frac{0}{1} = 0$$

(b)

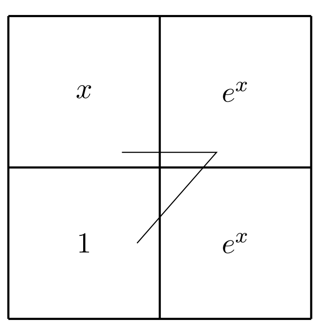
$$\frac{d}{dx}\left(\frac{xe^x}{x^2+1}\right) = \frac{(x^2+1)(e^x+xe^x)-(2x)(xe^x)}{(x^2+1)^2} = \frac{e^x\big((x^2+1)(1+x)-2x^2\big)}{(x^2+1)^2} = \frac{e^x(x^3-x^2+x+1)}{(x^2+1)^2}$$

(c)

$$\frac{d}{dx}(x^2+5)^{10} = 10(x^2+5)^9\cdot 2x = 20x(x^2+5)^9$$

$$\begin{aligned}\frac{d^2}{dx^2}(x^2+5)^{10} = \frac{d}{dx}\left(\frac{d}{dx}(x^2+5)^{10}\right) = \frac{d}{dx}\big(20x(x^2+5)^9\big) &= 20(x^2+5)^9 + 20x\cdot\big(9(x^2+5)^8\cdot 2x\big)\\ &= 20(x^2+5)^9 + 360x^2(x^2+5)^8\\ &= 20(x^2+5)^8(19x^2+5)\end{aligned}$$

(d)



$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

Problem 2. (10pts) One of the first 'non-trivial' approximation techniques one learns is the process of linearization. Recall that if $f(x)$ is differentiable at c, the linearization of $f(x)$ at c, denoted $L(x)$, is the tangent line of $f(x)$ at $x = c$. But then for $x \approx c$, we have $f(x) \approx L(x)$. Consider the function $f(x) = \sqrt{x}$.

(a) Find the linearization of $f(x)$ at $x = 144$.

(b) Use (a) to approximation $\sqrt{150}$. What is the error for your approximation?

(c) Is this generally a useful method for computing $f(x) = \sqrt{x}$? Explain.

Solution.

(a) The linearization of $f(x)$ at $x = 144$ is the tangent line of $f(x)$ at $x = 144$. We have...

$$f(144) = \sqrt{144} = 12$$

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(144) = \frac{1}{2\sqrt{144}} = \frac{1}{24}$$

The linearization is then...

$$L(x) = f(144) + f'(144)\big(x - 144\big) = 12 + \frac{1}{24}\big(x - 144\big) = \frac{x}{24} + 6$$

(b) We have...

$$\sqrt{150} = f(150) \approx L(150) = 12 + \frac{1}{24}\big(150 - 144\big) = 12 + \frac{6}{24} = 12 + \frac{1}{4} = 12.25$$

We know that $\sqrt{150} \approx 12.2474$. The error in the approximation is then $|12.2474 - 12.25| = 0.0026$ and the relative error is $\frac{|12.2474-12.25|}{|12.2474|} = 2.12\%$.

(c) The approximation in (b) requires us to be able to construct the tangent line to $f(x) = \sqrt{x}$ at $x = c$, which required us to know $f(c)$. But then we require c to be a perfect square. Then to approximate $\sqrt{r}$, we need r to be 'close' to a perfect square. However, as $n \to \infty$, the difference between the nth perfect square and the $(n + 1)$th perfect square tends to infinity. Therefore for 'most' r, r will not be 'close' to a perfect square. It is then likely that for such r, $\sqrt{r} = f(r) \not\approx L(r)$.[1]

[1]This is offset by the fact that $f'(x) = \frac{1}{2\sqrt{x}} \to 0$ as $x \to \infty$, so that the fact that r and nearest perfect square might have a large difference might be offset. In fact, this is still not the case. Suppose that $c = n^2$ is a perfect square. The difference between c and the next perfect square is $(n + 1)^2 - n^2 = 2n + 1$. Suppose r is one of the integers closest to $\frac{(n+1)^2+n^2}{2} = \frac{2n^2+2n+1}{2} = n^2 + n + \frac{1}{2}$, i.e. $n^2 + n$ or $n^2 + n + 1$. The tangent line to $f(x)$ at $x = n^2$ is $L(x) = f(n^2)+f'(n^2)(x-n^2) = n+\frac{1}{2n}(x-n^2)$. Take $r = n^2+n$. Then $L(r) = L(n^2+n) = n+\frac{1}{2n}(n^2+n-n^2) = n+\frac{1}{2}$. So if r is large (so that n must too be large), this has error $|\sqrt{r} - L(r)| = |\sqrt{n^2+n} - n - \frac{1}{2}| = |\sqrt{n^2(1+\frac{1}{n})} - n - \frac{1}{2}| = |n\sqrt{1+\frac{1}{n}} - n - \frac{1}{2}| \approx |n\sqrt{1+0} - n + \frac{1}{2}| = |n - n + \frac{1}{2}| = \frac{1}{2}$. But then we cannot use the linearization to approximate $\sqrt{r}$ for values between 'large' perfect squares with arbitrary accuracy.

Problem 3. (10pts) Another of the first 'non-trivial' approximation techniques one learns is Taylor series. The Taylor series of a function can be used to approximate values of the function. In fact, the (infinite) Taylor series can be exactly equal to the function. Consider the polynomial $f(x) = x^3 - 5x^2 + 7$.

(a) Find the Taylor Series for $f(x)$ at $x = 1$.

(b) Show your Taylor Series in (a) is exactly $f(x)$.

(c) Assuming that $(x-1)^n$ is 'negligible' whenever $n > 1$ and $x \approx 1$, use (b) to approximate $f(1.01)$. What is the error for this approximation?

Solution.

(a) Recall that the Taylor series of $f(x)$ at $x = c$ (if it exists) is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$. We have...

$$\begin{aligned}
f(1) &= 1 - 5 + 7 = 3 \\
f'(x) &= 3x^2 - 10x \Rightarrow f'(1) = 3 - 10 = -7 \\
f''(x) &= 6x - 10 \Rightarrow f''(1) = 6 - 10 = -4 \\
f'''(x) &= 6 \Rightarrow f'''(1) = 6
\end{aligned}$$

Clearly, $f^{(n)}(x) = 0$ for $n \geq 4$. But then we have...

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \sum_{n=4}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \\
&= 3 - 7(x-1) + \frac{-4}{2}(x-1)^2 + \frac{6}{6}(x-1)^3 + 0 \\
&= 3 - 7(x-1) - 2(x-1)^2 + (x-1)^3
\end{aligned}$$

(b) We have...

$$3 - 7(x-1) - 2(x-1)^2 + (x-1)^3 = 3 + (-7x+7) + (-2x^2+4x-2) + (x^3-3x^2+3x-1) = x^3 - 5x^2 + 7$$

(c) We have...

$$f(1.01) = 3 - 7(1.01-1) - 2(1.01-1)^2 + (1.01-1)^3 \approx 3 - 7(1.01-1) - 0 + 0 = 3 - 7(0.01) = 3 - 0.07 = 2.93$$

We know that $f(1.01) = 2.929801$. But then the error in our approximation is $|2.929800 - 2.93| = 0.000199$ and the relative error is $\dfrac{|2.929800 - 2.93|}{|2.929800|} = 0.00679\%$.

Problem 4. (10pts) Taylor series can also be used to approximate integrals that are not exactly computable. For instance, to find the percentage of values within one standard deviation of the mean for a normal distribution one would need to compute...

$$\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-x^2/2}\,dx$$

However, the integral $\displaystyle\int e^{-x^2/2}\,dx$ has no elementary antiderivative. Therefore, approximation must be used. Recall the Maclaurin series for e^x is $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and that this series has an infinite radius of convergence.

(a) Find the Maclaurin series for $e^{-x^2/2}$. Show that this series converges to $e^{-x^2/2}$ everywhere.

(b) Let $T_3(x)$ denote the first three nonzero terms from your series in (a). Approximate the integral above by using the fact that $e^{-x^2/2} \approx T_3(x)$ on $[-1, 1]$.

(c) It is a well-known fact in Statistics that approximately 68% of values in a normal distribution are within one standard deviation of the mean. Does your answer in (b) agree with this fact?

Solution.

(a) We know that the Maclaurin series for e^x is $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ with infinite radius of convergence. But then the Maclaurin series for $e^{-x^2/2}$ is $\displaystyle\sum_{n=0}^{\infty}\frac{(-x^2/2)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n n!}$. The radius of convergence is given by $R = 1/L$, where L is the following limit:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{2n+2}}{2^{n+1}(n+1)!}}{\frac{x^{2n}}{2^n n!}}\right| = \left|\frac{x^{2n+2}2^n n!}{x^{2n}2^{n+1}(n+1)!}\right| = \left|\frac{x^2}{2(n+1)}\right| = 0$$

In the case where $L = 0$, we take $R = \infty$. Therefore, this Taylor series converges to $e^{-x^2/2}$ for all x, i.e. $e^{-x^2/2} = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n n!}$ for all x.

(b) We have...

$$T_3(x) = \frac{x^0}{2^0\cdot 0!} - \frac{x^2}{2\cdot 1!} + \frac{x^4}{4\cdot 2!} = 1 - \frac{x^2}{2} + \frac{x^4}{8}$$

But then we have...

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-x^2/2}\,dx &\approx \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\left(1 - \frac{x^2}{2} + \frac{x^4}{8}\right)\,dx \\
&= \frac{1}{\sqrt{2\pi}}\left(x - \frac{x^3}{6} + \frac{x^5}{40}\right)\Big|_{-1}^{1} \\
&= \frac{1}{\sqrt{2\pi}}\big(0.858333 - (-0.858333)\big) \\
&= \frac{1.71667}{\sqrt{2\pi}} \\
&= 0.684851
\end{aligned}$$

(c) The integral from (b) computes the percentage of values within one standard deviation of the mean. We know that this is approximately 68.4851% from the work above. This agrees with the fact that approximately 68% of the values should be within one standard deviation of the mean.