Name: *Caleb McWhorter — Solutions*
MATH 308
Fall 2021
HW 4: Due 10/08

"I see the muscle shirt came today. Muscles coming tomorrow?"
–Wayne, Letterkenny

Problem 1. (10pt) Read Keith Conrad's "Advice on Mathematical Writing." What are some things that you learned about good mathematical exposition that you may have otherwise thought?

Problem 2. (10pt) Watch 3Blue1Brown's "The unexpectedly hard windmill question (2011 IMO, Q2)" and "The hardest problem on the hardest test." What proof strategies do you believe these videos exhibit?

Problem 3. (10pt) Prove that for $n \in \mathbb{N}$, if $n^2 + (n+1)^2 = (n+2)^2$, then $n = 3$.

Problem 4. (10pt) Recall that an integer n is called *even* if there is an integer k such that $n = 2k$ and called odd if there is an integer k such that $n = 2k + 1$. Prove that the product of two odd integers is odd.

Problem 5. (10pt) Rewrite the proof below to be shorter using either "without loss of generality" or "mutatis mutandis":

Theorem. *For all* $a, b \in \mathbb{R}$, $|ab| = |a|\,|b|$.

Proof.

Case 1 $(a, b \geq 0)$: Here $|a| = a$, $|b| = b$, and $ab \geq 0$. But then $|ab| = ab = |a|\,|b|$.

Case 2 $(a < 0, b \geq 0)$: Here $|a| = -a$ and $|b| = b$. If $b = 0$, then $|b| = 0$ and $ab = 0$. But then $|ab| = |0| = 0 = -ab = |a|\,|b|$. Otherwise, $b > 0$ and then $ab < 0$. Then $|ab| = -ab = |a|\,|b|$.

Case 3 $(a \geq 0, b < 0)$: Here $|a| = a$ and $|b| = -b$. If $a = 0$, then $|a| = 0$ and $ab = 0$. But then $|ab| = |0| = 0 = -ab = |a|\,|b|$. Otherwise, $a > 0$ and then $ab < 0$. Then $|ab| = -ab = |a|\,|b|$.

Case 4 $(a, b < 0)$: Here $|a| = -a$, $|b| = -b$, and $ab > 0$. Then $|ab| = ab = (-a)(-b) = |a|\,|b|$. $\square$

Problem 6. (10pt) By mimicking the proof that $\sqrt{2}$ is irrational, prove that $\sqrt{p}$ is irrational for any prime p.

Problem 7. (10pt) Consider the checkerboard below that has two squares from each corner removed from the board. Prove that this board cannot be covered with the 'T-shapes' (or its rotations) shown on the right.