

**Quiz 1. True/False:** If  $P$  is the proposition  $6 < 5$  and  $Q$  is the proposition, “Earth is a planet,” then the logical statement  $P \rightarrow Q$  is false.

**Solution.** The statement is *false*. Recall that the truth table for  $P \rightarrow Q$  is as follows:

$P$	$Q$	$P \rightarrow Q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$T$
$F$	$F$	$T$

Here,  $P$  is the proposition  $P : 6 < 5$  and  $Q$  is the proposition  $Q$ : “Earth is a planet.” It is clear that  $P$  is false and  $Q$  is true. But then examining the logic table above, we can see that  $P \rightarrow Q$  is true.

**Quiz 2. True/False:**  $\neg(P \rightarrow \neg Q) \equiv P \wedge Q$

**Solution.** The statement is *true*. To determine if two propositions are logically equivalent, one can either examine the truth table or apply logical rules to obtain one logical expression from the other. If we construct a truth table, we have...

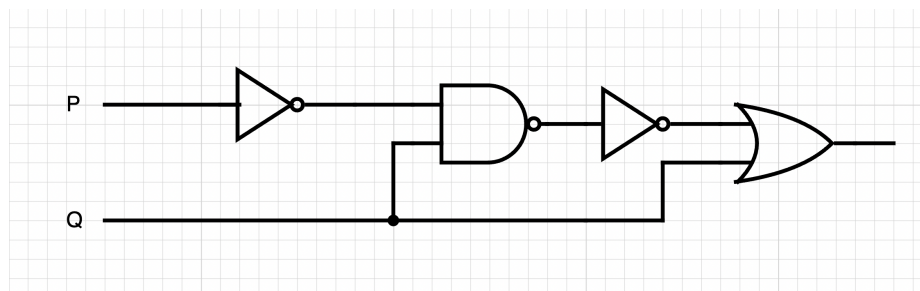
$P$	$Q$	$\neg Q$	$P \rightarrow \neg Q$	$\neg(P \rightarrow \neg Q)$	$P \wedge Q$
$T$	$T$	$F$	$F$	$T$	$T$
$T$	$F$	$T$	$T$	$F$	$F$
$F$	$T$	$F$	$T$	$F$	$F$
$F$	$F$	$T$	$T$	$F$	$F$

Because for each possible pair of choices for  $P$  and  $Q$  the outputs for  $\neg(P \rightarrow \neg Q)$  and  $P \wedge Q$  match,  $\neg(P \rightarrow \neg Q) \equiv P \wedge Q$ . Alternatively, we can transform one into the other by applying logical equivalences (recall  $P \rightarrow Q \equiv \neg P \vee Q$  or  $\neg(P \rightarrow Q) \equiv P \wedge \neg Q$ ):

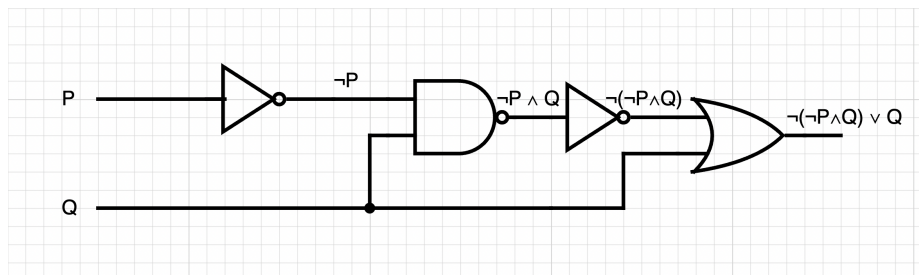
$$\neg(P \rightarrow \neg Q) \equiv \neg(\neg P \vee \neg Q) \equiv \neg(\neg P) \wedge \neg(\neg Q) \equiv P \wedge Q.$$

**Quiz 3. True/False:** The logic corresponding to the circuit shown below is the proposition:

$$(\neg P \wedge Q) \vee \neg Q.$$



**Solution.** The statement is *false*. We can trace through the circuit. We see that the current from  $P$  passes through a NOT gate and we obtain  $\neg P$ . This then feeds into an AND gate along with  $Q$  so that we obtain  $\neg P \wedge Q$ . The resulting current is then passed through a NOT gate, obtaining  $\neg(\neg P \wedge Q)$ . This finally reaches an OR gate—along with  $Q$ —to obtain  $\neg(\neg P \wedge Q) \vee Q$ . We can see a diagrammatic explanation below.



**Quiz 4.** *True/False:* Let the universe  $\mathcal{U}$  be the set of real numbers and define  $P(x)$  to be the predicate  $P(x) : x^2 + x - 4 \geq 0$ . Then  $(\forall x)(\neg P(x))$  is true.

**Solution.** The statement is *false*. If  $P(x) : x^2 + x - 4 \geq 0$ , then  $\neg P(x) : x^2 + x - 4 < 0$ . But then  $(\forall x)(\neg P(x))$  is the statement, “For all  $x$ ,  $x^2 + x - 4 < 0$ .” Now if  $x = 1$ , we have  $\neg P(1) : 1^2 + 1 - 4 < 0$ , i.e.  $-2 < 0$ , which is true. If  $x = 0$ , we have  $\neg P(0) : 0^2 + 0 - 4 < 0$ , i.e.  $-4 < 0$ , which is true. However, while  $(\forall x)(\neg P(x))$  is clearly true for *some* (we found at least two), it is not true *for all*  $x$ . As a counterexample, let  $x = 10$ . Then  $\neg P(10) : 10^2 + 10 - 4 < 0$ , which is  $104 < 0$ —clearly false. Therefore,  $\neg P(x)$  is not true for all  $x$ . But then  $(\forall x)(\neg P(x))$  is false.

**Quiz 5.** *True/False:* Let the domain of  $x, y$  be the integers. Then  $(\exists! x)(\forall y)(x + 2y = 5)$ .

**Solution.** The statement is *false*. The logical proposition  $(\exists! x)(\forall y)(x + 2y = 5)$  in words states, “There exists a unique  $x$  such that for all  $y$ ,  $x + 2y = 5$ .” Suppose that there were such a  $x$ , say  $x_0$ . Then we know that  $x_0 + 2y = 5$  for all  $y$ . In particular,  $x_0$  satisfies this equality when  $y = 0$ . But then we know that  $x_0 = 5$ . But also, it must satisfy the equality when  $x = 1$ . But then  $x_0 + 2 = 5$  so that  $x_0 = 3$ . Then there is not a unique  $x$  that works for all  $y$ ! Therefore, the statement is false. Note that if we reverse the quantifiers, the statement is true:  $(\forall y)(\exists! x)(x + 2y = 5)$ . In this case, this is the statement, “For all  $y$ , there exists a unique  $x$  such that  $x + 2y = 5$ .” If you were given any  $y$ , define  $x_0 := 5 - 2y$ . But then  $x + 2y = (5 - 2y) + 2y = 5$ . So there exists such an  $x$ . Is it unique? Well if there were two or more  $x$  values that worked for some  $y$ , say two of them are  $x_0$  and  $\tilde{x}_0$ , then we have  $x_0 + 2y = 5 = \tilde{x}_0 + 2y$ . But then  $x_0 + 2y = \tilde{x}_0 + 2y$ . Subtracting  $2y$ , we have  $x_0 = \tilde{x}_0$ . Therefore, there can only be one such  $x$ . Because we have found one, we know that the statement that for all  $y$ , there exists a unique  $y$  such that  $x + 2y = 5$  is true.