Name: *Caleb McWhorter — Solutions*
MATH 101
Fall 2023
HW 4: Due 09/20

"I'm not afraid of hard work. I just don't like it."
–Bob Belcher, Bob's Burgers

Problem 1. (10pt) Showing all your work, compute the following "without a calculator":

(a) $\sqrt[4]{256}$

(b) $\sqrt[3]{-125}$

(c) $\left(\dfrac{49}{36}\right)^{-1/2}$

(d) $\sqrt{\dfrac{1}{4}}$

(e) $216^{2/3}$

Solution.

(a) The prime factorization of 256 is $256 = 2^8$. But then...

$$\sqrt[4]{256} = \sqrt[4]{2^8} = (2^8)^{1/4} = 2^2 = 4$$

(b) The prime factorization of $125 = 5^3$. But then...

$$\sqrt[3]{-125} = \sqrt[3]{-5^3} = -5$$

(c) The prime factorizations of 49 and 36 are $49 = 7^2$ and $36 = 2^2 \cdot 3^2$. But then...

$$\left(\frac{49}{36}\right)^{-1/2} = \left(\frac{36}{49}\right)^{1/2} = \frac{36^{1/2}}{49^{1/2}} = \frac{(2^2 \cdot 3^2)^{1/2}}{(7^2)^{1/2}} = \frac{2 \cdot 3}{7} = \frac{6}{7}$$

(d) The prime factorization of 4 is $4 = 2^2$. But then...

$$\sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{\sqrt{2^2}} = \frac{1}{2}$$

(e) The prime factorization of 216 is $216 = 2^3 \cdot 3^3$. But then...

$$216^{2/3} = (2^3 \cdot 3^3)^{2/3} = \left((2^3 \cdot 3^3)^{1/3}\right)^2 = (2 \cdot 3)^2 = 6^2 = 36$$

Problem 2. (10pt) Showing all your work and completely justifying your reasoning, estimate $\sqrt[4]{101}$ without a calculator.

Solution. If $x = \sqrt[4]{101}$, then $x^4 = 101$. Now observe...

$$\begin{aligned} 1^4 &= 1 \\ 2^4 &= 16 \\ 3^4 &= 81 \\ 4^4 &= 256 \end{aligned}$$

But then $3^4 = 81 < x^4 = 101 < 256 = 4^4$. This shows that $3 < x < 4$, i.e. $3 < \sqrt[4]{101} < 4$.

We can further estimate $\sqrt[4]{101}$ by bisecting this interval. We have $3.5 = \frac{35}{10}$, so that $3.5^4 = \frac{35^4}{10^4} = \frac{1500620}{10000} = 150.0625 > 101$. But then we know that $3^4 = 81 < x^4 = 101 < 150.0625 = 3.5^4$, which implies $3 < \sqrt[4]{101} < 3.5$.

Yet again, we can further estimate $\sqrt[4]{101}$ by bisecting this interval. We have $3.25 = \frac{325}{100}$, so that $3.25^4 = \frac{325^4}{100^4} = \frac{11156640625}{100000000} = 111.56640625 > 101$. But then we know that $3^4 = 81 < x^4 = 101 < 111.56640625 = 3.25^4$, which implies $3 < \sqrt[4]{101} < 3.25$.

We can continue this process ad infinitum. However, stopping here, we can estimate $\sqrt[4]{101} \approx \dfrac{3 + 3.25}{2} = 3.125$. The true value of $\sqrt[4]{101}$ is ≈ 3.17015388. But then we have estimated $\sqrt[4]{101}$ with an error of only ≈ 0.0451539—a percentage error of only $\approx 1.42\%$.

Problem 3. (10pt) Simplify the following:

(a) $\sqrt{\dfrac{(xy^2)^3}{xy^{-8}}}$

(b) $\left(\dfrac{x^9y^{-1}(xy^5)^2}{x^{-1}y}\right)^{-1/2}$

(c) $\left(\sqrt[3]{\dfrac{xy(x^{-3}y^5)^{-2}}{x^{-2}y^5}}\right)^{-2}$

Solution.

(a)

$$\sqrt{\frac{(xy^2)^3}{xy^{-8}}} = \left(\frac{(xy^2)^3}{xy^{-8}}\right)^{1/2} = \left(\frac{x^3y^6}{xy^{-8}}\right)^{1/2} = \left(\frac{x^3y^6y^8}{x}\right)^{1/2} = (x^2y^{14})^{1/2} = x^{2/2}y^{14/2} = xy^7$$

(b)

$$\begin{aligned}
\left(\frac{x^9y^{-1}(xy^5)^2}{x^{-1}y}\right)^{-1/2} &= \left(\frac{x^9y^{-1}\cdot x^2y^{10}}{x^{-1}y}\right)^{-1/2} \\
&= \left(\frac{x^1\cdot x^9\cdot x^2y^{10}}{y^1\cdot y}\right)^{-1/2} \\
&= \left(\frac{x^{12}y^{10}}{y^2}\right)^{-1/2} \\
&= \left(\frac{y^2}{x^{12}y^{10}}\right)^{1/2} \\
&= \left(\frac{1}{x^{12}y^8}\right)^{1/2} \\
&= \frac{1}{x^{12/2}y^{8/2}} \\
&= \frac{1}{x^6y^4}
\end{aligned}$$

(c)

$$\left(\sqrt[3]{\frac{xy(x^{-3}y^5)^{-2}}{x^{-2}y^5}}\right)^{-2} = \left(\left(\frac{xy(x^{-3}y^5)^{-2}}{x^{-2}y^5}\right)^{1/3}\right)^{-2}$$

$$= \left(\frac{xy(x^{-3}y^5)^{-2}}{x^{-2}y^5}\right)^{-2/3}$$

$$= \left(\frac{x^{-2}y^5}{xy(x^{-3}y^5)^{-2}}\right)^{2/3}$$

$$= \left(\frac{x^{-2}y^5}{xy \cdot x^6y^{-10}}\right)^{2/3}$$

$$= \left(\frac{y^{10} \cdot y^5}{x^2 \cdot xy \cdot x^6}\right)^{2/3}$$

$$= \left(\frac{y^{15}}{x^9y}\right)^{2/3}$$

$$= \left(\frac{y^{14}}{x^9}\right)^{2/3}$$

$$= \frac{y^{14\cdot 2/3}}{x^{9\cdot 2/3}}$$

$$= \frac{y^{28/3}}{x^{3\cdot 2}}$$

$$= \frac{y^{28/3}}{x^6}$$

$$= \frac{\sqrt[3]{y^{28}}}{x^6}$$

Problem 4. (10pt) Simplify the following:

(a) $\dfrac{10}{\sqrt{72}}$

(b) $\sqrt{300}$

(c) $\sqrt[3]{360}$

(d) $\sqrt{2^{10} \cdot 3^5 \cdot 5^2 \cdot 11^3}$

(e) $\sqrt[5]{2^{12} \cdot 3^9 \cdot 5^1 \cdot 7^5}$

Solution.

(a)
$$\frac{10}{\sqrt{72}} = \frac{10}{\sqrt{2^3 \cdot 3^2}} = \frac{10}{\sqrt{2^2 2^1 \cdot 3^2}} = \frac{10}{2 \cdot 3\sqrt{2}} = \frac{5}{3\sqrt{2}}$$

(b)
$$\sqrt{300} = \sqrt{2^2 \cdot 3 \cdot 5^2} = 2 \cdot 5\sqrt{3} = 10\sqrt{3}$$

(c)
$$\sqrt[3]{360} = \sqrt[3]{2^3 \cdot 3^2 \cdot 5} = 2\sqrt[3]{3^2 \cdot 5} = 2\sqrt[3]{9 \cdot 5} = 2\sqrt[3]{45}$$

(d)
$$\sqrt{2^{10} \cdot 3^5 \cdot 5^2 \cdot 11^3} = \sqrt{2^{10} \cdot 3^4 3^1 \cdot 5^2 \cdot 11^2\, 11^1} = 2^5 \cdot 3^2 \cdot 5 \cdot 11\sqrt{3 \cdot 11} = 15840\sqrt{33}$$

(e)
$$\sqrt[5]{2^{12} \cdot 3^9 \cdot 5^1 \cdot 7^5} = \sqrt[5]{2^{10} 2^2 \cdot 3^5 3^4 \cdot 5^1 \cdot 7^5} = 2^2 \cdot 3^1 \cdot 7^1 \sqrt[5]{2^2 \cdot 3^4 \cdot 5^1} = 84\sqrt[5]{1620}$$