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MATH 101
Spring 2024
HW 10: Due 03/04

"I hate these nerds! Just 'cause I'm stupider than them they think they're smarter than me."
— Hubert J. Farnsworth, Futurama

Problem 1. (10pts) Let $f(x)$ be the function given by $f(x) = 4x - 5$.

(a) Find a value in the range of f. Be sure to justify why the value is in the range.

(b) Compute $f(-1)$. Is $(-1, -9)$ on the graph of f? Explain.

(c) Is there an x such that $f(x) = 11$? Explain.

(d) Is $2 \in f^{-1}(0)$? Explain.

(e) Assuming f^{-1} exists, what is $f(f^{-1}(\sqrt{2}))$ and $f^{-1}(f(\sqrt{2}))$?

Solution.

(a) The range of f is the set of outputs of f. So we can input any value of x in the domain of f into f to obtain an output. For instance, we have...

$$f(0) = 4(0) - 5 = -5$$

Therefore, -5 is in the range of f.

(b) We have...

$$f(-1) = 4(-1) - 5 = -4 - 5 = -9$$

This implies that $(-1, -9)$ is on the graph of f.

(c) Suppose there were an x_0 such that $f(x_0) = 11$. Then we have...

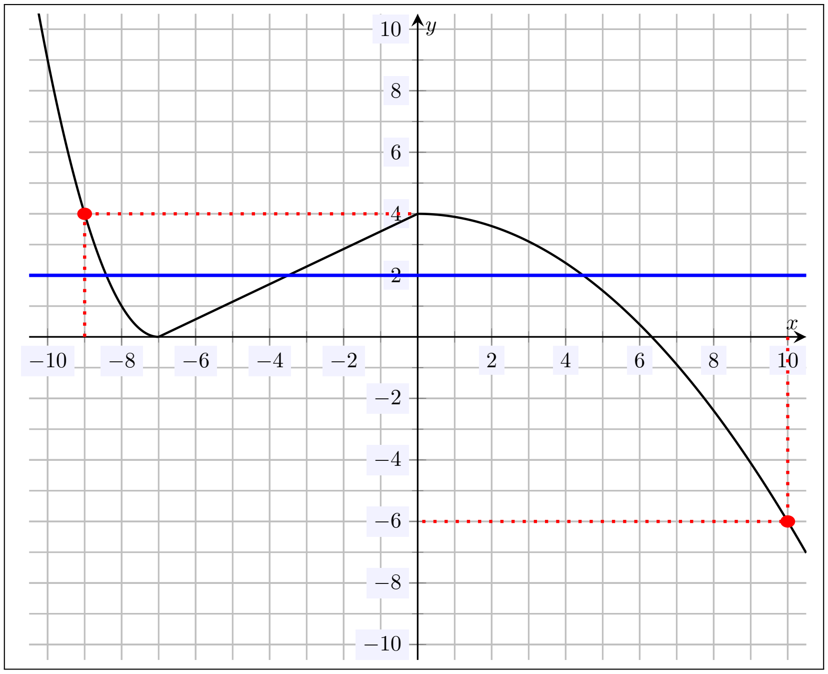
$$\begin{aligned} f(x_0) &= 11 \\ 4x_0 - 5 &= 11 \\ 4x_0 &= 16 \\ x_0 &= 4 \end{aligned}$$

Therefore, $x_0 = 4$ is a value such that $f(x_0) = 11$. We can confirm this: $f(4) = 4(4) - 5 = 16 - 5 = 11$.

(d) If $2 \in f^{-1}(0)$, then $f(2) = 0$. But we have $f(2) = 4(2) - 5 = 8 - 5 = 3 \neq 0$. Therefore, $2 \notin f^{-1}(0)$.

(e) If $f(x)$ has an inverse, then $f\big(f^{-1}(x)\big) = x$ and $f^{-1}\big(f(x)\big) = x$. But then $f(f^{-1}(\sqrt{2})) = \sqrt{2}$ and $f^{-1}(f(\sqrt{2})) = \sqrt{2}$.

Problem 2. (10pts) Consider the relation f plotted below.



(a) Compute $f(-9)$ and $f(10)$.

(b) Is $f(x)$ a function? Explain.

(c) Does $f(x)$ have an inverse? If so, sketch the inverse. If not, explain why.

Solution.

(a) Examining the graph of $f(x)$, we see that $f(-9) = 4$ and $f(10) = -6$.

(b) Yes, $f(x)$ is a function because $f(x)$ passes the Vertical Line Test, i.e. every vertical line intersects the graph of $f(x)$ at most once.

(c) The function $f(x)$ does not have an inverse, i.e. $f^{-1}(x)$ does not exist, because not every horizontal line intersects the graph of $f(x)$ at most once. For instance, the horizontal line at $y = 2$ (in blue) intersects the graph of $f(x)$ more than once.

Problem 3. (10pts) Showing all your work, verify that $g(x) = \frac{1-x}{5}$ is the inverse function for $f(x) = 1 - 5x$. Also, compute $g(6)$. What does the value of $g(6)$ tell you about the function $f(x)$?

Solution. If $g(x)$ is the inverse of $f(x)$, then $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. We have...

$$(f \circ g)(x) = f\big(g(x)\big) = f\left(\frac{1-x}{5}\right) = 1 - 5\left(\frac{1-x}{5}\right) = 1 - (1-x) = x$$

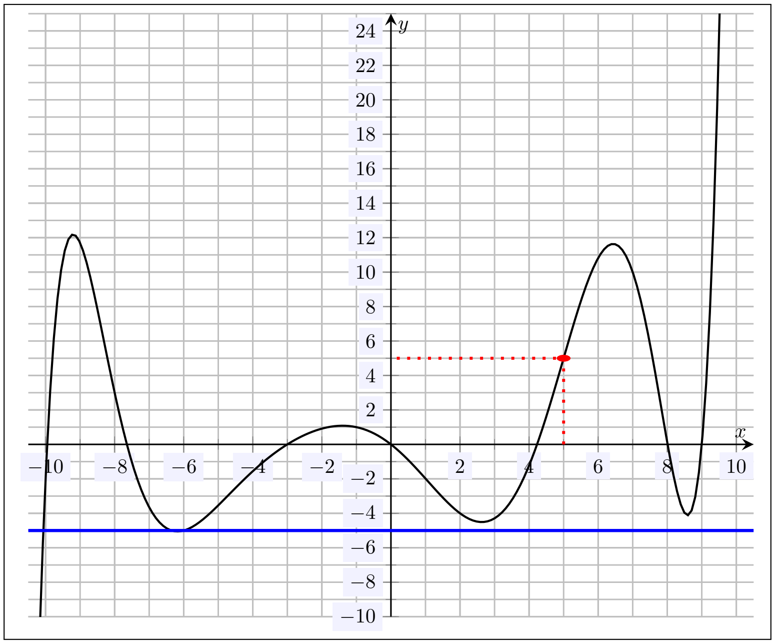
$$(g \circ f)(x) = g\big(f(x)\big) = g(1-5x) = \frac{1-(1-5x)}{5} = \frac{5x}{5} = x$$

Now we have...

$$g(6) = \frac{1-6}{5} = \frac{-5}{5} = -1$$

From our work above, we know that $g = f^{-1}$. But then $-1 = g(6) = f^{-1}(6)$. Recall that $x = f^{-1}(y)$ if and only if $f(x) = y$. Because $f^{-1}(6) = -1$, we must have $f(-1) = 6$.

Problem 4. (10pts) A relation ϕ is plotted below.



Using the plot above, answer the following:

(a) Compute $\phi(5)$.

(b) Find the y-intercept for $\phi(x)$.

(c) Find the x-intercepts for $\phi(x)$.

(d) As accurately as possible, compute the preimage of -5, i.e. $\phi^{-1}(-5)$.

(e) Explain why (d) implies that ϕ does not have an inverse function.

Solution.

(a) Examining the graph, we have $\phi(5) = 5$.

(b) The y-intercept of $\phi(x)$ is the point where the graph of $\phi(x)$ intersects the y-axis. Examining the graph of $\phi(x)$, the y-intercept is $(0, 0)$, i.e. 0.

(c) The x-intercept(s) are the point(s) (if there are any) where the graph of $\phi(x)$ intersects the x-axis. Examining the plot of $\phi(x)$, we see that the x-intercepts of $\phi(x)$ are $x \approx -9.95$, -7.65, -3, 0, 4.23, 8.01, 9.

(d) The preimage of -5 under ϕ, i.e. $\phi^{-1}(5)$ are the x-value(s) (if they exist) such that $\phi(x) = -5$. Examining the graph of $\phi(x)$ (using the blue line at $y = -5$), we see that $\phi^{-1}(5) = \{-10.06, -6.35, -6\}$.

(e) From (d), we know there is more than one possible input such that $\phi(x) = -5$. But then $\phi^{-1}(5)$ cannot be well defined so that ϕ^{-1} is not a function.