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MATH 308
Fall 2023
HW 2: Due 09/12

"I am very seldom interested in applications. I am more interested in the elegance of a problem. Is it a good problem, an interesting problem?"

–Claude Shannon

Problem 1. (10pt) Often, one encounters expressions along the lines, "This happens, unless this." In symbolic form, we could write this as, "P happens, unless Q." Most often, what is meant by this is, "if $\neg Q$, then P." That is, "P unless Q" means $\neg Q \to P$.

(a) Consider the statement, "Payment is due on fifth of the month, unless the fifth of the month is a weekend." Rewrite this statement in "if. . . then" form.

(b) Write the negation of "Payment is due on fifth of the month, unless the fifth of the month is a weekend" as a complete English sentence.

(c) Is payment not being due on the fifth of the month a sufficient condition for the fifth of the month being the weekend? Explain.

(d) Is the fifth of the month being a weekend sufficient for payment not being due on the fifth of the month? Explain.

Solution.

(a) From the problem statement (and from thinking about how we use language), we can see that the statement, "Payment is due on fifth of the month, unless it is a weekend," is equivalent to the sentence, "If the fifth of the month is not a weekend, then payment is due on the fifth of the month." Of course, this is equivalent to the statement, "If the fifth of the month is a weekday, then payment is due on the fifth of the month."

Alternatively, choose P to be the proposition, "Payment is due on the fifth of the month," and Q to be the proposition, "The fifth of the month is a weekend." Then the statement, "Payment is due on fifth of the month, unless the fifth is a weekend," is P unless Q. But from the problem statement, we know this is $\neg Q \to P$. As a sentence, this is the statement, "If the fifth of the month is not a weekend, then payment is due on the fifth of the month."

(b) From (a), we know that the statement of the problem is equivalent to $\neg Q \to P$, where P, Q are chosen as in (a). The negation of this is $\neg(\neg Q \to P) \equiv \neg Q \wedge \neg P$. This is the statement, "The fifth of the month is not a weekend and payment is not due on the fifth of the month." This is equivalent to the statement, "The fifth of the month is a weekday and payment is not due on the fifth of the month."

(c) Using the propositions P, Q chosen in (a), the statement "Payment not being due on the fifth of the month is a sufficient condition for the fifth of the month being the weekend," is $\neg P \to Q$. We have assumed that, "Payment is due on fifth of the month, unless the fifth of the month is a weekend," is true. This was the logical expression $\neg Q \to P$. A conditional statement is logically equivalent to its contrapositive. The contrapositive of the conditional $\neg Q \to P$ is $\neg P \to \neg(\neg Q) \equiv \neg P \to Q$. But then $\neg P$ is a sufficient condition for Q, i.e.

payment not being due on the fifth of the month is a sufficient condition for the fifth to be a weekend. This also shows that the fifth being a weekend is a necessary condition for payment not to be due on the fifth of the month (because Q is necessary for $\neg P$).

(d) Using the propositions P, Q chosen in (a), the statement, “The fifth of the month being a weekend is sufficient for payment not being due on the fifth of the month,” is $Q \to \neg P$. A conditional statement is logically equivalent to its contrapositive. The contrapositive of the conditional statement $Q \to \neg P$ is $\neg(\neg P) \to \neg Q \equiv P \to \neg Q$. The converse of $P \to \neg Q$ is $\neg Q \to P$, which is the expression in the statement of the problem in (a).[1] So although we have assumed $\neg \to P$ to be true, this does not necessarily convey any information about $Q \to \neg P$. So we cannot necessarily know one way or another whether payment is due or not. Payments might only be able to be processed during weekdays, so that the payment will not be due. There may be some exceptional reason that necessitates payment to be due on the weekend. We simply have assumed that if the fifth of the month is not a weekend, then payment will be due on the fifth of the month. We do not have any information about what occurs otherwise.

[1]The inverse of $Q \to \neg P$ is $\neg Q \to P$, which is the same because the inverse of a conditional statement is the contrapositive of the converse of the conditional statement.

Problem 2. (10pt) In everyday speech, “this or that” could mean, “this, that, or both” or it could mean, “this, that, but not both.” Both interpretations are common. “Mathematical OR” *always* allows the possibility for both, i.e. $P \vee Q$ is true if P is true, Q is true, or both P and Q are true. It would be advantageous to have an “exclusive or” for logical expressions. Define “mathematical exclusive or” to be $P \veebar Q$; that is, $P \veebar Q$ is true when exactly one of P, Q is true.

(a) Give the logic table for $P \veebar Q$

(b) Using a logic table, show that $P \veebar Q \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$.

(c) Without computing $P \veebar Q \iff (P \wedge \neg Q) \vee (\neg P \wedge Q)$, determine whether this proposition is a tautology. Explain.

(d) Use (b) to find a logical expression for $\neg(P \veebar Q)$ in terms of P’s, Q’s, $\neg$’s, $\wedge$’s, and $\vee$’s; that is, find a ‘rule’ for determining how to negate $P \veebar Q$. ‘Simplify’ your expression as much as possible.

Solution.

(a) We know that $P \veebar Q$ is true only when exactly one of P, Q is true, i.e. $P \veebar Q$ is false if P and Q are both true or when P and Q are both false. This gives us the following truth table:

P	Q	$P \veebar Q$
T	T	F
T	F	T
F	T	T
F	F	F

(b) We need only show that each output for $P \veebar Q$ and $(P \wedge \neg Q) \vee (\neg P \wedge Q)$, no matter the input for P, Q. Now we have…

P	Q	$\neg P$	$\neg Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$(P \wedge \neg Q) \vee (\neg P \wedge Q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	F	F

Because the output column here, i.e. $(P \wedge \neg Q) \vee (\neg P \wedge Q)$, matches the output column from (a), i.e. $P \veebar Q$, we know that $P \veebar Q \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$. Alternatively, we know that $P \veebar Q \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$ if both logical expressions are true simultaneously. We know $P \veebar Q$ is true only when P is true and Q is false, i.e. $P \wedge \neg Q$, or when P is false and Q is true, i.e. $\neg P \wedge Q$. But this is precisely the right-hand side of the equivalence.

(c) We know that $R \iff S$ is a tautology if and only if $R \equiv S$. From (b), we know that $P \veebar Q \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$. Therefore, it must be that $P \veebar Q \iff (P \wedge \neg Q) \vee (\neg P \wedge Q)$ is a tautology, which is easily verified via a logic table.

(d) We have...

$$\neg(P \veebar Q)$$

$$\neg\big((P \wedge \neg Q) \vee (\neg P \wedge Q)\big)$$

$$\neg(P \wedge \neg Q) \wedge \neg(\neg P \wedge Q)$$

$$\big(\neg P \vee \neg(\neg Q)\big) \wedge \big(\neg(\neg P) \vee \neg Q\big)$$

$$(\neg P \vee Q) \wedge (P \vee \neg Q)$$

$$\big(\neg P \wedge (P \vee \neg Q)\big) \vee \big(Q \wedge (P \vee \neg Q)\big)$$

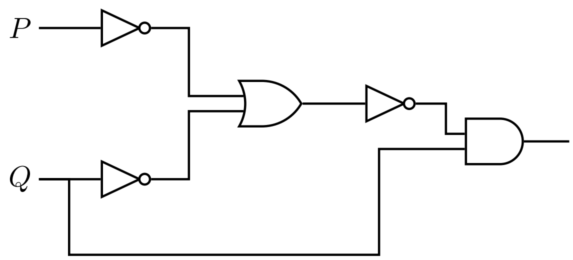
$$((\neg P \wedge P) \vee (\neg P \wedge \neg Q)) \vee ((Q \wedge P) \vee (Q \wedge \neg Q))$$

$$\big(F_0 \vee (\neg P \wedge \neg Q)\big) \vee \big((P \wedge Q) \vee F_0\big)$$

$$(\neg P \wedge \neg Q) \vee (P \wedge Q)$$

But then we have 'rule' $\neg(P \veebar Q) \equiv (\neg P \wedge \neg Q) \vee (P \wedge Q)$. This makes sense as if $P \veebar Q$ is true, then exactly one of P, Q is true. But then $\neg(P \veebar Q)$ is false. We know $P \veebar Q$ is false if P and Q are both false, i.e. $\neg P \wedge \neg Q$, or if P and Q are both true, i.e. $P \wedge Q$, but this is precisely the equivalence we have derived.

Problem 3. (10pt) We can use Boolean representations of circuits to simplify circuits and reduce the amount of wire, chips, etc. required. Consider the circuit below:



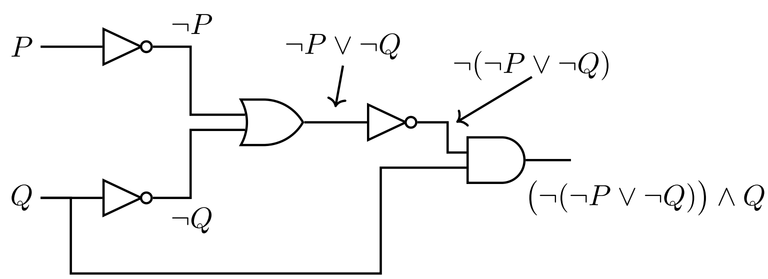
(a) Find the logical expression corresponding to the given circuit.

(b) 'Simplify' the logical expression from (a).

(c) Sketch the circuit from (b).

Solution.

(a) We can slowly 'move through' the circuit, carefully labeling along the way.



From the diagram above, we can see that the logical/Boolean expression equivalent to the circuit above is $\big(\neg(\neg P \vee \neg Q)\big) \wedge Q$.

(b) Simplifying the logical expression from (a), we have...

$$\big(\neg(\neg P \vee \neg Q)\big) \wedge Q$$

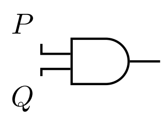
$$\big(\neg(\neg P) \wedge \neg(\neg Q)\big) \wedge Q$$

$$(P \wedge Q) \wedge Q$$

$$P \wedge (Q \wedge Q)$$

$$P \wedge Q$$

(c) Building the circuit from the simplified expression in (b), we have...

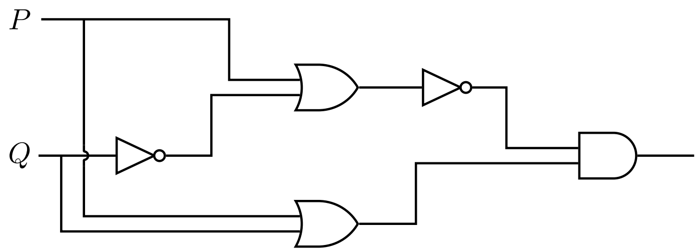


From the work in (a) and (b), we know that the circuit above has the exact same 'behavior', i.e. input-output table, as the circuit in (a)—but with a substantial savings on gates and wiring.

Problem 4. (10pt) Draw a circuit diagram corresponding to the logical expression below and construct its input/output table.

$$\neg(P \vee \neg Q) \wedge (P \vee Q)$$

Solution. Following 'order of operations,' we see that we need to feed Q into a NOT gate. This result must be fed into an OR gate with P. This result must then be put into an inverter, which is then fed into an AND gate. We need to construct the other input to the AND gate. For that, we need to feed P, Q into an OR gate. This result is the other input to the final AND gate. This gives us the circuit below.



Problem 5. (10pt) Watch at least one of the following videos:

- Exploring How Computers Work
- How Do Computers Remember?

Then as thoroughly as possible, comment on what you observed and learned from the video. Be sure to remark as much as possible on how these videos connect to the course content.

Solution.

Solutions will vary.