Problem 1. (10pt) Let $\{a_n\}_{n \in \mathbb{N}}$ be the sequence defined by $a_n := 2^n - 5$ and $\{b_m\}_{m \in \mathbb{Z}^\times}$ be defined by $b_m := \dfrac{m+1}{m}$. Showing all your work, compute the following:

(a) $\displaystyle\sum_{k=0}^{5} a_k$

(b) $\displaystyle\sum_{\substack{j=-3 \\ j \neq 0}}^{3} b_m$

(c) $\displaystyle\prod_{k=1}^{3} a_n$

(d) $\displaystyle\sum_{p=0}^{0} a_p$

(e) $\displaystyle\sum_{j=2}^{4} (a_j + b_j)$

(f) $\displaystyle\prod_{n=1}^{10^{50}} b_n$

Problem 2. (10pt) Let $a \in \mathbb{R}$. Consider the following sum defined for $n > 7$:

$$\sum_{k=7}^{n} (k + a - 7)^2$$

(a) Reindex the sum above so that it begins at $k = 0$.

(b) Using the given summation formulas below, find the sum from (a) in terms of n, a alone.

$$\sum_{k=0}^{n} 1 = n + 1, \qquad \sum_{k=0}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Problem 3. (10pt) Complete the proof of the given proposition below by filling in the corresponding blanks.

Proposition. For $n \geq 2$, $\displaystyle\prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) = \frac{n+1}{2n}$.

Proof. We prove this using _____. First, we establish a base case.

Base Case: Let $n = 2$. Then we have. . .

$$\underline{} = \underline{} = \underline{} = \frac{3}{4}$$

$$\left.\frac{n+1}{2n}\right|_{n=2} = \frac{2+1}{2(2)} = \frac{3}{4}$$

But then if $n = 2$, we know that $\displaystyle\prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) = \frac{n+1}{2n}$.

We know establish the induction step.

Induction Step: Assume that for $n = N$, $\displaystyle\prod_{k=2}^{N} \left(1 - \frac{1}{k^2}\right) = \frac{N+1}{2N}$. We show that the statement of

the proposition is then true for $n = $ _____. We have. . .

$$\prod_{k=2}^{N+1} \left(1 - \frac{1}{k^2}\right) = \underline{} \cdot \prod_{k=2}^{N} \left(1 - \frac{1}{k^2}\right)$$

$$= \underline{} \cdot \underline{}$$

$$= \underline{} \cdot \underline{}$$

$$= \underline{} \cdot \underline{}$$

$$= \frac{N+2}{2(N+1)}$$

$$= \frac{(N+1)+1}{2(N+1)}$$

But then we know that $\displaystyle\prod_{k=2}^{N+1} \left(1 - \frac{1}{k^2}\right) = \frac{(N+1)+1}{2(N+1)}$.

Therefore, by _____, we know that for $n \geq 2$, $\displaystyle\prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) = \frac{n+1}{2n}$.

$\square$

Problem 4. (10pt) Let $\{a_n\}_{n \in \mathbb{Z}^{\geq 0}}$ be the recursive sequence given by $a_0 = 1$, $a_1 = 3$, and $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$. A student observe that $a_0 = 1$, $a_1 = 3$, $a_2 = 5$, $a_3 = 7$, and $a_4 = 9$. They then predict that $a_n = 2n + 1$ for $n \geq 0$. Below is a proof of this conjecture, with parts of their proof removed. Complete the missing parts.

Proposition. Let $\{a_n\}_{n \in \mathbb{Z}^{\geq 0}}$ be the recursive sequence given by $a_0 = 1$, $a_1 = 3$, and $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$. Then for all $n \geq 0$, $a_n = 2n + 1$.

Proof. We prove this using _____. First, we establish a few bases cases.

Base Case: If _____, we have $a_0 = 1$ and $2n + 1 = 2(0) + 1 = 1$. Then if $n = 0$, we have

$a_n = 2n+1$. Now if $n = $ _____, we have _____ and _____.

But then if $n = 1$, we have _____.

We now establish the induction case.

Induction Case: Now assume that $a_k = 2k + 1$ for all $0 \leq k \leq n$. Now consider the term

_____.

We have...
$$a_{n+1} = 2a_n - a_{n-1}$$

$$= \underline{\hspace{5cm}}$$

$$= \underline{\hspace{5cm}}$$

$$= 2n + 3$$

$$= 2(n + 1) + 1$$

But then we know that $a_{n+1} = 2(n + 1) + 1$.

Therefore, by _____, we know that $a_n = 2n + 1$ for all $n \geq 0$.　　□