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MATH 108
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Written HW 4: Due 02/21

"You are braver than you believe, stronger than you seem, and smarter than you think."
–Christopher Robin, Winnie the Pooh

Problem 1. (10pt) The following augmented matrix is in reduced-row echelon form. Determine the solutions (if any).

$$\begin{pmatrix} 1 & 0 & -1 & 2 & 5 \\ 0 & 1 & 0 & 0 & -4 \end{pmatrix}$$

Solution. Using variables x_1, x_2, x_3, x_4, the rows of the matrix correspond to the equations:

$$\begin{aligned} x_1 - x_3 + 2x_4 &= 5 \\ x_2 &= -4 \end{aligned}$$

From the last equation, we see that $x_2 = -4$. Examining the matrix, we see that x_1 is 'fixed' (because it has a pivot entry), while x_3, x_4 are free (because they do not have pivot positions). Solving for x_1 in terms of x_3 and x_4 in the first equation, we find $x_1 = x_3 - 2x_4 + 5$. Therefore, the solutions are...

$$\begin{cases} x_1 = x_3 - 2x_4 + 5 \\ x_2 = -4 \\ x_3 : \text{ free} \\ x_4 : \text{ free} \end{cases}$$

Problem 2. (10pt) The following augmented matrix is in reduced-row echelon form. Determine the solutions (if any).

$$\begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution. Examining the equation corresponding to the last row, we see that $0 = 1$, which is impossible. Therefore, the original system of equations was inconsistent. But then there cannot be a solution to the original system of equations.

Problem 3. (10pt) The following augmented matrix is in reduced-row echelon form. Determine the solutions (if any).

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Solution. Using variables x_1, x_2, x_3, x_4, we write the equations corresponding to the rows of the matrix:

$$\begin{aligned} x_1 &= 4 \\ x_2 &= -5 \\ x_3 &= 0 \\ x_4 &= 1 \end{aligned}$$

Therefore, the solution to the original system of equations is...

$$\begin{cases} x_1 = 4 \\ x_2 = -5 \\ x_3 = 0 \\ x_4 = 1 \end{cases}$$

Problem 4. (10pt) Compute the following determinant:

$$\det\begin{pmatrix} 1 & -1 & 1 & 4 \\ 2 & 1 & 0 & 2 \\ 3 & 0 & 0 & -1 \\ 4 & 2 & -3 & 0 \end{pmatrix}$$

Solution. Because the third row has the greatest number of zero entries, we expand along this row (also, at each stage, we expand along the first row/column with the greatest number of zero entries):

$$\det\begin{pmatrix} 1 & -1 & 1 & 4 \\ 2 & 1 & 0 & 2 \\ 3 & 0 & 0 & -1 \\ 4 & 2 & -3 & 0 \end{pmatrix} = 3\begin{vmatrix} -1 & 1 & 4 \\ 1 & 0 & 2 \\ 2 & -3 & 0 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 & 4 \\ 2 & 0 & 2 \\ 4 & -3 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & -1 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -1 & 4 \\ 2 & 1 & 0 \\ 4 & 2 & -3 \end{vmatrix}$$

$$= 3\left(-1\begin{vmatrix} 1 & 4 \\ -3 & 0 \end{vmatrix} + 0\begin{vmatrix} -1 & 4 \\ 2 & 0 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix}\right) + 0 + 0 + 1\left(-2\begin{vmatrix} -1 & 4 \\ 2 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & 4 \\ 4 & -3 \end{vmatrix} + 0\begin{vmatrix} 1 & -1 \\ 4 & 2 \end{vmatrix}\right)$$

$$= 3\left(-1(0-(-12)) + 0 - 2(3-2)\right) + 1\left(-2(3-8) + 1(-3-16) + 0\right)$$

$$= 3\left(-1(12) + 0 - 2(1)\right) + 1\left(-2(-5) + 1(-19) + 0\right)$$

$$= 3(-12 + 0 - 2) + 1(10 - 19 + 0)$$

$$= 3(-14) + 1(-9)$$

$$= -42 - 9$$

$$= -51$$

Problem 5. (10pt) Consider the following system of equations:

$$\begin{cases} 2x + 3y = 0 \\ -x - 2y = 1 \end{cases}$$

(a) Show that the coefficient matrix has an inverse.

(b) Find the inverse of the coefficient matrix.

(c) Use the coefficient to solve the system of equations.

Solution.

(a) The coefficient matrix is...

$$\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$$

The determinant of this matrix is...

$$\det \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} = 2(-2) - 3(-1) = -4 - (-3) = -4 + 3 = -1 \neq 0$$

Because the determinant is *not* zero, we know that the matrix is invertible, i.e. has an inverse.

(b) If a 2×2 matrix has an inverse, we know that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Using...

$$A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$$

We know that...

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$$

(c) Writing the matrix in vector form, we have...

$$\begin{aligned} \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2(0) + 3(1) \\ -1(0) - 2(1) \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \end{aligned}$$

Therefore, the solution is $(x, y) = (3, -2)$, i.e. $x = 3$ and $y = -2$.

Problem 6. (10pt) Show that the matrix B is the inverse to A:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}$$

$$B = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Solution. Recall that a square $n \times n$ matrix B is an inverse to a square $n \times n$ matrix A if and only if $AB = I$ and $BA = I_n$, where I_n is the $n \times n$ identity matrix. If so, we write $B = A^{-1}$, i.e. B is the inverse of A. We simply check this for the matrix B:

$$\begin{aligned}
AB &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1(1)+0(-1)+1(1) & 1(-1)+0(1)+1(1) & 1(0)+0(1)+1(0) \\ -1(1)+0(-1)+1(1) & -1(-1)+0(1)+1(1) & -1(0)+0(1)+1(0) \\ 2(1)+2(-1)+0(1) & 2(-1)+2(1)+0(1) & 2(0)+2(1)+0(0) \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1+0+1 & -1+0+1 & 0+0+0 \\ -1+0+1 & 1+0+1 & 0+0+0 \\ 2-2+0 & -2+2+0 & 0+2+0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
BA &= \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1(1)+(-1)(-1)+0(2) & 1(0)+(-1)0+0(2) & 1(1)+(-1)1+0(0) \\ -1(1)+1(-1)+1(2) & -1(0)+1(0)+1(2) & -1(1)+1(1)+1(0) \\ 1(1)+1(-1)+0(2) & 1(0)+1(0)+0(2) & 1(1)+1(1)+0(0) \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1+1+0 & 0+0+0 & 1-1+0 \\ -1-1+2 & 0+0+2 & -1+1+0 \\ 1-1+0 & 0+0+0 & 1+1+0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Therefore, B is the inverse of A.

Problem 7. (10pt) Compute the following:

(a) $-3\begin{pmatrix}1 & -1\\ 0 & 3\end{pmatrix}$

(b) $\begin{pmatrix}1 & -1 & 5\\ 3 & 0 & 4\end{pmatrix} - \begin{pmatrix}0 & 5 & -6\\ 1 & 1 & -2\end{pmatrix}$

(c) $\begin{pmatrix}1 & 0\\ -1 & 2\\ 1 & 3\end{pmatrix}\begin{pmatrix}-1 & 4\\ 0 & 1\end{pmatrix}$

Solution.

(a)

$$-3\begin{pmatrix}1 & -1\\ 0 & 3\end{pmatrix} = \begin{pmatrix}-3 & 3\\ 0 & -9\end{pmatrix}$$

(b)

$$\begin{pmatrix}1 & -1 & 5\\ 3 & 0 & 4\end{pmatrix} - \begin{pmatrix}0 & 5 & -6\\ 1 & 1 & -2\end{pmatrix} = \begin{pmatrix}1-0 & -1-5 & 5-(-6)\\ 3-1 & 0-1 & 4-(-2)\end{pmatrix} = \begin{pmatrix}1 & -6 & 11\\ 2 & -1 & 6\end{pmatrix}$$

(c)

$$\begin{aligned}\begin{pmatrix}1 & 0\\ -1 & 2\\ 1 & 3\end{pmatrix}\begin{pmatrix}-1 & 4\\ 0 & 1\end{pmatrix} &= \begin{pmatrix}1(-1)+0(0) & 1(4)+0(1)\\ -1(-1)+2(0) & -1(4)+2(1)\\ 1(-1)+3(0) & 1(4)+3(1)\end{pmatrix}\\ &= \begin{pmatrix}-1+0 & 4+0\\ 1+0 & -4+2\\ -1+0 & 4+3\end{pmatrix}\\ &= \begin{pmatrix}-1 & 4\\ 1 & -2\\ -1 & 7\end{pmatrix}\end{aligned}$$