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MATH 308
Fall 2021
HW 8: Due 10/18

"I was never that great at math, but next to nothing is higher than nothing, right?"

–Dr. Gregory House, House

Problem 1. (20pt) Prove $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2} \right)^2$.

Solution. We prove this with induction. The base case is $n = 1$. We have...

$$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{1} i^3 = 1^3 = 1$$

$$\left(\frac{n(n+1)}{2} \right)^2 \bigg|_{n=1} = \left(\frac{1(1+1)}{2} \right)^2 = \left(\frac{2}{2} \right)^2 = 1^2 = 1$$

Therefore, the result is clearly true if $n = 1$. Now assume the result is true for $n = k$. We need to show the result is true for $n = k + 1$; that is, we want to prove $\displaystyle\sum_{i=1}^{k+1} i^3 = \left(\frac{(k+1)\big((k+1)+1\big)}{2} \right)^2$.

By assumption, we know that $\displaystyle\sum_{i=1}^{k} i^3 = \left(\frac{k(k+1)}{2} \right)^2$. But then...

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= (k+1)^3 + \sum_{i=1}^{k} i^3 \\
&= (k+1)^3 + \left(\frac{k(k+1)}{2} \right)^2 \\
&= (k+1)^3 + \frac{k^2(k+1)^2}{4} \\
&= (k+1)^2 \left((k+1) + \frac{k^2}{4} \right) \\
&= (k+1)^2 \left(\frac{4k+4}{4} + \frac{k^2}{4} \right) \\
&= (k+1)^2 \left(\frac{k^2+4k+4}{4} \right) \\
&= (k+1)^2 \left(\frac{(k+2)^2}{4} \right) \\
&= \left(\frac{(k+1)(k+2)}{2} \right)^2 \\
&= \left(\frac{(k+1)\big((k+1)+1\big)}{2} \right)^2
\end{aligned}$$

Therefore by induction, $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2} \right)^2$.

Problem 2. (20pt) Let $\{a_n\}_{n\in\mathbb{N}}$ be the sequence with $a_1 = 1$, $a_2 = 8$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$.

Solution. We prove this by induction. We test the base case of $n = 1, 2$:

$$n = 1\colon\ 3 \cdot 2^{1-1} + 2(-1)^1 = 3 \cdot 1 + 2(-1) = 3 - 2 = 1 = a_1$$
$$n = 2\colon\ 3 \cdot 2^{2-1} + 2(-1)^2 = 3 \cdot 2 + 2(1) = 6 + 2 = 8 = a_2$$

Now we assume the result is true for $n = 1, 2, \ldots, k$. We need to prove the result is true for $n = k + 1$; that is, we need to prove $a_{k+1} = 3 \cdot 2^{(k+1)-1} + 2(-1)^{k+1} = 3 \cdot 2^k + 2(-1)^{k+1}$. By the inductive hypothesis, we know that $a_k = 3 \cdot 2^{k-1} + 2(-1)^k$ and $a_{k-1} = 3 \cdot 2^{(k-1)-1} + 2(-1)^{k-1} = 3 \cdot 2^{k-2} + 2(-1)^{k-1}$. Observe...

$$\begin{aligned}
a_{k+1} &:= a_{(k+1)-1} + 2a_{(k+1)-2} \\
&= a_k + 2a_{k-1} \\
&= \left(3 \cdot 2^{k-1} + 2(-1)^k\right) + 2\left(3 \cdot 2^{k-2} + 2(-1)^{k-1}\right) \\
&= \left(3 \cdot 2^{k-1} + 2(-1)^k\right) + \left(3 \cdot 2^{k-1} + 2^2(-1)^{k-1}\right) \\
&= 3 \cdot 2^{k-1} + 2(-1)^k + 3 \cdot 2^{k-1} + 2^2(-1)^{k-1} \\
&= \left(3 \cdot 2^{k-1} + 3 \cdot 2^{k-1}\right) + \left(2(-1)^k + 2^2(-1)^{k-1}\right) \\
&= 2 \cdot (3 \cdot 2^{k-1}) + 2(-1)^k\left(1 + 2(-1)^1\right) \\
&= 3 \cdot 2^k + 2(-1)^{k+1}
\end{aligned}$$

Therefore by induction, $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$.

Problem 3. (20pt) Prove that for $n \geq 4$, $n^3 < 3^n$.

Solution. We prove this by induction. The base case is $n = 5$:

$$n^3 = 5^3 = 125$$
$$3^n = 3^5 = 243$$

Therefore, $n^3 < 3^n$ when $n = 5$. Now assume the result is true for $n = 5, 6, \dots, k$. We need to prove the result is true for $n = k + 1$; that is, we need to prove $(k+1)^3 < 3^{k+1}$. From the induction hypothesis, we know that $k^3 < 3^k$. Because $k \geq 5$, we know that $\frac{k}{4} \geq \frac{5}{4} > 1$. Observe that...

$$(k+1)^3 < \left(k + \frac{k}{4}\right)^3 = \left(\frac{5k}{4}\right)^3 = \frac{5^3}{4^3}\,k^3 = \frac{125}{64}\,k^3 < \frac{128}{64}\,k^3 = 2k^3 < 2(3^k) < 3(3^k) = 3^{k+1}$$

Therefore, the result follows by induction.[1]

[1] Alternatively, for the inductive step, we could observe that... $(k+1)^3 = k^3 + 3k^2 + 3k + 1 < 3^k + (3k^2 + 3k + 1)$. If one can prove $3k^2 + 3k + 1 < 2(3^k)$ for $k \geq 4$, then $(k+1)^3 < 3^k + 2(3^k) = 3(3^k) = 3^{k+1}$ and the result would follow. This itself can be proven by induction, which is left as a similar exercise.

Problem 4. (20pt) Recall that an integer m is divisible by 3 if $m = 3q$ for some $q \in \mathbb{Z}$. Prove that $7^n - 4^n$ is divisible by 3 for all $n \in \mathbb{Z}_{\geq 0}$.

Solution. We prove this by induction on n. For $n = 0$, we have $7^n - 4^n = 7^0 - 4^0 = 1 - 1 = 0$. Clearly, 0 is divisible by 3 because $0 = 3(0)$. Now assume the result is true for $n = k$. We need to prove the result is true for $n = k + 1$; that is, we need to prove that $7^{k+1} - 4^{k+1}$ is divisible by 3. From the induction hypothesis, we know that $7^k - 4^k$ is divisible by 3, i.e. there exists $q_0 \in \mathbb{Z}$ such that $7^k - 4^k = 3q_0$. Observe…

$$
\begin{aligned}
7^{k+1} - 4^{k+1} &= 7 \cdot 7^k - 4 \cdot 4^k \\
&= (4+3) \cdot 7^k - 4 \cdot 4^k \\
&= 4 \cdot 7^k + 3 \cdot 7^k - 4 \cdot 4^k \\
&= 4 \cdot 7^k - 4 \cdot 4^k + 3 \cdot 7^k \\
&= 4\left(7^k - 4^k\right) + 3 \cdot 7^k \\
&= 4(3q_0) + 3 \cdot 7^k \\
&= 3(4q_0 + 7^k)
\end{aligned}
$$

Letting $q := 4q_0 + 7^k$, which is an integer, we see that $7^{k+1} - 4^{k+1} = 3q$. Therefore, $7^{k+1} - 4^{k+1}$ is divisible by 3. Then by induction, we know that $7^n - 4^n$ is divisible by 3 for all $n \in \mathbb{Z}_{\geq 0}$.

Remark. If we had seen modular arithmetic, this is simple: we know an integer is divisible by 3 if and only if it is zero modulo 3. We know that $7 = 3(2) + 1 \equiv 1 \bmod 3$ and $4 = 3(1) + 1 \equiv 1 \bmod 3$. But then $7^n - 4^n \equiv 1^n - 1^n = 1 - 1 = 0 \bmod 3$. Therefore, $7^n - 4^n$ is divisible by 3 for all $n \in \mathbb{Z}_{\geq 0}$.

Problem 5. (20pt) Prove that $\mathbb{Z} = \{3x + 2y\colon x, y \in \mathbb{Z}\}$.

Solution. Let $S = \{3x + 2y\colon x, y \in \mathbb{Z}\}$. We know that $S \subseteq \mathbb{Z}$ because S only contains integers. We only need to show that $\mathbb{Z} \subseteq S$. Taking $x = y = 0$, we know that $3(0) + 2(0) = 0 \in S$. Furthermore, taking $x = 1$ and $y = -1$, observe that $3(1) + 2(-1) = 1 \in S$. We prove that each positive integer is in S. Let n be a positive integer. We know that $1 \in S$. Now assume that $n = 1, 2, \ldots, k$ is an element of S. We need to prove that $n = k + 1$ is an element of S. By the inductive hypothesis, we know there exists x_0, y_0 such that $3x_0 + 2y_0 = k$. But $x_0 + 1, y_0 - 1 \in \mathbb{Z}$ and...

$$3(x_0 + 1) + 2(y_0 - 1) = 3x_0 + 3 + 2y_0 - 2 = (3x_0 + 2y_0) + (3 - 2) = k + 1$$

This shows that taking $x = x_0 + 1$ and $y = y_0 - 1$, we know $k + 1$ is an element of S. Therefore by induction, every positive integer is an element of S.

Similarly, we can prove that every negative integer is an element of S using induction. Let n be a negative integer. We know that $n = -1$ is an element of S because choosing $x = -1$ and $y = 1$, we have $3(-1) + 2(1) = -3 + 2 = -1$ is an element of S. Now assume that $n = -1, -2, \ldots, -k$ is an element of S. We need to prove that $n = -k - 1$ is an element of S. By the inductive hypothesis, we know there exists x_0, y_0 such that $3x_0 + 2y_0 = -k$. But $x_0 - 1, y_0 + 1 \in \mathbb{Z}$ and...

$$3(x_0 - 1) + 2(y_0 + 1) = 3x_0 - 3 + 2y_0 + 2 = (3x_0 + 2y_0) + (-3 + 2) = -k - 1$$

This shows that taking $x = x_0 - 1$ and $y = y_0 + 1$, we know $-k - 1$ is an element of S. Therefore by induction, every negative integer is an element of S.

But then we know that $\mathbb{Z} \subseteq S$. Therefore, $\mathbb{Z} = S$.