

**Quiz 1. True/False:** If  $P$  is the proposition  $6 < 5$  and  $Q$  is the proposition, “Earth is a planet,” then the logical statement  $P \rightarrow Q$  is false.

**Solution.** The statement is *false*. Recall that the truth table for  $P \rightarrow Q$  is as follows:

$P$	$Q$	$P \rightarrow Q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$T$
$F$	$F$	$T$

Here,  $P$  is the proposition  $P : 6 < 5$  and  $Q$  is the proposition  $Q$ : “Earth is a planet.” It is clear that  $P$  is false and  $Q$  is true. But then examining the logic table above, we can see that  $P \rightarrow Q$  is true.

**Quiz 2. True/False:**  $\neg(P \rightarrow \neg Q) \equiv P \wedge Q$

**Solution.** The statement is *true*. To determine if two propositions are logically equivalent, one can either examine the truth table or apply logical rules to obtain one logical expression from the other. If we construct a truth table, we have...

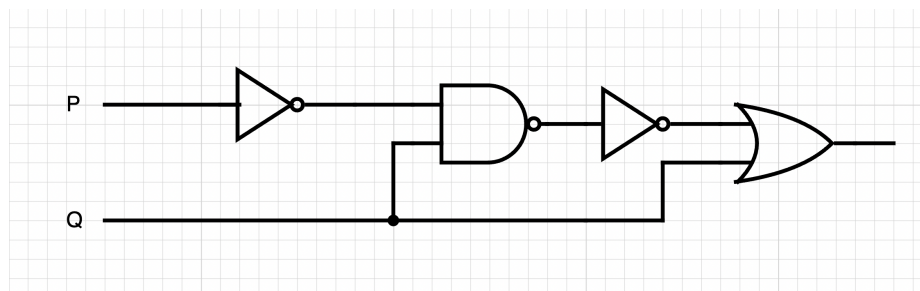
$P$	$Q$	$\neg Q$	$P \rightarrow \neg Q$	$\neg(P \rightarrow \neg Q)$	$P \wedge Q$
$T$	$T$	$F$	$F$	$T$	$T$
$T$	$F$	$T$	$T$	$F$	$F$
$F$	$T$	$F$	$T$	$F$	$F$
$F$	$F$	$T$	$T$	$F$	$F$

Because for each possible pair of choices for  $P$  and  $Q$  the outputs for  $\neg(P \rightarrow \neg Q)$  and  $P \wedge Q$  match,  $\neg(P \rightarrow \neg Q) \equiv P \wedge Q$ . Alternatively, we can transform one into the other by applying logical equivalences (recall  $P \rightarrow Q \equiv \neg P \vee Q$  or  $\neg(P \rightarrow Q) \equiv P \wedge \neg Q$ ):

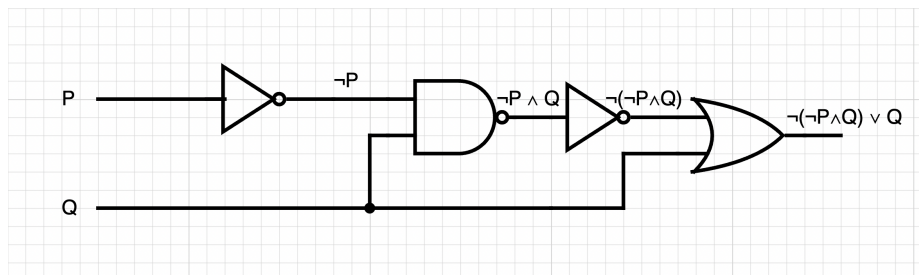
$$\neg(P \rightarrow \neg Q) \equiv \neg(\neg P \vee \neg Q) \equiv \neg(\neg P) \wedge \neg(\neg Q) \equiv P \wedge Q.$$

**Quiz 3. True/False:** The logic corresponding to the circuit shown below is the proposition:

$$(\neg P \wedge Q) \vee \neg Q.$$



**Solution.** The statement is *false*. We can trace through the circuit. We see that the current from  $P$  passes through a NOT gate and we obtain  $\neg P$ . This then feeds into an AND gate along with  $Q$  so that we obtain  $\neg P \wedge Q$ . The resulting current is then passed through a NOT gate, obtaining  $\neg(\neg P \wedge Q)$ . This finally reaches an OR gate—along with  $Q$ —to obtain  $\neg(\neg P \wedge Q) \vee Q$ . We can see a diagrammatic explanation below.



**Quiz 4. True/False:** Let the universe  $\mathcal{U}$  be the set of real numbers and define  $P(x)$  to be the predicate  $P(x) : x^2 + x - 4 \geq 0$ . Then  $(\forall x)(\neg P(x))$  is true.

**Solution.** The statement is *false*. If  $P(x) : x^2 + x - 4 \geq 0$ , then  $\neg P(x) : x^2 + x - 4 < 0$ . But then  $(\forall x)(\neg P(x))$  is the statement, “For all  $x$ ,  $x^2 + x - 4 < 0$ .” Now if  $x = 1$ , we have  $\neg P(1) : 1^2 + 1 - 4 < 0$ , i.e.  $-2 < 0$ , which is true. If  $x = 0$ , we have  $\neg P(0) : 0^2 + 0 - 4 < 0$ , i.e.  $-4 < 0$ , which is true. However, while  $(\forall x)(\neg P(x))$  is clearly true for *some* (we found at least two), it is not true *for all*  $x$ . As a counterexample, let  $x = 10$ . Then  $\neg P(10) : 10^2 + 10 - 4 < 0$ , which is  $104 < 0$ —clearly false. Therefore,  $\neg P(x)$  is not true for all  $x$ . But then  $(\forall x)(\neg P(x))$  is false.

**Quiz 5. True/False:** Let the domain of  $x, y$  be the integers. Then  $(\exists! x)(\forall y)(x + 2y = 5)$ .

**Solution.** The statement is *false*. The logical proposition  $(\exists! x)(\forall y)(x + 2y = 5)$  in words states, “There exists a unique  $x$  such that for all  $y$ ,  $x + 2y = 5$ .” Suppose that there were such a  $x$ , say  $x_0$ . Then we know that  $x_0 + 2y = 5$  for all  $y$ . In particular,  $x_0$  satisfies this equality when  $y = 0$ . But then we know that  $x_0 = 5$ . But also, it must satisfy the equality when  $x = 1$ . But then  $x_0 + 2 = 5$  so that  $x_0 = 3$ . Then there is not a unique  $x$  that works for all  $y$ ! Therefore, the statement is false. Note that if we reverse the quantifiers, the statement is true:  $(\forall y)(\exists! x)(x + 2y = 5)$ . In this case, this is the statement, “For all  $y$ , there exists a unique  $x$  such that  $x + 2y = 5$ .” If you were given any  $y$ , define  $x_0 := 5 - 2y$ . But then  $x + 2y = (5 - 2y) + 2y = 5$ . So there exists such an  $x$ . Is it unique? Well if there were two or more  $x$  values that worked for some  $y$ , say two of them are  $x_0$  and  $\tilde{x}_0$ , then we have  $x_0 + 2y = 5 = \tilde{x}_0 + 2y$ . But then  $x_0 + 2y = \tilde{x}_0 + 2y$ . Subtracting  $2y$ , we have  $x_0 = \tilde{x}_0$ . Therefore, there can only be one such  $x$ . Because we have found one, we know that the statement that for all  $y$ , there exists a unique  $y$  such that  $x + 2y = 5$  is true.

**Quiz 6.** True/False:  $\{1, 2\} \subseteq \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

**Solution.** The statement is *false*. We know that  $A \subseteq B$  if and only if for all  $a \in A$ , we have  $a \in B$ . We test every element of the set  $\{1, 2\}$ . The first element is 1. However,  $1 \notin \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ . [Note that  $1 \notin \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$  but  $\{1\} \in \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ .] However, we do have  $\{1, 2\} \notin \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ .

**Quiz 7.** True/False:  $\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \emptyset$

**Solution.** The statement is *false*. For  $n = 1$ , the set  $\left(-\frac{1}{n}, \frac{1}{n}\right)$  is the interval  $(-1, 1)$ . For  $n = 2$ , the set  $\left(-\frac{1}{n}, \frac{1}{n}\right)$  is the interval  $\left(-\frac{1}{2}, \frac{1}{2}\right)$ . For  $n = 3$ , the set  $\left(-\frac{1}{n}, \frac{1}{n}\right)$  is the interval  $\left(-\frac{1}{3}, \frac{1}{3}\right)$ . Note that 0 is an element of all these sets. Generally, we have  $0 \in \left(-\frac{1}{n}, \frac{1}{n}\right)$  for all  $n \in \mathbb{N}$ . But then we know that  $0 \in \bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$ . This is sufficient to demonstrate that this is not empty. [Note that it is actually true that  $\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ —though this takes more work to prove.]

**Quiz 8.** True/False: Let  $E(n)$  denote the relation from  $\mathbb{N}$  to  $\mathbb{Z}^{\geq 0}$  given by the rule that  $E(n)$  is the number of positive even integers less than or equal to  $n$ . Then this relation is a function with  $E(5) = 2$ , i.e. 2 is in the image of 5, and 10 in the preimage of 5.

**Solution.** The statement is *true*. There are several claims here. First, the claim that  $E(n) : \mathbb{N} \rightarrow \mathbb{Z}^{\geq 0}$  is a function. Given some  $n \in \mathbb{N}$ , there is a single number of positive even integers  $\leq n$ . But then for every input for  $E(n)$ , there is only one possible output. Therefore,  $E(n)$  is a function from  $\mathbb{N}$  to  $\mathbb{Z}^{\geq 0}$ . For 2 to be in the image of 5, we need  $E(5) = 2$ . There are two positive even integers  $\leq 5$  (namely, 2 and 4) so that 2 is in the image of 5. For 10 to be in the preimage of 5, we would have to have  $E(10) = 5$ . Note that there are 5 positive even integers  $\leq 10$  (namely 2, 4, 6, 8, 10). Therefore, 10 is in the preimage of 5.

**Quiz 9.** True/False: Let  $f : X \rightarrow Y$  be a function. Then  $f^{-1}$  will be a function if and only if the preimage set satisfies the following:  $(\forall y \in \text{im } f)(\exists x \in X)(f^{-1}(y) = x)$ .

**Solution.** The statement is *false*. Take for example the function  $f : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$  given by  $f(x) = x^2$ . For all  $y \in \mathbb{R}^{\geq 0}$ , there exists an  $x \in \mathbb{R}$  such that  $f(x) = y$ , namely  $\pm\sqrt{y}$ . But if  $y > 0$ , then there are two possibilities:  $+\sqrt{y}$  and  $-\sqrt{y}$ . But this function  $f(x)$  has  $f^{-1}$  with the property that  $(\forall y \in \text{im } f)(\exists x \in X)(f^{-1}(y) = x)$ . If we want  $f^{-1}$  to be a function, we require  $(\forall y \in \text{im } f)(\exists! x \in X)(f^{-1}(y) = x)$ .

**Quiz 10.** *True/False:* Suppose that  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are functions and that  $g \circ f$  is injective. Then it must be that  $f$  is injective.

**Solution.** The statement is *true*. Observe that  $g \circ f : A \rightarrow C$ . Suppose  $f$  were not injective. Then there are two values in  $A$ , say  $a_1, a_2$ , such that  $a_1 \neq a_2$  and  $f(a_1) = f(a_2)$ . But then we have...

$$\begin{aligned}f(a_1) &= f(a_2) \\g(f(a_1)) &= g(f(a_2)) \\(g \circ f)(a_1) &= (g \circ f)(a_2)\end{aligned}$$

But then there are two values in the domain of  $g \circ f$ , namely  $a_1, a_2$  such that  $a_1 \neq a_2$  but  $(g \circ f)(a_1) = (g \circ f)(a_2)$ . But then  $g \circ f$  is not injective, contrary to what we were told. Our assumption that  $f$  was not injective must then be wrong. Therefore, it must be that  $f$  is injective.

**Quiz 11.** *True/False:* Fix an integer  $n > 1$  and let  $\{a_n\}_{n \in \mathbb{N}}$  and  $\{b_n\}_{n \in \mathbb{N}}$  be sequences. Then  $\prod_{k=1}^n a_n b_n = \prod_{k=1}^n a_n \cdot \prod_{k=1}^n b_n$  but  $\sum_{k=1}^n a_n b_n \neq \sum_{k=1}^n a_n \cdot \sum_{k=1}^n b_n$ .

**Solution.** The statement is *true*. It is true that  $\prod_{k=1}^n a_n b_n = \prod_{k=1}^n a_n \cdot \prod_{k=1}^n b_n$ . For instance, if  $n = 2$ , we have...

$$\prod_{k=1}^2 a_n b_n = a_1 b_1 \cdot a_2 b_2 = (a_1 a_2) \cdot (b_1 b_2) = \prod_{k=1}^2 a_n \cdot \prod_{k=1}^2 b_n$$

We can always rearrange the terms in this way for any  $n$ . Therefore, the statement is true for products.

However, even in the case of  $n = 2$ , the statement is untrue for sums. For example, if  $n = 2$  in  $\sum_{k=1}^n a_n b_n \neq \sum_{k=1}^n a_n \cdot \sum_{k=1}^n b_n$ , then we have...

$$\begin{aligned}\sum_{k=1}^2 a_n b_n &= a_1 b_1 + a_2 b_2 \\ \sum_{k=1}^2 a_n \cdot \sum_{k=1}^2 b_n &= (a_1 + a_2) \cdot (b_1 + b_2) = a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2\end{aligned}$$

While this may be true for some sequences  $\{a_n\}_{n \in \mathbb{N}}$  and  $\{b_n\}_{n \in \mathbb{N}}$ , it will not generally be true. The ‘issues’ the distributive property cause for larger  $n$  make this even ‘more untrue.’

**Quiz 12. True/False:** Suppose that  $a_{n+2} = 6a_n - a_{n+1}$  with  $a_0 = 4$  and  $a_1 = 3$ . Then the characteristic polynomial is given by the equation  $x^2 = 6 - x$ , i.e. the characteristic polynomial is  $x^2 + x - 6$ . Because  $x^2 + x - 6 = (x + 3)(x - 2)$  has roots  $-3, 2$ , the general solution is  $a_n = c_1(-3)^n + c_2 2^n$ . The specific solution is then  $a_n = (-3)^n + 3 \cdot 2^n$ .

**Solution.** The statement is *true*. Because  $a_{n+2} = 6a_n - a_{n+1}$  and the ‘lowest’ term involved is  $n$ , we give the  $n$ th term power 0 for  $x$ . Then we have  $x^{0+2} = 6x^0 - x^{0+1}$ . This is  $x^2 = 6 - x$ . We then have  $x^2 + x - 6 = 0$ . Therefore, the characteristic polynomial for this homogeneous linear recurrence relation is  $x^2 + x - 6$ . This polynomial has roots  $-3$  and  $2$  because  $x^2 + x - 6 = 0$  is equivalent to  $(x + 3)(x - 2) = 0$ , which has solutions  $x = -3$  and  $x = 2$ . Therefore, we know that  $a_n = c_1(-3)^n + c_2 \cdot 2^n$ . Now we use the fact that when  $n = 0$ , we have  $a_0 = 4$ , and when  $n = 1$ , we have  $a_1 = 3$ . But then we have...

$$4 = a_0 = c_1(-3)^0 + c_2 \cdot 2^0 = c_1 + c_2$$

$$3 = a_1 = c_1(-3)^1 + c_2 \cdot 2^1 = -3c_1 + 2c_2$$

This is a linear system of two equations in two unknowns. Solving this system yields  $c_1 = 1$  and  $c_2 = 3$ . Therefore, we have  $a_n = (-3)^n + 3 \cdot 2^n$ .

**Quiz 13. True/False:**  $6^{2022} \equiv 1 \pmod{5}$

**Solution.** The statement is *true*. Using the division algorithm, we know that  $6 = 1(5) + 1$ . But then we know that  $6 \equiv 1 \pmod{5}$ . But then we have...

$$6^{2022} \equiv 1^{2022} \equiv 1 \pmod{5}$$

**Quiz 14. True/False:** There is a unique solution to the following system of linear congruences:

$$2x - 1 \equiv 2 \pmod{3}$$

$$x \equiv 0 \pmod{5}$$

$$6x \equiv 5 \pmod{7}$$

**Solution.** The statement is *true*. The first congruence is  $2x - 1 \equiv 2 \pmod{3}$ . Adding 1 to both sides, we see that this is equivalent to  $2x \equiv 3 \equiv 0 \pmod{3}$ . Because  $\gcd(2, 3) = 1$ , we know that  $2^{-1}$  exists mod 3. In fact, because  $2 \cdot 2 \equiv 4 \equiv 1 \pmod{3}$ . Therefore,  $2^{-1} \equiv 2 \pmod{3}$ . Therefore,  $2x \equiv 0 \pmod{3}$  implies  $2^{-1} \cdot 2x \equiv 2^{-1} \cdot 0 \pmod{3}$ . This is  $x \equiv 0 \pmod{3}$ . In the last congruence, because  $\gcd(6, 7) = 1$ , we know that  $6^{-1}$  exists mod 7. In fact, because  $6 \cdot 6 \equiv 36 \equiv 1 \pmod{7}$ , we know that  $6^{-1} \equiv 6 \pmod{7}$ . But then  $6x \equiv 5 \pmod{7}$  implies  $6^{-1} \cdot 6x \equiv 6^{-1} \cdot 5 \pmod{7}$ . But this is  $x \equiv 6 \cdot 5 \equiv 30 \equiv 2 \pmod{7}$ . The original system of congruences is then equivalent to...

$$x \equiv 0 \pmod{3}$$

$$x \equiv 0 \pmod{5}$$

$$x \equiv 2 \pmod{7}$$

This system now has the ‘proper form’ to apply the Chinese Remainder Theorem. Because  $\gcd(3, 5, 7) = 1$ , we know there exists a unique solution to the system of congruences. In fact, we have solution. . .

$$x = \sum a_i N_i M_i = 0 \cdot 2 \cdot 35 + 0 \cdot 1 \cdot 21 + 2 \cdot 1 \cdot 15 = 0 + 0 + 30 = 30$$

The solution is then the congruence class of 30 modulo  $3 \cdot 5 \cdot 7 = 105$ . Therefore, the solution is 30, i.e.  $[30] = \{\dots, -720, -570, -420, -270, -120, 30, 180, 330, 480, 630, 780, \dots\}$ .

**Quiz 15.** *True/False:* The number 1 is prime.

**Solution.** The statement is *false*. A prime number is an integer greater than 1 which has no proper divisors. Because 1 is not greater than 1, 1 cannot be prime. Note that 1 is also not composite. To be composite, an integer need have proper divisors. However, the only divisor of 1 is 1. Therefore, 1 also cannot be composite. This shows 1 is neither prime nor composite.

**Quiz 16.** *True/False:* Using the division algorithm to divide  $-10$  by 3, we have  $-10 = 3(3) + 1$ .

**Solution.** The statement is *false*. Recall that given  $a, b \in \mathbb{Z}$  with  $a \neq 0$ , we can write  $b = qa + r$  for some  $q, r \in \mathbb{Z}$  and  $0 \leq r < a$ . Clearly, the statement is false because  $3(3) + 1 = 9 + 1 = 10 \neq -10$ . The multiple of 3 that is less than or equal to  $-10$  is  $-12$ . Because  $-10 = -12 + 2$ , we have  $-10 = -4(3) + 2$ . Because  $-4, 2 \in \mathbb{Z}$  and  $0 \leq 2 < 3$ , we have expressed  $-10$  divided by 3 using the division algorithm. Note that one cannot use  $-10 = -3(3) - 1$  because it is not the case that  $0 \leq -1 < 3$ .

**Quiz 17.** *True/False:* The number  $2B002B$  in base-10 is 2818091.

**Solution.** The statement is *true*. The number  $2B002B$  is in hexadecimal. Converting this to base-10 (and recalling  $A = 10, B = 11, C = 12, D = 13, E = 14$ , and  $F = 15$ ), we have. . .

$$2B002B = 2 \cdot 16^5 + 11 \cdot 16^4 + 0 \cdot 16^3 + 0 \cdot 16^2 + 2 \cdot 16^1 + 11 \cdot 16^0 = 2097152 + 720896 + 0 + 0 + 32 + 11 = 2818091$$