Quiz 1. *True/False*: The following is a truth table for $P \to Q$:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	F
F	F	F

Solution. The statement is *false*. The correct truth table should be...

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

One way to think about this is as follows: imagine P is a guarantee. Namely, we promise that if P happens, Q must happen. For instance, P could represent the statement, "You do not tamper with your hardware," and Q could be the statement, "I will replace your broken computer." So $P \to Q$ is then the statement, "If you do not tamper with your hardware, then I will replace your broken computer." If both P and Q are true, then this should be true—because I promised to replace the computer if you left it alone. If P is true and Q is false, then the statement should be false because I broke my promise. However, my promise holds true whenever P is false. Why? Because you broke our agreement by tampering with the hardware. So while I may or may not replace the computer, my promise has not been broken in either case, i.e. it remains true. In an implication $P \to Q$, if P is false, then the statement $P \to Q$ is *always* true.

Quiz 2. *True/False*: $\forall x, \exists y,\ x^2 + y = 4$

Solution. The statement is *true*. The statement says that for all x there is a y such that $x^2 + y = 4$. If this is true (which it is), we need to prove it. Fix an x, say x_0. We need to find a y such that $x_0^2 + y = 4$. Define $y_0 := 4 - x_0^2$. But then we have

$$x_0^2 + y_0 = x_0^2 + (4 - x_0^2) = 4,$$

as desired.

Quiz 3. *True/False*: $\neg\left(\forall x, \exists y, P(x,y) \vee \neg Q(x,y)\right) = \exists x, \forall y, \neg P(x,y) \wedge Q(x,y)$

Solution. The statement is *true*. We can simply compute the negation step-by-step:

$$\begin{aligned}
\neg\left(\forall x, \exists y, P(x,y) \vee \neg Q(x,y)\right) &\equiv \exists x, \neg(\exists y, P(x,y) \vee \neg Q(x,y)) \\
&\equiv \exists x, \forall y, \neg(P(x,y) \vee \neg Q(x,y)) \\
&\equiv \exists x, \forall y, \neg P(x,y) \wedge \neg(\neg Q(x,y)) \\
&\equiv \exists x, \forall y, \neg P(x,y) \wedge Q(x,y)
\end{aligned}$$

Quiz 4. *True/False*: To prove $P \Rightarrow Q$, you can prove $Q \Rightarrow P$.

Solution. The statement is *false*. The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$. The converse of a logical statement is not necessarily logically equivalent to the original statement. So proving the converse does not necessarily prove the original statement. However, the contrapositive of $P \Rightarrow Q$, which is $\neg Q \Rightarrow \neg P$, is logically equivalent to $P \Rightarrow Q$. Therefore, to prove $P \Rightarrow Q$, one only need prove $\neg Q \Rightarrow \neg P$. This is called proof by contrapositive.

Quiz 5. *True/False*: Let $A = \{1\}$ and $B = \{3, \{1\}\}$. Then $A \subseteq B$.

Solution. The statement is *false*. Recall that $A \subseteq B$ if every element of A is an element of B. The only element of A is the element 1. However, $1 \notin B$, but rather $\{1\} \in B$, i.e. 1 is not in B but the set consisting of only the element of 1 is in B. However, note that $A \in B$ because $A = \{1\}$ and $\{1\} \in B$.

Quiz 6. *True/False*: Take the universal set to be the integers. Then the following two sets are equal:

$$\begin{aligned} A &= \{n \colon n \text{ odd}\} \\ B &= \{m \colon m \text{ prime and } m > 2\} \end{aligned}$$

Solution. The statement is *false*. We know that $9 \in A$ because 9 is odd. But $9 \notin B$ because $9 = 3 \cdot 3$ is not prime. Therefore, $A \not\subseteq B$ so that $A \neq B$.

Quiz 7. *True/False*: The sets $A \times B \times C$ and $(A \times B) \times C$ are not the same.

Solution. The statement is *true*. Elements in $A \times B \times C$ 'look like' (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. Whereas elements in $(A \times B) \times C$ 'look like' $((a, b), c)$, where $a \in A$, $b \in B$, and $c \in C$. Because elements in these sets are not of the same form, they cannot be the same. As an explicit example, take $A = \{1\}$, $B = \{2, 3\}$, and $C = \{4\}$. Then

$$\begin{aligned} A \times B \times C &= \{(1, 2, 4), (1, 3, 4)\} \\ (A \times B) \times C &= \{((1, 2), 4), ((1, 3), 4)\} \end{aligned}$$

Then $A \times B \times C \neq (A \times B) \times C$.

Quiz 8. *True/False*: There is a set S such that $\mathcal{P}(S)$ has 3 elements.

Solution. The statement is *false*. If S is an infinite set, then clearly there is a subset for each element $s \in S$, i.e. the subset $\{s\}$. Clearly, if there is such a set, it cannot be infinite. Now if S had 3 or more elements—having a subset for each element of S—we know that $\mathcal{P}(S)$ would have more than 3 subsets. Therefore, S must have 0, 1, or 2 elements. If $S = \emptyset$, then $\mathcal{P}(S) = \{\emptyset\}$. If $S = \{s_1\}$, then $\mathcal{P}(S) = \{\emptyset, \{s_1\}\}$. Finally, if $S = \{s_1, s_2\}$, then $\mathcal{P}(S) = \{\emptyset, \{s_1\}, \{s_2\}, S\}$. Therefore, there cannot be such a set S.

Quiz 9. *True/False*: The Principle of Induction is logically equivalent to the Well-Ordering Principle.

Solution. The statement is *true*. We saw in class that the Well-Ordering Principle implied the Principle of Induction. From the homework, we know that the Principle of Induction implies the Well-Ordering Principle.