Name: *Caleb McWhorter — Solutions*
MATH 101
Fall 2023
HW 1: Due 09/11

"I'm fine. It's just that life is pointless and nothing matters and I'm always tired."
–Andy Dwyer, Parks and Recreation

Problem 1. (10pt) Complete the following:

(a) List all the divisors of 80.

(b) List all the nonnegative multiples of 15 less than 180.

Solution.

(a) We know that 1 and 80 are divisors (factors) of 80—the improper divisors. Because 80 is even, we know it is divisible by 2. Because the last two digits of 80 are divisible by 4, we know 4 is a divisor of 80. Furthermore, because 80 ends in a 0, it is a multiple of 5 and 10; hence, 5 and 10 are divisors of 80. But then the divisors of 80 are...

$$\text{divisors of 80} = \{1, 2, 4, 5, 8, 10, 16, 20, 40, 80\}$$

(b) The nonnegative multiples of 15 will be integers of the form $15k$, where $k \geq 0$ is an integer. But then we have...

k	0	1	2	3	4	5	6	7	8	9	10	11
$15k$	0	15	30	45	60	75	90	105	120	135	150	165

Problem 2. (10pt) Showing all your work, find the prime factorizations of the following:

(a) 40

(b) 90

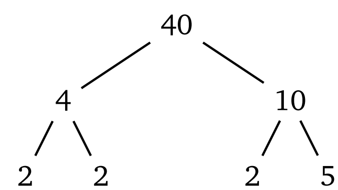
(c) 97

(d) 99

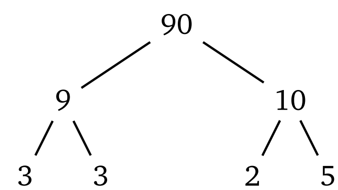
(e) 228

Solution.

(a) $40 = 2^3 \cdot 5$

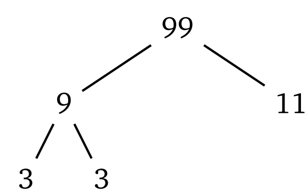


(b) $90 = 2 \cdot 3^2 \cdot 5$

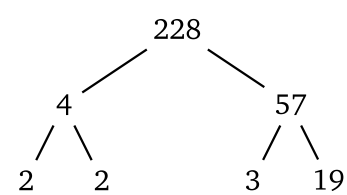


(c) Observe that $\sqrt{97} \approx 9.85$. If 97 is composite, then it has a (prime) factor between 2 and 9. We know that 2 does not divide 97 because it is not even. We know also that 3 and 9 do not divide 97 because $9 + 7 = 16$ is not divisible by 3 or 9, respectively. We know that 97 is not divisible by 5 because it does not end in 5 or 0. But then 97 does not have a prime factor between 2 and 9. Therefore, 97 is prime, so that the prime factorization is itself.

(d) $99 = 3^2 \cdot 11$



(e) $228 = 2^2 \cdot 3 \cdot 19$



Problem 3. (10pt) Without using a calculator, answer the following:

(a) Does 2 divide 8455? Explain.

(b) Does 3 divide 19436? Explain.

(c) Does 4 divide 764136? Explain.

(d) Does 5 divide 99999? Explain.

(e) Does 9 divide 331443? Explain.

Solution.

(a) No. We know that 8455 is not even, i.e. it is odd; therefore, 2 cannot divide 8455.

(b) No. We know that the sum of the digits is $1 + 9 + 4 + 3 + 6 = 23$, which is not divisible by 3; therefore, 19436 is not divisible by 3.

(c) Yes. Because the last two digits of 764136, i.e. 36, are divisible by 4, we know that 764136 is divisible by 4.

(d) No. We know that 99999 does not end in a 5 or 0; therefore, 99999 is not divisible by 5.

(e) Yes. We know that the sum of the digits is $3 + 3 + 1 + 4 + 4 + 3 = 18$, which is divisible by 9; therefore, 331443 is divisible by 9.

Problem 4. (10pt) Complete the following:

(a) List all the prime numbers up to 34.

(b) Compute $\sqrt{1223}$.

(c) Using (a) and (b), explain why 1223 is a prime number.

Solution.

(a) The primes up to 34 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, and 31.

(b) We have $\sqrt{1223} \approx 34.97$.

(c) Observe that none of 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, and 31 divide 1223 because...

$$\begin{array}{llll} \frac{1223}{2} \approx 611.5 & \frac{1223}{7} \approx 174.714 & \frac{1223}{17} \approx 71.9412 & \frac{1223}{29} \approx 42.1724 \\ \frac{1223}{3} \approx 407.667 & \frac{1223}{11} \approx 111.182 & \frac{1223}{19} \approx 64.3684 & \frac{1223}{31} \approx 39.4516 \\ \frac{1223}{5} \approx 244.6 & \frac{1223}{13} \approx 94.0769 & \frac{1223}{23} \approx 53.1739 & \end{array}$$

If 1223 were composite, it would have a (prime) divisor between 2 and 34 (because $\sqrt{1223} \approx 34.97$). But from the work above, we know that 1223 has no prime divisor between 2 and 34. Therefore, it must be that 1223 is not composite, i.e. that 1223 is prime.

Problem 5. (10pt) Showing all your work, compute the following:

(a) $\gcd(15, 33)$

(b) $\operatorname{lcm}(15, 33)$

(c) $\gcd(2^{70} \cdot 3^{40} \cdot 7^{60} \cdot 11^{20},\ 2^{90} \cdot 5^{48} \cdot 7^{50} \cdot 11^{20})$

(d) $\operatorname{lcm}(2^{70} \cdot 3^{40} \cdot 7^{60} \cdot 11^{20},\ 2^{90} \cdot 5^{48} \cdot 7^{50} \cdot 11^{20})$

Solution. We use the fact that the gcd of a collection of numbers is the product of the smallest possible power of the primes found in their prime factorizations and the lcm is the product of the largest possible power of the primes found in their prime factorizations.

(a)

$$\gcd(15, 33) = \gcd(3 \cdot 5, 3 \cdot 11) = 3^1 \cdot 5^0 \cdot 11^0 = 3$$

(b)

$$\operatorname{lcm}(15, 33) = \operatorname{lcm}(3 \cdot 5, 3 \cdot 11) = 3^1 \cdot 5^1 \cdot 11^1 = 165$$

(c)

$$\gcd(2^{70} \cdot 3^{40} \cdot 7^{60} \cdot 11^{20},\ 2^{90} \cdot 5^{48} \cdot 7^{50} \cdot 11^{20}) = 2^{70} \cdot 3^0 \cdot 5^0 \cdot 7^{50} \cdot 11^{20} =$$

$$142841828319599661183561398974776576045768348923759507176132717544737893231789 4475776$$

(d)

$$\operatorname{lcm}(2^{70} \cdot 3^{40} \cdot 7^{60} \cdot 11^{20},\ 2^{90} \cdot 5^{48} \cdot 7^{50} \cdot 11^{20}) = 2^{90} \cdot 3^{40} \cdot 5^{48} \cdot 7^{60} \cdot 11^{20} =$$

$$3597192753307258276485843647165367801057974556190310 4000000000000000000000000000000000000000000000000$$

Problem 6. (10pt) Isabella has two large wood strips. One is 70 ft long and the other is 98 ft long. She needs to cut them down into boards. The size of the boards does not matter, so long as they are all of equal size. She has a 'template' that will let her cut the wood to specific integer lengths, but it must be reset if you change the length—which is time consuming. To what length should she set the template in order to never have to reset it and cut each strip into pieces of equal length?

Solution. Suppose the length she can cut it into is ℓ. Because she is cutting the pieces of wood into some integer number of pieces, each of length ℓ, it must be that 70 and 98 are both multiples of ℓ; that is, ℓ is a divisor of both 70 and 98. But then ℓ is a common divisor of 70 and 98. We can list the divisors of both 70 and 98:

$$70\colon\ \mathbf{1}, \mathbf{2}, 5, \mathbf{7}, 10, \mathbf{14}, 35, 70$$
$$98\colon\ \mathbf{1}, \mathbf{2}, \mathbf{7}, \mathbf{14}, 49, 98$$

The common divisors are 1, 2, 7, and 14. Therefore, the possibilities for ℓ are 1, 2, 7, 14, and all are viable choices. To obtain the maximum number of pieces, she should choose $\ell = 1$ ft, which splits the 70 ft strip into 70 pieces, the 98 ft board into 98 pieces, for a total of 168 pieces of 1 ft length wood. If she wants to obtain the minimum number of pieces, she can choose $\ell = 14$ ft (the greatest common divisor), which splits the 70 ft strip into 5 pieces, the 98 ft board into 7 pieces, for a total of 12 pieces of 14 ft length wood. Note that choosing $\ell = 2$ ft cuts the boards into 35 and 49 pieces, respectively, for a total of 84 pieces of 2 ft length wood. Finally, choosing $\ell = 7$ ft cuts the boards into 10 and 14 pieces, respectively, for a total of 24 pieces of 7 ft length wood. We can summarize the data below:

Guide Setting	1	2	7	14
Number Pieces	168	84	24	12