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MATH 308
Fall 2023
HW 8: Due 10/12

"The study of Mathematics, like the Nile, begins in minuteness but ends in magnificence."

–Charles Caleb Colton

Problem 1. (10pt) Let $A = \{2, 6, 8, 10\}$, B be the set of nonnegative even numbers that are at most 10, and C be the set of perfect squares less than 10. Define $f : A \to \mathbb{Z}$ and $g : B \setminus C \to \mathbb{Z}$ via $x \to \frac{15(x+8)}{x}$ and $x \mapsto \frac{5(x^2-16x+88)}{4}$, respectfully. Fully justifying your answer, determine whether $f \equiv g$.

Solution. To show that two functions f, g are equal, i.e. $f = g$ or $f \equiv g$, we need to show that they have the same domain, the same codomain, and their outputs are the same everywhere on their 'common domain.'[1]

Equal Domains, $A = B$: We need to show $A = B$ that is, we need to show that A and B have all the same elements. We know that $A = \{2, 6, 8, 10\}$. Now B is the set of nonnegative even numbers less than 10, i.e. $B = \{0, 2, 4, 6, 8, 10\}$. Furthermore, C is the set of perfect squares less than 10, i.e. $C = \{0, 4, 9\}$. But then $B \setminus C = \{2, 6, 8, 10\}$. Therefore, $A = B \setminus C$.

Equal Codomains, $\mathbb{Z} = \mathbb{Z}$: It is immediately clear that f and g have the same codomain—namely, $\mathbb{Z}$.

Equivalent on their Common Domain: To check whether f and g have the same outputs for every element of their 'common domain', we can simply compute f, g for the values in $\{2, 6, 8, 10\}$:

$$\begin{aligned} f(2) &= \frac{15(2+8)}{2} = \frac{150}{2} = 75 & \qquad g(2) &= \frac{5(2^2-16(2)+88)}{4} = \frac{300}{4} = 75 \\ f(6) &= \frac{15(6+8)}{6} = \frac{210}{6} = 35 & g(6) &= \frac{5(6^2-16(6)+88)}{4} = \frac{140}{4} = 35 \\ f(8) &= \frac{15(8+8)}{8} = \frac{240}{8} = 30 & g(8) &= \frac{5(8^2-16(8)+88)}{4} = \frac{120}{4} = 30 \\ f(10) &= \frac{15(10+8)}{10} = \frac{270}{10} = 27 & g(10) &= \frac{5(10^2-16(10)+88)}{4} = \frac{140}{4} = 35 \end{aligned}$$

Observe that $f(2) = g(2) = 75$, $f(6) = g(6) = 35$, and $f(8) = g(8) = 30$. However, $f(10) = 27 \neq 35 = g(10)$. Therefore, f and g do not agree on their 'common domain.'

Because f and g do not agree on their 'common domain', f and g are not equal, i.e. $f \not\equiv g$.

[1]Note: This is not the same as the two functions having the same image. For example, take $A = \{1, 2\}$ and $B = \{a, b\}$. Define $f, g : A \to B$ via $f(1) = a$, $f(2) = b$, and $g(1) = b$ and $g(2) = a$. Clearly, f, g have the same domain and codomains. The image of both f and g are the same—namely, the set $\{a, b\}$, but observe $a = f(1) \neq g(1) = b$ and $b = f(2) \neq g(2) = a$.

Problem 2. (10pt) Define the following real-valued functions:

$$f(x) = 2x - 1 \qquad j(x) = \frac{x-1}{x+2}$$
$$g(x) = x^2 + x + 1 \qquad k(x) = \sin(\pi x)$$
$$h(x) = x2^x \qquad \ell(x) = 1 - x^2$$

Showing all your work, for each of the following, either compute the function at the specified value or find a general rule for the given function operation:

(a) $(f+g)(0)$

(b) $(j-\ell)(2)$

(c) $(gk)(5)$

(d) $\left(\frac{f}{j}\right)(3)$

(e) $(h \circ k)(1)$

(f) $(2f+\ell)(x)$

(g) $(fg)(x)$

(h) $\left(\frac{h}{f}\right)(x)$

(i) $(k \circ \ell)(x)$

(j) $(\ell \circ g \circ f)(x)$

Solution.

(a) $(f+g)(0)$

(b) $(j-\ell)(2)$

(c) $(gk)(5)$

(d) $\left(\frac{f}{j}\right)(3)$

(e) $(h \circ k)(1)$

(f) $(2f+\ell)(x)$

(g) $(fg)(x)$

(h) $\left(\frac{h}{f}\right)(x)$

(i) $(k \circ \ell)(x)$

(j) $(\ell \circ g \circ f)(x)$

Problem 3. (10pt) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $x \mapsto x^2 + 4x - 5$.

(a) Determine $f(-5)$.

(b) Compute $f([0, 1])$.

(c) Is $16 \in \operatorname{im} f$? Explain.

(d) Determine $f^{-1}(0)$.

(e) Find the domain, codomain, and range for $f(x)$.

Solution.

(a)

(b)

(c)

(d)

(e)

Problem 4. (10pt) Being sure to justify your answer, complete the following:

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 5 - x^2$. Is f an increasing function? Explain. Is f a decreasing function? Explain.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = 5x - 8$. Is g a positive function? Explain. Is g a negative function? Explain.

(c) Let g be as in (b) and define $A = [2, \infty)$ and $B = (-\infty, 0)$. Is $g\big|_A$ a positive function? Explain. Is $g\big|_B$ a negative function? Explain.

(d) Let $h : \mathbb{R} \to \mathbb{R}$ be given by...

$$h(x) = \begin{cases} 1 - x, & x < 2 \\ 3x + 5, & x \geq 2 \end{cases}$$

Find the largest possible interval $S \subseteq \mathbb{R}$ such that $h\big|_S$ is a nondecreasing function. Is h monotone on S? Is h strictly monotone on S?

Solution.

(a)

(b)

(c)

(d)