MAT 308: Exam 3 **Name:** _____
Fall – 2022
12/15/2022
'∞' Minutes

Write your name on the appropriate line on the exam cover sheet. This exam contains **??** pages (including this cover page) and **??** questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Run LaTeX again to produce the table

1. (10 points) Define the following vectors and matrices:

$$u = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \quad v = \begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 3 & 5 \\ -1 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 & 0 \\ -4 & 3 & 7 \\ 6 & 11 & 8 \end{pmatrix}$$

Showing all your work and fully justifying your responses, answer the following:

(a) Compute $-5u - 2v$.

(b) Compute $u \cdot v$.

(c) Compute A^2.

(d) Compute Bu

(e) Consider BC and CB. If the product is not defined, explain why. If the product is defined, compute it.

2. (10 points) Suppose that A, B, C, and D are events in a finite probability space. Define the following probabilities:

$$P(A) = 0.24 \quad P(B \mid A) = 0.90$$
$$P(B) = 0.61 \quad P(C \cap D) = 0$$
$$P(C) = 0.15$$
$$P(D) = 0.86$$

Showing all your work and fully justifying your responses, answer the following:

(a) What is $P(A^c)$?

(b) Are A^c and D disjoint? Explain.

(c) Assuming B and C are independent, compute $P(B \cup C)$.

(d) Compute $P(A \cap B)$.

(e) Are A and B independent? Explain. Are C and D independent? Explain.

3. (10 points) Showing all your work and fully justifying your reasoning, answer the following:

   (a) How many arrangements of letters are there using the letters from the word 'trichotillomania'?

   (b) How many arrangements of letters are there using the letters from 'assessment' such that 'a' and 't' appear alphabetically with 5 letters between them?

   (c) How many 4 character passwords with exactly one vowel can be formed using only English lower case letters and no repeated letters?

   (d) How many ways are there of breaking a room with 10 students into three groups with size five, three, and two, respectively.

4. (10 points) Showing all your work and fully justifying your reasoning, answer the following:

   (a) How many combinations of 15 chicken wings can be ordered at a restaurant that offers 20 different flavors of wings?

   (b) How many integer solutions are there to $x_1 + x_2 + x_3 = 20$ with $x_1, x_2, x_3 \geq 1$?

   (c) What is the coefficient of $x^{12}y^8$ in $\left(5y - \frac{x}{2}\right)^{20}$?

   (d) What is the coefficient of $x^3y^2z^5$ in $(x + y - z)^{10}$?

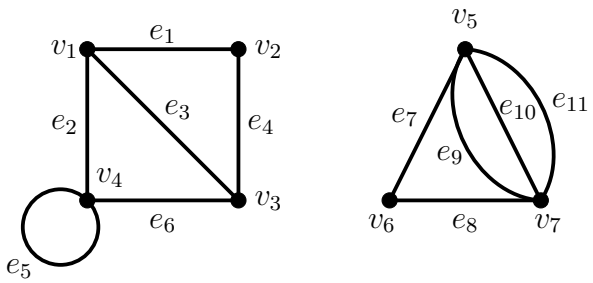
5. (10 points) Let S be the set $\{1, 2, 3, \ldots, 2022\}$. Showing all your work and fully explaining your reasoning, answer the following:

   (a) How many integers in S are divisible by 3 but are not divisible by 5 or 11?

   (b) How many integers in S are divisible by 2, 3, or 5?

   (c) How many integers in S are perfect squares but not perfect cubes?

6. (10 points) Let G be the graph given below.



(a) Is G simple? Explain. Is G a multigraph? Explain.

(b) Is G connected? If so, explain why. If not, explain why not and find the number of connected components.

(c) Find all vertices adjacent to v_2. Find all edges adjacent to e_{10}.

(d) Are there any parallel edges? If not, explain why. If so, find them.

(e) Find the degrees of vertices v_1, v_4, and v_7. What is the degree of G?

7. (10 points) Suppose G is an undirected graph whose adjacency matrix is given below.

$$\begin{pmatrix} 0 & 1 & 3 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Using only the adjacency matrix of G, i.e. without drawing the graph of G, answer the following:

(a) Does G have any loops? Explain.

(b) Is G connected? If so, explain why. If not, find the number of connected components.

(c) Does G have any multiple edges? If so, explain why and find the vertices having multiple edges between them. If not, explain why.

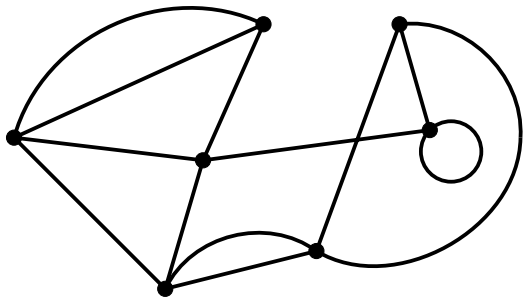
(d) Find the degree of G.

(e) Suppose the fifth power of the adjacency matrix is given below:

$$\begin{pmatrix} 126 & 139 & 369 & 0 & 0 \\ 139 & 78 & 139 & 0 & 0 \\ 369 & 139 & 126 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Find the number of walks of length five between v_2 and v_3.

8. (10 points) Let G be the graph given below.



(a) Does G have an Eulerian circuit? If so, give an example. If not, explain why.

(b) Does G have an Eulerian trial? If so, give an example. If not, explain why.

(c) Does G have a Hamiltonian circuit? If so, give an example. If not, explain why.

9. (10 points) Suppose you have a theoretical computer with infinite memory and infinite processing power. Given an input of 'size' n, let A be an algorithm with time complexity $O(n \log n)$, B be an algorithm with time complexity $O(n^2)$, and C be an algorithm with time complexity $O(\sqrt{n})$.

   (a) What is the time complexity of running A and B in parallel? What is the time complexity of running A and C in parallel?

   (b) What is the time complexity of running A and B sequentially? What is the time complexity of running A and B sequentially?

   (c) Suppose that given an input of size n, an algorithm D requires A to be performed n times, then B is performed, and then C is performed n^2 times. The algorithm then uses outputs from algorithms A, B, and C, performing an operation which runs in $O(1)$ time. What is the time complexity of D?

10. (10 points) Assume that each addition, subtraction, multiplication, division, and `print` 'costs' one flop while defining/redefining variables 'cost' no flops. Suppose you have an algorithm, whose pseudocode is given below, to compute a particular sum for some given fixed n.

```
sum = 0;
i = 0; j = 0;
while(i <= n):
   while(j <= i):
      term = i^2 * j + j − 1;
      sum = sum + term
      print([i, j, sum])
      j = j + 1;
i = i + 1;
```

(a) Write the outputs of this algorithm for $n = 2$.

(b) Find the total flops performing this algorithm. What is the O for this algorithm?

(c) (Bonus) Find sum using the smallest number of total flops for 'large' n.