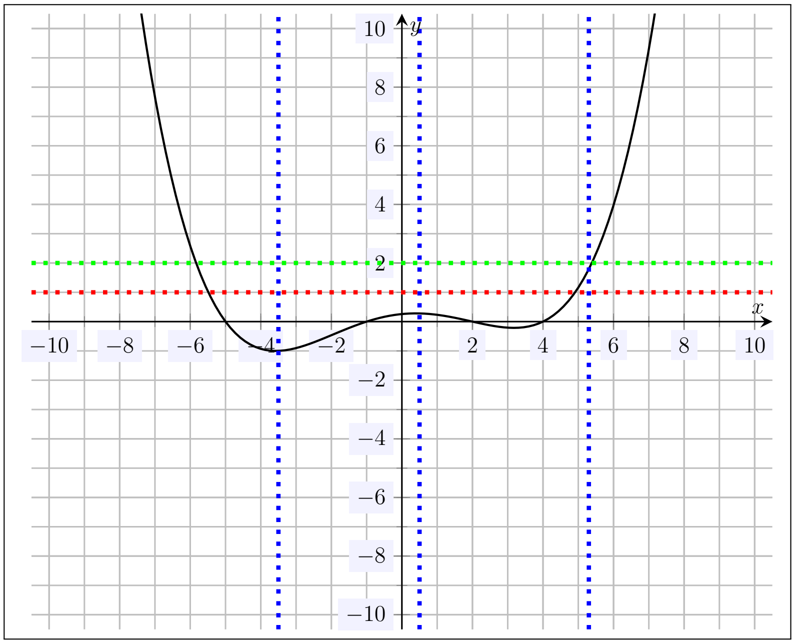
MAT 101: Exam 2
Fall – 2023
11/08/2023
85 Minutes

Name: *Caleb M^{c}Whorter — Solutions*

Write your name on the appropriate line on the exam cover sheet. This exam contains 11 pages (including this cover page) and 10 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) Consider the relation f shown below.



(a) Is the relation shown above a function of x? Explain.

(b) Assuming the relation is a function of x, does the relation above have an inverse that is a function of y? Explain.

(c) Find $f(6)$.

(d) Find the x-intercepts of $f(x)$.

(e) Is there an x such that $f(x) = 2$? Explain.

Solution.

(a) Yes, the relation shown is a function because $f(x)$ passes the Vertical Line Test; that is, every vertical line intersects the graph of $f(x)$ at most once—some sample vertical lines are shown in blue.

(b) No, the relation shown does not have an inverse function $f^{-1}(x)$ because $f(x)$ fails the Horizontal Line Test; that is, not every horizontal line intersects the graph of $f(x)$ at most once. For instance, the horizontal line $y = 1$ (shown in red) intersects the graph of $f(x)$ twice, i.e. $f^{-1}(1)$ cannot be well defined.

(c) From the graph of $f(x)$, we can see that the graph contains the point $(6, 4)$, which implies $f(6) = 4$.

(d) The x-intercepts are the point(s) where the graph of $f(x)$ intersects the x-axis. From the graph of $f(x)$, we can see that these are $(-5, 0)$, $(-1, 0)$, $(2, 0)$, and $(4, 0)$, i.e. $x = -5, -1, 2, 4$.

(e) There are such x-values. If there is an x such that $f(x) = 2$, then the line $y = 2$ (shown in green above) intersects the graph of $f(x)$. The x-value of such intersection points are x-values such that $f(x) = 2$. Examining the graph, we can see that if $x \approx -5.83284, 5.3828$, then $f(x) = 2$.

2. (10 points) Consider invertible functions f, g, whose values at several specified x-values are given below. Find the following:

x	-6	2	0	5	9
f	1	5	2	-6	3
g	0	2	9	1	6

(a) $(f+g)(9)$

(b) $(f \circ g)(0)$

(c) $\left(\frac{g}{f}\right)(2)$

(d) y-intercept of $f(x)$

(e) An x-intercept of $g(x)$

Solution.

(a)

$$(f+g)(9) = f(9) + g(9) = 3 + 6 = 9$$

(b)

$$(f \circ g)(0) = f\big(g(0)\big) = f(9) = 3$$

(c)

$$\left(\frac{g}{f}\right)(2) = \frac{g(2)}{f(2)} = \frac{2}{5}$$

(d) The y-intercept is the point/value where the graph of $f(x)$ intersects the y-axis. But this occurs when $x = 0$. So the y-intercept of $f(x)$ is the value $f(0)$:

$$f(0) = 2$$

Therefore, the y-intercept of $f(x)$ is $(0, 2)$.

(e) The x-intercept is the point(s)/value(s) where the graph of $g(x)$ intersects the x-axis. But this occurs when $g(x) = 0$. So the x-intercept(s) of $g(x)$ are the x-values for which $g(x) = 0$:

$$g(-6) = 0$$

Therefore, the x-intercept of $g(x)$ is $(-6, 0)$.

3. (10 points) Let $f(x) = x^2 + 2x - 1$, $g(x) = 3x + 8$, and c be a constant. Showing all your work and simplifying as much as possible, compute the following:

   (a) $(fg)(4)$

   (b) $f(-2) - g(1)$

   (c) $(f - g)(2)$

   (d) $(f \circ g)(0)$

   (e) $(g \circ f)(c)$

Solution. First, observe that...

$$\begin{aligned} f(-2) &= (-2)^2 + 2(-2) - 1 = 4 - 4 - 1 = -1 \\ f(2) &= 2^2 + 2(2) - 1 = 4 + 4 - 1 = 7 \\ f(4) &= 4^2 + 4(2) - 1 = 16 + 8 - 1 = 23 \\ f(8) &= 8^2 + 2(8) - 1 = 64 + 16 - 1 = 79 \end{aligned}$$

$$\begin{aligned} g(0) &= 3(0) + 8 = 0 + 8 = 8 \\ g(1) &= 3(1) + 8 = 3 + 8 = 11 \\ g(2) &= 3(2) + 8 = 6 + 8 = 14 \\ g(4) &= 3(4) + 8 = 12 + 8 = 20 \end{aligned}$$

Then we have...

(a)
$$(fg)(4) = f(4)g(4) = 23 \cdot 20 = 460$$

(b)
$$f(-2) - g(1) = -1 - 11 = -12$$

(c)
$$(f - g)(2) = f(2) - g(2) = 7 - 14 = -7$$

(d)
$$(f \circ g)(0) = f\big(g(0)\big) = f(8) = 79$$

(e)
$$(g \circ f)(c) = g\big(f(c)\big) = g(c^2 + 2c - 1) = 3(c^2 + 2c - 1) + 8 = 3c^2 + 6c - 3 + 8 = 3c^2 + 6c + 5$$

4. (10 points) Consider the function $\ell(x) = 4x - 6$.

   (a) Is $\ell(x)$ linear? Explain.
   (b) Find the slope and y-intercept of $\ell(x)$.
   (c) Compute $\ell(\frac{17}{2})$.
   (d) Is there an x such that $\ell(x) = 10$? Explain.
   (e) Find the x-intercept of $\ell(x)$.

Solution.

(a) The function $\ell(x) = 4x - 6$ is linear because it has the form $y = mx + b$ with $y = \ell$, $x = x$, $m = 4$, and $b = -6$.

(b) From (a), we have $m = 4$ and $b = -6$. Therefore, the slope is $m = 4$ and the y-intercept is -6, i.e. the point $(0, -6)$.

(c) We have...

$$\ell\left(\frac{17}{2}\right) = 4 \cdot \frac{17}{2} - 6 = 2 \cdot 17 - 6 = 34 - 6 = 28$$

(d) If there were such an x, we would have...

$$\begin{aligned} \ell(x) &= 10 \\ 4x - 6 &= 10 \\ 4x &= 16 \\ x &= 4 \end{aligned}$$

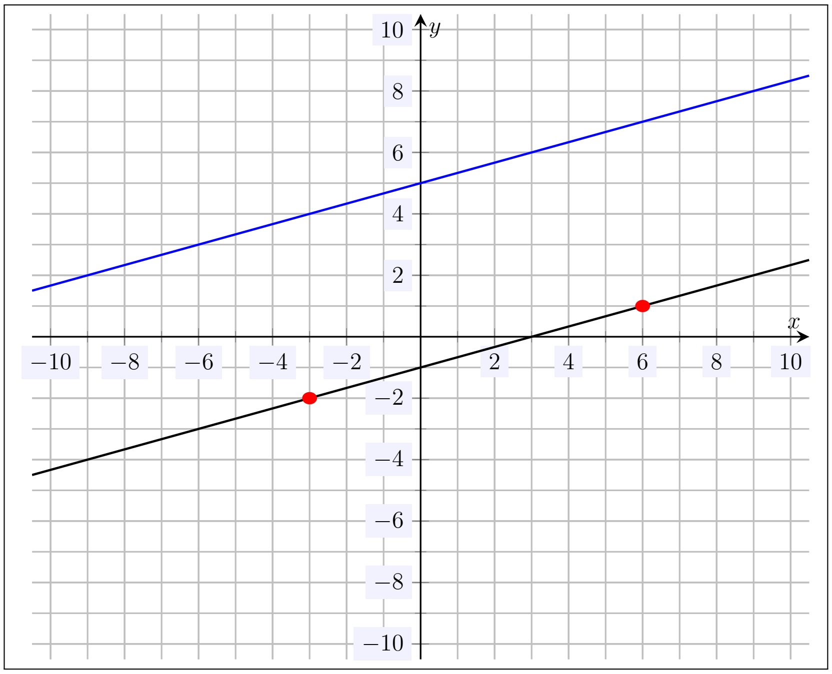
Finally, observe that $\ell(4) = 4(4) - 6 = 16 - 6 = 10$. Therefore, $x = 4$ is an x such that $\ell(x) = 10$.

(e) The x-intercept(s) of a function $f(x)$ are the x-values such that $f(x) = 0$. If $\ell(x)$ has an x-intercept, then $\ell(x) = 0$. But then...

$$\begin{aligned} \ell(x) &= 0 \\ 4x - 6 &= 0 \\ 4x &= 6 \\ x &= \frac{6}{4} \\ x &= \frac{3}{2} \end{aligned}$$

Because $\ell(\frac{3}{2}) = 4 \cdot \frac{3}{2} - 6 = 2 \cdot 3 - 6 = 6 - 6 = 0$. Therefore, the x-intercept of $\ell(x)$ is $x = \frac{3}{2}$, i.e. the point $(\frac{3}{2}, 0)$.

5. (10 points) Find the equation of the line that has y-intercept 5 that is parallel to the line shown below.



Solution. Because the given line is not vertical, any line parallel to this line will also not be vertical. Then the line in question must have the form $y = mx + b$ for some m, b. Because the line in question is parallel to the line shown, it must have the same slope as the given line. Using the points in red on the given line, we compute the slope of the line shown above:

$$m = \frac{\Delta y}{\Delta x} = \frac{1 - (-2)}{6 - (-3)} = \frac{1 + 2}{6 + 3} = \frac{3}{9} = \frac{1}{3}$$

But then we have $y = \frac{1}{3}x + b$. Because b is the y-intercept and the line in question must have y-intercept 5, we must have $y = \frac{1}{3}x + 5$. We plot this line in blue on the plot above.

6. (10 points) Find the equation of the line with x-intercept -6 that passes through the point of intersection of $y = 5x - 1$ and $y = 6 - 2x$.

Solution. We know the line in question has x-intercept -6, i.e. the graph of the line contains the point $(-6, 0)$. We know the line in question also contains the point of intersection of $y = 5x - 1$ and $y = 6 - 2x$. But at this point of intersection, the lines have the same x and y-value. But then...

$$\begin{aligned} y &= y \\ 5x - 1 &= 6 - 2x \\ 7x - 1 &= 6 \\ 7x &= 7 \\ x &= 1 \end{aligned}$$

But then $y(1) = 5(1) - 1 = 5 - 1 = 4$ (or $y(1) = 6 - 2(1) = 6 - 2 = 4$). Therefore, the line in question also contains the point $(1, 4)$. Because $1 \neq -6$, we know the line in question is not vertical; therefore, it must have the form $y = mx + b$ for some m, b. Using the points $(-6, 0)$ and $(1, 4)$, we can compute the slope of the line in question:

$$m = \frac{\Delta y}{\Delta x} = \frac{4 - 0}{1 - (-6)} = \frac{4 - 0}{1 + 6} = \frac{4}{7}$$

We then have $y = \frac{4}{7}\,x + b$. Because the line contains the point $(-6, 0)$, when $x = -6$, $y = 0$. But then...

$$\begin{aligned} y &= \frac{4}{7}\,x + b \\ 0 &= \frac{4}{7} \cdot -6 + b \\ 0 &= -\frac{24}{7} + b \\ b &= \frac{24}{7} \end{aligned}$$

Therefore, $y = \frac{4}{7}\,x + \frac{24}{7} = \frac{4x+24}{7} = \frac{4(x+6)}{7}$.

7. (10 points) Consider the lines $\ell_1(x) = 6x - 17$ and $\ell_2(x) = 8 - 11x$.

   (a) Determine whether the given lines are parallel, perpendicular, or neither. Justify your answer.

   (b) Do the lines intersect? If not, explain why. If so, find their point of intersection.

Solution.

(a) The slope of $\ell_1(x) = 6x - 17$ is $m_1 = 6$. The slope of $\ell_2(x) = 8 - 11x$ is $m_2 = -11$. Because $m_1 \neq m_2$, the lines are not parallel. Furthermore, because $m_1 = 6 \neq \frac{1}{11} = -\frac{1}{m_2}$, the lines are not perpendicular. Therefore, ℓ_1 and ℓ_2 are neither parallel nor perpendicular.

(b) Because ℓ_1 and ℓ_2 are not parallel, they intersect. At their point of intersection, the lines have the same output for the same input. But then...

$$\begin{aligned} \ell_1(x) &= \ell_2(x) \\ 6x - 17 &= 8 - 11x \\ 17x - 17 &= 8 \\ 17x &= 25 \\ x &= \frac{25}{17} \end{aligned}$$

But then...

$$\ell_1\left(\frac{25}{17}\right) = 6 \cdot \frac{25}{17} - 17 = \frac{150}{17} - 17 = \frac{150}{17} - \frac{289}{17} = -\frac{139}{17}$$

$$\ell_2\left(\frac{25}{17}\right) = 8 - 11 \cdot \frac{25}{17} = 8 - \frac{275}{17} = \frac{136}{17} - \frac{275}{17} = -\frac{139}{17}$$

Therefore, $\ell_1(x)$ and $\ell_2(x)$ intersect at the point $\left(\frac{25}{17}, -\frac{139}{17}\right)$.

8. (10 points) Consider the function given by $f(x) = 11 - 9x$.

   (a) Explain why $f^{-1}(x)$ exists.

   (b) Find $f^{-1}(x)$.

   (c) Use f^{-1} to solve the equation $f(x) = \frac{17}{9}$.

Solution.

(a) The function $f(x) = 11-9x$ is a linear function because it has the form $y = mx+b$, where $y = f$, $x = x$, $m = -9$, and $b = 11$. Because $f(x)$ is not a vertical line, it must pass the Horizontal Line Test. Therefore, $f^{-1}(x)$ must exist.

(b) We interchange the role of $y = f(x)$ and x and solve for y to find $f^{-1}(x)$. Interchanging the variables, we have $y = 11 - 9x \rightsquigarrow x = 11 - 9y$. But then…

$$\begin{aligned} x &= 11 - 9y \\ x - 11 &= -9y \\ y &= \frac{x - 11}{-9} \\ y &= \frac{11 - x}{9} \end{aligned}$$

Therefore, $f^{-1}(x) = \frac{11-x}{9}$. One can confirm this: $f^{-1}(f(x)) = f^{-1}(11 - 9) = \frac{11-(11-9x)}{9} = \frac{9x}{9} = x$ and $f(f^{-1}(x)) = f(\frac{11-x}{9}) = 11 - 9(\frac{11-x}{9}) = 11 - (11 - x) = x$.

(c) We have…

$$\begin{aligned} f(x) &= \frac{17}{9} \\ f^{-1}\big(f(x)\big) &= f^{-1}\left(\frac{17}{9}\right) \\ x &= \frac{11 - x}{9}\bigg|_{x=\frac{17}{9}} \\ x &= \frac{11 - \frac{17}{9}}{9} \\ x &= \frac{\frac{82}{9}}{9} \\ x &= \frac{82}{81} \end{aligned}$$

9. (10 points) An *arithmetic sequence* is a list of numbers where the difference between one number and the next is always the same. For instance, 2, 6, 10, 14, 18, . . . is an arithmetic sequence because the difference between sequential terms is always 4, while the sequence 1, 2, 3, 5, 7, 10, 13, . . . is *not* an arithmetic sequence because the difference between sequential terms is not constant. Let S be the sequence 34, 57, 80, 103, 126, ...

   (a) Find a function $S(n)$ that gives the nth term of the sequence.

   (b) Find the 835th term of the sequence.

   (c) Is 3,500 a term of this sequence? Explain.

Solution. Observe that the sequence S is arithmetic because $23 = 57 - 34 = 80 - 57 = 103 - 80 = 127 - 103 = \cdots$.

(a) Because the terms of the sequence change by a constant rate, we can represent the nth term of the sequence by a linear function. Let $S(n)$ denote the nth term of the sequence. Then we know $S(n) = mn + b$. Because each term of the sequence is 23 more than the last, we know that $m = 23$. But then $S(n) = 23n + b$. We know that $S(1) = 34$. But $34 = S(1) = 23(1) + b = 23 + b$. So $b + 23 = 34$, which implies $b = 11$. Therefore, $S(n) = 23n + 11$.

(b) This is $S(835)$. But we have. . .

$$S(835) = 23(835) + 11 = 19205 + 11 = 19216$$

(c) Every term of the sequence is of the form $S(n) = 3500$. If 3500 were a term of the sequence, then there would be a whole number n such that $S(n) = 3500$. But then. . .

$$\begin{aligned} S(n) &= 3500 \\ 23n + 11 &= 3500 \\ 23n &= 3489 \\ n &= 151.696 \end{aligned}$$

Because n is not a whole number, 3,500 cannot be a term of the sequence.

10. (10 points) A cleaning service does not have their prices listed on their website but the site does mention they charge a fixed amount per hour. You make some calls and have one friend that used their service and paid \$212.50 for a 3 hour cleaning while another friend paid \$400 for a 6 hour cleaning. Let $C(h)$ be the cost the service will charge for h hours of cleaning.

    (a) Explain why $C(h)$ is linear.
    (b) Find $C(h)$.
    (c) Interpret the slope and y-intercept for $C(h)$.
    (d) How many hours of cleaning can you get for \$950?

    Solution.

    (a) Because the cost of the cleaning increases at a constant rate per hour, $C(h)$ must be a linear function.

    (b) From (a), we know that $C(h)$ is linear. Then $C(h) = mh + b$ for some m, b. We know that a 3 hour cleaning costs \$212.50 and a 6 hour cleaning costs \$400, i.e. $(3, 212.50)$ and $(6, 400)$ are points on the graph of $C(h)$. But then...

    $$m = \frac{\Delta C}{\Delta h} = \frac{400 - 212.50}{6 - 3} = \frac{187.50}{3} = 62.5$$

    Thus, $C(h) = 62.5h + b$. Using the fact that $(6, 400)$ on the graph of $C(h)$, we know $C = 400$ when $h = 6$. But then...

    $$\begin{aligned} C(h) &= 62.5h + b \\ C(6) &= 62.5(6) + b \\ 400 &= 375 + b \\ b &= 25 \end{aligned}$$

    Therefore, $C(h) = 62.5h + 25$.

    (c) The slope of $C(h)$ is $m = 62.50$. Because $m = \frac{\Delta C}{\Delta h}$, we know that $m = \$62.50/\text{hr}$. But then we know every additional hour of cleaning increases the price charged by \$62.50, i.e. the service charges \$62.50 per hour. The y-intercept of $C(h)$ is $b = 25$. This is the cost of cleaning for $h = 0$ hours of cleaning. This must represent some type of service, processing, arrival, etc. charge for the cleaning. This could also represent a minimal charge.

    (d) If h_0 is the number of hours of cleaning one can get for \$950, then $C(h_0) = 950$. But then...

    $$\begin{aligned} C(h_0) &= 950 \\ 62.5h_0 + 25 &= 950 \\ 62.5h_0 &= 925 \\ h_0 &\approx 14.8 \end{aligned}$$

    Therefore, one can receive up to 14.8 hours of cleaning for \$950.