

MAT 101: Exam 1
Fall – 2022
10/07/2022
85 Minutes

Name: Caleb McWhorter — Solutions

Write your name on the appropriate line on the exam cover sheet. This exam contains 21 pages (including this cover page) and 20 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
15	10	
16	10	
17	10	
18	10	
19	10	
20	10	
Total:	200	

1. (10 points) Mark each of the following statements as True (T) or False (F).

(a) F : The number 1 is prime.

(b) T : Every rational number is a real number.

(c) F : The number $0.5 \cdot 10^{-3}$ is in scientific notation.

(d) F : Every real number is a rational number.

(e) T : The number 6 has four positive divisors.

(f) T : Every integer greater than 1 is either prime or can be written as a product of primes.

(g) T : $\left(\sqrt[20]{\pi^5}\right)^4 = \pi$

(h) T : A positive number decreased by 20% results in the same number as finding 80% of the number.

(i) F : All real numbers have a square root that is a real number.

(j) T : A real number is rational if and only if it has a decimal expansion that repeats.

2. (10 points) Showing all your work, find the prime factorizations of the following integers—if the integer is prime, simply state this:

(a) 49

(b) 54

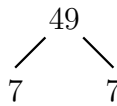
(c) 75

(d) 131

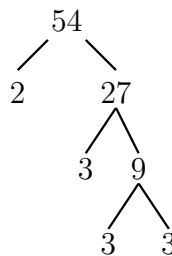
(e) 621

Solution.

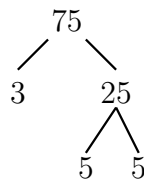
(a) $49 = 7^2$



(b) $54 = 2 \cdot 3^3$

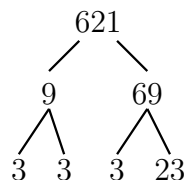


(c) $75 = 3 \cdot 5^2$



(d) 131 is prime, i.e. $131 = 131^1$

(e) $621 = 3^3 \cdot 23$



3. Without performing any explicit computation, e.g. saying $6/2 = 3$ is an integer or $7/2 = 3.5$ is not an integer, justify the following statements:
- (a) (3 points) The integer 5,549,787,074,121 is divisible by 3.
 - (b) (3 points) The integer 13,513,205,490,217,689,368 is not divisible by 5.
 - (c) (4 points) The integer 202,202,022 is divisible by 6.

Solution.

- (a) *An integer is divisible by 3 if and only if the sum of the digits is divisible by 3. We have...*

$$5 + 5 + 4 + 9 + 7 + 8 + 7 + 0 + 7 + 4 + 1 + 2 + 1 = 60$$

Because 60 is divisible by 3, we know that 5,549,787,074,121 is divisible by 3.

- (b) *An integer is divisible by 5 if and only if it ends in 0 or 5. Because the integer 13,513,205,490,217,689,368 does not end in 0 or 5, it is not divisible by 5.*

- (c) *An integer is divisible by 6 if and only if it is divisible by both 2 and 3. An integer is divisible by 2 if and only if it is even. Because 202,202,022 is even, it is divisible by 2. An integer is divisible by 3 if and only if the sum of the digits is divisible by 3. We have $2 + 0 + 2 + 2 + 0 + 2 + 0 + 2 + 2 = 12$, which is divisible by 3. But then 202,202,022 is divisible by 3. Because 202,202,022 is divisible by 2 and 3, it is divisible by 6.*

4. Showing any necessary work, answer the following:
- (a) (3 points) Find at least three multiples of 12.
 - (b) (3 points) Find at least two divisors of 140.
 - (c) (4 points) Find the least common multiple of the smallest and largest prime factor of 30.

Solution.

- (a) *The multiples of 12 are 0, 12, 24, 36, 48, 60, ... and $-12, -24, -36, -48, -60, \dots$*
- (b) *The divisors of 140 are 1, 2, 4, 5, 7, 10, 14, 20, 28, 35, 70, 140.*
- (c) *The factors of 30 are 1, 2, 3, 5, 6, 10, 15, 20. Of these factors, the prime factors of 30 are 2, 3, 5. The smallest is then 2 and the largest is 5. The least common multiple of 2 and 5 is 10.*

5. Showing any necessary work, find the following:

(a) (2 points) $\text{lcm}(20, 66)$

(b) (2 points) $\text{gcd}(20, 66)$

(c) (3 points) $\text{gcd}(2^{500} \cdot 3^{600} \cdot 7^{800} \cdot 11^{900}, 2^{400} \cdot 3^{800} \cdot 5^{600} \cdot 13^{700})$

(d) (3 points) $\text{lcm}(2^{500} \cdot 3^{600} \cdot 7^{800} \cdot 11^{900}, 2^{400} \cdot 3^{800} \cdot 5^{600} \cdot 13^{700})$

Solution.

(a)

$$\text{lcm}(20, 66) = \text{lcm}(2^2 \cdot 5, 2 \cdot 3 \cdot 11) = 2^2 \cdot 3 \cdot 5 \cdot 11 = 660$$

(b)

$$\text{gcd}(20, 66) = \text{gcd}(2^2 \cdot 5, 2 \cdot 3 \cdot 11) = 2$$

(c)

$$\text{gcd}(2^{500} \cdot 3^{600} \cdot 7^{800} \cdot 11^{900}, 2^{400} \cdot 3^{800} \cdot 5^{600} \cdot 13^{700}) = 2^{400} \cdot 3^{600}$$

(d)

$$\text{lcm}(2^{500} \cdot 3^{600} \cdot 7^{800} \cdot 11^{900}, 2^{400} \cdot 3^{800} \cdot 5^{600} \cdot 13^{700}) = 2^{500} \cdot 3^{800} \cdot 5^{600} \cdot 7^{800} \cdot 11^{900} \cdot 13^{700}$$

6. Showing all your work and reducing as much as possible, compute the following (be sure to give an exact value as your answer):

(a) (3 points) $\frac{3}{10} - \frac{11}{15}$

(b) (3 points) $\frac{6}{55} \cdot -\frac{22}{21}$

(c) (4 points) $\frac{\frac{20}{21}}{\frac{10}{49}}$

Solution.

(a)

$$\frac{3}{10} - \frac{11}{15} = \frac{9}{30} - \frac{22}{30} = \frac{9 - 22}{30} = -\frac{13}{30}$$

(b)

$$\frac{6}{55} \cdot -\frac{22}{21} = \frac{\cancel{6}^2}{\cancel{55}^5} \cdot -\frac{\cancel{22}^2}{\cancel{21}^7} = -\frac{4}{35}$$

(c)

$$\frac{\frac{20}{21}}{\frac{10}{49}} = \frac{\cancel{20}^2}{\cancel{21}^3} \cdot \frac{\cancel{49}^7}{\cancel{10}^1} = \frac{14}{3}$$

7. Showing all your work, either convert the given rational number to a decimal value or the given decimal number to a rational number—in either case, simplify as much as possible:

(a) (5 points) 0.18

(b) (5 points) $\frac{19}{11}$

Solution.

(a)

$$0.18 = \frac{18}{100} = \frac{9}{50}$$

(b)

$$\begin{array}{r} 11 \overline{) 19.00} \\ \underline{11} \\ 8.0 \\ \underline{7.7} \\ 30 \\ \underline{22} \\ 8 \end{array}$$

$$\frac{19}{11} = 1.\overline{72}$$

8. (10 points) Showing all your work, express the decimal number $0.17171717\overline{17}$ as a fully reduced rational number.

Solution.

$$\begin{array}{rcl} 100N & = & 17.17171717\overline{17} \\ - N & = & 0.17171717\overline{17} \\ \hline 99N & = & 17 \\ N & = & \frac{17}{99} \end{array}$$

$$0.\overline{17} = \frac{17}{99}$$

9. Showing all your work, simplify the expressions given below. Your answer should have each variable occurring at most once and should contain no negative powers.

(a) (3 points) $\frac{(xy^5)^{-2}}{x^{-8}(xy)^6y^5}$

(b) (3 points) $\left(\frac{x(x^{10}y^3)^0}{y^5}\right)^{-2}$

(c) (4 points) $\frac{x(x^3(y^{-2}z)^{-2})^{-1}}{y^{-5}z^2x^{-1}}$

Solution.

(a)

$$\frac{(xy^5)^{-2}}{x^{-8}(xy)^6y^5} = \frac{x^{-2}y^{-10}}{x^{-8}x^6y^6y^5} = \frac{x^{-2}y^{-10}}{x^{-2}y^{11}} = \frac{y^{-10}}{y^{11}} = \frac{1}{y^{10}y^{11}} = \frac{1}{y^{21}}$$

(b)

$$\left(\frac{x(x^{10}y^3)^0}{y^5}\right)^{-2} = \left(\frac{y^5}{x(x^{10}y^3)^0}\right)^2 = \frac{y^{10}}{x^2(x^{10}y^3)^0} = \frac{y^{10}}{x^2 \cdot 1} = \frac{y^{10}}{x^2}$$

(c)

$$\frac{x(x^3(y^{-2}z)^{-2})^{-1}}{y^{-5}z^2x^{-1}} = \frac{x \cdot x^{-3}(y^{-2}z)^2}{y^{-5}z^2x^{-1}} = \frac{x \cdot x^{-3} \cdot y^{-4} \cdot z^2}{y^{-5}z^2x^{-1}} = \frac{x^{-2}y^{-4}z^2}{x^{-1}y^{-5}z^2} = \frac{xy^5}{x^2y^4} = \frac{y}{x}$$

10. Showing all your work, simplify the expressions given below, being sure to give an exact value:

(a) (3 points) $\sqrt{24}$

(b) (3 points) $\sqrt[3]{54}$

(c) (4 points) $\sqrt[5]{2^3 \cdot 3^5 \cdot 5^{10} \cdot 7^6}$

Solution.

(a)

$$\sqrt{24} = \sqrt{2^3 \cdot 3} = \sqrt{2^2 \cdot (2 \cdot 3)} = 2\sqrt{2 \cdot 3} = 2\sqrt{6}$$

(b)

$$\sqrt[3]{54} = \sqrt[3]{2 \cdot 3^3} = 3\sqrt[3]{2}$$

(c)

$$\sqrt[5]{2^3 \cdot 3^5 \cdot 5^{10} \cdot 7^6} = \sqrt[5]{3^5 \cdot 5^{10} \cdot 7^5 \cdot (2^3 \cdot 7)} = 3^1 \cdot 5^2 \cdot 7^1 \sqrt{8 \cdot 7} = 525\sqrt{56}$$

11. Showing all your work, simplify the expressions given below. Your answer should have each variable occurring at most once, contain no negative powers, and contain no fractional powers:

(a) (3 points) $\frac{\sqrt{x^4 y^{11}}}{x\sqrt{y}}$

(b) (3 points) $\left(\frac{x^5 y^8}{x^{-2} y^5}\right)^{-1/3}$

(c) (4 points) $\frac{\sqrt[5]{x^{20} y^6}}{x^{10} \sqrt{y^3}}$

Solution.

(a)

$$\frac{\sqrt{x^4 y^{11}}}{x\sqrt{y}} = \frac{x^{4/2} y^{11/2}}{x y^{1/2}} = \frac{x^2 y^{11/2}}{x y^{1/2}} = x^{2-1} y^{11/2-1/2} = x y^{10/2} = x y^5$$

(b)

$$\left(\frac{x^5 y^8}{x^{-2} y^5}\right)^{-1/3} = \left(\frac{x^{-2} y^5}{x^5 y^8}\right)^{1/3} = \left(\frac{y^5}{x^2 x^5 y^8}\right)^{1/3} = \left(\frac{1}{x^7 y^3}\right)^{1/3} = \frac{1}{x^{7/3} y^{3/3}} = \frac{1}{y \sqrt[3]{x^7}}$$

(c)

$$\frac{\sqrt[5]{x^{20} y^6}}{x^{10} \sqrt{y^3}} = \frac{x^{20/5} y^{6/5}}{x^{10} y^{3/2}} = \frac{x^4 y^{6/5}}{x^{10} y^{3/2}} = \frac{y^{6/5}}{x^6 y^{3/2}} = \frac{y^{\frac{6}{5}-\frac{3}{2}}}{x^6} = \frac{y^{\frac{12}{10}-\frac{15}{10}}}{x^6} = \frac{y^{-3/10}}{x^6} = \frac{1}{x^6 \sqrt[10]{y^3}}$$

12. Showing all your work and being sure to give an exact answer that is simplified as much as possible, rationalize the following:

(a) (4 points) $\frac{4}{\sqrt{6}}$

(b) (4 points) $\frac{6}{5 + \sqrt{2}}$

(c) (2 points) $\frac{-5}{\sqrt[3]{7}}$

Solution.

(a)

$$\frac{4}{\sqrt{6}} = \frac{4}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{4\sqrt{6}}{6} = \frac{2\sqrt{6}}{3}$$

(b)

$$\frac{6}{5 + \sqrt{2}} = \frac{6}{5 + \sqrt{2}} \cdot \frac{5 - \sqrt{2}}{5 - \sqrt{2}} = \frac{6(5 - \sqrt{2})}{25 - 5\sqrt{2} + 5\sqrt{2} - 2} = \frac{6(5 - \sqrt{2})}{23}$$

(c)

$$\frac{-5}{\sqrt[3]{7}} = \frac{-5}{7^{1/3}} \cdot \frac{7^{2/3}}{7^{2/3}} = \frac{-5 \cdot 7^{2/3}}{7^1} = \frac{-5\sqrt[3]{7^2}}{7} = \frac{-5\sqrt[3]{49}}{7}$$

13. Showing all your work, compute the following:

(a) (3 points) 74% of 1450

(b) (3 points) 180% of 22

(c) (4 points) 1% of 60

Solution.

(a)

$$1450(0.74) = 1073$$

(b)

$$22(1.80) = 39.6$$

(c)

$$60(0.01) = 0.6$$

14. Showing all your work, compute the following:

- (a) (3 points) 550 increased by 40%
- (b) (3 points) 1200 decreased by 70%
- (c) (4 points) 70 increased by 150%

Solution.

(a)

$$550(1 + 0.40) = 550(1.40) = 770$$

(b)

$$1200(1 - 0.70) = 1200(0.30) = 360$$

(c)

$$70(1 + 1.50) = 70(2.50) = 175$$

15. (10 points) Suppose a student's course grade consists of the following weights:

Homework	50%	Exam 2	12%
Quizzes	5%	Final Exam	15%
Exam 1	8%	Project	10%

Suppose also that a student had a 80% homework average, 84.3% quiz average, 91% on exam 1, 95% on exam 2, 78% on the final, and 89% on the project. Compute the student's course average to the nearest tenth of a percent.

Solution. *The course grade is a weighted average of each of the grade components by what percentage was actually earned. We can compute this directly:*

$$\begin{aligned}\text{Course Average} &= \sum \text{values} \cdot \text{weights} \\ &= 50\%(0.80) + 5\%(0.843) + 8\%(0.91) + 12\%(0.95) + 15\%(0.78) + 10\%(0.89) \\ &= 40\% + 4.215\% + 7.28\% + 11.4\% + 11.7\% + 8.9\% \\ &= 83.495\% \\ &\approx 83.5\%\end{aligned}$$

16. (10 points) A certain student has a current GPA of 3.351 based on 86 credits. This semester, they take a number of courses with the courses, credits, and their letter grade in the course shown below.

Course	Credits	Grade				
Galois Representations	4	A	A	4.0	C+	2.3
Advanced Music Theory	3	A–	A–	3.7	C	2.0
Creative Writing	3	C	B+	3.3	C–	1.7
Business Ethics	3	B+	B	3.0	D	1.0
Common Law	3	B–	B–	2.7	F	0.0

Using the university's letter grade values show on the right above, what is the students current GPA to the nearest thousandth? [Be sure to show all your work.]

Solution. GPA is a weighted average, where the course value, i.e. grade (given by its numerical equivalent), is weighted by the percentage of the total credits. This can be written and computed more simply as follows:

$$\begin{aligned}
 \text{GPA} &= \frac{\sum \text{value} \cdot \text{credit}}{\text{total credits}} \\
 &= \frac{4.0(4) + 3.7(3) + 2.0(3) + 3.3(3) + 2.7(3)}{4 + 3 + 3 + 3 + 3} \\
 &= \frac{16 + 11.1 + 6 + 9.9 + 8.1}{16} \\
 &= \frac{51.1}{16} \\
 &\approx 3.194
 \end{aligned}$$

The students overall GPA is then the average of their previous GPA with their current GPA, weighted by credits:

$$\begin{aligned}
 \text{Overall GPA} &= \frac{\text{Prev. GPA} \cdot \text{Prev. Credits} + \text{New GPA} \cdot \text{New Credits}}{\text{Old Credits} + \text{New Credits}} \\
 &= \frac{3.351 \cdot 86 + 3.194 \cdot 16}{86 + 16} \\
 &= \frac{288.186 + 51.104}{102} \\
 &= \frac{339.29}{102} \\
 &\approx 3.326
 \end{aligned}$$

17. Showing all your work, convert the following numbers in scientific notation to decimal values:

(a) (3 points) $1.7 \cdot 10^{-6}$

(b) (3 points) $8.912 \cdot 10^8$

(c) (4 points) $-4.3 \cdot 10^0$

Solution.

(a)

$$1.7 \cdot 10^{-6} = 0.0000017$$

(b)

$$8.912 \cdot 10^8 = 891200000$$

(c)

$$-4.3 \cdot 10^0 = -4.3$$

18. Showing all your work, convert the following decimal numbers to scientific notation:

(a) (3 points) 0.0000000044

(b) (3 points) 15650000

(c) (4 points) -7.8

Solution.

(a)

$$0.0000000044 = 4.4 \cdot 10^{-9}$$

(b)

$$15650000 = 1.565 \cdot 10^7$$

(c)

$$-7.8 = -7.8 \cdot 10^0$$

19. (10 points) Showing all your work, convert 9.8 km/hr^2 to ft per square second. Note that 1 m is 3.28084 ft.

Solution.

9.8 km	1000 m	3.28084 ft	1 hr	1 min	1 hr	1 min
1 hr ²	1 km	1 m	60 min	60 s	60 min	60 s

$$= 0.0024808821 \text{ km/hr}^2$$

20. (10 points) A rectangular wall is 80 ft long and 15 ft wide. A painter uses paint cans that contain enough paint to cover 84 ft^2 of a surface. The painter is also able to paint 230 ft^2 of a surface every hour. What is the minimum number of paint cans they need to bring to finish the job and how long will it take them to then paint the wall? Be sure to fully justify your answers.

Solution. The area of the wall is $A = lw = (80 \text{ ft})(15 \text{ ft}) = 1200 \text{ ft}^2$. Because each can of paint can cover 84 ft^2 of wall, i.e. each the rate is $84 \text{ ft}^2/1 \text{ can}$, they need a minimum of $1200 \text{ ft}^2 / (84 \text{ ft}^2/1 \text{ can}) = 14.2857$ cans of paint. Because they cannot bring a fraction of a can of paint (and bringing 14 cans of paint would be too few), they need to bring a minimum of 15 cans of paint.

Now they are able to cover 230 ft^2 of surface every hour, i.e. they paint at a rate of $230 \text{ ft}^2/1 \text{ hr}$. The wall has a surface area of $1,200 \text{ ft}^2$. But then it will take $1200 \text{ ft}^2 / (230 \text{ ft}^2/1 \text{ hr}) \approx 5.21739$ hrs to paint the wall, i.e. 5 hours, 13 minutes, and 2.6 seconds.