Name: Solutions — Caleb McWhorter
MATH 308
Fall 2021
HW 14: Due 11/22

"If you're going to do something tonight that you'll be sorry for tomorrow morning, sleep late."

–Henny Youngman

Problem 1. (10pt) Do there exist integers a, b such that $2a+3b = 5$? Explain. Do there exist integers x, y such that $8x + 12y = 3$? Explain.

Solution. Let a, b be integers and let $d = \gcd(a, b)$. We know there exists integers x, y such that $ax + by = d$. In fact, there exist infinitely many integers x, y such that $ax + by = d$. This can be seen from the fact that if $ax + by = d$, then $d = a(x + k\,\frac{b}{d}) + b(y - k\,\frac{a}{d})$ for any integer k because…

$$a\left(x + k\,\frac{b}{d}\right) + b\left(y - k\,\frac{a}{d}\right) = \left(ax + k\,\frac{ab}{d}\right) + \left(by - k\,\frac{ab}{d}\right) = ax + by = d$$

In fact, all the solutions to $ax + by = d$ are of the form $x = x_0 + k\,\frac{b}{d}$, $y = y_0 - k\,\frac{a}{d}$ for some integer k, where x_0, y_0 are a solution to $ax_0 + by_0 = d$. Furthermore, if N is an integer with $ax + by = N$ for some x, y, then d divides N.

Observe that $\gcd(2, 3) = 1$ and 1 divides 5. Therefore, there exists integers a, b such that $2a + 3b = 5$. In fact, taking $a = b = 1$, we have $2a + 3b = 2(1) + 3(1) = 2 + 3 = 5$.

Observe that $\gcd(8, 12) = 4$ and 3 is not divisible by 4. Therefore, the equation $8x + 12y = 3$ has no integer solutions. Of course, we did not need the theory above to prove this. Observe that $8x + 12y = 2(4x + 6y)$ must be even. Because 3 is not even, there cannot be integers x, y with $8x + 12y = 3$.

Problem 2. (10pt) Compute $\gcd(2^8 \cdot 3^5 \cdot 7^{10} \cdot 11 \cdot 19^6,\ 2^5 \cdot 3^8 \cdot 5^3 \cdot 11^2 \cdot 13 \cdot 17^3)$.

Problem 3. (10pt) Compute $\operatorname{lcm}(2^8 \cdot 3^5 \cdot 7^{10} \cdot 11 \cdot 19^6,\ 2^5 \cdot 3^8 \cdot 5^3 \cdot 11^2 \cdot 13 \cdot 17^3)$.

Problem 4. (10pt) Prove that if $a, b \in \mathbb{Z}$, then $ab = \gcd(a, b) \cdot \mathrm{lcm}(a, b)$.

Problem 5. (10pt) Use the Euclidean Algorithm to compute $\gcd(36, 98)$.

Solution. We have...

$$\begin{aligned}
98 &= 2(36) + 26 \\
36 &= 1(26) + 10 \\
26 &= 2(10) + 6 \\
10 &= 1(6) + 4 \\
6 &= 1(4) + 2 \\
4 &= 2(2)
\end{aligned}$$

Therefore, $\gcd(36, 98) = 2$.

Problem 6. (10pt) Use the Euclidean Algorithm to find integers x, y such that $36x + 98y = \gcd(36, 98)$.

Solution. We use the Extended Euclidean Algorithm. From Problem 5, we have...

$$\begin{aligned} 98 &= 2(36) + 26 \\ 36 &= 1(26) + 10 \\ 26 &= 2(10) + 6 \\ 10 &= 1(6) + 4 \\ 6 &= 1(4) + 2 \\ 4 &= 2(2) \end{aligned}$$

But then we have...

$$\begin{aligned} 2 &= 6 - 1(4) \\ &= 6 - 1\big(10 - 1(6)\big) \\ &= 6 - 1(10) + 1(6) \\ &= 2(6) - 1(10) \\ &= 2\big(26 - 2(10)\big) - 1(10) \\ &= 2(26) - 4(10) - 1(10) \\ &= 2(26) - 5(10) \\ &= 2(26) - 5\big(36 - 1(26)\big) \\ &= 2(26) - 5(36) + 5(26) \\ &= 7(26) - 5(36) \\ &= 7\big(98 - 2(36)\big) - 5(36) \\ &= 7(98) - 14(36) - 5(36) \\ &= 7(98) - 19(36) \end{aligned}$$

Therefore, taking $x = -19$ and $y = 7$, we have $36x + 98y = 2 = \gcd(36, 98)$.