

MAT 308: Exam 1

Name: _____

Fall – 2023

10/12/2023

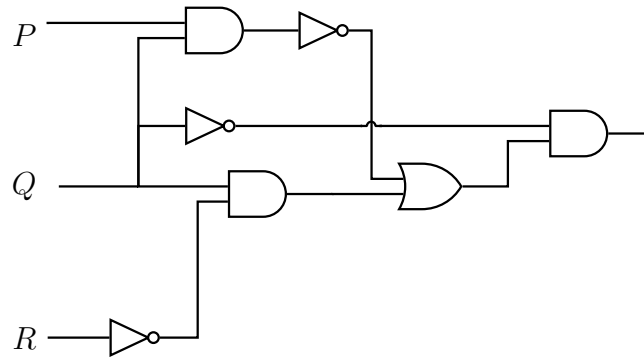
'∞' Minutes

Write your name on the appropriate line on the exam cover sheet. This exam contains 13 pages (including this cover page) and 10 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) Consider the logical expressions $\neg(P \wedge Q)$ and $(P \rightarrow \neg Q) \vee (\neg P \wedge \neg Q)$.
 - (a) Show that these two expressions are logically equivalent by constructing the truth table for both expressions.
 - (b) Show that $\neg(P \wedge Q) \equiv (P \rightarrow \neg Q) \vee (\neg P \wedge \neg Q)$ by ‘simplifying’ the expression $(P \rightarrow \neg Q) \vee (\neg P \wedge \neg Q)$.
 - (c) Without computing the truth table for $P \wedge Q \Leftrightarrow (P \rightarrow \neg Q) \vee (\neg P \wedge \neg Q)$, determine whether this expression is a tautology, contradiction, or neither. Explain how you came to your conclusion.

2. (10 points) Consider the digital logic circuit given below. We shall find a simpler circuit that has the same ‘behavior’, i.e. input-output table, as the given circuit.



- Find the logic expression corresponding to this circuit.
- Simplify the logical expression in (a) as much as possible.
- Construct the digital logic circuit corresponding to your answer in (b).

3. (10 points) Let z, q be free variables with universe \mathbb{Z} and \mathbb{Q} , respectively. Define the predicate $P(z, q): z \neq 0 \rightarrow qz = 1$. Consider the quantified statement. . .

$$(\forall z)(\exists q)P(z, q)$$

- (a) Write the quantified statement above as a complete English sentence.
- (b) Determine whether the given quantified statement is true or false. Be sure to justify your answer.
- (c) Write the negation, converse, and contrapositive of the given quantified statement as complete English sentences.

4. (10 points) Define sets A, B, C, D as given below with universe $\mathcal{U} = (-\infty, \infty)$.

$$A = (-10, 10) \qquad C = [5, 20)$$

$$B = (-15, 0] \qquad D = [-6, 6]$$

(a) B^c

(b) $A \Delta C$

(c) $D \setminus B$

(d) $(A \cup B)^c$

(e) $A \cap (B \cup C)$

5. (10 points) Let $X = \{a, b, c\}$ and $Y = \{1, 2\}$.

- (a) Compute $\mathcal{P}(X)$.
- (b) Compute $\mathcal{P}(X) \times Y$.
- (c) Compute $|\mathcal{P}(X) \times Y|$.
- (d) Is $(\{b, a, a\}, 1) \in \mathcal{P}(X) \times Y$?

6. (10 points) For natural numbers n , let $A_n = [-\frac{1}{n}, \frac{n+1}{n})$, and for real numbers x , let $B_x = (x - 1, x + 1)$. Compute the following:

(a) $\bigcup_{n \in \mathbb{N}} A_n$

(b) $\bigcap_{n \in \mathbb{N}} A_n$

(c) $\bigcup_{x \in \mathbb{R}} B_x$

(d) $\bigcap_{x \in \mathbb{R}} B_x$

7. (10 points) Let $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ be given by $x \mapsto 2x^2 + 11x - 6$.

- (a) Is $7 \in \text{im } \Phi$? Explain.
- (b) Is $-11 \in \Phi^{-1}(-5)$? Explain.
- (c) Compute $\text{im } \Phi$.
- (d) Compute $\Phi^{-1}(0)$.
- (e) Is $(3, 11)$ on the graph of Φ ? Explain.
- (f) Is Φ a decreasing function? Explain.
- (g) Is Φ a negative function? Explain.

— *Continued Space for Problem 7* —

8. (10 points) Complete the following parts, being sure to fully justify your reasoning:

- (a) Is the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x, y) = x^2 + y^3 - 7$ injective?
- (b) Is the function $g : \mathbb{R} \rightarrow (-\infty, 10]$ given by $x \mapsto 10 - (x - 3)^2$ surjective?
- (c) Is the function $h : \mathbb{R} \rightarrow \mathbb{R}$ given by $h(x) = \pi x - \sqrt{2}$ bijective?

9. (10 points) Below is a partial proof of the fact that if A, B, C are sets, then $A \times (B \setminus C) = (A \times B) - (A \times C)$. By filling in the missing portions, complete the partial proof below so that it is a correct, logically sound proof with ‘no gaps.’

Proposition. If A, B, C are sets, then $A \times (B \setminus C) = (A \times B) - (A \times C)$.

Proof. To show that $A \times (B \setminus C) = (A \times B) - (A \times C)$, we need to show that $A \times (B \setminus C) \subseteq (A \times B) - (A \times C)$ and $(A \times B) - (A \times C) \subseteq A \times (B \setminus C)$.

If $A \times (B \setminus C) = \emptyset$, we clearly have $A \times (B \setminus C) \subseteq (A \times B) - (A \times C)$. Similarly, if $(A \times B) - (A \times C) = \emptyset$, then we have $(A \times B) - (A \times C) \subseteq A \times (B \setminus C)$. Now assume that $A \times (B \setminus C)$ and $(A \times B) - (A \times C)$ are nonempty.

$A \times (B \setminus C) \subseteq (A \times B) - (A \times C)$: Let $(x, y) \in A \times (B \setminus C)$. We want to show that

$(x, y) \in (A \times B) - (A \times C)$. By definition, $(x, y) \in A \times (B \setminus C)$ implies that _____

and _____. Because $y \in B \setminus C$, we know that _____ and

_____. Now $x \in A$ and $y \in B$, so that $(x, y) \in A \times B$. Because $y \notin C$,

we know that $(x, y) \notin A \times C$. Because _____ and _____,

we know that $(x, y) \in (A \times B) - (A \times C)$. But then if $(x, y) \in A \times (B \setminus C)$, we know

that $(x, y) \in (A \times B) - (A \times C)$. Therefore, _____.

$(A \times B) - (A \times C) \subseteq A \times (B \setminus C)$: Let _____. We want to

show that $(x, y) \in A \times (B \setminus C)$. Because $(x, y) \in (A \times B) - (A \times C)$, we know that

_____ and _____. By definition, because $(x, y) \in$

$A \times B$, $x \in A$ and $y \in B$. We also know that $(x, y) \notin A \times C$. So either _____

or _____. But we know that $x \in A$, so that it must be that $y \notin C$. But

then $y \in B$ and _____, which implies that $y \in B \setminus C$. Because _____

and $y \in B \setminus C$, we know that $(x, y) \in A \times (B \setminus C)$. Then if $(x, y) \in (A \times B) - (A \times C)$,

we know _____. Therefore, $(A \times B) - (A \times C) \subseteq A \times (B \setminus C)$.

Because _____ and _____, we know

that $A \times (B \setminus C) = (A \times B) - (A \times C)$. □

10. (10 points) Below is a partial proof of results about the surjectivity of functions. By filling in the missing portions, complete the partial proof below so that it is a correct, logically sound proof with ‘no gaps.’

Proposition. Let A, B, C be sets and $f : A \rightarrow B, g : B \rightarrow C$ be functions.

- (i) If f, g are surjective, then $g \circ f$ is surjective.
- (ii) If $g \circ f$ is surjective, then g is surjective.

Proof.

- (i) Suppose that f, g are surjective. We want to show that _____.

We need to show that if _____, there exists _____ such that

$(g \circ f)(a) = c$. Because g is surjective, there exists $b \in B$ such that _____.

Because _____, there exists $a \in A$ such that $f(a) = b$. But then _____
_____. Therefore, $g \circ f$ is surjective.

- (ii) Assume that _____. We want to show that g is surjective.

We need to show that if _____, there exists _____ such that _____.

Let $c \in C$. Because $g \circ f : A \rightarrow C$ is surjective, there exists

$a \in A$ such that $(g \circ f)(a) = c$. We know that $(g \circ f)(a) = g(f(a))$. Because

$f : A \rightarrow B$, we know that $f(a) \in$ _____. Then there is _____

such that $b = f(a)$. Then $g(b) =$ _____. Therefore, g is surjective.

□