Name: ________________________________

MATH 361

Spring 2022

Written HW 6: Due 05/12

"Mathematics is a place where you can do things which you can't do in the real world."

–Marcus du Sautoy

Problem 1. (Exactness of Simpson's Rule) To create approximations to integrals, we used the idea of quadrature; that is, we approximated

$$\int_a^b f(x)\,dx = Q(f) + E(f)$$

where $Q(f) = \sum_{k=0}^{n} w_k f(x_k)$ and $E(f)$ was an error term. The degree of precision of a quadrature formula was a positive integer d such that the approximation was exact for polynomials of degree $\leq d$. For instance, we derived Simpson's Rule:

$$\frac{h}{3}\sum_{k=1}^{n}\big(f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})\big)$$

which had error term $E(f,h) = \dfrac{-(b-a)f^{(4)}(c)h^4}{180}$ for some $c \in (a,b)$. Hence, Simpson's Rule was exact for linear, quadratic, and cubic polynomials. Verify that Simpson's Rule is exact for cubic polynomials two ways: using the error term and directly applying the formula on an interval $[a,b]$ to x^3, x^2, x, and 1.

Problem 2. (Trapezoidal & Simpson's Rule) To approximate integrals, we had a number of different quadrature formulas. For instance, we created the Trapezoidal Rule and Simpson's Rule:

$$T(f,h) = \frac{h}{2}\sum_{k=1}^{n}\big(f(x_{k-1}) + f(x_k)\big)$$
$$S(f,h) = \frac{h}{3}\sum_{k=1}^{n}\big(f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})\big)$$

which had error terms

$$E_T(f,h) = -\frac{(b-a)f^{(2)}(c)h^2}{12}$$
$$E_S(f,h) = -\frac{(b-a)f^{(4)}(c)h^4}{180}$$

for some $h \in (a,b)$, respectively. For instance, let $f(x) = \dfrac{5}{2x+1}$.

(a) Find the exact value of $\displaystyle\int_0^1 f(x)\,dx$.

(b) Approximate the integral $\int_0^1 f(x)\,dx$ using step size $h = 0.5$ and $h = 0.2$.

(c) Find an upper bound for the error and show that your approximation in (b) is accurate to the guaranteed accuracy.

(d) If a step size of $h = 0.1$ were to approximate $\displaystyle\int_0^1 f(x)\,dx$ accurate to 8 decimal places, what should the step size be to obtain 20 digits of accuracy?

Problem 3. (Gaussian Quadrature) Quadrature allowed us to 'best' approximate an integral by finding an optimal choice of weights given a collection of nodes $\{x_i\}$. However, fixing an interval $[a, b]$, we could use Gaussian Quadrature to find both an optimal choice of weights and nodes. For instance, using a three point rule, we have...

$$\int_{-1}^{1} f(x)\, dx \approx \frac{5f(-\sqrt{3/5}) + 8f(0) + 5f(\sqrt{3/5})}{9}$$

with error term given by $\dfrac{f^{(6)}(c)}{15750}$.

(a) What are the weights and nodes in a Gaussian three-point rule?

(b) What is the precision for a Gaussian three-point rule? Explain.

(c) Use the Gaussian three-point rule to approximate

$$\int_{-2}^{6} \frac{2x-1}{x^4+1}\, dx$$

Problem 4. (Approximating Arclength) Recall from Calculus that if $f(x)$ is a differentiable function on the interval $[a, b]$, then the arclength of the curve given by $(t, f(t))$ from $t = a$ to $t = b$ is

$$\mathcal{L} = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}\, dx$$

However, even for 'reasonable' choice of $f(x)$, one could not obtain exact values for the arclength. Our only option is then to approximate the arclength. Consider the famous case of the complete elliptic integral

$$\int_0^2 \sqrt{1 + \cos^2(x)}\, dx$$

Using ten evenly spaced subintervals, apply composite Simpson's Rule to approximate the integral above. Find the absolute and relative error to the 'actual' value of 2.35168880740.

Evaluation.

Complete the following survey by rating each problem. Each area will be rated on a scale of 1 to 5. For interest, 1 is “mind-numbing” while a 5 is “mind-blowing.” For difficulty, 1 is “trivial/routine” while 5 is “brutal.” For learning, 1 means “nothing new” while 5 means ”profound awakening.” Then you to estimate the amount of time you spent on each problem (in minutes).

	Interest	Difficulty	Learning	Time Spent
Problem 1				
Problem 2				
Problem 3				
Problem 4				
Problem 5				

Finally, indicate whether you believe lectures were useful in completing this assignment and whether you believe the problems were useful enough/interesting enough to assign again to future students by checking the appropriate space.

	Lectures		Assign Again	
	Yes	No	Yes	No
Problem 1				
Problem 2				
Problem 3				
Problem 4				
Problem 5				