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MATH 308
Fall 2022
HW 10: Due 10/13

"I think that some intuition leaks out in every step of an induction proof."
–Jim Propp

Problem 1. (10pt) Let $\{a_n\}_{n\in\mathbb{N}}$ be the sequence defined by $a_n := 2^n - 5$ and $\{b_m\}_{m\in\mathbb{Z}^\times}$ be defined by $b_m := \dfrac{m+1}{m}$. Showing all your work, compute the following:

(a) $\displaystyle\sum_{k=0}^{5} a_k$

(b) $\displaystyle\sum_{\substack{j=-3\\ j\neq 0}}^{3} b_m$

(c) $\displaystyle\prod_{k=1}^{3} a_n$

(d) $\displaystyle\sum_{p=0}^{0} a_p$

(e) $\displaystyle\sum_{j=2}^{4}(a_j+b_j)$

(f) $\displaystyle\prod_{n=1}^{10^{50}} b_n$

Solution.

(a)
$$\sum_{k=0}^{5} a_k = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = -4 + (-3) + (-1) + 3 + 11 + 27 = 33$$

(b)
$$\sum_{\substack{j=-3\\ j\neq 0}}^{3} b_m = b_{-3} + b_{-2} + b_{-1} + b_1 + b_2 + b_3 = \frac{2}{3} + \frac{1}{2} + 0 + 2 + \frac{3}{2} + \frac{4}{3} = 6$$

(c)
$$\prod_{k=1}^{3} a_n = a_1 \cdot a_2 \cdot a_3 = -3 \cdot -1 \cdot 3 = 9$$

(d)
$$\sum_{p=0}^{0} a_p = a_0 = -3$$

(e)
$$\sum_{j=2}^{4}(a_j+b_j) = (a_2+b_2)+(a_3+b_3)+(a_4+b_4) = \left(-1+\frac{3}{2}\right)+\left(3+\frac{4}{3}\right)+\left(11+\frac{5}{4}\right) = \frac{1}{2}+\frac{13}{3}+\frac{49}{4} = \frac{205}{12}$$

(f)
$$\prod_{n=1}^{10^{50}} b_n = b_1\cdot b_2\cdot\cdots\cdot b_{10^{50}} = 2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\frac{6}{5}\cdot\cdots\cdot\frac{10^{50}}{10^{50}-1}\cdot\frac{10^{50}+1}{10^{50}} = \cancel{2}\cdot\frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{4}}{\cancel{3}}\cdot\frac{\cancel{5}}{\cancel{4}}\cdot\frac{\cancel{6}}{\cancel{5}}\cdot\cdots\cdot\frac{\cancel{10^{50}}}{\cancel{10^{50}-1}}\cdot\frac{10^{50}+1}{\cancel{10^{50}}} = 10^{50}+1$$

Problem 2. (10pt) Let $a \in \mathbb{R}$. Consider the following sum defined for $n > 7$:

$$\sum_{k=7}^{n} (k + a - 7)^2$$

(a) Reindex the sum above so that it begins at $k = 0$.

(b) Using the given summation formulas below, find the sum from (a) in terms of n, a alone.

$$\sum_{k=0}^{n} 1 = n + 1, \qquad \sum_{k=0}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution.

(a) We have...

$$\sum_{k=7}^{n} (k + a - 7)^2 = \sum_{k=7-7}^{n-7} \big((k+7) + a - 7\big)^2 = \sum_{k=0}^{n-7} (k+a)^2$$

(b) First, observe that...

$$\begin{aligned}
\sum_{k=7}^{n} (k + a - 7)^2 &= \sum_{k=0}^{n-7} (k+a)^2 \\
&= \sum_{k=0}^{n-7} (k^2 + 2ak + a^2) \\
&= \sum_{k=0}^{n-7} k^2 + \sum_{k=0}^{n-7} 2ak + \sum_{k=0}^{n-7} a^2 \\
&= \sum_{k=0}^{n-7} k^2 + 2a \sum_{k=0}^{n-7} k + a^2 \sum_{k=0}^{n-7} 1
\end{aligned}$$

Recall the following formulas:

$$\sum_{k=0}^{n} 1 = n + 1, \qquad \sum_{k=0}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

But then we have...

$$\begin{aligned}
\sum_{k=7}^{n} (k + a - 7)^2 &= \sum_{k=0}^{n-7} k^2 + 2a \sum_{k=0}^{n-7} k + a^2 \sum_{k=0}^{n-7} 1 \\
&= \frac{(n-7)(n-7+1)\big(2(n-7)+1\big)}{6} + 2a \cdot \frac{(n-7)(n-7+1)}{2} + a^2 \cdot (n - 7 + 1) \\
&= \frac{(n-7)(n-6)(2n-13)}{6} + a(n-7)(n-6) + a^2(n-6)
\end{aligned}$$

Problem 3. (10pt) Complete the proof of the given proposition below by filling in the corresponding blanks.

Proposition. For $n \geq 2$, $\displaystyle\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)=\frac{n+1}{2n}$.

Proof. We prove this using _____weak induction_____. First, we establish a base case.

Base Case: Let $n = 2$. Then we have...

$$\underline{\prod_{k=2}^{2}\left(1-\frac{1}{k^2}\right)} = \underline{1-\frac{1}{2^2}} = \underline{1-\frac{1}{4}} = \frac{3}{4}$$

$$\left.\frac{n+1}{2n}\right|_{n=2} = \frac{2+1}{2(2)} = \frac{3}{4}$$

But then if $n = 2$, we know that $\displaystyle\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)=\frac{n+1}{2n}$.

We know establish the induction step.

Induction Step: Assume that for $n = N$, $\displaystyle\prod_{k=2}^{N}\left(1-\frac{1}{k^2}\right)=\frac{N+1}{2N}$. We show that the statement of the proposition is then true for $n =$ ___$N+1$___. We have...

$$\begin{aligned}
\prod_{k=2}^{N+1}\left(1-\frac{1}{k^2}\right) &= \underline{\left(1-\frac{1}{(N+1)^2}\right)} \cdot \prod_{k=2}^{N}\left(1-\frac{1}{k^2}\right) \\
&= \underline{\frac{(N+1)^2}{(N+1)^2}-\frac{1}{(N+1)^2}} \cdot \underline{\frac{N+1}{2N}} \\
&= \underline{\frac{N^2+2N+1-1}{(N+1)^2}} \cdot \underline{\frac{N+1}{2N}} \\
&= \underline{\frac{N^2+2N}{(N+1)^2}} \cdot \underline{\frac{N+1}{2N}} \\
&= \frac{N+2}{2(N+1)} \\
&= \frac{(N+1)+1}{2(N+1)}
\end{aligned}$$

But then we know that $\displaystyle\prod_{k=2}^{N+1}\left(1-\frac{1}{k^2}\right)=\frac{(N+1)+1}{2(N+1)}$.

Therefore, by _____weak induction_____, we know that for $n \geq 2$, $\displaystyle\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)=\frac{n+1}{2n}$. $\square$

Problem 4. (10pt) Let $\{a_n\}_{n\in\mathbb{Z}^{\geq 0}}$ be the recursive sequence given by $a_0 = 1$, $a_1 = 3$, and $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$. A student observe that $a_0 = 1$, $a_1 = 3$, $a_2 = 5$, $a_3 = 7$, and $a_4 = 9$. They then predict that $a_n = 2n+1$ for $n \geq 0$. Below is a proof of this conjecture, with parts of their proof removed. Complete the missing parts.

Proposition. Let $\{a_n\}_{n\in\mathbb{Z}^{\geq 0}}$ be the recursive sequence given by $a_0 = 1$, $a_1 = 3$, and $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$. Then for all $n \geq 0$, $a_n = 2n + 1$.

Proof. We prove this using Strong Induction. First, we establish a few bases cases.

Base Case: If $n = 0$, we have $a_0 = 1$ and $2n + 1 = 2(0) + 1 = 1$. Then if $n = 0$, we have $a_n = 2n+1$. Now if $n =$ 1, we have $a_1 = 3$ and $a_1 = 2(1) + 1 = 3$.

But then if $n = 1$, we have $a_n = 2n + 1$.

We now establish the induction case.

Induction Case: Now assume that $a_k = 2k + 1$ for all $0 \leq k \leq n$. Now consider the term

$k = n + 1$.

We have...

$$
\begin{aligned}
a_{n+1} &= 2a_n - a_{n-1} \\
&= 2(2n+1) - (2(n-1)+1) \\
&= 4n + 2 - 2n + 2 - 1 \\
&= 2n + 3 \\
&= 2(n+1) + 1
\end{aligned}
$$

But then we know that $a_{n+1} = 2(n + 1) + 1$.

Therefore, by Strong Induction, we know that $a_n = 2n + 1$ for all $n \geq 0$. $\square$