MAT 101: Exam 4
Fall – 2022
12/14/2022
85 Minutes

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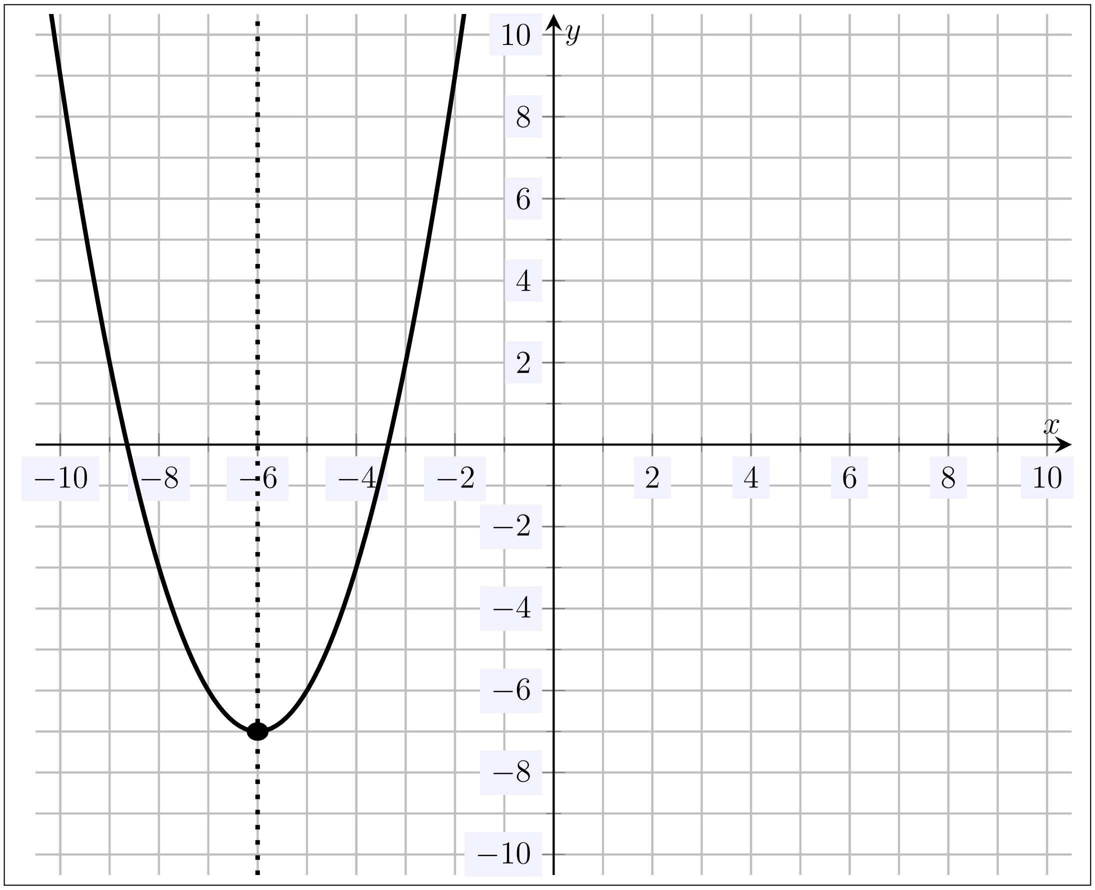
Write your name on the appropriate line on the exam cover sheet. This exam contains 15 pages (including this cover page) and 14 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work.

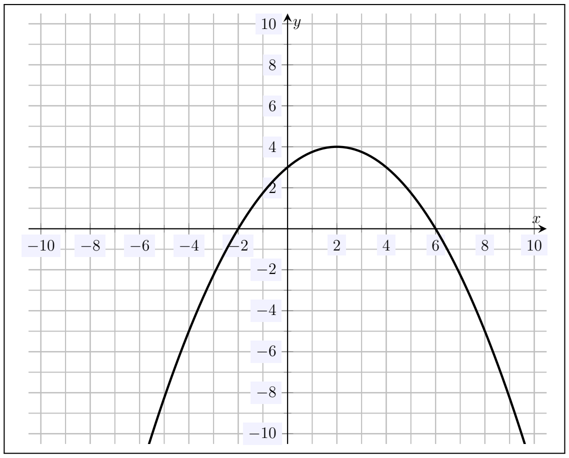
Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
Total:	140	

1. (10 points) Sketch the function $y = (x+6)^2 - 7$ as accurately as possible on the graph below. Your sketch should include the vertex and axis of symmetry.



Solution. Recall the vertex form of a quadratic function is $f(x) = a(x-P)^2+Q$, where (P,Q) is the vertex of the quadratic function a is the a in the standard form ax^2+bx+c of the quadratic function. We have $y = (x+6)^2 - 7 = 1\big(x-(-6)\big)^2 + (-7)$. But then the vertex of this quadratic function is $(-6,-7)$. Furthermore, the axis of symmetry is $x = -6$. Because $a = 1 > 0$, the quadratic function opens upwards, i.e. is convex. Using this gives the sketch above.

2. (10 points) A quadratic function $f(x) = ax^2 + bx + c$ is plotted below. Find a, b, and c for this function.



Solution. The vertex form of a quadratic function is $f(x) = a(x-P)^2+Q$, where (P, Q) is the vertex of the quadratic function a is the a in the standard form $ax^2 + bx + c$ of the quadratic function. From the plot, we see that the vertex is $(2, 4)$. But then $f(x) = a(x - P)^2 + Q = a(x - 2)^2 + 4$. Because the parabola opens downwards, we know $a < 0$. We see that the parabola contains the points $(-4, 5)$, $(-2, 0)$, $(0, 3)$, $(2, 4)$, $(4, 3)$, $(6, 0)$, and $(8, -5)$. Using the point $(0, 3)$, we have...

$$\begin{aligned} f(x) &= a(x-2)^2 + 4 \\ f(0) &= a(0-2)^2 + 4 \\ 3 &= 4a + 4 \\ 4a &= -1 \\ a &= -\frac{1}{4} \end{aligned}$$

Therefore, $f(x) = -\frac{1}{4}(x-2)^2 + 4$.

Alternatively, if $f(x)$ is a quadratic function with roots r_1, r_2, then $f(x) = a(x-r_1)(x-r_2)$, where a is the a in the standard form $ax^2 + bx + c$. From the plot, we can see the parabola has x-intercepts $(-2, 0)$ and $(6, 0)$, i.e. the roots are $x = -2, 6$. But then $f(x) = a(x-r_1)(x-r_2) = a\big(x-(-2)\big)(x-6) = a(x+2)(x-6)$. We see that the parabola contains the points $(-4, 5)$, $(-2, 0)$, $(0, 3)$, $(2, 4)$, $(4, 3)$, $(6, 0)$, and $(8, -5)$. Using the point $(0, 3)$, we have...

$$\begin{aligned} f(x) &= a(x+2)(x-6) \\ f(0) &= a(0+2)(0-6) \\ 3 &= a \cdot 2 \cdot -6 \\ 3 &= -12a \\ a &= -\frac{1}{4} \end{aligned}$$

Therefore, $f(x) = -\frac{1}{4}(x-2)^2 + 4$.

3. (10 points) Consider the quadratic function $f(x) = \dfrac{5}{3} - \left(x + \dfrac{3}{2}\right)^2$.

   (a) Find the vertex and axis of symmetry for $f(x)$.
   (b) Does this parabola open upwards or downwards?
   (c) Is this parabola concave or convex?
   (d) Does this parabola have a maximum or minimum?
   (e) Find the maximum or minimum of $f(x)$, if it exists.

Solution.

(a) The vertex form of a quadratic function is $f(x) = a(x-P)^2 + Q$, where (P,Q) is the vertex of the quadratic function a is the a in the standard form ax^2+bx+c of the quadratic function. This also forces the axis of symmetry to be the line $x = P$. Observe...

$$f(x) = \frac{5}{3} - \left(x + \frac{3}{2}\right)^2 = -\left(x - \left(-\frac{3}{2}\right)\right)^2 + \frac{5}{3}$$

Therefore, we have $(P,Q) = (-\frac{3}{2}, \frac{5}{3})$ and $a = -1$. The vertex is $(-\frac{3}{2}, \frac{5}{3})$ and the axis of symmetry is $x = -\frac{3}{2}$.

(b) Because $a = -1 < 0$, the parabola opens downwards.

(c) Because $a = -1 < 0$, the parabola is convex.

(d) Because $a = -1 < 0$, we know the parabola opens downwards. But then the parabola does not have a minimum value but does have a maximum value.

(e) The maximum value is the y-value of the vertex. From (a), we know the vertex is $(-\frac{3}{2}, \frac{5}{3})$. Therefore, the maximum value for $f(x)$ is $\frac{5}{3}$.

4. (10 points) Showing all your work, find the vertex form of $f(x) = 4x^2 + 4x - 6$.

Solution. By completing the square, we have...

$$
\begin{gathered}
4x^2 + 4x - 6 \\
4\left(x^2 + x - \frac{3}{2}\right) \\
4\left(x^2 + x + \left(\frac{1}{4} - \frac{1}{4}\right) - \frac{3}{2}\right) \\
4\left(\left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4} - \frac{3}{2}\right) \\
4\left(\left(x + \frac{1}{2}\right)^2 - \frac{7}{4}\right) \\
4\left(x + \frac{1}{2}\right)^2 - 7
\end{gathered}
$$

Using the 'evaluation method', we know the vertex occurs at $x = -\frac{b}{2a} = -\frac{4}{2(4)} = -\frac{4}{8} = -\frac{1}{2}$. We have...

$$f\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right)^2 + 4\left(-\frac{1}{2}\right) - 6 = 4 \cdot \frac{1}{4} - 2 - 6 = 1 - 2 - 6 = -7$$

Therefore, the vertex is $(-\frac{1}{2}, -7)$. We know the vertex from of a quadratic function is $f(x) = a(x-P)^2 + Q$, where (P, Q) is the vertex of the quadratic function a is the a in the standard form $ax^2 + bx + c$ of the quadratic function. But then $f(x) = a(x-P)^2 + Q = 4\left(x - (-\frac{1}{2})\right)^2 + (-7) = 4(x + \frac{1}{2})^2 - 7$.

5. (10 points) Find the y and x-intercepts for the function $f(x) = x^2 + 21x - 72$.

Solution. We know the y-intercept is the point where the graph of the function intersects the y-axis. But the y-axis is the line $x = 0$. Therefore, the y-intercept is the point on the graph of $f(x)$ with $x = 0$. We have $f(0) = 0^2 + 21(0) - 72 = 0 + 0 - 72 = -72$. Therefore, the y-intercept is -72, i.e. the point $(0, -72)$.

The x-intercept(s)—if they exist—is the point(s) where the graph of $f(x)$ intersects the x-axis. The x-axis is the line $y = 0$. But then the x-intercept(s) are the points on the graph of $f(x)$ with $y = f(x) = 0$. We have...

$$\begin{aligned} x^2 + 21x - 72 &= 0 \\ (x + 24)(x - 3) &= 0 \end{aligned}$$

This implies that either $x + 24 = 0$, i.e. $x = -24$, or $x - 3 = 0$, which implies $x = 3$. Therefore, the x-intercepts are $x = -24, 3$, i.e. the points $(-24, 0)$ and $(3, 0)$.

Alternatively, we could use the quadratic formula with $a = 1$, $b = 21$, and $c = -72$:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-21 \pm \sqrt{21^2 - 4(1)(-72)}}{2(1)} \\ &= \frac{-21 \pm \sqrt{441 + 288}}{2} \\ &= \frac{-21 \pm \sqrt{729}}{2} \\ &= \frac{-21 \pm 27}{2} \end{aligned}$$

Therefore, the x-intercepts are $x = \frac{-21-27}{2} = \frac{-48}{2} = -24$ and $x = \frac{-21+27}{2} = \frac{6}{2} = 3$, i.e. the points $(-24, 0)$ and $(3, 0)$.

6. (10 points) Use the discriminant of $f(x) = x^2 - 3x - 108$ to show that $f(x)$ has a 'nice' factorization and then find its factorization.

Solution. The discriminant of a quadratic function $ax^2 + bx + c$ is $b^2 - 4ac$. But then...

$$\operatorname{disc} f(x) = b^2 - 4ac = (-3)^2 - 4(1)(-108) = 9 + 432 = 441 = (21)^2$$

We know that a quadratic function has a 'nice' factorization if and only if its discriminant is a perfect square. Because $\operatorname{disc} f(x) = 21^2$ is a perfect square, $f(x)$ has a 'nice' factorization.

To factor $f(x)$, we find factors of 108 that sum to -3. Because $-108 < 0$, the factors must have opposite signs.

108	
$1 \cdot -108$	-107
$-1 \cdot 108$	107
$2 \cdot -54$	-52
$-2 \cdot 54$	52
$3 \cdot -36$	-33
$-3 \cdot 36$	33
$4 \cdot -27$	-23
$-4 \cdot 27$	23
$6 \cdot -18$	-12
$-6 \cdot 18$	12
$\boxed{9 \cdot -12}$	$\boxed{-3}$
$-9 \cdot 12$	3

Therefore, we have...

$$f(x) = (x + 9)(x - 12)$$

7. (10 points) Use the discriminant of some quadratic function to show that the equation given below does not have a 'nice' solution.

$$31 = x(14 - x)$$

Solution. We have...

$$31 = x(14 - x)$$

$$31 = 14x - x^2$$

$$x^2 - 14x + 31 = 0$$

Therefore, the original equation has a solution if and only if $x^2 - 14x + 31 = 0$. The solutions to $x^2 - 14x + 31 = 0$ correspond to roots of $f(x)$. We know that a quadratic function has 'nice' roots if and only if the discriminant of the quadratic function is a perfect square. The discriminant of a quadratic function $ax^2 + bx + c$ is $b^2 - 4ac$. But then...

$$\text{disc}(x^2 - 14x + 31) = (-14)^2 - 4(1)31 = 196 - 124 = 72$$

Because 72 is not a perfect square, $x^2 - 14x + 31$ does not have 'nice' roots so that $31 = x(14 - x)$ does not have a 'nice' solution.

8. (10 points) Showing all your work, factor the polynomial $x^2 - 23x - 24$. Verify that your factorization is correct.

Solution. To factor $x^2 - 23x - 24$, we find factors of 24 that sum to -23. Because $-24 < 0$, the factors must have opposite signs.

<u>24</u>

$\boxed{1 \cdot -24 \quad -23}$	
$-1 \cdot 24$	23
$2 \cdot -12$	-10
$-2 \cdot 12$	10
$3 \cdot -8$	-5
$-3 \cdot 8$	5
$4 \cdot -6$	-2
$-4 \cdot 6$	2

Therefore, we have...

$$x^2 - 23x - 24 = (x + 1)(x - 24)$$

We verify this factorization is correct:

$$(x + 1)(x - 24) = x^2 - 24x + x - 24 = x^2 - 23x - 24$$

9. (10 points) Showing all your work, factor the polynomial $x^2 + 17x - 84$.

Solution. To factor $x^2 + 17x - 84$, we find factors of 84 that sum to 17. Because $-84 < 0$, the factors must have opposite signs.

$$\begin{array}{cr}
\multicolumn{2}{c}{\underline{\mathbf{84}}} \\
1 \cdot -84 & -83 \\
-1 \cdot 84 & 83 \\
2 \cdot -42 & -40 \\
-2 \cdot 42 & 40 \\
3 \cdot -28 & -25 \\
-3 \cdot 28 & 25 \\
4 \cdot -21 & -17 \\
\boxed{-4 \cdot 21 \quad 17} & \\
6 \cdot -14 & -8 \\
-6 \cdot 14 & 8 \\
7 \cdot -12 & -5 \\
-7 \cdot 12 & 5
\end{array}$$

Therefore, we have...

$$x^2 + 17x - 84 = (x - 4)(x + 21)$$

10. (10 points) Showing all your work, factor the polynomial $7x^2 + 18x - 9$.

Solution. To factor $7x^2 + 18x - 9$, we find factors of $7 \cdot -9 = -63$ that sum to 18. Because $-63 < 0$, the factors must have opposite signs.

63	
$1 \cdot -63$	-62
$-1 \cdot 63$	62
$3 \cdot -21$	-18
$\boxed{-3 \cdot 21}$	$\boxed{18}$
$7 \cdot -9$	-2
$-7 \cdot 9$	2

Therefore, we have...

$$7x^2+18x-9 = 7x^2+(-3x+21x)-9 = (7x^2-3x)+(21x-9) = x(7x-3)+3(7x-3) = (7x-3)(x+3)$$

11. (10 points) Showing all your work, use the quadratic formula to factor the polynomial $288x^2 - 1524x + 935$.

Solution. If a quadratic function $ax^2 + bx + c$ has roots r_1, r_2, then it factors as $a(x - r_1)(x - r_2)$. We then use the quadratic function to find the roots of $288x^2 - 1524x + 935$:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-1524) \pm \sqrt{(-1524)^2 - 4(288)935}}{2(288)} \\ &= \frac{1524 \pm \sqrt{2322576 - 1077120}}{576} \\ &= \frac{1524 \pm \sqrt{1245456}}{576} \\ &= \frac{1524 \pm 1116}{576} \end{aligned}$$

Therefore, the roots are $r_1 = \frac{1524-1116}{576} = \frac{408}{576} = \frac{17}{24}$ and $r_2 = \frac{1524+1116}{576} = \frac{2640}{576} = \frac{55}{12}$. Therefore, we have...

$$288x^2 - 1524x + 935$$

$$288\left(x - \frac{17}{24}\right)\left(x - \frac{55}{12}\right)$$

$$(24 \cdot 12)\left(x - \frac{17}{24}\right)\left(x - \frac{55}{12}\right)$$

$$24\left(x - \frac{17}{24}\right) \cdot 12\left(x - \frac{55}{12}\right)$$

$$(24x - 17)(12x - 55)$$

12. (10 points) Showing all your work, solve the equation below then verify that your solution(s) are correct:

$$x^2 + 9 = 10x$$

Solution. By factoring, we have...

$$\begin{gathered} x^2 + 9 = 10x \\ x^2 - 10x + 9 = 0 \\ (x-1)(x-9) = 0 \end{gathered}$$

This implies $x - 1 = 0$, which implies $x = 1$, or $x - 9 = 0$, which implies $x = 9$. Alternatively, the equation has a solution if and only if $x^2 - 10x + 9 = 0$. We can then use the quadratic formula to solve this equation:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)9}}{2(1)} \\ &= \frac{10 \pm \sqrt{100 - 36}}{2} \\ &= \frac{10 \pm \sqrt{64}}{2} \\ &= \frac{10 \pm 8}{2} \end{aligned}$$

Therefore, the solutions are $x = \frac{10-8}{2} = \frac{2}{2} = 1$ and $x = \frac{10+8}{2} = \frac{18}{2} = 9$.

We now verify these solutions:

$$\begin{aligned} x^2 + 9 &= 10x \\ 1^2 + 9 &\stackrel{?}{=} 10(1) \\ 1 + 9 &\stackrel{?}{=} 10 \\ 10 &= 10 \\ &\checkmark \end{aligned} \qquad \begin{aligned} x^2 + 9 &= 10x \\ 9^2 + 9 &\stackrel{?}{=} 10(9) \\ 81 + 9 &\stackrel{?}{=} 90 \\ 90 &= 90 \\ &\checkmark \end{aligned}$$

13. (10 points) Showing all your work, solve the equation below:

$$6x^2 = 5 - 7x$$

Solution. We have...

$$6x^2 = 5 - 7x$$

$$6x^2 + 7x - 5 = 0$$

Therefore, the original equation has a solution if and only if $6x^2 + 7x - 5 = 0$. Solving this using factoring, we have...

$$6x^2 + 7x - 5 = 0$$

$$(3x + 5)(2x - 1) = 0$$

But then either $3x + 5 = 0$, which implies $x = -\frac{5}{3}$, or $2x - 1 = 0$, which implies $x = \frac{1}{2}$.

Alternatively, using the quadratic formula, we have...

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-7 \pm \sqrt{7^2 - 4(6)(-5)}}{2(6)} \\ &= \frac{-7 \pm \sqrt{49 + 120}}{12} \\ &= \frac{-7 \pm \sqrt{169}}{12} \\ &= \frac{-7 \pm 13}{12} \end{aligned}$$

Therefore, the solutions are $x = \frac{-7-13}{12} = \frac{-20}{12} = -\frac{5}{3}$ and $x = \frac{-7+13}{12} = \frac{6}{12} = \frac{1}{2}$.

14. (10 points) Showing all your work, solve the equation below:

$$-x^2 = 2(23 - 7x)$$

Solution. We have...

$$-x^2 = 2(23 - 7x)$$

$$-x^2 = 46 - 14x$$

$$0 = x^2 - 14x + 46$$

Observe that $b^2 - 4ac = (-14)^2 - 4(1)46 = 196 - 184 = 12$ is not a perfect square. Therefore, $x^2 - 14x + 46$ does not factor 'nicely.' Therefore, we use the quadratic formula to solve the equation:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)46}}{2(1)} \\ &= \frac{14 \pm \sqrt{196 - 164}}{2} \\ &= \frac{14 \pm \sqrt{12}}{2} \\ &= \frac{14 \pm \sqrt{4 \cdot 3}}{2} \\ &= \frac{14 \pm 2\sqrt{3}}{2} \\ &= 7 \pm \sqrt{3} \end{aligned}$$

Therefore, the solutions are $x = 7 - \sqrt{3} \approx 5.268$ and $x = 7 + \sqrt{3} \approx 8.732$.