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**MATH 101**
**Summer 2022**
**HW 9: Due 06/09**

*"Science is organized knowledge. Wisdom is organized life."*
*–Immanuel Kant*

**Problem 1.** (10pt) Find the domain of the rational function below. What are the vertical asymptotes of the given rational function? Also, simplify the rational function.

$$\frac{x^2 - 36}{x^2 - 2x - 24}$$

**Solution.** The domain of a rational function $\frac{p(x)}{q(x)}$ are the values for which $q(x) \neq 0$. We have...

$$q(x) = 0$$

$$x^2 - 2x - 24 = 0$$

$$(x+4)(x-6) = 0$$

$$x + 4 = 0 \text{ or } x - 6 = 0$$

$$x = -4 \text{ or } x = 6$$

Therefore, the domain of the function is $x \neq -4, 6$, i.e. all real numbers except for $x = -4$ and $x = 6$. For $x \neq -4, 6$, we have...

$$\frac{x^2 - 36}{x^2 - 2x - 24} = \frac{(x-6)(x+6)}{(x+4)(x-6)} = \frac{\cancel{(x-6)}(x+6)}{(x+4)\cancel{(x-6)}} = \frac{x+6}{x+4}$$

Therefore, the vertical asymptotes of $\frac{x^2-36}{x^2-2x-24}$ are $x = -4$. Because $\frac{x+6}{x+4}$ evaluated at $x = 6$ is $\frac{6+6}{6+4} = \frac{12}{10} = \frac{6}{5}$, the value $x = 6$ corresponds to the hole $(6, \frac{6}{5})$ for $\frac{x^2-36}{x^2-2x-24}$.

| | |
|---|---|
| Domain: | $x \in \mathbb{R}, x \neq -4, 6$ |
| Vertical Asymptotes: | $x = -4$ |

**Problem 2.** (10pt) Simplifying as much as possible, compute the following:

$$\frac{3x+1}{x^2-1} - \frac{x+5}{x+1}$$

**Solution.**

$$\begin{aligned}
\frac{3x+1}{x^2-1} - \frac{x+5}{x+1} &= \frac{3x+1}{(x-1)(x+1)} - \frac{x+5}{x+1} \\
&= \frac{3x+1}{(x-1)(x+1)} - \frac{(x-1)(x+5)}{(x-1)(x+1)} \\
&= \frac{3x+1}{(x-1)(x+1)} - \frac{x^2+5x-x-5}{(x-1)(x+1)} \\
&= \frac{3x+1-(x^2+4x-5)}{(x-1)(x+1)} \\
&= \frac{3x+1-x^2-4x+5}{(x-1)(x+1)} \\
&= \frac{-x^2-x+6}{(x-1)(x+1)} \\
&= \frac{-(x^2+x-6)}{(x-1)(x+1)} \\
&= \frac{-(x+3)(x-2)}{(x-1)(x+1)}
\end{aligned}$$

**Problem 3.** (10pt) Simplifying as much as possible, compute the following:

$$\frac{7-x}{x^2+8x+12}+\frac{x}{x^2+x-30}$$

**Solution.**

$$\begin{aligned}
\frac{7-x}{x^2+8x+12}+\frac{x}{x^2+x-30} &= \frac{7-x}{(x+2)(x+6)}+\frac{x}{(x+6)(x-5)} \\
&= \frac{(7-x)(x-5)}{(x+2)(x+6)(x-5)}+\frac{x(x+2)}{(x+6)(x-5)(x+2)} \\
&= \frac{7x-35-x^2+5x}{(x+2)(x+6)(x-5)}+\frac{x^2+2x}{(x+6)(x-5)(x+2)} \\
&= \frac{-x^2+12x-35}{(x+2)(x+6)(x-5)}+\frac{x^2+2x}{(x+6)(x-5)(x+2)} \\
&= \frac{-x^2+12x-35+x^2+2x}{(x+2)(x+6)(x-5)} \\
&= \frac{14x-35}{(x+2)(x+6)(x-5)} \\
&= \frac{7(2x-5)}{(x+2)(x+6)(x-5)}
\end{aligned}$$

**Problem 4.** (10pt) Simplifying as much as possible, compute the following:

$$\frac{x^2-4}{x^3-9x} \cdot \frac{x^2-2x-3}{x^2-3x-10}$$

**Solution.**

$$\begin{aligned}
\frac{x^2-4}{x^3-9x} \cdot \frac{x^2-2x-3}{x^2-3x-10} &= \frac{(x-2)(x+2)}{x(x^2-9)} \cdot \frac{(x-3)(x+1)}{(x-5)(x+2)} \\
&= \frac{(x-2)(x+2)}{x(x-3)(x+3)} \cdot \frac{(x-3)(x+1)}{(x-5)(x+2)} \\
&= \frac{(x-2)\cancel{(x+2)}}{x\cancel{(x-3)}(x+3)} \cdot \frac{\cancel{(x-3)}(x+1)}{(x-5)\cancel{(x+2)}} \\
&= \frac{(x-2)(x+1)}{x(x+3)(x-5)}
\end{aligned}$$

**Problem 5.** (10pt) Simplifying as much as possible, compute the following:

$$\frac{\dfrac{2x^2+8x}{x^2-6x-7}}{\dfrac{x^2+9x+20}{x^2-4x-5}}$$

**Solution.**

$$\begin{aligned}
\frac{\dfrac{2x^2+8x}{x^2-6x-7}}{\dfrac{x^2+9x+20}{x^2-4x-5}} &= \frac{2x^2+8x}{x^2-6x-7} \cdot \frac{x^2-4x-5}{x^2+9x+20} \\
&= \frac{2x(x+4)}{(x-7)(x+1)} \cdot \frac{(x-5)(x+1)}{(x+4)(x+5)} \\
&= \frac{2x\cancel{(x+4)}}{(x-7)\cancel{(x+1)}} \cdot \frac{(x-5)\cancel{(x+1)}}{\cancel{(x+4)}(x+5)} \\
&= \frac{2x(x-5)}{(x-7)(x+5)}
\end{aligned}$$

**Problem 6.** (10pt) Simplifying as much as possible, compute the following:

$$\frac{\dfrac{x^2+2x-3}{x^2+11x+10}}{\dfrac{x^2+8x-9}{x^2+8x-20}}$$

**Solution.**

$$\begin{aligned}
\frac{\dfrac{x^2+2x-3}{x^2+11x+10}}{\dfrac{x^2+8x-9}{x^2+8x-20}} &= \frac{x^2+2x-3}{x^2+11x+10} \cdot \frac{x^2+8x-20}{x^2+8x-9} \\
&= \frac{(x+3)(x-1)}{(x+1)(x+10)} \cdot \frac{(x-2)(x+10)}{(x+9)(x-1)} \\
&= \frac{(x+3)\cancel{(x-1)}}{(x+1)\cancel{(x+10)}} \cdot \frac{(x-2)\cancel{(x+10)}}{(x+9)\cancel{(x-1)}} \\
&= \frac{(x+3)(x-2)}{(x+1)(x+9)}
\end{aligned}$$

**Problem 7.** (10pt) Fully justifying your answer, determine if the point $(-1, 3)$ is a solution to the following system of equations:

$$\begin{cases} 4x - 7y = -8 \\ -3x + 5y = 5 \end{cases}$$

**Solution.** The point $(-1, 3)$ is a solution to the system of equations if and only if it satisfies both of the equations. We check this:

$$\begin{aligned} 4x - 7y &= -8 \\ 4(-1) - 7(3) &\stackrel{?}{=} -8 \\ -4 - 21 &\stackrel{?}{=} -8 \\ -25 &\neq -8 \\ &\boldsymbol{\times} \end{aligned}$$

and

$$\begin{aligned} -3x + 5y &= 5 \\ -3(-1) + 5(3) &\stackrel{?}{=} 5 \\ 3 + 15 &\stackrel{?}{=} 5 \\ 18 &\neq 5 \\ &\boldsymbol{\times} \end{aligned}$$

Because $(-1, 3)$ satisfies neither equation, $(-1, 3)$ is not a solution to the system of equations.

**Problem 8.** (10pt) Show that the following system of equations has a solution:

$$6x - 3y = 11$$

$$2x + 5y = 12$$

**Solution.** This a system of linear equations. The system will have a solution if and only if the lines intersect. But this will only happen if they are not parallel. We find the slopes of each line:

$$\begin{aligned} 6x - 3y &= 11 & \qquad 2x + 5y &= 12 \\ -3y &= -6x + 11 & \qquad 5y &= -2x + 12 \\ y &= 2x - \tfrac{11}{3} & \qquad y &= -\tfrac{2}{5}x + \tfrac{12}{5} \end{aligned}$$

The slope of the first line is $m_1 = 2$ while the slope of the second line is $m_2 = -\frac{2}{5}$. Because $m_1 \neq m_2$, the lines are not parallel. But then the lines intersect so that there is a solution to the system of equations.

**Problem 9.** (10pt) Solve the following system of equations and verify that your solution is valid:

$$\begin{cases} 6x + 4y = 0 \\ -12x + 6y = -7 \end{cases}$$

**Solution.** This is a system of linear equations. Assuming that there is a solution, it can be found with substitution or elimination. If we use substitution, we can solve for $y$ in the first equation. This yields $4y = -6x$ so that $y = -\frac{3}{2}\,x$. Using this in the second equation, we have...

$$\begin{aligned} -12x + 6y &= -7 \\ -12x + 6 \cdot -\frac{3}{2}\,x &= -7 \\ -12x - 9x &= -7 \\ -21x &= -7 \\ x &= \frac{1}{3} \end{aligned}$$

But then we have $y = -\frac{3}{2}\,x = -\frac{3}{2} \cdot \frac{1}{3} = -\frac{1}{2}$. Therefore, the solution is $(\frac{1}{3}, -\frac{1}{2})$.

Using elimination, suppose we eliminate $x$. Multiplying the first equation by 2 and adding this to the second equation, we find

$$\begin{aligned} 12x + 8y &= 0 \\ -12x + 6y &= -7 \\ \hline 14y &= -7 \\ y &= -\frac{1}{2} \end{aligned}$$

Using this in the first equation, we find

$$\begin{aligned} 6x + 4y &= 0 \\ 6x + 4 \cdot -\frac{1}{2} &= 0 \\ 6x - 2 &= 0 \\ 6x &= 2 \\ x &= \frac{1}{3} \end{aligned}$$

Therefore, the solution is $(\frac{1}{3}, -\frac{1}{2})$. We verify this in each of the two equations:

$$\begin{aligned} 6x + 4y &= 0 \\ 6 \cdot \tfrac{1}{3} + 4 \cdot -\tfrac{1}{2} &\stackrel{?}{=} 0 \\ 2 - 2 &\stackrel{?}{=} 0 \\ 0 &= 0 \\ &\checkmark \end{aligned} \qquad \begin{aligned} -12x + 6y &= -7 \\ -12 \cdot \tfrac{1}{3} + 6 \cdot -\tfrac{1}{2} &\stackrel{?}{=} -7 \\ -4 - 3 &\stackrel{?}{=} -7 \\ -7 &= -7 \\ &\checkmark \end{aligned}$$

Therefore, $(\frac{1}{3}, -\frac{1}{2})$ is the solution to the given system of equations.

**Problem 10.** (10pt) Solve the following system of equations and explain whether your solution is the only one possible:

$$\frac{1}{2}x + 4y = -1$$

$$\frac{1}{3}x - 5y = 7$$

**Solution.** This is a system of linear equations. Assuming that there is a solution, it can be found with substitution or elimination. If we use substitution, we can solve for $x$ in the first equation. This yields $\frac{1}{2}x = -4y - 1$ so that $x = -8y - 2$. Using this in the second equation, we have...

$$\begin{aligned} \frac{1}{3}x - 5y &= 7 \\ \frac{1}{3} \cdot (-8y - 2) - 5y &= 7 \\ -8y - 2 - 15y &= 21 \\ -23y &= 23 \\ y &= -1 \end{aligned}$$

But then we have $x = -8y - 2 = -8(-1) - 2 = 8 - 2 = 6$. Therefore, the solution is $(6, -1)$.

Using elimination, suppose we eliminate $x$. Multiplying the first equation by $2$, the second equation by $-3$, and adding these equations, we find

$$\begin{aligned} x + 8y &= -2 \\ -x + 15y &= -21 \\ \hline 23y &= -23 \\ y &= -1 \end{aligned}$$

Using this in the second equation, we find

$$\begin{aligned} \frac{1}{3}x - 5y &= 7 \\ x - 15y &= 21 \\ x - 15(-1) &= 21 \\ x + 15 &= 21 \\ x &= 6 \end{aligned}$$

Therefore, the solution is $(6, -1)$. We verify this in each of the two equations:

$$\begin{aligned} \tfrac{1}{2}x + 4y &= -1 \\ \tfrac{1}{2} \cdot 6 + 4 \cdot -1 &\stackrel{?}{=} -1 \\ 3 - 4 &\stackrel{?}{=} -1 \\ -1 &= -1 \\ &\checkmark \end{aligned} \qquad \begin{aligned} \tfrac{1}{3}x - 5y &= 7 \\ \tfrac{1}{3} \cdot 6 - 5 \cdot -1 &\stackrel{?}{=} 7 \\ 2 + 5 &\stackrel{?}{=} 7 \\ 7 &= 7 \\ &\checkmark \end{aligned}$$

Therefore, $(6, -1)$ is the solution to the given system of equations.