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MATH 308
Fall 2021
HW 13: Due 11/22

"All generalizations are false, including this one."

–Mark Twain

Problem 1. (10pt) Prove that the product of two even integers is even and that the product of an even integer with an odd integer is even.

Solution. Let n, m be even integers. But then because n, m are even, there exist integers k_n, k_m such that $n = 2k_n$ and $m = 2k_m$. But then...

$$nm = (2k_n)(2k_m) = 4k_n k_m = 2(2k_n k_m)$$

Now $k := 2k_n k_m$ is an integer because k_n, k_m are integers. But then $nm = 2k$. Therefore, nm is even.

Now let n be an even integer and m be an odd integer. Because n is even, there exists an integer k_n such that $n = 2k_n$. Because m is odd, there exists an integer k_m such that $m = 2k_m + 1$. But then...

$$nm = (2k_n)(2k_m + 1) = 4k_n k_m + 2k_n = 2(2k_n k_m + k_n)$$

Now $k := 2k_n k_m + k_n$ is an integer because k_n, k_m are integers. Therefore, $nm = 2k$ is even.

Problem 2. (10pt) Prove that if the square of an integer is even, then the integer is even. Use this to prove that if $n^2 + 1$ is a prime greater than 5, then the digit in the 1's place of n is 0, 4, or 6.

Solution. Let n be an integer such that its square is even. Because n^2 is even, we know there exists an integer k such that $n^2 = 2k$. Clearly, 2 divides $2k$. Therefore, 2 divides $n^2 = 2k$. By Euclid's Lemma, if a prime p divides ab, then p divides a or p divides b. We know that 2 divides $n^2 = n \cdot n$. But then by Euclid's Lemma, 2 divides n. Therefore, there exists an integer j such that $n = 2j$. Therefore, n is even.

Now suppose that $n^2 + 1$ is a prime greater than 5. All primes greater than 2 are odd (otherwise, they would be divisible by 2 and hence not prime). Therefore, $n^2 + 1$ is odd. But then there exists an integer s such that $n^2 + 1 = 2s + 1$. This implies $n^2 = 2s$, so that n^2 is even. By the work above, this implies that n is even. Because n is even, the digit in its 1's place must be 0, 2, 4, 6, or 8. It only remains to show that the digit in the 1's place cannot be 2 or 8.

Now use the division algorithm to write $n = 10q + r$, where q, r are integers and $0 \leq r \leq 9$. Clearly, r is the 1's digit of n. We prove that $r \neq 2, 8$ by contrapositive; that is, we prove that if the 1's digit of n is 2 or 8, then $n^2 + 1$ cannot be a prime greater than 5. Observe that...

$$r = 2\colon n^2 + 1 = (10q + r)^2 + 1 = (10q + 2)^2 + 1 = (100q^2 + 40q + 4) + 1 = 100q^2 + 40q + 5 = 5(20q^2 + 8q + 1)$$
$$r = 8\colon n^2 + 1 = (10q + r)^2 + 1 = (10q + 8)^2 + 1 = (100q^2 + 160q + 64) + 1 = 100q^2 + 160q + 65 = 5(20q^2 + 32q + 13)$$

In the case $r = 2$, we know that $n^2 + 1 = 5(20q^2 + 8q + 1)$ is divisible by 5 and since $n^2 + 1 > 5$, this implies that $n^2 + 1$ is not prime. In the case $r = 8$, $n^2 + 1 = 5(20q^2 + 32q + 13)$ is divisibly by 5 and since $n^2 + 1 > 5$, this implies that $n^2 + 1$ is not prime. Therefore, if $n^2 + 1$ is a prime greater than 5, the 1's digit of n cannot be 2 or 8. Putting this together with the information above, we know that if $n^2 + 1$ is a prime greater 5 that the 1's digit of n must be 0, 4, or 6.

Problem 3. (10pt) Use the division algorithm to write $180 = 7q + r$, where $q, r \in \mathbb{Z}$ and $0 \leq r < 7$.

Solution. Recall that the Division Algorithm states that for $a, b \in \mathbb{Z}$ with $a \neq 0$, there are unique $q, r \in \mathbb{Z}$ with $0 \leq r < |a|$ such that $b = qa + r$. If q is known, we can take $r = b - qa$. Recall that we can find q via...

$$q = \begin{cases} \left\lfloor \dfrac{b}{a} \right\rfloor, & a > 0 \\ \\ \left\lceil \dfrac{b}{a} \right\rceil, & a < 0 \end{cases}$$

Observe that in our case$b = 180$ and $a = 7$. Because $a = 7 > 0$, we have...

$$q = \left\lfloor \frac{180}{7} \right\rfloor = \lfloor 25.7143 \rfloor = 25$$

But then $r = 180 - 25(7) = 180 - 175 = 5$. Therefore, we have...

$$180 = 7(25) + 5$$

That is, $180 = 7q + r$ where $q = 25$ and $r = 5$.

Problem 4. (10pt) Use the division algorithm to prove that the 1's digit of a perfect square is never 2, 3, 7, or 8.

Solution. Suppose that N is a perfect square. Because N is a perfect square, there exists an integer k such that $N = k^2$. Using the division algorithm, we can write $k = 10q + r$, where q, r are integers and $0 \leq r < 10$. But then...

$$N = k^2 = (10q + r)^2 = 100q^2 + 20qr + r^2 = 10(10q^2 + 2qr) + r^2$$

Clearly, the 1's digit of N is then r^2, i.e. itself a perfect square. We can examine the 1's digit of the squares of r for $r = 0, 1, \ldots, 9$:

$$\begin{array}{ll} r = 0\colon 0^2 = 0 & r = 5\colon 5^2 = 25 \\ r = 1\colon 1^2 = 1 & r = 6\colon 6^2 = 36 \\ r = 2\colon 2^2 = 4 & r = 7\colon 7^2 = 49 \\ r = 3\colon 3^2 = 9 & r = 8\colon 8^2 = 64 \\ r = 4\colon 4^2 = 16 & r = 9\colon 9^2 = 81 \end{array}$$

Examining the possibilities above, we see the 1's digit of r^2 must be one of $0, 1, 4, 5, 6, 9$. Therefore, the 1's digit of r^2 cannot be $2, 3, 7, 8$. But then the 1's digit of N cannot be $2, 3, 7, 8$.

Problem 5. (10pt) Prove or disprove: Let $x, a, b \in \mathbb{Z}$. If x does not divide a and x does not divide b, then x does not divide ab.

Solution. The statement is *false*. For instance, let $x = 6$, $a = 2$, and $b = 3$. Clearly, $x = 6$ does not divide $a = 2$ or $b = 3$. However, $ab = 2(3) = 6$ and $x = 6$ does divide $ab = 6$.

The statement that if x does not divide a and x does not divide b, then x does not divide ab is the contrapositive of the statement if x divides ab, then x divides a or x divides b. We know this statement is true when x is a prime. By Euclid's Lemma, if x is a prime dividing ab, then x must divide a or x must divide b. Clearly, this need not hold when x is composite. However, there are examples when this does hold for composite integers. For instance, let $x = 4$, $a = 12$, and $b = 8$. Then $ab = 12(8) = 96$ and $x = 4$ divides $ab = 96$. Now $x = 4$ divides $a = 12$ and $x = 4$ divides $b = 8$.

Problem 6. (10pt) Prove that if n is composite, then n has a prime factor p with $p \leq \sqrt{n}$. Use this to show that 1321 is prime.

Solution. Suppose that $n = ab$, where a, b are integers. Without loss of generality, assume that $a \leq b$. But then…

$$n = ab \geq a \cdot a = a^2$$

But then $a \leq \sqrt{n}$. But if n is composite, let p be the smallest prime factor of n. Because p is a divisor of n, we can write $n = pb$ for some integer b. We need to show that $p \leq b$.

Any divisor of b must also divide $n = pb$. Let p_b denote the smallest prime divisor of b and write $b = qp_b$ for some integer q. Clearly, $p_b \geq p$; otherwise, this would contradict the fact that p is the smallest prime dividing n. But then $b = qp_b \geq qp \geq p$, as desired. From the work above, we know that $p \leq \sqrt{n}$. Therefore, if n is composite, it has a prime factor p with $p \leq \sqrt{n}$.

Now consider the fact where $n = 1321$. We have $\sqrt{1321} \approx 36.3456$. Therefore, if n is composite then n has a prime divisor less than 36.3. The only primes less than 36 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, and 31. However, we have…

$$\begin{aligned}
\frac{1321}{2} &\approx 660.5 & \frac{1321}{17} &\approx 77.7 \\
\frac{1321}{3} &\approx 440.3 & \frac{1321}{19} &\approx 69.5 \\
\frac{1321}{5} &\approx 264.2 & \frac{1321}{23} &\approx 57.4 \\
\frac{1321}{7} &\approx 188.7 & \frac{1321}{29} &\approx 45.6 \\
\frac{1321}{11} &\approx 120.1 & \frac{1321}{31} &\approx 42.6 \\
\frac{1321}{13} &\approx 101.6
\end{aligned}$$

Therefore, no prime less than 36 divides 1321. Therefore, 1321 cannot be composite. This implies that 1321 is prime.