Name: *Caleb McWhorter — Solutions*
MATH 308
Fall 2023
HW 10: Due 11/10

"Matrices act. They don't just sit there."
–Gilbert Strang

Problem 1. (10pt) Showing all your work, compute the following:

(a) $\displaystyle\sum_{k=0}^{5}(5k-3)$

(b) $\displaystyle\sum_{\substack{k=-2\\k\neq 0}}^{3}\frac{k+1}{k}$

(c) $\displaystyle\prod_{j=1}^{4}2j$

(d) $\displaystyle\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$ [Hint: Combine terms, factor, then write out some terms.]

(e) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2+3k}$ [Hint: Use partial fractions, then write out some terms.]

Solution.

(a)

$$\begin{aligned}\sum_{k=0}^{5}(5k-3) &= (5\cdot 0-3)+(5\cdot 1-3)+(5\cdot 2-3)+(5\cdot 3-3)+(5\cdot 4-3)+(5\cdot 5-3)\\ &= -3+2+7+12+17+22\\ &= 57\end{aligned}$$

OR

We use the fact that $\displaystyle\sum_{k=a}^{b}c=(b-a+1)c$ and $\displaystyle\sum_{k=0}^{n}k=\frac{k(k+1)}{2}$, where $c\in\mathbb{R}$ is a constant.

$$\sum_{k=0}^{5}(5k-3)=\sum_{k=0}^{5}5k-\sum_{k=0}^{5}3=5\sum_{k=0}^{5}k-\sum_{k=0}^{5}3=5\cdot\left.\frac{n(n+1)}{2}\right|_{n=5}-(5-0+1)3=5\cdot\frac{5(6)}{2}-6\cdot 3=57$$

(b)

$$\sum_{\substack{k=-2\\k\neq 0}}^{3}\frac{k+1}{k}=\frac{-2+1}{-2}+\frac{-1+1}{-1}+\frac{1+1}{1}+\frac{2+1}{2}+\frac{3+1}{3}=\frac{1}{2}+0+2+\frac{3}{2}+\frac{4}{3}=\frac{16}{3}$$

(c)

$$\prod_{j=1}^{4} 2j = 2(1) \cdot 2(2) \cdot 2(3) \cdot 2(4) = 2 \cdot 4 \cdot 6 \cdot 8 = 384$$

OR

$$\prod_{j=1}^{4} 2j = 2^{4-1+1} \prod_{j=1}^{4} j = 2^4 \cdot \big(1 \cdot 2 \cdot 3 \cdot 4\big) = 2^4 \cdot 4! = 16 \cdot 24 = 384$$

(d)

$$\begin{aligned}
\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) &= \prod_{n=2}^{\infty} \left(\frac{n^2-1}{n^2}\right) \\
&= \prod_{n=2}^{\infty} \left(\frac{(n-1)(n+1)}{n \cdot n}\right) \\
&= \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{4 \cdot 6}{5 \cdot 5} \cdots \\
&= \frac{1 \cdot 3}{2 \cdot \cancel{2}} \cdot \frac{\cancel{2} \cdot 4}{3 \cdot \cancel{3}} \cdot \frac{\cancel{3} \cdot 5}{4 \cdot \cancel{4}} \cdot \frac{\cancel{4} \cdot 6}{5 \cdot \cancel{5}} \cdots \\
&= \frac{1 \cdot 3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \frac{6}{5} \cdots \\
&= \frac{1 \cdot \cancel{3}}{2} \cdot \frac{\cancel{4}}{\cancel{3}} \cdot \frac{\cancel{5}}{\cancel{4}} \cdot \frac{\cancel{6}}{\cancel{5}} \cdots \\
&= \frac{1}{2}
\end{aligned}$$

Alternatively, to make this argument rigorous, we can define $a_k := \prod_{n=2}^{k} \left(1 - \frac{1}{n^2}\right)$. Using the 'cancellation trick' from above, we can observe that…

$$\begin{aligned}
a_k &:= \prod_{n=2}^{k} \left(1 - \frac{1}{n^2}\right) \\
&= \prod_{n=2}^{k} \frac{(n-1)(n+1)}{n \cdot n} \\
&= \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \cdots \cdot \frac{(k-2) \cdot k}{(k-1) \cdot (k-1)} \cdot \frac{(k-1)(k+1)}{k \cdot k} \\
&= \frac{1 \cdot \cancel{3}}{2 \cdot \cancel{2}} \cdot \frac{\cancel{2} \cdot 4}{\cancel{3} \cdot \cancel{3}} \cdot \frac{\cancel{3} \cdot \cancel{5}}{\cancel{4} \cdot \cancel{4}} \cdot \cdots \cdot \frac{\cancel{(k-2)} \cdot \cancel{k}}{\cancel{(k-1)} \cdot \cancel{(k-1)}} \cdot \frac{\cancel{(k-1)}(k+1)}{\cancel{k} \cdot k} \\
&= \frac{1}{2} \cdot \frac{k+1}{k} \\
&= \frac{1}{2}\left(1 + \frac{1}{k}\right)
\end{aligned}$$

But then we have...

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\lim_{k\to\infty}a_k=\lim_{k\to\infty}\frac{1}{2}\left(1+\frac{1}{k}\right)=\frac{1}{2}\,(1+0)=\frac{1}{2}$$

OR

Assume that $L=\lim\limits_{N\to\infty}\prod\limits_{n=2}^{N}\left(1-\dfrac{1}{n^2}\right)$.[1] Recall the following telescoping series:

$$\sum_{k=a}^{b}(\ln(k+1)-\ln k)$$

$$=\big(\ln(a+1)-\ln a\big)+\big(\ln(a+2)-\ln(a+1)\big)+\big(\ln(a+3)-\ln(a+2)\big)+\cdots+\big(\ln(b)-\ln(b-1)\big)+\big(\ln(b+1)-\ln b\big)$$

$$=\big(\cancel{\ln(a+1)}-\ln a\big)+\big(\cancel{\ln(a+2)}-\cancel{\ln(a+1)}\big)+\big(\cancel{\ln(a+3)}-\cancel{\ln(a+2)}\big)+\cdots+\big(\cancel{\ln(b)}-\cancel{\ln(b-1)}\big)+\big(\ln(b+1)-\cancel{\ln b}\big)$$

$$=-\ln a+\ln(b+1)$$

$$=\ln(b+1)-\ln a$$

Examining $\ln(L)$,[2] making use of the continuity of $\ln x$,[3], and making use of the telescoping series above, we can observe...

$$\begin{aligned}
\ln L&=\ln\left(\lim_{N\to\infty}\prod_{n=2}^{N}\left(1-\frac{1}{n^2}\right)\right)\\
&=\lim_{N\to\infty}\ln\left(\prod_{n=2}^{N}\left(1-\frac{1}{n^2}\right)\right)\\
&=\lim_{N\to\infty}\sum_{k=2}^{N}\ln\left(1-\frac{1}{n^2}\right)\\
&=\lim_{N\to\infty}\sum_{k=2}^{N}\ln\left(\frac{n^2-1}{n^2}\right)\\
&=\lim_{N\to\infty}\sum_{k=2}^{N}\big(\ln(n^2-1)-\ln(n^2)\big)\\
&=\lim_{N\to\infty}\sum_{k=2}^{N}\big(\ln\big((n+1)(n-1)\big)-2\ln(n)\big)
\end{aligned}$$

[1]The limit exists so that L is well-defined: let $a_N=\prod_{n=2}^{N}\left(1-\frac{1}{n^2}\right)$. Now $1<n^2$ for $n>1$, so that $1>\frac{1}{n^2}$ which implies $1-\frac{1}{n^2}>0$ for $n>1$. Furthermore, $1-\frac{1}{n^2}<1$, so that $0<1-\frac{1}{n^2}<1$. Using this work and a simple induction argument, one can show that $0<a_N<1$ for all N. Moreover, $\{a_N\}$ is a decreasing sequence because $a_{N+1}=a_N\left(1-\frac{1}{N+1}\right)<a_N\cdot 1=a_N$. But then $\{a_N\}$ is a bounded, decreasing sequence. Therefore, by the Monotone Convergence Theorem, $\{a_N\}$ converges.

[2]Using the Monotone Convergence Theorem, we know that $\inf a_N=\lim_{a_N}$. A priori, we need worry that $\inf a_N=0$. But we can see from the 'cancellation trick' from previous work that $a_N=\frac{1}{2}\left(1+\frac{1}{N}\right)>\frac{1}{2}$. Therefore, $\inf a_N\geq\frac{1}{2}$, so that $\ln(L)$ is well-defined.

[3]For continuous functions, $f(x)$, we have $\lim f(x_n)=f(\lim x_n)$.

$$
\begin{aligned}
&= \lim_{N\to\infty} \sum_{k=2}^{N} \big(\ln(n+1) + \ln(n-1) - 2\ln(n)\big) \\
&= \lim_{N\to\infty} \sum_{k=2}^{N} \big(\ln(n+1) - \ln n + \ln(n-1) - \ln(n)\big) \\
&= \lim_{N\to\infty} \left(\sum_{k=2}^{N} \big(\ln(n+1) - \ln n\big) + \sum_{k=2}^{N} \big(\ln(n-1) - \ln(n)\big) \right) \\
&= \lim_{N\to\infty} \left(\sum_{k=2}^{N} \big(\ln(n+1) - \ln n\big) - \sum_{k=2}^{N} \big(\ln(n) - \ln(n-1)\big) \right) \\
&= \lim_{N\to\infty} \Big(\big(\ln(N+1) - \ln 2\big) - \big(\ln N - \ln(2-1)\big)\Big) \\
&= \lim_{N\to\infty} \Big(\big(\ln(N+1) - \ln 2\big) - \big(\ln N - \ln 1\big)\Big) \\
&= \lim_{N\to\infty} \left(-\ln 2 + \ln(N+1) - \ln N\right) \\
&= \lim_{N\to\infty} \left(-\ln 2 + \ln\left(\frac{N+1}{N}\right)\right) \\
&= \lim_{N\to\infty} \left(-\ln 2 + \ln\left(1 + \frac{1}{N}\right)\right) \\
&= -\ln 2 + \ln(1+0) \\
&= -\ln 2 + \ln(1) \\
&= -\ln 2
\end{aligned}
$$

But then $\ln L = -\ln 2 = \ln\left(\frac{1}{2}\right)$. Therefore, $L = e^{\ln(1/2)} = \dfrac{1}{2}$.

(e) First, observe that the partial fraction decomposition of $\frac{1}{k^2+3k}$ is $\frac{1}{3k} - \frac{1}{3(k+3)} = \frac{1}{3}\left(\frac{1}{k} - \frac{1}{k+3}\right)$. One can find this via the following:

$$\frac{1}{k^2+3k} = \frac{1}{k(k+3)} = \frac{A}{k} + \frac{B}{k+3} = \frac{A(k+3)}{k(k+3)} + \frac{Bk}{k(k+3)} = \frac{Ak+3A+Bk}{k(k+3)} = \frac{(A+B)k+3A}{k(k+3)}$$

Comparing numerators on the far left and far right, we must have $3A = 1$ and $A+B = 0$. But then $A = \frac{1}{3}$ and $B = -A = -\frac{1}{3}$, which gives the partial fraction decomposition given above: $\frac{1/3}{k} + \frac{-1/3}{k+3} = \frac{1}{3}\left(\frac{1}{k} - \frac{1}{k+3}\right)$. Alternatively, we can use Heaviside's Method:

$$\frac{1}{k^2+3k} = \frac{1}{k(k+3)} = \frac{A}{k} + \frac{B}{k+3}$$

$$A = \left.\frac{1}{\boxed{}(k+3)}\right|_{k=0} = \frac{1}{3}, \qquad B = \left.\frac{1}{k\boxed{}}\right|_{k=-3} = -\frac{1}{3}$$

We then have…

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k^2+3k} &= \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{k^2+3k} \\
&= \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{3}\left(\frac{1}{k} - \frac{1}{k+3}\right) \\
&= \lim_{n\to\infty} \frac{1}{3} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+3}\right) \\
&= \frac{1}{3} \cdot \lim_{n\to\infty} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+3}\right) \\
&= \frac{1}{3} \cdot \lim_{n\to\infty} \left(\left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) + \cdots + \right. \\
&\qquad \left. \left(\frac{1}{n-3} - \frac{1}{n}\right) + \left(\frac{1}{n-2} - \frac{1}{n+1}\right) + \left(\frac{1}{n-1} - \frac{1}{n+2}\right) + \left(\frac{1}{n} - \frac{1}{n+3}\right) \right) \\
&= \frac{1}{3} \cdot \lim_{n\to\infty} \left(\left(\frac{1}{1} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{5}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{7}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{8}}\right) + \cdots + \right. \\
&\qquad \left. \left(\cancel{\frac{1}{n-3}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n-2}} - \frac{1}{n+1}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+2}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+3}\right) \right) \\
&= \frac{1}{3} \cdot \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right) \\
&= \frac{1}{3} \cdot \left(1 + \frac{1}{2} + \frac{1}{3} - 0 - 0 - 0\right) \\
&= \frac{1}{3} \cdot \frac{11}{6} \\
&= \frac{11}{18}
\end{aligned}
$$

Problem 2. (10pt) Define $\mathbf{u} = \langle 2, 0, -1, 3\rangle$ and $\mathbf{v} = \langle 1, -1, 5, 6\rangle$. Showing all your work, complete the following:

(a) $\mathbf{u} - 2\mathbf{v}$

(b) $\|\mathbf{u} - 2\mathbf{v}\|$

(c) $\mathbf{u} \cdot \mathbf{v}$

(d) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|\cos\theta$, where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Using this fact, compute the angle between $\mathbf{u}$ and $\mathbf{v}$.

Solution.

(a)

$$\mathbf{u}-2\mathbf{v} = \langle 2,0,-1,3\rangle - 2\langle 1,-1,5,6\rangle = \langle 2,0,-1,3\rangle - \langle 2,-2,10,12\rangle = \langle 2-2, 0-(-2), -1-10, 3-12\rangle = \langle 0,2,-11,-9\rangle$$

(b)

$$\|\mathbf{u}-2\mathbf{v}\| = \|\langle 0,2,-11,-9\rangle\| = \sqrt{0^2+2^2+(-11)^2+(-9)^2} = \sqrt{0+4+121+81} = \sqrt{206} \approx 14.3527$$

(c)

$$\mathbf{u}\cdot\mathbf{v} = \langle 2,0,-1,3\rangle \cdot \langle 1,-1,5,6\rangle = 2(1)+0(-1)+(-1)5+3(6) = 2+0-5+18 = 15$$

(d) First, observe that…

$$\|\mathbf{u}\| = \|\langle 2,0,-1,3\rangle\| = \sqrt{2^2+0^2+(-1)^2+3^2} = \sqrt{4+0+1+9} = \sqrt{14}$$
$$\|\mathbf{v}\| = \|\langle 1,-1,5,6\rangle\| = \sqrt{1^2+(-1)^2+5^2+6^2} = \sqrt{1+1+25+36} = \sqrt{63} = 3\sqrt{7}$$

We know from (c) that $\mathbf{u}\cdot\mathbf{v} = 25$. But then…

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

$$25 = \sqrt{14}\cdot 3\sqrt{7}\cos\theta$$

$$25 = (\sqrt{2}\sqrt{7})\cdot 3\sqrt{7}\cos\theta$$

$$25 = 21\sqrt{2}\cos\theta$$

$$\cos\theta = \frac{25}{21\sqrt{2}}$$

$$\theta = \cos^{-1}\left(\frac{25}{21\sqrt{2}}\right)$$

$$\theta \approx 32.67^\circ$$

Problem 3. (10pt) Define the following:

$$A = \begin{pmatrix} 0 & -2 \\ 6 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & -1 & 3 \\ 5 & 1 & 0 & 4 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 4 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Showing all your work, compute the following:

(a) $BC - 2A$

(b) CB

(c) $B^T\mathbf{u}$

Solution.

(a)

$$\begin{aligned}
BC - 2A &= \begin{pmatrix} 1 & 0 & -1 & 3 \\ 5 & 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 4 & 1 \\ -1 & 1 \end{pmatrix} - 2\begin{pmatrix} 0 & -2 \\ 6 & 5 \end{pmatrix} \\
&= \begin{pmatrix} 1(2)+0(-1)+(-1)4+3(-1) & 1(1)+0(0)+(-1)1+3(1) \\ 5(2)+1(-1)+0(4)+4(-1) & 5(1)+1(0)+0(1)+4(1) \end{pmatrix} - \begin{pmatrix} 2\cdot 0 & 2\cdot -2 \\ 2\cdot 6 & 2\cdot 5 \end{pmatrix} \\
&= \begin{pmatrix} 2+0-4-3 & 1+0-1+3 \\ 10-1+0-4 & 5+0+0+4 \end{pmatrix} - \begin{pmatrix} 0 & -4 \\ 12 & 10 \end{pmatrix} \\
&= \begin{pmatrix} -5 & 3 \\ 5 & 9 \end{pmatrix} - \begin{pmatrix} 0 & -4 \\ 12 & 10 \end{pmatrix} \\
&= \begin{pmatrix} -5-0 & 3-(-4) \\ 5-12 & 9-10 \end{pmatrix} \\
&= \begin{pmatrix} -5 & 7 \\ -7 & -1 \end{pmatrix}
\end{aligned}$$

(b)

$$\begin{aligned}
CB &= \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 4 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 3 \\ 5 & 1 & 0 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 2(1)+1(5) & 2(0)+1(1) & 2(-1)+1(0) & 2(3)+1(4) \\ -1(1)+0(5) & -1(0)+0(1) & -1(-1)+0(0) & -1(3)+0(4) \\ 4(1)+1(5) & 4(0)+1(1) & 4(-1)+1(0) & 4(3)+1(4) \\ -1(1)+1(5) & -1(0)+1(1) & -1(-1)+1(0) & -1(3)+1(4) \end{pmatrix} \\
&= \begin{pmatrix} 2+5 & 0+1 & -2+0 & 6+4 \\ -1+0 & 0+0 & 1+0 & -3+0 \\ 4+5 & 0+1 & -4+0 & 12+4 \\ -1+5 & 0+1 & 1+0 & -3+4 \end{pmatrix}
\end{aligned}$$

$$
= \begin{pmatrix} 7 & 1 & -2 & 10 \\ -1 & 0 & 1 & -3 \\ 9 & 1 & -4 & 16 \\ 4 & 1 & 1 & 1 \end{pmatrix}
$$

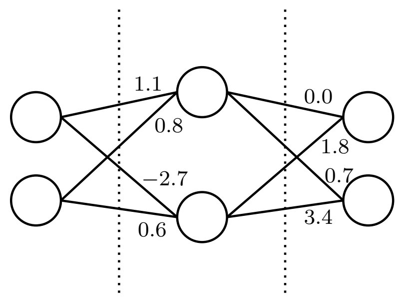
(c)

$$
\begin{aligned}
B^T\mathbf{u} &= \begin{pmatrix} 1 & 0 & -1 & 3 \\ 5 & 1 & 0 & 4 \end{pmatrix}^T \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 5 \\ 0 & 1 \\ -1 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 1(1) + 5(-1) \\ 0(1) + 1(-1) \\ -1(1) + 0(-1) \\ 3(1) + 4(-1) \end{pmatrix} \\
&= \begin{pmatrix} 1 - 5 \\ 0 - 1 \\ -1 + 0 \\ 3 - 4 \end{pmatrix} \\
&= \begin{pmatrix} -4 \\ -1 \\ -1 \\ -1 \end{pmatrix}
\end{aligned}
$$

Problem 4. (10pt) A *neural network* is a computational model resembling how the human brain works and they are used to create predictive models in data science. There are many types of neural networks: feed-forward neural networks, recurrent neural networks, convolutional neural networks, etc.

(a) Watch 3Blue1Brown's "But what is a neural network?" and then comment about what you learned and how it relates to the course material.

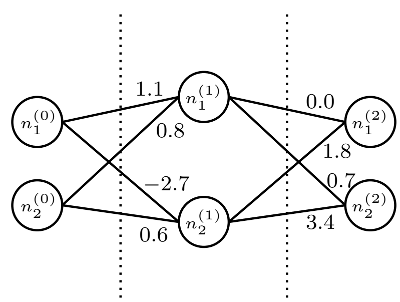
(b) Using the (logistic) sigmoid function $\sigma(x) = \frac{1}{1+e^{-x}}$, bias vectors $\mathbf{b}_1 = \begin{pmatrix} 1.5 \\ -0.4 \end{pmatrix}$ and $\mathbf{b}_2 = \begin{pmatrix} 0.3 \\ 2.0 \end{pmatrix}$, and initial input $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, compute the output of the single hidden layer neural network given below.



Solution.

(a) *Solutions will vary.*

(b) For ease of notation, we label our neural network as below:



We know that for $i \geq 1$, the 'value' of each node is given by $\mathbf{n}_i = \sigma(W^{(i)}\mathbf{n}_{i-1} + \mathbf{b}_i)$, where $\mathbf{n}_i$ is the vector of the 'values' for the ith node, $n_j^{(i)}$ is the jth entry of the ith node (that is $\mathbf{n}_i = (n_j^{(i)})$), $W^{(i)}$ is the weight matrix (the transition matrix from $\mathbf{n}_{i-1}$ to $\mathbf{n}_i$, i.e. from the $(i-1)$th layer to the ith layer), $\mathbf{b}_i$ is the ith bias vector, and σ is a choice of activation function (which we shall take to be the (logistic) sigmoid function $\sigma(x) = \frac{1}{1+e^{-x}}$). Note that if $\mathbf{v}$ is a vector, $\sigma(\mathbf{v})$ is taken to mean $\mathbf{v}$ with σ applied to each component. The matrix $W^{(i)}$ is given by $W^{(i)} = (w_{j,k})$, where $w_{j,k}$ is the weight from the kth node of the $(i-1)$th layer to the jth node of the ith layer.[4] In the notation of the video, we have $\mathbf{a}^{(i)} = \mathbf{n}^{(i)}$, $\mathbf{b} = \mathbf{b}_i$, and

[4] We can derive this directly for this case. Observe that we must have $n_1^{(1)} = 1.1n_1^{(0)} + 0.8n_2^{(0)}$ and $n_2^{(1)} = -2.7n_1^{(0)} + 0.6n_2^{(0)}$. From this we can immediately see that $W^{(1)} = \begin{pmatrix} 1.1 & 0.8 \\ -2.7 & 0.6 \end{pmatrix}$.

$W = W^{(i)}$. We have $\mathbf{n}^{(0)} = \mathbf{a} = \langle 3, -1, \rangle^T$.

$$\mathbf{n}^{(0)} = \mathbf{a} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

We can now compute the values of the nodes for the first layer:

$$\begin{aligned}
\mathbf{n}^{(1)} &= \sigma(W^{(1)}\mathbf{n}_0 + \mathbf{b}_1) \\
&= \sigma\left(\begin{pmatrix} 1.1 & 0.8 \\ -2.7 & 0.6 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1.5 \\ -0.4 \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 2.5 \\ -8.7 \end{pmatrix} + \begin{pmatrix} 1.5 \\ -0.4 \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 4.0 \\ -9.1 \end{pmatrix}\right) \\
&= \begin{pmatrix} \sigma(4.0) \\ \sigma(-9.1) \end{pmatrix} \\
&= \begin{pmatrix} 0.982014 \\ 0.000111653 \end{pmatrix}
\end{aligned}$$

We can then compute the output of the given neural network.

$$\begin{aligned}
\mathbf{n}^{(2)} &= \sigma(W^{(2)}\mathbf{n}_1 + \mathbf{b}_2) \\
&= \sigma\left(\begin{pmatrix} 0.0 & 1.8 \\ 0.7 & 3.4 \end{pmatrix}\begin{pmatrix} 0.982014 \\ 0.000111653 \end{pmatrix} + \begin{pmatrix} 0.3 \\ 2.0 \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 0.000200975 \\ 0.687789 \end{pmatrix} + \begin{pmatrix} 0.3 \\ 2.0 \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 0.300200975 \\ 2.687789 \end{pmatrix}\right) \\
&= \begin{pmatrix} \sigma(0.300200975) \\ \sigma(2.687789) \end{pmatrix} \\
&= \begin{pmatrix} 0.574492 \\ 0.936302 \end{pmatrix}
\end{aligned}$$