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MATH 308
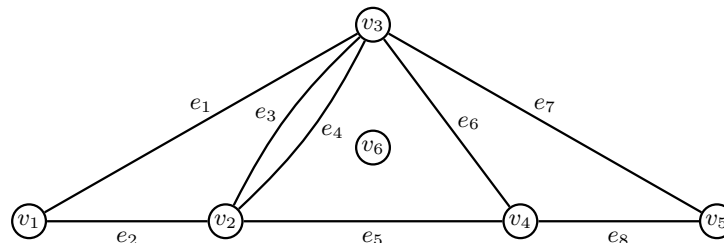
Fall 2021

HW 19: Due 12/15

Problem 1. (10pt) Consider the following graph:

(a) Is the graph directed or undirected?

(b) Give the vertex set and the edge set for the graph.

(c) Give the adjacency matrix of the graph.

(d) Is the graph simple?

(e) Are there any isolated vertices?

(f) List all pairs of parallel edges.

(g) Compute the degree of the graph.

(h) Are the vertices v_1 and v_6 connected? What about the vertices v_1 and v_4?

(i) Does the graph $G \setminus \{v_6\}$ have an Eulerian circuit? Find one or explain why none exists.

(j) Does the graph $G \setminus \{v_6\}$ have an Hamiltonian circuit? Find one or explain why none exists.

Solution.

(a) This graph is undirected as there is no direction/orientation to the edges.

(b) We have. . .

$$V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

$$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$$
$$= \{\{v_1, v_3\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}\}$$

(c) The adjacency matrix is $A = (a_{ij})$, where a_{ij} is the number of edges from vertex i to vertex j.

But then the adjacency matrix is...

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observe that this matrix is symmetric, i.e. $A = A^T$, because the graph is undirected.

(d) A simple graph is a graph without loops or multiple edges. There are multiple edges in this graph because there are two edges between v_2 and v_3—namely e_3 and e_4. Therefore, the graph is not simple.

(e) An isolated vertex has no edges coming into or going out of it. Therefore, the only isolated vertex is v_6.

(f) Two distinct edges are parallel if they have the same endpoints. The only pair of edges that share the same endpoints are e_3 and e_4, which share the endpoints v_2 and v_3. Therefore, e_3 and e_4 are the only parallel edges. [Not that other edges share *an* endpoint but not *all* endpoints, e.g. e_1 and e_2 share the endpoint v_1 but are not parallel because they do not share the end point v_2 or v_3.]

(g) The degree of an undirected graph is the sum of the degrees of its vertices. For an undirected graph, the degree of a vertex is the number of edges incident to it. We can compute the degree of each of the vertices for this graph:

$$\begin{aligned} \deg v_1 &= 2 & \deg v_4 &= 3 \\ \deg v_2 &= 4 & \deg v_5 &= 2 \\ \deg v_3 &= 5 & \deg v_6 &= 0 \end{aligned}$$

Then the degree of G is...

$$\deg G = \sum_i \deg(v_i) = 2 + 4 + 5 + 3 + 2 + 0 = 16$$

We could also have computed the degree of the graph using the Handshake Theorem. From this theorem, we know that the degree of the graph is twice the number of edges. But then $\deg G = 2 \# E(G) = 2 \cdot 8 = 16$.

(h) Two vertices are connected if there exists a walk connecting them. There is no walk from v_1 to v_6; therefore, v_1 and v_6 are not connected. However, there is a walk from v_1 to v_4, e.g. e_2, e_5 or $e_1, e_3, e_2, e_1, e_7, e_8$; therefore, v_1 and v_4 are connected.

(i) Recall that if a graph G is connected and the degree of every vertex of G is a positive even integer, then G has an Euler circuit. Therefore, if G does not have an Eulerian circuit then either G is not connected or not every vertex of G has positive even degree. The graph $G \setminus \{v_6\}$ is connected but not every vertex has positive even degree. For instance, the vertex v_4 has degree 3. Therefore, G does not have an Eulerian circuit.

(j) The graph $G \setminus \{v_6\}$ has a Hamiltonian circuit. For instance, the walk $v_1, e_1, e_7, e_8, e_5, e_2, v_1$ is a Hamiltonian circuit.

Problem 2. (10pt) Give vertex and edge set for a graph (directed) and tell if....
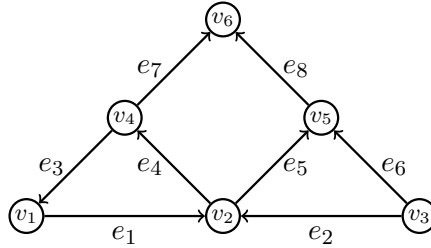
(a) Is the graph directed or undirected?

(b) Give the vertex set and the edge set for the graph.

(c) Give the adjacency matrix of the graph.

(d) Is the graph simple?

(e) Are there any isolated vertices?

(f) Compute the in- and out-degree of each vertex.

(g) Compute the degree of the graph.

(h) Is the vertex v_1 connected to v_6? Is the vertex v_6 connected to v_1?

(i) Does the graph have an Eulerian circuit? Find one or explain why none exists.

(j) Does the graph have an Hamiltonian circuit? Find one or explain why none exists.

Solution.

(a) This graph is directed because the edges have an orientation/direction.

(b) We have...

$$V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

$$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$$
$$= \{(v_1, v_2), (v_3, v_2), (v_4, v_1), (v_2, v_4), (v_2, v_5), (v_3, v_5), (v_4, v_6), (v_5, v_8)\}$$

(c) Recall that the adjacency matrix is $A = (a_{ij})$, where a_{ij} is the number of edges from vertex i to vertex j. But then the adjacency matrix is...

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(d) A simple graph is a graph without loops or multiple edges. Observe that this graph has no loops or multiple edges. Therefore, this graph is simple.

(e) An isolated vertex has no edges coming into or going out of it. Therefore, the graph has no isolated vertices.

(f) The in-degree of a vertex, $\deg^- v_i$, is the number of edges 'entering' the vertex while the out-degree of a vertex, $\deg^+ v_i$, is the number of edges 'leaving' the vertex. [Observe these are the sum of the ith column or row of the adjacency matrix, respectively.] Therefore, we have...

$$
\begin{aligned}
\deg^- v_1 &= 1 & \deg^+ v_1 &= 1 \\
\deg^- v_2 &= 2 & \deg^+ v_2 &= 2 \\
\deg^- v_3 &= 0 & \deg^+ v_3 &= 2 \\
\deg^- v_4 &= 1 & \deg^+ v_4 &= 2 \\
\deg^- v_5 &= 2 & \deg^+ v_5 &= 1 \\
\deg^- v_6 &= 2 & \deg^+ v_6 &= 0
\end{aligned}
$$

(g) While not a standard definition, we defined the degree of a directed graph to be the sum of its in- and out-degrees. But because every edge goes out of some vertex and into some vertex, this results in twice the sum of the edges. This is exactly the same as the degree of the 'underlying' undirected graph, i.e. the graph 'forgetting' about the orientation of the edges. Therefore, $\deg G = 2 \# E(G) = 2 \cdot 8 = 16$.

(h) Vertex v_i is connected to v_j if and only if there is a walk from v_i to v_j. The vertices v_1 and v_6 are connected because there is a walk from v_1 to v_6, e.g. v_1, e_1, e_4, e_7, v_6. However, v_6 is *not* connected to v_1. There is no edge out of vertex v_6 so that v_6 cannot be a walk from v_6 to v_1.

(i) Recall that a directed graph has an Euler circuit if and only if G is connected and $\deg^- v = \deg^+ v$ for all $v \in V(G)$. [A directed graph has an Euler trail if the graph if there is are unique vertices s, t with $\deg^+ s = \deg^- s + 1$ (the starting vertex) and $\deg^- t = \deg^+ t + 1$ (the ending vertex), $\deg^- v = \deg^+ v$ for all $v \in V(G)$ with $v \neq s$ or $v \neq t$, and all vertices are in the same strongly connected component.] Observe that $2 = \deg^- v_6 \neq \deg^+ v_6 = 0$. Therefore, G cannot have an Euler circuit.

(j) The graph cannot have a Hamiltonian circuit. A Hamiltonian circuit is a walk that begins and ends at the same vertex, visits every vertex of the graph exactly once (except for the start/end vertex), and never repeats an edge. Observe such a walk must include the vertex v_6. However, once one 'enters' v_6, one cannot travel to any other vertex because there are no edges out of v_6. Therefore, any such walk must begin/end at v_6. But if the walk starts at v_6, no other vertices can be visited from v_6. Therefore, no Hamiltonian circuit can exist.

Problem 3. (10pt) Draw a graph that has the adjacency matrix given below. How can you tell from this matrix if the graph is undirected or directed?

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Solution. The adjacency matrix is $A = (a_{ij})$, where a_{ij} is the number of edges from vertex i to vertex j. For an undirected graph, the adjacency matrix must be symmetric, i.e. $A = A^T$. Therefore, if an adjacency matrix is not symmetric, then the graph must be directed. However, a directed graph may or may not have a symmetric adjacency matrix. The adjacency matrix above is symmetric. Therefore, the graph may or may not be directed. There are many possible answers (but 'only one' possible undirected graph). For instance, drawing the unique (up to isomorphism) undirected graph associated to this adjacency matrix, we have...
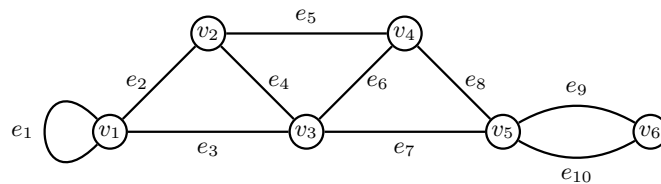

We can create a directed graph from this undirected graph by replacing each non-loop edge with a pair of edges connecting the given vertices in each direction. The resulting matrix would have the same adjacency matrix. Therefore, the matrix below have the given adjacency matrix...
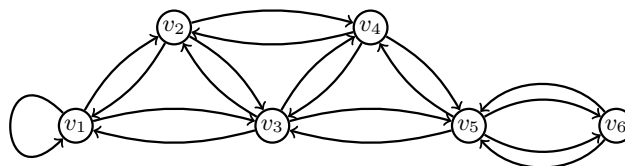
Problem 4. (10pt) Determine which of the following graphs are isomorphic. In each case, explain the isomorphism or explains why an isomorphism cannot exist.
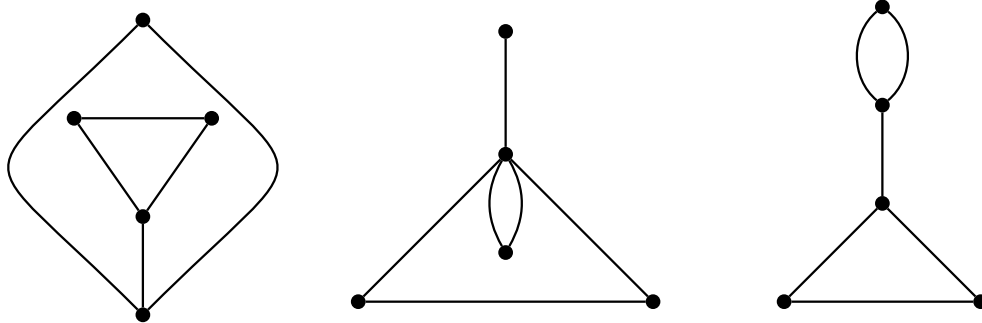

Solution. We label these graphs from left to right as Graph I, II, and III, respectively. Graph I and Graph III are isomorphic. To see this, observe that we can 'pick up' the 'uppermost' vertex and its edges and place it below the graph. We can then rotate the graph $180°$, which results in Graph I. We illustrate this process below.
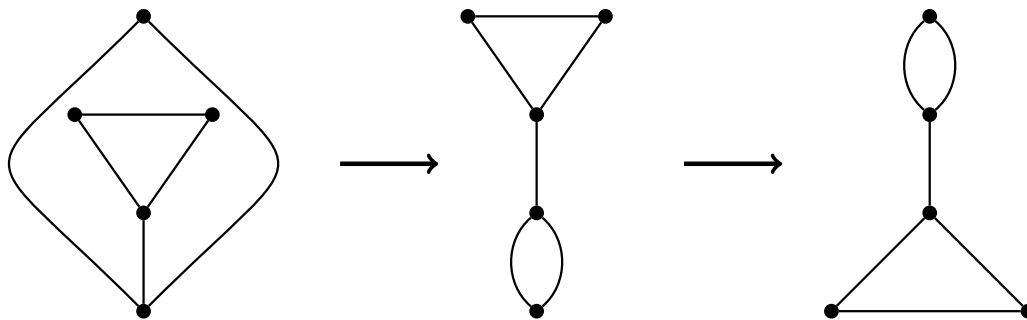
This shows that Graph I and Graph III are isomorphic.

We now need to determine whether Graph III is isomorphic to Graph II or Graph III. However, because being isomorphic is a transitive property, if Graph III is isomorphic to Graph II or Graph III, it is isomorphic to the other. Similarly, if Graph III is not isomorphic to Graph II or Graph III, it is not isomorphic to the other. Observe each graph has the same number of vertices and edges. This alone does not determine whether or not they are isomorphic. Observe the 'uppermost' vertex of Graph II had degree 1. However, Graphs II and III do not have a vertex of degree 1. Because isomorphisms preserve the degrees of vertices, Graph II is neither isomorphic to Graph I nor Graph III. Alternatively, we can see that Graph II has a vertex of degree 5 while Graph I and III do not. Alternatively, Graph I and III both have a vertex of degree 3 while Graph II does not.

Problem 5. (10pt) Let G be an undirected graph. The *degree sequence* of G is a monotonic non-increasing sequence of the degrees for vertices of G. For instance, consider the graph from Problem i:
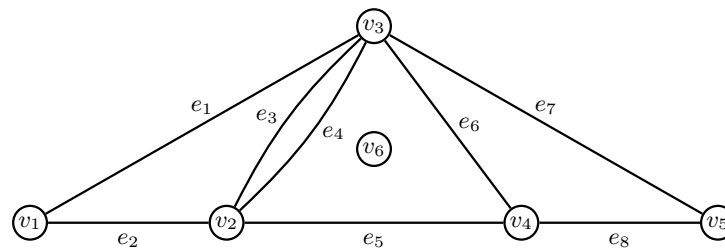
This graph has degree sequence $(5, 4, 3, 2, 2, 0)$. For each of the monotonic non-increasing sequences below, determine whether they are the degree sequence for some simple graph G. If there is such a simple graph G, give an example. If no such simple graph is possible, explain why.

(a) $(4, 3, 2, 2, 2)$

(b) $(2, 2, 2, 0)$

(c) $(5, 4, 3, 2, 1, 0)$

(d) $(4, 3, 3, 3, 3)$

(e) $(4, 3, 2, 1)$

**Solution.** We first make a few observations about a degree sequence for a simple (undirected) graph G:

- *The sum of the degree sequence, $\sum_i d_i$, must be even:* We know the degree of G is the sum of the degrees of its vertices. By the Handshake Theorem, we know that $\deg G$ is even. The sum of the terms of the degree sequence is the sum of the degrees of the vertices of G, which is $\deg G$. Therefore, this sum must be even.

- *Each term of the degree sequence must be smaller than the length of the degree sequence:* Suppose a vertex v had a degree $d \geq n$, where n is the number of vertices. If v has a loop incident to it, then G is not simple, a contradiction. Otherwise, each edge of v must connect v to one of the other $n - 1$ vertices of G. Observe that $d \geq n > n - 1$. By the Pigeonhole Principle, 'assigning' each of the d edges incident to v to a vertex of the graph, there must be a vertex with more than one edge connecting it to v. But then G has multiple edges so that it cannot be simple, a contradiction. Therefore, $d < n$.

- *If the degree sequence has length n, both $n - 1$ and 0 cannot be terms in the degree sequence:* A simple graph cannot have a vertex of both degree 0 and of degree $n - 1$. Suppose G had a vertex of degree $n - 1$, say v. There cannot be a loop at v because then G would not be simple. Because G is simple, there are not multiple edges between v and other vertices, i.e. there is at most 1 edge connecting v to any other vertex of G. If v was not connected to every other vertex of G, then there is a vertex with no edge connecting it to v. But then $\deg v \leq 1 + \underbrace{1 + 1 + \cdots + 1}_{n-2 \text{ times}} < 0 + \underbrace{1 + 1 + \cdots + 1}_{n-2 \text{ times}} = n - 2$, contradicting the fact that $\deg v = n - 1$. Therefore, v is connected to every other vertex of G. But then $\deg v \geq 1$ for all

$v \in V(G)$. But then G has no vertex of degree 0. Conversely, if there is a vertex of degree 0, there cannot be a vertex of degree $n - 1$ because (by the previous reasoning) this vertex of degree $n - 1$ would then be connected to the degree 0 vertex, contradicting the fact that it has degree 0.

- *The sum of the degree sequence, $\sum_i d_i$, is at most $n(n-1)$:* We know that $\deg G = \sum_i d_i$. Because G is simple, we know that G has no loops or multiple edges. We can construct a new graph, $\widetilde{G}$, as follows: $\widetilde{G}$ has the same vertex set as G, i.e. $V(\widetilde{G}) := V(G)$ and if $e \in E(G)$ connects vertices $v, w \in V(G)$, then $e \in E(\widetilde{G})$ connects $v, w \in V(\widetilde{G})$, i.e. $E(G) \subseteq E(\widetilde{G})$. But if $v, w \in V(G)$ are distinct vertices that do not have an edge connecting them, we add a single edge $e \in E(\widetilde{G})$ connecting v, w. Because G had no loops and we added none through this process, $\widetilde{G}$ has no loops. Because there were no multiple edges in G and we added none through this process, $\widetilde{G}$ has no multiple edges. Observe also that $\widetilde{G}$ has precisely one edge connecting any two vertices, $\widetilde{G}$ is a complete graph containing G has a subgraph. Clearly, $\deg G \le \deg \widetilde{G}$. We know that $\widetilde{G}$ has $\frac{n(n-1)}{2}$ edges, where n is the number of vertices of G (which is the same number of vertices as $\widetilde{G}$). By the Handshake Theorem, $\deg \widetilde{G} = 2|E(\widetilde{G})| = 2 \cdot \frac{n(n-1)}{2} = n(n-1)$. But then $\sum_i d_i = \deg G \le \deg \widetilde{G} = n(n-1)$.

We will now address each of the given degree sequences.

(a) Observe that $\sum_i d_i = 4 + 3 + 2 + 2 + 2 = 13$, which is not even. Therefore, this sequence cannot be the degree sequence of a simple undirected graph G.

(b) There exists an undirected simple graph G with degree sequence $(2, 2, 2, 0)$. For instance, the graph G below has the given degree sequence.
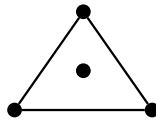
(c) Observe that $\sum_i d_i = 5 + 4 + 3 + 2 + 1 + 0 = 15$, which is not even. Therefore, this sequence cannot be the degree sequence of a simple undirected graph G. Alternatively, observe the sequence $(5, 4, 3, 2, 1, 0)$ has length $n = 6$ and contains a vertex with degree $n - 1 = 6 - 1 = 5$ and 0, which is impossible.

(d) There exists an undirected simple graph G with degree sequence $(4, 3, 3, 3, 3)$. For instance, the graph G below has the given degree sequence.
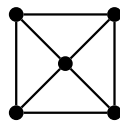
(e) Observe that the sequence $(4, 3, 2, 1)$ has length 4. Clearly, there exists a vertex with degree 4. But then not every vertex has degree less than the length of the sequence. Therefore, there cannot be a simple undirected graph G with degree sequence $(4, 3, 2, 1)$.

In fact, there is a general theorem that can be used to determine whether a given nonincreasing sequence of nonnegative integers is the degree sequence of a simple, undirected graph.

Theorem. (*Erdős–Gallai*) A sequence of nonincreasing, nonnegative integers $d_1 \geq d_2 \geq \cdots \geq d_n$ can be represented as the degree sequence of a finite, simple, undirected graph with n vertices if and only if

- $d_1 + d_2 + \cdots + d_n$ is even

- $\displaystyle\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$ for all $1 \leq k \leq n$.

Of course, the statement of this theorem merely states when there exists a graph G with degree sequence $(d_1, d_2, \ldots, d_n)$ and not how to construct such a graph. However, there exist algorithms to construct such a graph. For instance, the Havel-Hakimi algorithm allows one to construct a graph from a degree sequence.

Problem 6. (10pt) Prove that every simple graph with $n \geq 2$ vertices has at least two distinct vertices of the same degree. [Hint: Pigeonhole Principle]

Solution. Let G be a simple, finite (undirected) graph with n vertices, i.e. $|V(G)| = n$. Note that if $n = 1$, then G only has one vertex. Regardless of whether G is simple or not, if G has only one vertex, then every vertex of G has the same degree—there's only one degree!

Assume then that $n \geq 2$. Because G is simple, it has no loops or multiple edges between vertices. We first show that if $v \in V(G)$, then $\deg v < n$. Suppose that $v \in V(G)$ with $\deg v \geq n$, i.e. v is incident to n or more edges. Because G is simple, there cannot be a loop at v. But then these $\deg v \geq n$ edges incident to v connect v to one of the other $|V(G)| - 1 = n - 1$ vertices. But then the $\deg v \geq n$ edges each connect v to one of $n - 1$ possible vertices. By the Pigeonhole Principle, there must vertex $w \in V(G)$ such that there are at least two edges connecting w to v. But then G has multiple edges, contradicting the fact that G is simple.

Therefore, if $v \in V(G)$, then $\deg v \in \{0, 1, 2, \ldots, n-1\}$ for all $v \in V(G)$—a total of $(n-1) - 0 + 1 = n$ possible values for $\deg v$. However, a simple graph cannot have a vertex of both degree 0 and of degree $n - 1$. By the reasoning above, if G had a vertex of degree $n - 1$, then this vertex is connected to every other vertex. But then $\deg v \geq 1$ for all $v \in V(G)$ so that there is no vertex in G with degree 0. Conversely, if there is a vertex of degree 0, there cannot be a vertex of degree $n - 1$ because (by the reasoning above) this vertex of degree $n - 1$ would then be connected to the degree 0 vertex, contradicting the fact that it has degree 0.

Therefore, the possible degrees for vertices in G are either $\{0, 1, \ldots, n - 2\}$ or $\{1, 2, \ldots, n - 1\}$. In either case, there are a total of $n - 1$ possible values for the degree of $v \in V(G)$. But then there are only $n - 1$ possible values for the degrees of the $|V(G)| = n$ vertices of G. By the Pigeonhole Principle, there must be two vertices that have the same degree.

Remark. What if the graph is not simple? If G is not simple, then G either has loops or multiple edges. If a graph has loops, then there may not be two distinct vertices with the same degree. An example of this is the graph below:
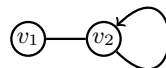
Clearly, $\deg v_1 = 1$ and $\deg v_2 = 3$. Furthermore, if G has multiple edges, then there may not be two distinct vertices with the same degree. An example of this is the graph below:
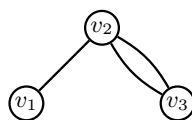
Clearly, $\deg v_1 = 1$, $\deg v_2 = 3$, and $\deg v_3 = 2$.

Problem 7. (10pt) Prove that in a tree, T, every pair of vertices u and v has a unique path connecting them.

Solution. In fact, a graph T is a tree if and only if there is a unique path between every distinct pair of vertices u, v. We prove this stronger statement.

If T is a tree, then there is a unique walk between every pair of distinct vertices u, v: Assume that T is a tree, i.e. a connected circuit-free graph. We want to show every pair of vertices u and v has a unique walk connecting them. Because T is connected, there exists a path between u and v. We need only show that this path is unique.

We prove there is a unique path between vertices of T by contradiction.

Suppose that $ue_1w_1e_2w_2\cdots e_nw_nv$ and $ue'_1w'_1e'_2w'_2\cdots e'_mw'_mv$ were two distinct walks from u to v.

We shall use these two paths to construct a circuit in T, contradicting the fact that T is circuit-free.[1]

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[1]One might think we can simply use $ue_1w_1e_2w_2\cdots e_nw_nvw'_me'_mw'_{m-1}e'_{m-1}\cdots w'_1e'_1u$. However, while this is a closed walk from u to u, there may be repeated edges, i.e. there may exist i, j such that $w_i = w_j$, $w'_i = w'_j$, or $w_i = w'_j$. If this occurs, then this walk is not a circuit.

If every pair of vertices u and v has a unique path connecting them, then the graph is a tree: Suppose that G is a graph such that every pair of vertices has a unique path connecting them. To prove that G is a tree, we need to show that G is connected and circuit-free. Because there is a path—in fact a unique path—between u and v for all $u, v \in V(G)$, we know that G is connected. Now we only need to show that G is circuit-free.

Suppose that G contained a circuit, say $v e_1 v_2 e_2 v_3 e_3 \cdots e_{n-1} v_n e_n v$. Because this walk is a circuit, the walk contains at least one edge, i.e. $n \geq 1$, and the walk contains no repeated edges, i.e. the edges are distinct so that $e_i = e_j$ if and only if $i = j$. Suppose that $n > 1$, i.e. the walk contains more than one edge. Let $j \in \{2, 3, \ldots, n\}$ (which is nonempty because $n \geq 2$) and consider the walks $W_1 : v e_1 v_2 e_2 v_3 e_3 \cdots e_{j-1} v_j$ and $W_2 : v e_{n-1} v_{n-1} e_{n-2} \cdots e_j v_j$. Both W_1, W_2 are walks from v to v_j.[2] We know that $j \neq 1$ because $j \in \{2, 3, \ldots, n\}$. But then $e_j \neq e_1$ because $j \neq 1$. Therefore, W_1 and W_2 are distinct walks from v to v_j because W_1 contains the edge e_1, W_2 contains the edge e_j, and $e_1 \neq e_j$. This contradicts the fact that there is a unique walk between vertices of G. Therefore, no such j can exist. But then it must be that $\{2, 3, \ldots, n\}$ is empty, which implies that $n = 1$. Therefore, the circuit $v e_1 v_2 e_2 v_3 e_3 \cdots e_{n-1} v_n e_n v$ is the circuit $v e_1 v$, i.e. e_1 is a loop at v. But then v and $v e_1 v$ are both distinct walks from v to v, contradicting the fact that there is a unique walk between vertices of G. Therefore, G cannot contain a circuit.

Because G is connected and circuit-free, G is a tree.

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[2] W_1 can be considered the 'forward' walk from v to v_j while W_2 can be considered the 'backward' walk from v to v_j.

Problem 8. (10pt) Suppose the ground floor plan of a building is given below. Is it possible to walk through every door on the first floor exactly once, ending up in your starting room? Explain. Is it possible to visit every room exactly once, ending up in your starting room? Explain.
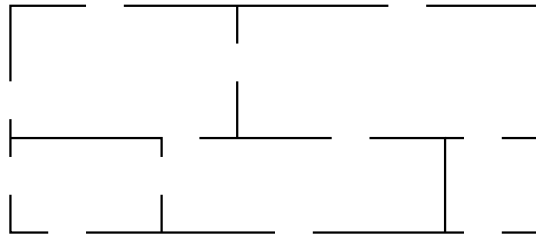
Problem 9. (10pt) Define the following matrices:

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \\ 0 & 3 \\ -2 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

For each of the following operations, either compute the given expression or explain why it is undefined.

(a) $2A - B$

(b) AB

(c) $A + C$

(d) Av

(e) AC

(f) CA

Solution. One cannot only add or subtract matrices with the exact same dimension. If A, B are $n \times m$ and $r \times s$ matrices, respectively, then AB is defined if and only if $m = r$. If so, the product has dimension $n \times s$.

(a)
$$2A - B = 2\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -4 \\ 6 & 2 & 10 \end{pmatrix} - \begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 4 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -3 \\ 6 & 4 & 6 \end{pmatrix}$$

(b) The product AB is not defined because A has 3 columns while B only has 2 rows.

(c) The sum $A + C$ is not defined because A and C have different dimensions.

(d)
$$Av = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1(3) + 0(0) + (-2)(-1) \\ 3(3) + 1(0) + 5(-1) \end{pmatrix} = \begin{pmatrix} 3 + 0 + 2 \\ 9 + 0 - 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

(e)
$$AC = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 3 \\ -2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1(1) + 0(0) + (-2)(-2) & 1(-1) + 0(3) + (-2)1 \\ 3(1) + 1(0) + 5(-2) & 3(-1) + 1(3) + 5(1) \end{pmatrix}$$
$$= \begin{pmatrix} 1 + 0 + 4 & -1 + 0 - 2 \\ 3 + 0 - 10 & -3 + 3 + 5 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & -3 \\ -7 & 5 \end{pmatrix}$$

(f)

$$CA = \begin{pmatrix} 1 & -1 \\ 0 & 3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1(1) + (-1)3 & 1(0) + (-1)1 & 1(-2) + (-1)5 \\ 0(1) + 3(3) & 0(0) + 3(1) & 0(-2) + 3(5) \\ -2(1) + 1(3) & -2(0) + 1(1) & -2(-2) + 1(5) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 3 & 0 - 1 & -2 - 5 \\ 0 + 9 & 0 + 3 & 0 + 15 \\ -2 + 3 & 0 + 1 & 4 + 5 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -1 & -7 \\ 9 & 3 & 15 \\ 1 & 1 & 9 \end{pmatrix}$$

Problem 10. (10pt) Consider the graph G given below:
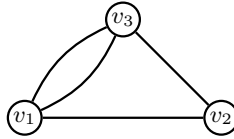
(a) Compute the adjacency matrix, A.

(b) Using a computer system, compute A, A^2, A^3, and A^4.

(c) Compute the number of walks from v_1 to v_3 of lengths one, two, three, and four, respectively.

(d) Compare your answers from (b) and (c). Make a conjecture on what the a_{ij} entry of A^k represents.

Solution.

(a) The adjacency matrix is $A = (a_{ij})$, where a_{ij} is the number of edges from vertex i to vertex j. But then the adjacency matrix is...

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

(b) We find that...

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 4 & 6 & 12 \\ 6 & 4 & 6 \\ 12 & 6 & 4 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 5 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix} \qquad A^4 = \begin{pmatrix} 30 & 16 & 14 \\ 16 & 12 & 16 \\ 14 & 17 & 30 \end{pmatrix}$$

(c)

(d) The number of walks of length $k > 0$ from v_i to v_j is the a_{ij}-entry of A^k, where A is the adjacency matrix of the graph (whether directed or undirected).