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MATH 308
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HW 9: Due 11/05

"Computer Science is no more about computers than astronomy is about telescopes."

–Edsger W. Dijkstra

Problem 1. (10pt) Let $f: A \to \mathbb{R}$ be defined by $f(x) := x^3 - 9x^2 + 23x - 12$, where $A = \{1, 3, 6\}$. Let $g: B \to \mathbb{R}$ be defined by $g(x) = x^2 - 4x + 6$, where

$$B = \{x \in \mathbb{N} \mid x \text{ divides } 6\} \setminus \{x \colon x \text{ is an even prime number}\}$$

Prove that $f = g$.

Solution. To prove two functions are equal, we need to show that...

• *The functions have the same domain:*

The domain for f is A and the domain for g is B. So we need to show that $A = B$. Observe that...

$$\begin{aligned} B &= \{x \in \mathbb{N} \mid x \text{ divides } 6\} \setminus \{x \colon x \text{ is an even prime number}\} \\ &= \{1, 2, 3, 6\} \setminus \{2\} \\ &= \{1, 3, 6\} \end{aligned}$$

But then it is clear that $A = B$.

•*The functions have the same codomain.*

This is immediate because the codomain of f and g are the same—namely $\mathbb{R}$.

•*The functions are equal everywhere on their common domain.*

The domain for both functions is $\{1, 3, 6\}$. Observe...

$$\begin{aligned} f(1) &= 1^3 - 9(1^2) + 23(1) - 12 = 1 - 9 + 23 - 12 = 3 \\ g(1) &= 1^2 - 4(1) + 6 = 1 - 4 + 6 = 3 \end{aligned}$$

$$\begin{aligned} f(3) &= 3^3 - 9(3^2) + 23(3) - 12 = 27 - 81 + 69 - 12 = 3 \\ g(3) &= 3^2 - 4(3) + 6 = 9 - 12 + 6 = 3 \end{aligned}$$

$$\begin{aligned} f(6) &= 6^3 - 9(6^2) + 23(6) - 12 = 216 - 324 + 138 - 12 = 18 \\ g(6) &= 6^2 - 4(6) + 6 = 36 - 24 + 6 = 18 \end{aligned}$$

Therefore, we know that $f = g$.

Problem 2. (10pt) Recall the absolute value function, $f(x) = |x|$, is given by

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Considering $f : \mathbb{R} \to \mathbb{R}$, determine the following sets:

(a) $f((-2, 1])$

(b) $f(\mathbb{Z})$

(c) $f^{-1}((-2, 1])$

(d) $f^{-1}(\{-5\})$

(e) $f^{-1}(\mathbb{Z})$

Solution. Let $f(x) = |x|$. If S is a set of real numbers, let $\pm|S| := \{\pm|s| \colon s \in S\}$. Observe that because $f(P) = \{f(p) \colon p \in P\}$, if P is a set of nonnegative real numbers, then $f(P) = P$. Moreover, because $f(N) = \{f(n) \colon n \in N\}$, if N is a set of negative real numbers, we know that $f(N) = \{f(n) \colon n \in N\} = \{f(|n|) \colon n \in N\} = f(|N|) = |N|$. But then given a set S of real numbers, we can decompose $S = P \cup N$ into a set of nonnegative numbers, P, and negative numbers, N, respectively. But then we have $f(S) = f(P \cup N) = f(P) \cup f(N) = P \cup |N|$.

Now recall that $f^{-1}(S)$ is the preimage of S under f, i.e. the set of $x \in \mathbb{R}$ such that $f(x) \in S$. As above, let P contain only nonnegative real numbers and N consists of only negative real numbers. Clearly, $f^{-1}(N) = \varnothing$ because $f(x) \geq 0$ for all $x \in \mathbb{R}$, i.e. $f(x) \notin N$ for all $x \in \mathbb{R}$. Now let $p \in P$. Clearly, $f(\pm p) = |\pm p| = p$. But then $f^{-1}(P) = \{\pm p \colon p \in P\}$.

Solution.

(a) We have...

$$f\big((-2, 1]\big) = f\big((-2, 0) \cup [0, 1]\big) = f\big((-2, 0)\big) \cup f\big([0, 1]\big) = (0, 2) \cup [0, 1] = [0, 2)$$

(b)

$$f(\mathbb{Z}) = f\left(\mathbb{Z}_{<0} \cup \mathbb{Z}_{\geq 0}\right) = f\left(\mathbb{Z}_{<0}\right) \cup f\left(\mathbb{Z}_{\geq 0}\right) = |\mathbb{Z}_{<0}| \cup \mathbb{Z}_{\geq 0} = \mathbb{Z}_{>0} \cup \mathbb{Z}_{\geq 0} = \mathbb{Z}_{\geq 0}$$

(c)

$$f^{-1}\big((-2, 1]\big) = f^{-1}\big((-2, 0) \cup [0, 1]\big) = f^{-1}\big((-2, 0)\big) \cup f^{-1}\big([0, 1]\big) = \varnothing \cup \{\pm r \colon r \in [0, 1]\} = [-1, 1]$$

(d)

$$f^{-1}\big(\{-5\}\big) = \varnothing$$

(e)

$$f^{-1}(\mathbb{Z}) = f^{-1}\left(\mathbb{Z}_{<0} \cup \mathbb{Z}_{\geq 0}\right) = f^{-1}\left(\mathbb{Z}_{<0}\right) \cup f^{-1}\left(\mathbb{Z}_{\geq 0}\right) = \varnothing \cup \{\pm z \colon z \in \mathbb{Z}_{\geq 0}\} = \mathbb{Z}$$

Problem 3. (10pt) Let $f : \mathbb{Z} \to \mathbb{R}$ be given by $f(n) = 2^n$, and let $g : \mathbb{Z} \to \mathbb{R}$ be given by $g(n) = 100 - 3^n$.

(a) Compute $f(1)$.

(b) Compute $g(1)$.

(c) Compute $(fg)(1)$.

(d) Compute $(f \circ g)(1)$.

(e) Find the rule for $(fg)(x)$.

Solution.

(a)

$$f(1) = 2^1 = 2$$

(b)

$$g(1) = 100 - 3^1 = 100 - 3 = 97$$

(c)

$$(fg)(1) = f(1) \cdot g(1) = 2 \cdot 97 = 194$$

(d)

$$(f \circ g)(1) = f\big(g(1)\big) = f(97) = 2^{97} = 158{,}456{,}325{,}028{,}528{,}675{,}187{,}087{,}900{,}672$$

(e)

$$(fg)(x) = f(x) \cdot g(x) = 2^x(100 - 3^x)$$

Problem 4. (10pt) Recall that given a function $f : S \to S$, we say that $x \in S$ is a fixed point of f if $f(x) = x$. Let $S = \mathbb{R}$ and let f be the function given by $x \mapsto x^2 + 4x - 10$. Find the fixed points of f. How does the answer change if $S = \mathbb{N}$?

Solution. We know that $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2 + 4x - 10$. If x is a fixed point for f, then $f(x) = x$. But then...

$$\begin{aligned} f(x) &= x \\ x^2 + 4x - 10 &= x \\ x^2 + 3x - 10 &= 0 \\ (x+5)(x-2) &= 0 \end{aligned}$$

This implies that either $x + 5 = 0$, so that $x = -5$, or $x - 2 = 0$, so that $x = 2$. Observe that $f(-5) = (-5)^2 + 4(-5) - 10 = 25 - 20 - 10 = -5$ and $f(2) = 2^2 + 4(2) - 10 = 4 + 8 - 10 = 2$, which shows that $-5, 2$ are fixed points for $f(x)$.

Now consider $f(x)$ as a function $f : \mathbb{N} \to \mathbb{N}$. The solution $x = -5$ is no longer valid because $-5 \notin \mathbb{N}$. But then the only fixed point for $f(x)$ would be $x = 2$.

Problem 5. (10pt) Recall that the image of a function $f : S \to S$ (also called the range) is the set $\operatorname{im} f = \{f(s) \colon s \in S\}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{1+x^2}$.

(a) Determine the error in the following 'proof' that $\operatorname{im} f = \mathbb{R}$:

We need prove that $\operatorname{im} f \subseteq \mathbb{R}$ *and* $\mathbb{R} \subseteq \operatorname{im} f$. *Clearly,* $f(x) \in \mathbb{R}$ *so that* $\operatorname{im} f \subseteq \mathbb{R}$. *Now let* $y \in \mathbb{R}$. *Define* $x := \sqrt{\dfrac{1-y}{y}}$. *Then*

$$f(x) = \frac{1}{1+x^2} = \frac{1}{1 + \dfrac{1-y}{y}} = \frac{1}{\dfrac{y+1-y}{y}} = \frac{1}{1/y} = y.$$

But then $x \in \mathbb{R}$ *and* $f(x) = y$. *Therefore,* $\mathbb{R} \subseteq \operatorname{im} f$. *Because* $\operatorname{im} f \subseteq \mathbb{R}$ *and* $\mathbb{R} \subseteq \operatorname{im} f$, $\operatorname{im} f = \mathbb{R}$.

(b) Determine $\operatorname{im} f$ and prove that your answer is correct.

Solution.

(a) The first sentence is correct—two sets are equal if and only if they are subsets of each other. The second sentence is also true—the outputs of $f(x)$ are real numbers (because the inputs are real) so that $\operatorname{im} f \subseteq \mathbb{R}$. The third sentence is fine, one can declare y to be some real number. However, the next sentence has an error. We took $y \in \mathbb{R}$ to be *any* real number and defined a (hopefully) real number x. But the definition of x should then work for any y-value. If $y = -1$, then $x = \sqrt{\dfrac{1-(-1)}{-1}} = \sqrt{\dfrac{2}{-1}} = \sqrt{-2}$ is not a real number. The next sentence is true, if x is a real number (so that it is in the domain of f), then the evaluation of $f(x) = y$ is correct so that $y \in \mathbb{R}$ is in the range of $f(x)$. Therefore, the error in the proof is the fact that the defined x may not be a real number.

(b) Based on (a), we see that the proof fails because x may not be a real number. For x to be defined, we need $\frac{1-y}{y} \geq 0$. Clearly, $y \neq 0$. Furthermore, if $y < 0$, then $-y > 0$ so that $1 - y > 1 > 0$ and then $\frac{1-y}{y} < 0$ (because $y < 0$). Finally, suppose $y \geq 0$. Then $\frac{1-y}{y} \geq 0$ implies that $1 - y \geq 0$ so that $1 \geq y$. As $y > 0$, this implies $0 < y \leq 1$. We would then conjecture that $\operatorname{im} f = (0, 1]$, which we shall prove. In fact, the proof above works with little modification!

We need prove that $\operatorname{im} f \subseteq (0, 1]$ and $(0, 1] \subseteq \operatorname{im} f$. Because $x \in \mathbb{R}$, we know that $x^2 \geq 0$. But then $1 + x^2 \geq 1$, so that $f(x) = \frac{1}{1+x^2} \leq 1$. Finally, because $1 + x^2 > 0$, we know that $f(x) = \frac{1}{1+x^2} > 0$. Therefore, $0 < f(x) \leq 1$. This proves that $\operatorname{im} f \subseteq (0, 1]$.

Now let $y \in (0, 1]$. Then $1 \geq y$ so that $1 - y \geq 0$. But because $y > 0$, this implies that $\frac{1-y}{y} \geq 0$. Now because $\frac{1-y}{y} \geq 0$, we can define $x := \sqrt{\dfrac{1-y}{y}}$. Then

$$f(x) = \frac{1}{1+x^2} = \frac{1}{1 + \dfrac{1-y}{y}} = \frac{1}{\dfrac{y+1-y}{y}} = \frac{1}{1/y} = y.$$

But then $x \in \mathbb{R}$ and $f(x) = y$. Therefore, $(0, 1] \subseteq \operatorname{im} f$. Because $\operatorname{im} f \subseteq (0, 1]$ and $(0, 1] \subseteq \operatorname{im} f$, $\operatorname{im} f = (0, 1]$.

We might have predicted this to be the image of $\mathbb{R}$ under f by examining the graph of $f(x)$: