MAT 101: Exam 2
Winter – 2021
01/14/2021
95 Minutes

Name: *Caleb M^c Whorter — Solutions*

Write your name on the appropriate line on the exam cover sheet. This exam contains 18 pages (including this cover page) and 17 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work.

Question	Points	Score
1	6	
2	6	
3	4	
4	6	
5	6	
6	6	
7	6	
8	6	
9	6	
10	4	
11	6	
12	6	
13	6	
14	6	
15	6	
16	4	
17	10	
Total:	100	

1. (6 points) A table of values for a function $f(x)$ is given below.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$	6	3	0	-2	4	6	3	0	5	6	8

Determine the y-intercepts and x-intercepts for the function $f(x)$.

The y-intercept is the point where $f(x)$ intersects the y-axis, i.e. the point $(0, f(0))$. From the table, we see that this is $(0, 6)$.

The x-intercepts are the points where $f(x)$ passes through the x-axis, i.e. the values x_0 such that $f(x_0) = 0$. From the table, we see that these are $(-3, 0)$ and $(2, 0)$.

y-intercepts: $(0, 6)$

x-intercepts: $(-3, 0), (2, 0)$

2. (6 points) A table of values for a function $f(x)$ is given below. Determine whether the function $f(x)$ is linear or not. Be sure to fully justify your answer.

x	0	1	2	4	5	6
$f(x)$	-5	-2	1	7	11	13

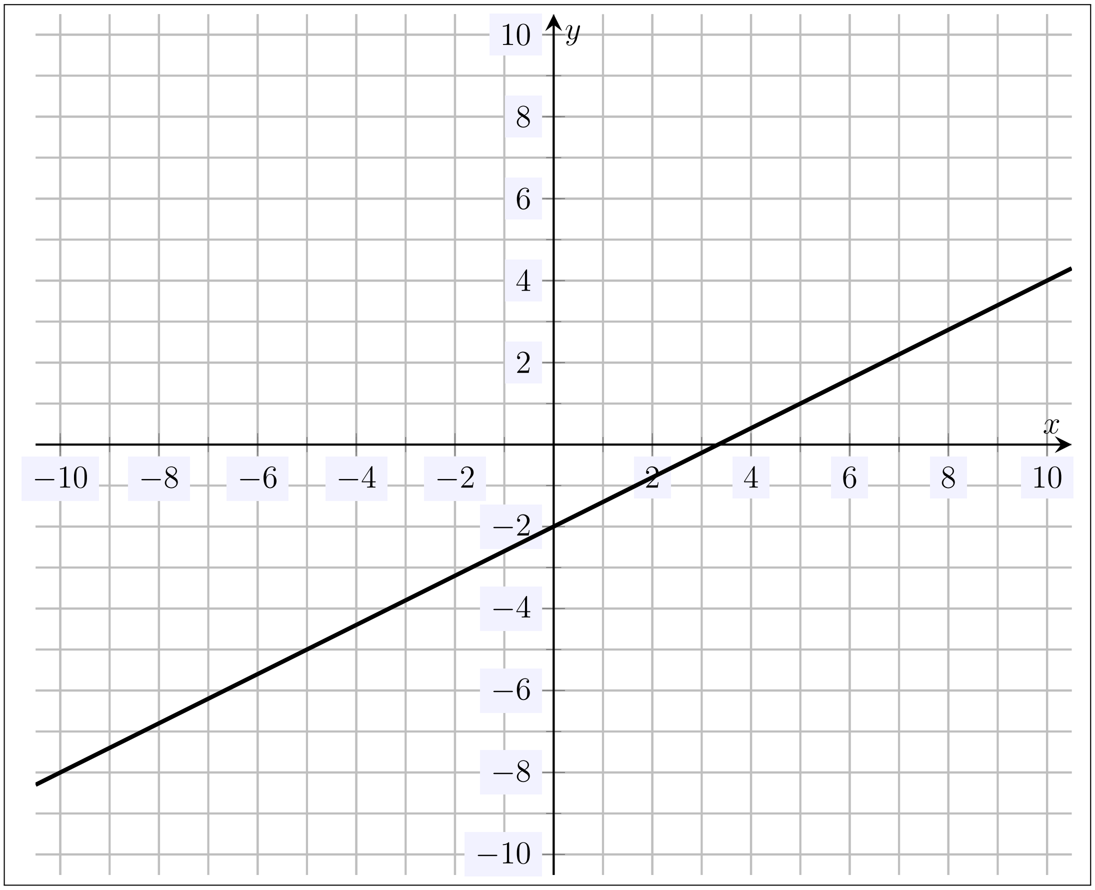
If $f(x)$ is a linear function, then the slope of $f(x)$ is constant. We compute the slopes using the points $(0,-5),(1,-2)$ and $(4,7),(5,11)$:

$$m = \frac{-2-(-5)}{1-0} = \frac{3}{1} = 3$$

$$m = \frac{11-7}{5-4} = \frac{4}{1} = 4$$

Because the slope of this function is not constant, $f(x)$ cannot be linear.

3. (4 points) Sketch the line $3x - 5y = 10$ on the plot below.



Solving for y, we have...

$$3x - 5y = 10$$

$$-5y = -3x + 10$$

$$y = \frac{3}{5}x - 2$$

We can then easily find two points on the line to connect with a straight line, as in the plot above:

$$y(5) = \frac{3}{5} \cdot 5 - 2 = 3 - 2 = 1 \rightsquigarrow (5, 1)$$

$$y(-5) = \frac{3}{5} \cdot -5 - 2 = -3 - 2 = -5 \rightsquigarrow (-5, -5)$$

4. (6 points) Two lines are given below. Determine whether these lines are the same or parallel. Determine also whether these lines intersect or not. If so, determine whether they intersect perpendicularly.

$$\begin{aligned} \ell_1: &\quad y = 8 - \frac{5}{6}\,x \\ \ell_2: &\quad 6x + 5y = 15 \end{aligned}$$

Solving for y in the second line, we have...

$$\begin{aligned} 6x + 5y &= 15 \\ 5y &= -6x + 15 \\ y &= -\frac{6}{5}\,x + 3 \end{aligned}$$

Therefore, the lines are...

$$\begin{aligned} \ell_1: &\quad y = 8 - \frac{5}{6}\,x \\ \ell_2: &\quad y = -\frac{6}{5}\,x + 3 \end{aligned}$$

The slope of the first line is $m_1 = -\frac{5}{6}$ and the slope of the second line is $m_2 = -\frac{6}{5}$. Because $m_1 \neq m_2$, the lines cannot be the same or parallel. Therefore, the lines must intersect. However, the negative reciprocal of $m_1 = -\frac{5}{6}$ is $\frac{6}{5} \neq m_2$. Therefore, the lines do not intersect perpendicularly.

5. (6 points) A table of values for a linear function $f(x)$ is given below.

x	-12	-8	-4	4	8	12
$f(x)$	21	18	15	9	6	3

Determine the equation for $f(x)$.

Clearly, the line is not vertical so that we know $f(x) = mx + b$. First, we compute the slope of $f(x)$ using the points $(4, 9)$ and $(8, 6)$ (although any two distinct points would suffice):

$$m = \frac{9-6}{4-8} = \frac{3}{-4} = -\frac{3}{4}$$

Then $f(x) = -\frac{3}{4}x + b$. Because $(4, 9)$ is on the line, it satisfies the equation for the line:

$$f(x) = -\frac{3}{4}\,x + b$$

$$f(4) = -\frac{3}{4} \cdot 4 + b$$

$$9 = -3 + b$$

$$12 = b$$

Therefore, $f(x) = -\frac{3}{4}\,x + 12$.

6. (6 points) Find the equation of the line that contains the points $(-10, 15)$ and $(2, -1)$.

Clearly, the line is not vertical so that we know $y = mx + b$. *First, we compute the slope of the line:*

$$m = \frac{15 - (-1)}{-10 - 2} = \frac{16}{-12} = -\frac{4}{3}$$

Then $y = -\frac{4}{3}x + b$. *Because* $(2, -1)$ *is on the line, it satisfies the equation for the line:*

$$\begin{aligned} y &= -\frac{4}{3}\,x + b \\ -1 &= -\frac{4}{3} \cdot 2 + b \\ -1 &= -\frac{8}{3} + b \\ b &= -1 + \frac{8}{3} \\ b &= \frac{-3}{3} + \frac{8}{3} \\ b &= \frac{5}{3} \end{aligned}$$

Therefore, $f(x) = -\frac{4}{3}\,x + \frac{5}{3} = \frac{5-4x}{3}$.

7. (6 points) Determine the equation of the line that contains the point $(6, 10)$ and is perpendicular to the line $y = -1$.

Because the line $y = -1$ is horizontal, a line perpendicular to this line must be vertical, i.e. of the form $x = M$ for some M. Because the line passes through the point $(6, 10)$, it must be that $x = 6$.

8. (6 points) Find the equation of the line that is perpendicular to the line $y = 4 - 5x$ and passes through the y-intercept of the line $y = 2x + 3$.

Because the line is perpendicular to $y = 4 - 5x$ *(which is not horizontal), the line is not vertical. Therefore, the line has the form* $y = mx + b$*. Because this line is perpendicular to* $y = 4 - 5x$*, which has slope* -5*, the slope of the line must be* $m = -(\frac{1}{-5}) = \frac{1}{5}$*. Then we know that* $y = \frac{1}{5}\,x + b$*.*

Now because $y(0) = 2(0) + 3 = 0 + 3 = 3$*, the* y*-intercept of* $y = 2x + 3$ *is the point* $(0, 3)$*. Because this point lies along the line* $y = \frac{1}{5}\,x + b$*, we know that it satisfies the equation of the line, i.e.*

$$y = \frac{1}{5}\,x + b$$

$$3 = \frac{1}{5} \cdot 0 + b$$

$$3 = b$$

Therefore, $y = \frac{1}{5}\,x + 3 = \frac{x+15}{5}$*.*

9. (6 points) A researcher creates a model to predict adolescent male's weight (in lbs) from their height (in cm). The model is $W(h) = 2.6h - 17.3$.

   (a) Is the model linear? Explain.

   (b) Determine the slope of $W(h)$. Interpret the slope in context.

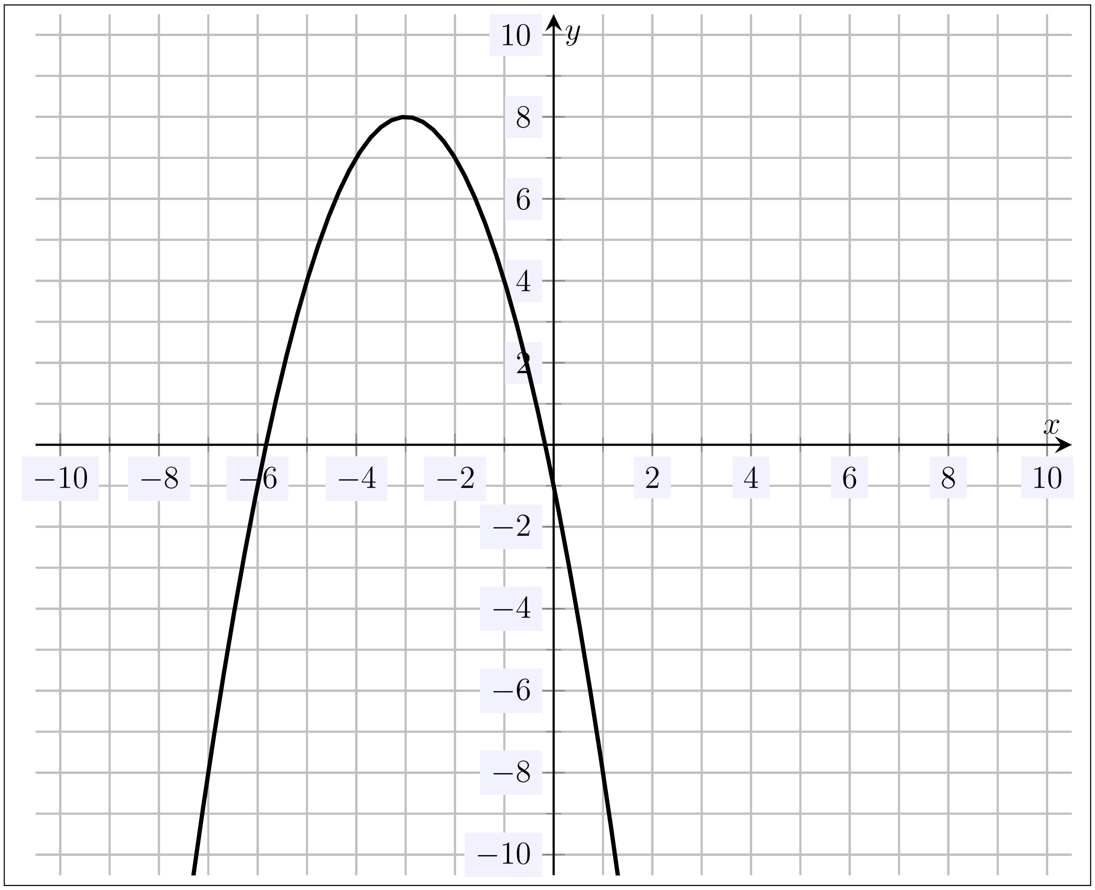
   (c) Determine the y-intercept of $W(h)$. Does the y-intercept have meaning in this context? Explain.

(a) The model $W(h) = 2.6h - 17.3$ has the form $y = mx + b$, where $x = t$, $m = 2.6$, and $b = -17.3$. Therefore, $W(h)$ is linear.

(b) From (a), we know that the slope is $m = 2.6$. Interpreting $m = 2.6 = \frac{2.6}{1}$ as $\frac{\Delta y}{\Delta x}$ and using the fact that x has units of cm and y has units of lbs, we see that the slope represents that the model says that for each addition centimeter of height, an adolescent male should weigh 2.6 lbs more.

(c) Because $W(0) = 2.6(0) - 17.3 = -17.3$, the y-intercept is $(0, -17.3)$. However, $h = 0$ would represent a male adolescent with heigh 0 cm—which is impossible. Moreover, $W(0) = -17.3$ lb would imply that an adolescent male with height 0 cm would weigh -17.3 lb—equally nonsensical. Therefore, the y-intercept likely does not have meaning in this context.

10. (4 points) Sketch the function $f(x) = 8 - (x+3)^2$ on the plot below.



Because $f(x) = 8 - (x+3)^2$ is in vertex form, we see that the vertex is $(-3, 8)$. Furthermore, because $a = -1 < 0$, we know that the parabola opens downwards. This gives the sketch above.

11. (6 points) Showing all your work, find the vertex form of $y = 2x^2 - 12x + 23$.

We complete the square. First, we factor out the 2 to obtain $y = 2(x^2 - 6x + \frac{23}{2})$. *Now observe that* $\frac{-6}{2} = -3$ *and* $(-3)^2 = 9$. *But then...*

$$y = 2x^2 - 12x + 23$$

$$y = 2\left(x^2 - 6x + \frac{23}{2}\right)$$

$$y = 2\left(x^2 - 6x + (9 - 9) + \frac{23}{2}\right)$$

$$y = 2\left((x^2 - 6x + 9) - 9 + \frac{23}{2}\right)$$

$$y = 2\left((x^2 - 6x + 9) - \frac{18}{2} + \frac{23}{2}\right)$$

$$y = 2\left((x - 3)^2 + \frac{5}{2}\right)$$

$$y = 2(x - 3)^2 + 5$$

12. (6 points) Showing all your work, factor the polynomial $x^2 - 22x - 48$.

48

$1 \cdot -48$	-47
$-1 \cdot 48$	47
$\boxed{2 \cdot -24}$	$\boxed{-22}$
$-2 \cdot 24$	22
$3 \cdot -16$	-13
$-3 \cdot 16$	13
$4 \cdot -12$	-8
$-4 \cdot 12$	8
$6 \cdot -8$	-2
$-6 \cdot 8$	2

Therefore,

$$x^2 - 22x - 48 = (x + 2)(x - 24)$$

13. (6 points) Showing all your work, factor the polynomial $5x^2 + 19x - 4$.

$$\begin{array}{c} \underline{\mathbf{4}} \\ 1 \cdot -4 \\ -1 \cdot 4 \\ 2 \cdot -2 \end{array}$$

Then as $5 = 1 \cdot 5$, we have...

$$\begin{array}{ccccccccc} & 1 \cdot -4 & & & \boxed{-1 \cdot 4} & & & 2 \cdot -2 & \\ 1,5 & & 5,1 & 1,5 & & 5,1 & 1,5 & & 5,1 \\ 1,-20 & & 5,-4 & \boxed{-1,20} & & -5,4 & 2,-10 & & 10,-2 \end{array}$$

Therefore,

$$5x^2 + 19x - 4 = (5x - 1)(x + 4)$$

14. (6 points) Showing all your work, solve the equation $2x = 24 - x^2$.

$$\begin{aligned} 2x &= 24 - x^2 \\ x^2 + 2x - 24 &= 0 \\ (x+6)(x-4) &= 0 \end{aligned}$$

Then either $x + 6 = 0$*, which implies* $x = -6$*, or* $x - 4 = 0$*, which implies that* $x = 4$*. Therefore, we have. . .*

$$x = -6,\ 4$$

15. (6 points) Showing all your work, use the quadratic equation to solve $2x^2 = 4x - 10$.

First, we rewrite the equation:

$$2x^2 = 4x - 10$$

$$2x^2 - 4x + 10 = 0$$

Then we have...

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(10)}}{2(2)}$$

$$x = \frac{4 \pm \sqrt{16 - 80}}{4}$$

$$x = \frac{4 \pm \sqrt{-64}}{4}$$

$$x = \frac{4 \pm \sqrt{-64}}{4}$$

$$x = \frac{4 \pm \sqrt{64}i}{4}$$

$$x = \frac{4 \pm 8i}{4}$$

$$x = 1 \pm 2i$$

Then either $x = 1 + 2i$ *or* $x = 1 - 2i$. *Therefore,* $x = 1 - 2i, 1 + 2i$.

16. (4 points) Consider the quadratic function $f(x) = x^2 - 6x + 4$. Use the discriminant of $f(x)$ to show that $f(x)$ does not factor 'nicely', then use the quadratic formula to factor $f(x)$.

We know that $D = b^2 - 4ac$. For $f(x)$, we have $a = 1$, $b = -6$, and $c = 4$. But then

$$D = b^2 - 4ac = (-6)^2 - 4(1)4 = 36 - 16 = 20 = 2^2 \cdot 5$$

Because 20 is not a perfect square, the polynomial $f(x) = x^2 - 6x + 4$ does not factor 'nicely.' To find a factorization of $f(x)$, we find the roots of $f(x)$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{6 \pm \sqrt{36 - 16}}{2}$$

$$x = \frac{6 \pm \sqrt{20}}{2}$$

$$x = \frac{6 \pm \sqrt{4 \cdot 5}}{2}$$

$$x = \frac{6 \pm 2\sqrt{5}}{2}$$

$$x = 3 \pm \sqrt{5}$$

Then the roots are $r_1 = 3 + \sqrt{5}$ and $r_2 = 3 - \sqrt{5}$. Then we have...

$$f(x) = x^2 - 6x + 4 = a(x - r_1)(x - r_2) = 1\big(x - (3 + \sqrt{5})\big)\big(x - (3 - \sqrt{5})\big) = \big(x - (3 + \sqrt{5})\big)\big(x - (3 - \sqrt{5})\big)$$

17. (10 points) Consider the function $f(x) = 3x^2 - 15x + 20$.

(a) Determine if the parabola is concave up or concave down.

(b) Determine the axis of symmetry of $f(x)$.

(c) Determine the vertex of $f(x)$.

(d) Does $f(x)$ have a maximum or minimum value? Explain.

(e) Find the maximum or minimum value of $f(x)$ from (d).

(a) Because $a = 3 > 0$, the parabola opens upwards, i.e. is concave up (or convex).

(b) We know the x-coordinate of the vertex occurs when $x = \frac{-b}{2a}$. But then...

$$x = \frac{-b}{2a} = \frac{-(-15)}{2(3)} = \frac{15}{2(3)} = \frac{5}{2}$$

Therefore, the axis of symmetry is $x = \frac{5}{2}$.

(c) We know the x-coordinate of the vertex is $x = \frac{5}{2}$. Then the y-value is...

$$f\left(\frac{5}{2}\right) = 3\left(\frac{5}{2}\right)^2 - 15 \cdot \frac{5}{2} + 20 = 3 \cdot \frac{25}{4} - \frac{75}{2} + 20 = \frac{75}{4} - \frac{150}{4} + \frac{80}{4} = \frac{5}{4}$$

Therefore, the vertex is $(\frac{5}{2}, \frac{5}{4})$.

(d) Because $a = 3 > 0$, the parabola opens upwards, i.e. is concave up or convex. But then $f(x)$ has a minimum value.

(e) We know the minimum value for $f(x)$ occurs at the vertex, which from (c) is $(\frac{5}{2}, \frac{5}{4})$. But then the minimum value for $f(x)$ is $\frac{5}{4}$.