MATH 108

Fall 2021

HW 7: Due 11/04

Problem 1. (10pt) Write down the tableau associated to the following linear programming problem:

$$\max z = 3x_1 + x_2$$
$$2x_1 + 3x_2 \leq 6$$
$$4x_1 + 2x_2 \leq 8$$
$$-3x_1 + 4x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

Solution. Before creating the tableau, we want to be sure the linear programming problem is in standard form (or as close as possible). Because this is a maximization problem, we require all the inequalities to be '$\leq$.' [This is excluding the final inequality $x_1, x_2 \geq 0$, because we require the variables to be nonnegative.] So we multiply both sides of the third inequality by -1 to obtain $3x_1 - 4x_2 \leq -2$. Then our system is

$$\max z = 3x_1 + x_2$$
$$2x_1 + 3x_2 \leq 6$$
$$4x_1 + 2x_2 \leq 8$$
$$3x_1 - 4x_2 \leq -2$$
$$x_1, x_2 \geq 0$$

We introduce slack variables s_1, s_2, s_3 so that. . .

$$2x_1 + 3x_2 + s_1 \qquad\qquad = 6$$
$$4x_1 + 2x_2 \qquad + s_2 \qquad = 8$$
$$3x_1 - 4x_2 \qquad\qquad + s_3 = -2$$

Moving everything to the left side in $z = 3x_1 + x_2$, we have $z - 3x_1 - x_2 = 0$. Then the associated tableau is. . .

x_1	x_2	s_1	s_2	s_3		
2	3	1	0	0	6	s_1
4	2	0	1	0	8	s_2
3	-4	0	0	1	-2	s_3
-3	-1	0	0	0	0	

Problem 2. (10pt) Assume the following is a tableau associated to a standard maximization problem. Write down the function being maximization and the system of constraints.

1	2	1	1	0	0	100
2	8	2	0	1	0	150
1	1	1	0	0	1	200
-5	-4	-5	0	0	0	0

Solution. It is clear that we had three inequalities. Therefore, we had three slack variables, s_1, s_2, s_3. Because there are seven columns, one of which must correspond to the b's, there must be three variables, x_1, x_2, x_3. Looking at the last row and assuming we are trying to maximize a variable z, we must have $z - 5x_1 - 4x_2 - 5_x3 = 0$. But then $z = 5x_1 + 4x_2 + 5x_3$. Therefore, the standard maximization problem was...

$$\max z = 5x_1 + 4x_2 + 5x_3$$
$$x_1 + 2x_2 + x_3 \le 100$$
$$2x_1 + 8x_2 + 2x_3 \le 150$$
$$x_1 + x_2 + x_3 \le 200$$
$$x_1, x_2, x_3 \ge 0$$

Problem 3. (10pt) Solve the following linear programming problem:

$$\max z = x_1 + 6x_2 + 3x_3$$
$$x_1 + x_2 + 2x_3 \leq 4$$
$$x_1 + 2x_2 + x_3 \leq 4$$
$$x_1, x_2, x_3 \geq 0$$

Solution. This linear programming problem is already in standard form for a maximization. We introduce slack variables s_1 and s_2 so that...

$$x_1 + x_2 + 2x_3 + s_1 \quad\;\; = 4$$
$$x_1 + 2x_2 + x_3 \quad\;\; + s_2 = 4$$

Moving everything to the left side in $z = x_1 + 6x_2 + 3x_3$, we have $z - x_1 - 6x_2 - 3x_3 = 0$. Then the associated tableau is...

x_1	x_2	x_3	s_1	s_2		
1	1	2	1	0	4	s_1
1	2	1	0	1	4	s_2
-1	-6	-3	0	0	0	

We find our first pivot position:

x_1	x_2	x_3	s_1	s_2			
1	1	2	1	0	4	s_1	$4/1 = 4$
1	2	1	0	1	4	s_2	$4/2 = 2$
-1	-6	-3	0	0	0		

So x_x is the entering variable and s_2 is the exiting variable. We make this pivot entry 1 by $\frac{1}{2}R_2 \to R_2$.

x_1	x_2	x_3	s_1	s_2		
1	1	2	1	0	4	s_1
$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	2	x_2
-1	-6	-3	0	0	0	

Now we perform the first step of the simplex method: $-R_2 + R_1 \to R_1$ and $6R_2 + R_3 \to R_3$.

x_1	x_2	x_3	s_1	s_2		
$\frac{1}{2}$	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	2	s_1
$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	2	x_2
2	0	0	0	3	12	

Because there are no negatives in the bottom row, the simplex method is complete. We see that the maximum value for z is 12 (the bottom-rightmost entry) and occurs at $(x_1, x_2, x_3, s_1, s_2) = (0, 2, 0, 2, 0)$, i.e. $x_1 = 0, x_2 = 2, x_3 = 0, s_1 = 2, s_2 = 0$.

Problem 4. (10pt) Find the dual problem to...

$$\min z = 5x_1 + 4x_2$$
$$x_1 + 7x_2 \geq 7$$
$$x_1 + 3x_2 \geq 9$$
$$x_1, x_2 \geq 0$$

Solution. First, observe that this linear programming problem is a minimization and is already in standard form. Therefore, the associated matrix is...

$$\begin{pmatrix} 1 & 7 & 7 \\ 1 & 3 & 9 \\ 5 & 4 & 0 \end{pmatrix}$$

Taking the transpose, we find...

$$\begin{pmatrix} 1 & 1 & 5 \\ 7 & 3 & 4 \\ 7 & 9 & 0 \end{pmatrix}$$

Therefore, the dual problem is...

$$\max w = 7y_1 + 9y_2$$
$$y_1 + y_2 \leq 5$$
$$7y_1 + 3y_2 \leq 4$$
$$y_1, y_2 \geq 0$$

Problem 5. (10pt) Solve the following linear programming problem:

$$\min z = 2x_1 + 3x_2$$
$$2x_1 + x_2 \geq 1$$
$$x_1 + 3x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

Solution. This is linear programming problem is a minimization and it is already in standard form. Therefore, the associated matrix is...

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 2 & 3 & 0 \end{pmatrix}$$

The transpose is then...

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \\ 1 & 1 & 0 \end{pmatrix}$$

Therefore, the associated maximization problem (the dual problem) is...

$$\max w = y_1 + y_2$$
$$2y_1 + y_2 \leq 2$$
$$y_1 + 3y_2 \leq 3$$
$$y_1, y_2 \geq 0$$

Because $w = y_1 + y_2$, we know that $w - y_1 - y_2 = 0$. We introduce slack variables s_1, s_2 so that...

$$2y_1 + y_2 + s_1 \quad = 2$$
$$y_1 + 3y_2 \quad + s_2 = 3$$

Therefore, the associated tableau is...

y_1	y_2	s_1	s_2		
2	1	1	0	2	s_1
1	3	0	1	3	s_2
−1	−1	0	0	0	

We find our first pivot position:

y_1	y_2	s_1	s_2			
2	1	1	0	2	s_1	$2/2 = 1$
1	3	0	1	3	s_2	$3/1 = 3$
−1	−1	0	0	0		

So y_1 is the entering variable and s_1 is the exiting variable. We make this pivot entry 1 by $\frac{1}{2}R_1 \to R_1$.

$$
\begin{array}{cccc|c c}
y_1 & y_2 & s_1 & s_2 & & \\
1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & y_1 \\
1 & 3 & 0 & 1 & 3 & s_2 \\
\hline
-1 & -1 & 0 & 0 & 0 &
\end{array}
$$

Now we perform the first step of the simplex method: $-R_1 + R_2 \to R_2$ and $R_1 + R_3 \to R_3$.

$$
\begin{array}{cccc|c c}
y_1 & y_2 & s_1 & s_2 & & \\
1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & y_1 \\
0 & \frac{5}{2} & -\frac{1}{2} & 1 & 2 & s_2 \\
\hline
0 & -\frac{1}{2} & \frac{1}{2} & 0 & 1 &
\end{array}
$$

There are still negatives in the last row. So we proceed with the next step of the simplex method. We find our pivot position:

$$
\begin{array}{cccc|c c l}
y_1 & y_2 & s_1 & s_2 & & & \\
1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & y_1 & 1/(1/2) = 2 \\
0 & \boxed{\frac{5}{2}} & -\frac{1}{2} & 1 & 2 & s_2 & 2/(5/2) = 4/5 \\
\hline
0 & -\frac{1}{2} & \frac{1}{2} & 0 & 1 & &
\end{array}
$$

So y_2 is the entering variable and s_2 is the exiting variable. We make the pivot position 1 by $\frac{2}{5}R_2 \to R_2$.

$$
\begin{array}{cccc|c c}
y_1 & y_2 & s_1 & s_2 & & \\
1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & y_1 \\
0 & 1 & -\frac{1}{5} & \frac{2}{5} & \frac{4}{5} & s_2 \\
\hline
0 & -\frac{1}{2} & \frac{1}{2} & 0 & 1 &
\end{array}
$$

Now we perform the next step of the simplex method: $-\frac{1}{2}R_2 + R_1 \to R_1$ and $\frac{1}{2}R_2 + R_3 \to R_3$:

$$
\begin{array}{cccc|c c}
y_1 & y_2 & s_1 & s_2 & & \\
1 & 0 & \frac{3}{5} & -\frac{1}{5} & \frac{3}{5} & y_1 \\
0 & 1 & -\frac{1}{5} & \frac{2}{5} & \frac{4}{5} & y_2 \\
\hline
0 & 0 & \frac{2}{5} & \frac{1}{5} & \frac{7}{5} &
\end{array}
$$

Because there are no remaining negative entries in the bottom row, the simplex method is complete. We have maximum value $w = \frac{7}{5} = 1.4$ (the bottom-rightmost entry) occurring at $(y_1, y_2, s_1, s_2) = (\frac{3}{5}, \frac{4}{5}, 0, 0) = (0.6, 0.8, 0, 0)$, i.e. $y_1 = \frac{3}{5} = 0.6, y_2 = \frac{4}{5} = 0.8, s_1 = 0, s_2 = 0$.

But this maximum value is the minimum value for our original minimization problem (the dual problem). Therefore, the minimum value is $z = \frac{7}{5} = 1.4$ occurring at $(x_1, x_2, s_1, s_2) = (\frac{2}{5}, \frac{1}{5}, 0, 0) = (0.4, 0.2, 0, 0)$, i.e. $x_1 = \frac{2}{5} = 0.4, x_2 = \frac{1}{5} = 0.2, s_1 = 0, s_2 = 0$