MATH 122: Final Exam **Name:** *Caleb M^c Whorter — Solutions*
Fall — 2024
12/10/2024
150 Minutes

Write your name on the appropriate line on the exam cover sheet. This exam contains 12 pages (including this cover page) and 10 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

’Twas the night before Christmas, up at the Pole,
Santa was frazzled—he’d lost all control!
The reindeer were restless, the sleight off its track,
And the toy distribution? A logistical smack!

Santa scratched his beard, eyes weary and red,
“The math here is tricky. I’m in over my head!
The children are waiting, and I need to take flight.
I need a Calculus wizard to save Christmas tonight!”

1. (10 points) *Santa's workshop is busy, the costs start to climb,*

   "Help me with this function—I'm running out of time!"

   The cost to produce toys for a small town for the North Pole toy manufacturing production line is given by $C(q) = 0.01q^2 + 1.7q + 15760$, where q is the number of toys produced and C is measured in dollars. Showing all your work...

   (a) Find the cost to produce 3,700 toys.

   $$C(3700) = 0.01(3700)^2 + 1.7(3700) + 15760 = 136900 + 6290 + 15760 = \$158{,}950$$

   (b) Find the fixed costs to produce the toys for this small town.

   $$C(0) = 0.01(0)^2 + 1.7(0) + 15760 = \$15{,}760$$

   (c) Find and interpret the marginal cost to produce 3,700 toys.

   $$C'(q) = 0.02q + 1.7$$

   $$C'(3700) = 75.7$$

   Therefore, the cost to produce the 3,701st toy is approximately \$75.70.

2. (10 points) *Santa's gifts bring joy, but his budget's not lit,*

"Help me crunch the numbers—I need to make a profit!"

To help produce enough toys during the year, Santa has Bernard the Elf sell snow globes. Each snow globe is sold for \$119.99 and costs approximately \$8.34 to produce. However, the costs to maintain the North Pole snow globe manufacturing plant are huge—approximately \$250 million. Showing all your work, find the minimal number of snow globes Santa needs to sell to turn a profit.

Solution. *Each globe sells for \$119.99, which makes the revenue function* $R(q) = 119.99q$*. The variable costs are* $8.34q$ *because each snow globe costs \$8.34 to produce. The fixed costs are \$250 million. Therefore, the cost function is* $C(q) = 8.34q + 250000000$*. The breakeven point occurs when revenue equals cost. We have...*

$$R(q) = C(q)$$

$$119.99q = 8.34q + 250000000$$

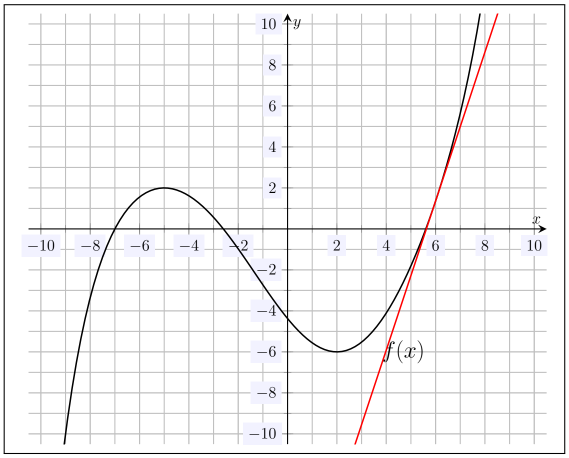
$$111.65q = 250000000$$

$$q \approx 2{,}239{,}140.17$$

Therefore, Bernard would have to sell at least 2,239141 snow globes to turn a profit for Santa.

3. (10 points) *Santa's sleigh is in peril, the night's looking rough,*
"Help me read this graph—Christmas math is tough!"

Consider the **plot of a function** $f(x)$ shown below.



Based on the plot above, answer the following:

(a) On what interval(s)—if any—is $f'(x) > 0$?

$$(-\infty, -5) \cup (2, \infty)$$

(b) On what interval(s)—if any—is $f'(x) < 0$?

$$(-5, 2)$$

(c) Find the critical values—if any—for $f(x)$.

$$x = -5,\ 2$$

(d) Find the points of inflection—if any—for $f(x)$.

$$x = -1.3$$

(e) On what interval(s)—if any—is $f''(x) > 0$?

$$(-1.3, \infty)$$

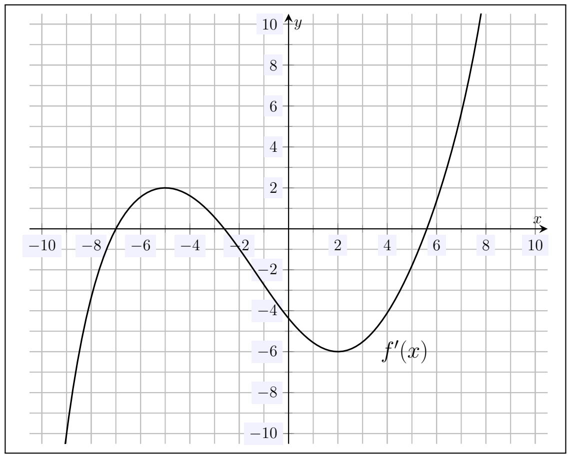
(f) On what interval(s)—if any—is $f''(x) < 0$?

$$(-\infty, -1.3)$$

(g) Sketch the tangent line to $f(x)$ at $x = 6$ on the graph above.

4. (10 points) *Santa's old graph was flawed, it led him astray,*
"Help me solve this new one, or Christmas won't stay!"

Consider the **plot of a derivative** $f'(x)$ shown below. **This is not the graph of the function,** $f(x)$**.**



Based on the plot above, answer the following questions:

(a) On what interval(s)—if any—is $f(x)$ increasing?

$$(-7, -2.6) \cup (5.6, \infty)$$

(b) On what interval(s)—if any—is $f(x)$ decreasing?

$$(-\infty, -7) \cup (-2.6, 5.6)$$

(c) On what interval(s)—if any—is $f(x)$ concave up?

$$(-\infty, -5) \cup (2, \infty)$$

(d) On what interval(s)—if any—is $f(x)$ concave down?

$$(-5, 2)$$

(e) Find any critical values for $f(x)$—if any.

$$x = -7, \; -2.6, \; 5.6$$

(f) Classify any critical values you found in (e).

$$\text{Local Min}: x = -7, 5.6 \qquad \text{Local Max}: x = -2.6$$

(g) Find any points of inflection for $f(x)$.

$$x = -5, \; 2$$

5. (10 points) *Santa's sleigh's gone off course, the math isn't quite right,*
"Quick, help me compute these derivatives this Christmas night!"

Showing all your work, compute the following derivatives:

(a) $\dfrac{d}{dx}\left(\dfrac{\sqrt[3]{\pi^2}\ln(15)}{19^{-3}}\right) = 0$

(b) $\dfrac{d}{dx}\left(4x^3 - 5^x + \sqrt[3]{x}\right) = \dfrac{d}{dx}\left(4x^3 - 5^x + x^{1/3}\right) = 12x^2 - 5^x \ln 5 + \dfrac{1}{3}\,x^{-2/3}$

(c) $\dfrac{d}{dx}\left((x^2-7)e^x\right) = 2xe^x + (x^2-7)e^x$

(d) $\dfrac{d}{dx}\ln(5x^2-9) = \dfrac{1}{5x^2-9}\cdot 10x$

(e) $\dfrac{d}{dx}\left(\dfrac{3x^4}{x^2-x}\right) = \dfrac{12x^3(x^2-x)-(2x-1)3x^4}{(x^2-x)^2}$

6. (10 points) *Santa's sleigh is tilting, the path's tough to find,*
"Help me with the tangent—Christmas is on the line!"

Consider the function $f(x) = x^2 - 6x + 4$. Showing all your work...

(a) Find the tangent line to $f(x)$ at $x = -2$.

We find the derivative to find the slope.

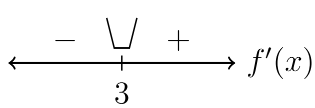
$$f'(x) = 2x - 6$$
$$f'(-2) = 2(-2) - 6 = -4 - 6 = -10$$

Using the fact that $f(-2) = (-2)^2 - 6(-2) + 4 = 4 + 12 + 4 = 20$*, we have the point* $(-2, 20)$*. Therefore, the tangent line is...*

$$y = y_0 + m(x - x_0) = 20 + (-10)\big(x - (-2)\big) = 20 - 10(x + 2)$$

(b) Find the absolute maximum and absolute minimum of $f(x)$ on $[0, 8]$.

We have $f'(x) = 2x - 6$*. We need to find the critical values, e.g. the values where* $f'(x) = 0$*. But then* $2x - 6 = 0$*. This implies that* $2x = 6$*, so that* $x = 3$*. We can check whether this is a maximum or a minimum using the first derivative test:*



Therefore, we can see that $x = 3$ *is a local minimum. We compare the value of* $f(x)$ *at* $x = 3$ *to its value at the endpoints:*

$$f(3) = 3^2 - 6(3) + 4 = 9 - 18 + 4 = -5$$
$$f(0) = 0^2 - 6(0) + 4 = 0 - 0 + 4 = 4$$
$$f(8) = 8^2 - 6(8) + 4 = 64 - 48 + 4 = 20$$

Therefore, the absolute minimum is -5 *and occurs at* $x = 3$ *and the absolute maximum is* 20 *and occurs at* $x = 8$.

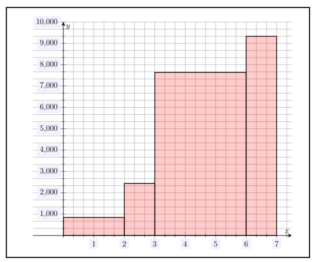
7. (10 points) *Santa's sleigh is on point, but his math's in a tizz,*
"Quick, estimate this integral—I can't lose my rizz!"

The speed of Santa's sleigh in mph, $v(t)$, at a time t (in hours) as he races across the globe at several different times since the start of Christmas are given in the table below.

Time, t	0	2	3	6	7
Velocity, $v(t)$	844	2,440	7,640	9,330	6,210

Showing all your work and being as accurate as possible, use a left-hand sum to estimate and interpret $\int_0^7 v(t)\,dt$.

Solution. *We can create a rough sketch of the left-hand sum:*



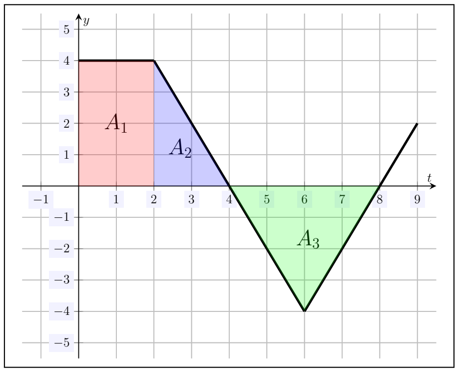
Computing the total area of these rectangles, we have...

$$\int_0^7 v(t)\,dt \approx 2(844)+1(2440)+3(7640)+1(9330) = 1{,}688+2{,}440+22{,}920+9{,}330 = 36{,}378$$

We know that $\int_0^7 v(t)\,dt$ *represents the net change in the distance traveled over the next seven hours. Therefore, Santa has traveled a net distance of 36,378 mi over the past seven hours.*

8. (10 points) *Santa's graph is confusing, the numbers don't show,*
"Help me compute this integral, so I know where to go!"

A function $f(x)$ is plotted below.



Showing all your work, based on the graph above, compute the following:

(a) $\displaystyle\int_0^4 f(x)\,dx = A_1 + A_2 = 2(4) + \frac{1}{2}(2)4 = 8 + 4 = 12$

(b) $\displaystyle\int_4^8 f(x)\,dx = -A_3 = -\frac{1}{2}(4)4 = -8$

(c) $\displaystyle\int_0^8 f(x)\,dx = A_1 + A_2 - A_3 = 8 + 4 - 8 = 4$

(d) $\displaystyle\int_5^5 f(x)\,dx = 0$

(e) The area between $f(x)$ and the x-axis.

$$\text{Area} = A_1 + A_2 + A_3 = 8 + 4 + 4 = 16$$

9. (10 points) *Addressing Christmas challenges, Santa has another request.*
"Help me solve this problem—then we'll pass this tough test!"

Showing all your work, compute the following:

(a) $\int (x^5 - e^x + 7)\, dx$

$$\int (x^5 - e^x + 7)\, dx = \frac{1}{6}\, x^6 - e^x + 7x + C$$

(b) $\int \frac{x^6 - x^2 + 9}{x^2}\, dx$

$$\int \frac{x^6 - x^2 + 9}{x^2}\, dx = \int \left(\frac{x^6}{x^2} - \frac{x^2}{x^2} + \frac{9}{x^2}\right)\, dx = \int \left(x^4 - 1 + 9x^{-2}\right)\, dx = \frac{1}{5}\, x^5 - x - 9x^{-1} + C$$

(c) $\int_{-1}^{3} (4 - 3x^2)\, dx$

$$\int_{-1}^{3} (4 - 3x^2)\, dx = 4x - x^3 \Big|_{-1}^{3} = \left(4(3) - 3^3\right) - \left(4(-1) - (-1)^3\right) = -15 - (-3) = -12$$

10. (10 points) *Christmas almost over, you're nearly clear.*

"Help me find these integrals, then we're done for this year."

Showing all your work, use u-substitution to find the following:

(a) $\displaystyle\int \frac{x}{x^2+1}\,dx$

Making a u-substitution, we have...

$$u = x^2+1 \qquad du = 2x\,dx \qquad dx = \frac{1}{2x}\,du$$

$$\int \frac{x}{x^2+1}\,dx$$

$$\int \frac{x}{u}\cdot\frac{1}{2x}\,du$$

$$\int \frac{1}{2u}\,du$$

$$\frac{1}{2}\int \frac{1}{u}\,du$$

$$\frac{1}{2}\ln|u|+C$$

$$\frac{1}{2}\ln|x^2+1|+C$$

(b) $\displaystyle\int_0^1 (1-2x)^{10}\,dx$

Making the u-substitution and changing bounds, we have...

$$\begin{aligned} u &= 1-2x & \qquad x=1 &: u = 1-2(1) = 1-2 = -1 \\ du &= -2\,dx & x=0 &: u = 1-2(0) = 1-0 = 1 \\ dx &= -\frac{1}{2}\,du \end{aligned}$$

$$\int_0^1 (1-2x)^{10}\,dx = \int_1^{-1} u^{10}\cdot -\frac{1}{2}\,du = -\frac{1}{2}\int_1^{-1} u^{10}\,du = -\frac{1}{2}\cdot\left.\frac{u^{11}}{11}\right|_1^{-1}$$

But then we have...

$$-\frac{1}{2}\cdot\left.\frac{u^{11}}{11}\right|_1^{-1} = -\frac{1}{22}u^{11}\Big|_1^{-1} = -\frac{1}{22}\cdot\left((-1)^{11}-1^{11}\right) = -\frac{1}{22}\cdot(-1-1) = -\frac{1}{22}\cdot -2 = \frac{1}{11} \approx 0.0909$$