Lecture π-Day:

Polar Form of Complex Numbers

Things to Review:

- Complex Arithmetic:
  - The Organic Chemistry Tutor
  - 3Blue1Brown
  - The A+ Tutor
  - Khan Academy

- Unit Circle & Trig. Values:
  - The Organic Chemistry Tutor
  - Maths Genie
  - Khan Academy
  - Professor Dave Explains

You Should Be Able To...

- Define the absolute value/modulus of a complex number
- Compute the absolute value of a complex number.
- Define and find the argument of a complex number.
- Define the rectangular and polar form of a complex number.
- Find the polar form of a complex number.
- Find the rectangular form of a complex number.
- State DeMoivre's Theorem.
- Use DeMoivre's Theorem to find powers and roots of complex numbers.

Recalling Complex Arithmetic

You have seen how to perform arithmetic with complex numbers, i.e. numbers of the form $a + bi$:

Example.

- *Addition:* $(3 - 4i) + (2 + i) = (3 + 2) + (-4 + 1)i = 5 - 3i$
- *Subtraction:* $(5 - i) - (1 - 4i) = (5 - 1) + \big(-i - (-4)\big)\,i = 4 + 3i$
- *Multiplication:* $(2 - i)(1 + 4i) = 2 + 8i - i - 4i^2 = 2 + 7i - 4(-1) = 6 + 7i$
- *Division:*

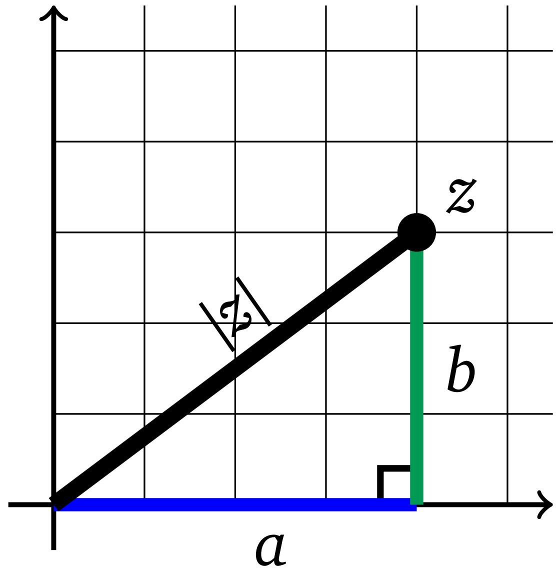
$$\frac{20 + 10i}{3 + 4i} = \frac{20 + 10i}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \frac{(20 + 10i)(3 - 4i)}{3^2 + 4^2} = \frac{100 - 50i}{25} = 4 - 2i$$

- *Powers:* $(2 + i)^3 = (2 + i)(2 + i)(2 + i) = (3 + 4i)(2 + i) = 2 + 11i$

ABSOLUTE VALUE/MODULUS

The *absolute value/modulus* of a complex number $z = a + bi$ is...

$$|z| := \sqrt{a^2 + b^2}$$



Imagine plotting the complex number $z = a + bi$.

The absolute value of z is simply the length of the line segment connecting z to the origin.

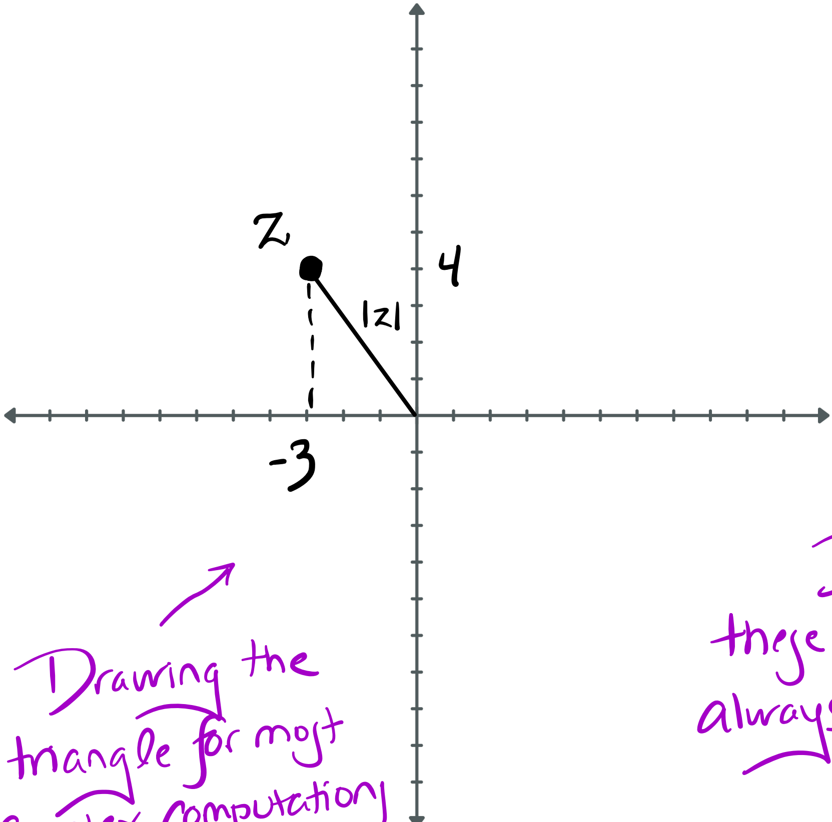
Example.

If $z = 1 - 4i$, then...

$$|z| = \sqrt{1^2 + (-4)^2} = \sqrt{1 + 16} = \sqrt{17} \approx 4.12311$$

Try It!

Let $z = -3 + 4i$. Plot the complex number z and compute $|z|$.



Drawing the triangle for most complex computations can really help.

$$|z|$$
$$|-3+4i|$$
$$\sqrt{(-3)^2+4^2}$$
$$\sqrt{9+16}$$
$$\sqrt{25}$$
$$5$$

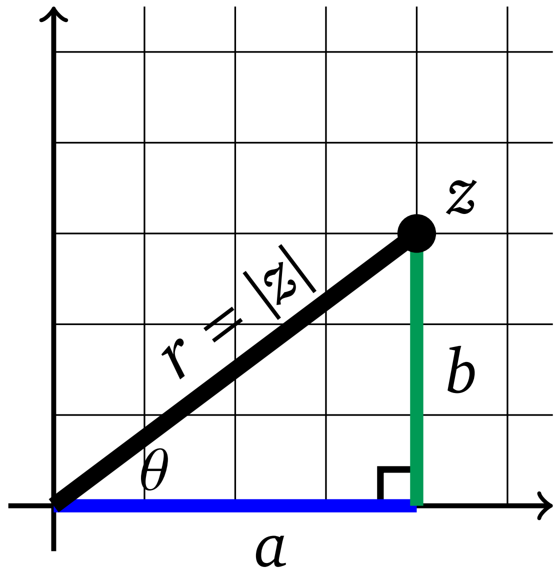
Be sure these numbers always work out ≥ 0.

This is the length of the "point" z to the origin

$$\boxed{|z| = 5}$$

Deriving the Polar Form

Imagine plotting the complex number $z = a + bi$.



We know that $z = a + bi$. But using right-triangle trig:

$$\cos\theta = \frac{a}{r} \implies a = r\cos\theta$$

$$\sin\theta = \frac{b}{r} \implies b = r\sin\theta$$

But then $z = x + yi = r\cos\theta + i(r\sin\theta)\,i = r\,(\cos\theta + i\sin\theta)$. This is the *polar representation* of z and the angle θ is called the *argument* of z.

Definition. (Polar Form)

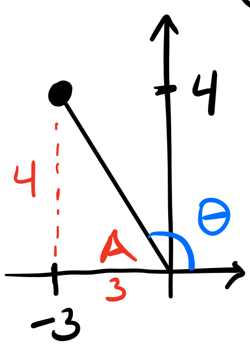
Given the rectangular form $z = a + bi$, the polar form of z is...

$$z = r\,(\cos\theta + i\sin\theta)$$

Example

Find the polar representation of $z = -3 + 4i$.

∗ We saw that $|z| = 5$ and plotted z.



$\tan A = \frac{4}{3}$

$A = \tan^{-1}(4/3)$

$A \approx 53.1^\circ$

$\Downarrow$

$|z| = 5$

$\rightsquigarrow$

$z = r(\cos\theta + i\sin\theta)$

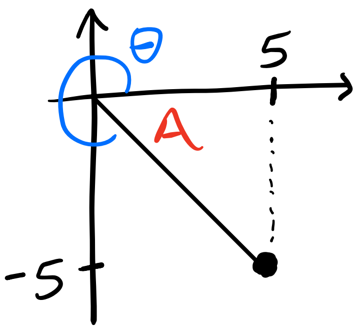
$\boxed{z = 5(\cos 53.1^\circ + i \sin 53.1^\circ)}$

∗ If you compute $\cos 53.1^\circ$ and $\sin 53.1^\circ$, and then expand the above expression, you should get $-3+4i$; that is the point! We haven't changed z – just reexpressed it!

Try It!

Find the polar representation of $5 - 5i$.

1. Plot the point & draw Δ.
2. Compute $|z|$ & θ
3. Write the expression.

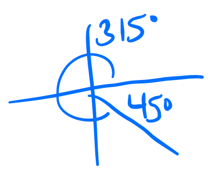


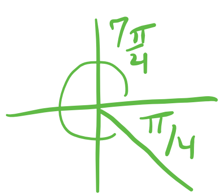
$$|5-5i| = \sqrt{5^2+(-5)^2}$$
$$= \sqrt{25+25}$$
$$= \sqrt{50}$$

$\sqrt{25\cdot 2}$

$5\sqrt{2}$

$$\tan A = \frac{5}{5} = 1$$
$$A = \arctan(1)$$
$$A = \underbrace{\frac{\pi}{4}}_{45^\circ}$$





$$z = r(\cos\theta + i\sin\theta)$$

$$\boxed{5-5i = \sqrt{50}\left(\cos \underbrace{\frac{7\pi}{4}}_{315^\circ} + i \sin \underbrace{\frac{7\pi}{4}}_{315^\circ}\right)}$$

OR

$$\boxed{5-5i = \sqrt{50}\left(\cos\underbrace{\left(\frac{-\pi}{4}\right)}_{-45^\circ} + i \sin\underbrace{\left(\frac{-\pi}{4}\right)}_{-45^\circ}\right)}$$

Example

Find the rectangular representation of $z = 10\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$.

$$z = 10\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$$

$$z = 10\left(-\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right)$$

$$\boxed{z = -5\sqrt{3} + 5i}$$

$$\overset{5}{\cancel{10}} \cdot \frac{-\sqrt{3}}{\cancel{2}_1} \qquad \overset{5}{\cancel{10}} \cdot i\,\frac{1}{\cancel{2}_1}$$

Try It!

* Evaluate & expand

Find the rectangular representation of $3\,(\cos(330^\circ) + i\sin(330^\circ))$.

$$3\left(\cos(330^\circ) + i\sin(330^\circ)\right)$$

$$3\left(\frac{\sqrt{3}}{2} + i \cdot \frac{-1}{2}\right)$$

$$\boxed{\frac{3\sqrt{3}}{2} - \frac{3}{2}i}$$

De Moivre’s Theorem

De Moivre's Theorem

De Moivre's Theorem allows one to compute powers of complex numbers easily.

Theorem. (De Moivre's Theorem)

If $z = a + bi$, has polar form $z = r(\cos\theta + i\sin\theta)$ then...

$$z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$

Computing Complex Powers: To compute a power of a complex number z, say z^n, you need to...

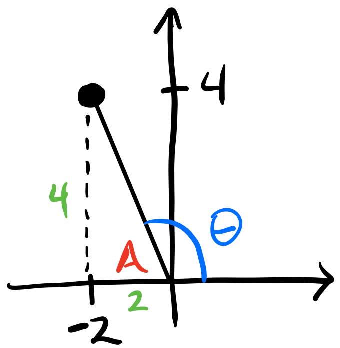
1. Find the polar form of z.
2. Use De Moivre's Theorem to write

   $$z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$

   That is, compute the nth power of its absolute value and multiply the angle by n.
3. Simplify this expression (if possible).

Example

Compute $(\underbrace{-2+4i}_{z})^4$.



$$|z| = \sqrt{(-2)^2+4^2}$$
$$= \sqrt{4+16}$$
$$= \sqrt{20} \leftarrow \sqrt{4\cdot 5} \quad 2\sqrt{5}$$

$$\tan A = \frac{4}{2} = 2$$
$$A = \tan^{-1}(2)$$
$$A \approx 63.43^\circ$$
$$\theta = 180^\circ - 63.43^\circ = 116.57^\circ$$

↑ Be sure to use θ, not A ♡

$$z^4 = r^4\left(\cos(4\theta) + i\sin(4\theta)\right)$$
$$z^4 = (\sqrt{20})^4\left(\cos(4\cdot 116.57^\circ) + i\sin(4\cdot 116.57^\circ)\right)$$
$$z^4 = 400\left(\cos(466.28^\circ) + i\sin(466.28^\circ)\right)$$
$$z^4 = 400(-0.2803 + i\cdot 0.9599)$$
$$\boxed{z^4 = -112.133 + 383.961i}$$

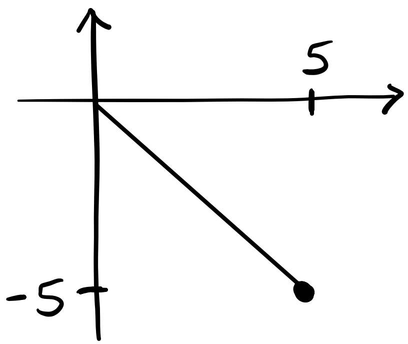
* Exact answer: $-112+384i$

* Saved us having to FOIL $(-2+4i)(-2+4i)(-2+4i)(-2+4i)$

* Do you see how you could use A by using reference angles? ♡

TRY IT!

If $z = 5 - 5i$, compute z^3.



$*$ We already saw $|z| = \sqrt{50}$

$*$ We already saw $\theta = 315°$ (or $-45°$)

We use $\theta = -45°$ just to make numbers easier.

$$z^3 = r^3\left(\cos(3\theta) + i\sin(3\theta)\right)$$

$$z^3 = (\sqrt{50})^3\left(\cos(3\cdot -45°) + i\sin(3\cdot -45°)\right)$$

$\underbrace{\sqrt{50}\sqrt{50}}_{50}\sqrt{50}$

$$z^3 = 50\sqrt{50}\left(\cos(\underbrace{-135°}_{225°}) + i\sin(\underbrace{-135°}_{225°})\right)$$

$$z^3 = 50\sqrt{50}\left(-\frac{\sqrt{2}}{2} + i\cdot -\frac{\sqrt{2}}{2}\right)$$

$\sqrt{50}\sqrt{2} = \sqrt{100}$

$$z^3 = -25\sqrt{100} - 25\sqrt{100}\,i$$

$$\boxed{z^3 = -250 - 250i}$$

Roots of Complex Numbers

De Moivre's Theorem

De Moivre's Theorem also allows one to compute roots of complex numbers 'easily.'

But just like $\sqrt{4} = \pm 2$, i.e. 4 has two possible square roots, there will be n possible nth roots for a complex number.

Theorem. (De Moivre's Theorem)

If $z = a + bi$, has polar form $z = r(\cos\theta + i\sin\theta)$ then...

$$\underbrace{\sqrt[n]{z}}_{\text{Or } z^{1/n}} = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right) \quad \textit{(radians)}$$

$$\underbrace{\sqrt[n]{z}}_{\text{Or } z^{1/n}} = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i\sin\left(\frac{\theta + 360^\circ k}{n}\right)\right) \quad \textit{(degrees)}$$

where $k = 0, 1, 2, \ldots, n - 1$.

Note. This says the complex nth roots of z are equally spaced points on the circle at the origin with radius $\sqrt[n]{z}$—each $\frac{360^\circ}{n}$ degrees apart.

Computing Complex Roots

Theorem. (De Moivre's Theorem)

If $z = a + bi$, has polar form $z = r(\cos\theta + i\sin\theta)$ then…

$$\underbrace{\sqrt[n]{z}}_{Or\ z^{1/n}} = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right) \qquad \textit{(radians)}$$

$$\underbrace{\sqrt[n]{z}}_{Or\ z^{1/n}} = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i\sin\left(\frac{\theta + 360^\circ k}{n}\right)\right) \qquad \textit{(degrees)}$$

where $k = 0, 1, 2, \ldots, n-1$.

Computing Complex Roots: To compute the nth roots of a complex number z, say $\sqrt[n]{z}$, you need to…

1. Find the polar form of z.
2. Use De Moivre's Theorem to write

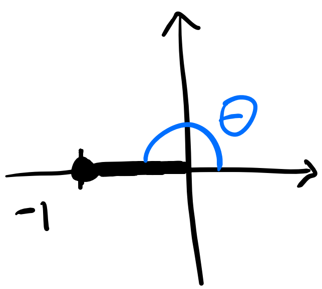
   $$\sqrt[n]{z} = \sqrt[n]{r}\,(\cos(n\theta) + i\sin(n\theta))$$

   That is, compute the nth root of its absolute value, divide the angle by n, and write out each possible root by plugging in $k = 0, 1, \ldots, n-1$.
3. Simplify these expressions (if possible).

EXAMPLE

Find all the complex cube roots of $-1 = \overbrace{1}^{r}(\cos \overbrace{180^\circ}^{\theta} + i \sin \overbrace{180^\circ}^{\theta})$.

$\ast$ We know $\sqrt[3]{-1} = -1$ is a real cube root of -1, but also....



We see that $r=1$ and $\theta = 180^\circ$.

$$\sqrt[3]{-1} = \sqrt[3]{1}\left(\cos\left(\frac{180^\circ + 360^\circ k}{3}\right) + i\sin\left(\frac{180^\circ + 360^\circ k}{3}\right)\right)$$

$$\sqrt[3]{-1} = 1\left(\cos(60^\circ + 120^\circ k) + i\sin(60^\circ + 120^\circ k)\right)$$

$\ast \sqrt[3]{\ }$ number of complex roots

$60^\circ + 120^\circ(0)$

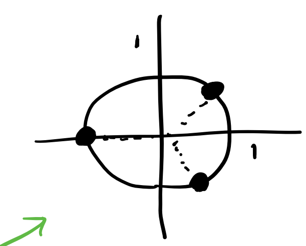
$$K=0:\ 1\left(\cos(60^\circ) + i\sin(60^\circ)\right) = \boxed{\frac{1}{2} + \frac{\sqrt{3}}{2} i}$$

$60^\circ + 120^\circ(1)$

$$K=1:\ 1\left(\cos(180^\circ) + i\sin(180^\circ)\right) = -1 + i\cdot 0 = \boxed{-1}$$

$60^\circ + 120^\circ(2)$

$$K=2:\ 1\left(\cos(300^\circ) + i\sin(300^\circ)\right) = \boxed{\frac{1}{2} - \frac{\sqrt{3}}{2} i}$$



Complex cube roots of -1

Try It!

Find all the complex fourth roots of $\overbrace{\underbrace{16}_{r}(\cos \underbrace{120^\circ}_{\theta} + i \sin \underbrace{120^\circ}_{\theta})}^{z_1}$.

$$\sqrt[4]{z} = \sqrt[4]{16}\left(\cos\left(\frac{120^\circ + 360k}{4}\right) + i \sin\left(\frac{120^\circ + 360k}{4}\right)\right)$$

$$\sqrt[4]{z} = 2\left(\cos(30^\circ + 90^\circ k) + i \sin(30^\circ + 90^\circ k)\right)$$

$$k=0: 2\left(\cos(30^\circ) + i \sin(30^\circ)\right) = 2\left(\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right) = \boxed{\sqrt{3} + i}$$

$$k=1: 2\left(\cos(120^\circ) + i \sin(120^\circ)\right) = 2\left(\frac{-1}{2} + i\,\frac{\sqrt{3}}{2}\right) = \boxed{-1 + \sqrt{3}\,i}$$

$$k=2: 2\left(\cos(210^\circ) + i \sin(210^\circ)\right) = 2\left(-\frac{\sqrt{3}}{2} + i \cdot \frac{-1}{2}\right) = \boxed{-\sqrt{3} - i}$$

$$k=3: 2\left(\cos(300^\circ) + i \sin(300^\circ)\right) = 2\left(\frac{1}{2} + i \cdot -\frac{\sqrt{3}}{2}\right) = \boxed{1 - \sqrt{3}\,i}$$

Extra Facts

Euler's Identity

All of this is related to Euler's Identity:

Theorem. (Euler's Identity)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This leads to one of the most famous identities of all time:

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i(0) = -1$$

But then adding 1 to both sides, we have...

$$e^{i\pi} + 1 = 0$$

This has nearly every 'important' constant all in one equation: $1, 0, \pi, e$, and i!

This allows allows one to write the polar form of a complex number z as...

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Computing with the Alternative Polar Form

Using the polar form $z = re^{i\theta}$, we can perform complex multiplication, division, and powers quite easily!

Example. Let $z = -\sqrt{3} + i$ and $w = 3 + 3\sqrt{3}i$.

One can find (*Try It!*) that z and w have polar forms...

$$z = 6\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) = 6e^{i\frac{\pi}{3}}$$

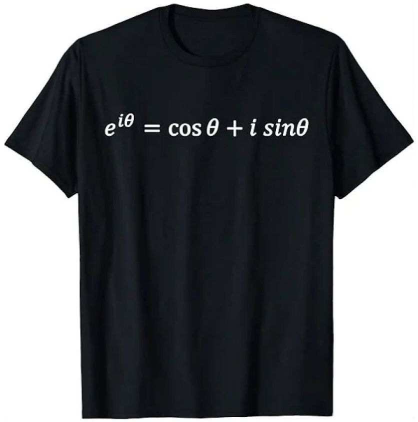
$$w = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) = 2e^{i\frac{5\pi}{6}}$$

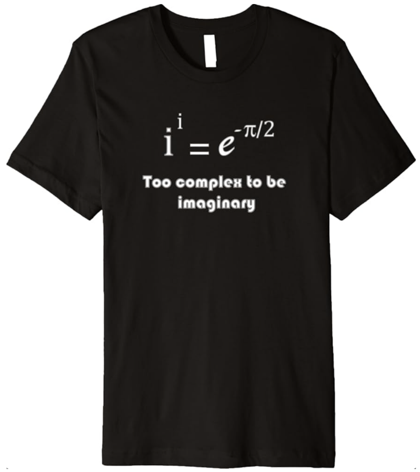
But then, for example, we have...

- $zw = 6e^{i\frac{\pi}{3}} \cdot 2e^{i\frac{5\pi}{6}} \cdot = 12\,e^{i\left(\frac{\pi}{3}+\frac{5\pi}{6}\right)} = 12e^{i\frac{7\pi}{6}}$
- $\dfrac{z}{w} = \dfrac{6e^{i\frac{\pi}{3}}}{2e^{i\frac{5\pi}{6}}} = 3e^{i\left(\frac{\pi}{3}-\frac{5\pi}{6}\right)} = 3\,e^{-i\frac{\pi}{2}}$
- $z^3 = (6e^{i\frac{\pi}{3}})^3 = 6^3 e^{i\frac{\pi}{3}\cdot 3} = 216\,e^{i\pi}$

Note. One can then convert these to their regular representation. For instance, $zw = 12e^{i\frac{7\pi}{6}} = 12(\cos(\frac{7\pi}{6}) + i\sin(\frac{7\pi}{6})) = 12(-\frac{\sqrt{3}}{2} - i\frac{1}{2}) = -6\sqrt{3} - 6i.$

Most importantly, now you can understand all them nerd shirts!





Be sure to check out the textbook to see how all this is related to the 'mysterious' Mandelbrot set!