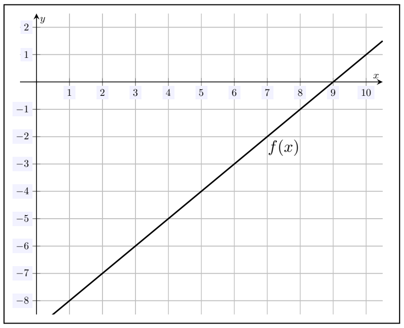
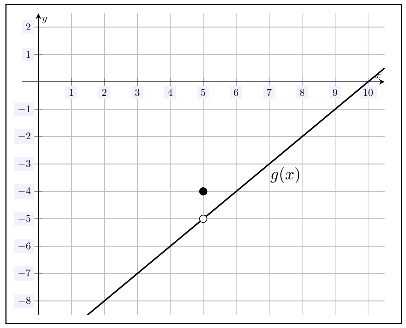
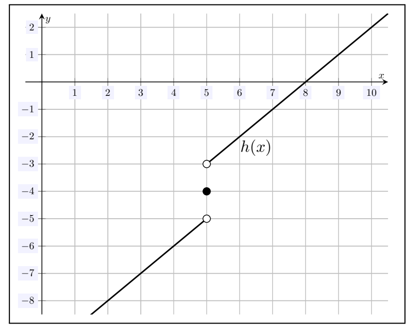
Check-In 08/22. *(True/False)* If $f(x)$ is a function with $f(5) = -4$, then it must be that $\lim_{x\to 5} f(x) = -4$.

Solution. The statement is *false*. Limits are about happens 'near' an input—not what happens at the input. A limit may or may not exist at a given x-value even when the function is defined for that x-value. Moreover, even if the limit exists, it may not be equal to the function value there!



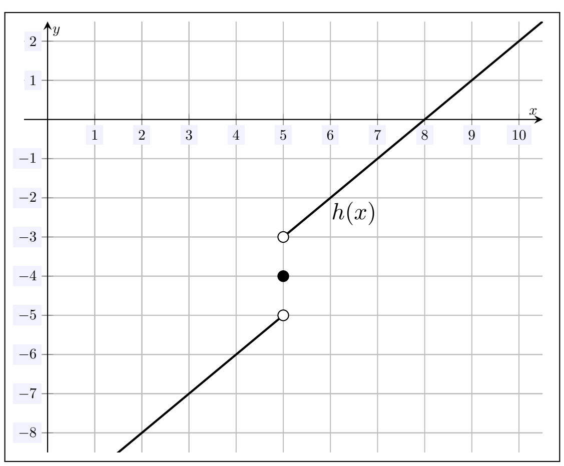




For instance, for the function $f(x)$ on the left, we have $f(5) = -4$ and $\lim_{x\to 5} f(x) = -4$. However, for the function $g(x)$ in the middle, we have $g(5) = -4$ but $\lim_{x\to 5} g(x) = -5$. But for $h(x)$ on the right, we have $h(5) = -4$ but $\lim_{x\to 5} h(x)$ does not exist because the left and right hand limits are not equal.

Check-In 08/26. *(True/False)* If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, then $\lim_{x\to a} f(x)$ exists.

Solution. The statement is *false*. We know that if $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a^-} f(x)$, exists, $\lim_{x\to a^+} f(x)$ exists, and $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$. This is because if $\lim_{x\to a} f(x)$ exists, then $f(x)$ is getting 'close' to a single number, say L, whenever x is 'close' to a—no matter if it is 'below' or 'above' $x = a$. However, just because $f(x)$ is getting 'close' to a particular output 'on the left' does not mean $f(x)$ is getting 'close' to the same output from the right. Take the example from the previous quiz!



For this function, we have $\lim_{x\to 5^-} h(x) = -5$, $\lim_{x\to 5^+} h(x) = -3$, but $\lim_{x\to 5^-} h(x) \neq \lim_{x\to 5^+} h(x)$. However, if the left and right hand limits exist *and* are equal, then $\lim_{x\to a} f(x)$ exists.

Check-In 08/26. *(True/False)* $\lim\limits_{\theta\to 0} \dfrac{\sin(3\theta)}{2\theta} = \dfrac{3}{2}$

Solution. The statement is *true*. Recall that $\lim\limits_{\square\to 0} \dfrac{\sin(\square)}{\square} = 1$. But then…

$$\lim_{\theta\to 0} \frac{\sin(3\theta)}{2\theta} = \frac{1}{2}\lim_{\theta\to 0} \frac{\sin(3\theta)}{\theta} = \frac{1}{2}\lim_{\theta\to 0} \frac{3\sin(3\theta)}{3\theta} = \frac{3}{2}\lim_{\theta\to 0} \underbrace{\frac{\sin(3\theta)}{3\theta}}_{\rightsquigarrow 1} = \frac{3}{2}\cdot 1 = \frac{3}{2}$$

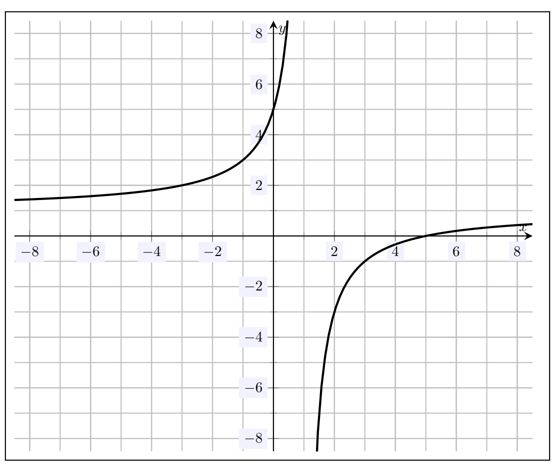
Check-In 08/28. *(True/False)* $\lim\limits_{x\to 5} \dfrac{x^2-3x-10}{x-5} = 7$

Solution. The statement is *true*. We have…

$$\lim_{x\to 5} \frac{x^2-3x-10}{x-5} = \lim_{x\to 5} \frac{\cancel{(x-5)}(x+2)}{\cancel{x-5}} = \lim_{x\to 5}(x+2) = 5+2 = 7$$

Check-In 08/29. *(True/False)* $\lim\limits_{x\to 1^-} \dfrac{x-5}{x-1} = -\infty$

Solution. The statement is *false*. 'Plugging in' $x = 1$, we obtain $\frac{-4}{0}$—so certainly this limit is either $-\infty$, $+\infty$, or DNE. Because we approach 1 from the left, we know that $x < 1$. But then $x - 1 < 0$. But then $\frac{1}{x-1}$ approaches $-\infty$ as x tends to 1 from the left. But the numerator is also negative because when x is 'close' to 1, $x - 5 < 0$. Therefore, the limit tends to ∞. We can see this from the plot of $\frac{x-5}{x-1}$.



The given answer failed to take the sign of the numerator into account.

Check-In 09/04. *(True/False)* $\lim_{x\to\infty} \dfrac{9x^2-5x+7}{8-3x^2} = 3$

Solution. The statement is *true*. We know that $\lim_{x\to\pm\infty} \dfrac{\text{polynomial}}{\text{polynomial}}$ is 0 if $\deg$ den. $>$ $\deg$ num., $\pm\infty$ (depending on the limit and sign of the leading coefficient in the numerator) if $\deg$ num. $>$ $\deg$ den., and is the ratio of the leading coefficients if $\deg$ den. $=$ $\deg$ num.. The degree of the numerator and denominator is 2. Therefore, we know that

$$\lim_{x\to\infty} \frac{9x^2-5x+7}{8-3x^2} = \frac{9}{-3} = -3$$

The given answer did not correctly identify the leading coefficient in the denominator. Alternatively, we can multiply by $\frac{1/x^{\deg\ \text{denom}}}{1/x^{\deg\ \text{denom}}}$:

$$\lim_{x\to\infty} \frac{9x^2-5x+7}{8-3x^2} = \lim_{x\to\infty} \frac{9x^2-5x+7}{8-3x^2} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{9-\frac{5}{x}+\frac{7}{x}}{\frac{8}{x^2}-3} = \frac{9-0+0}{0-3} = \frac{9}{-3} = -3$$

Check-In 09/05. *(True/False)* If $f(x)$ is defined to be the following function:

$$f(x) = \begin{cases} x^2+x-6, & x < -1 \\ x-5, & x \geq -1 \end{cases}$$

Then $f(x)$ is everywhere continuous.

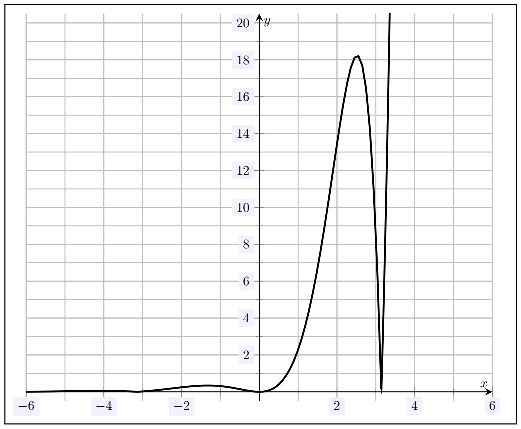
Solution. The statement is *true*. If $x < -1$, then $f(x) = x^2+x-6$. We know that x^2+x-6 is a polynomial, which are everywhere continuous. If $x > -1$, then $f(x) = x-5$, which is a polynomial. We know that polynomials are everywhere continuous. Therefore, we know $f(x)$ is continuous when $x < -1$ and when $x > -1$. We only need to check if $f(x)$ is continuous at $x = -1$. For $f(x)$ to be continuous at $x = -1$, we need to check that $f(-1) = \lim_{x\to -1} f(x)$:

- $f(-1) = -1-5 = -6$
- $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x^2+x-6) = (-1)^2 + (-1) - 6 = -6$
- $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (x-5) = -1-5 = -6$

Because $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$, we know that $\lim_{x\to -1} f(x) = -6$. Therefore, $f(-1) = \lim_{x\to -1} f(x)$. But then $f(x)$ is continuous at $x = -1$. Therefore, $f(x)$ is continuous for all x, i.e. $f(x)$ is everywhere continuous.

Check-In 09/09. *(True/False)* The function $f(x) = |xe^x \sin x|$ is continuous. Therefore, $\lim_{x \to \pi} f(x) = f\left(\lim_{x \to \pi}\right) = f(\pi) = 0$.

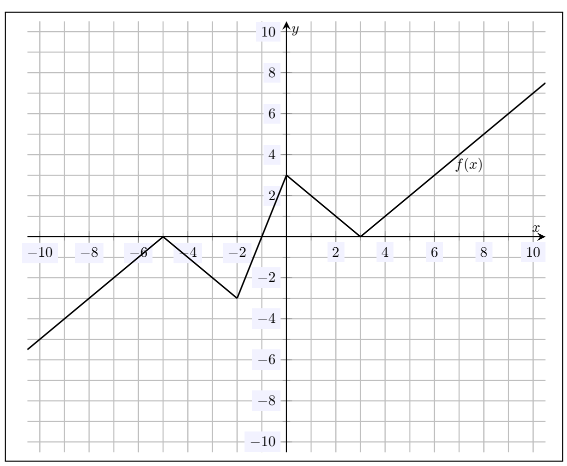
Solution. The statement is *true*. We know that x, e^x, and $\sin x$ are everywhere continuous. Therefore, their product—$g(x) := xe^x \sin x$—is continuous. We also know the function $h(x) = |x|$ is everywhere continuous. But then the composition $(h \circ g)(x)$ is continuous. But $(h \circ g)(x) = h\big(g(x)\big) = h(xe^x \sin x) = |xe^x \sin x|$. We can see the continuity from a plot of this function.



Finally, we know that if a function $f(x)$ is continuous at $x = a$, then $\lim_{x \to a} f(x) = f(a)$. But we know that the given $f(x)$ is continuous at $x = \pi$—it is everywhere continuous. But then...

$$f(\pi) = |\pi \cdot e^\pi \sin \pi| = |\pi \cdot e^\pi \cdot 0| = |0| = 0$$

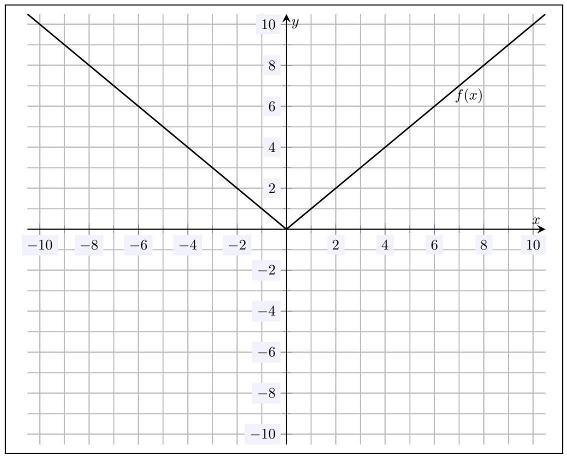
Check-In 09/10. *(True/False)* The function $f(x)$, plotted below, is *not* differentiable at $x = -2$ but *is differentiable* at $x = 6$.



Solution. The statement is *true*. At $x = -2$, we can see that $f(x)$ has a cusp. [The derivative somehow 'wants' to be -1 and 3 at the same time.] Therefore, $f(x)$ is not differentiable at $x = -2$. However, we can see that $f(x)$ is linear at $x = 6$. We know linear functions are differentiable—the derivative is the slope of the function. Therefore, $f(x)$ is differentiable at $x = 6$. In fact, the value of the derivative at $x = 6$ is the slope of the line through $\big(6, f(x)\big)$—which is $3x + 3$ so that $f'(6) = 3$.

Check-In 09/11. *(True/False) Every* differentiable function is continuous, but not every continuous function is differentiable.

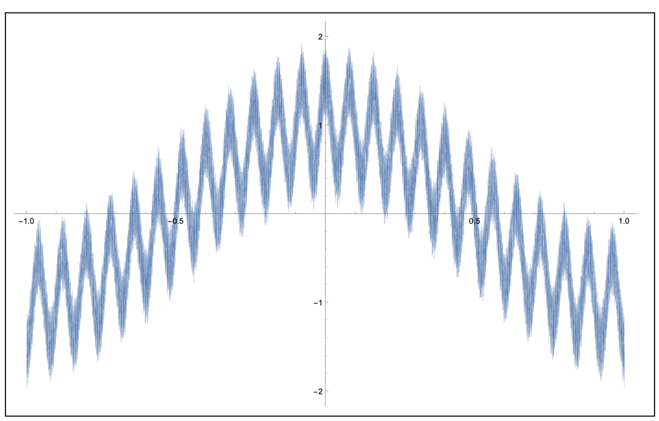
Solution. The statement is *true*. We know that every differentiable function is continuous. However, not every continuous function is necessarily differentiable. For instance, consider the function $f(x) = |x|$, shown below.



We see that $f(x)$ has a cusp at $x = 0$. Therefore, $f(x)$ is not differentiable at $x = 0$. We can check this directly:

$$f'(0) := \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \frac{|h| - 0}{h} = \begin{cases} \dfrac{h}{h} = 1, h > 0 \\ \\ \dfrac{-h}{h} = -1, \qquad h < 0 \end{cases}$$

This limit does not exist. Therefore, $f'(0)$ does not exist. There are other functions, e.g. the Weierstrass function shown below, that are *everywhere* continuous but *nowhere* differentiable.



Check-In 09/12. *(True/False)* $\dfrac{d}{dx}\left(e^{-x}\cos x\right) = -e^{-x}\cos x - e^{-x}\sin x$

Solution. The statement is *true*. We use the product rule and the chain rule. We have...

$$\begin{aligned}
\frac{d}{dx}\left(e^{-x}\cos x\right) &= \frac{d}{dx}(e^{-x})\cos x + e^{-x}\frac{d}{dx}(\cos x) \\
&= (-e^{-x})\cos x + e^{-x}(-\sin x) \\
&= -e^{-x}\cos x - e^{-x}\sin x \\
&= -e^{-x}(\sin x + \cos x)
\end{aligned}$$