Lecture π-Day:

Polar Form of Complex Numbers

Things to Review:

- Complex Arithmetic:
  - The Organic Chemistry Tutor
  - 3Blue1Brown
  - The A+ Tutor
  - Khan Academy

- Unit Circle & Trig. Values:
  - The Organic Chemistry Tutor
  - Maths Genie
  - Khan Academy
  - Professor Dave Explains

You Should Be Able To. . .

- Define the absolute value/modulus of a complex number
- Compute the absolute value of a complex number.
- Define and find the argument of a complex number.
- Define the rectangular and polar form of a complex number.
- Find the polar form of a complex number.
- Find the rectangular form of a complex number.
- State DeMoivre's Theorem.
- Use DeMoivre's Theorem to find powers and roots of complex numbers.

Recalling Complex Arithmetic

You have seen how to perform arithmetic with complex numbers, i.e. numbers of the form $a + bi$:

Example.

- *Addition:* $(3 - 4i) + (2 + i) = (3 + 2) + (-4 + 1)i = 5 - 3i$

- *Subtraction:* $(5 - i) - (1 - 4i) = (5 - 1) + \big(-i - (-4)\big)\, i = 4 + 3i$

- *Multiplication:* $(2 - i)(1 + 4i) = 2 + 8i - i - 4i^2 = 2 + 7i - 4(-1) = 6 + 7i$

- *Division:*

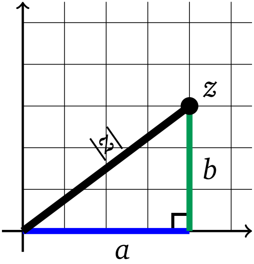
$$\frac{20 + 10i}{3 + 4i} = \frac{20 + 10i}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \frac{(20 + 10i)(3 - 4i)}{3^2 + 4^2} = \frac{100 - 50i}{25} = 4 - 2i$$

- *Powers:* $(2 + i)^3 = (2 + i)(2 + i)(2 + i) = (3 + 4i)(2 + i) = 2 + 11i$

Absolute Value/Modulus

The *absolute value/modulus* of a complex number $z = a + bi$ is. . .

$$|z| := \sqrt{a^2 + b^2}$$



Imagine plotting the complex number $z = a + bi$.

The absolute value of z is simply the length of the line segment connecting z to the origin.

Example.

If $z = 1 - 4i$, then. . .

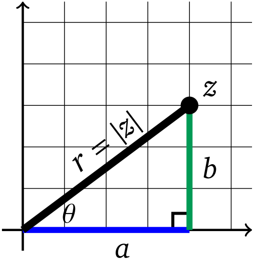
$$|z| = \sqrt{1^2 + (-4)^2} = \sqrt{1 + 16} = \sqrt{17} \approx 4.12311$$

Try It!

Let $z = -3 + 4i$. Plot the complex number z and compute $|z|$.

Deriving the Polar Form

Imagine plotting the complex number $z = a + bi$.



We know that $z = a + bi$. But using right-triangle trig:

$$\cos\theta = \frac{a}{r} \quad \Longrightarrow \quad a = r\cos\theta$$

$$\sin\theta = \frac{b}{r} \quad \Longrightarrow \quad b = r\sin\theta$$

But then $z = x + yi = r\cos\theta + i(r\sin\theta)\,i = r\,(\cos\theta + i\sin\theta)$. This is the *polar representation* of z and the angle θ is called the *argument* of z.

Definition. (Polar Form)

Given the rectangular form $z = a + bi$, the polar form of z is...

$$z = r\,(\cos\theta + i\sin\theta)$$

Example

Find the polar representation of $z = -3 + 4i$.

Try It!

Find the polar representation of $5 - 5i$.

Example

Find the rectangular representation of $z = 10\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$.

Try It!

Find the rectangular representation of $3\left(\cos(330^\circ) + i\sin(330^\circ)\right)$.

De Moivre’s Theorem

De Moivre's Theorem

De Moivre's Theorem allows one to compute powers of complex numbers easily.

Theorem. (De Moivre's Theorem)

If $z = a + bi$, has polar form $z = r(\cos\theta + i\sin\theta)$ then...

$$z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$

Computing Complex Powers: To compute a power of a complex number z, say z^n, you need to...

1. Find the polar form of z.
2. Use De Moivre's Theorem to write

   $$z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$

   That is, compute the nth power of its absolute value and multiply the angle by n.
3. Simplify this expression (if possible).

Example

Compute $(-2 + 4i)^4$.

Try It!

If $z = 5 - 5i$, compute z^3.

Roots of Complex Numbers

De Moivre's Theorem

De Moivre's Theorem also allows one to compute roots of complex numbers 'easily.'

But just like $\sqrt{4} = \pm 2$, i.e. 4 has two possible square roots, there will be n possible nth roots for a complex number.

Theorem. (De Moivre's Theorem)

If $z = a + bi$, has polar form $z = r(\cos\theta + i\sin\theta)$ then...

$$\underbrace{\sqrt[n]{z}}_{\text{Or } z^{1/n}} = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right) \qquad \text{(radians)}$$

$$\underbrace{\sqrt[n]{z}}_{\text{Or } z^{1/n}} = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i\sin\left(\frac{\theta + 360^\circ k}{n}\right)\right) \qquad \text{(degrees)}$$

where $k = 0, 1, 2, \ldots, n-1$.

Note. This says the complex nth roots of z are equally spaced points on the circle at the origin with radius $\sqrt[n]{z}$—each $\frac{360^\circ}{n}$ degrees apart.

Computing Complex Roots

Theorem. (De Moivre's Theorem)

If $z = a + bi$, has polar form $z = r(\cos\theta + i\sin\theta)$ then...

$$\underbrace{\sqrt[n]{z}}_{\text{Or } z^{1/n}} = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right) \qquad \textit{(radians)}$$

$$\underbrace{\sqrt[n]{z}}_{\text{Or } z^{1/n}} = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i\sin\left(\frac{\theta + 360^\circ k}{n}\right)\right) \qquad \textit{(degrees)}$$

where $k = 0, 1, 2, \ldots, n-1$.

Computing Complex Roots: To compute the nth roots of a complex number z, say $\sqrt[n]{z}$, you need to...

1. Find the polar form of z.
2. Use De Moivre's Theorem to write

$$\sqrt[n]{z} = \sqrt[n]{r}\,(\cos(n\theta) + i\sin(n\theta))$$

   That is, compute the nth root of its absolute value, divide the angle by n, and write out each possible root by plugging in $k = 0, 1, \ldots, n-1$.
3. Simplify these expressions (if possible).

Example

Find all the complex cube roots of $-1 = 1(\cos 180^\circ + i \sin 180^\circ)$.

Try It!

Find all the complex fourth roots of $16(\cos 120^\circ + i \sin 120^\circ)$.

Extra Facts

Euler's Identity

All of this is related to Euler's Identity:

Theorem. (Euler's Identity)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This leads to one of the most famous identities of all time:

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i(0) = -1$$

But then adding 1 to both sides, we have...

$$e^{i\pi} + 1 = 0$$

This has nearly every 'important' constant all in one equation: $1, 0, \pi, e$, and i!

This allows allows one to write the polar form of a complex number z as...

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Computing with the Alternative Polar Form

Using the polar form $z = re^{i\theta}$, we can perform complex multiplication, division, and powers quite easily!

Example. Let $z = -\sqrt{3} + i$ and $w = 3 + 3\sqrt{3}i$.

One can find (*Try It!*) that z and w have polar forms...

$$z = 6\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) = 6e^{i\frac{\pi}{3}}$$

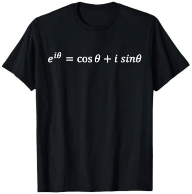
$$w = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) = 2e^{i\frac{5\pi}{6}}$$

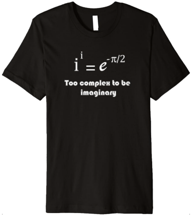
But then, for example, we have...

- $zw = 6e^{i\frac{\pi}{3}} \cdot 2e^{i\frac{5\pi}{6}} \cdot = 12\, e^{i\left(\frac{\pi}{3}+\frac{5\pi}{6}\right)} = 12e^{i\frac{7\pi}{6}}$
- $\dfrac{z}{w} = \dfrac{6e^{i\frac{\pi}{3}}}{2e^{i\frac{5\pi}{6}}} = 3e^{i\left(\frac{\pi}{3}-\frac{5\pi}{6}\right)} = 3\, e^{-i\frac{\pi}{2}}$
- $z^3 = (6e^{i\frac{\pi}{3}})^3 = 6^3 e^{i\frac{\pi}{3}\cdot 3} = 216\, e^{i\pi}$

Note. One can then convert these to their regular representation. For instance, $zw = 12e^{i\frac{7\pi}{6}} = 12(\cos(\frac{7\pi}{6}) + i\sin(\frac{7\pi}{6})) = 12(-\frac{\sqrt{3}}{2} - i\frac{1}{2}) = -6\sqrt{3} - 6i$.

Most importantly, now you can understand all them nerd shirts!





Be sure to check out the textbook to see how all this is related to the ‘mysterious’ Mandelbrot set!