MATH 122: Exam 2
Fall — 2024
10/15/2024
75 Minutes

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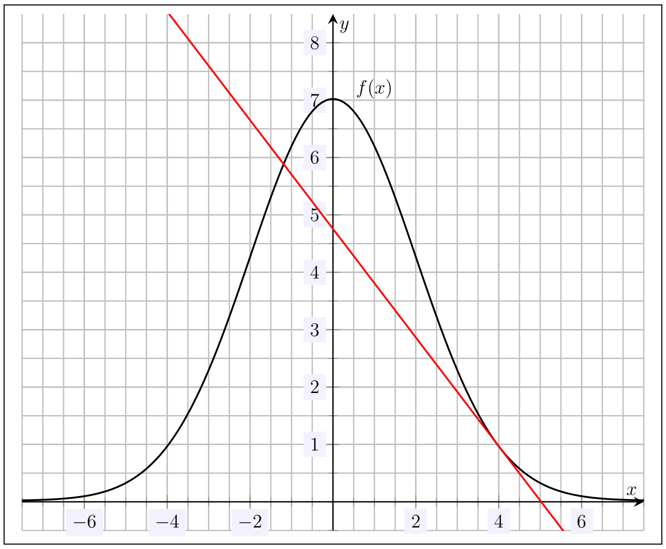
Write your name on the appropriate line on the exam cover sheet. This exam contains 8 pages (including this cover page) and 8 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	20	
2	15	
3	15	
4	15	
5	15	
6	5	
7	8	
8	7	
Total:	100	

1. (20 points) Consider the function $f(x)$ plotted below.



Based on the plot above, answer the following:

(a) On what interval(s)—if any—is $f'(x) > 0$?

If $f'(x) > 0$, then $f(x)$ is increasing. But then $f'(x) > 0$ on $(-\infty, 0)$.

(b) On what interval(s)—if any—is $f'(x) < 0$?

If $f'(x) < 0$, then $f(x)$ is decreasing. But then $f'(x) < 0$ on $(0, \infty)$.

(c) Find any points of inflection—if any.

We can see points of inflection at $x = -2$ and $x = 2$, i.e. $(-2, 4.27)$ and $(2, 4.27)$.

(d) On what interval(s)—if any—is $f''(x) > 0$?

If $f''(x) > 0$, then f is concave up. From (c) and the graph, this is $(-\infty, -2) \cup (2, \infty)$.

(e) On what interval(s)—if any—is $f''(x) < 0$?

If $f''(x) < 0$, then f is concave down. From (c) and the graph, this is $(-2, 2)$.

(f) Determine whether the following are positive (> 0), negative (< 0), or zero ($= 0$):

- $f(-2)$ $\underline{\quad > \quad}$ 0
- $f'(-3)$ $\underline{\quad > \quad}$ 0
- $f'(0)$ $\underline{\quad = \quad}$ 0
- $f'(5)$ $\underline{\quad < \quad}$ 0
- $f''(5)$ $\underline{\quad > \quad}$ 0
- $f''(0)$ $\underline{\quad < \quad}$ 0

(g) Sketch the tangent line to $f(x)$ at $x = 4$ on the graph above.

See the drawn red line.

2. (15 points) Showing all your work, compute the derivatives given below. *Do not simplify your answer.*

(a) $\dfrac{d}{dx}\left(x^6 - 3x + \dfrac{1}{\sqrt{x}} - 11\right) = \dfrac{d}{dx}\left(x^6 - 3x + x^{-1/2} - 11\right) = 6x^5 - 3 - \frac{1}{2}\,x^{-3/2}$

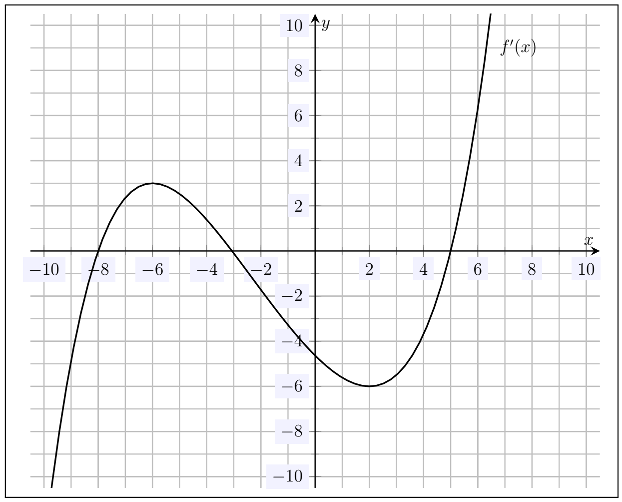
(b) $\dfrac{d}{dx}(x^{10}\,2^x) = 10x^9 \cdot 2^x + x^{10} \cdot 2^x \ln 2$

(c) $\dfrac{d}{dx}\ \ln(5x^2 - x) = \dfrac{1}{5x^2 - x} \cdot (10x - 1)$

(d) $\dfrac{d}{dx}\,(5\pi^3) = 0$

(e) $\dfrac{d}{dx}\left(\dfrac{6x - 1}{x^2 + 1}\right) = \dfrac{6(x^2 + 1) - 2x(6x - 1)}{(x^2 + 1)^2}$

3. (15 points) Consider the *derivative* of a function, $f'(x)$, plotted below.



Based on the plot above, answer the following questions:

(a) On what interval(s)—if any—is $f(x)$ increasing?

If $f'(x) > 0$, then $f(x)$ is increasing. Therefore, $f(x)$ is increasing on $(-8, -3) \cup (5, \infty)$.

(b) On what interval(s)—if any—is $f(x)$ decreasing?

If $f'(x) > 0$, then $f(x)$ is decreasing. Therefore, $f(x)$ is decreasing on $(-\infty, -8) \cup (-3, 5)$.

(c) On what interval(s)—if any—is $f(x)$ concave up?

If $f(x)$ is concave up, $f''(x) > 0$. But then $f'(x)$ is increasing. Therefore, $f(x)$ is concave up on $(-\infty, -6) \cup (2, \infty)$.

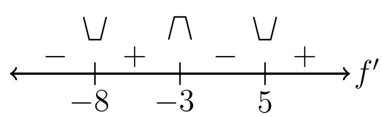
(d) On what interval(s)—if any—is $f(x)$ concave down?

If $f(x)$ is concave down, $f''(x) < 0$. But then $f'(x)$ is decreasing. Therefore, $f(x)$ is concave down on $(-6, 2)$.

(e) Find any critical values for $f(x)$—if any.

The critical values are where $f'(x) = 0$. But these values are $x = -8, -3, 5$.

(f) Classify any critical values you found in (e).



Alternatively, we know $f''(x) > 0$ at $x = -8, 5$ and $f''(x) < 0$ at $x = -3$. Therefore, $x = -8, 5$ are minima and $x = -3$ is a maxima.

(g) Find any points of inflection for $f(x)$.

An inflection point is where $f''(x)$ changes sign. From (c) and (d), we can see this is at $x = -6, 2$.

4. (15 points) Showing all your work, compute the derivatives given below. *Do not simplify your answer.*

(a) $\dfrac{d}{dx}\big(3x^2\, e^{-x}\, \log_5(x)\big) = 6x \cdot e^{-x} \log_5 x + -e^{-x} \cdot 3x^2 \log_5 x + \dfrac{1}{x \ln 5} \cdot 3x^2 e^{-x}$

(b) $\dfrac{d}{dx}\big(9^{x^3} - x\big)^7 = 7\left(9^{x^3} - x\right)^6 \cdot \left(9^{x^3} \ln 9 \cdot 3x^2 - 1\right)$

(c) $\dfrac{d}{dx}\left(\dfrac{(3x-1)e^{2x}}{\ln(1-x)}\right) = \dfrac{\big(3 \cdot e^{2x} + 2e^{2x} \cdot (3x-1)\big)\ln(1-x) - \left(\dfrac{1}{1-x} \cdot -1\right) \cdot \big((3x-1)e^{2x}\big)}{\big(\ln(1-x)\big)^2}$

5. (15 points) A company produces widgets. They hire financial analysts to examine their production costs. The analysts determine that if q items are produced, then the total production cost for the widgets is given by $C(q) = 223{,}000 + 1{,}000q - q^2$.

   (a) Find the fixed costs for producing these widgets.

   (b) Find the marginal costs at a production level of $q = 180$.

   (c) What level of production maximizes the total cost of producing these widgets? Be sure to justify your answer with either the first or second derivative test.

Solution.

(a) The fixed costs are the cost not associated with the production of the widgets. But these are the costs at a production level of $q = 0$. Therefore, the fixed costs are...

$$C(0) = 223{,}000 + 1{,}000(0) - 0^2 = 223{,}000 + 0 + 0 = \$223{,}000$$

(b) We know the marginal costs are the (approximate) costs of producing the next item. But this cost is approximately $C'(q)$. But then the marginal costs at a production level of $q = 180$ are $C'(180)$. We have...

$$C'(q) = 1{,}000 - 2q$$

$$C'(180) = 1{,}000 - 2(180) = 1{,}000 - 360 = \$640$$

Therefore, the marginal costs at a production level of $q = 180$ items is \$640.

(c) We know maxima or minima occur when the derivative is zero, i.e. at critical values. We know from (b) that $C'(q) = 1000 - 2q$. Setting $C'(q) = 0$, we have...

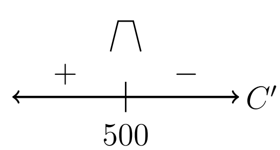
$$C'(q) = 0$$

$$1000 - 2q = 0$$

$$2q = 1000$$

$$q = 500$$

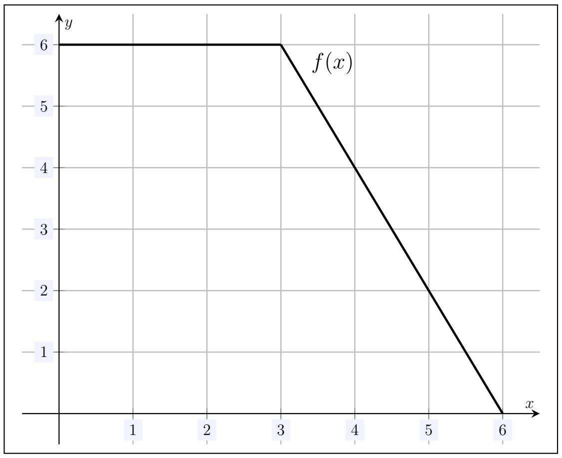
Using the first derivative test, we have...

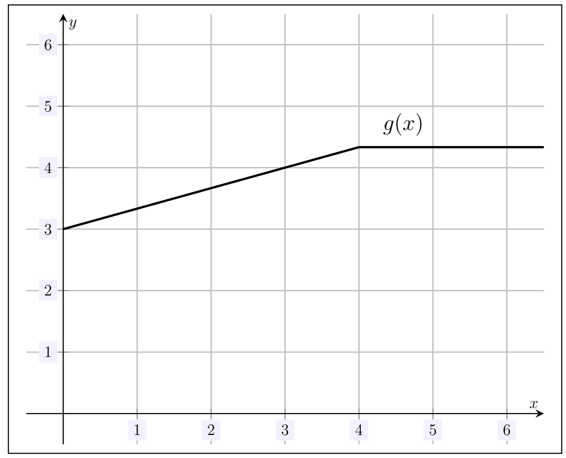


Then $q = 500$ is a maximum. Alternatively, using the fact that $C''(q) = -2$, we know that $C''(150) = -2 < 0$, so that $q = 500$ is a maximum.

Therefore, the level of production that maximizes the cost of production is $q = 500$.

6. (5 points) Consider the graphs of the functions $f(x)$ and $g(x)$ given below.





Based on these plot and showing all your work, compute $\dfrac{d}{dx} f\big(g(x)\big)\Big|_{x=3}$.

Observe that the slope of $f(x)$ *at* $x = 4$ *is* $\frac{6-0}{3-6} = \frac{6}{-3} = -2$ *and the slope of* $g(x)$ *at* $x = 3$ *is* $\frac{3-4}{0-3} = \frac{-1}{-3} = \frac{1}{3}$. *But then...*

$$\frac{d}{dx} f\big(g(x)\big)\Big|_{x=3} = f'\big(g(x)\big) \cdot g'(x)\Big|_{x=3} = f'\big(g(3)\big) \cdot g'(3) = f'(4) \cdot g'(3) = -2 \cdot \frac{1}{3} = -\frac{2}{3}$$

7. (8 points) Suppose $f(x)$ is a function with $f(10) = -2$, $f'(10) = 8$, and $f''(10) = 12$.

(a) Find the tangent line to $f(x)$ at $x = 10$.

$$\ell(x) = y_0 + m(x - x_0) = f(10) + f'(10)\big(x - 10\big) = -2 + 8(x - 10)$$

(b) Use (a) to find an approximation for $f(10.3)$.

$$f(10.3) \approx \ell(10.3) = -2 + 8(10.3 - 10) = -2 + 8(0.3) = -2 + 2.4 = 0.4$$

(c) Is your approximation (b) more likely an under-approximation or an over-approximation? Explain.

Because $f''(10) = 12 > 0$, *it is likely that (b) is an under-approximation.*

8. (7 points) Let $f(x) = 2x^2 + 8x - 5$. Using the definition of the derivative, approximate $f'(-1)$. *While you may check your answer using the derivative shortcuts, you will receive no credit for using derivative shortcuts to find this value.*

Solution. *The definition of $f'(a)$ is...*

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Here $a = -1$. We choose $h = 0.001$. Then...

$$\begin{aligned} f(a+h) &= f(-1 + 0.001) = f(-0.999) = 2(-0.999)^2 + 8(-0.999) - 5 = -10.996 \\ f(a) &= f(-1) = 2(-1)^2 + 8(-1) - 5 = -11 \end{aligned}$$

Therefore, we have...

$$f'(1) := \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} \approx \frac{-10.996 - (-11)}{0.001} = \frac{-10.996 + 11}{0.001} = \frac{0.004}{0.001} = 4$$

Note. Observe that finding $f'(x) = 4x + 8$, the exact value is $f(-1) = 4(-1) + 8 = -4 + 8 = 4$.