Check-In 01/16. *(True/False)* Given $\displaystyle\int_0^{\pi} e^{\sin x} \cos x \; dx$, the u-substitution $u = \sin x$ transforms this integral into $\displaystyle\int_0^{\pi} e^u \; du$.

Solution. The statement is *false*. If $u = \sin x$, then $du = \cos x \, dx$. So indeed, this u-substitution would transform the integral $\displaystyle\int e^{\sin x} \cos x \, dx$ into the integral $\displaystyle\int e^u \, du$. However with definite integrals, one needs to remember to transform the limits as well. If $x = 0$, then $u = \sin(0) = 0$. If $x = \pi$, then $u = \sin(\pi) = 0$. Therefore, the correct substitution is $\displaystyle\int_0^{\pi} e^{\sin x} \cos x \, dx = \int_0^0 e^u \, du = 0$.

Check-In 01/21. *(True/False)* To integrate $\displaystyle\int \operatorname{arccot} \theta \, d\theta$, one can use integration-by-parts by choosing $u = \operatorname{arccot} \theta$ and $dv = 1$.

Solution. The statement is *true*. Using LIATE, it is likely that the choice of $u = \operatorname{arccot} \theta$ will work. With 'nothing left' in the integrand, this means that $dv = 1$. We fill in our box as follows:

$$\begin{array}{|c|c|}\hline \operatorname{arccot}\theta & \\ \hline & 1 \\ \hline\end{array} \quad\Longrightarrow\quad \begin{array}{|c|c|}\hline \operatorname{arccot}\theta & \theta \\ \hline \dfrac{-1}{1+\theta^2} & 1 \\ \hline\end{array}$$

Then using the 'Rule of 7', we find that...

$$\int \operatorname{arccot} \theta \, d\theta = \theta \operatorname{arccot} \theta - \int \frac{-\theta}{1+\theta^2} \, d\theta = \theta \operatorname{arccot} \theta + \int \frac{\theta}{1+\theta^2} \, d\theta$$

Using the u-substitution $u = 1 + \theta^2$, we see that $\displaystyle\int \frac{\theta}{1+\theta^2} \, d\theta = \frac{1}{2} \ln |1 + \theta^2| + C$. Therefore, we have...

$$\int \operatorname{arccot} \theta \, d\theta = \theta \operatorname{arccot} \theta + \frac{1}{2} \ln |1 + \theta^2| + C$$

Check-In 01/23. *(True/False)* The integral $\displaystyle\int e^{x/2} \sin(3x) \; dx$ is a 'looping' integral.

Solution. The statement is *true*. Recall that integrals whose integrand is the product of an exponential function with $\sin x$ or $\cos x$ 'loop.' We can see this directly: choose $u = \sin(3x)$. Using the ‘box method’, we have...

$\sin(3x)$	$2e^{x/2}$
$3\cos(3x)$	$e^{x/2}$

Therefore, we have...

$$\int e^{x/2}\sin(3x)\;dx = 2e^{x/2}\sin(3x) - \int 6e^{x/2}\cos(3x)\;dx$$

But this integral on the right also requires integration-by-parts: we choose $u = \cos(3x)$ and then...

$\cos(3x)$	$12e^{x/2}$
$-3\sin(3x)$	$6e^{x/2}$

So then we have...

$$\begin{aligned}
\int e^{x/2}\sin(3x)\;dx &= 2e^{x/2}\sin(3x) - \int 6e^{x/2}\cos(3x)\;dx \\
&= 2e^{x/2}\sin(3x) - \left(12e^{x/2}\cos(3x) - \int -36e^{x/2}\sin(3x)\;dx\right) \\
&= 2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x) + \int -36e^{x/2}\sin(3x)\;dx \\
&= 2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x) - 36\int e^{x/2}\sin(3x)\;dx
\end{aligned}$$

Observe that we have ‘looped’—obtaining a multiple of the original integral on the right. Adding $36\int e^{x/2}\sin(3x)\;dx$ to both sides, we have...

$$37\int e^{x/2}\sin(3x)\;dx = 2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x)$$

Therefore, we have...

$$\int e^{x/2}\sin(3x)\;dx = \frac{2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x)}{37} + C$$

We can shortcut this work by adjusting tabular integration:

$$\begin{array}{ccc} u & & dv \\ \hline \sin(3x) & & e^{x/2} \\ & + & \\ 3\cos(3x) & & 2e^{x/2} \\ & - & \\ -9\sin(3x) & \xrightarrow{+} & 4e^{x/2} \end{array}$$

Therefore, we have...

$$\begin{aligned} \int e^{x/2}\sin(3x)\;dx &= 2e^{x/2} - 12e^{x/2}\cos(3x) - 36\int e^{x/2}\sin(3x)\;dx \\ 37\int e^{x/2}\sin(3x)\;dx &= 2e^{x/2} - 12e^{x/2}\cos(3x) \\ \int e^{x/2}\sin(3x)\;dx &= \frac{2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x)}{37} + C \end{aligned}$$

Check-In 01/28. *(True/False)* To integrate $\displaystyle\int \cos^8\theta\sin^5\theta\;d\theta$, one should choose $u = \cos\theta$.

Solution. The statement is *true*. Observe that if we choose $u = \cos\theta$, then $\cos^8\theta$ becomes u^8. We know du will then produce a $\sin\theta$—specifically $-\sin\theta$. This 'uses' one of the $\sin\theta$'s in the integrand. This leaves $\sin^4\theta$ remaining in the integrand. But we can replace even powers of $\sin\theta$ in terms of $\cos\theta$. So we can carry out this substitution. Observe that if $u = \cos\theta$, then $du = -\sin\theta\;d\theta$. But then using the fact that $\sin^2\theta = 1 - \cos^2\theta$ (so that $\sin^4\theta = (\sin^2\theta)^2 = (1-\cos^2\theta)^2$), we have...

$$\begin{aligned} \int \cos^8\theta\sin^5\theta\;d\theta &= \int \cos^8\theta\sin^4\theta\cdot\sin\theta\;d\theta \\ &= -\int \cos^8\theta\sin^4\theta\cdot -\sin\theta\;d\theta \\ &= -\int \cos^8\theta(1-\cos^2\theta)^2\cdot -\sin\theta\;d\theta \\ &= -\int u^8(1-u^2)^2\;du \\ &= -\int u^8(1-2u^2+u^4)\;du \\ &= \int -u^8 + 2u^{10} - u^{12}\;du \\ &= -\frac{u^9}{9} + \frac{2u^{11}}{11} - \frac{u^{13}}{13} + C \\ &= -\frac{\cos^9\theta}{9} + \frac{2\cos^{11}\theta}{11} - \frac{\cos^{13}\theta}{13} + C \end{aligned}$$

Alternatively, observe that if we had chosen $u = \sin\theta$, then $\sin^5\theta$ becomes u^5. We know that du produces a $\cos\theta$. This 'uses' one of the $\cos\theta$'s in the integrand. This leaves $\cos^7\theta$ remaining in the integrand. However, we can only replace even powers of $\cos\theta$ in terms of $\sin\theta$ using $\cos^2\theta = 1 - \sin^2\theta$. So without using other identities or using other techniques, we cannot 'simply' choose $u = \sin\theta$ for this integral.