Check-In 08/21. *(True/False)* The integral $\int x\sqrt[3]{x-2}\,dx$ can be treated as a 'shifting integral' by using the u-substitution $u = x - 2$.

Solution. The statement is *true*. We 'want' to be able to distribute the x across the cube-root but we cannot—this is not a valid operation. However, if we make the u-substitution $u = x - 2$, then we will be able to distribute in a way that makes this integral 'routine.' So, let $u = x - 2$, then $du = dx$. Moreover, because $u = x - 2$, we know that $x = u + 2$. But then...

$$\int x\sqrt[3]{x-2}\,dx = \int (u+2)\sqrt[3]{u}\,du = \int \left(u^{4/3} + 2u^{1/3}\right)\,du = \tfrac{3}{7}\,u^{7/3} + \tfrac{3}{4}\cdot 2u^{4/3} + C = \tfrac{3}{7}(x-2)^{7/3} + \tfrac{3}{2}(x-2)^{4/3} + C$$

Note that a computer algebra system may write the answer (though you will *not* be expected to) like this:

$$\tfrac{3}{7}(x-2)^{7/3} + \tfrac{3}{2}(x-2)^{4/3} + C = (x-2)^{4/3}\left(\tfrac{3}{7}(x-2) + \tfrac{3}{2}\right) + C = (x-2)^{4/3}\left(\tfrac{3}{7}x + \tfrac{9}{14}\right) + C = \tfrac{3}{14}(x-2)^{4/3}\,(2x+3) + C$$

Check-In 08/26. *(True/False)* Using integration-by-parts to evaluate $\int x\tan^{-1}(x)\,dx$, one chooses $u = \tan^{-1} x$ and $dv = x$.

Solution. The statement is *true*. Using LIATE, the first term that appears is 'I' for inverse trig. Therefore, we choose $u = \tan^{-1} x$. But then $dv = x$. We then fill in our box:

$\tan^{-1} x$	$\dfrac{x^2}{2}$
$\dfrac{1}{1+x^2}$	x

Using the 'rule of 7', we have...

$$\int x\tan^{-1} x\,dx = \frac{1}{2}\,x^2\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

We now need only evaluate the integral on the right. Dividing $1 + x^2$ into x^2, we have a remainder of -1, i.e. $\dfrac{x^2}{1+x^2} = 1 + \dfrac{-1}{1+x^2}$. Therefore, we have...

$$\frac{1}{2}\int \frac{x^2}{1+x^2}\,dx = \frac{1}{2}\int \left(1 + \frac{-1}{1+x^2}\right)\,dx = \frac{1}{2}\left(x - \tan^{-1} x\right) + C$$

But then...

$$\begin{aligned}\int x\tan^{-1}x\,dx &= \frac{1}{2}\,x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx\\ &= \frac{1}{2}\,x^2\tan^{-1}x - \frac{1}{2}\left(x - \tan^{-1}x\right) + C\\ &= \frac{1}{2}\,x^2\tan^{-1}x - \frac{1}{2}\,x + \frac{1}{2}\,\tan^{-1}x + C\\ &= \frac{x^2\tan^{-1}x - x + \tan^{-1}x}{2} + C\\ &= \frac{(x^2+1)\tan^{-1}x - x}{2} + C\end{aligned}$$

Check-In 08/28. *(True/False)* The integral $\int e^x\sin(3x)\,dx$ can be treated as an integration-by-parts 'looping' integral.

Solution. The statement is *true*. Using integration-by-parts for $\int e^x\sin(3x)\,dx$ would result in an integral that would 'loop' back to itself. Generally, an integrand of the form exponential $\cdot$ (sin or cos) or trig $\cdot$ trig will have this property. Using traditional integration-by-parts, by LIATE, we choose $u = \sin(3x)$ and $dv = e^x$. Filling out our box, we have...

$\sin(3x)$	e^x
$3\cos(3x)$	e^x

Using the 'rule of seven', we then have...

$$\int e^x\sin(3x)\,dx = e^x\sin(3x) - \int 3e^x\cos(3x)\,dx$$

To integrate $\int 3e^x\cos(3x)\,dx$, we again use integration-by-parts. Using LIATE, we choose $u =$

$3\cos(3x)$ and $dv = e^x$. Filling out the box, we have...

$3\cos(3x)$	e^x
$-9\sin(3x)$	e^x

Using the 'rule of seven', we then have

$$\int 3e^x \cos(3x)\ dx = 3e^x \cos(3x) - \int -9e^x \sin(3x)\ dx = 3e^x \cos(3x) + 9 \int e^x \sin(3x)\ dx$$

But then we have...

$$\int e^x \sin(3x)\ dx = e^x \sin(3x) - \int 3e^x \cos(3x)\ dx = \int e^x \sin(3x)\ dx = e^x \sin(3x) - \left(3e^x \cos(3x) + 9 \int e^x \sin(3x)\ dx\right)$$

Therefore, we have...

$$\int e^x \sin(3x)\ dx = e^x \sin(3x) - \left(3e^x \cos(3x) + 9 \int e^x \sin(3x)\ dx\right)$$

$$\int e^x \sin(3x)\ dx = e^x \sin(3x) - 3e^x \cos(3x) - 9 \int e^x \sin(3x)\ dx$$

$$10 \int e^x \sin(3x)\ dx = e^x \sin(3x) - 3e^x \cos(3x)$$

$$\int e^x \sin(3x)\ dx = \frac{e^x \sin(3x) - 3e^x \cos(3x)}{10} + C$$

$$\int e^x \sin(3x)\ dx = \frac{e^x}{10} \big(\sin(3x) - 3\cos(3x)\big) + C$$

Alternatively, we can use an alteration of the tabular method of integration-by-parts. We choose $u = \sin(3x)$ and $dv = e^x$. We then have...

u		dv
$\sin(3x)$		e^x
	$+$	
$3\cos(3x)$	$-$	e^x
$-9\sin(3x)$	$+$	e^x

Therefore, we have...

$$\int e^x \sin(3x)\ dx = e^x \sin(3x) - 3\cos(3x)e^x - 9 \int e^x \sin(3x)\ dx$$

Solving for our integral, we have...

$$\int e^x \sin(3x)\ dx = e^x \sin(3x) - 3\cos(3x)e^x - 9\int e^x \sin(3x)\ dx$$

$$10\int e^x \sin(3x)\ dx = e^x \sin(3x) - 3e^x\cos(3x)$$

$$\int e^x \sin(3x)\ dx = \frac{e^x \sin(3x) - 3e^x \cos(3x)}{10} + C$$

$$\int e^x \sin(3x)\ dx = \frac{e^x}{10}\big(\sin(3x) - 3\cos(3x)\big) + C$$

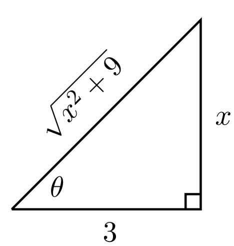
Check-In 09/02. *(True/False)* To integrate $\displaystyle\int \cot^2 x \csc^2 x\ dx$ as a trigonometric integral, one could make the substitution $u = \cot x$.

Solution. The statement is *true*. If one let $u = \csc x$, we would have $du = -\csc x \cot x$. Setting aside a $\cot x$ for the du, this would leave only a single $\cot x$ term in the integrand—which we cannot replace with a Pythagorean identity. However, this is not an issue if we let $u = \cot x$. If $u = \cot x$, then $du = -\csc^2 x\, dx$. But then...

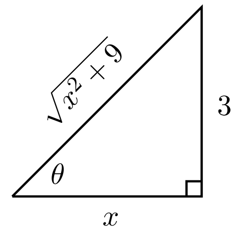
$$\int \cot^2 x \csc^2 x\ dx = -\int \cot^2 x \cdot -\csc^2 x\ dx = -\int u^2\ du = -\frac{u^3}{3} + C = -\frac{\cot^3 x}{3} + C$$

Check-In 09/04. *(True/False)* To integrate $\displaystyle\int \frac{x^2}{\sqrt{x^2+9}}\ dx$, one can make the substitution $x = 3\cot\theta$.

Solution. The statement is *true*. Observe that the $x^2 + 9$ 'resembles' a term from the Pythagorean Theorem. This suggests that a trig. substitution might be useful. Because $a^2 + b^2 = c$, we see that $a^2 + b^2$ corresponds to $x^2 + 9$, i.e. a^2 corresponds to x^2 and b^2 corresponds to 9. This implies that $a = x$ and $b = 3$. But then $c^2 = x^2 + 9$, i.e. $c = \sqrt{x^2+9}$. We construct a right triangle with these legs and hypotenuse:



But then $\tan\theta = \frac{x}{3}$, which implies $x = 3\tan\theta$. While this seems like it makes the statement of the problem false, this is not the only right triangle we could have constructed. If we had instead drawn



We would have $\tan\theta = \frac{3}{x}$, which implies that $x\tan\theta = 3$, so that $x = \frac{3}{\tan\theta} = 3\cot\theta$. This is the substitution in the problem statement. Both the substitutions $x = 3\cot\theta$ and $x = 3\tan\theta$ are viable trig. substitutions to compute this integral.

Check-In 09/09. *(True/False)* The partial fraction decomposition of $\dfrac{x+4}{x^2(x-3)}$ has the form $\dfrac{Ax+B}{x^2}+\dfrac{C}{x-3}$.

Solution. The statement is *false*. For a partial fraction decomposition, one first needs to be sure that the degree of the numerator is smaller than the degree of the denominator—which is the case here. One then needs to be sure that the denominator is factored completely—which is the case here. One then needs to 'run' through each power of the factored terms of the denominator—being sure that the numerator term for quadratic factors is linear. In this case, the denominator terms are x (up to power 2) and $x-3$. Therefore, the partial fraction decomposition is...

$$\frac{x+4}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}$$

Although the term x^2 is quadratic, the base—x—is linear. Hence, its numerator terms will always be constant—never linear. This is the mistake in the decomposition given in the problem statement.

Check-In 09/11. *(True/False)* The partial fraction decomposition of $\dfrac{7x-5}{x^2-x}$ is $\dfrac{5}{x}+\dfrac{2}{x-1}$.

Solution. The statement is *true*. First, observe that $\dfrac{7x-5}{x^2-x} = \dfrac{7x-5}{x(x-1)}$. Therefore, we have a decomposition of the form

$$\frac{7x-5}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

Observe that Heaviside's can be used to find both A and B. So, we have...

$$A = \left.\frac{7x-5}{\boxed{\cancel{x}}(x-1)}\right|_{x=0} = \frac{0-5}{0-1} = \frac{-5}{-1} = 5$$

$$B = \left.\frac{7x-5}{x\,\boxed{\cancel{x-1}}}\right|_{x=1} = \frac{7-5}{1} = \frac{2}{1} = 2$$

Therefore, we have...

$$\frac{7x-5}{x^2-x} = \frac{5}{x} + \frac{2}{x-1}$$

Check-In 09/18. *(True/False)* The integral $\int_{-1}^{1} \frac{dx}{\sqrt[3]{x}} = \int_{-1}^{1} x^{-1/3}\ dx = \frac{3}{2}\,x^{2/3}\Big|_{-1}^{1} = \frac{3}{2}\,(1^{2/3} - (-1)^{2/3}) = \frac{2}{3}\,(1 - \sqrt[3]{(-1)^2}) = \frac{2}{3}\,(1-1) = 0.$

Solution. The statement is *false*. Observe that the integrand is undefined at $x = 0$. In fact, the integrand has a vertical asymptote at $x = 0$. Therefore, the integral is improper. We need split the integral at this x-value. So, we write...

$$\int_{-1}^{1} \frac{dx}{\sqrt[3]{x}} = \int_{-1}^{0} \frac{dx}{\sqrt[3]{x}} + \int_{0}^{1} \frac{dx}{\sqrt[3]{x}}$$

We need to take the limit as the integral limits approach 0:

$$\int_{-1}^{1} \frac{dx}{\sqrt[3]{x}} := \lim_{b\to 0^-} \int_{-1}^{b} \frac{dx}{\sqrt[3]{x}} + \lim_{b\to 0^+} \int_{b}^{1} \frac{dx}{\sqrt[3]{x}}$$

Observe that...

$$\int \frac{dx}{\sqrt[3]{x}} = \int x^{-1/3}\ dx = \frac{3}{2}\,x^{2/3} + C$$

But then...

$$\lim_{b\to 0^-} \int_{-1}^{b} \frac{dx}{\sqrt[3]{x}} = \lim_{b\to 0^-} \frac{3}{2}\,x^{2/3}\Big|_{-1}^{b} = \lim_{b\to 0^-} \frac{3}{2}\,b^{2/3} - \frac{3}{2} = -\frac{3}{2}$$

$$\lim_{b\to 0^+} \int_{b}^{1} \frac{dx}{\sqrt[3]{x}} = \lim_{b\to 0^+} \frac{3}{2}\,x^{2/3}\Big|_{b}^{1} = \frac{3}{2} - \lim_{b\to 0^+} \frac{3}{2}\,b^{2/3} = \frac{3}{2}$$

Therefore, we have...

$$\int_{-1}^{1} \frac{dx}{\sqrt[3]{x}} := \lim_{b\to 0^-} \int_{-1}^{b} \frac{dx}{\sqrt[3]{x}} + \lim_{b\to 0^+} \int_{b}^{1} \frac{dx}{\sqrt[3]{x}} = -\frac{3}{2} + \frac{3}{2} = 0$$

So, while the given answer is correct, the approach is entirely incorrect. In other cases, using the given approach for improper integrals will result in incorrect solutions.