Check-In 01/16. *(True/False)* Given $\displaystyle\int_0^{\pi} e^{\sin x} \cos x \; dx$, the u-substitution $u = \sin x$ transforms this integral into $\displaystyle\int_0^{\pi} e^u \; du$.

Solution. The statement is *false*. If $u = \sin x$, then $du = \cos x \, dx$. So indeed, this u-substitution would transform the integral $\displaystyle\int e^{\sin x} \cos x \, dx$ into the integral $\displaystyle\int e^u \, du$. However with definite integrals, one needs to remember to transform the limits as well. If $x = 0$, then $u = \sin(0) = 0$. If $x = \pi$, then $u = \sin(\pi) = 0$. Therefore, the correct substitution is $\displaystyle\int_0^{\pi} e^{\sin x} \cos x \, dx = \int_0^0 e^u \, du = 0$.

Check-In 01/21. *(True/False)* To integrate $\displaystyle\int \operatorname{arccot} \theta \, d\theta$, one can use integration-by-parts by choosing $u = \operatorname{arccot} \theta$ and $dv = 1$.

Solution. The statement is *true*. Using LIATE, it is likely that the choice of $u = \operatorname{arccot} \theta$ will work. With 'nothing left' in the integrand, this means that $dv = 1$. We fill in our box as follows:

$$
\begin{array}{|c|c|}
\hline
\operatorname{arccot} \theta & \\
\hline
 & 1 \\
\hline
\end{array}
\quad \Longrightarrow \quad
\begin{array}{|c|c|}
\hline
\operatorname{arccot} \theta & \theta \\
\hline
\dfrac{-1}{1+\theta^2} & 1 \\
\hline
\end{array}
$$

Then using the 'Rule of 7', we find that...

$$\int \operatorname{arccot} \theta \, d\theta = \theta \operatorname{arccot} \theta - \int \frac{-\theta}{1+\theta^2} \, d\theta = \theta \operatorname{arccot} \theta + \int \frac{\theta}{1+\theta^2} \, d\theta$$

Using the u-substitution $u = 1 + \theta^2$, we see that $\displaystyle\int \frac{\theta}{1+\theta^2} \, d\theta = \frac{1}{2} \ln|1+\theta^2| + C$. Therefore, we have...

$$\int \operatorname{arccot} \theta \, d\theta = \theta \operatorname{arccot} \theta + \frac{1}{2} \ln|1+\theta^2| + C$$

Check-In 01/23. *(True/False)* The integral $\displaystyle\int e^{x/2} \sin(3x) \; dx$ is a 'looping' integral.

Solution. The statement is *true*. Recall that integrals whose integrand is the product of an exponential function with $\sin x$ or $\cos x$ 'loop.' We can see this directly: choose $u = \sin(3x)$. Using the ‘box method’, we have...

$\sin(3x)$	$2e^{x/2}$
$3\cos(3x)$	$e^{x/2}$

Therefore, we have...

$$\int e^{x/2}\sin(3x)\,dx = 2e^{x/2}\sin(3x) - \int 6e^{x/2}\cos(3x)\,dx$$

But this integral on the right also requires integration-by-parts: we choose $u = \cos(3x)$ and then...

$\cos(3x)$	$12e^{x/2}$
$-3\sin(3x)$	$6e^{x/2}$

So then we have...

$$\begin{aligned}
\int e^{x/2}\sin(3x)\,dx &= 2e^{x/2}\sin(3x) - \int 6e^{x/2}\cos(3x)\,dx \\
&= 2e^{x/2}\sin(3x) - \left(12e^{x/2}\cos(3x) - \int -36e^{x/2}\sin(3x)\,dx\right) \\
&= 2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x) + \int -36e^{x/2}\sin(3x)\,dx \\
&= 2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x) - 36\int e^{x/2}\sin(3x)\,dx
\end{aligned}$$

Observe that we have ‘looped’—obtaining a multiple of the original integral on the right. Adding $36\int e^{x/2}\sin(3x)\,dx$ to both sides, we have...

$$37\int e^{x/2}\sin(3x)\,dx = 2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x)$$

Therefore, we have...

$$\int e^{x/2}\sin(3x)\,dx = \frac{2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x)}{37} + C$$

We can shortcut this work by adjusting tabular integration:

u		dv
$\sin(3x)$	$+$	$e^{x/2}$
$3\cos(3x)$	$-$	$2e^{x/2}$
$-9\sin(3x)$	$+$	$4e^{x/2}$

Therefore, we have...

$$\begin{aligned}\int e^{x/2}\sin(3x)\;dx &= 2e^{x/2} - 12e^{x/2}\cos(3x) - 36\int e^{x/2}\sin(3x)\;dx\\ 37\int e^{x/2}\sin(3x)\;dx &= 2e^{x/2} - 12e^{x/2}\cos(3x)\\ \int e^{x/2}\sin(3x)\;dx &= \frac{2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x)}{37} + C\end{aligned}$$

Check-In 01/28. *(True/False)* To integrate $\displaystyle\int \cos^8\theta\sin^5\theta\;d\theta$, one should choose $u = \cos\theta$.

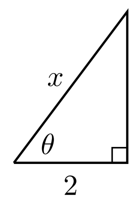
Solution. The statement is *true*. Observe that if we choose $u = \cos\theta$, then $\cos^8\theta$ becomes u^8. We know du will then produce a $\sin\theta$—specifically $-\sin\theta$. This 'uses' one of the $\sin\theta$'s in the integrand. This leaves $\sin^4\theta$ remaining in the integrand. But we can replace even powers of $\sin\theta$ in terms of $\cos\theta$. So we can carry out this substitution. Observe that if $u = \cos\theta$, then $du = -\sin\theta\;d\theta$. But then using the fact that $\sin^2\theta = 1 - \cos^2\theta$ (so that $\sin^4\theta = (\sin^2\theta)^2 = (1-\cos^2\theta)^2$), we have...

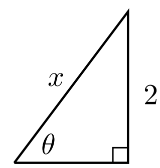
$$\begin{aligned}\int \cos^8\theta\sin^5\theta\;d\theta &= \int \cos^8\theta\sin^4\theta\cdot\sin\theta\;d\theta\\ &= -\int \cos^8\theta\sin^4\theta\cdot -\sin\theta\;d\theta\\ &= -\int \cos^8\theta(1-\cos^2\theta)^2\cdot -\sin\theta\;d\theta\\ &= -\int u^8(1-u^2)^2\;du\\ &= -\int u^8(1-2u^2+u^4)\;du\\ &= \int -u^8+2u^{10}-u^{12}\;du\\ &= -\frac{u^9}{9} + \frac{2u^{11}}{11} - \frac{u^{13}}{13} + C\\ &= -\frac{\cos^9\theta}{9} + \frac{2\cos^{11}\theta}{11} - \frac{\cos^{13}\theta}{13} + C\end{aligned}$$

Alternatively, observe that if we had chosen $u = \sin\theta$, then $\sin^5\theta$ becomes u^5. We know that du produces a $\cos\theta$. This 'uses' one of the $\cos\theta$'s in the integrand. This leaves $\cos^7\theta$ remaining in the integrand. However, we can only replace even powers of $\cos\theta$ in terms of $\sin\theta$ using $\cos^2\theta = 1 - \sin^2\theta$. So without using other identities or using other techniques, we cannot 'simply' choose $u = \sin\theta$ for this integral.

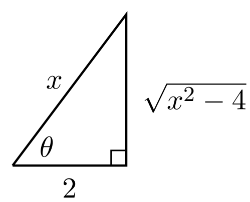
Check-In 01/30. *(True/False)* To compute $\displaystyle\int \frac{x^3}{\sqrt{x^2-4}}\,dx$, one can make the substitution $x = 2\sec\theta$.

Solution. The statement is *true*. Observe we have the term $x^2 - 4$, which resembles a term from the Pythagorean Theorem. For a right triangle, we know that $a^2 + b^2 = c^2$. This implies that $a^2 = c^2 - b^2$, which is $x^2 - 4$ if $c^2 = x^2$ and $b^2 = 4$, i.e. $c = x$ and $b = 2$. We construct a triangle with $c = x$ and $b = 2$. This only gives two possibilities:





In the first, we would have $\cos\theta = \frac{2}{x}$, which implies that $x = \frac{2}{\cos\theta} = 2\sec\theta$. This is the given substitution. In the second, we would have $\sin\theta = \frac{2}{x}$, which implies that $x = \frac{2}{\sin\theta} = 2\csc\theta$. Choosing the former substitution, we would have $x = 2\sec\theta$, so that $dx = 2\sec\theta\tan\theta$. Calling the vertical side s and using the Pythagorean Theorem, we see that $2^2 + s^2 = x^2$, i.e. $4 + s^2 = x^2$. But then $s = \sqrt{x^2-4}$. This gives us the following triangle:



We need to find x^3 and $\sqrt{x^2-4}$. Because $x = 2\sec\theta$, we know that $x^3 = (2\sec\theta)^3 = 2^3\sec^3\theta$. To find $\sqrt{x^2-4}$, observe that $\tan\theta = \frac{\sqrt{x^2-4}}{2}$, which implies that $\sqrt{x^2-4} = 2\tan\theta$. Therefore, we have...

$$\int \frac{x^3}{\sqrt{x^2-4}}\,dx = \int \frac{2^3\sec^3\theta}{2\tan\theta}\cdot 2\sec\theta\tan\theta\;d\theta = \int 8\sec^4\theta\;d\theta$$

Now using the fact that $\tan^2\theta + 1 = \sec^2\theta$, we know that

$$\sec^4\theta = (\sec^2\theta)^2 = (\tan^2\theta + 1)^2 = \tan^4\theta + 2\tan^2\theta + 1$$

But then...

$$\int 8\sec^4\theta\;d\theta = \int 8\sec^2\theta\sec^2\theta\;d\theta = \int 8\sec^2\theta(\tan^2\theta+1)\;d\theta = \int 8(\tan^2\theta+1)\cdot\sec^2\theta\;d\theta$$

Now letting $u = \tan\theta$, we have $du = \sec^2\theta\;d\theta$. But then...

$$\int 8(\tan^2\theta+1)\cdot\sec^2\theta\;d\theta = \int 8(u^2+1)\;du = 8\int(u^2+1)\;du = 8\left(\frac{u^3}{3}+u\right)+C$$

Replacing u, we have...

$$8\left(\frac{u^3}{3}+u\right)+C=8\left(\frac{\tan^3\theta}{3}+\tan\theta\right)+C$$

But from the triangle, we know that $\tan\theta=\frac{\sqrt{x^2-4}}{2}$. Therefore, we have...

$$8\left(\frac{\tan^3\theta}{3}+\tan\theta\right)+C=.8\left(\frac{1}{3}\cdot\left(\frac{\sqrt{x^2-4}}{2}\right)^3+\frac{\sqrt{x^2-4}}{2}\right)+C$$

If we simplify this, we have...

$$8\left(\frac{1}{3}\cdot\left(\frac{\sqrt{x^2-4}}{2}\right)^3+\frac{\sqrt{x^2-4}}{2}\right)+C$$
$$8\cdot\frac{\sqrt{x^2-4}}{2}\left(\frac{1}{3}\cdot\left(\frac{\sqrt{x^2-4}}{2}\right)^2+1\right)+C$$
$$4\sqrt{x^2-4}\left(\frac{1}{3}\cdot\frac{x^2-4}{4}+1\right)+C$$
$$4\sqrt{x^2-4}\left(\frac{x^2-4}{12}+1\right)+C$$
$$4\sqrt{x^2-4}\left(\frac{x^2-4}{12}+\frac{12}{12}\right)+C$$
$$4\sqrt{x^2-4}\left(\frac{x^2+8}{12}\right)+C$$
$$\frac{1}{3}\sqrt{x^2-4}\,(x^2+8)+C$$

Therefore, we have...

$$\int\frac{x^3}{\sqrt{x^2-4}}\,dx=\frac{1}{3}\sqrt{x^2-4}\,(x^2+8)+C$$