**MATH 115: Exam 2**
**Fall — 2024**
**10/28/2024**
**75 Minutes**

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Write your name on the appropriate line on the exam cover sheet. This exam contains 9 pages (including this cover page) and 11 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 8 | |
| 3 | 8 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 8 | |
| 8 | 8 | |
| 9 | 10 | |
| 10 | 10 | |
| 11 | 8 | |
| Total: | 100 | |

1. (10 points) Complete the table of exact values for the trigonometric figures given below.

| $\theta$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $\sin\theta$ | $0$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | $1$ |
| $\cos\theta$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | $0$ |
| $\tan\theta$ | $0$ | $\frac{1}{\sqrt{3}}$ | $1$ | $\sqrt{3}$ | undef. |

2. (8 points) Compute the following—no work is required:

   (a) $\log_3(27) = 3$

   (b) $\log_5\left(\frac{1}{25}\right) = -2$

   (c) $\log_4(2) = \frac{1}{2}$

   (d) $\log_{\sqrt{2}}(1) = 0$

   (e) $\ln\left(e^{2/3}\right) = \frac{2}{3}$

   (f) $\pi^{2\log_\pi(3)} = \pi^{\log_\pi(3^2)} = 3^2 = 9$

   (g) $\log_{25}(125) = \frac{\log_5 125}{\log_5 25} = \frac{3}{2}$

3. Consider the function $f(x) = 5\left(3^{2-x}\right)$.

   (a) (5 points) Write $f(x)$ in the form $Ab^x$ for some $A$ and $b$.

$$f(x) = 5\left(3^{2-x}\right) = 5\left(3^2 \cdot 3^{-x}\right) = (5\cdot 9)3^{-x} = 45\left(3^{-1}\right)^x = 45\left(\frac{1}{3}\right)^x$$

   (b) (3 points) Determine whether $f(x)$ is exponentially increasing or decreasing. Use (a) to justify your answer.

   *Observe that $f(x)$ has the form $Ab^x$ with $A = 45$ and $b = \frac{1}{3}$. Because $A > 0$ and $b = \frac{1}{3}$, we know that $f(x)$ is exponentially decreasing.*

4. (10 points) Find the quotient and remainder when $2x^3 + x^2 + 8$ is divided by $x + 2$.

***Solution.*** *We have...*

$$\begin{array}{r|llll}
 & 2x^2 & -3x & +6 & \\
\hline
x+2 & 2x^3 & +x^2 & & +8 \\
 & -2x^3 & -4x^2 & & \\
\cline{2-3}
 & & -3x^2 & & \\
 & & 3x^2 & +6x & \\
\cline{3-4}
 & & & 6x & +8 \\
 & & & -6x & -12 \\
\cline{4-5}
 & & & & -4
\end{array}$$

*Therefore, the quotient is* $2x^2 - 3x + 6$ *and the remainder is* $-4$.

*Quotient:* $2x^2 - 3x + 6$

*Remainder:* $-4$

*Alternatively, because* $x + 2$ *is linear, we can instead use synthetic division. For synthetic division, we write* $x + 2$ *in the form* $x - a$*, i.e.* $x + 2 = x - (-2)$. *Being sure to include the zero terms in* $2x^3 + x^2 + 8$*, i.e. writing* $2x^3 + x^2 + 8 = 2x^3 + x^2 + 0x + 8$*, we have...*

$$\begin{array}{r|rrrr}
 & 2 & 1 & 0 & 8 \\
-2 & & -4 & 6 & -12 \\
\hline
 & 2 & -3 & 6 & -4
\end{array}$$

*Therefore, we can see that the quotient is* $2x^2 - 3x + 6$ *and the remainder is* $-4$.

*Quotient:* $2x^2 - 3x + 6$

*Remainder:* $-4$

5. (10 points) Showing all your work, solve the following:

(a) $x(2x-3)=5$

$$x(2x-3)=5$$
$$2x^2-3x=5$$
$$2x^2-3x-5=0$$
$$(2x-5)(x+1)=0$$
$$2x-5=0 \quad \text{or} \quad x+1=0$$
$$x=\frac{5}{2} \quad \text{or} \quad x=-1$$

(b) $\dfrac{x}{x^2-4}=\dfrac{2}{x+2}$

$$\frac{x}{x^2-4}=\frac{2}{x+2}$$
$$x(x+2)=2(x^2-4)$$
$$x^2+2x=2x^2-8$$
$$0=x^2-2x-8$$
$$0=(x-4)(x+2)$$
$$x-4=0 \quad \text{or} \quad x+2=0$$
$$x=4 \quad \text{or} \quad x=-2$$
$$x\neq-2 \textit{ (Extraneous) } \implies x=4$$

*OR*

$$\frac{x}{x^2-4}=\frac{2}{x+2}$$
$$\frac{x}{(x-2)(x+2)}=\frac{2}{x+2}$$
$$\frac{x}{x-2}=2$$
$$x=2(x-2)$$
$$x=2x-4$$
$$-x=-4$$
$$x=4$$

6. (10 points) Showing all your work, solve the following:

(a) $|2x - 5| = 7$

*Because* $|x| = x$ *or* $|x| = -x$*, depending on the value of* $x$*, if* $|2x-5| = 7$*, then either* $2x - 5 = 7$ *or* $-(2x-5) = 7$*. But then...*

$$2x - 5 = 7$$
$$2x = 12$$
$$x = 6$$

*OR*

$$-(2x-5) = 7$$
$$-2x + 5 = 7$$
$$-2x = 2$$
$$x = -1$$

(b) $3 - x \geq 4x - 7$

$$3 - x \geq 4x - 7$$
$$3 \geq 5x - 7$$
$$10 \geq 5x$$
$$2 \geq x$$

OR

$$3 - x \geq 4x - 7$$
$$3 - 5x \geq -7$$
$$-5x \geq -10$$
$$x \leq 2$$

7. (8 points) Showing all your work, write the following as a single logarithm:

$$3\log_2(x) - \frac{1}{4}\log_2(y) + 5$$

$$3\log_2(x) - \frac{1}{4}\log_2(y) + 5$$

$$\log_2(x^3) - \log_2(y^{1/4}) + \log_2(2^5)$$

$$\log_2\left(\frac{2^5x^3}{y^{1/4}}\right)$$

$$\log_2\left(\frac{32x^3}{\sqrt[4]{y}}\right)$$

8. (8 points) Using the fact that $\log_b(x) = -3$, $\log_b(y) = 4$, and $\log_b(z) = 1$, find the exact value of the expression below—be sure to show all your work.

$$\log_b\left(\frac{x\sqrt{y}}{z}\right)$$

$$\log_b\left(\frac{x\sqrt{y}}{z}\right)$$

$$\log_b\left(x\sqrt{y}\right) - \log_b(z)$$

$$\log_b(x) + \log_b(\sqrt{y}) - \log_b(z)$$

$$\log_b(x) + \log_b(y^{1/2}) - \log_b(z)$$

$$\log_b(x) + \frac{1}{2}\log_b(y) - \log_b(z)$$

$$-3 + \frac{1}{2}\cdot 4 - 1$$

$$-3 + 2 - 1$$

$$-2$$

9. (10 points) Showing all your work, solve the following equations:

(a) $3e^{-5x} - 11 = 19$

$$3e^{-5x} - 11 = 19$$
$$3e^{-5x} = 30$$
$$e^{-5x} = 10$$
$$\ln(e^{-5x}) = \ln(10)$$
$$-5x = \ln(10)$$
$$x = -\frac{\ln(10)}{5} \approx -0.460517$$

(b) $2\log_{11} x - \log_{11} 4 = \log_{11} 25$

$$2\log_{11} x - \log_{11} 4 = \log_{11} 25$$
$$\log_{11} x^2 - \log_{11} 4 = \log_{11} 25$$
$$\log_{11}\left(\frac{x^2}{4}\right) = \log_{11} 25$$
$$11^{\log_{11}\left(\frac{x^2}{4}\right)} = 11^{\log_{11} 25}$$
$$\frac{x^2}{4} = 25$$
$$x^2 = 100$$
$$x = \pm 10$$

Because the domain of $\log_{11} x$ is $x > 0$, we have...

$$x = 10$$

OR

$$2\log_{11} x - \log_{11} 4 = \log_{11} 25$$
$$2\log_{11} x = \log_{11} 4 + \log_{11} 25$$
$$2\log_{11} x = \log_{11}(100)$$
$$\log_{11} x = \frac{1}{2}\log_{11}(100)$$
$$\log_{11} x = \log_{11} 10$$
$$11^{\log_{11} x} = 11^{\log_{11} 10}$$
$$x = 10$$

10. (10 points) Showing all your work, use the quadratic formula to solve the following:

$$3x^2 = 1 - 2x$$

***Solution.*** *We have...*

$$3x^2 = 1 - 2x$$

$$3x^2 + 2x - 1 = 0$$

*Therefore, we have $a = 3$, $b = 2$, and $c = -1$. But then...*

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(3)(-1)}}{2(3)} \\ &= \frac{-2 \pm \sqrt{4 + 12}}{6} \\ &= \frac{-2 \pm \sqrt{16}}{6} \\ &= \frac{-2 \pm 4}{6} \end{aligned}$$

*Therefore, either $x = \dfrac{-2 - 4}{6} = \dfrac{-6}{6} = -1$ or $x = \dfrac{-2 + 4}{6} = \dfrac{2}{6} = \dfrac{1}{3}$.*

11. (8 points) Consider the following function:

$$g(x) = \frac{(2x+5)(x-4)}{(x+2)(x-3)}$$

(a) Determine the domain of $g(x)$.

*The domain of a 'reduced' rational function $\frac{f(x)}{g(x)}$ is the $x$-values for which $g(x) \neq 0$. We see that $g(x)$ is reduced and $(x+2)(x-3) = 0$ if and only if $x = -2, 3$. Therefore, the domain is the set of all reals such that $x \neq -2, 3$, i.e. $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$.*

(b) Find the $x$-intercepts for $g(x)$.

*The $x$-intercepts of a 'reduced' rational function $\frac{f(x)}{g(x)}$ are the $x$-values such that $f(x) = 0$. Because $g(x)$ is 'reduced' and $(2x+5)(x-4) = 0$ if and only if $x = -\frac{5}{2}, 4$, the $x$-intercepts of $g(x)$ are $x = -\frac{5}{2}, 4$.*

(c) Find any vertical asymptotes for $g(x)$.

*The vertical asymptotes of a 'reduced' rational function $\frac{f(x)}{g(x)}$ are the $x$-values for which $g(x) = 0$. We see that $g(x)$ is reduced and $(x+2)(x-3) = 0$ if and only if $x = -2, 3$. Therefore, the vertical asymptotes of $g(x)$ are $x = -2$ and $x = 3$.*

(d) Find any horizontal asymptotes for $g(x)$.

*We know the horizontal asymptote (if any) of a rational function $\frac{f(x)}{g(x)}$ depends only on the degrees of $f(x)$ and $g(x)$. The numerator and denominator of $g(x)$ have degree two. Therefore, the horizontal asymptote is the ratio of the leading coefficients. We have $g(x) = \frac{(2x+5)(x-4)}{(x+2)(x-3)} = \frac{2x^2-3x-20}{x^2-x-6}$. Therefore, the horizontal asymptotes is $y = 2$.*