Elliptic Tales:
A Story of Bitcoin, Moonshine, String Theory, and Clocks

March 14, 2025

Reference Sheet

- **Arithmetic Geometry.** Broadly speaking, arithmetic geometry is essentially the study of 'nice solutions' to 'nice equations.' What is typically meant by this is Diophantine equations, i.e. integer solutions to equations in polynomials with integer coefficients. However, the term is often meant broader to systems of equations in polynomials with coefficients coming from rings with a lot of 'interesting arithmetic structure.'

- Common questions asked in the study of equations to arithmetic geometers: are there integer, rational, etc. solutions to an equation, if there are solutions then how many, can we find the solutions, can we describe (parametrize) the solutions, can we find an algorithm to compute all the solutions, etc.

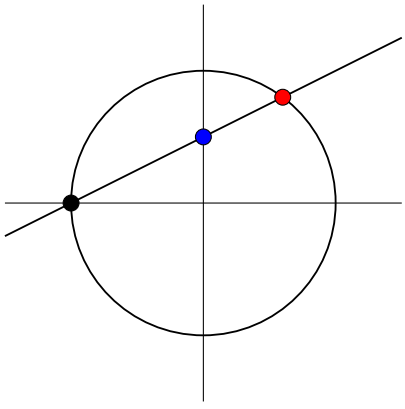
  Theorem (Rational Roots Theorem)**.** *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_i \in \mathbb{Z}$ and $a_0, a_n \neq 0$. Then the only rational solutions to $f(x) = 0$ have $x = p/q$, where p is an integer factor of a_0 and q is an integer factor of a_n.*

  Theorem (Linear Diophantine Equations in Two Variables)**.** *Let a, b, c be integers with $a, b \neq 0$ and let $d = \gcd(a, b)$. The equation $ax + by = c$ has integer solutions if and only if $d \mid c$. If so, the equation has infinitely many solutions and all solutions have the form...*

  $$x = x_0 + \frac{b}{d}\,k, \quad y = y_0 - \frac{a}{d}\,k,$$

  where (x_0, y_0) is a solution and k is an integer.

- **Finding Rational Points on Conics.** If there is a rational point on a conic section, we can find all the rational points on the conic and parametrize them all (except one) by 'projecting' this point onto a line. Take the rational parametrization of the circle $x^2 + y^2 = 1$ using the point $(-1, 0)$ as our rational point and projecting onto the y-axis (the line $x = 0$); that is, connect $(-1, 0)$ to a point $(0, t)$ on the line $x = 0$ using a line, where $t \in \mathbb{Q}$.



$$C(\mathbb{Q}) = \{(-1,0)\} \cup \left\{ \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) : t \in \mathbb{Q} \right\}$$

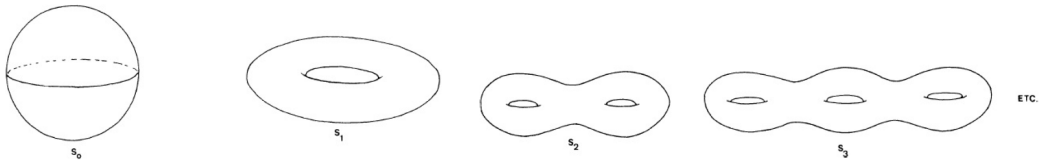
Principle (Hasse, Local-Global Principle)**.** A collection of equations has a solution 'if and only if' it has a solution in $\mathbb{R}$ and $\mathbb{Q}_p$ for all p.

Theorem 0.1 (Mordell, 1922, Faltings, 1983)**.** *If $\mathcal{C}$ is a curve over $\mathbb{Q}$ of genus g at least 2, then $\mathcal{C}$ has at most finitely many rational points.*

- **Genus.** If $\mathcal{C}$ is a nonsingular, projective plane curve of degree d is given by...

$$g = \frac{(d-1)(d-2)}{2}$$

For a non-singular hypersurface H of degree d in $\mathbb{P}^n$, we have $g = \binom{d-1}{n}$. Topologically, the genus is the number of 'holes.'



- **Elliptic Curve.** An elliptic curve is...
  - A nonsingular projective curve of genus 1.
  - An abelian variety of dimension 1.
  - A nonempty smooth variety, $V(F)$, with $\deg F = 3$.
  - A compact Riemann surface of genus 1.
  - The set of solutions to...

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

    along with a specified 'distinguished point ∞' and an addition law given by the chord-tangent law.

- One can simply the last definition of an elliptic curve (called the Weierstrass form) to what is called the short Weierstrass form:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

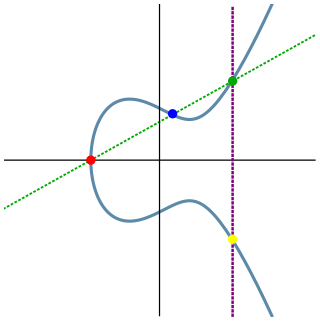
  - Make the substitution $y \mapsto y + \dfrac{a_1x + a_3}{2}$.
  - Obtain $y^2 = x^3 + a_2'x^2 + a_4'x + a_6'$
  - Make the substitution $x \mapsto x + \dfrac{a_2'}{3}$

$$\star \quad \boxed{E_{A,B} : y^2 = x^3 + Ax + B} \quad \star$$

- The set of rational points on an elliptic curve (ignoring the point at infinity) can be empty, finite, or infinite.
- **CM (Complex Multiplication).** We say that an elliptic curve E has complex multiplication (CM) or is CM if the endomorphism ring of E is strictly larger than $\mathbb{Z}$. [The endomorphism ring, i.e. the ring of ring maps $E \to E$, always contains $\mathbb{Z}$ because we always have a map $E \to E$ give by multiplication by n, i.e. the map $P \mapsto nP$.]
- **j-invariant.** Take an elliptic curve $y^2 = x^3 + Ax + B$. The transformations which preserve this equations are: $x = \mu^2 x$ and $y = \mu^3 y$ for $\mu \in \overline{K}^\times$. We then define the j-invariant

  $$j = 1728\frac{4A^3}{4A^4 + 27B^2}$$

  These classify elliptic curves up to isomorphism over $\overline{K}$.
- **Addition Law for Elliptic Curves.** To add two rational points on an elliptic curve (with the identity being the point at infinity and in short Weierstrass form), draw the line through the two points (or the tangent line in the case of a single distinct point) and find the intersection of the elliptic curve with this line (if the line is vertical the sum is the point at infinity). Reflect this point across the x-axis. This final point is the sum of the two rational points. By construction, it is immediate that this operation is abelian (you get the same line no matter the order of the points!).



- **Field.** A field is a collection of 'numbers' where one can perform addition, subtraction, multiplication, and (nonzero) division and these operations are 'nice.'
- **Galois Field.** A Galois field is a 'number field' whose numbers satisfy extra symmetries.

  Theorem (Mordell-Weil-Néron, 1952)**.** *Let K be a field that is finitely generated over its prime field, and let A/K be an abelian variety. Then the group of K-rational points on A, denoted $A(K)$, is a finitely generated abelian group. In particular,*

  $$A(K) \cong \mathbb{Z}^{r_K} \oplus A(K)_{tors},$$

  where $r_K \geq 0$ is the rank and $A(K)_{tors}$ is the torsion subgroup.
- By the above, every elliptic curve is isomorphic to a group of the form $\mathbb{Z}^r \oplus E_{\text{tors}}$, where r is the rank of E and E_{tors} is the torsion subgroup. In fact, every elliptic curve is isomorphic to a group of the form...

  $$E \cong \mathbb{Z}^r \oplus \left(\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/nm\mathbb{Z}\right)$$

  for some r, n, m.

- The largest known rank for an elliptic curve is 28 due to Noam Elkies. It is known that 50% of elliptic curves have rank 0 and 50% have rank 1. 'Most' elliptic curves are torsion-free, meaning the only point of finite order on E is the identity.

- The collection of all points of order dividing n over $\mathbb{C}$ on E is denoted $E[n]$. In fact, $E[n] \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$.

  Theorem (Levi-Ogg Conjecture; Mazur, 1977)**.** *If $E/\mathbb{Q}$ is a rational elliptic curve, then the possible torsion subgroups $E(\mathbb{Q})_{tors}$ are precisely:*

  $$\begin{cases} \mathbb{Z}/n\mathbb{Z}, & \text{with } n = 1, 2, \ldots, 10, 12 \text{ or} \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2n\mathbb{Z}, & \text{with } n = 1, \ldots, 4 \end{cases}$$

  Furthermore, each possibility occurs infinitely often.

- **Isogeny.** Let E_1, E_2 be elliptic curves. An isogeny from E_1 to E_2 is a morphism $\phi : E_1 \to E_2$ with $\phi(\mathcal{O}) = \mathcal{O}$. If $|\ker \phi| = n$, we say ϕ is an n-isogeny.

  Theorem (Fricke, Kenku, Klein, Kubert, Ligozat, Mazur, Ogg, et al.)**.** *If $E/\mathbb{Q}$ has an n-isogeny over $\mathbb{Q}$, then*

  $$n \in \{1, 2, \ldots, 19, 21, 25, 27, 37, 43, 67, 163\}.$$

  If E does not have CM, then $n \leq 18$ or $n \in \{21, 25, 37\}$.