MATH 122: Exam 3 **Name:** *Caleb M^{c}Whorter — Solutions*

Fall — 2024

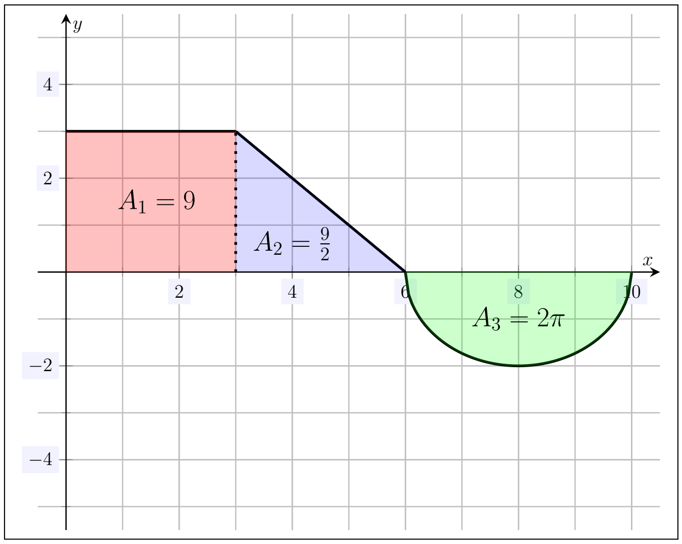
11/21/2024

75 Minutes

Write your name on the appropriate line on the exam cover sheet. This exam contains 9 pages (including this cover page) and 8 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	15	
2	15	
3	15	
4	10	
5	10	
6	10	
7	15	
8	10	
Total:	100	

1. (15 points) Consider the plot of a function $f(x)$ given below.



Using the above plot, compute the following:

(a) $\displaystyle\int_0^3 f(x)\,dx = A_1 = bh = 3(3) = 9$

(b) $\displaystyle\int_6^{10} f(x)\,dx = -A_3 = -\frac{1}{2}\pi r^2 = -\frac{1}{2}\cdot\pi(2^2) = -\frac{1}{2}\cdot 4\pi = -2\pi \approx -6.28319$

(c) $\displaystyle\int_0^{10} f(x)\,dx = A_1 + A_2 + (-A_3) = 9 + \frac{9}{2} - 2\pi = \frac{27}{2} - 2\pi \approx 7.21681$

(d) $\displaystyle\int_5^5 f(x)\,dx = 0$

(e) The area between $f(x)$ and the x-axis.

$$A_1 + A_2 + A_3 = 9 + \frac{9}{2} + 2\pi = \frac{27}{2} + 2\pi \approx 19.7832$$

2. (15 points) A jet plane is descending for a landing. The velocity in mph at a time t minutes from the start of its descent, $v(t)$, is given in the table below.

Time, t	0	5	10	15	20
Velocity, $v(t)$	550	520	500	490	410

(a) Using the table above and a right-hand sum, approximate $\int_0^{20} v(t)\,dt$ as accurately as possible.

We can create a rough sketch of the right-hand sum:

Computing the total area of these rectangles, we have...

$$\int_0^{20} v(t)\,dt \approx 5(520)+5(500)+5(490)+5(410) = 2600+2500+2450+2050 = 9600$$

(b) What does your value in (a) represent?

We know that $v(t)$ has units of miles per hour. But then $\int v(t)\,dt$ has units of mph · hr = miles. We know the integral of a rate over an interval gives the net change of that function over that interval. Therefore, $\int_0^{20} v(t)\,dt \approx 9600$ represents that the plane has approximately traveled an additional 9,600 miles.

(c) Based on the table is your approximation in (a) likely an under- or over-estimate?

Because the speed seems to be consistently decreasing, we make the assumption the speed is only decreasing. But then using the right-endpoint would use the lowest possible speed on each time interval. Therefore, we get the smallest possible area, i.e. the smallest possible increase to the net distance traveled. Therefore, the approximation in (a) is likely an underestimate.

3. Showing all your work, compute the following:

(a) (5 points) $\int (\sqrt{x} + e^x)\, dx$

$$\int (\sqrt{x} + e^x)\, dx = \int \left(x^{1/2} + e^x\right)\, dx = \frac{x^{3/2}}{3/2} + e^x + C = \frac{2}{3}x^{3/2} + e^x + C$$

(b) (5 points) $\int_{-1}^{1} \left(x^3 - 6x^2\right)\, dx$

$$\int_{-1}^{1} \left(x^3 - 6x^2\right)\, dx = \frac{x^4}{4} - \frac{6x^3}{3}\Big|_{-1}^{1} = \frac{x^4}{4} - 2x^3\Big|_{-1}^{1} = \left(\frac{1}{4} - 2(1)\right) - \left(\frac{1}{4} - 2(-1)\right) = -\frac{7}{4} - \frac{9}{4} = -\frac{16}{4} = -4$$

(c) (5 points) $\int \left(5^x - \frac{2}{x}\right)\, dx$

$$\int \left(5^x - \frac{2}{x}\right)\, dx = \frac{5^x}{\ln 5} - 2\ln|x| + C$$

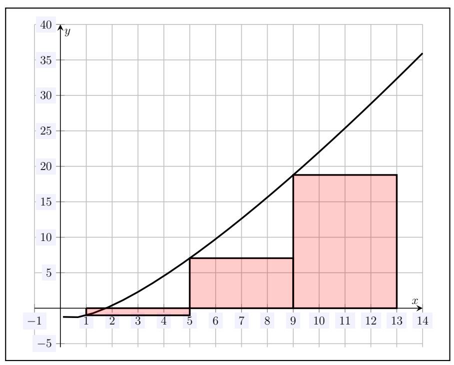
4. (10 points) Showing all your work, use a left-hand sum with three evenly spaced rectangles to approximate the following:

$$\int_1^{13} (x \ln x - 1)\ dx$$

Solution. *Because the integral begins at $a = 1$, ends at $b = 13$, and we will use $n = 3$ evenly spaced rectangles, each rectangle must have width...*

$$\Delta x = \frac{b-a}{n} = \frac{13-1}{3} = \frac{12}{3} = 4$$

We can even create a sketch to help visualize the integral—the accuracy of the actual function curve is not important:



Letting $f(x) = x \ln x - 1$, the left-hand sum is...

$$\int_1^{13} (x \ln x - 1)\ dx \approx 4f(1) + 4f(5) + 4f(9)$$

$$= 4(1\ln(1) - 1) + 4(5\ln(5) - 1) + 4(9\ln(9) - 1)$$

$$= 4(-1) + 4(7.04719) + 4(18.775)$$

$$= -4 + 28.1888 + 75.1$$

$$= 99.2888$$

Note. The actual value of the given integral is approximately $\int_1^{13} (x \ln x - 1)\ dx \approx 162.738$. The approximation is above has only a 38.99% error using only three rectangles.

5. Showing all your work, compute the following:

(a) (5 points) $\displaystyle\int \frac{x^3 - 5x^2 + 6}{x^2}\, dx$

$$\int \frac{x^3 - 5x^2 + 6}{x^2}\, dx = \int \left(\frac{x^3}{x^2} - \frac{5x^2}{x^2} + \frac{6}{x^2}\right)\, dx = \int \left(x - 5 + 6x^{-2}\right)\, dx = \frac{1}{2}\,x^2 - 5x - 6x^{-1} + C$$

(b) (5 points) $\displaystyle\int (x^4 + 3)^2\, dx$

$$\int (x^4+3)^2\, dx = \int (x^4+3)(x^4+3)\, dx = \int (x^8+6x^4+9)\, dx = \frac{1}{9}\,x^9 + \frac{6}{5}\,x^5 + 9x + C$$

6. Showing all your work, use u-substitution to compute the following:

(a) (5 points) $\displaystyle\int x\sqrt{3x^2+5}\,dx$

We have...

$$u = 3x^2+5 \qquad \begin{aligned} du &= 6x\;dx \\ dx &= \frac{1}{6x}\,du \end{aligned}$$

Therefore, we have...

$$\begin{aligned}
\int x\sqrt{3x^2+5}\;dx &= \int x\sqrt{u}\cdot\frac{1}{6x}\;du \\
&= \int \frac{1}{6}\sqrt{u}\;du \\
&= \frac{1}{6}\int u^{1/2}\;du \\
&= \frac{1}{6}\cdot\frac{2}{3}u^{3/2}+C \\
&= \frac{1}{9}u^{3/2}+C \\
&= \frac{1}{9}\left(3x^2+5\right)^{3/2}+C
\end{aligned}$$

(b) (5 points) $\displaystyle\int_0^4 (5-2x)^3\;dx$

We have...

$$u = 5-2x$$

$$du = -2\;dx \qquad x=0\colon\; u = 5-2(0) = 5-0 = 5$$

$$dx = -\frac{1}{2}\,du \qquad x=4\colon\; u = 5-2(4) = 5-8 = -3$$

Therefore, we have...

$$\int_0^4 (5-2x)^3\;dx = \int_5^{-3} u^3\cdot -\frac{1}{2}\;du = -\frac{1}{2}\int_5^{-3} u^3\;du = -\frac{1}{2}\cdot\frac{u^4}{4}\bigg|_5^{-3} = -\frac{1}{8}\,u^4\bigg|_5^{-3}$$

But then, we have...

$$\int_0^4 (5-2x)^3\;dx == -\frac{1}{8}\left((-3)^4-5^4\right) = -\frac{1}{8}\,(81-625) = -\frac{1}{8}\cdot -544 = 68$$

7. (15 points) The marginal cost of producing q dill pickle scented candles is given by $C'(q) = 1.5 + \dfrac{100}{2q+1}$. The total cost to produce 6,000 candles is \$34,500.

   (a) Find the cost function, $C(q)$.

   (b) What are the fixed costs?

   (c) Find the cost to produce 20,000 candles?

(a) *We know that up to a constant, the total cost function is given by* $C(q) = \int C(q)\, dq$. *We have...*

$$C(q) = \int C'(q)\, dq = \int \left(1.5 + \frac{100}{2q+1}\right) dq = \int 1.5\, dq + \int \frac{100}{2q+1}\, dq$$

We know that $\int 1.5\, dq = 1.5q$. *For the second integral, we use* u*-substitution: choose* $u = 2q+1$, *so that* $du = 2\, dq$. *But then* $dq = \frac{1}{2}\, du$. *But then...*

$$\int \frac{100}{2q+1}\, dq = \int \frac{100}{u} \cdot \frac{1}{2}\, du = \int \frac{50}{q}\, dq = 50 \ln|u| = 50 \ln|2q+1|$$

Therefore, we have...

$$C(q) = \int C'(q)\, dq = 1.5q + 50 \ln|2q+1| + K$$

where K *is a constant. But we know that 6,000 candles has a total production cost of #34,500, i.e.* $C(6000) = 34500$. *Therefore, we have...*

$$\begin{aligned} C(6000) &= 34500 \\ 1.5(6000) + 50 \ln|2(6000)+1| + K &= 34500 \\ 9000 + 50 \ln(12001) + K &= 34500 \\ 9000 + 469.64 + K &= 34500 \\ 9469.64 + K &= 34500 \\ K &= 25030.36 \end{aligned}$$

Therefore, we know...

$$C(q) = 1.5q + 50 \ln|2q+1| + 25030.36$$

(b) *We know the fixed costs are* $C(0)$. *But then...*

$$\textit{Fixed Costs} = C(0) = 1.5(0) + 50 \ln(1) + 25030.36 = \$25{,}030.36$$

(c) *The cost to produce 20,000 candles is...*

$$C(20000) = 1.5(20000) + 50 \ln|40001| + 25030.36 = \$55{,}560.19$$

8. (10 points) A fresh brewed, decaf venti cup of coffee with 3 squirts of vanilla, 8 pumps of caramel, 5 sugars, and almond milk is at 195°F. The rate of change in temperature of the coffee, measured in degrees Fahrenheit per minute, t minutes from now is given by $r(t) = -15e^{-0.25t}$. Estimate the coffee's temperature after 30 minutes. Be sure to show all your work.

Solution. *We know that…*

$$\textit{Current Temperature} = \textit{Initial Temperature} + \textit{Change in Temperature}$$

We know the net change in the temperature of the coffee can be computed using the integral. Using the fact that $\int e^{kt}\,dt = \frac{1}{k}e^{kt} + C$, *we have…*

$$\begin{aligned}
\textit{Change in Temperature} &= \int_0^{30} r(t)\,dt \\
&= \int_0^{30} -15e^{-0.25t}\,dt \\
&= -15\int_0^{30} e^{-0.25t}\,dt \\
&= -15 \cdot \frac{1}{-0.25}\,e^{-0.25t}\Big|_0^{30} \\
&= 60 \cdot e^{-0.25}\Big|_0^{30} \\
&= 60 \cdot \left(e^{-0.25\cdot 30} - e^{-0.25\cdot 0}\right) \\
&= 60 \cdot \left(e^{-7.5} - e^{0}\right) \\
&= 60 \cdot \left(e^{-7.5} - e^{0}\right) \\
&= 60 \cdot (0.000553084 - 1) \\
&= 60 \cdot - - 0.999446916 \\
&= -59.96681496 \approx -59.97
\end{aligned}$$

Therefore, we have…

$$\begin{aligned}
\textit{Current Temperature} &= \textit{Initial Temperature} + \textit{Change in Temperature} \\
&= 195^\circ\text{F} - 59.97^\circ\text{F} \\
&= 135.03^\circ\text{F}
\end{aligned}$$