Bonus Problems

Santa now gone, Krampus comes with a sneer.
"Forget New Years, tough problems are here!
Panic sets in, and there's no joy in sight.
Face these hard problems, which require great might.

Problem A. Let $\mathcal{R}$ be the region bounded by the curves $y = 5x^2$ and $y = 5x^3$. Consider the volume given by...

(a) an object whose base is $\mathcal{R}$ and whose cross-sections perpendicular to the x-axis are squares.

(b) an object formed by rotating $\mathcal{R}$ about the x-axis.

(c) an object formed by rotating $\mathcal{R}$ about the y-axis.

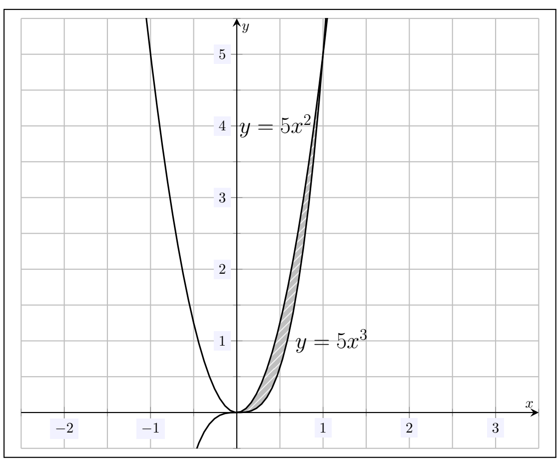
Choose ***one*** of (a), (b), or (c) and set-up—***but do not evaluate***—an integral which computes the volume of the object in the part you have chosen. Indicate which one you have chosen by circling the part.

Problem B. Showing all your work, compute the following...

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \sin\left(\frac{\pi k}{n}\right)$$

Solutions.

Problem A.



$$\begin{aligned} 5x^2 &= 5x^3 \\ 0 &= 5x^3 - 5x^2 \\ 0 &= 5x^2(x-1) \end{aligned}$$

But then either $5x^2 = 0$, which implies $x = 0$, or $x - 1 = 0$, which implies $x = 1$. Therefore, the curves intersect at $x = 0$ and $x = 1$. If $x = 0$, then $y = 5(0^2) = 0$, i.e. $(0,0)$, and if $x = 1$, then $y = 5(1^2) = 5$, i.e. $(1,5)$. Choosing $x = \frac{1}{2}$, we can see that $5(\frac{1}{2})^2 = \frac{5}{4} = \frac{10}{8} > 5(\frac{1}{2})^3 = \frac{5}{8}$. Therefore, the curve $y = 5x^2$ is 'on top.' Finally, observe that for this region, $x, y \geq 0$. Therefore, $y = 5x^2$ if and only if $x = \frac{y}{5}$, and $y = 5x^3$ if and only if $x = \sqrt[3]{\frac{y}{5}}$.

(a)

$$V = \int_0^1 (5x^2 - 5x^3)^2 \, dx$$

(b)

$$V = \pi \int_0^1 (5x^2)^2 - (5x^3)^2 \, dx \qquad \textit{OR} \qquad V = 2\pi \int_0^5 y\left(\sqrt[3]{\frac{y}{5}} - \sqrt{\frac{y}{5}}\right) dy$$

(c)

$$V = 2\pi \int_0^1 x(5x^2 - 5x^3) \, dx \qquad \textit{OR} \qquad V = \pi \int_0^5 \left(\sqrt[3]{\frac{y}{5}}\right)^2 - \left(\sqrt{\frac{y}{5}}\right)^2 dy$$

Problem B. *First, we rewrite the summation:*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\sin\left(\frac{\pi k}{n}\right)=\lim_{n\to\infty}\frac{1}{\pi}\cdot\frac{\pi}{n}\sum_{k=0}^{n-1}\sin\left(\frac{\pi k}{n}\right)=\lim_{n\to\infty}\frac{1}{\pi}\cdot\frac{\pi-0}{n}\sum_{k=0}^{n-1}\sin\left(\frac{\pi-0}{n}k\right)$$

Recognizing $\Delta x := \frac{\pi-0}{n}$ *as* $\Delta x := \frac{b-a}{n}$ *as a step-size, i.e. taking* $b=\pi$, $a=0$, *and* $n=n$, *and given the argument of* $\sin(x)$ *is* $\Delta x\cdot k$ *from* $k=0$ *to* $k=n-1$, *we can recognize this limit as the left-hand Riemann sum with equal widths for* $\sin(x)$ *from* $x=0$ *to* $x=\pi$. *Therefore, we have...*

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\sin\left(\frac{\pi k}{n}\right)&=\lim_{n\to\infty}\frac{1}{\pi}\cdot\frac{\pi-0}{n}\sum_{k=0}^{n-1}\sin\left(\frac{\pi-0}{n}k\right)\\&=\frac{1}{\pi}\int_0^{\pi}\sin(x)\,dx\\&=\frac{1}{\pi}\cdot-\cos x\Bigg|_{x=0}^{x=\pi}\\&=\frac{1}{\pi}\cdot\Big(-\cos\pi-(-\cos 0)\Big)\\&=\frac{1}{\pi}\cdot\Big(-(-1)-(-1)\Big)\\&=\frac{1}{\pi}\Big(1+1\Big)\\&=\frac{2}{\pi}\end{aligned}$$