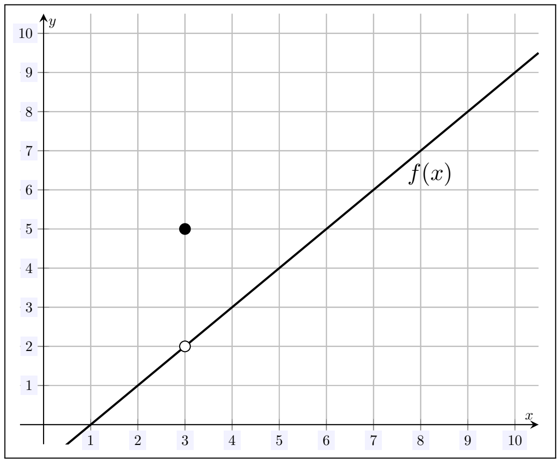
Check-In 01/15. *(True/False) True/False*: If $f(3) = 5$, then $\lim_{x \to 3} f(x) = 5$.

Solution. The statement is *false*. Recall that the limit of a function (if it exists) is what the output gets 'close' to as the input gets 'close' to its limiting value. The fact that $f(3) = 5$ does not mean the outputs are all 'close' to 5 when x is 'close' to 3. For instance, consider the function $f(x)$ plotted below.



Despite the fact that $f(3) = 5$, $\lim_{x \to 3} f(x) = 2$ because all the outputs are 'close' to 2 when the inputs are 'close' to 3.

Check-In 01/17. *(True/False) True/False*: Let $f(x)$ be a function defined on all real numbers such that $\lim_{x \to \pi} f(x) = 10$. Then it must be that $\lim_{x \to \pi^+} f(x) = 10$.

Solution. The statement is *true*. Recall that the limit (if it exists) is what the output gets 'close' to as the input gets 'close' to its limiting value. Because $\lim_{x \to \pi} f(x) = 10$, the outputs of $f(x)$ are all 'close' to 10 whenever x is 'close' to π—no matter how x is 'close' to π. The right-hand limit $\lim_{x \to \pi^+} f(x)$ asks what the outputs are 'close' to if x is 'close' to π—but bigger than π. But we already know that the outputs are 'close' to 10. Therefore, it must be that $\lim_{x \to \pi^+} f(x) = 10$. Recall that $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

Check-In 01/22. *(True/False)* $\lim_{x \to \infty} \left(1 + \frac{1}{3x}\right)^x = e^3$

Solution. The statement is *false*. Recall that $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$. But then...

$$\lim_{x \to \infty} \left(1 + \frac{1}{3x}\right)^x = \lim_{x \to \infty} \left(1 + \frac{1}{3x}\right)^{x \cdot 3/3} = \lim_{x \to \infty} \left[\left(1 + \frac{1}{3x}\right)^{3x}\right]^{1/3} = e^{1/3} = \sqrt[3]{e}$$

Check-In 01/24. *(True/False)* $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = (1+0)^\infty = 1^\infty = 1$

Solution. The statement is *false*. One does obtain 1^∞ after naïvely plugging in $x=\infty$. However, ∞ is not a number; moreover, although one might feel otherwise, it is simply need not be the case that $1^\infty = 1$. Indeed, 1^∞ is an indeterminant form. One could correctly recall that...

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$$

Check-In 01/27. *(True/False)* The function $f(x) = \dfrac{e^x \sin(\sqrt[3]{x})}{x^2+6x+9}$ is continuous on any interval which does not contain $x=-3$.

Solution. The statement is *true*. We know that e^x, $\sin x$, $\sqrt[3]{x}$, and x^2+6x+9 are everywhere continuous. But then $\sin(\sqrt[3]{x})$ is everywhere continuous, because it is a composition of continuous functions. This makes $e^x \sin(\sqrt[3]{x})$ continuous, because it is the product of continuous functions. But then $f(x) = \dfrac{e^x \sin(\sqrt[3]{x})}{x^2+6x+9}$ is continuous so long as $x^2+6x+9 \neq 0$, because it would be a quotient of continuous functions. Observe that $x^2+6x+9 = (x+3)^2$. Therefore, if $x^2+6x+9=0$, then $(x+3)^2 = 0$ so that $x=-3$. Therefore, $f(x)$ is continuous on any interval not containing -3.

Check-In 01/29. *(True/False)* The limit $\lim_{x\to 0} \dfrac{\sqrt{x+9}-3}{x}$ represents $f'(9)$, where $f(x)=\sqrt{x}$.

Solution. The statement is *true*. The definition of the derivative at $x=a$ is $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$. Taking $f(x)=\sqrt{x}$ and $a=9$, we would have $f'(9) = \lim_{h\to 0} \dfrac{f(9+h)-f(9)}{h} = \lim_{h\to 0} \dfrac{\sqrt{9+h}-\sqrt{9}}{h} = \lim_{h\to 0} \dfrac{\sqrt{h+9}-3}{h}$. This is the same as the given limit with the role of h and x interchanged.

Check-In 01/31. *(True/False)* $\dfrac{d}{dx} \sin(\ln x) = \cos\left(\dfrac{1}{x}\right)$

Solution. The statement is *false*. We have a derivative of a composition of functions. This requires chain rule: $\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big) \cdot g'(x)$. Here, we have $f(x)=\sin x$ and $g(x)=\ln x$. The correct derivative should be...

$$\frac{d}{dx} \sin(\ln x) = \cos(\ln x) \cdot \frac{1}{x}$$

Here, the 'rule' $\frac{d}{dx} f\big(g(x)\big) = f'\big(g'(x)\big)$ has been applied, which is incorrect.

Check-In 02/03. *(True/False)* $\dfrac{d}{dx}\, x^3 \sin(x) = 3x^2 \cos x$

Solution. The statement is *incorrect*. We have a derivative of a product of functions. This requires the product rule: $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$. Here, we have $f(x) = x^3$ and $g(x) = \sin x$. The correct derivative should be…

$$\frac{d}{dx}\, x^3 \sin x = 3x^2 \cdot \sin x + x^3 \cdot \cos x$$

Here, the 'rule' $\frac{d}{dx} f(x)g(x) = f'(x)g'(x)$ has been applied, which is incorrect.

Check-In 02/05. *(True/False)* If θ_d is an angle measured in degrees, then $\dfrac{d}{d\theta}\, \sin(\theta_d) = \cos(\theta_d)$.

Solution. The statement is *false*. This is a problem which can arise in working with derivatives with 'mixed units' or especially when programming computer systems to perform the computations and one is not paying attention to the units. Trigonometric functions should be computed using radians. Even if one wishes to use degrees, the units would need to be consistent. Clearly, $\frac{d}{d\theta}$ is the derivative with respect to θ—measured in radians. However, the input to the trigonometric function is in degrees. We would need to convert this input to radians by multiplying by $\frac{\pi}{180}$. But then…

$$\frac{d}{d\theta}\, \sin(\theta_d) = \frac{d}{d\theta}\, \sin\left(\theta \cdot \frac{\pi}{180}\right) = \frac{\pi}{180} \cdot \cos\left(\theta \cdot \frac{\pi}{180}\right) = \frac{\pi}{180} \cos\left(\theta_d\right)$$

One can work out the derivation of $\frac{d}{d\theta} \sin\theta$ to see the reliance on radians to get a deeper—non-chain rule—reason for why this is the case.

Check-In 02/12. *(True/False)* If $f'(1) = 0$ and $f''(1) < 0$, then $f(x)$ has a local minima at $x = 1$.

Solution. The statement is *false*. Suppose that $f(x)$ is a twice-differentiable function. If $f'(a) = 0$, then $x = a$ is a critical value for $f(x)$. By the Second Derivative Test, if $f''(a) < 0$, then $x = a$ is a local maxima. If $f''(a) > 0$, then $x = a$ is a local minima. However, if $f''(a) = 0$, then the test is inconclusive. Because $f'(1) = 0$, we know that $x = 1$ is a critical value. Because $f''(1) < 0$, then $x = 1$ is a local maxima.

Check-In 02/14. *(True/False)* If $f(x)$ is differentiable at $x = a$, then the linearization of $f(x)$ at $x = a$ and the tangent line to $f(x)$ at $x = a$ are the same thing.

Solution. The statement is *true*. Recall that the tangent line for $f(x)$ at $x = x_0$ (if it exists) is the line $y = f(x_0) + m(x - x_0)$, where $m = f'(x_0)$. However, the linearization of $f(x)$ at $x = a$ is $\ell(x) = f(x_0) + f'(x_0)\big(x - x_0\big)$. Observe that these are the same line.

Check-In 02/17. *(True/False)* A point where $f''(x) = 0$ is called an inflection point.

Solution. The statement is *false*. An inflection point is a point where $f(x)$ changes concavity. Consider the function $f(x) = x^4$. Observe that $f'(x) = 4x^3$ and $f''(x) = 12x^2$. We have $f''(0) = 12(0^2) = 0$. However, $f''(x) = 12x^2 \geq 0$ for all x. But then $f''(x)$ can never change sign, so that $f(x)$ has no points of inflection. This contradicts the assertion in the given statement. While points where $f''(x) = 0$ *can* be points of inflection, they do not *have* to be.

Check-In 02/19. *(True/False)* $\displaystyle\lim_{x\to 2} \frac{x-2}{x^2-4} \overset{\text{L.H.}}{=} \lim_{x\to 2} \frac{1}{2x} = \frac{1}{4}$

Solution. The statement is *true*. Recall that by l'Hôpital's Rule, if $\lim \dfrac{f(x)}{g(x)}$ is $\dfrac{0}{0}$ or $\pm\dfrac{\infty}{\infty}$ and $\lim \dfrac{f'(x)}{g'(x)}$ exists, then $\lim \dfrac{f(x)}{g(x)} = \lim \dfrac{f'(x)}{g'(x)}$. Observe that 'plugging in' $x = 2$ into $\dfrac{x-2}{x^2-4}$, we obtain $\dfrac{0}{0}$. By l'Hôpital's Rule, we then have that this limit is the same as $\displaystyle\lim_{x\to 2} \frac{1}{2x} = \frac{1}{2(2)} = \frac{1}{4}$.

Check-In 02/21. *(True/False)* $\displaystyle\lim_{x\to 1} \frac{4-x}{x^2-1} \overset{\text{L.H.}}{=} \lim_{x\to 1} \frac{-1}{2x} = \frac{-1}{2(1)} = -\frac{1}{2}$

Solution. The statement is *false*. Recall that by l'Hôpital's Rule, if $\lim \dfrac{f(x)}{g(x)}$ is $\dfrac{0}{0}$ or $\pm\dfrac{\infty}{\infty}$ and $\lim \dfrac{f'(x)}{g'(x)}$ exists, then $\lim \dfrac{f(x)}{g(x)} = \lim \dfrac{f'(x)}{g'(x)}$. However, 'plugging in' $x = 1$, we obtain $\frac{3}{0}$, which is neither of these forms. In fact, $\displaystyle\lim_{x\to 1} \frac{4-x}{x^2-1}$ does not exist because $\displaystyle\lim_{x\to 1^+} \frac{4-x}{x^2-1} = -\infty$ and $\displaystyle\lim_{x\to 1^+} \frac{4-x}{x^2-1} = \infty$.

Check-In 02/24. *(True/False)* Implicitly differentiating $4x+y^2+e^{xy} = y$ yields $4+2y+e^{xy}\cdot y+y' = y'$.

Solution. The statement is *false*. Recall that when implicitly differentiating y with respect to x, we have $1 \cdot \dfrac{dy}{dx} = y'$. Therefore, we have...

$$4x + y^2 + e^{xy} = y$$

$$\frac{d}{dx}\left(4x + y^2 + e^{xy}\right) = \frac{d}{dx}\, y$$

$$4 + 2y \cdot \frac{dy}{dx} + e^{xy} \cdot \left(1 \cdot y + x \cdot \frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$4 + 2y \cdot y' + e^{xy} \cdot \left(y + xy'\right) = y'$$

Check-In 02/26. *(True/False)* If the radius of a sphere is decreasing at a rate of 1 meter per minute and is currently 2 m across, the rate of change of the volume of the sphere is 16π.

Solution. The statement is *false*. Recall that the volume of a sphere is $V = \frac{4}{3}\pi r^3$. Because the sphere is 2 m across, the diameter is 2 m, i.e. the radius is $r = 1$ m. We know that the radius of the sphere is decreasing at a rate of 1 m per minute, i.e. $\frac{dr}{dt} = -1$. But then...

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ \frac{d}{dt}V &= \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \\ \frac{dV}{dt} &= 4\pi r^2 \cdot \frac{dr}{dt} \\ \frac{dV}{dt} &= 4\pi(1^2) \cdot -1 \\ \frac{dV}{dt} &= -4\pi \end{aligned}$$

Therefore, the volume of the sphere is decreasing at 4π m^3 per minute. The given answer must be incorrect as the rate of change of the volume of the sphere is positive—whereas we know the volume must be decreasing. The given answer used $r = 2$ rather than the correct value $r = 1$.

Check-In 03/03. *(True/False)* If one is optimizing a function $f(x)$ on an interval $[a, b]$, if there are not critical values, then $f(x)$ has no absolute maximum or minimum on $[a, b]$.

Solution. The statement is *false*. Consider the function $f(x) = x$ on the interval $[-1, 1]$. Because $f'(x) = 1 \neq 0$, $f(x)$ has no critical values. However, observe that $f'(x) = 1 > 0$, so that $f(x)$ is increasing. Therefore, the smallest value for $f(x)$ must be at $x = -1$ and the largest value must be at $x = 1$. This shows that $f(-1) = -1$ is the absolute minimum for $f(x)$ on $[-1, 1]$ and $f(1) = 1$ is the absolute maximum for $f(x)$ on $[-1, 1]$.

Check-In 03/05. *(True/False)* If $f'(x)$ is increasing, then $f(x)$ is convex.

Solution. The statement is *true*. We assume that $f''(x)$ exists. Because $f'(x)$ is increasing, we know that its derivative is positive. But $\frac{d}{dx} f'(x) = f''(x)$. But then $f''(x) > 0$. Because $f''(x) > 0$, we know that $f(x)$ is concave up, i.e. $f(x)$ is convex.

Check-In 03/19. *(True/False)* Let $p(x)$ be an odd degree polynomial. Because $p(x)$ is everywhere continuous and $\lim\limits_{x \to \pm\infty} p(x) = \pm\infty$, by the Intermediate Value Theorem, it must be that $p(x)$ has a real root. That is, every odd degree polynomial has a real root.

Solution. The statement is *true*. A polynomial is everywhere continuous, so that the Intermediate Value Theorem applies to $p(x)$. If $p(x)$ has odd degree, then either $\lim\limits_{x \to \infty} p(x) = \infty$ and $\lim\limits_{x \to -\infty} p(x) =$

$-\infty$ or $\lim_{x \to \infty} p(x) = -\infty$ and $\lim_{x \to -\infty} p(x) = \infty$. In either case, by the Intermediate Value Theorem, there must be an $x_0 \in (-\infty, \infty)$ such that $p(x_0) = 0$. But then $p(x)$ has a root.

Check-In 03/21. *(True/False)* Starting from a stop, if you drive 250 miles in 5 hours, there had to have been a time when you drove 30 mph.

Solution. The statement is *true*. We assume that one's position in time, $p(t)$, is continuous and differentiable—in fact, continuously differentiable. [This should logically be the case.] Therefore, $p(t)$ is continuous on $[0, 5]$ and differentiable on $(0, 5)$. We know that the average speed over this time period is $\frac{250}{5} = 50$ mph. By the Mean Value Theorem, there is a time between 0 and 5 hours where one's speed is 50 mph. But to accelerate to 50 mph, one had to first have traveled at 30 mph (by the Intermediate Value Theorem).

Check-In 03/24. *(True/False)* $\sum_{n=1}^{3} n^2 = 14$

Solution. The statement is *true*. Recall that $\sum_{n=a}^{b} f(n) = f(a) + f(a+1) + \cdots + f(b)$. Therefore, we have...

$$\sum_{n=1}^{3} n^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$$

Check-In 03/26. *(True/False)* The function $\sec^2(\theta) - 5$ is an antiderivative for the function $f(\theta) = \tan(\theta)$.

Solution. The statement is *false*. A function $F(x)$ is an antiderivative for $f(x)$ if $F'(x) = f(x)$. Observe that $\frac{d}{d\theta}\big(\sec^2(\theta) - 5\big) = 2\sec\theta \cdot \sec\theta\tan\theta - 0 = 2\sec^2\theta\tan\theta \neq f(\theta)$. But then $\sec^2(\theta) - 5$ cannot be an antiderivative for $f(\theta)$. In fact, the antiderivative for $f(\theta)$ is $\ln|\sec\theta| + C$, where C is a constant. To see this, observe that...

$$\frac{d}{d\theta}\left(\ln|\sec\theta| + C\right) = \frac{1}{\sec\theta} \cdot \sec\theta\tan\theta + 0 = \tan\theta$$

Check-In 03/28—I. *(True/False)* $\int \frac{x+1}{x^2}\,dx = \frac{\frac{1}{2}x^2 + x}{\frac{1}{3}x^3} + C$

Solution. The statement is *false*. Recall that integrals *do not* distribute over multiplication, powers, and division, i.e.

$$\int f(x)g(x)\,dx \neq \int f(x)\,dx \cdot \int g(x)\,dx$$
$$\int \big(f(x)\big)^n\,dx \neq \left(\int f(x)\,dx\right)^n$$
$$\int \frac{f(x)}{g(x)}\,dx \neq \frac{\displaystyle\int f(x)\,dx}{\displaystyle\int g(x)\,dx}$$

So, to integrate $\dfrac{x+1}{x^2}$, one *cannot* simply integrate the numerator and denominator. Instead, we divide x^2 into $x+1$ so that we can apply the power rule for integration:

$$\int \frac{x+1}{x^2}\,dx = \int \left(\frac{x}{x^2} + \frac{1}{x^2}\right)\,dx = \int \left(\frac{1}{x} + x^{-2}\right)\,dx = \ln|x| + \frac{x^{-1}}{-1} + C = \ln|x| - \frac{1}{x} + C$$

Check-In 03/28—II. *(True/False)* $\dfrac{d}{dx}\displaystyle\int_0^{\sqrt{x}} \cos(t^2)\,dt = \cos(x)\cdot\frac{1}{2\sqrt{x}}$

Solution. The statement is *true*. We know from the Fundamental Theorem of Calculus that...

$$\frac{d}{dx}\int_c^{g(x)} f(t)\,dt = f\big(g(x)\big)\cdot g'(x)$$

But then, we have...

$$\frac{d}{dx}\int_0^{\sqrt{x}} \cos(t^2)\,dt = \cos\big((\sqrt{x})^2\big)\cdot\frac{d}{dx}\sqrt{x} = \cos(x)\cdot\frac{1}{2\sqrt{x}} = \frac{\cos x}{2\sqrt{x}}$$

Check-In 03/28—III. *(True/False)* If $\displaystyle\int_0^2 f(x)\,dx = 4$ and $\displaystyle\int_1^0 f(x)\,dx = 3$, then $\displaystyle\int_1^2 f(x)\,dx = 1$.

Solution. The statement is *false*. Recall the following two properties of integrals:

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

From the first property above, we know that...

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

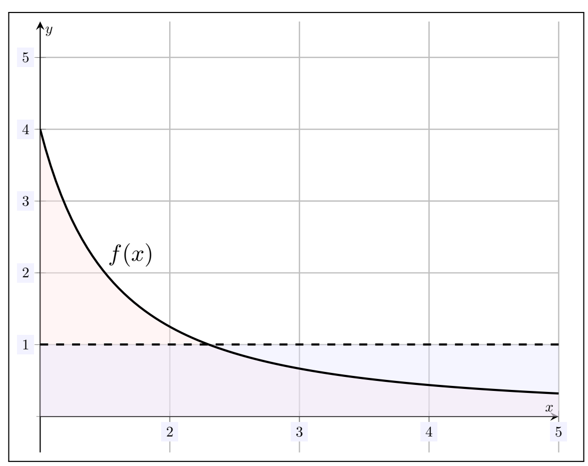
We know that $\int_1^0 f(x) = 3$. Using the second integral property above, we then know that $\int_0^1 f(x) = -\int_1^0 f(x)\,dx = -3$. But then...

$$\begin{aligned}\int_0^2 f(x)\,dx &= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx \\ 4 &= -3 + \int_1^2 f(x)\,dx \\ \int_1^2 f(x)\,dx &= 7\end{aligned}$$

Check-In 04/02. *(True/False)* The average value of a continuous function $f(x)$ on an interval $[a, b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

Solution. The statement is *true*. This is the definition of average value for a continuous function $f(x)$ on an interval $[a, b]$. The typical discrete average value is $\dfrac{\text{sum of values}}{\text{number values}}$. Instead, the traditional numerator 'sum' of the values is replaced with $\int_a^b f(x)\,dx$, while the traditional denominator of 'total number of values' is replaced by the 'interval worth of values' $[a, b]$. For instance, the average value of $f(x) = \dfrac{x+3}{x^2}\,dx$ on $[1, 5]$ is...

$$\begin{aligned}\text{Avg}_{[1,5]}\big(f(x)\big) &= \frac{1}{5-1}\int_1^5 \frac{x+3}{x^2}\,dx \\ &= \frac{1}{4}\int_1^5 \left(\frac{1}{x} + 3x^{-2}\right)\,dx \\ &= \frac{1}{4}\left(\ln|x| - 3x^{-1}\right)\Big|_{x=1}^{x=5} \\ &= \frac{1}{4}\left[\left(\ln 5 - \frac{3}{5}\right) - \left(\ln 1 - \frac{3}{1}\right)\right] \\ &= \frac{1}{4}\left[\ln 5 - \frac{3}{5} - 0 + \frac{15}{5}\right] \\ &= \frac{1}{4}\left(\ln 5 + \frac{12}{5}\right) \approx 1.00236\end{aligned}$$



We can see this in the plot on the right above. Recall also that the average value is also the 'height' of the box so that the 'area under the curve' of $f(x)$ on $[a, b]$ (shown in red) is the same as the area of the box on $[a, b]$ (shown in blue), also shown above.

Check-In 04/04. *(True/False)* The area between the curves $y = -3x$ and $y = x(1-x)$ is given by $\int_0^4 x(1-x) - 3x \; dx$.

Solution. The statement is *false*. We need to find the intersection between the two curves:

$$\begin{aligned} -3x &= x(1-x) \\ -3x &= x - x^2 \\ x^2 - 4x &= 0 \\ x(x-4) &= 0 \end{aligned}$$

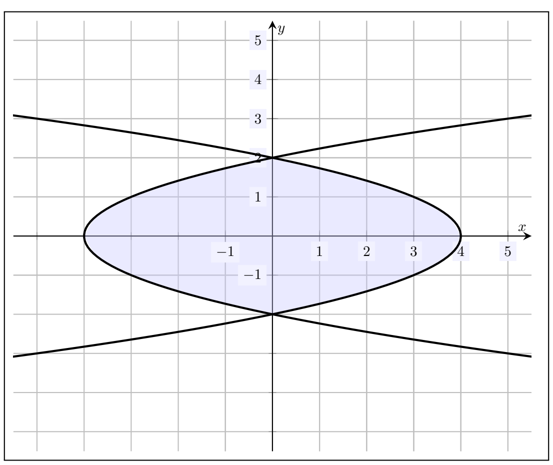
Therefore, either $x = 0$ or $x - 4 = 0$, which implies $x = 4$. Of course, we could have simply verified this from the beginning as the given integral implies the intersections are $x = 0$ and $x = 4$: we have $y = -3(0) = 0$ and $y = 0(1-0) = 0$, and we have $y = -3(4) = -12$ and $y = 4(1-4) = 4(-3) = -12$. Because the given curves are a line and parabola, they can only intersect in at most two points. These must be the two given intersection points. Choosing a value in-between $x = 0$ and $x = 4$, say $x = 1$, we see that $y = -3(1) = -3$ and $y = 1(1-1) = 0$. This shows that the parabola is 'on-top.' Alternatively, we know that the parabola opens downwards and picturing the intersection, it must be the case that the parabola is 'on top'. [Otherwise, the line would intersect the parabola at a point higher than the vertex—which is impossible.] Therefore, the area between the curves is. . .

$$A = \int_0^4 x(1-x) - (-3x) \; dx = \int_0^4 x(1-x) + 3x \; dx$$

The given integral has incorrectly subtracted the 'lower' function, $y = -3x$.

Check-In 04/07. *(True/False)* The area between $x = y^2 - 4$ and $x = 4 - y^2$ is 'best' computed using an integral in y.

Solution. The statement is *true*. The function $x = y^2 - 4$ is a parabola that opens to the right, while the function $x = 4 - y^2$ is a parabola that opens to the left. Clearly, these curves intersect at $y = \pm 2$. We plot these functions below.



Observe that integrating in x, one would need an integral for the 'left side' of the region (where the slices go back and forth between the upper and lower portions of the curve $x = y^2 - 4$) and an integral for the 'right side' of the region (where the slices go back and forth between the upper and lower portions of the curve $x = 4 - y^2$). This would be...

$$\int_{-4}^{0} \sqrt{x+4} - (-\sqrt{x+4}) \, dx + \int_{0}^{4} \sqrt{4-x} - (-\sqrt{4-x}) \, dx$$

Whereas, using an integral in y, each slice in y consistently goes between the curves $x = y^2 - 4$ and $x = 4 - y^2$—requiring only one integral. This integral would be...

$$\int_{-2}^{2} (4 - y^2) - (y^2 - 4) \, dy$$

Check-In 04/09. *(True/False)* The 'best' choice for a u-substitution to integrate $\displaystyle\int 5x^2 e^{x^3} \, dx$ would be $u = x^3$.

Solution. The statement is *true*. Observe that choosing $u = x^3$, we have $du = 3x^2 \, dx$. This implies that $dx = \frac{1}{3x^2} \, du$. But then...

$$\int 5x^2 e^{x^3} \, dx = \int 5x^2 e^u \cdot \frac{1}{3x^2} \, du = \frac{5}{3} \int e^u \, du = \frac{5}{3} \, e^u + C = \frac{5}{3} \, e^{x^3} + C$$

Alternatively, choosing $u = x^3$, we have $du = 3x^2 \, dx$. But then...

$$\int 5x^2 e^{x^3} \, dx = 5 \int x^2 e^{x^3} \, dx = \frac{5}{3} \int 3x^2 e^{x^3} \, dx = \frac{5}{3} \int e^{x^3} \cdot 3x^2 \, dx = \frac{5}{3} \int e^u \, du = \frac{5}{3} \, e^u + C = \frac{5}{3} \, e^{x^3} + C$$

Check-In 04/11. *(True/False)* The resulting integral from making the substitution $u = x^2 - 1$ in $\displaystyle\int_1^2 x(x^2 - 1)^3 \, dx$ is the integral $\displaystyle\frac{1}{2} \int_1^2 u^3 \, du$.

Solution. The statement is *false*. When making a u-substitution in a definite integral, one must also remember to change the bounds for the integral as well. We use the substitution $u = x^2 - 1$, so that $du = 2x \, dx$. Observe that if $x = 1$, then $u = 1^2 - 1 = 0$. If $x = 2$, then $u = 2^2 - 1 = 3$. But then...

$$\int_1^2 x(x^2 - 1)^3 \, dx = \frac{1}{2} \int_1^2 2x(x^2 - 1)^3 \, dx = \frac{1}{2} \int_0^3 u^3 \, du$$

Check-In 04/14. *(True/False)* To integrate $\int x\sqrt{x+3}\,dx$, one makes the substitution $u = x+2$.

Solution. The statement is *true*. This is a 'shifting integral.' Choosing $u = x+3$, we have $du = dx$. Observe that we also know that $x = u - 3$. But then...

$$\begin{aligned}
\int x\sqrt{x+3}\,dx &= \int (u-3)\sqrt{u}\,du \\
&= \int \left(u^{3/2} - 3u^{1/2}\right)\,du \\
&= \frac{2}{5}\,u^{5/2} - 2u^{3/2} + C \\
&= \frac{2}{5}\,(x+3)^{5/2} - 2(x+3)^{3/2} + C \\
&= \frac{2}{5}\,(x+3)^{3/2}\left((x+3)^{2/2} - 5\right) + C \\
&= \frac{2}{5}\,(x+3)^{3/2}\left(x+3-5\right) + C \\
&= \frac{2}{5}\,(x+3)^{3/2}(x-2) + C
\end{aligned}$$

Check-In 04/16. *(True/False)* $\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$

Solution. The statement is *true*. One can always verify this by taking a derivative:

$$\frac{d}{d\theta}\,(\ln|\sec\theta + \tan\theta| + C) = \frac{1}{\sec\theta + \tan\theta}\cdot(\sec\theta\tan\theta + \sec^2\theta) = \frac{\sec\theta(\tan\theta + \sec\theta)}{\sec\theta + \tan\theta} = \sec\theta$$

Check-In 04/23. *(True/False)* If a three-dimensional region has regular cross sections perpendicular to the x-axis between $x = a$ and $x = b$ with area $A(x)$, then the volume of the resulting region is $\int_a^b A(x)\,dx$.

Solution. The statement is *true*. This is volume by cross sections. Suppose we take n 'slices' at $x_0 = a, x_2, x_3, \ldots, x_n = b$. The area of these 'slices' is $A_i = A(x_i)$. If extend these slices 'through space' for a width $\Delta x_i = x_i - x_{i-1}$, the volume of the resulting figure is $V_i = A_i\Delta x_i = A(x_i)\Delta x_i$. But then the volume of the figure is approximately $V \approx \sum_{i=1}^n V_i = \sum_{i=1}^n A(x_i)\Delta x_i$. Taking infinitely many such slices, we see that we have a Riemann sum that converges to a Riemann integral:

$$V = \lim_{n\to\infty}\sum_{i=1}^n V_i = \lim_{n\to\infty}\sum_{i=1}^n A(x_i)\Delta x_i = \int_a^b A(x)\,dx$$

Check-In 04/25. *(True/False)* If the region between the curve $y = f(x)$ for $x = a$ and $x = b$ is revolved around the x-axis, then the volume of the solid obtained is $2\pi \int_a^b x f(x)\, dx$.

Solution. The statement is *false*. From the '2π' term, we can see that the individual is attempting to use the method of shells. For the shell method, 'slices' are parallel to the axis around which the region is being rotated. As this is the x-axis, slices must be the result of intersecting a line $y = a$ with the region, i.e. the slices are horizontal. These must be 'slid' up and down to 'cover' the region, so that the integral must be in terms of y—not x. Furthermore, the radius term for a slice at y would be y—not x as in the integrand. There are further problems, but these reasons are already more than sufficient to show that the statement must be false.