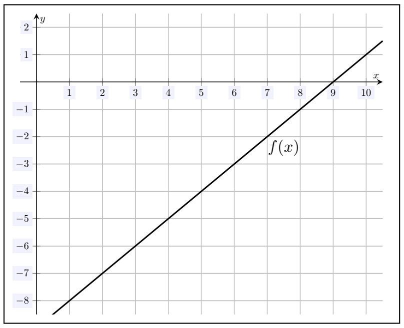
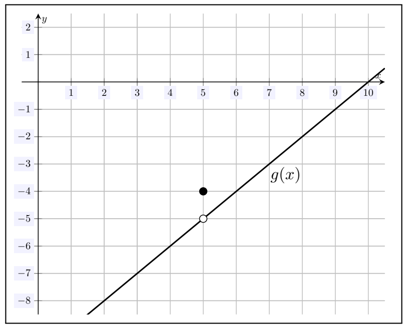
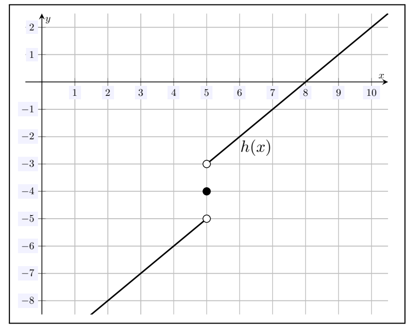
Check-In 08/22. *(True/False)* If $f(x)$ is a function with $f(5) = -4$, then it must be that $\lim_{x\to 5} f(x) = -4$.

Solution. The statement is *false*. Limits are about happens 'near' an input—not what happens at the input. A limit may or may not exist at a given x-value even when the function is defined for that x-value. Moreover, even if the limit exists, it may not be equal to the function value there!



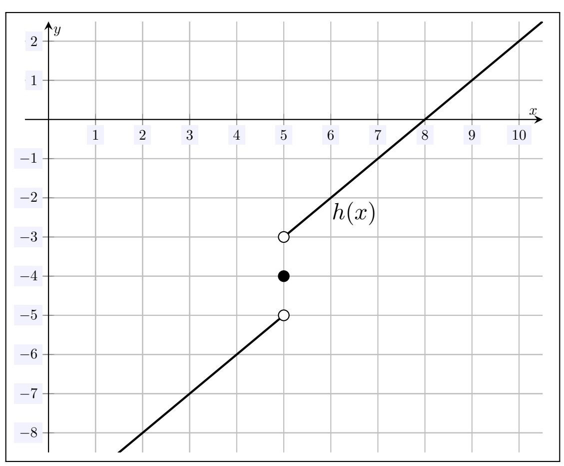




For instance, for the function $f(x)$ on the left, we have $f(5) = -4$ and $\lim_{x\to 5} f(x) = -4$. However, for the function $g(x)$ in the middle, we have $g(5) = -4$ but $\lim_{x\to 5} g(x) = -5$. But for $h(x)$ on the right, we have $h(5) = -4$ but $\lim_{x\to 5} h(x)$ does not exist because the left and right hand limits are not equal.

Check-In 08/26. *(True/False)* If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, then $\lim_{x\to a} f(x)$ exists.

Solution. The statement is *false*. We know that if $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a^-} f(x)$, exists, $\lim_{x\to a^+} f(x)$ exists, and $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$. This is because if $\lim_{x\to a} f(x)$ exists, then $f(x)$ is getting 'close' to a single number, say L, whenever x is 'close' to a—no matter if it is 'below' or 'above' $x = a$. However, just because $f(x)$ is getting 'close' to a particular output 'on the left' does not mean $f(x)$ is getting 'close' to the same output from the right. Take the example from the previous quiz!



For this function, we have $\lim_{x\to 5^-} h(x) = -5$, $\lim_{x\to 5^+} h(x) = -3$, but $\lim_{x\to 5^-} h(x) \neq \lim_{x\to 5^+} h(x)$. However, if the left and right hand limits exist *and* are equal, then $\lim_{x\to a} f(x)$ exists.

Check-In 08/26. *(True/False)* $\lim_{\theta\to 0} \dfrac{\sin(3\theta)}{2\theta} = \dfrac{3}{2}$

Solution. The statement is *true*. Recall that $\lim_{\square\to 0} \dfrac{\sin(\square)}{\square} = 1$. But then...

$$\lim_{\theta\to 0} \frac{\sin(3\theta)}{2\theta} = \frac{1}{2} \lim_{\theta\to 0} \frac{\sin(3\theta)}{\theta} = \frac{1}{2} \lim_{\theta\to 0} \frac{3\sin(3\theta)}{3\theta} = \frac{3}{2} \lim_{\theta\to 0} \underbrace{\frac{\sin(3\theta)}{3\theta}}_{\rightsquigarrow 1} = \frac{3}{2}\cdot 1 = \frac{3}{2}$$

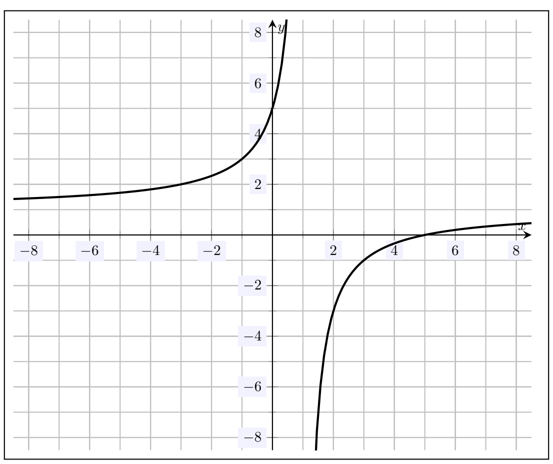
Check-In 08/28. *(True/False)* $\lim_{x\to 5} \dfrac{x^2-3x-10}{x-5} = 7$

Solution. The statement is *true*. We have...

$$\lim_{x\to 5} \frac{x^2-3x-10}{x-5} = \lim_{x\to 5} \frac{\cancel{(x-5)}(x+2)}{\cancel{x-5}} = \lim_{x\to 5} (x+2) = 5+2 = 7$$

Check-In 08/29. *(True/False)* $\lim_{x\to 1^-} \dfrac{x-5}{x-1} = -\infty$

Solution. The statement is *false*. 'Plugging in' $x=1$, we obtain $\frac{-4}{0}$—so certainly this limit is either $-\infty$, $+\infty$, or DNE. Because we approach 1 from the left, we know that $x<1$. But then $x-1<0$. But then $\frac{1}{x-1}$ approaches $-\infty$ as x tends to 1 from the left. But the numerator is also negative because when x is 'close' to 1, $x-5<0$. Therefore, the limit tends to ∞. We can see this from the plot of $\frac{x-5}{x-1}$.



The given answer failed to take the sign of the numerator into account.

Check-In 09/04. *(True/False)* $\lim_{x\to\infty} \dfrac{9x^2-5x+7}{8-3x^2} = 3$

Solution. The statement is *true*. We know that $\lim_{x\to\pm\infty} \dfrac{\text{polynomial}}{\text{polynomial}}$ is 0 if $\deg$ den. $>$ $\deg$ num., $\pm\infty$ (depending on the limit and sign of the leading coefficient in the numerator) if $\deg$ num. $>$ $\deg$ den., and is the ratio of the leading coefficients if $\deg$ den. $=$ $\deg$ num.. The degree of the numerator and denominator is 2. Therefore, we know that

$$\lim_{x\to\infty} \frac{9x^2-5x+7}{8-3x^2} = \frac{9}{-3} = -3$$

The given answer did not correctly identify the leading coefficient in the denominator. Alternatively, we can multiply by $\frac{1/x^{\deg\text{ denom}}}{1/x^{\deg\text{ denom}}}$:

$$\lim_{x\to\infty} \frac{9x^2-5x+7}{8-3x^2} = \lim_{x\to\infty} \frac{9x^2-5x+7}{8-3x^2} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{9-\frac{5}{x}+\frac{7}{x}}{\frac{8}{x^2}-3} = \frac{9-0+0}{0-3} = \frac{9}{-3} = -3$$

Check-In 09/05. *(True/False)* If $f(x)$ is defined to be the following function:

$$f(x) = \begin{cases} x^2+x-6, & x < -1 \\ x-5, & x \geq -1 \end{cases}$$

Then $f(x)$ is everywhere continuous.

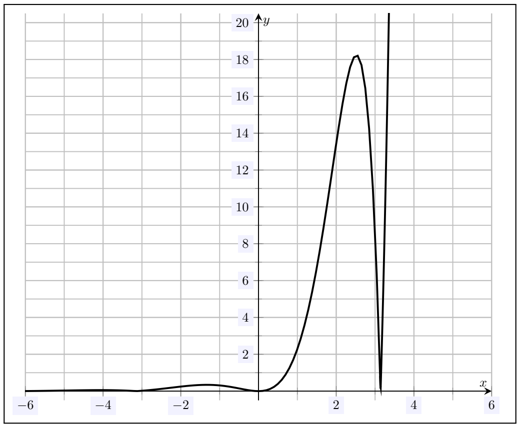
Solution. The statement is *true*. If $x < -1$, then $f(x) = x^2+x-6$. We know that x^2+x-6 is a polynomial, which are everywhere continuous. If $x > -1$, then $f(x) = x-5$, which is a polynomial. We know that polynomials are everywhere continuous. Therefore, we know $f(x)$ is continuous when $x < -1$ and when $x > -1$. We only need to check if $f(x)$ is continuous at $x = -1$. For $f(x)$ to be continuous at $x = -1$, we need to check that $f(-1) = \lim_{x\to -1} f(x)$:

- $f(-1) = -1-5 = -6$
- $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x^2+x-6) = (-1)^2 + (-1) - 6 = -6$
- $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (x-5) = -1-5 = -6$

Because $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$, we know that $\lim_{x\to -1} f(x) = -6$. Therefore, $f(-1) = \lim_{x\to -1} f(x)$. But then $f(x)$ is continuous at $x = -1$. Therefore, $f(x)$ is continuous for all x, i.e. $f(x)$ is everywhere continuous.

Check-In 09/09. *(True/False)* The function $f(x) = |xe^x \sin x|$ is continuous. Therefore, $\lim\limits_{x \to \pi} f(x) = f\left(\lim\limits_{x\to\pi}\right) = f(\pi) = 0$.

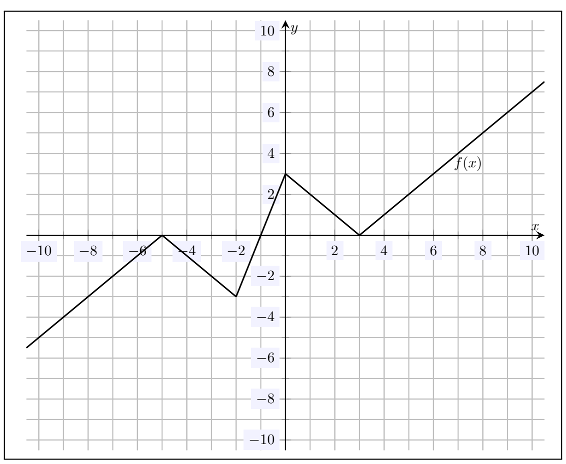
Solution. The statement is *true*. We know that x, e^x, and $\sin x$ are everywhere continuous. Therefore, their product—$g(x) := xe^x \sin x$—is continuous. We also know the function $h(x) = |x|$ is everywhere continuous. But then the composition $(h \circ g)(x)$ is continuous. But $(h \circ g)(x) = h\big(g(x)\big) = h(xe^x \sin x) = |xe^x \sin x|$. We can see the continuity from a plot of this function.



Finally, we know that if a function $f(x)$ is continuous at $x = a$, then $\lim\limits_{x\to a} f(x) = f(a)$. But we know that the given $f(x)$ is continuous at $x = \pi$—it is everywhere continuous. But then...

$$f(\pi) = |\pi \cdot e^\pi \sin \pi| = |\pi \cdot e^\pi \cdot 0| = |0| = 0$$

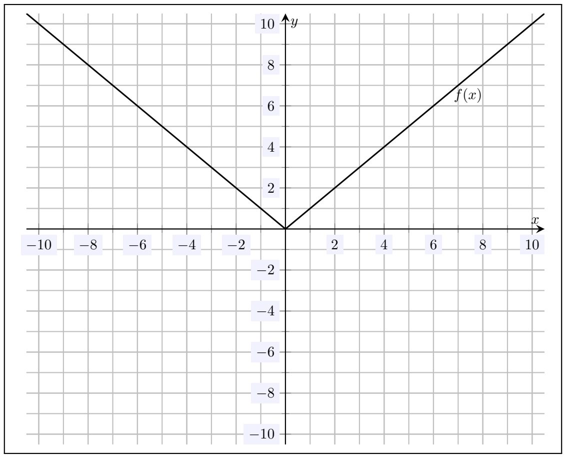
Check-In 09/10. *(True/False)* The function $f(x)$, plotted below, is *not* differentiable at $x = -2$ but *is differentiable* at $x = 6$.



Solution. The statement is *true*. At $x = -2$, we can see that $f(x)$ has a cusp. [The derivative somehow 'wants' to be -1 and 3 at the same time.] Therefore, $f(x)$ is not differentiable at $x = -2$. However, we can see that $f(x)$ is linear at $x = 6$. We know linear functions are differentiable—the derivative is the slope of the function. Therefore, $f(x)$ is differentiable at $x = 6$. In fact, the value of the derivative at $x = 6$ is the slope of the line through $\big(6, f(x)\big)$—which is $3x + 3$ so that $f'(6) = 3$.

Check-In 09/11. *(True/False) Every* differentiable function is continuous, but not every continuous function is differentiable.

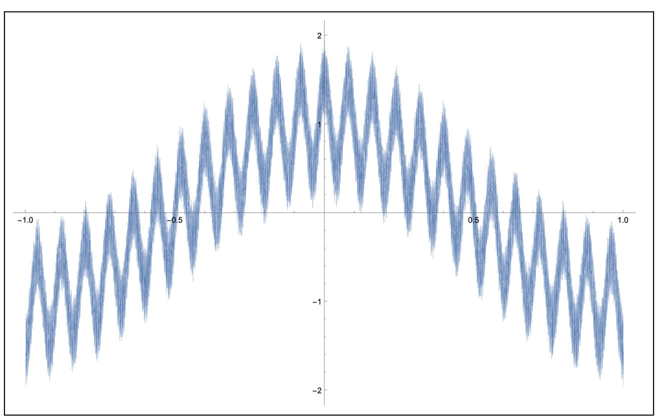
Solution. The statement is *true*. We know that every differentiable function is continuous. However, not every continuous function is necessarily differentiable. For instance, consider the function $f(x) = |x|$, shown below.



We see that $f(x)$ has a cusp at $x = 0$. Therefore, $f(x)$ is not differentiable at $x = 0$. We can check this directly:

$$f'(0) := \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \frac{|h| - 0}{h} = \begin{cases} \dfrac{h}{h} = 1, h > 0 \\ \\ \dfrac{-h}{h} = -1, \qquad h < 0 \end{cases}$$

This limit does not exist. Therefore, $f'(0)$ does not exist. There are other functions, e.g. the Weierstrass function shown below, that are *everywhere* continuous but *nowhere* differentiable.



Check-In 09/12. *(True/False)* $\dfrac{d}{dx}\left(e^{-x}\cos x\right) = -e^{-x}\cos x - e^{-x}\sin x$

Solution. The statement is *true*. We use the product rule and the chain rule. We have…

$$\begin{aligned}
\frac{d}{dx}\left(e^{-x}\cos x\right) &= \frac{d}{dx}(e^{-x})\cos x + e^{-x}\frac{d}{dx}(\cos x) \\
&= (-e^{-x})\cos x + e^{-x}(-\sin x) \\
&= -e^{-x}\cos x - e^{-x}\sin x \\
&= -e^{-x}(\sin x + \cos x)
\end{aligned}$$

Check-In 09/16. *(True/False)* $\dfrac{d}{dx}\left(x3^x\right)^{10} = 10\left(3^x + x3^x\right)^9$

Solution. The statement is *false*. The chain rule has not been properly applied. The individual began to apply the chain rule by beginning with the power rule—but then let the base be result of the next step in the chain rule—while also incorrectly taking the derivative of 3^x. [The derivative of b^x is $b^x \ln b$.] We have…

$$\frac{d}{dx}\left(x3^x\right)^{10} = 10\left(x3^x\right)^9 \cdot \frac{d}{dx}\left(x3^x\right) = 10(x3^x)\cdot(3^x + x3^x\ln 3)$$

Of course, one could perform some arithmetic first to avoid the chain rule—mostly:

$$\frac{d}{dx}\left(x3^x\right)^{10} = \frac{d}{dx}x^103^{10x} = 10x^9\cdot 3^{10x} + x^{10}\cdot\left(3^{10x}\ln 3\cdot 10\right)$$

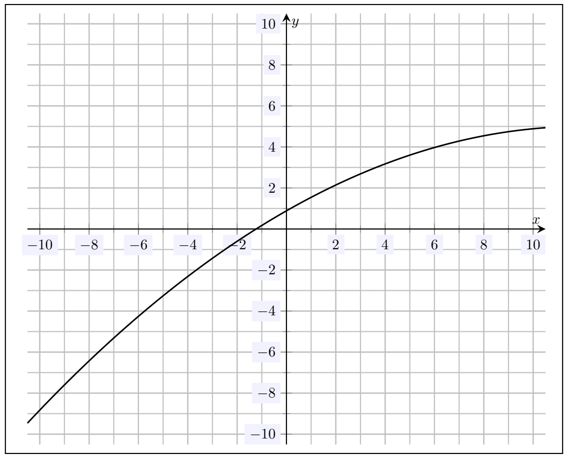
Check-In 09/18. *(True/False)* $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{f'g - g'f}{g^2}$

Solution. The statement is *true*. This is the quotient rule! Indeed, we can derive this using the power and chain rules:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{d}{dx}\left(f\cdot(g)^{-1}\right) = f'\cdot(g)^{-1} + f\cdot(-g^{-2}\cdot g') = \frac{f'}{g} - \frac{fg'}{g^2} = \frac{f'g}{g^2} - \frac{fg'}{g^2} = \frac{f'g - fg'}{g^2}$$

Check-In 09/19. *(True/False)* Let $f(x)$ be a twice-differentiable function. If $f'(x) > 0$, then it must be that $f''(x) > 0$.

Solution. The statement is *false*. Recall that if $f'(x) > 0$, the function $f(x)$ is increasing at that x-value, and if $f'(x) < 0$, the function $f(x)$ is decreasing at that x-value. Furthermore, recall that if $f''(x) > 0$, the function $f(x)$ is concave up at that x-value, and if $f''(x) < 0$, the function $f(x)$ is concave down at that x-value. Therefore, the question is asking if a function is increasing, does it have to be concave up. This is certainly not the case. For instance, consider the function $f(x)$ shown below.



This function is clearly everywhere increasing, so that $f'(x) > 0$. However, observe that the function is concave down, so that $f''(x) < 0$. The sign of f' and f'' do indeed give you information about $f(x)$. However, the signs of f, f', and f'' *do not* need to be the same.

Check-In 10/07. *(True/False)* $\dfrac{d}{dx}\left(x^2 + y^2 - \sin^2(xy)\right) = 2x + 2yy' - \cos^2(xy)(y + x)$

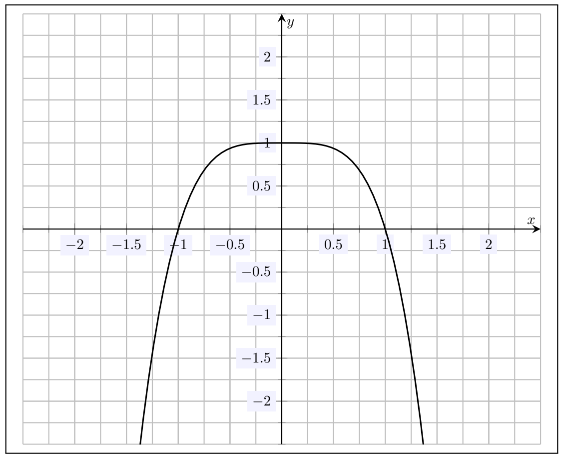
Solution. The statement is *false*. This is implicit differentiation. One must remember that $\dfrac{d}{dx}\, y = y'$—and to properly apply derivative rules:

$$\frac{d}{dx}\left(x^2 + y^2 - \sin^2(xy)\right) = 2x + 2y\,\frac{dy}{dx} - 2\sin(xy)\cdot\cos(xy)\cdot\left(1\cdot y + x\cdot\frac{dy}{dx}\right)$$

Check-In 10/07. *(True/False)* Suppose that $f'(a) = 0$ and $f(x)$ has a local maximum at $x = a$. Then $f'(x)$ changes sign at $x = a$ and $f''(a) < 0$.

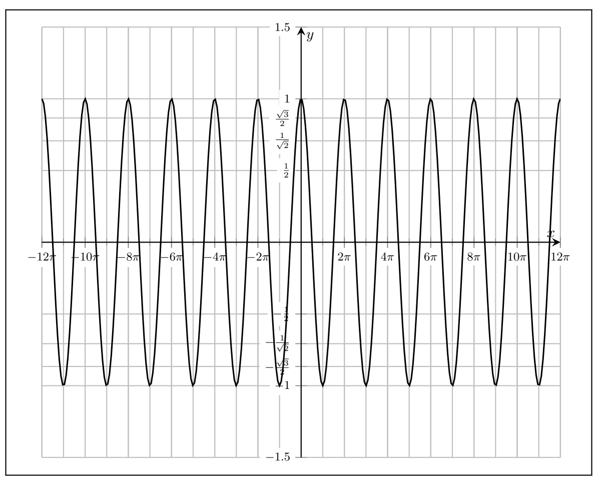
Solution. The statement is *false*. We know that $f'(a) = 0$, i.e. $x = a$ is a critical value, and $f(x)$ has a local maximum at $x = a$. Because the first derivative test applies, it must be that $f'(x)$ changes sign at $x = a$—from positive to negative across $x = a$. We can also determine that $x = a$ is a local maxima by the second derivative test if $f''(a) < 0$. However, if $x = a$ is a local maxima, the second derivative test may not apply. For instance, the function may not be twice differentiable—even if it is differentiable. Furthermore, it could be that $f''(a) = 0$—so that the second derivative test is inconclusive. For instance, consider the function $f(x) = 1 - x^4$ and take $a = 0$. We have $f'(x) = -4x^3$ so that $f'(0) = 0$. We know that $f'(-0.1) > 0$ and $f'(0.1) < 0$, i.e. $f'(x)$ changes sign from positive to negative. Therefore, $x = 0$ is a local maximum, which we can see in the plot below.

However, $f''(x) = -12x^2$ and $f''(0) = 0$. Therefore, the second derivative test is inconclusive.

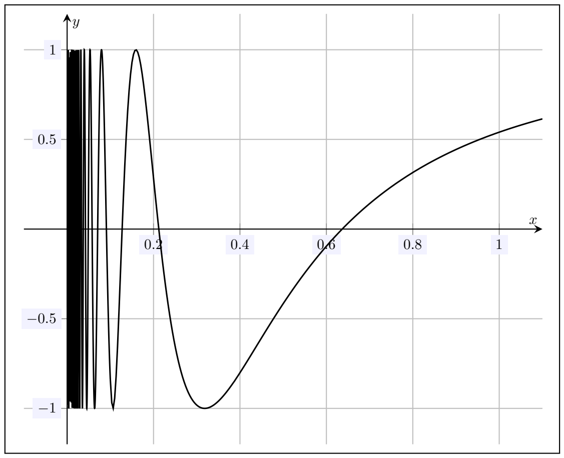


Check-In 10/14. *(True/False)* A function can have infinitely many local maxima or local minima.

Solution. The statement is *true*. Consider the function $f(\theta) = \cos(\theta)$, plotted below.



We know that $f(\theta) = \cos(\theta)$ has a local maxima at each even integer multiple of π and local minima at each odd integer multiple of π. In fact, a function can have infinitely many local maxima/minima on a finite interval. Consider the function $f(x) = \cos\left(\frac{1}{x}\right)$ for $0 < x \leq 1$ and $f(0) = 1$ if $x = 0$. This function has infinitely many local maxima and minima on $[0, 1]$. For instance, it has a local maxima whenever $x = \frac{1}{2k\pi}$ for any integer $k > 0$ and a local minima whenever $x = \frac{1}{(2k+1)\pi}$ for any integer $k > 0$.



Check-In 10/15. *(True/False)* The *Froude number*, F_r, is a dimensionless value that is used in the description of open channel flows. Specifically, it is the ratio of inertial and gravitational forces. Using this, we have $F_r = \dfrac{v}{\sqrt{gD}}$, where v is the water velocity, D is the hydraulic depth, and g is the constant gravitational attraction. Given this, their rates of changes with respect to time are related by...

$$\frac{dF_r}{dt} = \frac{\frac{dv}{dt}\,\sqrt{gD} - v\,\frac{\sqrt{g}}{2\sqrt{D}}\,\frac{dD}{dt}}{gD}$$

Solution. The statement is *true*. Implicitly differentiating with respect to t, we have...

$$F_r = \frac{v}{\sqrt{gD}}$$

$$\frac{d}{dt}\,F_r = \frac{d}{dt}\left(\frac{v}{\sqrt{gD}}\right)$$

$$\frac{dF_r}{dt} = \frac{\dfrac{d}{dt}\,v \cdot \sqrt{gD} - \dfrac{d}{dt}(\sqrt{g}\sqrt{D}) \cdot v}{\sqrt{gD}^2}$$

$$\frac{dF_r}{dt} = \frac{\dfrac{dv}{dt}\,\sqrt{gD} - \left(\sqrt{g}\,\dfrac{D'}{2\sqrt{D}}\right) \cdot v}{gD}$$

$$\frac{dF_r}{dt} = \frac{\frac{dv}{dt}\,\sqrt{gD} - v\,\frac{\sqrt{g}}{2\sqrt{D}}\,\frac{dD}{dt}}{gD}$$

Check-In 10/16. *(True/False)* If one wants to find the rate of change of the distance between two objects, one at $\big(x_1(t), y_1(t)\big)$ and the other at $\big(x_2(t), y_2(t)\big)$, then one can implicitly differentiation $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ with respect to time, t.

Solution. The statement is *true*. The distance between the objects at time t is given by...

$$d = \sqrt{\big(x_1(t) - x_2(t)\big)^2 + \big(y_1(t) - y_2(t)\big)^2}$$

One can then implicitly differentiate with respect to time, t, to find the rate of change in the distance between the objects.

$$\begin{aligned}
d &= \sqrt{\big(x_1(t) - x_2(t)\big)^2 + \big(y_1(t) - y_2(t)\big)^2} \\
\frac{d}{dt}\, d &= \frac{d}{dt}\sqrt{\big(x_1(t) - x_2(t)\big)^2 + \big(y_1(t) - y_2(t)\big)^2} \\
d' &= \frac{1}{2}\Big(\big(x_1(t) - x_2(t)\big)^2 + \big(y_1(t) - y_2(t)\big)^2\Big)^{-1/2} \cdot \\
&\qquad \Big(2\big(x_1(t) - x_2(t)\big) \cdot \big(x_1'(t) - x_2'(t)\big) + 2\big(y_1(t) - y_2(t)\big) \cdot \big(y_1'(t) - y_2'(t)\big)\Big) \\
d' &= \frac{\big(x_1(t) - x_2(t)\big) \cdot \big(x_1'(t) - x_2'(t)\big) + \big(y_1(t) - y_2(t)\big) \cdot \big(y_1'(t) - y_2'(t)\big)}{\sqrt{\big(x_1(t) - x_2(t)\big)^2 + \big(y_1(t) - y_2(t)\big)^2}}
\end{aligned}$$

Alternatively, we can square to avoid the square root and then implicitly differentiate:

$$\begin{aligned}
d &= \sqrt{\big(x_1(t) - x_2(t)\big)^2 + \big(y_1(t) - y_2(t)\big)^2} \\
d^2 &= \big(x_1(t) - x_2(t)\big)^2 + \big(y_1(t) - y_2(t)\big)^2 \\
\frac{d}{dt}\, d^2 &= \frac{d}{dt}\big(x_1(t) - x_2(t)\big)^2 + \big(y_1(t) - y_2(t)\big)^2 \\
2d \cdot d' &= 2\big(x_1(t) - x_2(t)\big) \cdot \big(x_1'(t) - x_2'(t)\big) + 2\big(y_1(t) - y_2(t)\big) \cdot \big(y_1'(t) - y_2'(t)\big) \\
d' &= \frac{\big(x_1(t) - x_2(t)\big) \cdot \big(x_1'(t) - x_2'(t)\big) + \big(y_1(t) - y_2(t)\big) \cdot \big(y_1'(t) - y_2'(t)\big)}{d} \\
d' &= \frac{\big(x_1(t) - x_2(t)\big) \cdot \big(x_1'(t) - x_2'(t)\big) + \big(y_1(t) - y_2(t)\big) \cdot \big(y_1'(t) - y_2'(t)\big)}{\sqrt{\big(x_1(t) - x_2(t)\big)^2 + \big(y_1(t) - y_2(t)\big)^2}}
\end{aligned}$$

Check-In 10/21. *(True/False)* If $f(x)$ is a continuous function with $f(1) = 1$ and $f(5) = 4$, then there is an $x \in [1, 5]$ with $f(x)^2 + 4 = 10$.

Solution. The statement is *true*. Because $f(x)$ is continuous, we know that $f(x)^2$ is continuous—being the product of continuous functions. Now $f(1)^2 = 1^2 = 1$ and $f(5)^2 = 4^2 = 16$. Observe that $f(x)^2 + 4 = 10$ has a solution if and only if $f(x)^2 = 6$. But $f(1)^2 = 1 < 6 < 16 = f(5)^2$. Therefore, by the Intermediate Value Theorem, there is a $c \in [1, 5]$ such that $f(c)^2 = 6$. But then $f(c)^2 + 4 = 6 + 4 = 10$, i.e. there is a solution to $f(x)^2 + 4 = 10$ on $[1, 5]$.

Check-In 10/30. *(True/False)* $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}$

Solution. The statement is *true*. Let $y = \sqrt{1-x^2}$. We want to compute $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx = \int_{-1}^{1} y(x)\,dx$. But if $y = \sqrt{1-x^2}$, then $y^2 = 1-x^2$, which implies that $x^2+y^2=1$. But this is the circle with radius 1 centered at the origin. This shows us that $y = \sqrt{1-x^2}$ on $[-1,1]$ is the 'upper half' of this unit circle. We know that $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$ is the (signed) area between the curve and the x-axis. We know that the area of a circle is πr^2. For the whole unit circle, $A = \pi r^2 = \pi(1^2) = \pi$. But then the area of the half circle is $\frac{\pi}{2}$. Therefore, $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}$.

Check-In 11/04. *(True/False)* Both $\sin^2 x$ and $-\frac{1}{2}\cos(2x)$ are antiderivatives for $\sin(2x)$, i.e. they are solutions to $\displaystyle\int \sin(2x)\,dx$.

Solution. The statement is *true*. Recall that an antiderivative (if it exists) for a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. Observe that...

$$\frac{d}{dx}\,\sin^2 x = 2\sin x \cdot \cos x = \sin(2x)$$

$$\frac{d}{dx}\, -\frac{1}{2}\,\cos(2x) = -\frac{1}{2}\,\cdot -\sin(2x)\cdot 2 = \sin(2x)$$

Therefore, these are both antiderivatives of $\sin(2x)$, i.e. solutions to $\displaystyle\int \sin(2x)\,dx$. For the most general form of a antiderivative for $\sin(2x)$ is $\displaystyle\int \sin(2x)\,dx = -\frac{\cos(2x)}{2} + C$. Observe that this also shows there is some C such that $-\frac{1}{2}\cos(2x) + C = \sin^2 x$. Recalling that $\sin^2 x = \frac{1-\cos(2x)}{2}$, we see that $C = \frac{1}{2}$. One can also substitute any x value and then solve for C to find that $C = \frac{1}{2}$.

Check-In 11/06. *(True/False)* $\displaystyle\frac{d}{dx}\int_{\sin x}^{9} e^{t^3}\;dt = 3x^2 e^{\sin^3 x} + C$

Solution. The statement is *false*. Recall that from the Second Fundamental Theorem of Calculus, we have...

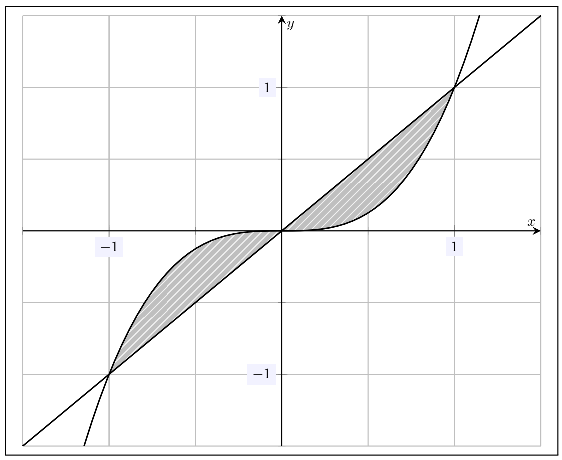
$$\frac{d}{dx}\int_{c}^{g(x)} f(t)\,dt = f\big(g(x)\big)\cdot g'(x)$$

But observe that the *lower limit* is the constant and one differentiates the upper limit—not the integrant. Moreover, there is never a '$+C$' in these problems. We must first rewrite the integral and then properly apply the theorem:

$$\frac{d}{dx}\int_{\sin x}^{9} e^{t^3}\;dt = -\frac{d}{dx}\int_{9}^{\sin x} e^{t^3}\;dt = -e^{\sin^3 x}\cdot \frac{d}{dx}\,(\sin x) = -e^{\sin^3 x}\cos x$$

Check-In 11/11. *(True/False)* The area bound by the curves $y = x^3$ and $y = x$ is given by $\int_{-1}^{1} (x^3 - x)\, dx$.

Solution. The statement is *false*. Recall that when we say the area bound by curves, we mean that to be the *area* not the *signed/directed area*; that is, the area bound by curves must be zero or bigger, i.e. nonnegative. We first find the intersection of these curves. If $x^3 = x$, then $x^3 - x = 0$. But $x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$. Therefore, the solutions are $x = -1$, $x = 0$, and $x = 1$. If one naïvely integrates, we find that $\int_{-1}^{1} (x^3 - x)\, dx = 0$. But yet, we can see on the plot below that there is indeed area between them!



The issue is that some of this area was treated as negative because the area was not computed properly. To make sure that the area is treated as 'positive', we need to be sure that we integrate the 'top' curve minus the 'bottom' curve. But which is the 'top' flips across this region. On $[-1, 0]$, $y = x^3$ is the 'top' curve, while on $[0, 1]$, $y = x$ is the 'top' curve. Therefore, to compute the area correctly, we need to compute the following:

$$A = \int_{-1}^{0} (x^3 - x)\, dx + \int_{0}^{1} (x - x^3)\, dx$$

$$A = \left(\frac{1}{4}\, x^4 - \frac{1}{2}\, x^2 \right) \Bigg|_{-1}^{0} + \left(\frac{1}{2}\, x^2 - \frac{1}{4}\, x^4 \right) \Bigg|_{0}^{1}$$

$$A = \left[(0 - 0) - \left(\frac{1}{4} - \frac{1}{2} \right) \right] + \left[\left(\frac{1}{2} - \frac{1}{4} \right) - (0 - 0) \right]$$

$$A = \left[0 - \left(-\frac{1}{4} \right) \right] + \left[\frac{1}{4} - 0 \right]$$

$$A = \frac{1}{4} + \frac{1}{4}$$

$$A = \frac{1}{2}$$

Check-In 11/3. *(True/False)* The substitution $u = \sqrt{x}$ allows one to write the integral $\displaystyle\int \sqrt{x}\, e^{\sqrt{x}}\, dx$ as $\displaystyle\frac{1}{2} \int e^u\, du$. Therefore, $\displaystyle\int \sqrt{x}\, e^{\sqrt{x}}\, dx = \frac{1}{2}\, e^{\sqrt{x}} + C$.

Solution. The statement is *false*. If this were the case, then $\displaystyle\frac{d}{dx}\left(\tfrac{1}{2}\, e^{\sqrt{x}} + C\right) = \tfrac{1}{2}\, e^{\sqrt{x}} \cdot \tfrac{1}{2\sqrt{x}} = \tfrac{e^{\sqrt{x}}}{4\sqrt{x}} \neq \sqrt{x}\, e^{\sqrt{x}}$. Observe that if one chooses $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}\, dx$. But then $dx = 2\sqrt{x}\, du$. Therefore, the correctly substituted integral is…

$$\int \sqrt{x}\, e^{\sqrt{x}}\, dx = \int \sqrt{x}\, e^u \cdot 2\sqrt{x}\, du = \int (u \cdot e^u \cdot 2u)\, du = \int 2u^2 e^u\, du$$

We have—although we do not know this yet—that…

$$\int 2u^2 e^u\, du = 2u^2 e^u - 4ue^u + 4e^u + C = 2(\sqrt{x})^2 e^{\sqrt{x}} - 4\sqrt{x}\, e^{\sqrt{x}} + 4e^{\sqrt{x}} + C = 2xe^{\sqrt{x}} - 4\sqrt{x}\, e^{\sqrt{x}} + 4e^{\sqrt{x}} + C$$