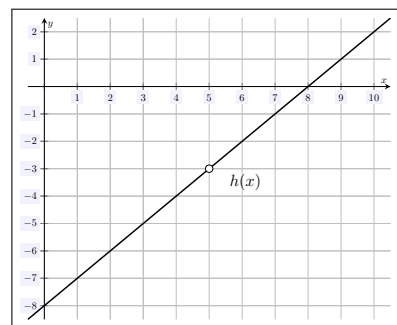
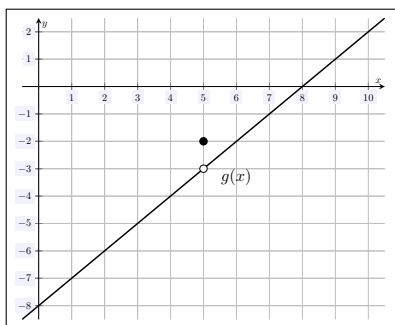
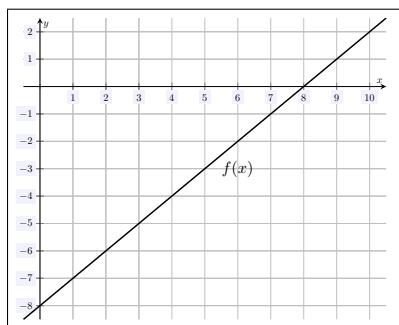


Check-In 08/22. (True/False) If $\lim_{x \rightarrow 5} f(x) = -3$, then $f(5) = -3$.

Solution. The statement is *false*. A function's limit (if it even exists) *does not* have to be the same as the function's value at that limiting value—the function does not even have to be defined there! Consider the three examples below.



For the graph of $f(x)$ on the left, $\lim_{x \rightarrow 5} f(x) = -3$, so they are equal. However, observe that for $g(x)$ (the middle graph), we have $\lim_{x \rightarrow 5} g(x) = -3$ but $g(5) = -2$, so that $\lim_{x \rightarrow 5} g(x) \neq g(5)$. Similarly, in the graph of $h(x)$ on the right, $\lim_{x \rightarrow -3} h(x) = -3$ but $h(-3)$ is not defined, so that $\lim_{x \rightarrow -3} h(x) \neq h(-3)$. A function's value (if even defined) need not be related to its limit (if the limit even exists).

Check-In 08/25. (True/False) The limit $\lim_{x \rightarrow 0} \frac{\sin x}{x} = \text{DNE}$ because $\frac{\sin x}{x}$ becomes $\frac{0}{0}$ when one 'plugs-in' $x = 0$ and $\frac{0}{0}$ is undefined.

Solution. The statement is *false*. In fact, $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. Yes, $\frac{0}{0}$, $\pm\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0 , 1^∞ , and ∞^0 are indeterminate/undefined expressions. However, that does not mean that the limit they arise from does not exist. The limit could exist or not—just like *any* limit. 'Running into' one of these expressions when evaluating a limit at its limiting value only means that one needs to try a different approach to determine if the limit exists or not.

Check-In 08/27. (True/False) $\lim_{x \rightarrow 0} \frac{\tan(5x)}{5x} = 1$

Solution. The statement is *true*. Recall that $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$. We should think of this as $\lim_{\square \rightarrow 0} \frac{\sin \square}{\square} = 1$. But then...

$$\lim_{x \rightarrow 0} \frac{\tan(5x)}{5x} = \lim_{x \rightarrow 0} \frac{\frac{\sin(5x)}{\cos(5x)}}{5x} = \lim_{x \rightarrow 0} \frac{\sin(5x)}{5x \cos x} = \lim_{x \rightarrow 0} \left(\frac{\sin(5x)}{5x} \cdot \frac{1}{\cos(5x)} \right)$$

Using the fact that $\lim_{\square \rightarrow 0} \frac{\sin \square}{\square} = 1$, we then have...

$$\lim_{x \rightarrow 0} \frac{\tan(5x)}{5x} = \lim_{x \rightarrow 0} \left(\frac{\sin(5x)}{5x} \cdot \frac{1}{\cos(5x)} \right) = 1 \cdot \frac{1}{\cos(0)} = 1 \cdot \frac{1}{1} = 1$$

Check-In 08/29. (True/False) $\lim_{x \rightarrow 2^+} \frac{x-3}{x-2} = \infty$

Solution. The statement is *false*. Observe that plugging-in $x = 2$, we obtain $\frac{-1}{0}$, which is undefined. However, this indicates that this is a ‘thinking limit’. Because in the limit, x is ‘close’ to 2, $x - 3$ is negative. As we approach 2 from the right, $x > 2$ so that $x - 2$ is positive. But then...

$$\lim_{x \rightarrow 2^+} \frac{\overbrace{x-3}^{-}}{\underbrace{x-2}_{+}} = -\infty$$

Check-In 09/03. (True/False) $\lim_{x \rightarrow \infty} \frac{x^3 - 1}{1 - x^2} = -\infty$

Solution. The statement is *true*. This is a rational limit. Observe that the degree of the numerator is 3 while the degree of the denominator is 2. Therefore, we know the limit will be $\pm\infty$. Observe that in the numerator, $x \rightarrow \infty$, so we know that $x^3 \rightarrow \infty$. In the denominator, we know that $x \rightarrow \infty$, so that $-x^2 \rightarrow -\infty$. Therefore, it should be that the limit tends to $-\infty$. We can confirm this intuition with the appropriate algebraic ‘trick’: we divide the numerator and denominator by the ‘dominating’ term in the denominator, which in this case is x^2 :

$$\lim_{x \rightarrow \infty} \frac{x^3 - 1}{1 - x^2} = \lim_{x \rightarrow \infty} \frac{x^3 - 1}{1 - x^2} \cdot \frac{1/x^2}{1/x^2} = \lim_{x \rightarrow \infty} \frac{\frac{x^3}{x^2} - \frac{1}{x^2}}{\frac{1}{x^2} - \frac{x^2}{x^2}} = \lim_{x \rightarrow \infty} \frac{x - \frac{1}{x^2}}{\frac{1}{x^2} - 1} = -\infty$$

Check-In 09/05. (True/False) If $f(x)$ is a function and $\lim_{x \rightarrow 1} f(x) = f(1)$, then $f(x)$ must be continuous at $x = 1$.

Solution. The statement is *true*. A function $f(x)$ is continuous at $x = a$ if $f(a) = \lim_{x \rightarrow a} f(x)$. The statement of the problem states that $\lim_{x \rightarrow 1} f(x) = f(1)$, i.e. $f(1) = \lim_{x \rightarrow 1} f(x)$. Therefore, $f(x)$ is continuous at $x = 1$. Similarly, if we knew $f(x)$ was continuous at $x = 1$, then we would know that $f(1) = \lim_{x \rightarrow 1} f(x)$.

Check-In 09/07. (True/False) If $f(x)$ is given by...

$$f(x) = \begin{cases} x + 1, & x \geq 0 \\ 4, & x < 0 \end{cases}$$

Then $f(x)$ is differentiable at $x = 0$.

Solution. The statement is *false*. Recall that $f(x)$ is continuous at $x = a$ if $f(a) = \lim_{x \rightarrow a} f(x)$ and that if a function is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$. Therefore, $f(x)$ is continuous

at $x = 0$ if and only if $f(0) = \lim_{x \rightarrow 0} f(x)$. Observe that $f(0) = 0 + 1 = 1$, $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} 4 = 4$, and $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x + 1) = 0 + 1 = 1$. Because $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$, i.e. the left and right hand limits are not equal, $\lim_{x \rightarrow 0} f(x)$ does not exist. But then $f(0) \neq \lim_{x \rightarrow 0} f(x)$, so that $f(x)$ is not continuous at $x = 0$. Because $f(x)$ is not continuous at $x = 0$, $f(x)$ cannot be differentiable at $x = 0$.

We can also verify this directly from the definition of the derivative. Recall that $f'(a) := \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$. Here, we have $a = 0$ and $f(0) = 0 + 1 = 1$. We compute this limit by computing the left and right hand limits. We have...

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{4 - 1}{h} = \lim_{h \rightarrow 0^-} \frac{3}{h} = -\infty$$

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{(h+1) - 1}{h} = \lim_{h \rightarrow 0^+} 1 = 1$$

Because $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$ does not exist, $f'(0)$ does not exist, i.e. $f(x)$ is not differentiable at $x = 0$.