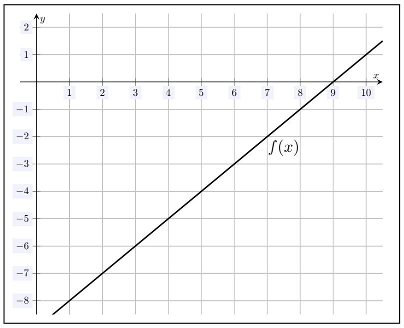
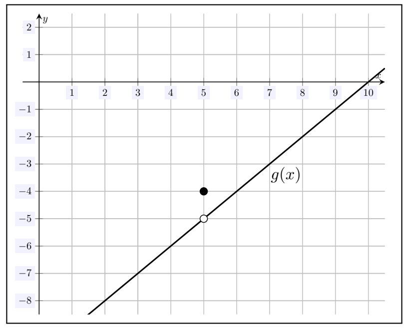
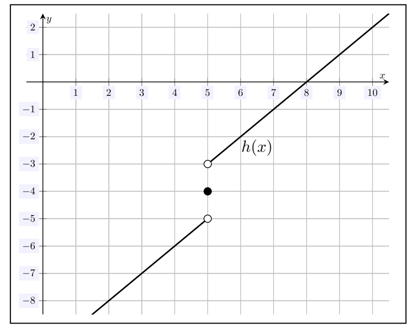
Check-In 08/22. *(True/False)* If $f(x)$ is a function with $f(5) = -4$, then it must be that $\lim_{x\to 5} f(x) = -4$.

Solution. The statement is *false*. Limits are about happens 'near' an input—not what happens at the input. A limit may or may not exist at a given x-value even when the function is defined for that x-value. Moreover, even if the limit exists, it may not be equal to the function value there!



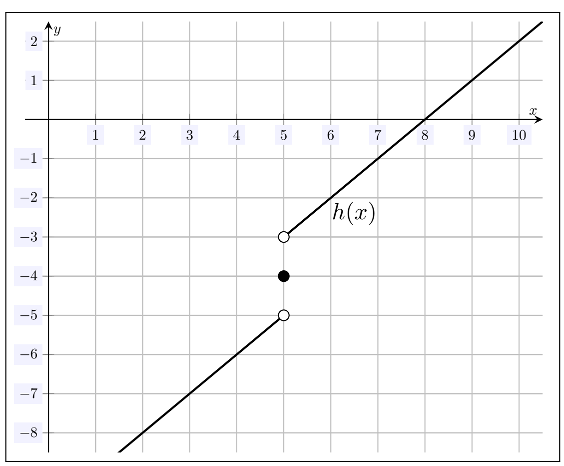




For instance, for the function $f(x)$ on the left, we have $f(5) = -4$ and $\lim_{x\to 5} f(x) = -4$. However, for the function $g(x)$ in the middle, we have $g(5) = -4$ but $\lim_{x\to 5} g(x) = -5$. But for $h(x)$ on the right, we have $h(5) = -4$ but $\lim_{x\to 5} h(x)$ does not exist because the left and right hand limits are not equal.

Check-In 08/26. *(True/False)* If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, then $\lim_{x\to a} f(x)$ exists.

Solution. The statement is *false*. We know that if $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a^-} f(x)$, exists, $\lim_{x\to a^+} f(x)$ exists, and $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$. This is because if $\lim_{x\to a} f(x)$ exists, then $f(x)$ is getting 'close' to a single number, say L, whenever x is 'close' to a—no matter if it is 'below' or 'above' $x = a$. However, just because $f(x)$ is getting 'close' to a particular output 'on the left' does not mean $f(x)$ is getting 'close' to the same output from the right. Take the example from the previous quiz!



For this function, we have $\lim_{x\to 5^-} h(x) = -5$, $\lim_{x\to 5^+} h(x) = -3$, but $\lim_{x\to 5^-} h(x) \neq \lim_{x\to 5^+} h(x)$. However, if the left and right hand limits exist *and* are equal, then $\lim_{x\to a} f(x)$ exists.

Check-In 08/26. *(True/False)* $\lim_{\theta \to 0} \frac{\sin(3\theta)}{2\theta} = \frac{3}{2}$

Solution. The statement is *true*. Recall that $\lim_{\Box \to 0} \frac{\sin(\Box)}{\Box} = 1$. But then…

$$\lim_{\theta \to 0} \frac{\sin(3\theta)}{2\theta} = \frac{1}{2} \lim_{\theta \to 0} \frac{\sin(3\theta)}{\theta} = \frac{1}{2} \lim_{\theta \to 0} \frac{3\sin(3\theta)}{3\theta} = \frac{3}{2} \lim_{\theta \to 0} \underbrace{\frac{\sin(3\theta)}{3\theta}}_{\rightsquigarrow 1} = \frac{3}{2} \cdot 1 = \frac{3}{2}$$

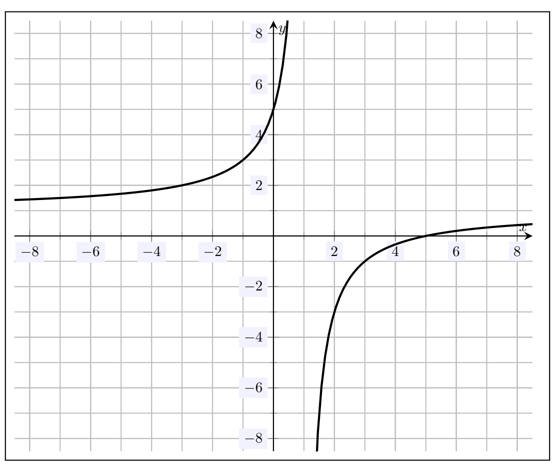
Check-In 08/28. *(True/False)* $\lim_{x \to 5} \frac{x^2 - 3x - 10}{x - 5} = 7$

Solution. The statement is *true*. We have…

$$\lim_{x \to 5} \frac{x^2 - 3x - 10}{x - 5} = \lim_{x \to 5} \frac{\cancel{(x-5)}(x+2)}{\cancel{x-5}} = \lim_{x \to 5} (x + 2) = 5 + 2 = 7$$

Check-In 08/29. *(True/False)* $\lim_{x \to 1^-} \frac{x - 5}{x - 1} = -\infty$

Solution. The statement is *false*. ‘Plugging in’ $x = 1$, we obtain $\frac{-4}{0}$—so certainly this limit is either $-\infty$, $+\infty$, or DNE. Because we approach 1 from the left, we know that $x < 1$. But then $x - 1 < 0$. But then $\frac{1}{x-1}$ approaches $-\infty$ as x tends to 1 from the left. But the numerator is also negative because when x is ‘close’ to 1, $x - 5 < 0$. Therefore, the limit tends to ∞. We can see this from the plot of $\frac{x-5}{x-1}$.



The given answer failed to take the sign of the numerator into account.

Check-In 09/04. *(True/False)* $\lim_{x\to\infty} \dfrac{9x^2-5x+7}{8-3x^2} = 3$

Solution. The statement is *true*. We know that $\lim_{x\to\pm\infty} \dfrac{\text{polynomial}}{\text{polynomial}}$ is 0 if $\deg$ den. $>$ $\deg$ num., $\pm\infty$ (depending on the limit and sign of the leading coefficient in the numerator) if $\deg$ num. $>$ $\deg$ den., and is the ratio of the leading coefficients if $\deg$ den. $=$ $\deg$ num.. The degree of the numerator and denominator is 2. Therefore, we know that

$$\lim_{x\to\infty} \frac{9x^2-5x+7}{8-3x^2} = \frac{9}{-3} = -3$$

The given answer did not correctly identify the leading coefficient in the denominator. Alternatively, we can multiply by $\frac{1/x^{\deg\ \text{denom}}}{1/x^{\deg\ \text{denom}}}$:

$$\lim_{x\to\infty} \frac{9x^2-5x+7}{8-3x^2} = \lim_{x\to\infty} \frac{9x^2-5x+7}{8-3x^2} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{9-\frac{5}{x}+\frac{7}{x}}{\frac{8}{x^2}-3} = \frac{9-0+0}{0-3} = \frac{9}{-3} = -3$$

Check-In 09/05. *(True/False)* If $f(x)$ is defined to be the following function:

$$f(x) = \begin{cases} x^2+x-6, & x < -1 \\ x-5, & x \geq -1 \end{cases}$$

Then $f(x)$ is everywhere continuous.

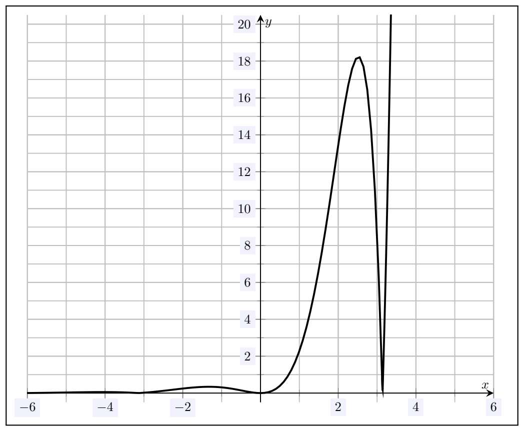
Solution. The statement is *true*. If $x < -1$, then $f(x) = x^2+x-6$. We know that x^2+x-6 is a polynomial, which are everywhere continuous. If $x > -1$, then $f(x) = x-5$, which is a polynomial. We know that polynomials are everywhere continuous. Therefore, we know $f(x)$ is continuous when $x < -1$ and when $x > -1$. We only need to check if $f(x)$ is continuous at $x = -1$. For $f(x)$ to be continuous at $x = -1$, we need to check that $f(-1) = \lim_{x\to -1} f(x)$:

- $f(-1) = -1 - 5 = -6$
- $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x^2+x-6) = (-1)^2 + (-1) - 6 = -6$
- $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (x-5) = -1 - 5 = -6$

Because $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$, we know that $\lim_{x\to -1} f(x) = -6$. Therefore, $f(-1) = \lim_{x\to -1} f(x)$. But then $f(x)$ is continuous at $x = -1$. Therefore, $f(x)$ is continuous for all x, i.e. $f(x)$ is everywhere continuous.

Check-In 09/09. *(True/False)* The function $f(x) = |xe^x \sin x|$ is continuous. Therefore, $\lim\limits_{x \to \pi} f(x) = f\left(\lim\limits_{x \to \pi}\right) = f(\pi) = 0$.

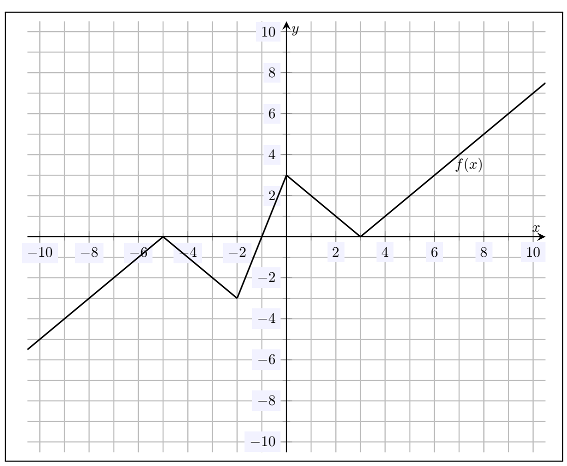
Solution. The statement is *true*. We know that x, e^x, and $\sin x$ are everywhere continuous. Therefore, their product—$g(x) := xe^x \sin x$—is continuous. We also know the function $h(x) = |x|$ is everywhere continuous. But then the composition $(h \circ g)(x)$ is continuous. But $(h \circ g)(x) = h\big(g(x)\big) = h(xe^x \sin x) = |xe^x \sin x|$. We can see the continuity from a plot of this function.



Finally, we know that if a function $f(x)$ is continuous at $x = a$, then $\lim\limits_{x \to a} f(x) = f(a)$. But we know that the given $f(x)$ is continuous at $x = \pi$—it is everywhere continuous. But then...

$$f(\pi) = |\pi \cdot e^\pi \sin \pi| = |\pi \cdot e^\pi \cdot 0| = |0| = 0$$

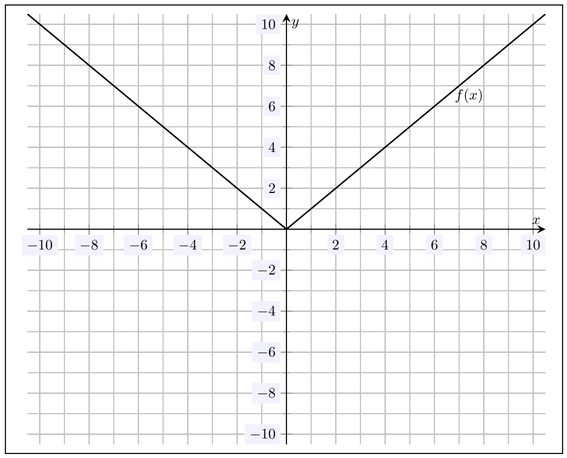
Check-In 09/10. *(True/False)* The function $f(x)$, plotted below, is *not* differentiable at $x = -2$ but *is differentiable* at $x = 6$.



Solution. The statement is *true*. At $x = -2$, we can see that $f(x)$ has a cusp. [The derivative somehow 'wants' to be -1 and 3 at the same time.] Therefore, $f(x)$ is not differentiable at $x = -2$. However, we can see that $f(x)$ is linear at $x = 6$. We know linear functions are differentiable—the derivative is the slope of the function. Therefore, $f(x)$ is differentiable at $x = 6$. In fact, the value of the derivative at $x = 6$ is the slope of the line through $\big(6, f(x)\big)$—which is $3x + 3$ so that $f'(6) = 3$.

Check-In 09/11. *(True/False) Every* differentiable function is continuous, but not every continuous function is differentiable.

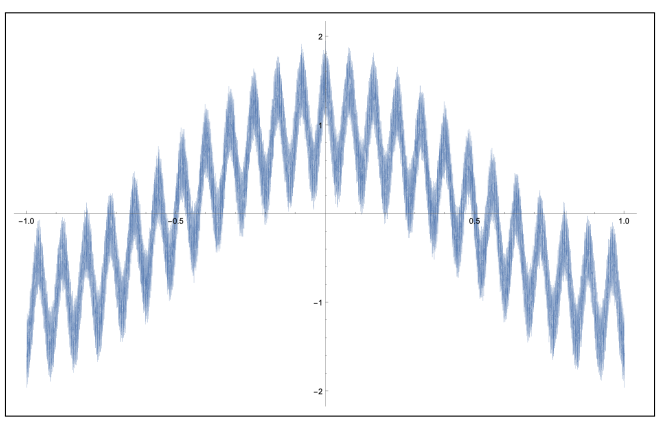
Solution. The statement is *true*. We know that every differentiable function is continuous. However, not every continuous function is necessarily differentiable. For instance, consider the function $f(x) = |x|$, shown below.



We see that $f(x)$ has a cusp at $x = 0$. Therefore, $f(x)$ is not differentiable at $x = 0$. We can check this directly:

$$f'(0) := \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \frac{|h| - 0}{h} = \begin{cases} \dfrac{h}{h} = 1, h > 0 \\ \\ \dfrac{-h}{h} = -1, \qquad h < 0 \end{cases}$$

This limit does not exist. Therefore, $f'(0)$ does not exist. There are other functions, e.g. the Weierstrass function shown below, that are *everywhere* continuous but *nowhere* differentiable.



Check-In 09/12. *(True/False)* $\dfrac{d}{dx}\left(e^{-x}\cos x\right) = -e^{-x}\cos x - e^{-x}\sin x$

Solution. The statement is *true*. We use the product rule and the chain rule. We have...

$$\begin{aligned}\frac{d}{dx}\left(e^{-x}\cos x\right) &= \frac{d}{dx}(e^{-x})\cos x + e^{-x}\frac{d}{dx}(\cos x)\\ &= (-e^{-x})\cos x + e^{-x}(-\sin x)\\ &= -e^{-x}\cos x - e^{-x}\sin x\\ &= -e^{-x}(\sin x + \cos x)\end{aligned}$$

Check-In 09/16. *(True/False)* $\dfrac{d}{dx}\left(x3^x\right)^{10} = 10\left(3^x + x3^x\right)^9$

Solution. The statement is *false*. The chain rule has not been properly applied. The individual began to apply the chain rule by beginning with the power rule—but then let the base be result of the next step in the chain rule—while also incorrectly taking the derivative of 3^x. [The derivative of b^x is $b^x \ln b$.] We have...

$$\frac{d}{dx}\left(x3^x\right)^{10} = 10\left(x3^x\right)^9 \cdot \frac{d}{dx}\left(x3^x\right) = 10(x3^x)\cdot(3^x + x3^x\ln 3)$$

Of course, one could perform some arithmetic first to avoid the chain rule—mostly:

$$\frac{d}{dx}\left(x3^x\right)^{10} = \frac{d}{dx}x^103^{10x} = 10x^9\cdot 3^{10x} + x^{10}\cdot\left(3^{10x}\ln 3\cdot 10\right)$$

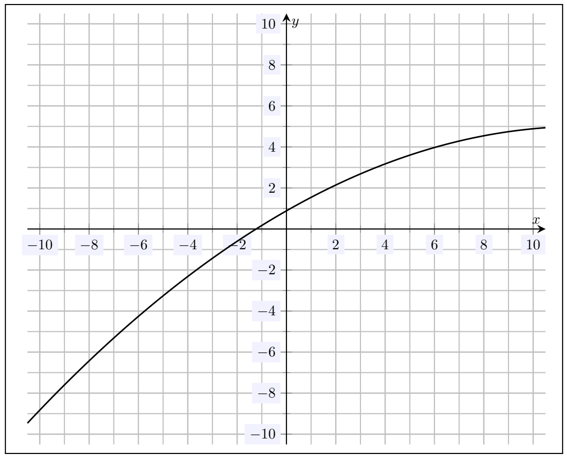
Check-In 09/18. *(True/False)* $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{f'g - g'f}{g^2}$

Solution. The statement is *true*. This is the quotient rule! Indeed, we can derive this using the power and chain rules:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{d}{dx}\left(f\cdot(g)^{-1}\right) = f'\cdot(g)^{-1} + f\cdot(-g^{-2}\cdot g') = \frac{f'}{g} - \frac{fg'}{g^2} = \frac{f'g}{g^2} - \frac{fg'}{g^2} = \frac{f'g - fg'}{g^2}$$

Check-In 09/19. *(True/False)* Let $f(x)$ be a twice-differentiable function. If $f'(x) > 0$, then it must be that $f''(x) > 0$.

Solution. The statement is *false*. Recall that if $f'(x) > 0$, the function $f(x)$ is increasing at that x-value, and if $f'(x) < 0$, the function $f(x)$ is decreasing at that x-value. Furthermore, recall that if $f''(x) > 0$, the function $f(x)$ is concave up at that x-value, and if $f''(x) < 0$, the function $f(x)$ is concave down at that x-value. Therefore, the question is asking if a function is increasing, does it have to be concave up. This is certainly not the case. For instance, consider the function $f(x)$ shown below.



This function is clearly everywhere increasing, so that $f'(x) > 0$. However, observe that the function is concave down, so that $f''(x) < 0$. The sign of f' and f'' do indeed give you information about $f(x)$. However, the signs of f, f', and f'' *do not* need to be the same.