MATH 141: Final Exam
Fall — 2024
12/09/2024
150 Minutes

Name: *Caleb McWhorter — Solutions*

Write your name on the appropriate line on the exam cover sheet. This exam contains 15 pages (including this cover page) and 10 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
8	15	
9	15	
10	15	
Total:	150	

'Twas the night before Christmas, up at the Pole,
Santa was frazzled—he'd lost all control!
The reindeer were restless, the sleight off its track,
And the toy distribution? A logistical smack!

Santa scratched his beard, eyes weary and red,
"The math here is tricky. I'm in over my head!
The children are waiting, and I need to take flight.
I need a Calculus wizard to save Christmas tonight!"

1. (15 points) *Santa missing Christmas would be a great crime,*
      "Help me dear students, before I run out of time!

   Showing all your work and using any method, compute the following limits:

   (a) $\lim\limits_{x \to 2} \dfrac{x^2 + x - 6}{x - 2}$

$$\lim_{x \to 2} \frac{(x-2)(x+3)}{x-2} = \lim_{x \to 2}(x+3) = 2 + 3 = 5$$

OR

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} \stackrel{\text{L.H.}}{=} \lim_{x \to 2} \frac{2x + 1}{1} = 2(2) + 1 = 5$$

   (b) $\lim\limits_{h \to 0} \dfrac{3 - \sqrt{h + 9}}{h}$

$$\lim_{h \to 0} \frac{3 - \sqrt{h+9}}{h} \cdot \frac{3 + \sqrt{h+9}}{3 + \sqrt{h+9}} = \lim_{h \to 0} \frac{9 - (h+9)}{h(3 + \sqrt{h+9})} = \lim_{h \to 0} \frac{-1}{3 + \sqrt{h+9}} = \frac{-1}{3 + \sqrt{9}} = -\frac{1}{6}$$

OR

$$\lim_{h \to 0} \frac{3 - \sqrt{h+9}}{h} \stackrel{\text{L.H.}}{=} \lim_{h \to 0} \frac{-\frac{1}{2}(h+9)^{-1/2}}{1} = \lim_{h \to 0} \frac{-1}{2\sqrt{h+9}} = \frac{-1}{2\sqrt{9}} = -\frac{1}{6}$$

OR

$$\lim_{h \to 0} \frac{3 - \sqrt{h+9}}{h} = -\frac{d}{dx}\sqrt{x}\bigg|_{x=9} = -\frac{1}{2\sqrt{x}}\bigg|_{x=9} = -\frac{1}{2\sqrt{9}} = -\frac{1}{6}$$

   (c) $\lim\limits_{a \to 0} \dfrac{3a}{\sin(4a)}$

$$\lim_{a \to 0} \frac{3a}{\sin(4a)} = \lim_{a \to 0} 3 \frac{a}{\sin(4a)} = \lim_{a \to 0} \frac{3}{4} \frac{4a}{\sin(4a)} = \frac{3}{4} \cdot 1 = \frac{3}{4}$$

OR

$$\lim_{a \to 0} \frac{3a}{\sin(4a)} \stackrel{\text{L.H.}}{=} \lim_{a \to 0} \frac{3}{4\cos(4a)} = \frac{3}{4\cos(0)} = \frac{3}{4(1)} = \frac{3}{4}$$

(d) $\displaystyle\lim_{b\to\infty}\left(1+\frac{2}{b}\right)^b$

Use $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$:

$$\lim_{b\to\infty}\left(1+\frac{2}{b}\right)^b$$

$$\lim_{b\to\infty}\left(1+\frac{1}{b/2}\right)^b$$

$$\lim_{b\to\infty}\left[\left(1+\frac{1}{b/2}\right)^{b/2}\right]^2$$

$$(e^1)^2$$

$$e^2$$

OR

$$y := \lim_{b\to\infty}\left(1+\frac{2}{b}\right)^b$$

$$\ln y = \lim_{b\to\infty}\ln\left(1+\frac{2}{b}\right)^b$$

$$\ln y = \lim_{b\to\infty} b\ln\left(1+\frac{2}{b}\right)$$

$$\ln y = \lim_{b\to\infty}\frac{\ln\left(1+\frac{2}{b}\right)}{1/b}$$

$$\ln y \overset{\text{L.H.}}{=} \lim_{b\to\infty}\frac{\frac{1}{1+\frac{2}{b}}\cdot-\frac{2}{b^2}}{-\frac{1}{b^2}}$$

$$\ln y = \lim_{b\to\infty}\frac{2}{1+\frac{2}{b}}$$

$$\ln y = \frac{2}{1+0}$$

$$\ln y = 2$$

$$y = e^2$$

(e) $\displaystyle\lim_{y\to\infty}\frac{3y^2-4y+9}{6y^2+5y-1}$

$$\lim_{y\to\infty}\frac{3y^2-4y+9}{6y^2+5y-1} = \lim_{y\to\infty}\frac{3y^2-4y+9}{6y^2+5y-1}\cdot\frac{1/y^2}{1/y^2} = \lim_{y\to\infty}\frac{3-\frac{4}{y}+\frac{9}{y^2}}{6+\frac{5}{y}-\frac{1}{y^2}} = \frac{3-0+0}{6+0-0} = \frac{3}{6} = \frac{1}{2}$$

OR

$$\lim_{y\to\infty}\frac{3y^2-4y+9}{6y^2+5y-1} \overset{\text{L.H.}}{=} \lim_{y\to\infty}\frac{6y-4}{12y+5} \overset{\text{L.H.}}{=} \lim_{y\to\infty}\frac{6}{12} = \frac{1}{2}$$

2. (15 points) *Rudolph's math looked quite queer, and Santa's in doubt.*

   "Did you check the derivative? Help me figure it out!"

   Use the definition of the derivative to find $f'(x)$, where $f(x) = 2x - 3x^2$. You may **not** use derivative shortcuts to find $f'(x)$. **Do not** use l'Hôpital's rule to compute any limits.

   Solution. We have...

$$
\begin{aligned}
f(x + h) &= 2(x + h) - 3(x + h)^2 \\
&= (2x + 2h) - 3(x^2 + 2xh + h^2) \\
&= 2x + 2h - 3x^2 - 6xh - 3h^2 \\
&= -3x^2 + 2x - 6xh - 3h^2 + 2h
\end{aligned}
$$

$$
f(x) = 2x - 3x^2
$$

   Therefore, we have...

$$
\begin{aligned}
f'(x) &:= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(-3x^2 + 2x - 6xh - 3h^2 + 2h) - (2x - 3x^2)}{h} \\
&= \lim_{h \to 0} \frac{-3x^2 + 2x - 6xh - 3h^2 + 2h - 2x + 3x^2}{h} \\
&= \lim_{h \to 0} \frac{-6xh - 3h^2 + 2h}{h} \\
&= \lim_{h \to 0} (-6x - 3h + 2) \\
&= -6x - 3(0) + 2 \\
&= -6x + 2 \\
&= 2 - 6x
\end{aligned}
$$

3. (15 points) *Santa's sleigh is on point, but his math's in a tizz,*

      "Quick, take these derivatives—I can't lose my rizz!"

   Showing all your work, compute the following derivatives—*do not simplify your results*:

   (a) $\dfrac{d}{dx}\left(\log_6(x) - \arcsin x + \sec x\right) = \dfrac{1}{x\ln 6} - \dfrac{1}{\sqrt{1-x^2}} + \sec x \tan x$

   (b) $\dfrac{d}{dx}\left(x^5\, 2^x \ln x\right) = 5x^4 \cdot 2^x \ln x \; + \; 2^x \ln 2 \cdot x^5 \ln x \; + \; \dfrac{1}{x} \cdot x^5\, 2^x$

   (c) $\dfrac{d}{dx}\sin^2(1 - e^x) = \dfrac{d}{dx}\left[\sin(1 - e^x)\right]^2 = 2\sin(1 - e^x) \cdot \cos(1 - e^x) \cdot -e^x$

   (d) $\dfrac{d}{dx}\left(\dfrac{x - \tan^{-1} x}{1 + \cos x}\right) = \dfrac{\left(1 - \frac{1}{1+x^2}\right) - (-\sin x)(x - \tan^{-1} x)}{(1 + \cos x)^2}$

4. **(15 points)** *Santa's sleigh is in motion, but the curve's hard to trace,*

   "Find the tangent for me—I'm racing through space!"

   Consider the curve $\mathcal{C}$ given by $xy + y^2 = 6$. Showing all your work, find the tangent line to $\mathcal{C}$ at the point $(5, 1)$.

   Solution. We know the tangent line is given by $y = y_0 + m(x - x_0)$, where m is the slope. We have...

   $$xy + y^2 = 6$$

   $$\frac{d}{dx}\left(xy + y^2\right) = \frac{d}{dx}(6)$$

   $$\left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$

   We know that $(x, y) = (5, 1)$. Therefore, we have...

   $$\left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$

   $$\left(1 + 5\frac{dy}{dx}\right) + 2(1)\frac{dy}{dx} = 0$$

   $$7\frac{dy}{dx} + 1 = 0$$

   $$7\frac{dy}{dx} = -1$$

 $$\frac{dy}{dx} = -\frac{1}{7}$$

   Therefore, the tangent line is...

   $$y = y_0 + m(x - x_0)$$

   $$y = 1 - \frac{1}{7}(x - 5)$$

5. (15 points) *Oh no, Elone has taken to Y—mischief his goal,*

   But Santa saw through it and stuffed his stocking with coal.

   Elone fires a missile at Santa in retaliation as Santa flies overhead. At this instant, Santa is 8 miles due south of Elone's SpaceY base, traveling directly south at 70 mph. The missile, having missed, is 6 miles due east of the base, traveling eastward at 60 mph. How fast is the distance between Santa and the missile changing at this moment?

   Solution. *First, we draw a picture of the situation, shown below.*
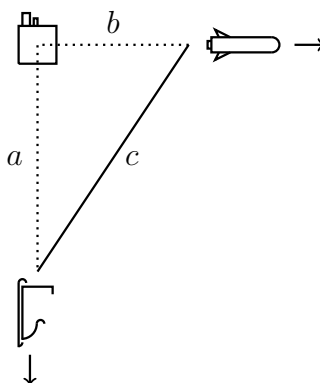
   We know that the distance between Santa and the missile is given by c, where $c^2 = a^2 + b^2$ by the Pythagorean Theorem. We want to find the rate of change of the distance between Santa and the missile, i.e. $\frac{dc}{dt}$. We know at this moment in time, $a = -8$ and $b = 6$. But then $c^2 = (-8)^2 + 6^2 = 64 + 36 = 100$. But then $c = \sqrt{100} = 10$. Furthermore, we know that $\frac{da}{dt} = -70$, $\frac{db}{dt} = 60$. Therefore, we know...

   $$c^2 = a^2 + b^2$$

   $$\frac{d}{dt}c^2 = \frac{d}{dt}\left(a^2 + b^2\right)$$

   $$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt}$$

   $$c\frac{dc}{dt} = a\frac{da}{dt} + b\frac{db}{dt}$$

   $$10 \cdot \frac{dc}{dt} = -8 \cdot -70 + 6 \cdot 60$$

   $$10\frac{dc}{dt} = 560 + 360$$

   $$10\frac{dc}{dt} = 920$$

   $$\frac{dc}{dt} = 92 \text{ mph}$$

   Therefore, at this moment, the distance between Santa and the missile is increasing at a rate of 92 mi every hour.

6. (15 points) *Santa's workshop is booming, but expenses are a fright.*

   "Help me minimize costs, or I'm bankrupt tonight!"

A rectangular gift shipping container must have an open top and a volume of 450 cubic meters. The container must also be three times as long as it is wide. Through Christmas magic, the cost of materials for the base is $1 per square meter, while the material for the sides costs $2 per square meter. Showing all your work, find the dimensions of the shipping container having minimal cost. *You do not need to prove that your dimensions give the minimum cost.*

Solution. *We want the dimensions of the box that minimize the costs. Observe that the volume $V = \ell w h$ must be 450 cubic meters, i.e. $\ell w h = 450$.*

$$\ell = 3w$$
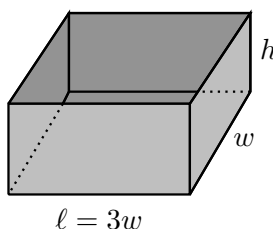
Because the box is three times as long as wide, we know that $\ell = 3w$. But then $450 = \ell w h = (3w)wh = 3w^2 h$, which implies $h = \frac{450}{3w^2} = \frac{150}{w^2}$. The area of bottom of the box is $\ell w = (3w)w = 3w^2$. As it costs \$1 per square meter, the bottom costs $3w^2 \cdot 1 = 3w^2$ dollars to construct. The sides have a total area of $2\ell h + 2wh = 2(3w)\frac{150}{w^2} + 2w \cdot \frac{150}{w^2} = \frac{900}{w} + \frac{300}{w} = \frac{1200}{w}$. But then the cost of the sides is $2 \cdot \frac{1200}{w} = \frac{2400}{w}$. Therefore, the total cost of constructing the container is...

$$C(w) = 3w^2 + \frac{2400}{w}$$

We have $C'(w) = 6w - \frac{2400}{w^2}$. We know $w \neq 0$ because $V \neq 0$. But then the only critical values occur when $C'(w) = 0$. But then...

$$C'(w) = 0$$
$$6w - \frac{2400}{w^2} = 0$$
$$6w = \frac{2400}{w^2}$$
$$6w^3 = 2400$$
$$w^3 = 400$$
$$w = \sqrt[3]{400}$$

But then $\ell = 3w = 3\sqrt[3]{400}$ and $h = \frac{150}{w^2} = \frac{150}{(\sqrt[3]{400})^2} = \frac{150}{\sqrt[3]{160000}} = \frac{150}{\sqrt[3]{8000 \cdot 20}} = \frac{150}{20\sqrt[3]{20}}$. Therefore, the dimensions of the box are...

$$\sqrt[3]{400}\ m \ \times \ 3\sqrt[3]{400}\ m \ \times \ \frac{150}{20\sqrt[3]{20}}\ m$$

Note. These dimensions are approximately $7.368 \times 22.104 \times 2.763$.

7. (15 points) *Stuck on a slope, the math's in a twist,*

   "Help me analyze the derivative or gifts will be missed!"

   Below is the plot of the ***derivative of a function,*** $f'(x)$. **This is *not* the graph of** $f(x)$.
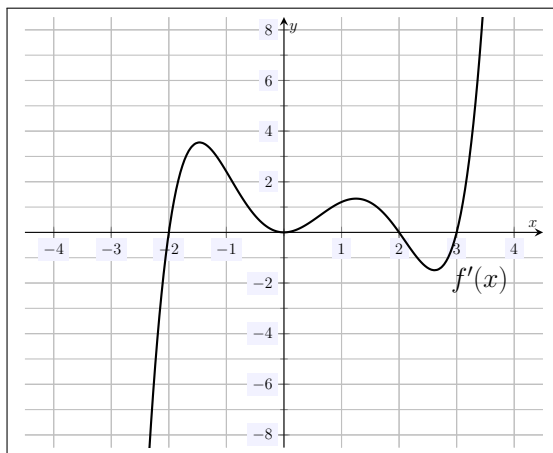

   Based on the plot above, answer the following:

   (a) Identify the intervals where $f(x)$ is increasing and decreasing.

   If $f'(x) > 0$, then $f(x)$ is increasing. We can see that $f'(x) > 0$ on $(-2, 0)$, $(0, 2)$, and $(3, \infty)$. Therefore, $f(x)$ is increasing on $(-2, 0) \cup (0, 2) \cup (3, \infty)$.

   (b) Classify the x-values of any local maxima or local minima for $f(x)$.

   We search for critical values for $f(x)$, i.e. values where $f'(x)$ is undefined or 0. Clearly, $f'(x)$ is defined everywhere. We see that $f'(x) = 0$ at $x = -2, 0, 2, 3$. We can use the first derivative test to classify these critical values:
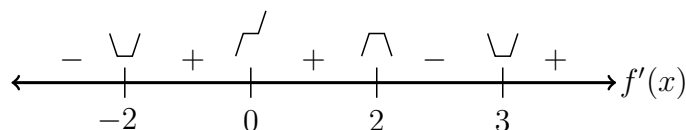
   Therefore, $x = -2, 3$ are local minima, $x = 2$ is a local maxima, and $x = 0$ is neither a local maxima nor a local minima. Alternatively, we can prove this using the second derivative by observing $f''(-2) > 0$, $f''(2) < 0$, and $f''(3) > 0$ (see below). But we cannot determine whether $x = 0$ is a local min/max from the second derivative test because $f''(0) = 0$.

   (c) Is $f(x)$ concave up, down, or neither on the interval $(0, 1)$? Justify your answer.

   Observe that on $(0, 1)$, $f'(x)$ is increasing, which implies that $f''(x) > 0$. But if $f''(x) > 0$, then $f(x)$ is concave up. Therefore, $f(x)$ is concave up (convex) on $(0, 1)$.

8. (15 points) *Addressing Christmas challenges, Santa has another request.*

   "Help me solve this problem—then we'll pass this tough test!"

Let $g(x) = \int_0^x f(t)\, dt$, where $f(t)$ is the function whose graph on the interval $[0, 9]$ is shown below.
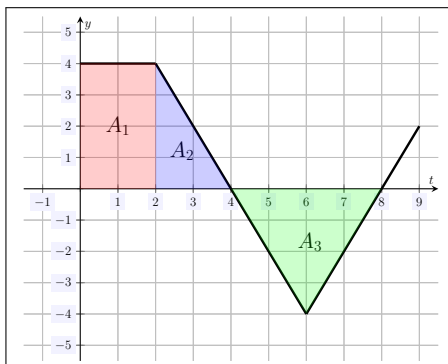
(a) Showing all your work, compute $g(8)$.

$$g(8) = \int_0^8 f(t)\, dt = A_1 + A_2 - A_3 = 2(4) + \frac{1}{2}(2)\,4 - \frac{1}{2}(4)\,4 = 8 + 4 - 8 = 4$$

(b) On what interval(s) is $g(x)$ decreasing? Justify your answer.

   If $g'(x) < 0$, then $g(x)$ is decreasing. But using the Second Fundamental Theorem of Calculus,

   $$g'(x) = \frac{d}{dx}\, g(x) = \frac{d}{dx} \int_0^x f(t)\, dt = f(x)$$

   Therefore, if $f(x) < 0$, then $g'(x) < 0$. We know $f(x) < 0$ on $(4, 8)$. Therefore, $g'(x)$ is decreasing on $(4, 8)$.

(c) Find and classify any local minima/maxima for $g(x)$. Justify your answer.

   We find the critical values, i.e. where $g'(x) = 0$ or is undefined. We know that $g'(x) = f(x)$. We know $f(x)$ is defined on $[0, 9]$ and $f(x) = 0$ only when $x = 4$ or $x = 8$. The critical values are $x = 4$ and $x = 8$. Using the first derivative test (the second will fail) below, we see that $x = 4$ is a local max and $x = 8$ is a local min.
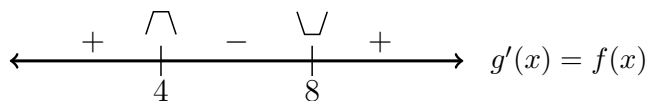
(d) Showing all your work, compute $g''(4)$.

   We know that $g'(x) = f(x)$. Therefore, $g''(x) = \dfrac{d}{dx}\, g'(x) = \frac{d}{dx} f(x) = f'(x)$. At $x = 4$, we can see that $f(x)$ is linear. Therefore, $g''(4) = f'(4)$ is the slope of the line at $x = 4$, which is $\frac{4-(-4)}{2-6} = \frac{8}{-4} = -2$, so $g''(4) = -2$.

9. (15 points) *Snow's on the way, and Santa's on a tight schedule today.*

> *Find the maxima and minima, there's no time to—you know what?*
>
> *Don't question Santa. Just find the maxima and minima.*

Let $f(x) = 2x - 3x^{2/3}$. Showing all your work, find the absolute maxima and minima for $f(x)$ on the interval $[-1, 3]$. Be sure to justify your answers with an appropriate derivative test.

Solution. First, we find the critical values for $f(x)$. We know that $f'(x) = 2 - 2x^{-1/3}$. Observe that $f'(x)$ is undefined at $x = 0$. Furthermore, we know that $f'(x) = 0$ when...
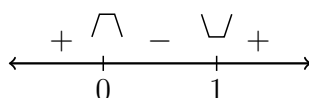
$$2 - 2x^{-1/3} = 0$$
$$2x^{-1/3} = 2$$
$$x^{-1/3} = 1$$
$$x = 1$$

Therefore, the critical values are $x = 0, 1$. Using the first derivative test, we have...

$$+ \quad \wedge \quad - \quad \vee \quad +$$
$$\xleftarrow{\qquad\qquad\qquad} \underset{0}{\quad} \underset{1}{\quad} \xrightarrow{\qquad\qquad}$$

Therefore, $x = 0$ is a local maximum and $x = 1$ is a local minimum. Observe...

$$f(0) = 2(0) - 3(0) = 0$$
$$f(1) = 2(1) - 3(1) = 2 - 3 = -1$$

$$f(-1) = 2(-1) - 3(-1)^{2/3} = -2 - 3 = -5$$
$$f(3) = 2(3) - 3(3^{2/3}) = 3(2 - 3^{2/3})$$

Note that $f(3) = 3(2 - 3^{2/3}) = 3(2 - 9^{1/3}) = 3(8^{1/3} - 9^{1/3}) < 0$ and $f(3) = 3(2 - 3^{2/3}) > 3(2 - 3) = 3(-1) = -3$. Therefore, the absolute maximum for $f(x)$ on $[-1, 3]$ is 0 and the absolute minimum for $f(x)$ on $[-1, 3]$ is -5.

> Absolute Maximum: 0 at $x = 0$
> Absolute Minimum: -5 at $x = -1$

10. (15 points) *Christmas almost over, you're nearly clear.*

   "Help me find these integrals, then we're done for this year."

   Showing all your work, compute the following:

(a) $\displaystyle \int \left(\sin x - e^x + \frac{1}{x^2 + 1} \right) dx$

$$\int \left(\sin x - e^x + \frac{1}{x^2 + 1} \right) dx = -\cos x - e^x + \arctan x + C$$

(b) $\displaystyle \int_1^2 \left(\frac{1 - x}{x} \right)^2 dx$

$$\int_1^2 \left(\frac{1}{x} - 1 \right)^2 dx$$

$$\int_1^2 \left(\frac{1}{x^2} - 2 \cdot \frac{1}{x} + 1 \right) dx$$

$$-\frac{1}{x} - 2 \ln x + x \Big|_1^2$$

$$\left(-\frac{1}{2} - 2 \ln 2 + 2 \right) - \left(-\frac{1}{1} - 2 \ln 1 + 1 \right)$$

$$\left(\frac{3}{2} - 2 \ln 2 \right) - 0$$

$$\frac{3}{2} - 2 \ln 2$$

(c) $\displaystyle\int \frac{dx}{x \ln x}$

$u = \ln x$ $\qquad \displaystyle\int \frac{dx}{x \ln x}$

$du = \dfrac{1}{x} \, dx$

$dx = x \, du \qquad \displaystyle\int \frac{x \, du}{xu}$

$\displaystyle\int \frac{du}{u}$

$\ln |u| + C$

$\ln |\ln x| + C$

OR

$u = \ln x \qquad \displaystyle\int \frac{dx}{x \ln x}$

$du = \dfrac{1}{x} \, dx$

$\displaystyle\int \frac{1}{\ln x} \cdot \frac{dx}{x}$

$\displaystyle\int \frac{du}{u}$

$\ln |u| + C$

$\ln |\ln x| + C$

(d) $\displaystyle\int_0^{\pi/2} \cos^3(x) \sin(x) \, dx$

$u = \cos x \qquad \displaystyle\int_0^{\pi/2} \cos^3(x) \sin(x) \, dx$

$du = -\sin x \, dx$

$dx = -\dfrac{1}{\sin x} \, du \qquad \displaystyle\int_1^0 u^3 \sin(x) \cdot \frac{-1}{\sin x} \, du$

$x = \dfrac{\pi}{2} : u = \cos(\dfrac{\pi}{2}) = 0 \qquad -\displaystyle\int_1^0 u^3 \, du$

$x = 0 : u = \cos(0) = 1 \qquad \displaystyle\int_0^1 u^3 \, du$

$\dfrac{u^4}{4} \Big|_0^1$

$\dfrac{1}{4} \left(1^4 - 0^4 \right)$

$\dfrac{1}{4}$

OR

$u = \cos x \qquad \displaystyle\int_0^{\pi/2} \cos^3(x) \sin(x) \, dx$

$du = -\sin x \, dx$

$-\displaystyle\int_0^{\pi/2} \cos^3(x) \cdot (-\sin x) \, dx$

$x = \dfrac{\pi}{2} : u = \cos(\dfrac{\pi}{2}) = 0$

$x = 0 : u = \cos(0) = 1 \qquad -\displaystyle\int_1^0 u^3 \, du$

$\displaystyle\int_0^1 u^3 \, du$

$\dfrac{u^4}{4} \Big|_0^1$

$\dfrac{1}{4} \left(1^4 - 0^4 \right)$

$\dfrac{1}{4}$

Santa exclaimed, at the end of Christmas night,
"Thanks to you, the math's worked out right!
Stay curious, dear student, keeping Mathematics in sight,
And merry Christmas to all, and to all a good night!