MATH 141: Exam 2
Fall — 2024
10/23/2024
75 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 11 pages (including this cover page) and 10 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) L'Hôpital's rule will not compute the limit below—the resulting limits essentially 'cycle.' Compute the limit below *without the use of l'Hôpital's*.

$$\lim_{x\to\infty} \frac{2x+5}{\sqrt{3x^2+1}}$$

Solution. *Observe that when x is 'large', then $3x^2+1 \approx 3x^2$ so that $\sqrt{3x^2+1} \approx \sqrt{3x^2} = \sqrt{3}\,x$. But then we have...*

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x+5}{\sqrt{3x^2+1}} &= \lim_{x\to\infty} \frac{2x+5}{\sqrt{3x^2+1}} \cdot \frac{1/x}{1/x} \\
&= \lim_{x\to\infty} \frac{\frac{2x}{x}+\frac{5}{x}}{\frac{\sqrt{3x^2+1}}{x}} \\
&= \lim_{x\to\infty} \frac{2+\frac{5}{x}}{\frac{\sqrt{3x^2+1}}{\sqrt{x^2}}} \\
&= \lim_{x\to\infty} \frac{2+\frac{5}{x}}{\sqrt{\frac{3x^2+1}{x^2}}} \\
&= \lim_{x\to\infty} \frac{2+\frac{5}{x}}{\sqrt{3+\frac{1}{x^2}}} \\
&= \frac{2+0}{\sqrt{3+0}} \\
&= \frac{2}{\sqrt{3}}
\end{aligned}$$

Note. A rather clever approach which requires some non-obvious (but nevertheless true) facts: suppose that $\lim_{x\to\infty} f(x)^2 = L \geq 0$ and $f(x) \geq 0$ on $[0,\infty)$. Because $f(x) \geq 0$ and $\sqrt{x}$ is continuous, we know that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} |f(x)| = \lim_{x\to\infty} \sqrt{f(x)^2} = \sqrt{\lim_{x\to\infty} f(x)^2} = \sqrt{L}$. Because $\left(\frac{2x+5}{\sqrt{3x^2+1}}\right)^2 = \frac{(2x+5)^2}{3x^2+1} = \frac{4x^2+20x+25}{3x^2+1}$, we know that $\lim_{x\to\infty} \frac{4x^2+20x+25}{3x^2+1} = \frac{4}{3}$. But then from our remarks, $\lim_{x\to\infty} \frac{2x+5}{\sqrt{3x^2+1}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$.

2. (10 points) Let f be the function $f(x) = 2x^3 - 3x^2 - 12x + 7$. Show that $f(x)$ has a root on $[0, 1]$. Be sure to show all your work and fully justify your response.

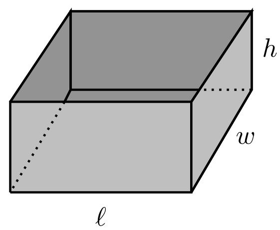
Solution. *If $f(x)$ has a root on $[0, 1]$, then there is a $c \in [0, 1]$ such that $f(c) = 0$. Now observe...*

- *The function $f(x) = 2x^3 - 3x^2 - 12x + 7$ is continuous on $[0, 1]$ because it is a polynomial and polynomials are everywhere continuous.*
- $f(0) = 2(0^3) - 3(0^2) - 12(0) + 7 = 2(0) - 3(0) - 12(0) + 7 = 0 - 0 - 0 + 7 = 7.$
- $f(1) = 2(1^3) - 3(1^2) - 12(1) + 7 = 2(1) - 3(1) - 12(1) + 7 = 2 - 3 - 12 + 7 = -6.$
- $f(1) < 0 < f(0)$

Therefore, by the Intermediate Value Theorem, there exists $c \in [0, 1]$ such that $f(c) = 0$. But then c is a root of $f(x)$ in $[0, 1]$.

3. (10 points) An open rectangular storage box is going to be designed to hold 36,000 cubic inches. The base of the box will be twice as long as it is wide. Find the dimensions of the box that will require the least amount of material, i.e. the box with minimal surface area. *For full credit, show all your work and justify that your dimensions actually give the minimal surface area.*

Solution. *We first draw a picture.*



We want the dimensions of the box which minimize the surface area $\mathrm{SA} = \ell w + 2\ell h + 2wh$ *(notice there is no top because it is an open box). We know the volume is 36,000 cubic inches, i.e.* $V = 36000$*. But we know also that* $V = \ell wh$*. But then...*

$$V = \ell wh \Longrightarrow 36000 = \ell wh \Longrightarrow h = \frac{36000}{\ell w}$$

But then...

$$\mathrm{SA} = \ell w + 2\ell h + 2wh = \ell w + 2\ell \cdot \frac{36000}{\ell w} + 2w \cdot \frac{36000}{\ell w} = \ell w + \frac{72000}{w} + \frac{72000}{\ell}$$

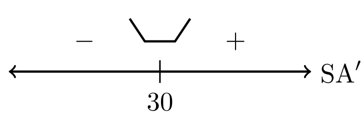
We are told that the length is twice the width, i.e. $\ell = 2w$*. But then...*

$$\mathrm{SA} = \ell w + \frac{72000}{w} + \frac{72000}{\ell} = (2w)w + \frac{72000}{w} + \frac{72000}{2w} = 2w^2 + \frac{72000}{w} + \frac{36000}{w} = 2w^2 + \frac{108000}{w}$$

We want to find the minimum of SA *with respect to* w *for* $w \in (0, \infty)$*. We have* $\mathrm{SA}' = 4w - \frac{108000}{w^2}$*. We know that* $w \neq 0$ *(because* $V \neq 0$*), so that* $w = 0$ *is not a critical value. Setting* SA *to zero, we have...*

$$\begin{aligned}
\mathrm{SA} &= 0 \\
4w - \frac{108000}{w^2} &= 0 \\
4w &= \frac{108000}{w^2} \\
4w^3 &= 108000 \\
w^3 &= 27000 \\
w &= \sqrt[3]{27000} \\
w &= 30 \textit{ in}
\end{aligned}$$

We confirm that this is a minimum with the first derivative test:



Alternatively, we have $\mathrm{SA}''(w) = 4 + \frac{216000}{w^3}$*, so that* $\mathrm{SA}''(30) = 4 + \frac{216000}{30^3} = 4 + \frac{216000}{27000} = 4 + 8 = 12 > 0$*. Now if* $w = 30$ *in, then* $\ell = 2w = 2(30) = 60$ *in. But then* $h = \frac{36000}{\ell w} = \frac{36000}{60(30)} = \frac{36000}{1800} = 20$ *in. Therefore, the dimensions of the box with minimal surface area, i.e. minimal material, is...*

$$20 \textit{ in} \times 30 \textit{ in} \times 60 \textit{ in}$$

4. (10 points) Showing all your work, compute the following limits:

(a) $$\lim_{x\to 0}\frac{\sin^2(x)}{\cos(x)-1}\overset{\text{L.H.}}{=}\lim_{x\to 0}\frac{2\sin x\cos x}{-\sin x}=\lim_{x\to 0}-2\cos x=-2\cos(0)=-2$$

OR

$$\lim_{x\to 0}\frac{\sin^2(x)}{\cos(x)-1}\overset{\text{L.H.}}{=}\lim_{x\to 0}\frac{2\sin x\cos x}{-\sin x}\overset{\text{L.H.}}{=}\lim_{x\to 0}\frac{2\cos^2 x-2\sin^2 x}{-\cos x}=\frac{2(1)-2(0)}{-1}=-2$$

OR

$$\lim_{x\to 0}\frac{\sin^2(x)}{\cos(x)-1}=\lim_{x\to 0}\frac{1-\cos^2 x}{\cos(x)-1}=\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{\cos x-1}=\lim_{x\to 0}-(1+\cos x)=-(1+\cos 0)=-2$$

(b) $$\lim_{x\to\infty}2x\tan\left(\frac{3}{x}\right)=\lim_{x\to\infty}\frac{2\tan\left(\frac{3}{x}\right)}{\frac{1}{x}}\overset{\text{L.H.}}{=}\lim_{x\to\infty}\frac{2\sec^2\left(\frac{3}{x}\right)\cdot\frac{-3}{x^2}}{\frac{-1}{x^2}}=\lim_{x\to\infty}6\sec^2\left(\frac{3}{x}\right)=6\sec^2(0)=6$$

OR

$$\lim_{x\to\infty}2x\tan\left(\frac{3}{x}\right)=\lim_{x\to\infty}\frac{2x\sin\left(\frac{3}{x}\right)}{\cos\left(\frac{3}{x}\right)}=\lim_{x\to\infty}2\cdot 3\,\frac{\sin\left(\frac{3}{x}\right)}{\frac{3}{x}}\cdot\frac{1}{\cos\left(\frac{3}{x}\right)}=6\cdot 1\cdot\frac{1}{\cos(0)}=6$$

(c) $$\lim_{x\to 0^-}\frac{2x-e^x+1}{x^2}\overset{\text{L.H.}}{=}\lim_{x\to 0^-}\underbrace{\frac{2-e^x}{2x}}_{\substack{2x\to 0\\ 2x<0\\ 2-e^x>0}}=-\infty$$

5. (10 points) A trisectrix of Maclaurin is a curve that can be given by the equation below:

$$x^3 + xy^2 = 3x^2 - y^2$$

Find the tangent line to this curve at the point $(1, -1)$.

Solution. *First, we implicitly differentiate:*

$$x^3 + xy^2 = 3x^2 - y^2$$

$$\frac{d}{dx}\left(x^3 + xy^2\right) = \frac{d}{dx}\left(3x^2 - y^2\right)$$

$$3x^2 + \left(y^2 + 2xy\,\frac{dy}{dx}\right) = 6x - 2y\,\frac{dy}{dx}$$

We then use the point $(1, -1)$*:*

$$3x^2 + \left(y^2 + 2xy\,\frac{dy}{dx}\right) = 6x - 2y\,\frac{dy}{dx}$$

$$3(1^2) + \left((-1)^2 + 2(1)(-1)\,\frac{dy}{dx}\right) = 6(1) - 2(-1)\,\frac{dy}{dx}$$

$$4 - 2\,\frac{dy}{dx} = 6 + 2\,\frac{dy}{dx}$$

$$4\,\frac{dy}{dx} = -2$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

Therefore, the tangent line is...

$$\ell(x) = y_0 + m(x - x_0)$$

$$\ell(x) = -1 - \frac{1}{2}(x - 1)$$

Note. Before using the point $(1, -1)$*, one could explicitly solve for* $\frac{dy}{dx}$ *and find that* $\frac{dy}{dx} = \frac{6x - y^2 - 3x^2}{2(y + xy)}\Big|_{(x,y)=(1,-1)} = \frac{6(1) - (-1)^2 - 3(1^2)}{2(-1 + 1(-1))} = \frac{6 - 1 - 3}{2(-1-1)} = \frac{2}{-4} = -\frac{1}{2}$.

6. (10 points) Consider the function $f(x) = \sqrt{x}$.

   (a) Find the linearization of $f(x)$ at $x = 64$. *Do not simplify your answer.*

$$f(64) = \sqrt{64} = 8$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$
$$f'(64) = \frac{1}{2\sqrt{64}} = \frac{1}{2(8)} = \frac{1}{16}$$

Therefore, the linearization is...

$$\ell(x) = y_0 + m(x - x_0)$$

$$\ell(x) = 8 + \frac{1}{16}(x - 64)$$

   (b) Use (a) to approximate $\sqrt{60}$. Express your answer as a decimal.

We have...

$$\sqrt{60} \approx \ell(60) = 8 + \frac{1}{16}(60 - 64) = 8 - \frac{4}{16} = 8 - \frac{1}{4} = 8 - 0.25 = 7.75$$

Note. The actual value is $\sqrt{60} \approx 7.74596669\ldots$. *Therefore, our approximation has only a* 0.0520% *error!*

7. (10 points) Showing all your work, compute the following limits:

(a) $\lim\limits_{x\to\infty} \dfrac{\ln(3+e^{2x})}{5x} \overset{\text{L.H.}}{=} \lim\limits_{x\to\infty} \dfrac{\frac{1}{3+e^{2x}} \cdot 2e^{2x}}{5} = \lim\limits_{x\to\infty} \dfrac{2e^{2x}}{15+5e^{2x}} \overset{\text{L.H.}}{=} \lim\limits_{x\to\infty} \dfrac{4e^{2x}}{10e^{2x}} = \lim\limits_{x\to\infty} \dfrac{4}{10} = \dfrac{2}{5}$

(b) $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{e^x-1}\right) = \dfrac{1}{2}$

$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{e^x-1}\right) = \lim_{x\to 0^+} \left(\frac{e^x-1}{x(e^x-1)} - \frac{x}{x(e^x-1)}\right) = \lim_{x\to 0^+} \frac{e^x-1-x}{x(e^x-1)} \overset{\text{L.H.}}{=} \lim_{x\to 0^+} \frac{e^x-1}{(e^x-1)+xe^x} \overset{\text{L.H.}}{=} \lim_{x\to 0^+} \frac{e^x}{e^x+(e^x+xe^x)} = \frac{1}{1+1+0} = \frac{1}{2}$$

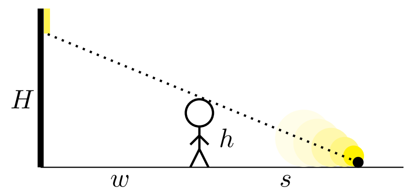
(c) $\lim\limits_{x\to\infty} \left(1 - \dfrac{2}{x}\right)^x = \dfrac{1}{e^2}$

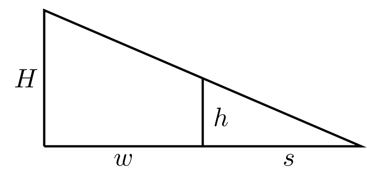
$$\begin{aligned}
y &= \lim_{x\to\infty} \left(1-\frac{2}{x}\right)^x \\
\ln y &= \lim_{x\to\infty} \ln\left(1-\frac{2}{x}\right)^x \\
\ln y &= \lim_{x\to\infty} x\ln\left(1-\frac{2}{x}\right) \\
\ln y &= \lim_{x\to\infty} \frac{\ln\left(1-\frac{2}{x}\right)}{\frac{1}{x}} \\
\ln y &\overset{\text{L.H.}}{=} \lim_{x\to\infty} \frac{\frac{1}{1-\frac{2}{x}} \cdot \frac{2}{x^2}}{\frac{-1}{x^2}} \\
\ln y &= \lim_{x\to\infty} \frac{-2}{1-\frac{2}{x}} \\
\ln y &= \frac{-2}{1-0} \\
\ln y &= -2 \\
y &= e^{-2}
\end{aligned}$$

Alternatively, use the fact that $\lim\limits_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x = e$, *so that* $\lim\limits_{x\to\infty} \left(1-\dfrac{2}{x}\right)^x = \lim\limits_{x\to\infty} \left(1+\dfrac{-2}{x}\right)^x =$ $\lim\limits_{x\to\infty} \left[\left(1+\dfrac{1}{x/-2}\right)^{x/-2}\right]^{-2} = e^{-2}$.

8. (10 points) A spotlight on the ground is illuminating a wall that is 40 ft away from the spotlight. A person that is 6 ft tall begins walking from the spotlight towards the wall at a speed of 6 ft/sec. How fast is the height of the resulting shadow along the wall changing when the person is 15 ft away from the spotlight?

Solution. *We first diagram the situation, letting H be the height of the shadow along the wall, h be the height of the person, s be the distance to the spotlight, and w be the distance to the wall:*





We want the rate of change of the height of the shadow, i.e. $\frac{dH}{dt}$. By similar triangles, we have...

$$\frac{h}{s} = \frac{H}{w+s}$$

$$sH = h(w+s)$$

Now implicitly differentiating with respect to time, we have...

$$\frac{d}{dt}(sH) = \frac{d}{dt}\left[h(w+s)\right]$$

$$\frac{ds}{dt}\,H + s\,\frac{dH}{dt} = \frac{dh}{dt}\,(w+s) + h\left(\frac{dw}{dt} + \frac{ds}{dt}\right)$$

When the person is 15 ft from the spotlight, i.e. $s = 15$, they are 25 ft from the wall (because the total distance to the spotlight is 40 ft), i.e. $w = 25$. The person is 6 ft tall, i.e. $h = 6$. But then...

$$\frac{h}{s} = \frac{H}{w+s} \implies \frac{6}{15} = \frac{H}{25+15} \implies \frac{2}{5} = \frac{H}{40} \implies H = 40 \cdot \frac{2}{5} = 16$$

The person is not growing/shrinking and we know that the person is walking towards the spotlight at 6 ft/s. But then...

$$h = 6\,\mathit{ft}, \qquad s = 15\,\mathit{ft}, \qquad w = 25\,\mathit{ft}, \qquad H = 16\,\mathit{ft}$$
$$\frac{dh}{dt} = 0\,\mathit{ft/s}, \quad \frac{ds}{dt} = 6\,\mathit{ft/s}, \quad \frac{dw}{dt} = -6\,\mathit{ft/s}, \quad \frac{dH}{dt} = ?$$

Therefore, we know...

$$\frac{ds}{dt}\,H + s\,\frac{dH}{dt} = \frac{dh}{dt}\,(w+s) + h\left(\frac{dw}{dt} + \frac{ds}{dt}\right)$$

$$6(16) + 15\,\frac{dH}{dt} = 0(25+15) + 6\,(-6+6)$$

$$6(16) + 15\,\frac{dH}{dt} = 0$$
$$\frac{dH}{dt} = -\frac{6(16)}{15}$$
$$\frac{dH}{dt} = -\frac{32}{5} \approx -6.4\,\mathit{ft/s}$$

Therefore, the height of the shadow is shrinking at 6.4 ft per second.

9. (10 points) Suppose that $f(x)$ is a function such that $f(x)$ is differentiable on $[1, 5]$, $f(1) = 10$, and $-1 \leq f'(x) \leq 3$ on $[1, 5]$. Find the smallest and largest possible values for $f(5)$. Be sure to show all your work and fully justify your solutions.

Solution. *Observe that...*

- $f(x)$ *is continuous on* $[1, 5]$*: Differentiable functions are continuous. Therefore,* $f(x)$ *is continuous on* $[1, 5]$.
- $f(x)$ *is differentiable on* $(1, 5)$*: We know* $f(x)$ *is differentiable on the entirety of* $[1, 5]$.

Therefore, by the Mean Value Theorem, there exists $c \in (1, 5)$ *such that...*

$$f(5) - f(1) = f'(c)(5 - 1)$$

$$f(5) - f(1) = 4f'(c)$$

$$f(5) = f(1) + 4f'(c)$$

$$f(5) = 10 + 4f'(c)$$

But then using the fact that $-1 \leq f'(x) \leq 3$ *on* $[1, 5]$*, we have...*

$$f(5) = 10 + 4f'(c) \geq 10 + 4(-1) = 10 - 4 = 6$$

$$f(5) = 10 + 4f'(c) \leq 10 + 4(3) = 10 + 12 = 22$$

Therefore, we know...

$$6 \leq f(5) \leq 22$$

10. (10 points) let $f(x) = 3x^4 - 4x^3 - 2$. Find the absolute minimum and absolute maximum *values* of $f(x)$ on the interval $[-1, 2]$. Be sure to show all your work and fully your answers. You may use any derivative test to justify your answers.

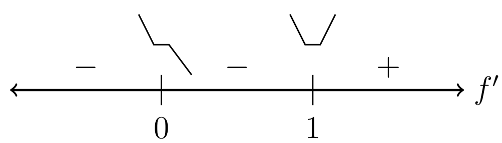
Solution. *We have* $f'(x) = 12x^3 - 12x^2$. *But then setting* $f'(x) = 0$, *we have...*

$$f'(x) = 0$$

$$12x^3 - 12x^2 = 0$$

$$12x^2(x-1) = 0$$

But then either $12x^2 = 0$, *which implies* $x = 0$, *or* $x - 1 = 0$, *which implies* $x = 1$. *Therefore, the only critical values are* $x = 0, 1$, *which are both in the interval* $[-1, 2]$. *Now observe...*



Therefore, $x = 1$ *is a local minimum—but not necessarily an absolute minimum on* $[-1, 2]$. *Observe...*

$$f(1) = 3(1^4) - 4(1^3) - 2 = 3 - 4 - 2 = -3$$

$$f(-1) = 3(-1)^4 - 4(-1)^3 - 2 = 3 + 4 - 2 = 5$$

$$f(2) = 3(2^4) - 4(2^3) - 2 = 3(16) - 4(8) - 2 = 48 - 32 - 2 = 14$$

Therefore, the absolute minimum on $[-1, 2]$ *is* -3 *at* $x = 1$ *and the absolute maximum on* $[-1, 2]$ *is* 14 *at* $x = 2$.

Absolute Minimum $= -3$ *at* $x = 1$

Absolute Maximum $= 14$ *at* $x = 2$