Check-In 01/16. *(True/False)* Given $\int_0^{\pi} e^{\sin x} \cos x \, dx$, the u-substitution $u = \sin x$ transforms this integral into $\int_0^{\pi} e^u \, du$.

Solution. The statement is *false*. If $u = \sin x$, then $du = \cos x \, dx$. So indeed, this u-substitution would transform the integral $\int e^{\sin x} \cos x \, dx$ into the integral $\int e^u \, du$. However with definite integrals, one needs to remember to transform the limits as well. If $x = 0$, then $u = \sin(0) = 0$. If $x = \pi$, then $u = \sin(\pi) = 0$. Therefore, the correct substitution is $\int_0^{\pi} e^{\sin x} \cos x \, dx = \int_0^0 e^u \, du = 0$.

Check-In 01/21. *(True/False)* To integrate $\int \operatorname{arccot} \theta \, d\theta$, one can use integration-by-parts by choosing $u = \operatorname{arccot} \theta$ and $dv = 1$.

Solution. The statement is *true*. Using LIATE, it is likely that the choice of $u = \operatorname{arccot} \theta$ will work. With 'nothing left' in the integrand, this means that $dv = 1$. We fill in our box as follows:

$$\begin{array}{|c|c|} \hline \operatorname{arccot} \theta & \\ \hline & 1 \\ \hline \end{array} \quad \Longrightarrow \quad \begin{array}{|c|c|} \hline \operatorname{arccot} \theta & \theta \\ \hline \dfrac{-1}{1+\theta^2} & 1 \\ \hline \end{array}$$

Then using the 'Rule of 7', we find that...

$$\int \operatorname{arccot} \theta \, d\theta = \theta \operatorname{arccot} \theta - \int \frac{-\theta}{1+\theta^2} \, d\theta = \theta \operatorname{arccot} \theta + \int \frac{\theta}{1+\theta^2} \, d\theta$$

Using the u-substitution $u = 1 + \theta^2$, we see that $\int \frac{\theta}{1+\theta^2} \, d\theta = \frac{1}{2} \ln|1+\theta^2| + C$. Therefore, we have...

$$\int \operatorname{arccot} \theta \, d\theta = \theta \operatorname{arccot} \theta + \frac{1}{2} \ln|1+\theta^2| + C$$

Check-In 01/23. *(True/False)* The integral $\int e^{x/2} \sin(3x) \, dx$ is a 'looping' integral.

Solution. The statement is *true*. Recall that integrals whose integrand is the product of an exponential function with $\sin x$ or $\cos x$ 'loop.' We can see this directly: choose $u = \sin(3x)$. Using the 'box method', we have...

$\sin(3x)$	$2e^{x/2}$
$3\cos(3x)$	$e^{x/2}$

Therefore, we have...

$$\int e^{x/2}\sin(3x)\;dx = 2e^{x/2}\sin(3x) - \int 6e^{x/2}\cos(3x)\;dx$$

But this integral on the right also requires integration-by-parts: we choose $u = \cos(3x)$ and then...

$\cos(3x)$	$12e^{x/2}$
$-3\sin(3x)$	$6e^{x/2}$

So then we have...

$$\begin{aligned}
\int e^{x/2}\sin(3x)\;dx &= 2e^{x/2}\sin(3x) - \int 6e^{x/2}\cos(3x)\;dx \\
&= 2e^{x/2}\sin(3x) - \left(12e^{x/2}\cos(3x) - \int -36e^{x/2}\sin(3x)\;dx\right) \\
&= 2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x) + \int -36e^{x/2}\sin(3x)\;dx \\
&= 2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x) - 36\int e^{x/2}\sin(3x)\;dx
\end{aligned}$$

Observe that we have 'looped'—obtaining a multiple of the original integral on the right. Adding $36\int e^{x/2}\sin(3x)\;dx$ to both sides, we have...

$$37\int e^{x/2}\sin(3x)\;dx = 2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x)$$

Therefore, we have...

$$\int e^{x/2}\sin(3x)\;dx = \frac{2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x)}{37} + C$$

We can shortcut this work by adjusting tabular integration:

u		dv
$\sin(3x)$		$e^{x/2}$
	$+$	
$3\cos(3x)$		$2e^{x/2}$
	$-$	
$-9\sin(3x)$	$+$	$4e^{x/2}$

Therefore, we have. . .

$$\begin{aligned}
\int e^{x/2}\sin(3x)\;dx &= 2e^{x/2} - 12e^{x/2}\cos(3x) - 36\int e^{x/2}\sin(3x)\;dx \\
37\int e^{x/2}\sin(3x)\;dx &= 2e^{x/2} - 12e^{x/2}\cos(3x) \\
\int e^{x/2}\sin(3x)\;dx &= \frac{2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x)}{37} + C
\end{aligned}$$

Check-In 01/28. *(True/False)* To integrate $\displaystyle\int \cos^8\theta\sin^5\theta\;d\theta$, one should choose $u = \cos\theta$.

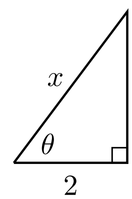
Solution. The statement is *true*. Observe that if we choose $u = \cos\theta$, then $\cos^8\theta$ becomes u^8. We know du will then produce a $\sin\theta$—specifically $-\sin\theta$. This 'uses' one of the $\sin\theta$'s in the integrand. This leaves $\sin^4\theta$ remaining in the integrand. But we can replace even powers of $\sin\theta$ in terms of $\cos\theta$. So we can carry out this substitution. Observe that if $u = \cos\theta$, then $du = -\sin\theta\;d\theta$. But then using the fact that $\sin^2\theta = 1 - \cos^2\theta$ (so that $\sin^4\theta = (\sin^2\theta)^2 = (1-\cos^2\theta)^2$), we have. . .

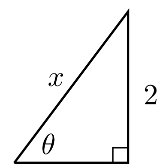
$$\begin{aligned}
\int \cos^8\theta\sin^5\theta\;d\theta &= \int \cos^8\theta\sin^4\theta\cdot\sin\theta\;d\theta \\
&= -\int \cos^8\theta\sin^4\theta\cdot -\sin\theta\;d\theta \\
&= -\int \cos^8\theta(1-\cos^2\theta)^2\cdot -\sin\theta\;d\theta \\
&= -\int u^8(1-u^2)^2\;du \\
&= -\int u^8(1-2u^2+u^4)\;du \\
&= \int -u^8 + 2u^{10} - u^{12}\;du \\
&= -\frac{u^9}{9} + \frac{2u^{11}}{11} - \frac{u^{13}}{13} + C \\
&= -\frac{\cos^9\theta}{9} + \frac{2\cos^{11}\theta}{11} - \frac{\cos^{13}\theta}{13} + C
\end{aligned}$$

Alternatively, observe that if we had chosen $u = \sin\theta$, then $\sin^5\theta$ becomes u^5. We know that du produces a $\cos\theta$. This 'uses' one of the $\cos\theta$'s in the integrand. This leaves $\cos^7\theta$ remaining in the integrand. However, we can only replace even powers of $\cos\theta$ in terms of $\sin\theta$ using $\cos^2\theta = 1 - \sin^2\theta$. So without using other identities or using other techniques, we cannot 'simply' choose $u = \sin\theta$ for this integral.

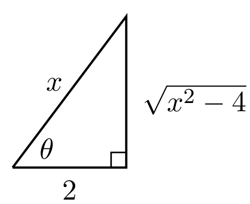
Check-In 01/30. *(True/False)* To compute $\displaystyle\int \frac{x^3}{\sqrt{x^2-4}}\,dx$, one can make the substitution $x = 2\sec\theta$.

Solution. The statement is *true*. Observe we have the term $x^2 - 4$, which resembles a term from the Pythagorean Theorem. For a right triangle, we know that $a^2 + b^2 = c^2$. This implies that $a^2 = c^2 - b^2$, which is $x^2 - 4$ if $c^2 = x^2$ and $b^2 = 4$, i.e. $c = x$ and $b = 2$. We construct a triangle with $c = x$ and $b = 2$. This only gives two possibilities:





In the first, we would have $\cos\theta = \frac{2}{x}$, which implies that $x = \frac{2}{\cos\theta} = 2\sec\theta$. This is the given substitution. In the second, we would have $\sin\theta = \frac{2}{x}$, which implies that $x = \frac{2}{\sin\theta} = 2\csc\theta$. Choosing the former substitution, we would have $x = 2\sec\theta$, so that $dx = 2\sec\theta\tan\theta$. Calling the vertical side s and using the Pythagorean Theorem, we see that $2^2 + s^2 = x^2$, i.e. $4 + s^2 = x^2$. But then $s = \sqrt{x^2-4}$. This gives us the following triangle:



We need to find x^3 and $\sqrt{x^2-4}$. Because $x = 2\sec\theta$, we know that $x^3 = (2\sec\theta)^3 = 2^3\sec^3\theta$. To find $\sqrt{x^2-4}$, observe that $\tan\theta = \frac{\sqrt{x^2-4}}{2}$, which implies that $\sqrt{x^2-4} = 2\tan\theta$. Therefore, we have...

$$\int \frac{x^3}{\sqrt{x^2-4}}\,dx = \int \frac{2^3\sec^3\theta}{2\tan\theta} \cdot 2\sec\theta\tan\theta\;d\theta = \int 8\sec^4\theta\;d\theta$$

Now using the fact that $\tan^2\theta + 1 = \sec^2\theta$, we know that

$$\sec^4\theta = (\sec^2\theta)^2 = (\tan^2\theta + 1)^2 = \tan^4\theta + 2\tan^2\theta + 1$$

But then...

$$\int 8\sec^4\theta\;d\theta = \int 8\sec^2\theta\sec^2\theta\;d\theta = \int 8\sec^2\theta(\tan^2\theta+1)\;d\theta = \int 8(\tan^2\theta+1)\cdot\sec^2\theta\;d\theta$$

Now letting $u = \tan\theta$, we have $du = \sec^2\theta\;d\theta$. But then...

$$\int 8(\tan^2\theta+1)\cdot\sec^2\theta\;d\theta = \int 8(u^2+1)\;du = 8\int(u^2+1)\;du = 8\left(\frac{u^3}{3} + u\right) + C$$

Replacing u, we have...

$$8\left(\frac{u^3}{3}+u\right)+C=8\left(\frac{\tan^3\theta}{3}+\tan\theta\right)+C$$

But from the triangle, we know that $\tan\theta=\frac{\sqrt{x^2-4}}{2}$. Therefore, we have...

$$8\left(\frac{\tan^3\theta}{3}+\tan\theta\right)+C=.8\left(\frac{1}{3}\cdot\left(\frac{\sqrt{x^2-4}}{2}\right)^3+\frac{\sqrt{x^2-4}}{2}\right)+C$$

If we simplify this, we have...

$$\begin{gathered}
8\left(\frac{1}{3}\cdot\left(\frac{\sqrt{x^2-4}}{2}\right)^3+\frac{\sqrt{x^2-4}}{2}\right)+C \\
8\cdot\frac{\sqrt{x^2-4}}{2}\left(\frac{1}{3}\cdot\left(\frac{\sqrt{x^2-4}}{2}\right)^2+1\right)+C \\
4\sqrt{x^2-4}\left(\frac{1}{3}\cdot\frac{x^2-4}{4}+1\right)+C \\
4\sqrt{x^2-4}\left(\frac{x^2-4}{12}+1\right)+C \\
4\sqrt{x^2-4}\left(\frac{x^2-4}{12}+\frac{12}{12}\right)+C \\
4\sqrt{x^2-4}\left(\frac{x^2+8}{12}\right)+C \\
\frac{1}{3}\sqrt{x^2-4}\,(x^2+8)+C
\end{gathered}$$

Therefore, we have...

$$\int\frac{x^3}{\sqrt{x^2-4}}\,dx=\frac{1}{3}\sqrt{x^2-4}\,(x^2+8)+C$$

Check-In 02/04. *(True/False)* The rational function $\dfrac{x^6-4x}{(x-1)(x+2)(x^2+4)}$ can be decomposed as $\dfrac{A}{x-1}+\dfrac{B}{x+2}+\dfrac{Cx+D}{x^2+4}$.

Solution. The statement is *false*. We know that given a rational function has a partial fraction decomposition given in the 'traditional' way so long as the degree of the numerator is *less* than the degree of the denominator. Observe that the degree of the numerator in the original function is 6 while the degree of the denominator is $1+1+2=4$. So, this cannot be broken down in the 'traditional' way. Alternatively, find the common denominator of $(x-1)(x+2)(x^2+4)$ for the given decomposition, observe that the numerator will have at most degree 4, which could not possibly yield the numerator of x^6-4x with degree 6. One would first need to long divide x^6-4x by

$(x-1)(x+2)(x^2+4)$ to find the quotient and remainder before trying to give a partial fraction decomposition. Observe...

$$(x-1)(x+2)(x^2+4) = (x^2+x-2)(x^2+4) = x^4+x^3+2x^2+4x-8$$

But then...

$$\begin{array}{r|rrrrrrr}
 & & & x^2 & & -x & -1 & \\ \hline
x^4+x^3+2x^2+4x-8 & x^6 & & & & & -4x & \\
 & -x^6 & -x^5 & -2x^4 & -4x^3 & +8x^2 & & \\ \cline{2-6}
 & & -x^5 & -2x^4 & -4x^3 & +8x^2 & -4x & \\
 & & x^5 & +x^4 & +2x^3 & +4x^2 & -8x & \\ \cline{3-7}
 & & & -x^4 & -2x^3 & +12x^2 & -12x & \\
 & & & x^4 & +x^3 & +2x^2 & +4x & -8 \\ \cline{4-8}
 & & & & -x^3 & +14x^2 & -8x & -8
\end{array}$$

Therefore, we have...

$$\frac{x^6-4x}{(x-1)(x+2)(x^2+4)} = x^2-x-1+\frac{-x^3+14x^2-8x-8}{x^4+x^3+2x^2+4x-8} = x^2-x-1+\frac{-x^3+14x^2-8x-8}{(x-1)(x+2)(x^2+4)}$$

We can then find the 'traditional' partial fraction decomposition of the resulting rational function above because the degree of the numerator, which is 3, is *less* than the degree of the denominator, which is 4. We would have...

$$\frac{-x^3+14x^2-8x-8}{(x-1)(x+2)(x^2+4)} = \frac{A}{x-1}+\frac{B}{x+2}+\frac{Cx+D}{x^2+4}$$

Using Heaviside's Method/Cover-Up Method, we can find A and B 'instantly':

$$A = \frac{-1^3+14(1^2)-8(1)-8}{(1+2)(1^2+4)} = -\frac{3}{15} = -\frac{1}{5}$$

$$B = \frac{-(-2^3)+14(-2)^2-8(-2)-8}{(-2-1)\big((-2)^2+4\big)} = \frac{72}{-24} = -3$$

To find C, D, we can get a common denominator:

$$\frac{A}{x-1}+\frac{B}{x+2}+\frac{Cx+D}{x^2+4} = \frac{A(x+2)(x^2+4)+B(x-1)(x^2+4)+(Cx+D)(x-1)(x+2)}{(x-1)(x+2)(x^2+4)}$$

Using the fact that the rational functions now have equal denominators, we can equate their numerators:

$$-x^3+14x^2-8x-8 = A(x+2)(x^2+4)+B(x-1)(x^2+4)+(Cx+D)(x-1)(x+2)$$

$$-x^3+14x^2-8x-8 = -\frac{1}{5}(x+2)(x^2+4)-3(x-1)(x^2+4)+(Cx+D)(x-1)(x+2)$$

We multiply the last equation by 5 to clear fractions:

$$5(-x^3+14x^2-8x-8) = -(x+2)(x^2+4)-15(x-1)(x^2+4)+5(Cx+D)(x-1)(x+2)$$

Rather than expand this and relate coefficients to obtain a system of equations, we will simply substitute two x-values (ones not used in Heaviside's Method): using $x = 0$, we have...

$$\begin{aligned}
5(-x^3 + 14x^2 - 8x - 8) &= -(x+2)(x^2+4) - 15(x-1)(x^2+4) + 5(Cx+D)(x-1)(x+2) \\
-40 &= 52 - 10D \\
-92 &= -10D \\
\frac{46}{5} &= D
\end{aligned}$$

and using $x = -1$ and the value for D above, we have...

$$\begin{aligned}
5(-x^3 + 14x^2 - 8x - 8) &= -(x+2)(x^2+4) + 15(x-1)(x^2+4) + 5(Cx+D)(x-1)(x+2) \\
75 &= 145 - 10(-C + D) \\
-70 &= 10C - 10D \\
-70 &= 10C - 10 \cdot \frac{46}{5} \\
-70 &= 10C - 92 \\
22 &= 10C \\
\frac{11}{5} &= C
\end{aligned}$$

Therefore, we have...

$$\frac{x^6 - 4x}{(x-1)(x+2)(x^2+4)} = x^2 - x - 1 + \frac{-1/5}{x-1} + \frac{-3}{x+2} + \frac{\frac{11}{5}x + \frac{46}{5}}{x^2+4} = x^2 - x - 1 - \frac{1}{5(x-1)} - \frac{3}{x+2} + \frac{11x+46}{5(x^2+4)}$$

Check-In 02/06. *(True/False)*

$$\int \frac{5}{x+1} - \frac{1}{(x+1)^2} - \frac{x+1}{x^2+1}\, dx = 5\ln(x+1) - \frac{1}{x+1} - \frac{1}{2}\ln|x^2+1| + \arctan x + K$$

Solution. The statement is *false*. First, we know that...

$$\int \frac{5}{x+1}\, dx = 5\ln|x+1| + K$$

The integral of the second term is also incorrect:

$$\int -\frac{1}{(x+1)^2}\, dx = \int -(x+1)^{-2}\, dx = \frac{-(x+1)^{-1}}{-1} + K = \frac{1}{x+1} + K$$

Moreover, the third term is also incorrect as the negative has been improperly distributed:

$$-\int \frac{x+1}{x^2+1}\, dx = -\int \frac{x}{x^2+1} + \frac{1}{x^2+1}\, dx = -\left(\frac{1}{2}\ln|x^2+1| + \arctan x\right) + K = -\frac{1}{2}\ln|x^2+1| - \arctan x + K$$

Check-In 02/11. *(True/False)* The integral $\displaystyle\int_1^\infty \frac{dx}{\sqrt[5]{x^{12}}}$ converges.

Solution. The statement is *true*. Recall that $\displaystyle\int_1^\infty \frac{dx}{x^p}$ converges if $p > 1$ and diverges otherwise. Observe that $\displaystyle\int_1^\infty \frac{dx}{\sqrt[5]{x^{12}}} = \int_1^\infty \frac{dx}{x^{12/5}}$. Because the power of x, $\frac{12}{5}$, is greater than 1, we know that this integral converges. We can see this directly:

$$\int_1^\infty \frac{dx}{\sqrt[5]{x^{12}}} := \lim_{N\to\infty} \int_1^N \frac{dx}{x^{12/5}} = \lim_{N\to\infty} -\frac{5}{7x^{7/5}}\bigg|_1^N = \lim_{N\to\infty} -\frac{5}{7N^{7/5}} - \frac{-5}{7(1^{7/5})} = 0 + \frac{5}{7} = \frac{5}{7}$$

Therefore, the integral converges to $\frac{5}{7}$.

Check-In 02/20. *(True/False)* Considering the infinite series $\displaystyle\sum_{n=1}^\infty \frac{n-1}{n^3}$ using the Divergence Test, because $\displaystyle\lim_{n\to\infty} \frac{n-1}{n^3} = 0$, the series converges converges.

Solution. The statement is *false*. The Divergence Test can *never* determine that a series converges; otherwise, they would have called it the Convergence Test. The Divergence Test states that for a series $\displaystyle\sum_{n=1}^\infty a_n$, the series diverges if $\displaystyle\lim_{n\to\infty} a_n \neq 0$. If $\displaystyle\lim_{n\to\infty} a_n = 0$, we know zero information about whether the series converges or diverges. Because $\displaystyle\lim_{n\to\infty} \frac{n-1}{n^3} = 0$, we cannot yet determine whether the series $\displaystyle\sum_{n=1}^\infty \frac{n-1}{n^3}$ converges or diverges. In fact, Because for 'large' n, $\displaystyle\frac{n-1}{n^3} \approx \frac{n}{n^3} = \frac{1}{n^2}$ and the fact that the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$ converges, we suspect that the series $\sum_{n=1}^\infty \displaystyle\frac{n-1}{n^3}$ converges.

Check-In 02/25. *(True/False)* The series $\displaystyle\sum_{n=1}^\infty 5\left(\frac{2^{2n+1}}{3^n}\right)$ diverges.

Solution. The statement is *true*. This series is geometric, i.e. the series can be put in the form $\sum ar^n$. Observe that...

$$\sum_{n=1}^\infty 5\left(\frac{2^{2n+1}}{3^n}\right) = \sum_{n=1}^\infty 5\left(\frac{2^{2n}\cdot 2}{3^n}\right) = \sum_{n=1}^\infty 10\left(\frac{(2^2)^n}{3^n}\right) = \sum_{n=1}^\infty 10\left(\frac{2^2}{3}\right)^n = \sum_{n=1}^\infty 10\left(\frac{4}{3}\right)^n$$

Observe that this series is geometric with $a = 10$ and $r = \frac{4}{3}$. Because $|r| = \frac{4}{3} \geq 1$, the series diverges by the Geometric Series Test.

Check-In 02/27. *(True/False)* The series $\sum_{n=2}^{\infty} \dfrac{n^2+1}{\sqrt{n^3-5}}$ diverges.

Solution. The statement is *true*. Observe that for 'large' n, $\dfrac{n^2+1}{\sqrt{n^3-5}} \approx \dfrac{n^2}{\sqrt{n^3}} = \dfrac{n^2}{n^{3/2}} = n^{1/2}$. Because the series $\sum_{n=2}^{\infty} \sqrt{n}$ diverges (by the Divergence Test), we suspect that the series $\sum_{n=2}^{\infty} \dfrac{n^2+1}{\sqrt{n^3-5}}$ diverges. We can prove this with the Direct Comparison Test:

$$\sum_{n=2}^{\infty} \frac{n^2+1}{\sqrt{n^3-5}} > \sum_{n=2}^{\infty} \frac{n^2}{\sqrt{n^3}} = \sum_{n=2}^{\infty} \sqrt{n}$$

Because $\lim_{n\to\infty} \sqrt{n} = \infty \neq 0$, the series $\sum_{n=2}^{\infty} \sqrt{n}$ diverges by the Divergence Test. Therefore, $\sum_{n=2}^{\infty} \dfrac{n^2+1}{\sqrt{n^3-5}}$ diverges by the Direct Comparison Test. Alternatively, we know that because $\lim_{n\to\infty} \sqrt{n} = \infty \neq 0$, the series $\sum_{n=2}^{\infty} \sqrt{n}$ diverges by the Divergence Test. But then because...

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{n^2+1}{\sqrt{n^3-5}}}{\sqrt{n}} &= \lim_{n\to\infty} \frac{n^2+1}{\sqrt{n}\cdot\sqrt{n^3-5}} \\
&= \lim_{n\to\infty} \frac{n^2+1}{\sqrt{n^4-5n}} \\
&= \lim_{n\to\infty} \frac{n^2+1}{\sqrt{n^4-5n}} \cdot \frac{1/n^2}{1/n^2} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{\frac{\sqrt{n^4-5n}}{n^2}} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{\frac{\sqrt{n^4-5n}}{\sqrt{n^4}}} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{\sqrt{\frac{n^4-5n}{n^4}}} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{\sqrt{1-\frac{5}{n^3}}} \\
&= \frac{1+0}{\sqrt{1-0}} \\
&= 1 < \infty
\end{aligned}$$

Because this limit is not also 0, by the Limit Comparison Test, the series $\sum_{n=2}^{\infty} \dfrac{n^2+1}{\sqrt{n^3-5}}$ also diverges.

Check-In 03/04. *(True/False)* The series $\displaystyle\sum_{n=-5}^{\infty} \frac{n^2+5}{2^n-6}$ converges.

Solution. The statement is *true*. Because for 'large' n, $\dfrac{n^2+5}{2^n-6} \approx \dfrac{n^2}{2^n}$ and the fact that 2^n grows much faster than n^2, we suspect that $\displaystyle\sum_{n=-5}^{\infty} \frac{n^2}{2^n}$ converges. Hence, we suspect $\displaystyle\sum_{n=-5}^{\infty} \frac{n^2+5}{2^n-6}$ converges. Because the series $\displaystyle\sum_{n=-5}^{\infty} \frac{n^2+5}{2^n-6}$ converges if and only if the series $\displaystyle\sum_{n=4}^{\infty} \frac{n^2+5}{2^n-6}$ (whose terms are all positive),[1] it suffices to show that $\displaystyle\sum_{n=4}^{\infty} \frac{n^2+5}{2^n-6}$ converges. We know...

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{2} = 1^2 \cdot \frac{1}{2} = \frac{1}{2} < 1$$

Therefore, by the Ratio Test, the series $\displaystyle\sum_{n=4}^{\infty} \frac{n^2}{2^n}$ converges absolutely. Alternatively, we have...

$$\lim_{n\to\infty} \left|\frac{n^2}{2^n}\right|^{1/n} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^2}{2} = \frac{1^2}{2} = \frac{1}{2} < 1$$

Therefore, by the Root Test, the series $\displaystyle\sum_{n=4}^{\infty} \frac{n^2}{2^n}$ converges absolutely. But then...

$$\lim_{n\to\infty} \frac{\frac{n^2+5}{2^n-6}}{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{2^n(n^2+5)}{n^2(2^n-6)} = \lim_{n\to\infty} \frac{2^n}{2^n-6} \cdot \frac{n^2+5}{n^2} = 1 \cdot 1 = 1 < \infty$$

Because this limit is also not 0, the series $\displaystyle\sum_{n=4}^{\infty} \frac{n^2+5}{2^n-6}$ converges by the Limit Comparison Test. Alternatively, using the fact that $2^{n-1} > 6$ for $n \geq 4$, we have...

$$\sum_{n=4}^{\infty} \frac{n^2+5}{2^n-6} < \sum_{n=4}^{\infty} \frac{n^2+5n^2}{2^n-2^{n-1}} = \sum_{n=4}^{\infty} \frac{6n^2}{2^{n-1}} = 12\sum_{n=4}^{\infty} \frac{n^2}{2^n}$$

Therefore, the series $\displaystyle\sum_{n=4}^{\infty} \frac{n^2+5}{2^n-6}$ converges by the Direct Comparison Test.

[1] We only need $n \geq 3$ for the Limit Comparison Test. However, we need $n \geq 4$ for our Limit Comparison Test.

Check-In 03/06. *(True/False)* The series $\displaystyle\sum_{n=1}^{\infty} \frac{5n-2}{n^3+1}$ converges absolutely.

Solution. The statement is *true*. For large n, we have $\dfrac{5n-2}{n^3+1} \approx \dfrac{5n}{n^3} = \dfrac{5}{n^2}$. Because the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we suspect this series converges. First, observe that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test with $p = 2 > 1$. But then…

$$\lim_{n\to\infty} \frac{\frac{5n-2}{n^3+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2(5n-2)}{n^3+1} = \lim_{n\to\infty} \frac{5n^3-2n^2}{n^3+1} = \frac{5}{1} = \underbrace{5}_{\neq 0} < \infty$$

Therefore, by the Limit Comparison Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{5n-2}{n^3+1}$ converges. Alternatively, observe that…

$$\sum_{n=1}^{\infty} \frac{5n-2}{n^3+1} < \sum_{n=1}^{\infty} \frac{5n}{n^3} = 5\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Therefore, by the Direct Comparison Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{5n-2}{n^3+1}$ converges.

But all the terms of $\displaystyle\sum_{n=1}^{\infty} \frac{5n-2}{n^3+1}$ are positive. [Observe that $n^3+1 > 0$ for $n \geq 1$ and $5n-2 > 0$ so long as $n > \frac{2}{5}$.] Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{5n-2}{n^3+1}$ converges absolutely.

Check-In 03/18. *(True/False)* The series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+3}\right)^n$ converges.

Solution. The statement is *true*. Observe that…

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left| \left(\frac{n+1}{2n+3}\right)^n \right|^{1/n} = \lim_{N\to\infty} \frac{n+1}{2n+3} = \frac{1}{2} < 1$$

Therefore, by the Root Test, the series converges absolutely. Alternatively, we can use the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{\left(\dfrac{(n+1)+1}{2(n+1)+3} \right)^{n+1}}{\left(\dfrac{n+1}{2n+3} \right)^n} \\
&= \lim_{n\to\infty} \frac{\left(\dfrac{n+2}{2n+5} \right)^{n+1}}{\left(\dfrac{n+1}{2n+3} \right)^n} \\
&= \lim_{n\to\infty} \frac{\left(\dfrac{n+2}{2n+5} \right)^{n} \left(\dfrac{n+2}{2n+5} \right)}{\left(\dfrac{n+1}{2n+3} \right)^n} \\
&= \lim_{n\to\infty} \frac{\left(\dfrac{n+2}{2n+5} \right)^{n}}{\left(\dfrac{n+1}{2n+3} \right)^n} \cdot \frac{n+2}{2n+5} \\
&= \lim_{n\to\infty} \left(\frac{\dfrac{n+2}{2n+5}}{\dfrac{n+1}{2n+3}} \right)^n \cdot \frac{n+2}{2n+5} \\
&= \lim_{n\to\infty} \left(\frac{(n+2)(2n+3)}{(2n+5)(n+1)} \right)^n \cdot \frac{n+2}{2n+5} \\
&= 1 \cdot \frac{1}{2} \\
&= \frac{1}{2} < 1
\end{aligned}$$

Therefore, the series converges absolutely.[2]

Check-In 03/27. *(True/False)* If a power series has an interval of convergence of $(-1, 3]$, then the center is $x = 1$ and the radius of convergence is $R = 2$.

Solution. The statement is *true*. We know that the center of the interval must be $c = \dfrac{3 + (-1)}{2} = \dfrac{2}{2} = 1$. Therefore, the center is $x = 1$. The radius of convergence is half the width of the interval. But then the radius of convergence is $R = \dfrac{3 - (-1)}{2} = \dfrac{3+1}{2} = \dfrac{4}{2} = 2$.

[2] Note. To show that $\lim_{n\to\infty} \left(\frac{(n+2)(2n+3)}{(2n+5)(n+1)} \right)^n = 1$, let $L = \lim_{n\to\infty} \left(\frac{(n+2)(2n+3)}{(2n+5)(n+1)} \right)^n$. But then $\ln L = \lim_{n\to\infty} \ln \left(\frac{(n+2)(2n+3)}{(2n+5)(n+1)} \right)^n = \lim_{n\to\infty} n \ln \left(\frac{(n+2)(2n+3)}{(2n+5)(n+1)} \right) = \lim_{n\to\infty} \frac{\ln \left(\frac{(n+2)(2n+3)}{(2n+5)(n+1)} \right)}{1/n} \overset{\text{L.H.}}{=} \lim_{n\to\infty} \frac{n^2(4n+7)}{4n^4 + 28n^3 + 71n^2 + 77n + 30} = 0$. But then $\ln L = 0$, so that $L = e^0 = 1$.

Check-In 04/01. *(True/False)* The series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2 3^n}$ has center $x = 5$ and radius of convergence 3. Therefore, the series converges for $x = 7$.

Solution. The statement is *true*. Clearly, $x = 5$ 'kills' the entire series, i.e. makes the summand 0. So, the center is $x = 5$. We can confirm that the series has radius of convergence 3. For instance, using the root test, we have...

$$\lim_{n\to\infty} \left| \frac{(x-5)^n}{n^2 3^n} \right|^{1/n} = \lim_{n\to\infty} \frac{|x-5|}{n^{2/n} 3} = \frac{|x-5|}{1 \cdot 3} = \frac{|x-5|}{3}$$

By the root test, we know that this series converges absolutely if this limit is less than 1. But then...

$$\begin{gathered} \frac{|x-5|}{3} < 1 \\ |x-5| < 3 \\ -3 < x - 5 < 3 \\ 2 < x < 8 \end{gathered}$$

The radius of convergence is then $R = \dfrac{8-2}{2} = \dfrac{6}{2} = 3$. But because $2 < 7 < 8$, we know that the series converges absolutely for $x = 7$.

Check-In 04/03. *(True/False)* The first-order Taylor series for $f(x)$ at x_0 is the tangent line to $f(x)$ at x_0. Furthermore, the Taylor series for $f(x)$ at x_0 is an 'infinite degree' polynomial approximating $f(x)$ 'near' x_0.

Solution. The statement is *true*. The Taylor series for $f(x)$ centered at $x = x_0$ is...

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n = f(x_0) + f'(x_0)\,(x - x_0) + \frac{f''(x_0)}{2}\,(x - x_0)^2 + \frac{f^{(3)}(x_0)}{6}\,(x - x_0) + \cdots$$

Observe if one only uses the first-order Taylor series, i.e. $T_1(x)$, then we have...

$$T_1(x) = f(x_0) + f'(x_0)\,(x - x_0)$$

This is exactly the tangent line to $f(x)$ at $x = x_0$. For the second part, observe that the above Taylor series terminated at any fixed n, i.e. considering only $T_n(x)$, is a polynomial in x. Because the Taylor series is a infinite series consisting of infinitely many polynomial terms, we can consider the Taylor series as an infinite degree polynomial which converges to (not just approximates) $f(x)$ on the interval of convergence centered at $x = x_0$.

Check-In 04/08. *(True/False)* If a Taylor series for $f(x)$ at $x = a$ converges to $f(x)$ on an interval I containing $x = a$, then the error in approximating $f(x)$ by $T_n(x)$ for $x \in I$ is given by the $(n+1)$st term of the Taylor series, i.e. $\dfrac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{n+1}$, where c is some number between a and x.

Solution. The statement is *true*. This is the Taylor Series Remainder Theorem. For instance, the third Taylor polynomial for $f(x) = x^{10}$ centered at $x = 1$ is...

$$T_3(x) = 1 + 10(x-1) + 45(x-1)^2 + 120(x-1)^3$$

Suppose one uses this to approximate $f(1.3) = (1.3)^{10}$. The Taylor Remainder Theorem states there is a c between $x = 1$ and $x = 1.3$ such that the error for this approximation is...

$$\frac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{n+1} = \frac{f^{(3+1)}(c)}{(3+1)!}\,(1.3-1)^{3+1} = \frac{f^{(4)}(c)}{4!}\,(0.3)^4 = 0.0003375 \cdot f^{(4)}(c)$$

However, we know that $f^{(4)}(x) = 5040x^6$ and is increasing on the interval $[1, 1.3]$. Therefore, we have...

$$0.0003375 \cdot f^{(4)}(c) \leq 0.0003375 \cdot f^{(4)}(1.3) = 0.0003375 \cdot 24327.11736 = 8.2104$$

So the error in saying $1.3^{10} = f(1.3) \approx T_3(1.3)$ is at most 8.2104. In fact, we have...

$$|f(1.3) - T_3(1.3)| \approx 2.4958492$$

Check-In 04/10. *(True/False)* The value of $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!\,2^n}$ is e^2.

Solution. The statement is *false*. Recall that the Taylor series for e^x centered at $x = 0$ is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and that this series converges to e^x for all $x \in (-\infty, \infty)$. But then...

$$\sqrt{e} = e^{1/2} = e^x\bigg|_{x=\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{x^n}{n!}\bigg|_{x=\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!\,2^n}$$

The given value incorrectly substitutes $x = 2$ for the Taylor series for e^x.

Check-In 04/15. *(True/False)* The Taylor series for a function always converges to that function.

Solution. The statement is *false*. For instance, the Taylor series of $f(x) = \dfrac{1}{1-x}$ is $\displaystyle\sum_{n=0}^{\infty} x^n$, which only converges for $(-1, 1)$. On this interval, it is true that the Taylor series converges to $f(x)$. However, $f(x)$ is defined for all $x \neq 1$ while the Taylor series is only defined for $x \in (-1, 1)$. There are more extreme examples. For instance, consider the function $g(x)$ given by...

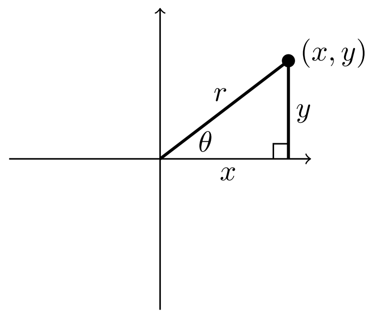
$$g(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

One can show that $g(x)$ is infinitely differentiable at $x = 0$. However, the Taylor series for $g(x)$ centered at $x = 0$ is 0—the zero function. But $g(x) > 0$ for all $x > 0$. Therefore, the Taylor series for

$g(x)$ centered at $x = 0$ does not converge to $g(x)$ at all (except for at its center, obviously). Another such example is the function $h(x) = e^{-1/x^2}$ for $x \neq 0$ and we take $h(0) = 0$. So, while a Taylor series *should* approximate a function 'near' its center, this may not always be the case. However, for sufficiently 'nice' functions, this indeed will be the case.

Check-In 04/24. *(True/False)* In polar coordinates, $x = r\sin\theta$ and $y = r\cos\theta$.

Solution. The statement is *true*. Recall that in polar coordinates, r is the distance from a point to the origin and θ is the angle the ray from the origin to the point makes with the positive x-axis (measured counterclockwise). We can construct a right triangle and use basic trigonometry to derive these formulas.



Observe that $\cos\theta = \frac{x}{r}$, so that $x = r\cos\theta$, and $\sin\theta = \frac{y}{r}$, so that $y = r\sin\theta$. Furthermore, observe that from the Pythagorean Theorem, we know that $x^2 + y^2 = r^2$. Finally, observe that $\tan\theta = \frac{y}{x}$, which implies that $\theta = \tan^{-1}(\frac{y}{x})$.