MATH 142: Exam 1
Spring — 2025
02/13/2025
75 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 9 pages (including this cover page) and 8 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Choose any of the seven questions in this exam. Only these seven problems will be graded. Indicate which you do not want to be graded by circling that problem number on this cover page.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

1. (10 points) Showing all your work, integrate the following:

(a) $\int \sin^2 \theta \; d\theta$

$$\int \sin^2 \theta \; d\theta = \int \left(\frac{1 - \cos 2\theta}{2} \right) \; d\theta = \frac{1}{2} \int (1 - \cos 2\theta) \; d\theta = \frac{1}{2} \left(\theta - \frac{\sin 2\theta}{2} \right) + C$$

(b) $\int \frac{3x - 7}{x^2 + 1} \; dx$

$$\int \frac{3x - 7}{x^2 + 1} \; dx = \int \left(\frac{3x}{x^2 + 1} - \frac{7}{x^2 + 1} \right) \; dx = \frac{3}{2} \ln |x^2 + 1| - 7 \arctan x + C$$

2. (10 points) Showing all your work, integrate the following:

$$\int_0^{\pi/4} \sec^4\theta\ \tan^3\theta\ d\theta$$

Solution. *Let $u = \tan\theta$, so that $du = \sec^2\theta\ d\theta$. If $\theta = 0$, then $u = \tan 0 = 0$. If $\theta = \frac{\pi}{4}$, then $u = \tan\frac{\pi}{4} = 1$. Using the fact that $\sec^2\theta = \tan^2\theta + 1$, we have...*

$$\begin{aligned}
\int_0^{\pi/4} \sec^4\theta\ \tan^3\theta\ d\theta &= \int_0^{\pi/4} \sec^2\theta\ \tan^3\theta \cdot \sec^2\theta\ d\theta \\
&= \int_0^{\pi/4} (\tan^2\theta + 1)\ \tan^3\theta \cdot \sec^2\theta\ d\theta \\
&= \int_0^1 (u^2+1)u^3\ du \\
&= \int_0^1 (u^5+u^3)\ du \\
&= \left(\frac{u^6}{6} + \frac{u^4}{4}\right)\Big|_0^1 \\
&= \left(\frac{1}{6} + \frac{1}{4}\right) - \left(\frac{0}{6} + \frac{0}{4}\right) \\
&= \frac{2}{12} + \frac{3}{12} \\
&= \frac{5}{12}
\end{aligned}$$

OR

Let $u = \sec\theta$, so that $du = \sec\theta\tan\theta\ d\theta$. If $\theta = 0$, then $u = \sec 0 = 1$. If $\theta = \frac{\pi}{4}$, then $u = \sec\frac{\pi}{4} = \sqrt{2}$. Using the fact that $\tan^2\theta = \sec^2\theta - 1$, we have...

$$\begin{aligned}
\int_0^{\pi/4} \sec^4\theta\ \tan^3\theta\ d\theta &= \int_0^{\pi/4} \sec^3\theta \tan^2\theta \cdot \sec\theta\tan\theta\ d\theta \\
&= \int_0^{\pi/4} \sec^3\theta\,(\sec^2\theta - 1) \cdot \sec\theta\tan\theta\ d\theta \\
&= \int_1^{\sqrt{2}} u^3(u^2-1)\ du \\
&= \int_1^{\sqrt{2}} (u^5-u^3)\ du \\
&= \left(\frac{u^6}{6} - \frac{u^4}{4}\right)\Big|_1^{\sqrt{2}} \\
&= \left(\frac{\sqrt{2}^6}{6} - \frac{\sqrt{2}^4}{4}\right) - \left(\frac{1}{6} - \frac{1}{4}\right) \\
&= \left(\frac{8}{6} - \frac{4}{4}\right) - \frac{-1}{12} \\
&= \frac{16}{12} - \frac{12}{12} + \frac{1}{12} \\
&= \frac{5}{12}
\end{aligned}$$

$$\int_0^{\pi/4} \sec^4\theta\ \tan^3\theta\ d\theta$$

Interesting enough, there are two other possibilities for approaching this integral.

Taking $u = \sec^2\theta$*, we have* $du = 2\sec\theta\cdot\sec\theta\tan\theta\ d\theta = 2\sec^2\theta\tan\theta\ d\theta$*. We then also have bounds given by* $u = \sec^2(0) = 1^2 = 1$ *and* $u = \sec^2\frac{\pi}{4} = (\sqrt{2})^2 = 2$*. Using the fact that* $\tan^2\theta = \sec^2\theta - 1$*, we have...*

$$\begin{aligned}
\int_0^{\pi/4} \sec^4\theta\ \tan^3\theta\ d\theta &= \int_0^{\pi/4} \sec^2\theta\tan^2\theta\cdot\sec^2\theta\tan\theta\ d\theta\\
&= \frac{1}{2}\int_0^{\pi/4} \sec^2\theta\,(\sec^2\theta - 1)\cdot 2\sec^2\theta\tan\theta\ d\theta\\
&= \frac{1}{2}\int_1^2 u\,(u-1)\ du\\
&= \frac{1}{2}\int_1^2 (u^2 - u)\ du\\
&= \frac{1}{2}\left[\frac{u^3}{3} - \frac{u^2}{2}\right]\Big|_1^2\\
&= \frac{1}{2}\left[\left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]\\
&= \frac{1}{2}\left[\frac{2}{3} - \left(-\frac{1}{6}\right)\right]\\
&= \frac{1}{2}\cdot\frac{5}{6}\\
&= \frac{5}{12}
\end{aligned}$$

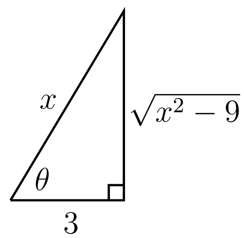
We can also choose $u = \tan^2\theta$*, which implies that* $du = 2\tan\theta\cdot\sec^2\theta\ d\theta$*. The bounds are then* $u = \tan^2(0) = 0^2 = 0$ *and* $u = \tan^2\frac{\pi}{4} = 1^2 = 1$*. Using the fact that* $\sec^2\theta = \tan^2\theta + 1$*, we have...*

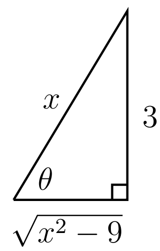
$$\begin{aligned}
\int_0^{\pi/4} \sec^4\theta\ \tan^3\theta\ d\theta &= \int_0^{\pi/4} \sec^2\theta\tan^2\theta\cdot\tan\theta\sec^2\theta\ d\theta\\
&= \frac{1}{2}\int_0^{\pi/4} (\tan^2\theta + 1)\tan^2\theta\cdot 2\tan\theta\sec^2\theta\ d\theta\\
&= \frac{1}{2}\int_0^1 (u+1)u\ du\\
&= \frac{1}{2}\int_0^1 (u^2 + u)\ du\\
&= \frac{1}{2}\left[\frac{u^3}{3} + \frac{u^2}{2}\right]\Big|_0^1\\
&= \frac{1}{2}\left[\left(\frac{1}{3} + \frac{1}{2}\right) - (0+0)\right]\\
&= \frac{1}{2}\left[\frac{5}{6} - 0\right]\\
&= \frac{5}{12}
\end{aligned}$$

3. (10 points) Showing all your work, integrate the following:

$$\int \frac{dx}{\sqrt{x^2-9}}$$

Solution. *We know from the Pythagorean Theorem that* $a^2+b^2=c^2$*, which implies that* $a^2=c^2-b2$*. Taking* $c^2=x^2$*, i.e.* $c=x$*, and* $b^2=9$*, i.e.* $b=3$*, we have* $a^2=c^2-b^2=x^2-9$*. This also shows that* $a=\sqrt{x^2-9}$*. There are two possible right triangles corresponding to these sides we could draw:*





Using the right triangle on the left, we have $\cos\theta=\frac{3}{x}$*, so that* $x=\frac{3}{\cos\theta}=3\sec\theta$*. But then* $dx=3\sec\theta\tan\theta\;d\theta$*. Observe that* $\tan\theta=\frac{\sqrt{x^2-9}}{3}$*, so that* $\sqrt{x^2-9}=3\tan\theta$*. But then…*

$$\int \frac{dx}{\sqrt{x^2-9}}=\int \frac{3\sec\theta\tan\theta}{3\tan\theta}\;d\theta=\int \sec\theta\;d\theta=\ln|\sec\theta+\tan\theta|+C$$

But we know that $\sec\theta=\frac{x}{3}$ *and* $\tan\theta=\frac{\sqrt{x^2-9}}{3}$*. Therefore,*

$$\int \frac{dx}{\sqrt{x^2-9}}=\ln|\sec\theta+\tan\theta|+C=\ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right|+C$$

Alternatively, using the right triangle on the right, we have $\sin\theta=\frac{3}{x}$*, so that* $x=\frac{3}{\sin\theta}=3\csc\theta$*. But then* $dx=-3\csc\theta\cot\theta$*. Observe that* $\tan\theta=\frac{3}{\sqrt{x^2-9}}$*, so that* $\sqrt{x^2-9}=\frac{3}{\tan\theta}=3\cot\theta$*. But then…*

$$\int \frac{dx}{\sqrt{x^2-9}}=\int \frac{-3\csc\theta\cot\theta}{3\cot\theta}\;d\theta=-\int \csc\theta\;d\theta=-\ln|\csc\theta-\cot\theta|+C$$

But we know that $\csc\theta=\frac{x}{3}$ *and* $\cot\theta=\frac{\sqrt{x^2-9}}{3}$*. Therefore,*

$$\int \frac{dx}{\sqrt{x^2-9}}=-\ln|\csc\theta-\cot\theta|+C=-\ln|\csc\theta-\cot\theta|+C=-\ln\left|\frac{x}{3}-\frac{\sqrt{x^2-9}}{3}\right|+C$$

Note. Observing that $\ln(\frac{1}{3})$ *is a constant, we can write the solution as* $\ln\left|\frac{x}{3}\pm\frac{\sqrt{x^2-9}}{3}\right|+C=\ln\left|\frac{1}{3}\left(x\pm\sqrt{x^2-9}\right)\right|+C=\ln(\frac{1}{3})+\ln\left|x\pm\sqrt{x^2-9}\right|+C=\ln\left|x\pm\sqrt{x^2-9}\right|+C$*. Furthermore, to see that the solutions are equivalent, observe that by 'rationalizing', we have* $-\ln|x-\sqrt{x^2-9}|=\ln|(x-\sqrt{x^2-9})^{-1}|=\ln|\frac{1}{9}(x+\sqrt{x^2-9})|=\ln(\frac{1}{9})+\ln|x+\sqrt{x^2-9}|$*. Therefore, the solutions differ only by a constant. One could also have integrated* $\csc\theta$ *using the antiderivative* $-\ln|\csc\theta+\cot\theta|+C$*, which used in the integral above results in the same solution as using the substitution* $x=3\sec\theta$*.*

4. (10 points) Showing all your work, integrate the following:

$$\int_0^4 \frac{dx}{\sqrt{4-x}}$$

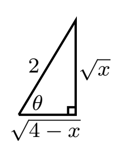
Solution. *Observe that* $\frac{1}{\sqrt{4-x}}$ *is undefined at* $x = 4$*, so that* $\frac{1}{\sqrt{4-x}}$ *is not continuous at* $x = 4$*. Therefore, this is an improper integral. We first find the indefinite integral:*

$$\int \frac{dx}{\sqrt{4-x}} = \int (4-x)^{-1/2}\, dx = (4-x)^{1/2} \cdot \frac{1}{\frac{1}{2}} \cdot \frac{1}{-1} = -2\sqrt{4-x} + C$$

But then, we have. . .

$$\begin{aligned}
\int_0^4 \frac{dx}{\sqrt{4-x}} &:= \lim_{b\to 4^-} \int_0^b \frac{dx}{\sqrt{4-x}} \\
&= \lim_{b\to 4^-} -2\sqrt{4-x}\Big|_0^b \\
&= -2 \lim_{b\to 4^-} \sqrt{4-x}\Big|_0^b \\
&= -2\left(\lim_{b\to 4^-} \sqrt{4-b} - \sqrt{4-0}\right) \\
&= -2\left(\sqrt{0} - \sqrt{4}\right) \\
&= -2 \cdot -2 \\
&= 4
\end{aligned}$$

Note. To compute $\int \frac{dx}{\sqrt{4-x}}$*, one needs the* u*-substitution* $u = 4-x$*, which implies* $du = -dx$*. Then* $\int \frac{dx}{\sqrt{4-x}} = \int \frac{-du}{\sqrt{u}} = \int -u^{-1/2}\, du = -2u^{1/2} + C = -2\sqrt{4-x} + C$*. Interestingly, one can also choose* $u = \sqrt{4-x}$*. This implies that* $du = \frac{-1}{2\sqrt{4-x}}\, dx = \frac{-1}{2u}\, dx$*, i.e.* $dx = -2u\, du$*. But then* $\int \frac{dx}{\sqrt{4-x}} = \int \frac{-2u}{u}\, du = -2\int du = -2u + C = -2\sqrt{4-x} + C$*. One could also integrate this using trig substitution: construct the triangle given below. One then has* $\sin\theta = \frac{\sqrt{x}}{2}$*, so that* $\sqrt{x} = 2\sin\theta$*. But then* $x = 4\sin^2\theta$*, so that* $dx = 8\sin\theta \cdot \cos\theta\, d\theta$*. Observe that* $\cos\theta = \frac{\sqrt{4-x}}{2}$*, which implies* $\sqrt{4-x} = 2\cos\theta$*. But then* $\int \frac{dx}{\sqrt{4-x}} = \int \frac{8\sin\theta\cos\theta}{2\cos\theta}\, d\theta = 4\int \sin\theta\, d\theta = -4\cos\theta + C = -4 \cdot \frac{\sqrt{4-x}}{2} + C = -2\sqrt{4-x} + C$*.*



5. (10 points) Showing all your work, integrate the following:

$$\int x^3 e^{2x}\,dx$$

Solution. *Using tabular integration, we have...*

u		dv
x^3	$+$	e^{2x}
$3x^2$	$-$	$\frac{1}{2}\,e^{2x}$
$6x$	$+$	$\frac{1}{4}\,e^{2x}$
6	$-$	$\frac{1}{8}\,e^{2x}$
0		$\frac{1}{16}\,e^{2x}$

Therefore, we have...

$$\begin{aligned}\int x^3 e^{2x}\,dx &= \frac{1}{2}\,x^3e^{2x} - \frac{3}{4}\,x^2e^{2x} + \frac{6}{8}\,xe^{2x} - \frac{6}{16}e^{2x} + C\\ &= \frac{1}{2}\,x^3e^{2x} - \frac{3}{4}\,x^2e^{2x} + \frac{3}{4}\,xe^{2x} - \frac{3}{8}\,e^{2x} + C\end{aligned}$$

Note. We can also express the answer as $\int x^3e^{2x}\,dx = e^{2x}\left(\frac{1}{2}\,x^3 - \frac{3}{4}\,x^2 + \frac{3}{4}\,x - \frac{3}{8}\right) + C = \frac{e^{2x}}{8}\,\left(4x^3 - 6x^2 + 6x - 3\right) + C.$

6. (10 points) Showing all your work, integrate the following:

$$\int e^x \cos(2x)\ dx$$

Solution. *Using a modified tabular integration, we have...*

u		dv
$\cos(2x)$	$+$	e^x
$-2\sin(2x)$	$-$	e^x
$-4\cos(2x)$	$+$	e^x

But then, we have...

$$\int e^x \cos(2x)\ dx = e^x \cos(2x) + 2e^x \sin(2x) - \int 4e^x \cos(2x)\ dx$$

$$\int e^x \cos(2x)\ dx = e^x \cos(2x) + 2e^x \sin(2x) - 4\int e^x \cos(2x)\ dx$$

$$5\int e^x \cos(2x)\ dx = e^x \cos(2x) + 2e^x \sin(2x) + C$$

$$\int e^x \cos(2x)\ dx = \frac{1}{5}\left(e^x \cos(2x) + 2e^x \sin(2x)\right) + C$$

Note. We can also express the answer as $\int e^x \cos(2x)\ dx = \frac{e^x}{5}\left(\cos(2x) + 2\sin(2x)\right) + C$.

7. (10 points) Showing all your work, integrate the following:

$$\int_1^e \frac{\ln x}{x^2}\,dx$$

Solution. *Using LIATE, we choose* $u = \ln x$*, which forces* $dv = \dfrac{1}{x^2}$*. But then we have...*

$\ln x$	$-\dfrac{1}{x}$
$\dfrac{1}{x}$	$\dfrac{1}{x^2}$

Therefore, we have...

$$\begin{aligned}
\int_1^e \frac{\ln x}{x^2}\,dx &= -\frac{\ln x}{x}\bigg|_1^e - \int_1^e \frac{-1}{x^2}\,dx \\
&= \left(-\frac{\ln e}{e} - \frac{-\ln 1}{1}\right) + \int_1^e \frac{dx}{x^2} \\
&= \left(-\frac{1}{e} - 0\right) + \frac{-1}{x}\bigg|_1^e \\
&= -\frac{1}{e} + \left(\frac{-1}{e} - \frac{-1}{1}\right) \\
&= -\frac{1}{e} - \frac{1}{e} + 1 \\
&= -\frac{2}{e} + 1 \\
&= 1 - \frac{2}{e}
\end{aligned}$$

Note. Using a common denominator, we can express this answer as $1 - \frac{2}{e} = \frac{e}{e} - \frac{2}{e} = \frac{e-2}{e}$.

8. (10 points) Showing all your work, integrate the following:

$$\int \frac{x^2+5x+6}{(x-1)(x+1)^2}\,dx$$

Solution. *Observe that the degree of the denominator, which is 3, is strictly larger than the degree of the top, which is 2. We can then find the partial fraction decomposition of the integrand.*

$$\frac{x^2+5x+6}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Using the original common denominator, we have...

$$\begin{aligned}
\frac{x^2+5x+6}{(x-1)(x+1)^2} &= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \\
&= \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2} \\
&= \frac{A(x^2+2x+1) + B(x^2-1) + C(x-1)}{(x-1)(x+1)^2} \\
&= \frac{Ax^2 + 2Ax + A + Bx^2 - B + Cx - C}{(x-1)(x+1)^2} \\
&= \frac{(A+B)x^2 + (2A+C)x + (A-B-C)}{(x-1)(x+1)^2}
\end{aligned}$$

Equating the numerators and relating coefficients, we have...

$$\begin{aligned}
x^2&: \quad 1 = A + B \\
x&: \quad 5 = 2A + C \\
1&: \quad 6 = A - B - C
\end{aligned}$$

From the last equation, we have $C = A - B - 6$. But using this in the second equation, we have $5 = 2A + C = 2A + A - B - 6 = 3A - B - 6$. This implies that $11 = 3A - B$. Adding this to the first equation, we find that $12 = 4A$, which implies that $A = 3$. But because $1 = A + B = 3 + B$, we know that $B = -2$. Finally, we know that $C = A - B - 6 = 3 - (-2) - 6 = -1$. Alternatively, we can use Heaviside's/Cover-Up Method to find both A and C:

$$A = \left.\frac{x^2+5x+6}{\boxed{\cancel{(x-1)}}\,(x+1)^2}\right|_{x=1} = \frac{1^2+5(1)+6}{(1+1)^2} = \frac{12}{4} = 3$$

$$C = \left.\frac{x^2+5x+6}{(x-1)\,\boxed{\cancel{(x+1)^2}}}\right|_{x=1} = \frac{(-1)^2+5(-1)+6}{-1-1} = \frac{2}{-2} = -1$$

One could then use any x-value (other than $x = \pm 1$) to find the value of B. For instance, using $x = 0$, we know that $\frac{x^2+5x+6}{(x-1)(x+1)^2} = \frac{0+0+6}{-1\cdot 1} = -6$ and $\frac{3}{0-1} + \frac{B}{0+1} + \frac{-1}{(0+1)^2} = -3 + B - 1 = B - 4$. But then $B - 4 = -6$ so that $B = -2$. In any case, we then have...

$$\begin{aligned}
\int \frac{x^2+5x+6}{(x-1)(x+1)^2}\,dx &= \int \frac{3}{x-1} + \frac{-2}{x+1} + \frac{-1}{(x+1)^2}\,dx \\
&= 3\ln|x-1| - 2\ln|x+1| + \frac{1}{x+1} + K
\end{aligned}$$