Check-In 08/21. *(True/False)* The integral $\displaystyle\int x\sqrt[3]{x-2}\,dx$ can be treated as a 'shifting integral' by using the u-substitution $u = x-2$.

Solution. The statement is *true*. We 'want' to be able to distribute the x across the cube-root but we cannot—this is not a valid operation. However, if we make the u-substitution $u = x-2$, then we will be able to distribute in a way that makes this integral 'routine.' So, let $u = x-2$, then $du = dx$. Moreover, because $u = x-2$, we know that $x = u+2$. But then...

$$\int x\sqrt[3]{x-2}\,dx = \int (u+2)\sqrt[3]{u}\,du = \int \left(u^{4/3} + 2u^{1/3}\right)\,du = \tfrac{3}{7}\,u^{7/3} + \tfrac{3}{4}\cdot 2u^{4/3} + C = \tfrac{3}{7}(x-2)^{7/3} + \tfrac{3}{2}(x-2)^{4/3} + C$$

Note that a computer algebra system may write the answer (though you will *not* be expected to) like this:

$$\tfrac{3}{7}(x-2)^{7/3} + \tfrac{3}{2}(x-2)^{4/3} + C = (x-2)^{4/3}\left(\tfrac{3}{7}(x-2) + \tfrac{3}{2}\right) + C = (x-2)^{4/3}\left(\tfrac{3}{7}x + \tfrac{9}{14}\right) + C = \tfrac{3}{14}(x-2)^{4/3}\,(2x+3) + C$$

Check-In 08/26. *(True/False)* Using integration-by-parts to evaluate $\displaystyle\int x\tan^{-1}(x)\,dx$, one chooses $u = \tan^{-1} x$ and $dv = x$.

Solution. The statement is *true*. Using LIATE, the first term that appears is 'I' for inverse trig. Therefore, we choose $u = \tan^{-1} x$. But then $dv = x$. We then fill in our box:

$\tan^{-1} x$	$\dfrac{x^2}{2}$
$\dfrac{1}{1+x^2}$	x

Using the 'rule of 7', we have...

$$\int x\tan^{-1} x\,dx = \frac{1}{2}\,x^2\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

We now need only evaluate the integral on the right. Dividing $1+x^2$ into x^2, we have a remainder of -1, i.e. $\dfrac{x^2}{1+x^2} = 1 + \dfrac{-1}{1+x^2}$. Therefore, we have...

$$\frac{1}{2}\int \frac{x^2}{1+x^2}\,dx = \frac{1}{2}\int \left(1 + \frac{-1}{1+x^2}\right)\,dx = \frac{1}{2}\left(x - \tan^{-1} x\right) + C$$

But then...

$$\begin{aligned}
\int x \tan^{-1} x \, dx &= \frac{1}{2}\, x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2}\, dx \\
&= \frac{1}{2}\, x^2 \tan^{-1} x - \frac{1}{2}\left(x - \tan^{-1} x\right) + C \\
&= \frac{1}{2}\, x^2 \tan^{-1} x - \frac{1}{2}\, x + \frac{1}{2}\, \tan^{-1} x + C \\
&= \frac{x^2 \tan^{-1} x - x + \tan^{-1} x}{2} + C \\
&= \frac{(x^2+1) \tan^{-1} x - x}{2} + C
\end{aligned}$$

Check-In 08/28. *(True/False)* The integral $\int e^x \sin(3x)\; dx$ can be treated as an integration-by-parts 'looping' integral.

Solution. The statement is *true*. Using integration-by-parts for $\int e^x \sin(3x)\; dx$ would result in an integral that would 'loop' back to itself. Generally, an integrand of the form exponential $\cdot$ (sin or cos) or trig $\cdot$ trig will have this property. Using traditional integration-by-parts, by LIATE, we choose $u = \sin(3x)$ and $dv = e^x$. Filling out our box, we have...

$\sin(3x)$	e^x
$3\cos(3x)$	e^x

Using the 'rule of seven', we then have...

$$\int e^x \sin(3x)\; dx = e^x \sin(3x) - \int 3e^x \cos(3x)\; dx$$

To integrate $\int 3e^x \cos(3x)\; dx$, we again use integration-by-parts. Using LIATE, we choose $u =$

$3\cos(3x)$ and $dv = e^x$. Filling out the box, we have...

$3\cos(3x)$	e^x
$-9\sin(3x)$	e^x

Using the 'rule of seven', we then have

$$\int 3e^x \cos(3x)\, dx = 3e^x \cos(3x) - \int -9e^x \sin(3x)\, dx = 3e^x \cos(3x) + 9 \int e^x \sin(3x)\, dx$$

But then we have...

$$\int e^x \sin(3x)\, dx = e^x \sin(3x) - \int 3e^x \cos(3x)\, dx = \int e^x \sin(3x)\, dx = e^x \sin(3x) - \left(3e^x \cos(3x) + 9 \int e^x \sin(3x)\, dx\right)$$

Therefore, we have...

$$\int e^x \sin(3x)\, dx = e^x \sin(3x) - \left(3e^x \cos(3x) + 9 \int e^x \sin(3x)\, dx\right)$$

$$\int e^x \sin(3x)\, dx = e^x \sin(3x) - 3e^x \cos(3x) - 9 \int e^x \sin(3x)\, dx$$

$$10 \int e^x \sin(3x)\, dx = e^x \sin(3x) - 3e^x \cos(3x)$$

$$\int e^x \sin(3x)\, dx = \frac{e^x \sin(3x) - 3e^x \cos(3x)}{10} + C$$

$$\int e^x \sin(3x)\, dx = \frac{e^x}{10} \big(\sin(3x) - 3\cos(3x)\big) + C$$

Alternatively, we can use an alteration of the tabular method of integration-by-parts. We choose $u = \sin(3x)$ and $dv = e^x$. We then have...

u		dv
$\sin(3x)$	$+$	e^x
$3\cos(3x)$	$-$	e^x
$-9\sin(3x)$	$+$	e^x

Therefore, we have...

$$\int e^x \sin(3x)\, dx = e^x \sin(3x) - 3\cos(3x)e^x - 9 \int e^x \sin(3x)\, dx$$

Solving for our integral, we have...

$$\int e^x \sin(3x)\ dx = e^x \sin(3x) - 3\cos(3x)e^x - 9\int e^x \sin(3x)\ dx$$

$$10\int e^x \sin(3x)\ dx = e^x \sin(3x) - 3e^x \cos(3x)$$

$$\int e^x \sin(3x)\ dx = \frac{e^x \sin(3x) - 3e^x \cos(3x)}{10} + C$$

$$\int e^x \sin(3x)\ dx = \frac{e^x}{10}\big(\sin(3x) - 3\cos(3x)\big) + C$$

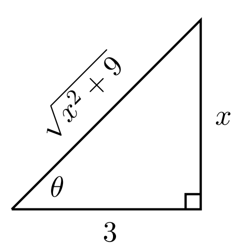
Check-In 09/02. *(True/False)* To integrate $\int \cot^2 x \csc^2 x\ dx$ as a trigonometric integral, one could make the substitution $u = \cot x$.

Solution. The statement is *true*. If one let $u = \csc x$, we would have $du = -\csc x \cot x$. Setting aside a $\cot x$ for the du, this would leave only a single $\cot x$ term in the integrand—which we cannot replace with a Pythagorean identity. However, this is not an issue if we let $u = \cot x$. If $u = \cot x$, then $du = -\csc^2 x\, dx$. But then...

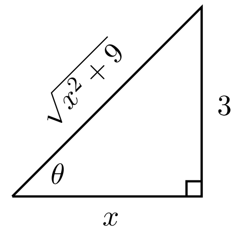
$$\int \cot^2 x \csc^2 x\ dx = -\int \cot^2 x \cdot -\csc^2 x\ dx = -\int u^2\ du = -\frac{u^3}{3} + C = -\frac{\cot^3 x}{3} + C$$

Check-In 09/04. *(True/False)* To integrate $\int \frac{x^2}{\sqrt{x^2+9}}\ dx$, one can make the substitution $x = 3\cot\theta$.

Solution. The statement is *true*. Observe that the $x^2 + 9$ 'resembles' a term from the Pythagorean Theorem. This suggests that a trig. substitution might be useful. Because $a^2 + b^2 = c$, we see that $a^2 + b^2$ corresponds to $x^2 + 9$, i.e. a^2 corresponds to x^2 and b^2 corresponds to 9. This implies that $a = x$ and $b = 3$. But then $c^2 = x^2 + 9$, i.e. $c = \sqrt{x^2+9}$. We construct a right triangle with these legs and hypotenuse:



But then $\tan\theta = \frac{x}{3}$, which implies $x = 3\tan\theta$. While this seems like it makes the statement of the problem false, this is not the only right triangle we could have constructed. If we had instead drawn



We would have $\tan\theta = \frac{3}{x}$, which implies that $x\tan\theta = 3$, so that $x = \frac{3}{\tan\theta} = 3\cot\theta$. This is the substitution in the problem statement. Both the substitutions $x = 3\cot\theta$ and $x = 3\tan\theta$ are viable trig. substitutions to compute this integral.

Check-In 09/09. *(True/False)* The partial fraction decomposition of $\dfrac{x+4}{x^2(x-3)}$ has the form $\dfrac{Ax+B}{x^2} + \dfrac{C}{x-3}$.

Solution. The statement is *false*. For a partial fraction decomposition, one first needs to be sure that the degree of the numerator is smaller than the degree of the denominator—which is the case here. One then needs to be sure that the denominator is factored completely—which is the case here. One then needs to 'run' through each power of the factored terms of the denominator—being sure that the numerator term for quadratic factors is linear. In this case, the denominator terms are x (up to power 2) and $x-3$. Therefore, the partial fraction decomposition is...

$$\frac{x+4}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}$$

Although the term x^2 is quadratic, the base—x—is linear. Hence, its numerator terms will always be constant—never linear. This is the mistake in the decomposition given in the problem statement.

Check-In 09/11. *(True/False)* The partial fraction decomposition of $\dfrac{7x-5}{x^2-x}$ is $\dfrac{5}{x} + \dfrac{2}{x-1}$.

Solution. The statement is *true*. First, observe that $\dfrac{7x-5}{x^2-x} = \dfrac{7x-5}{x(x-1)}$. Therefore, we have a decomposition of the form

$$\frac{7x-5}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

Observe that Heaviside's can be used to find both A and B. So, we have...

$$A = \left.\frac{7x-5}{\boxed{\cancel{x}}\,(x-1)}\right|_{x=0} = \frac{0-5}{0-1} = \frac{-5}{-1} = 5$$

$$B = \left.\frac{7x-5}{x\,\boxed{\cancel{x-1}}}\right|_{x=1} = \frac{7-5}{1} = \frac{2}{1} = 2$$

Therefore, we have...

$$\frac{7x-5}{x^2-x} = \frac{5}{x} + \frac{2}{x-1}$$

Check-In 09/18. *(True/False)* The integral $\displaystyle\int_{-1}^{1} \frac{dx}{\sqrt[3]{x}} = \int_{-1}^{1} x^{-1/3}\ dx = \frac{3}{2}\,x^{2/3}\Big|_{-1}^{1} = \frac{3}{2}\,(1^{2/3} - (-1)^{2/3}) = \frac{2}{3}\,(1 - \sqrt[3]{(-1)^2}) = \frac{2}{3}\,(1-1) = 0.$

Solution. The statement is *false*. Observe that the integrand is undefined at $x = 0$. In fact, the integrand has a vertical asymptote at $x = 0$. Therefore, the integral is improper. We need split the integral at this x-value. So, we write...

$$\int_{-1}^{1} \frac{dx}{\sqrt[3]{x}} = \int_{-1}^{0} \frac{dx}{\sqrt[3]{x}} + \int_{0}^{1} \frac{dx}{\sqrt[3]{x}}$$

We need to take the limit as the integral limits approach 0:

$$\int_{-1}^{1} \frac{dx}{\sqrt[3]{x}} := \lim_{b \to 0^-} \int_{-1}^{b} \frac{dx}{\sqrt[3]{x}} + \lim_{b \to 0^+} \int_{b}^{1} \frac{dx}{\sqrt[3]{x}}$$

Observe that...

$$\int \frac{dx}{\sqrt[3]{x}} = \int x^{-1/3}\ dx = \frac{3}{2}\,x^{2/3} + C$$

But then...

$$\lim_{b \to 0^-} \int_{-1}^{b} \frac{dx}{\sqrt[3]{x}} = \lim_{b \to 0^-} \frac{3}{2}\,x^{2/3}\Big|_{-1}^{b} = \lim_{b \to 0^-} \frac{3}{2}\,b^{2/3} - \frac{3}{2} = -\frac{3}{2}$$

$$\lim_{b \to 0^+} \int_{b}^{1} \frac{dx}{\sqrt[3]{x}} = \lim_{b \to 0^+} \frac{3}{2}\,x^{2/3}\Big|_{b}^{1} = \frac{3}{2} - \lim_{b \to 0^+} \frac{3}{2}\,b^{2/3} = \frac{3}{2}$$

Therefore, we have...

$$\int_{-1}^{1} \frac{dx}{\sqrt[3]{x}} := \lim_{b \to 0^-} \int_{-1}^{b} \frac{dx}{\sqrt[3]{x}} + \lim_{b \to 0^+} \int_{b}^{1} \frac{dx}{\sqrt[3]{x}} = -\frac{3}{2} + \frac{3}{2} = 0$$

So, while the given answer is correct, the approach is entirely incorrect. In other cases, using the given approach for improper integrals will result in incorrect solutions.

Check-In 09/30. *(True/False)* The series $\displaystyle\sum_{n=17}^{\infty} \frac{n^2}{n^2+1}$ diverges.

Solution. The statement is *true*. Recall that the Divergence Test (nth Term Test) states: if $\displaystyle\lim_{n \to \infty} a_n \neq 0$, then $\displaystyle\sum^{\infty} a_n$ diverges. Observe that here $a_n = \dfrac{n^2}{n^2+1}$ and...

$$\lim_{n \to \infty} \frac{n^2}{n^2+1} = 1 \neq 0$$

Therefore, the series diverges by the Divergence Test. However, it is important to note two things: first, one can *never* say that a series converges by the Divergence Test; otherwise, they would have called it the Convergence Test. Second, if $\lim_{n\to\infty} a_n = 0$, then this *does not* mean that the series converges. It could be that the series diverges—even though the Divergence Test does not tell you this. It could also be that the series converges—the Divergence Test also not telling you this. For example, the series $\sum^{\infty} \frac{1}{n}$ (the Harmonic Series) diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$, while the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (to $\frac{\pi^2}{6}$) even though $\lim_{n\to\infty} \frac{1}{n^2} = 0$. That is, if $\lim_{n\to\infty} a_n = 0$, then one knows *zero* information as to whether the series converges or diverges (at least from the Divergence Test).

Check-In 10/02. *(True/False)* The series $\sum_{n=1}^{\infty} 3640\, \frac{7^n}{4^{2n}}$ diverges.

Solution. The statement is false. This is a geometric series, although it is not written in the traditional form of a geometric series: $\sum^{\infty} ar^n$. We first do some algebra:

$$\sum_{n=1}^{\infty} 3640\, \frac{7^n}{4^{2n}} = \sum_{n=1}^{\infty} 3640\, \frac{7^n}{(4^2)^n} = \sum_{n=1}^{\infty} 3640\, \left(\frac{7}{4^2}\right)^n = \sum_{n=1}^{\infty} 3640\, \left(\frac{7}{16}\right)^n$$

Observe that this now has the form $\sum^{\infty} ar^n$ with $a = 3640$ and $r = \frac{7}{16}$. Because $|r| = \left|\frac{7}{16}\right| = \frac{7}{16} < 1$, this geometric series converges by the Geometric Series Test. In fact, we can then find the sum of this series:

$$\sum_{n=1}^{\infty} 3640\, \frac{7^n}{4^{2n}} = \sum_{n=1}^{\infty} 3640\, \left(\frac{7}{16}\right)^n = \frac{\text{first term}}{1-r} = \frac{3640 \cdot \frac{7}{16}}{1-\frac{7}{16}} = \frac{3640\, \frac{7}{16}}{\frac{9}{16}} = \frac{3640 \cdot 7}{9} = \frac{25480}{9}$$

Check-In 10/07. *(True/False)* Because the series $\sum_{n=1}^{\infty} (-1)^n\, \frac{\sqrt{n}-1}{n+1}$ 'behaves like' $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges by the p-test, one can use the Limit Comparison Test to show that the series $\sum_{n=1}^{\infty} (-1)^n\, \frac{\sqrt{n}-1}{n+1}$ diverges.

Solution. The statement is *false*. In fact, the series $\sum_{n=1}^{\infty} (-1)^n\, \frac{\sqrt{n}-1}{n+1}$ converges to approximately 0.05271877311. Recall that the Limit Comparison Test states: if $\{a_n\}, \{b_n\}$ are sequences of *positive* numbers, i.e. $a_n, b_n > 0$ for all n, and $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite and not 0, then $\sum^{\infty} a_n$ and $\sum^{\infty} b_n$ either both converge or both diverge. Notice the test requires that both series be (eventually) positive. However, while the sequence $\left\{\frac{1}{\sqrt{n}}\right\}$ is always positive, the sequence $\left\{(-1)^n\, \frac{\sqrt{n}-1}{n+1}\right\}$ is 0 when $n = 1$ and alternates infinitely between positive and negative values. Therefore, the Limit Comparison

Test does not apply. One would need another series test, such as the Alternating Series Test, to determine whether the series $\sum_{n=1}^{\infty}(-1)^n \dfrac{\sqrt{n}-1}{n+1}$ converges or diverges. Note that if it were the series $\sum_{n=1}^{\infty}(-1)^n \dfrac{\sqrt{n}-1}{n+1}$ were replaced with $\sum_{n=1}^{\infty} \dfrac{\sqrt{n}-1}{n+1}$ in the statement of the problem, then the statement would be true.

Check-In 10/14. *(True/False)* Because $\sum_{n=5}^{\infty} \dfrac{n-1}{n^2+1} \leq \sum_{n=5}^{\infty} \dfrac{n}{n^2} = \sum_{n=5}^{\infty} \dfrac{1}{n}$, the series $\sum_{n=5}^{\infty} \dfrac{n-1}{n^2+1}$ diverges.

Solution. The statement is *false*. It is true that

$$\sum_{n=5}^{\infty} \frac{n-1}{n^2+1} \leq \sum_{n=5}^{\infty} \frac{n}{n^2} = \sum_{n=5}^{\infty} \frac{1}{n}$$

We make the original fraction bigger by making the numerator bigger (by removing the -1) and making the denominator smaller (by removing the $+1$), which makes the fraction bigger. The series $\sum_{n=5}^{\infty} \dfrac{n-1}{n^2+1}$ does 'behave like' the series $\sum_{n=5}^{\infty} \dfrac{1}{n}$ because when n is large $\dfrac{n-1}{n^2+1} \approx \dfrac{n}{n^2} = \dfrac{1}{n}$, and the series $\sum_{n=5}^{\infty} \dfrac{1}{n}$ diverges (by the p-test with $p = 1$). However, a series being less than a series which diverges does not show that the original series diverges. This is not one of the conclusions of the Direct Comparison Test. For instance, $\sum_{n=1}^{\infty} \dfrac{1}{n} \leq \sum_{n=1}^{\infty} \dfrac{2}{n}$ and both series diverge while $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6} \leq \sum_{n=1}^{\infty} \dfrac{1}{n}$. To show that the series $\sum_{n=5}^{\infty} \dfrac{n-1}{n^2+1}$ diverges, we need to find a series with a summand that is always smaller than $\dfrac{n-1}{n^2+1}$ whose series also diverges. We could do this by making the fraction $\dfrac{n-1}{n^2+1}$ smaller, which can be done by making the denominator smaller or the denominator larger. Observe that because $\frac{n}{2} > 1$ and $1 \leq n^2$ for $n \geq 5$, we have...

$$\sum_{n=5}^{\infty} \frac{n-1}{n^2+1} \geq \sum_{n=5}^{\infty} \frac{n-\frac{n}{2}}{n^2+n^2} = \sum_{n=5}^{\infty} \frac{\frac{n}{2}}{2n^2} = \frac{1}{4} \sum_{n=5}^{\infty} \frac{1}{n}$$

Because the series $\sum_{n=5}^{\infty} \dfrac{1}{n}$ diverges by the p-test with $p = 1$ and $\sum_{n=5}^{\infty} \dfrac{n-1}{n^2+1} \geq \dfrac{1}{4} \sum_{n=5}^{\infty} \dfrac{1}{n}$, the series $\sum_{n=5}^{\infty} \dfrac{n-1}{n^2+1}$ diverges by the Direct Comparison Test.

Check-In 10/16. *(True/False)* The series $\sum_{n=1}^{\infty}(-1)^n \dfrac{\sin^4(1/n)}{n}$ converges.

Solution. The statement is *true*. Recall that an alternating series is a series of the form $\sum^{\infty}(-1)^n a_n$, where a_n. Choosing $a_n = \dfrac{\sin^4(n)}{n}$, observe that $a_n > 0$. Therefore, $\sum_{n=1}^{\infty}(-1)^n \dfrac{\sin^4(n)}{n}$ is an alternating series. The Alternating Series Test states that if $\sum^{\infty}(-1)^n a_n$ is an alternating series, then $\lim_{n\to\infty} a_n = 0$ and the sequence $\{a_n\}$ is decreasing, then $\sum^{\infty}(-1)^n a_n$ converges. Observe that $\lim_{n\to\infty} \dfrac{\sin^4(1/n)}{n} = 0$ and because

$$\begin{aligned}\frac{d}{dn}\left(\frac{\sin^4(1/n)}{n}\right) &= \frac{n\cdot(4\sin^3(1/n)\cdot\cos(1/n)\cdot -\frac{1}{n^2}) - 1\cdot\sin^4(1/n)}{n^2} \\ &= \frac{-\frac{1}{n}\cdot 4\sin^3(1/n)\cos(1/n) - \sin^4(1/n)}{n^2} \\ &= -\frac{\sin(1/n)^3\,(4\cos(1/n) + n\sin(1/n))}{n^3} < 0\end{aligned}$$

Therefore, the sequence is decreasing. Therefore, by the Alternating Series Test, the series $\sum_{n=1}^{\infty}(-1)^n \dfrac{\sin^4(1/n)}{n}$ converges.

Check-In 10/21. *(True/False)* The series $\sum_{n=1}^{\infty}\left(\dfrac{99n}{100n+1}\right)^n$ converges.

Solution. The statement is *true*. Using the Ratio Test, we have...

$$L := \lim_{n\to\infty}\left[\left(\frac{99n}{100n+1}\right)^n\right]^{1/n} = \lim_{n\to\infty}\frac{99n}{100n+1} = \frac{99}{100} < 1$$

Because $L < 1$, the series converges absolutely.

Check-In 10/30. *(True/False)* If a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges on the interval $(-5, 3]$, then the center of the power series is $c = -1$ and the radius of convergence is 4. Moreover, the power series diverges for every x with $x \leq -5$ or $x > 3$.

Solution. The statement is *true*. In the case of a finite interval, the center should be the center of the interval. The center of $(-5, 3]$ is $\frac{-5+3}{2} = \frac{-2}{2} = -1$. Therefore, the center is $c = -1$. The radius of convergence should be half the length of this interval, which is $\frac{3-(-5)}{2} = \frac{8}{2} = 4$. We know that the power series converges on the interval of convergence and diverges for all other x. The x-values which are not in $(-5, 3]$ are precisely $x \leq -5$ or $x > 3$.

Check-In 11/04. *(True/False)* If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = -5$ and diverges for $x = 10$, then the power series converges when $x = 4$.

Solution. The statement is *true*. We can see that the center of the power series is $x = 0$. We know the interval of convergence of a power series is an interval about its center. We know the interval of convergence is larger than just $\{0\}$. So, the interval of convergence must at least contain $(-a, a)$ for some a—an interval of some size contained about a center of $x = 0$. The radius of convergence of power series is the distance from the center for which the power series is guaranteed to converge. Because the power series converges for $x = -5$, we know that $R \geq 5$—the distance from $x = -5$ to $x = 0$. But we know also that the series diverges at $x = 10$. So, we know that $R \leq 10$—the distance from $x = 0$ to $x = 10$. Therefore, $5 \leq R \leq 10$. But then the interval of convergence contains at least $(-5, 5)$ and is contained within $[-10, 10)$. [Do you see why?] But then the power series must converge for $x = 4$ because $4 \in (-5, 5)$.

Check-In 11/06. *(True/False)* The Taylor series for $f(x) = \dfrac{1}{1-x}$ centered at $x = 0$ is $\sum_{n=0}^{\infty} x^n$ and has interval of convergence $(-1, 1)$. Furthermore, the Taylor series converges to $f(x)$ on the interval $(-1, 1)$.

Solution. The statement is *true*. We know from our table of Taylor series, that the Taylor series for $\frac{1}{1-x}$ centered at $x = 0$ is $\sum_{n=0}^{\infty} x^n$. Of course, one can find this: we know that $\frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = \frac{n!}{(1-x)^{n+1}}$, so $f^{(n)}(0) = \frac{n!}{(1-0)^{n+1}} = n!$. Therefore, the Taylor series is...

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{n!}{n!}x^n = \sum_{n=0}^{\infty} x^n$$

One can easily use the Ratio or Root Test to show that it must be that $|x| < 1$, i.e. $-1 < x < 1$. If $x = -1$, the series is $\sum_{n=0}^{\infty}(-1)^n$, which diverges, and if $x = 1$, the series is $\sum_{n=0}^{\infty} 1^n$, which diverges. Therefore, the interval of convergence is $(-1, 1)$. We can see that this series is geometric with common ratio x. But then for $|x| < 1$, we have $\sum_{n=0}^{\infty} x^n = \dfrac{x^0}{1-x} = \dfrac{1}{1-x}$.

Check-In 11/11. *(True/False)*

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \cdots$$

Solution. The statement is *true*. First, observe that...

$$4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \cdots = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right)$$

Recall the Taylor series for $\arctan(x)$ centered at $x = 0$ is $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ and is valid for $x \in [-1, 1]$. But then...

$$\frac{\pi}{4} = \arctan(1) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\bigg|_{x=1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

Therefore, we have...

$$\pi = 4 \cdot \frac{\pi}{4} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right) = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \cdots$$

Check-In 11/13. *(True/False)* Using more terms of the Taylor series for $f(x)$ always results in a better approximation for x values 'near' the center.

Solution. The statement is *false*. The approximation will never improve if we do not use x-values in the interval of convergence for our Taylor series. Even if we consider such an x-value, there is no guarantee that the Taylor series converges to the function. From the Taylor Remainder Theorem, we know the error in replacing $f(x)$ with $T_n(x)$ is given by $R(x) = \dfrac{f^{(n+1)}(a)}{n!}(x-c)^{n+1}$, where a is some value between x and c (the center). But as one uses more and more terms, i.e. increases n, there is no guarantee that for a given x the error shrinks, i.e. $\displaystyle\lim_{n\to\infty} \frac{f^{(n+1)}(a)}{n!}(x-c)^{n+1} = 0$. This depends on the growth of the $x - c$ term and the values of the $(n+1)$-st derivative. Some Taylor series do not converge to the function. For instance, consider the function

$$f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

This is a function whose Taylor polynomial at $x = 0$ is $T(x) = 0$. But $f(x) > 0$ for all $x > 0$. Therefore, $T(x)$ fails to converge to $f(x)$ for all positive values. No amount of terms of the Taylor series will improve the error. We can prove this fact. First, observe that all the derivatives of $f(x)$ exist. We claim that for $x > 0$, $f^{(n)}(x) = p_n(1/x)e^{-1/x}$, where $p_n(x)$ is a polynomial. For $n = 0$, we have $f^{(0)}(x) = e^{-1/x} = 1 \cdot e^{-1/x}$ so that we can take $p_0(x) = 1$. Assume the result is true for $n = 0, 1, 2, \ldots, k$. For $x > 0$, we have...

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx} f^{(k)}(x) \\ &= \frac{d}{dx}\left(p_k(1/x)e^{-1/x}\right) \\ &= \left(p_k'(1/x) \cdot -\frac{1}{x^2}\right) \cdot e^{-1/x} + p_k(1/x) \cdot \left(e^{-1/x} \cdot \frac{1}{x^2}\right) \\ &= e^{-1/x}\left(\frac{p_k(1/x) - p_k'(1/x)}{x^2}\right) \end{aligned}$$

But $\dfrac{p_k(1/x) - p_k'(1/x)}{x^2}$ is also a polynomial in $1/x$, so we can take $p_{k+1}(x) = \dfrac{p_k(1/x) - p_k'(1/x)}{x^2}$. Therefore, $f^{(k+1)}(x) = p_{k+1}e^{-1/x}$. The result then follows by induction. We have $\displaystyle\lim_{x\to 0^+} p_n(1/x)e^{-1/x} =$

$\lim_{x\to\infty} p_n(x)e^{-x}$. It follows easily from l'Hôpital's rule that $\lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty} \dfrac{x^n}{e^x} = 0$ for any integer $n \geq 0$. But then $\lim_{x\to 0^+} p_n(1/x)e^{-1/x} = \lim_{x\to\infty} p_n(x)e^{-x} = 0$. This matches the derivative of $f(x)$ for $x < 0$. Therefore, $f(x)$ is smooth at $x = 0$, i.e. infinitely differentiable with continuous derivatives at $x = 0$. [This uses the fact that $e^{-1/x}$ is smooth for $x > 0$.] Now consider the Taylor series for $f(x)$ centered at $x = 0$. From the work above, we know all the nth derivatives of $f(x)$ at $x = 0$ are 0. But this forces $T(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = 0$ because $f^{(n)}(0) = 0$ for all n.