The Mathical Adventures of Robbie the Red
Exponential Functions, Compound Interest, and Half-Life

To receive full or partial credit, you must show all work on your own paper.

Robbie the Red Robot has decided to open a savings account at Gamecock Bank. Recall that the formula for compound interest is given by

$$P(t) = P_0(1+r)^t \qquad\qquad P(t) = P_0 e^{rt}$$

depending on whether or not interest is compounded annually or compounded continuously, respectively. Here, P_0 is the initial deposit or principle. The annual interest rate is r, and $P(t)$ is the balance after t years.

Problem 1. (6 points) Use the formulas given above to find a function giving the balance in each situation in order to answer the given questions.

(a) If Robbie invests $\$5,000$ in an account earning 3% interest compounded annually, how much money will be in the account after 8 years?

(b) Suppose an account earns 4.5% interest compounded annually. How much money must Robbie invest up-front in order for this account to reach $\$12,000$ in 10 years?

(c) Suppose Robbie invests $300 into an account earning 5.25% compounded continuously. How long will it take for his account balance to reach $\$1,400$?

Explore. Robbie is curious about how the formulas for compound interest seem so different depending on how often compounding happens. To compound n times per year, we can use the following formula:

$$P(t) = P_0\left(1+\frac{r}{n}\right)^{nt}$$

(a) If the initial deposit is $200 with a 4% annual interest rate, find the balance after 8 years if you compound every year, every six months, every month, every day, and every minute (Hint: 1 year = 525,600 minutes)

(b) Rewrite the formula for $P(t)$ above without using n by making the substitution $z = \frac{n}{r}$, i.e. $n = z \cdot r$.

(c) As the number of compoundings per year, n, grows, z also grows. Make a table of the decimal values for $\left(1+\frac{1}{z}\right)^z$ for $z = 100$, $z = 500$, $z = 1,000$, $z = 10,000$, and $z = 100,000$. Is there a pattern? What is a decimal approximation for e?

After his trip to the bank, Robbie has been tasked with managing an amount of a radioactive material present in a container. The amount that should remain after t years is given by

$$P(t) = P_0 e^{kt}$$

where P_0 is the initial amount, t is the number of years, and k is a constant. This material has a half-life of t^* years. This means that after t^* number of years, there will be half as much material as what was present at the beginning of the time period. Namely for the specific t^* amount of time,

$$P(t^*) = \frac{1}{2}P_0$$

Problem 2. (6 points) The half-life of the substance is 9 years. There are 5 grams remaining after 4 years.

(a) Find a function $P(t)$ that gives the amount of substance remaining after t years.

(b) What sign did you get for k? Is this surprising?

(c) When will there be 2 grams of material remaining?

After a long day, Robbie is preparing his evening coffee. (Yes, robots can enjoy coffee.) The coffeemaker's carafe maintains the coffee's temperature at 170°F, while Robbie keeps his home at 72°F. After pouring some coffee into a non-insulated cup, the coffee's temperature drops to 165°F after 1 minute.

To frame this scenario, we recall **Newton's Law of Cooling**. This law dictates how an object's temperature changes with respect to the temperature of its surrounding environment. It states that

$$T(t) = T_s + (T_0 - T_s)e^{kt}$$

where $T(t)$ is the temperature of the object at time t, T_s is the surrounding temperature, T_0 is the object's temperature when $t = 0$ (i.e. the object's temperature when placed in the new environment), and k is a constant depending on the object.

Explore. Use Newton's Law of Cooling to answer the following questions.

(a) Find a function $T(t)$ that gives the temperature of the coffee after t minutes.

(b) When will the coffee be exactly $100°F$?

(c) What will the temperature of the coffee approach when t is large? Explain your answer in graphical terms with respect to horizontal asymptotes.