Check-In 01/16. *(True/False)* Given $\displaystyle\int_0^\pi e^{\sin x}\cos x\,dx$, the u-substitution $u = \sin x$ transforms this integral into $\displaystyle\int_0^\pi e^u\,du$.

Solution. The statement is *false*. If $u = \sin x$, then $du = \cos x\,dx$. So indeed, this u-substitution would transform the integral $\displaystyle\int e^{\sin x}\cos x\,dx$ into the integral $\displaystyle\int e^u\,du$. However with definite integrals, one needs to remember to transform the limits as well. If $x = 0$, then $u = \sin(0) = 0$. If $x = \pi$, then $u = \sin(\pi) = 0$. Therefore, the correct substitution is $\displaystyle\int_0^\pi e^{\sin x}\cos x\,dx = \int_0^0 e^u\,du = 0$.

Check-In 01/21. *(True/False)* To integrate $\displaystyle\int \operatorname{arccot}\theta\,d\theta$, one can use integration-by-parts by choosing $u = \operatorname{arccot}\theta$ and $dv = 1$.

Solution. The statement is *true*. Using LIATE, it is likely that the choice of $u = \operatorname{arccot}\theta$ will work. With 'nothing left' in the integrand, this means that $dv = 1$. We fill in our box as follows:

$\operatorname{arccot}\theta$	
	1

$\Longrightarrow$

$\operatorname{arccot}\theta$	θ
$\dfrac{-1}{1+\theta^2}$	1

Then using the 'Rule of 7', we find that...

$$\int \operatorname{arccot}\theta\,d\theta = \theta\operatorname{arccot}\theta - \int \frac{-\theta}{1+\theta^2}\,d\theta = \theta\operatorname{arccot}\theta + \int \frac{\theta}{1+\theta^2}\,d\theta$$

Using the u-substitution $u = 1+\theta^2$, we see that $\displaystyle\int \frac{\theta}{1+\theta^2}\,d\theta = \frac{1}{2}\ln|1+\theta^2| + C$. Therefore, we have...

$$\int \operatorname{arccot}\theta\,d\theta = \theta\operatorname{arccot}\theta + \frac{1}{2}\ln|1+\theta^2| + C$$

Check-In 01/23. *(True/False)* The integral $\displaystyle\int e^{x/2}\sin(3x)\,dx$ is a 'looping' integral.

Solution. The statement is *true*. Recall that integrals whose integrand is the product of an exponential function with $\sin x$ or $\cos x$ 'loop.' We can see this directly: choose $u = \sin(3x)$. Using the 'box method', we have...

$$\begin{array}{|c|c|}\hline \sin(3x) & 2e^{x/2} \\ \hline 3\cos(3x) & e^{x/2} \\ \hline \end{array}$$

Therefore, we have...

$$\int e^{x/2}\sin(3x)\;dx = 2e^{x/2}\sin(3x) - \int 6e^{x/2}\cos(3x)\;dx$$

But this integral on the right also requires integration-by-parts: we choose $u = \cos(3x)$ and then...

$$\begin{array}{|c|c|}\hline \cos(3x) & 12e^{x/2} \\ \hline -3\sin(3x) & 6e^{x/2} \\ \hline \end{array}$$

So then we have...

$$\begin{aligned}
\int e^{x/2}\sin(3x)\;dx &= 2e^{x/2}\sin(3x) - \int 6e^{x/2}\cos(3x)\;dx \\
&= 2e^{x/2}\sin(3x) - \left(12e^{x/2}\cos(3x) - \int -36e^{x/2}\sin(3x)\;dx\right) \\
&= 2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x) + \int -36e^{x/2}\sin(3x)\;dx \\
&= 2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x) - 36\int e^{x/2}\sin(3x)\;dx
\end{aligned}$$

Observe that we have 'looped'—obtaining a multiple of the original integral on the right. Adding $36\int e^{x/2}\sin(3x)\;dx$ to both sides, we have...

$$37\int e^{x/2}\sin(3x)\;dx = 2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x)$$

Therefore, we have...

$$\int e^{x/2}\sin(3x)\;dx = \frac{2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x)}{37} + C$$

We can shortcut this work by adjusting tabular integration:

$$\begin{tikzcd} u & dv \\ \sin(3x) \arrow[dr, "+"] & e^{x/2} \\ 3\cos(3x) \arrow[dr, "-"] & 2e^{x/2} \\ -9\sin(3x) \arrow[r, "+"] & 4e^{x/2} \end{tikzcd}$$

Therefore, we have...

$$\begin{aligned} \int e^{x/2}\sin(3x)\;dx &= 2e^{x/2} - 12e^{x/2}\cos(3x) - 36\int e^{x/2}\sin(3x)\;dx \\ 37\int e^{x/2}\sin(3x)\;dx &= 2e^{x/2} - 12e^{x/2}\cos(3x) \\ \int e^{x/2}\sin(3x)\;dx &= \frac{2e^{x/2}\sin(3x) - 12e^{x/2}\cos(3x)}{37} + C \end{aligned}$$

Check-In 01/28. *(True/False)* To integrate $\displaystyle\int \cos^8\theta\sin^5\theta\;d\theta$, one should choose $u = \cos\theta$.

Solution. The statement is *true*. Observe that if we choose $u = \cos\theta$, then $\cos^8\theta$ becomes u^8. We know du will then produce a $\sin\theta$—specifically $-\sin\theta$. This 'uses' one of the $\sin\theta$'s in the integrand. This leaves $\sin^4\theta$ remaining in the integrand. But we can replace even powers of $\sin\theta$ in terms of $\cos\theta$. So we can carry out this substitution. Observe that if $u = \cos\theta$, then $du = -\sin\theta\;d\theta$. But then using the fact that $\sin^2\theta = 1 - \cos^2\theta$ (so that $\sin^4\theta = (\sin^2\theta)^2 = (1-\cos^2\theta)^2$), we have...

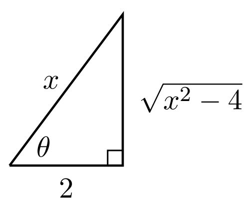
$$\begin{aligned} \int \cos^8\theta\sin^5\theta\;d\theta &= \int \cos^8\theta\sin^4\theta\cdot\sin\theta\;d\theta \\ &= -\int \cos^8\theta\sin^4\theta\cdot -\sin\theta\;d\theta \\ &= -\int \cos^8\theta(1-\cos^2\theta)^2\cdot -\sin\theta\;d\theta \\ &= -\int u^8(1-u^2)^2\;du \\ &= -\int u^8(1-2u^2+u^4)\;du \\ &= \int -u^8 + 2u^{10} - u^{12}\;du \\ &= -\frac{u^9}{9} + \frac{2u^{11}}{11} - \frac{u^{13}}{13} + C \\ &= -\frac{\cos^9\theta}{9} + \frac{2\cos^{11}\theta}{11} - \frac{\cos^{13}\theta}{13} + C \end{aligned}$$

Alternatively, observe that if we had chosen $u = \sin\theta$, then $\sin^5\theta$ becomes u^5. We know that du produces a $\cos\theta$. This 'uses' one of the $\cos\theta$'s in the integrand. This leaves $\cos^7\theta$ remaining in the integrand. However, we can only replace even powers of $\cos\theta$ in terms of $\sin\theta$ using $\cos^2\theta = 1 - \sin^2\theta$. So without using other identities or using other techniques, we cannot 'simply' choose $u = \sin\theta$ for this integral.

Check-In 01/30. *(True/False)* To compute $\displaystyle\int \frac{x^3}{\sqrt{x^2-4}}\,dx$, one can make the substitution $x = 2\sec\theta$.

Solution. The statement is *true*. Observe we have the term $x^2 - 4$, which resembles a term from the Pythagorean Theorem. For a right triangle, we know that $a^2 + b^2 = c^2$. This implies that $a^2 = c^2 - b^2$, which is $x^2 - 4$ if $c^2 = x^2$ and $b^2 = 4$, i.e. $c = x$ and $b = 2$. We construct a triangle with $c = x$ and $b = 2$. This only gives two possibilities:

In the first, we would have $\cos\theta = \frac{2}{x}$, which implies that $x = \frac{2}{\cos\theta} = 2\sec\theta$. This is the given substitution. In the second, we would have $\sin\theta = \frac{2}{x}$, which implies that $x = \frac{2}{\sin\theta} = 2\csc\theta$.
Choosing the former substitution, we would have $x = 2\sec\theta$, so that $dx = 2\sec\theta\tan\theta$. Calling the vertical side s and using the Pythagorean Theorem, we see that $2^2 + s^2 = x^2$, i.e. $4 + s^2 = x^2$. But then $s = \sqrt{x^2-4}$. This gives us the following triangle:



We need to find x^3 and $\sqrt{x^2-4}$. Because $x = 2\sec\theta$, we know that $x^3 = (2\sec\theta)^3 = 2^3\sec^3\theta$. To find $\sqrt{x^2-4}$, observe that $\tan\theta = \frac{\sqrt{x^2-4}}{2}$, which implies that $\sqrt{x^2-4} = 2\tan\theta$. Therefore, we have...

$$\int \frac{x^3}{\sqrt{x^2-4}}\,dx = \int \frac{2^3\sec^3\theta}{2\tan\theta}\cdot 2\sec\theta\tan\theta\;d\theta = \int 8\sec^4\theta\;d\theta$$

Now using the fact that $\tan^2\theta + 1 = \sec^2\theta$, we know that

$$\sec^4\theta = (\sec^2\theta)^2 = (\tan^2\theta+1)^2 = \tan^4\theta + 2\tan^2\theta + 1$$

But then...

$$\int 8\sec^4\theta\;d\theta = \int 8\sec^2\theta\sec^2\theta\;d\theta = \int 8\sec^2\theta(\tan^2\theta+1)\;d\theta = \int 8(\tan^2\theta+1)\cdot\sec^2\theta\;d\theta$$

Now letting $u = \tan\theta$, we have $du = \sec^2\theta\;d\theta$. But then...

$$\int 8(\tan^2\theta+1)\cdot\sec^2\theta\;d\theta = \int 8(u^2+1)\;du = 8\int(u^2+1)\;du = 8\left(\frac{u^3}{3}+u\right)+C$$

Replacing u, we have...

$$8\left(\frac{u^3}{3}+u\right)+C=8\left(\frac{\tan^3\theta}{3}+\tan\theta\right)+C$$

But from the triangle, we know that $\tan\theta=\frac{\sqrt{x^2-4}}{2}$. Therefore, we have...

$$8\left(\frac{\tan^3\theta}{3}+\tan\theta\right)+C=.8\left(\frac{1}{3}\cdot\left(\frac{\sqrt{x^2-4}}{2}\right)^3+\frac{\sqrt{x^2-4}}{2}\right)+C$$

If we simplify this, we have...

$$\begin{gathered}
8\left(\frac{1}{3}\cdot\left(\frac{\sqrt{x^2-4}}{2}\right)^3+\frac{\sqrt{x^2-4}}{2}\right)+C \\
8\cdot\frac{\sqrt{x^2-4}}{2}\left(\frac{1}{3}\cdot\left(\frac{\sqrt{x^2-4}}{2}\right)^2+1\right)+C \\
4\sqrt{x^2-4}\left(\frac{1}{3}\cdot\frac{x^2-4}{4}+1\right)+C \\
4\sqrt{x^2-4}\left(\frac{x^2-4}{12}+1\right)+C \\
4\sqrt{x^2-4}\left(\frac{x^2-4}{12}+\frac{12}{12}\right)+C \\
4\sqrt{x^2-4}\left(\frac{x^2+8}{12}\right)+C \\
\frac{1}{3}\sqrt{x^2-4}\,(x^2+8)+C
\end{gathered}$$

Therefore, we have...

$$\int\frac{x^3}{\sqrt{x^2-4}}\,dx=\frac{1}{3}\sqrt{x^2-4}\,(x^2+8)+C$$

Check-In 02/04. *(True/False)* The rational function $\dfrac{x^6-4x}{(x-1)(x+2)(x^2+4)}$ can be decomposed as $\dfrac{A}{x-1}+\dfrac{B}{x+2}+\dfrac{Cx+D}{x^2+4}$.

Solution. The statement is *false*. We know that given a rational function has a partial fraction decomposition given in the 'traditional' way so long as the degree of the numerator is *less* than the degree of the denominator. Observe that the degree of the numerator in the original function is 6 while the degree of the denominator is $1+1+2=4$. So, this cannot be broken down in the 'traditional' way. Alternatively, find the common denominator of $(x-1)(x+2)(x^2+4)$ for the given decomposition, observe that the numerator will have at most degree 4, which could not possibly yield the numerator of x^6-4x with degree 6. One would first need to long divide x^6-4x by

$(x-1)(x+2)(x^2+4)$ to find the quotient and remainder before trying to give a partial fraction decomposition. Observe...

$$(x-1)(x+2)(x^2+4) = (x^2+x-2)(x^2+4) = x^4+x^3+2x^2+4x-8$$

But then...

$$\begin{array}{r|l}
 & \qquad\qquad\qquad\qquad\quad x^2 \quad -x-1 \\
\hline
x^4+x^3+2x^2+4x-8 & x^6 \qquad\qquad\qquad\qquad\qquad -4x \\
 & -x^6-x^5-2x^4-4x^3+8x^2 \\
\cline{2-2}
 & \quad -x^5-2x^4-4x^3+8x^2-4x \\
 & \qquad x^5+x^4+2x^3+4x^2-8x \\
\cline{2-2}
 & \qquad\quad -x^4-2x^3+12x^2-12x \\
 & \qquad\qquad x^4+x^3+2x^2+4x-8 \\
\cline{2-2}
 & \qquad\qquad\quad -x^3+14x^2-8x-8
\end{array}$$

Therefore, we have...

$$\frac{x^6-4x}{(x-1)(x+2)(x^2+4)} = x^2-x-1+\frac{-x^3+14x^2-8x-8}{x^4+x^3+2x^2+4x-8} = x^2-x-1+\frac{-x^3+14x^2-8x-8}{(x-1)(x+2)(x^2+4)}$$

We can then find the 'traditional' partial fraction decomposition of the resulting rational function above because the degree of the numerator, which is 3, is *less* than the degree of the denominator, which is 4. We would have...

$$\frac{-x^3+14x^2-8x-8}{(x-1)(x+2)(x^2+4)} = \frac{A}{x-1}+\frac{B}{x+2}+\frac{Cx+D}{x^2+4}$$

Using Heaviside's Method/Cover-Up Method, we can find A and B 'instantly':

$$A = \frac{-1^3+14(1^2)-8(1)-8}{(1+2)(1^2+4)} = -\frac{3}{15} = -\frac{1}{5}$$

$$B = \frac{-(-2^3)+14(-2)^2-8(-2)-8}{(-2-1)\big((-2)^2+4\big)} = \frac{72}{-24} = -3$$

To find C, D, we can get a common denominator:

$$\frac{A}{x-1}+\frac{B}{x+2}+\frac{Cx+D}{x^2+4} = \frac{A(x+2)(x^2+4)+B(x-1)(x^2+4)+(Cx+D)(x-1)(x+2)}{(x-1)(x+2)(x^2+4)}$$

Using the fact that the rational functions now have equal denominators, we can equate their numerators:

$$-x^3+14x^2-8x-8 = A(x+2)(x^2+4)+B(x-1)(x^2+4)+(Cx+D)(x-1)(x+2)$$

$$-x^3+14x^2-8x-8 = -\frac{1}{5}(x+2)(x^2+4)-3(x-1)(x^2+4)+(Cx+D)(x-1)(x+2)$$

We multiply the last equation by 5 to clear fractions:

$$5(-x^3+14x^2-8x-8) = -(x+2)(x^2+4)-15(x-1)(x^2+4)+5(Cx+D)(x-1)(x+2)$$

Rather than expand this and relate coefficients to obtain a system of equations, we will simply substitute two x-values (ones not used in Heaviside's Method): using $x = 0$, we have…

$$\begin{aligned} 5(-x^3 + 14x^2 - 8x - 8) &= -(x+2)(x^2+4) - 15(x-1)(x^2+4) + 5(Cx+D)(x-1)(x+2) \\ -40 &= 52 - 10D \\ -92 &= -10D \\ \frac{46}{5} &= D \end{aligned}$$

and using $x = -1$ and the value for D above, we have…

$$\begin{aligned} 5(-x^3 + 14x^2 - 8x - 8) &= -(x+2)(x^2+4) + 15(x-1)(x^2+4) + 5(Cx+D)(x-1)(x+2) \\ 75 &= 145 - 10(-C + D) \\ -70 &= 10C - 10D \\ -70 &= 10C - 10 \cdot \frac{46}{5} \\ -70 &= 10C - 92 \\ 22 &= 10C \\ \frac{11}{5} &= C \end{aligned}$$

Therefore, we have…

$$\frac{x^6 - 4x}{(x-1)(x+2)(x^2+4)} = x^2 - x - 1 + \frac{-1/5}{x-1} + \frac{-3}{x+2} + \frac{\frac{11}{5}x + \frac{46}{5}}{x^2+4} = x^2 - x - 1 - \frac{1}{5(x-1)} - \frac{3}{x+2} + \frac{11x + 46}{5(x^2+4)}$$

Check-In 02/06. *(True/False)* $\int \frac{5}{x+1} - \frac{1}{(x+1)^2} - \frac{x+1}{x^2+1}\,dx = 5\ln(x+1) - \frac{1}{x+1} - \frac{1}{2}\ln|x^2 + 1| + \arctan x + K$

Solution. The statement is *false*. First, we know that…

$$\int \frac{5}{x+1}\,dx = 5\ln|x+1| + K$$

The integral of the second term is also incorrect:

$$\int -\frac{1}{(x+1)^2}\,dx = \int -(x+1)^{-2}\,dx = \frac{-(x+1)^{-1}}{-1} + K = \frac{1}{x+1} + K$$

Moreover, the third term is also incorrect as the negative has been improperly distributed:

$$-\int \frac{x+1}{x^2+1}\,dx = -\int \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx = -\left(\frac{1}{2}\ln|x^2+1| + \arctan x\right) + K = -\frac{1}{2}\ln|x^2+1| - \arctan x + K$$