Visualising Special Relativity

Chris Nicholls

November 1, 2016

Contents

1 Introduction

We explore the connection between Möbius transformations, or fractional linear transformations, defined on a sphere, and the visual observations of relativistically moving observers. This set of notes accompanies a program written by the author to demonstrate this visually.

1.1 Special Relativity

We first introduce the subject of Special Relativity by the two postulates, as is standard in the literature:

P1: The laws of physics are the same in all inertial reference frames.
P2: The speed of light in a vacuum is always c in any inertial reference frame.

By inertial reference frame, it is meant an observer moving without acceleration.
We can ask the question: *what is fundamental?*
And we allow ourselves to think of the following as fundamental:

- Clocks - e.g. vibrations of atoms. Any inertial observer can measure these vibrations.
- Light signals - an observer can send and receive light signals.

These fundamental things will be used to define other measurements that an observer can make, like distance.

1.2 Operational definitions

In the spirit of [Woodhouse, 2003], we introduce distance with an operational definition; that is, distance is defined by the operation that an observer would perform to measure it.

The operation we consider for measuring distance is as follows: the observer is permitted to send light signals in particular directions, and subsequently receive them after they have been reflected off distant objects. Under **P2**, we see that if the time that the photon took to make the return trip was measured by the observer's clock as t, then the object must have been a distance $ct/2$ away. This is how we define distance.

Assuming that the observer knew which direction this photon was sent in, then the observer may map out coordinates in his frame. That is, to each event in spacetime, the observer can assign coordinates, (t, x, y, z). It should be noted that the choice of origins of t and (x, y, z) is arbitrary, but it makes sense to choose the origin of the spatial coordinates to be the position of the observer.

It should also be noted that the observer can only map out coordinates once he has received all the returning photons, and so cannot immediately give coordinates to events that are non-zero distance away. Indeed, until the observer receives a photon in reply, the observer doesn't know that anything has happened.

1.3 Measurement vs. Visual Appearance

There is an important distinction between a *measurement* of an object by an observer and the *visual appearance* of an object to an observer.

An example of a measurement would be the question: *how long is this rod?*, while an example of a visual observation would be: *what does the rod look like?*

In a measurement, the observer will choose a plane of simultaneity in his frame, say $t = 0$, and then consider the object to be made up of all the points on the object that satisfy $t = 0$. Here, we imagine an object to in fact be a set of worldlines, and then from each worldline we choose the point on the plane $t = 0$.

In contrast to this, the visual appearance of an object consists of all the photons arriving at the observer's position at a given time, that emanated from the object.

We can see that these cannot be equivalent. Indeed, in a measurement of the object we are considering the intersection of the object with some plane of constant time, whereas in the visual appearance of an object, the photons arriving at the observer's eye at some given time will not in general have been emitted at the same time from the object. To see this, note that photons arriving from further away on the object must have been emitted at an earlier time (reckoned by the observer) than photons arriving from closer points, in order to arrive at the same time at the observer.

1.4 Our Aim

Primarily, we will be concerned with relating the visual appearances of two observers.
The first thing we discuss, however, is the relation between the measurements of two observers, as this will help us to develop the language to talk about visual appearances.

2 Lorentz Transformations

Given two observers, O and O', each of whom has mapped out coordinates (t, x, y, z), (t', x', y', z'), it is a natural question to ask how these coordinate systems are related.

2.1 Two-dimensions, in standard configuration

We simplify to start with, and assume that there is just a two-dimensional spacetime (t, x), and we wish to work out how the coordinates (t, x) measured by O, and the coordinates (t', x'), measured by O', are related.

Suppose that O' moves with velocity v with respect to O, as measured by O. By this, it is meant that O measures the position of O' over a short period of time, and it is found to be given by $x(O) = x_0 + vt$, for some fixed x_0. Thus O says that O' moves with velocity v.

Let us now suppose that at time $t = 0$, O and O' instantaneously occupy the same position in space (however impossible this actually is).
We may ask the question: *how is* (t, x) *related to* (t', x')*?*

As is derived in the Appendix, from the two postulates **P1** and **P2**, they are in fact related by the Lorentz transformation:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v/c^2 \\ -\gamma v & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

where γ is the all-important Lorentz factor, given by:

$$\gamma(v) = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Intuitively, if O' moves with velocity v, then γ is approximately 1 for small v, and grows large for v near c.
It should be noted that when v is small, then the transformation is approximately:

$$\begin{aligned} t &\mapsto \gamma t - \gamma v/c^2 \approx t \\ x &\mapsto -\gamma vt + \gamma x \approx x + vt \end{aligned}$$

which is the classical transformation.

Rapidity: We note that it often proves useful stylistically to introduce the *rapidity*, ϕ, equal to

$$\phi = \sinh^{-1}(\beta\gamma).$$

where $\beta = v/c$ is the ratio of the velocity, v of O' relative to O, and the speed of light, c. It is also easier to work with ct rather than t.

Indeed, working with ct and rapidity, the Lorentz transformation now takes the form:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\phi) & -\sinh(\phi) \\ -\sinh(\phi) & \cosh(\phi) \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

which is much neater.

2.2 Addition of Velocities

As an example of Lorentz transformations in use, consider the problem of velocity addition in special relativity.

Classically, if O is standing on a moving train and throws a ball, then relative to the ground, the ball will have speed which is simply the sum of the speed of the train and the speed of the ball relative to the train. This could not be the case relativistically, else a train moving with velocity $c/2$ could propel a ball with velocity $c/2$ and then an observer on the ground would claim the ball to move at speed c, which is impossible for a massive object.
So *how do we actually add velocities?*

We can pose the velocity addition problem using three observers: O, O' and O'', by supposing that O sees O' moving with velocity v, and O' sees O'' moving with velocity w. The question is then simply: *with what velocity does O observe O'' to move?*
It is easiest to simply write down the Lorentz transformations and then multiply the matrices. Let ϕ be the rapidity corresponding to v, and ψ the rapidity corresponding to w. Then we have the following equations.

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\phi) & -\sinh(\phi) \\ -\sinh(\phi) & \cosh(\phi) \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$\begin{pmatrix} ct'' \\ x'' \end{pmatrix} = \begin{pmatrix} \cosh(\psi) & -\sinh(\psi) \\ -\sinh(\psi) & \cosh(\psi) \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

Multiplying the two matrices, we see:

$$\begin{pmatrix} ct'' \\ x'' \end{pmatrix} = \begin{pmatrix} \cosh(\phi+\psi) & -\sinh(\phi+\psi) \\ -\sinh(\phi+\psi) & \cosh(\phi+\psi) \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

Thus O observes the rapidity of O'' to be $\phi + \psi$. A quick rearrangement then shows that the corresponding velocity, u, is equal to:

$$u = \frac{v+w}{1+vw/c^2}$$

Note that in the case $v, w \ll c$, this reduces to the classical case, $v + w$.

2.3 The Sphere of Sight

Now that we have discussed the relationship between the *measurements* of two observers, we proceed to focus on the relationship between their *visual observations*.
We can imagine ourselves being at a particular point in spacetime, and taking a picture which captures our view in every direction. More explicitly, at some fixed time, we imagine having a spherical film and instantaneously opening, then closing, a shutter, letting light in for an instant of time. The photons are then allowed to paint the surface of the sphere. This is what we mean by the **sphere of sight**.

A very basic question is: how are the spheres of sight of two observers, O and O', moving relative to each other, but at the same instantaneous position in spacetime, related?

A first observation, which is not immediately obvious, is that the two observers will in fact have received precisely the same photons, and will only have assigned different directions to them. This is the case because, if we imagine that O and O' occupy the origin at time $t = 0$, then the world line of any photon intersecting O at time $t = 0$ will also intersect O', and vice versa. Thus there is a bijection between the sphere of sight, S, of O and the sphere of sight, S' of O'. This is what we shall now investigate.

3 Möbius Transformations

It will be helpful to introduce Möbius transformations after we have talked about homogeneous coordinates.

Before we embark on some technical definitions, I will explain the point of what we are about to do.

It turns out that there is a natural identification of Lorentz transformations on the sphere of sight with Möbius transformations defined on what we shall shortly call the Riemann sphere.
We would thus like a nice way of describing the Riemann sphere. To start with, picturing a sphere will be sufficient, but to proceed further it is necessary to provide coordinates so that we can talk about points on the Riemann sphere. We will also refer to the Riemann sphere as the complex projective line, for reasons which will hopefully become clear shortly.

After the next few sections, hopefully it will have been brought out that there is a natural identification of a sphere in three real dimensions with the extended complex plane, which consists of the complex plane, $\mathbb{C}$ along with an extra point, denoted ∞.

3.1 Parameterising the Sphere

The problem of putting coordinates on a sphere is not a new one, and without thought it is possible to approach it very badly. We first note that, as a surface, a sphere should be parameterised by two coordinates. The first approach one might attempt is probably some form of spherical polar coordinates. Here, we would have an angle, θ, measuring angular displacement from the vertical axis, and an angle ϕ, measuring rotation around the vertical axis, as in Figure 1.
At first sight this looks like a reasonable coordinate system: certainly, every point on the sphere has a description in (θ, ϕ) coordinates. However, a serious issue with this set up is that the north pole and south pole of the sphere (that is, $\theta = 0$ or $\theta = \pi$) do not take into account the ϕ coordinate, so that the coordinate system breaks down at the poles. Moreover, the coordinate ϕ may just as well be replaced with $\phi + 2\pi$, so that on traversing a line of constant θ, we have the problem of describing which ϕ to choose.

We shall discuss the issue at the poles first. Intuitively, an unmarked sphere has no distinguished points; indeed, if you were to look away, and I was to rotate the sphere, then you would have no idea which points have ended up where. So the fact that the spherical polar coordinate system distinguishes two points on the sphere is rather strange in itself. More than this, it becomes rather difficult to talk about points near each pole, because the coordinate system actually breaks down.

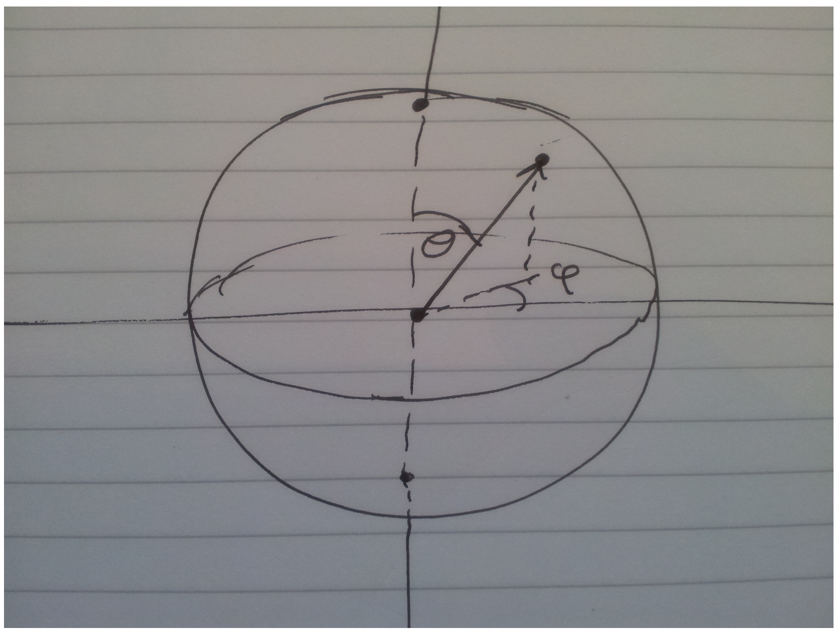

Figure 1: Spherical polar coordinates (θ, ϕ) on the sphere.

3.2 Homogeneous Coordinates and the Complex Projective Line

Over the field of complex numbers, $\mathbb{C}$, we can define homogeneous coordinates in any dimension. We shall only be interested in the complex projective line, $\mathbb{CP}^1$, where homogeneous coordinates take the form of a pair of complex numbers, not both zero, under the equivalence relation of scaling by a non-zero complex number.
Equivalently, we can consider the set of all lines through the origin in $\mathbb{C}^2$,

where a line through the origin is of the form $w\xi + z\eta = 0$, for some w, z not both zero. It is clear that multiplying through this equation by any $\lambda \in \mathbb{C} \setminus \{0\}$ leaves the same set of ξ, η.

More formally, we define

$$\mathbb{CP}^1 = \{(w, z) \in \mathbb{C} \times \mathbb{C} \setminus (0, 0)\} / \sim$$

where the equivalence relation $\sim$ is defined by: $(w, z) \sim (w', z')$ if and only if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $w = \lambda w'$ and $z = \lambda z'$.

There is in fact a much more intuitive way of thinking about $\mathbb{CP}^1$ in terms of the very readily available sphere, which involves two overlapping coordinate systems.

Here, the aim is to visualise what $\mathbb{CP}^1$ looks like topologically. We shall see that it is, in fact, a sphere.

We first note that, given a sphere, we can use stereographic projection from any point to set up a bijection between the rest of the sphere and a plane. The canonical stereographic projection is from the north pole of a unit sphere centred at the origin, to the horizontal plane through the origin, as in Figure 2. This takes the following form:

$$(x, y, z) \mapsto (\frac{x}{1-z}, \frac{y}{1-z})$$

where we note that the only point on the sphere that is not defined under this mapping is that for which $z = 1$, i.e. $(0, 0, 1)$.

So we have almost parameterised the sphere by the plane, but not quite. It turns out that an excellent way to proceed is to in fact consider two coordinate maps: one stereographic projection from the north pole of the sphere, and one stereographic projection from the south pole of the sphere.

3.3 Transformations on the complex projective line

We can think of transformations on the complex projective line. A simple class of such transformations is given by considering the action of invertible

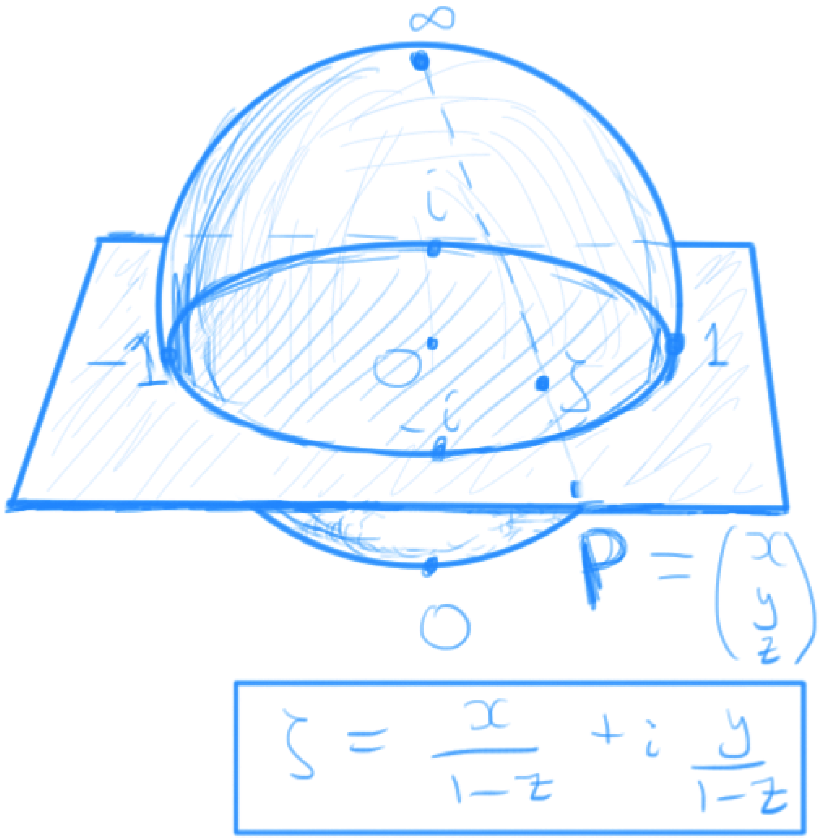

Figure 2: Stereographic projection from the north pole. We identify the plane $z = 0$ with the complex plane, $\mathbb{C}$, and map the point P on the unit sphere to the point ζ on the complex plane, shown.

matrices on the homogeneous coordinates. These are in fact the Möbius transformations:

$$[\xi, \eta] \mapsto [a\xi + b\eta, c\xi + d\eta]$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}).$$

Since we are working with homogeneous coordinates, the action of a matrix is only up to scaling, so in fact the action is of $SL(2, \mathbb{C})$, which means complex invertible matrices of determinant 1. Now we are only working up to a minus sign, as opposed to any complex scaling.

Since the complex projective line is identified with the Riemann sphere, any transformation on $\mathbb{CP}^1$ is also naturally a transformation of the sphere. Shown in Figure 3 is an example of a Möbius transformation on the sphere. The blue lines represent the path of certain points on the sphere under the transformation. We have opted to take points on a grid of latitude and longitude, and then plot these lines for each point.

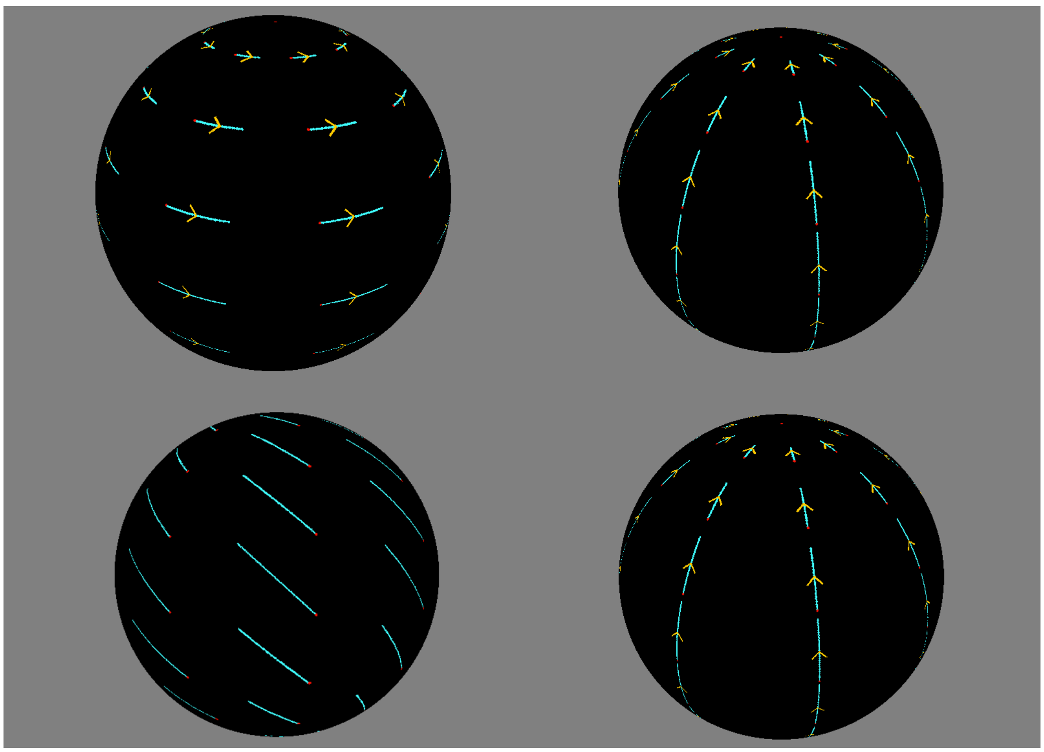

Figure 3: Some examples of Möbius transformations on the sphere.

3.4 Stereographic Coordinates

Here is a lovely bijection between spherical polar coordinates (θ, ϕ) on the sphere and homogeneous coordinates $[\xi, \eta]$ of $\mathbb{CP}^1$:

$$(\theta, \phi) \mapsto (\xi, \eta) \in \mathbb{C}^2$$
$$(\theta, \phi) \mapsto (\cos(\frac{\theta}{2})e^{\frac{i\phi}{2}}, \sin(\frac{\theta}{2})e^{\frac{-i\phi}{2}})$$

This satisfies the property that $|\xi|^2 + |\eta|^2 = 1$.

3.5 A Classification of Möbius Transformations

We now provide a classification of Möbius transformations. Before we do this, we must first make clear what in fact we are actually classifying. In this case, we are interested in Möbius transformations up to conjugacy. By this, we mean that we are considering the classes of Möbius transformations, up to the equivalence relation of being conjugate.
In this way, we can see that, for example:

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and

$$B = \begin{pmatrix} -i/2 & i\sqrt{3}/2 \\ i\sqrt{3}/2 & i/2 \end{pmatrix}$$

are conjugate. Indeed a quick calculation will verify that $CAC^{-1} = B$, with C given by:

$$\begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix}$$

Our classification will thus put A and B in the same equivalence class, and will not distinguish between them. We will see shortly why this is sensible.

Fixed Points

An important feature of Möbius transformations that will be used in the classification is their fixed points.

Observation: We first observe that given a Möbius transformation, with action given by $A \in SL(2, \mathbb{C})$, then a fixed point of M is actually an eigenvector of A.
To see this, note that a fixed point of the transformation M:

$$[\xi, \eta] \mapsto [a\xi + b\eta, c\xi + d\eta]$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}).$$

is a point $[\xi, \eta]$ in homogeneous coordinates, representing the direction through the corresponding point on the unit sphere, such that the action of A takes it to the same direction. That is, A takes the point to a scalar multiple of itself. This is precisely what it means to be an eigenvector. □

We are now in a position to show a rather interesting result about Möbius transformations; namely,

Proposition 1. *Any non-identity Möbius transformation has either one, or two fixed points.*

Proof. Given the above observation, it is enough to show that any matrix in $SL(2, \mathbb{C})$, that is, an invertible matrix with coefficients in $\mathbb{C}$ and determinant 1, has either one or two linearly independent eigenvectors, up to a scalar multiple.
Now, there exists a nonzero vector $v \in \mathbb{C}^2$ such that v is an eigenvector of the matrix $A \in SL(2, \mathbb{C})$ with eigenvalue $\lambda \neq 0$, if, and only if, the kernel of the transformation $A - \lambda I$ is nonzero. This condition is equivalent to the determinant of the matrix $A - \lambda I$ being zero, since the determinant is nonzero if, and only if, the transformation is invertible.
Therefore, if we find a solution, λ, to the equation $det(A - \lambda I) = 0$, the characteristic equation, we shall have found an eigenvector.
Explicitly, this equation is:

$$det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$$

and after expanding, and replacing $ad - bc$ by 1, we have our equation for λ as:

$$\lambda^2 - (a + d)\lambda + 1 = 0$$

As a quadratic over $\mathbb{C}$, this factorises as $(\lambda-\alpha)(\lambda-\beta)$ for some $\alpha, \beta \in \mathbb{C}$, and we either have two distinct solutions, when $\alpha \neq \beta$, or we have one repeated solution, $\alpha = \beta$. The discriminant, $(a+d)^2 - 4$ separates the cases, with the discriminant being nonzero and zero respectively for the two cases. We note that $a+d$ is the trace of A.
It should be noted that the product of the roots of any monic quadratic equation is the constant coefficient, so that $\beta = 1/\alpha$.

Case $(a+d)^2 \neq 4$. There are two distinct eigenvalues, and each corresponds to a unique eigenvector, up to a scalar multiple. Thus there are two fixed points.

Case $(a+d)^2 = 4$. There is a single, repeated, eigenvalue. In fact, we can determine it, because $\beta = 1/\alpha$ implies that $\lambda^2 = 1$, and so $\lambda = \pm 1$.
Now, we claim that either there is a single fixed point, or the matrix A is a scalar multiple of the identity. To see this, suppose that there is a second, linearly independent eigenvector, u, as well as the original eigenvector, v. In this case, the set $\{u, v\}$ forms a basis for $\mathbb{C}^2$, and A acts by $w \mapsto \lambda w$ for all $w \in \mathbb{C}^2$, which is simply a scalar multiple of the identity, as required.

$\square$

We now get for free:

Corollary 1. *Any Möbius transformation with (at least) three fixed points is the identity transformation.*

For certainly the identity transformation has at least three fixed points, and conversely, a non-identity transformation has at most two.

Our achievement thus far is that from the trace of the representing matrix, $A \in SL(2, \mathbb{C})$ we can read off the number of fixed points. In particular, there is a single fixed point if $tr(A)^2 = 4$, and there are two fixed points otherwise, for any non-identity Möbius transformation.

We can do better.

The remainder of this section will build up to a classification of those transformations with precisely two fixed points.
Suppose then, that we have such a transformation, with distinct fixed points

$[\xi_1, \eta_1]$ and $[\xi_2, \eta_2]$. That the fixed points are distinct in $\mathbb{CP}^1$ is equivalent to the representing vectors $\binom{\xi_1}{\eta_1}$, $\binom{\xi_2}{\eta_2}$ in $\mathbb{C}^2$ being linearly independent. Thus they form a basis, and so the matrix in $GL(2, \mathbb{C})$ with these as columns is invertible, and we can consider the conjugate of A by this matrix:

$$\begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix}$$

Having determinant 1, this is still in $SL(2, \mathbb{C})$, and it's columns are determined by its action on the vectors $\binom{1}{0}$ and $\binom{0}{1}$. We have constructed the matrix so as to send the standard basis, $\{\binom{1}{0}, \binom{0}{1}\}$ to the basis of eigenvectors of A, $\{\binom{\xi_1}{\eta_1}, \binom{\xi_2}{\eta_2}\}$, which A scales by the respective eigenvalues, and then simply send the scaled eigenvectors back to the standard basis.
Explicitly, the matrix acts by: $\binom{1}{0} \mapsto \binom{\lambda}{0}$, $\binom{0}{1} \mapsto \binom{0}{1/\lambda}$, and thus is equal to:

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$$

Our achievement now is that any matrix A with trace not squaring to 4 is equivalent to a diagonal matrix with entries $\lambda, 1/\lambda$. It is now prudent to show that $diag(\lambda, 1/\lambda)$ is equivalent to $diag(\pm\mu, \pm 1/\mu)$ if, and only if, $\lambda = \mu, 1/\mu$.
We now use the fact that the trace of a diagonal matrix is unchanged under a conjugation. Thus, if $diag(\lambda, 1/\lambda)$ is conjugate to $diag(\mu, 1/\mu)$, then λ satisfies:

$$(\lambda + 1/\lambda)^2 = (\mu + 1/\mu)^2$$

which is a quartic in λ, and thus has at most four solutions. But we can immediately see that $\mu, -\mu, 1/\mu, -1/\mu$ are four distinct solutions (for $\mu \neq \pm 1$, which corresponds to the identity transformation), and so we are done.

In summary, any Möbius transformation with precisely two fixed points is equivalent to a diagonal matrix of the form:

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$$

for some nonzero $\lambda \in \mathbb{C}$, up to a minus sign, and an inverse.
In fact, if we allow ourselves to move out of $SL(2, \mathbb{C})$, then we can consider

the matrix

$$\begin{pmatrix} \lambda^2 & 0 \\ 0 & 1 \end{pmatrix}$$

which represents the same Möbius transformation. We define the **characteristic constant**, $k = \lambda^2$, and then our achievement is that any two Möbius transformations with precisely two fixed points are equivalent if, and only if, they have the same characteristic constant, k.

We thus study Möbius transformations of this **normal form**.
They have fixed points at $[0, 1]$ and $[1, 0]$, and the characteristic constant controls the motion of points away from 0 and towards ∞. In particular, we can see:

Case $k = e^{i\beta}$: The corresponding Möbius transformation is a rotation of the sphere about the vertical axis, as shown in Figure 4.

Case $k = e^{\alpha}$: The corresponding Möbius transformation sends points along lines of constant longitude with speed α, as shown in Figure 5.

Case $k = e^{\alpha+i\beta}$: This class of transformations is termed **loxodromic**, and send points along lines that spiral from 0 to ∞ on the sphere, as shown in Figure 6.

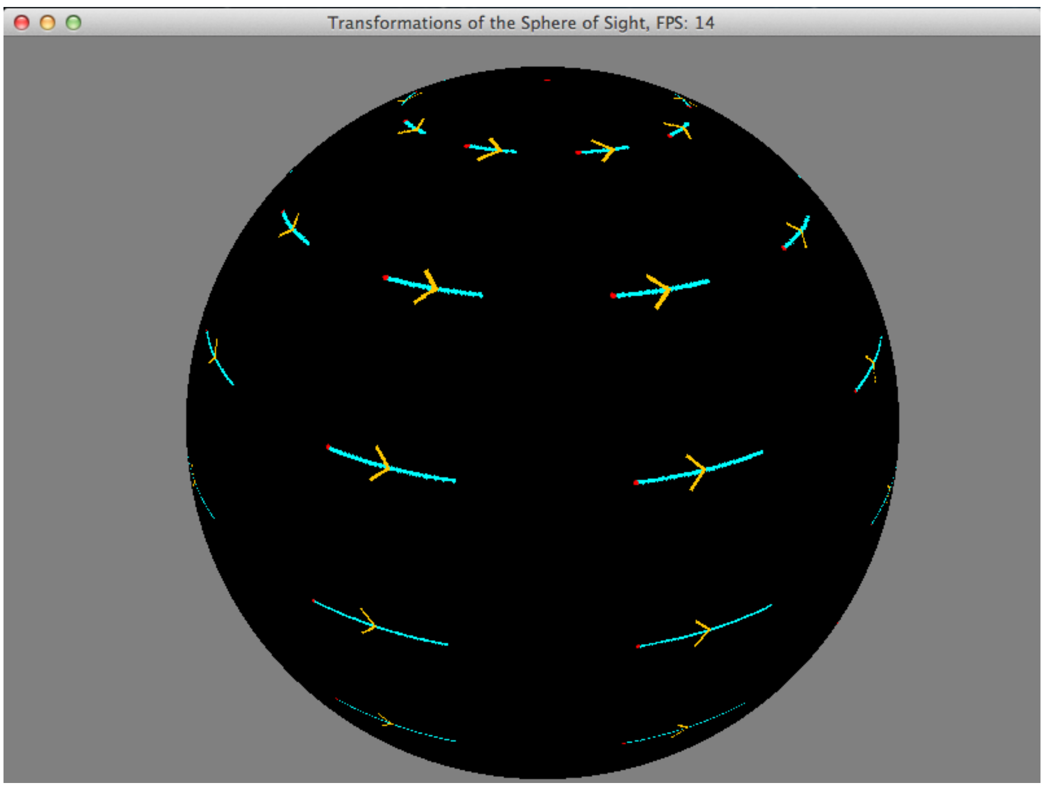

Figure 4: An elliptic Möbius transformation, with $k = e^{i\pi/10}$.

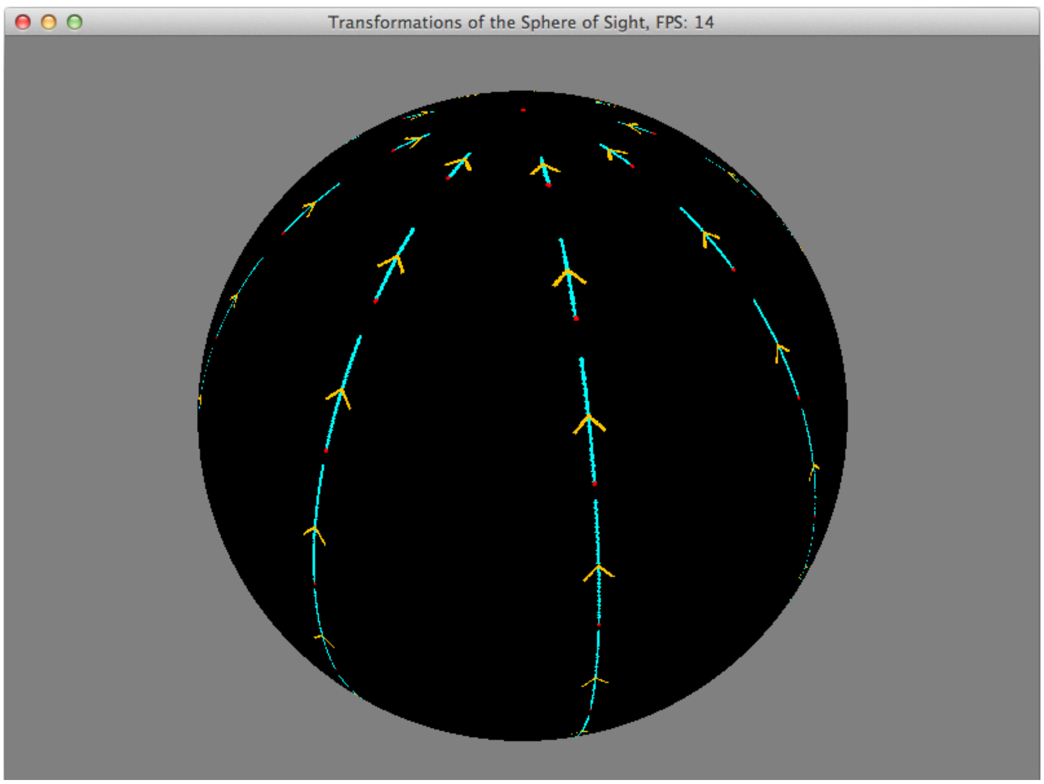

Figure 5: A hyperbolic Möbius transformation, with $k = e^{1/3}$.

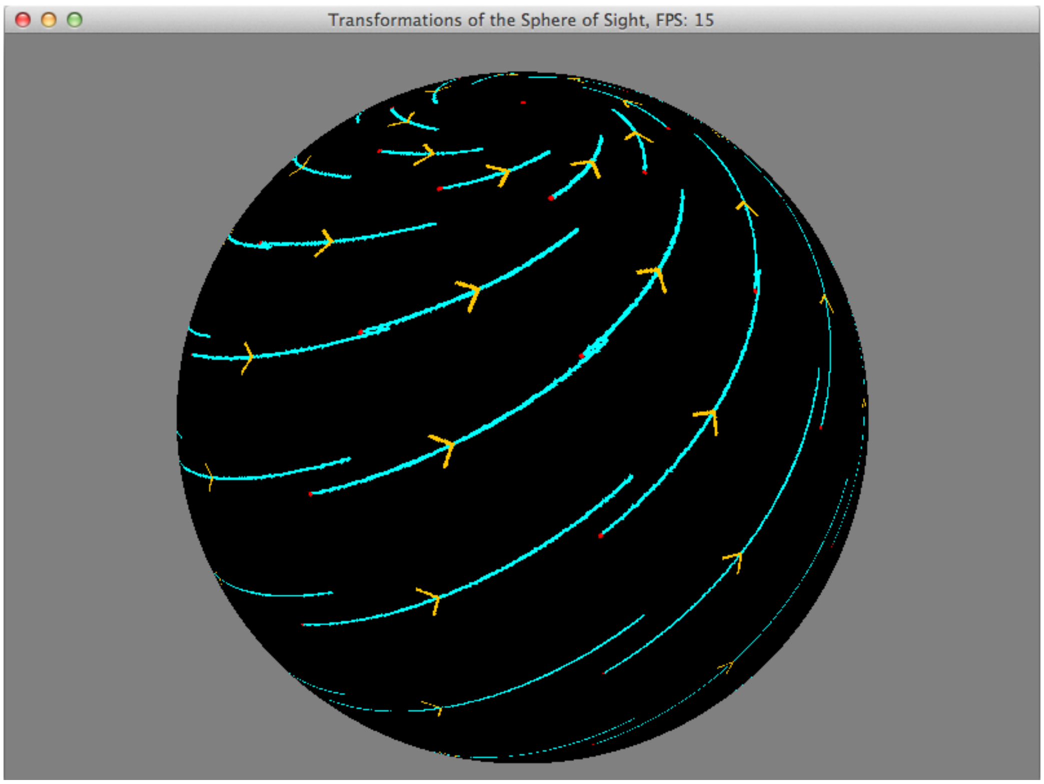

Figure 6: A loxodromic Möbius transformation, with $k = e^{1/2+i\pi/4}$.

3.6 The Parabolic Case

The remaining case to deal with is when there is only one fixed point. Here, our previous approach of diagonalising the matrix is impossible, and the normal form of the matrix is:

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

for some non-zero $b \in \mathbb{C}$.
To see this, note that we can choose the location of one of the fixed points to be ∞ on the sphere; that is, $[1, 0]$ in homogeneous coordinates. That amounts to putting the first column as $\binom{1}{0}$. Then the condition of the determinant being 1 forces the bottom right entry to be 1 also, and this leaves a general $b \in \mathbb{C}$ for the top right entry, though $b = 0$ corresponds to the identity.

4 The Connection between Lorentz transformations and Möbius transformations

We have now introduced both Lorentz transformations and Möbius transformations in some detail. There is a remarkable connection between proper, orthochronous Lorentz transformations and Möbius transformations, which we shall now proceed to illustrate.
Note that a proper Lorentz transformation preserves the orientation of space; so that both observers still agree about which hand is their right hand. And an orthochronous Lorentz transformation preserves the direction of time; so that both observers still agree on the rather important matter of which way time is flowing.

4.1 Sphere of Sight

The basic question we want to answer is: *How do I relate what I see to what you see, when we are moving relative to each other, and at the same spatial position?*
In the language that we have been using up till now, this means that we are interested in the transformation of the spheres of sight of two observers moving relative to each other. Recall that we defined the sphere of sight of an observer as the sphere centred at the observer's position, and with each point painted by the colour that the observer can see in that direction at that time.

We shall now be more formal about this.

First, we discuss light rays. In Figure 7 we can see the past null cone of the observer situated at the origin, with one of the light rays lying on the cone shown in blue. Note that the null cone of the observer is the set of null rays passing through the observer's position, which forms a cone with a lower and upper half, corresponding to the past and present respectively.

A Lorentz transformation transforms the light rays on the observer's null cone, so that they have different directions; see Figure 8. It should be noted that since a Lorentz transformation is invertible, then there is a one-to-one correspondence between the directions of light rays as reckoned by the two observers. Intuitively, this means that if I look in a certain direction, then

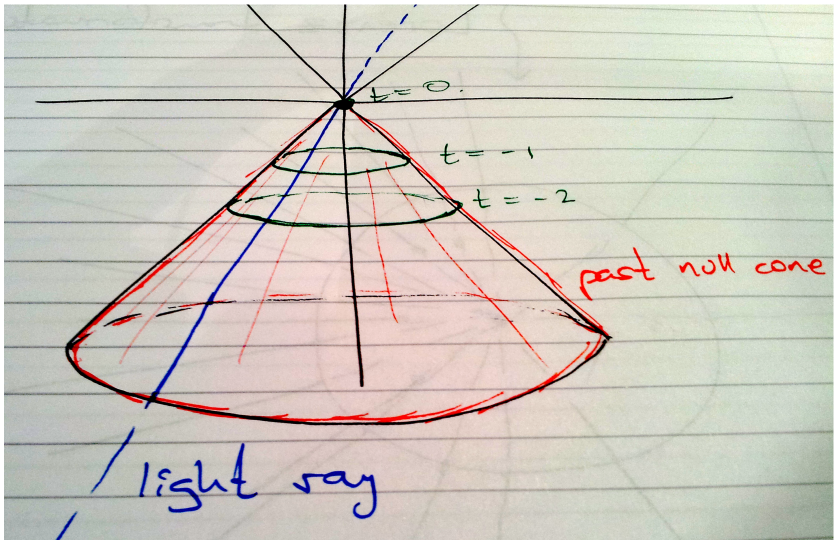

Figure 7: The null cone of an observer situated at the origin, with one light ray highlighted. Here we are using coordinates (ct, x, y), with the z coordinate omitted for visual purposes. There is a unique light ray through every direction on the unit circle $x^2 + y^2 = 1$.

there is a, in general different, direction that you can look in to see the same colour as me.

So the visual effect of a Lorentz transformation is just a transformation of the directions of light rays. But a nicer description of this is to do with spheres, which is where the sphere of sight comes in.

A light ray is determined by a single point on the past null cone of the observer, since there is a unique light ray passing through the origin and a given point on the null cone. We can choose the representative points to lie on the plane $t = -1/c$, so that the points are all of the form

$$\begin{pmatrix} -1 \\ x \\ y \\ z \end{pmatrix}$$

with $x^2 + y^2 + z^2 = 1$, due to the null condition. This is simply the sphere of sight, once we have associated the colour that the observer sees in each

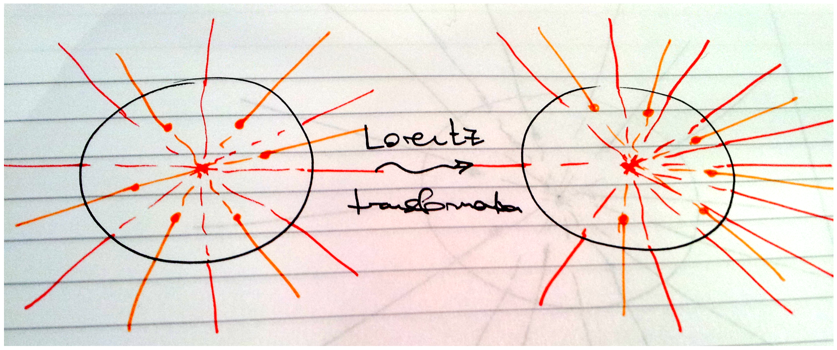

Figure 8: The light rays that an observer O sees (left) are transformed by a Lorentz transformation (right). An observer O' related to O by the Lorentz transformation will see any light ray that O sees, but in the direction given by this transformation. This particular example demonstrates relativistic aberration, whereby the directions of the corresponding light rays for O' are more towards the front of his sphere of sight, the greater the relative speed.

direction with the point on the sphere. That is, the sphere of sight is this sphere representing directions of light rays along with the information about the colour in each direction.

Upon transforming these points (and thus the light rays) by a Lorentz transformation, the ct component will not, in general, still equal -1. But we can simply choose a different point on the transformed light ray, namely that for which $ct' = -1$. Then we have succeeded in mapping the sphere of sight of O to the sphere of sight of O'.

The question remains: *what type of transformation is this?*

4.2 The connection

The key observation is that we can write a four-vector (ct, x, y, z) in the form:

$$\begin{pmatrix} -ct+z & x+iy \\ x-iy & -ct-z \end{pmatrix}$$

which is a hermitian matrix. That is, a matrix equal to its conjugate transpose.

Moreover, any hermitian matrix is of this form. So we have a bijection between four-vectors and hermitian matrices.

We now consider an action of the general linear group, $GL(2,\mathbb{C})$, (invertible 2x2 matrices with complex coefficients) on hermitian matrices:

$$X \mapsto AXA^*$$

where A^* denotes the conjugate transpose of A. Note that if A is invertible, then A^* is also invertible, as it has the same determinant. And moreover, if X is hermitian, (that is $X = X^*$) then $(AXA^*)^* = (A^*)^* X^* A^* = AXA^*$ and so the action preserves the hermitian structure.

The four-vectors of particular interest are the past-pointing null four-vectors, since these represent points on the observer's sphere of sight. Recall that these are characterised by having inner product zero. The hermitian matrix representation records this as having zero determinant, since:

$$\det \begin{pmatrix} -ct+z & x+iy \\ x-iy & -ct-z \end{pmatrix} = c^2t^2 - x^2 - y^2 - z^2$$

If we choose a representative four-vector for each direction, say with $t = -1/c$, then we have a correspondence with the points of the sphere. In fact, we can write:

$$\begin{pmatrix} -ct+z & x+iy \\ x-iy & -ct-z \end{pmatrix} = 2 \begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix}$$

The values of $\xi, \eta \in \mathbb{C}$ can be found explicitly, as follows.
Our intuition is that we would like to start as before, by associating each unit vector (x,y,z) on the sphere with a complex coordinate pair: (ξ, η). Recall that we previously used stereographic projection from the north pole and south pole to find that one way of doing this is:

$$[\xi, \eta] = \begin{cases} [x+iy, 1-z] & \text{if } z \leq 0 \\ [1+z, x-iy] & \text{if } z \geq 0 \end{cases}$$

Our equation for the matrix forces us to impose the condition $2(\xi\bar{\xi}+\eta\bar{\eta}) = -2ct = 2$, so we simply need to scale the two coordinates to satisfy this.

We shall need two coordinate maps, in order to satisfy this. One will be defined near the point 0 and one will be defined near the point ∞ on the sphere.

Near 0, that is $z < 0$, we define:

$$(\xi, \eta) = \left(\frac{x + iy}{\sqrt{2 - 2z}}, \frac{\sqrt{1 - z}}{\sqrt{2}} \right)$$

Near ∞, that is $z \geq 0$, we define:

$$(\xi, \eta) = \left(\frac{\sqrt{1 + z}}{\sqrt{2}}, \frac{x - iy}{\sqrt{2 + 2z}} \right)$$

Note that the first one breaks down when z is near 1, and the second one breaks down when z is near -1. Moreover, as homogeneous coordinates, these two are equivalent. To see this, note that when both are defined, so away from $z = \pm 1$, we have ξ/η given by:

$$\frac{x + iy}{1 - z}$$

in both cases.

Now note that if we write:

$$X = \begin{pmatrix} -ct + z & x + iy \\ x - iy & -ct - z \end{pmatrix} = 2 \begin{pmatrix} \xi \\ \eta \end{pmatrix} (\bar{\xi}, \bar{\eta})$$

then the action of $A \in SL(2,)$ is by $X \mapsto AXA^*$, which is given by:

$$2A \begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix} A^* = 2A \begin{pmatrix} \xi \\ \eta \end{pmatrix} \left(A \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right)^*$$

What we have succeded in demonstrating is that the action of Lorentz transformations on null four-vectors $(-1, x, y, z)$ via the action of $SL(2, \mathbb{C})$ on the space of hermitian matrices is equivalent to the action of $SL(2, \mathbb{C})$ on the space of homogeneous coordinates representing the direction (x, y, z).

The remarkable consequence is that **the action of a proper, orthochronous Lorentz transformation on the sphere of sight is by a Möbius transformation**.

It also happens that the converse is also true; so that the action of a Möbius transformation on the sphere of sight is by a Lorentz transformation. This can be seen by noticing that the action of $SL(2, \mathbb{C})$ on the hermitian matrices preserves the determinant, and then using the fact that any linear transformation on four vectors which preserves the inner product is a Lorentz transformation. An alternative, and more sophisticated, proof uses facts about the topologies of $SL(2, \mathbb{C})$ and the group of proper, orthochronous Lorentz transformations, but we do not concern ourselves too much with this.

4.3 Generators

We would now like a simple way of describing this action explicitly.
One way of doing this is to notice that the group of proper, orthochronous Lorentz transformations; that is, those that preserve the orientation of space and the direction of time, is generated by the rotations and the boosts in standard configuration. Indeed, any Lorentz transformation of this form can be written:

$$L = R\Lambda S$$

where R, S are orthogonal matrices with determinant one, and Λ is a boost in some arbitrary direction.

Lemma 1. *The group of proper, orthochronous Lorentz transformations is generated by the rotations and the boosts in the z-direction.*

Proof. An arbitrary proper, orthochronous Lorentz transformation, Λ, acts to preserve the direction of time, the orientation of space, and the metric $c^2t^2 - x^2 - y^2 - z^2 = 0$. Moreover, since Λ is invertible, then there is some four-vector (ct, x, y, z) which is sent to the four-vector $(c, 0, 0, 0)$, which represents the four-velocity of a stationary observer, O'.
Recall that a four-velocity of (ct, x, y, z) in some frame is better written as $(\gamma c, \gamma \mathbf{v})$, where $\mathbf{v} = (x, y, z)$. Hence O thinks that the four-velocity represents a second observer moving with velocity $\frac{1}{\gamma}(x, y, z)$ where γ^2 satisfies $\gamma^2(c^2 - x^2 - y^2 - z^2) = -c^2$.
Hence (ct, x, y, z) represents the four-velocity of an observer, O' moving relative to O. Now, up to a rotation, S, to align the direction of the boost

with the vertical axis in the frame of O, and up to a rotation at the end, R, representing the direction that O' is facing, the Lorentz transformation is simply a boost in the z-direction by velocity $v(0,0,1)$.
Hence, any proper, orthochronous Lorentz transformation is of the form $R\Lambda S$ for rotations R and S and a boost Λ in the z-direction. $\square$

It will thus suffice to provide Möbius transformations which correspond to rotations and boosts, which we now proceed to do.

Rotations: Any non-identity rotation of the sphere has precisely two fixed points, which are given by the points of intersection of the axis of rotation and the sphere, and is elliptic. We can specify the axis of rotation by a single point on the sphere, since the other point of intersection of the sphere and the axis is the antipodal point. So we can specify a rotation by the point, $[\xi, \eta]$, that the positive direcion of the axis intersects the sphere with, together with an angle, θ.
Given $[\xi, \eta]$ in normalised form (so that $\xi\bar{\xi} + \eta\bar{\eta} = 1$), the antipodal point is given by $[-\bar{\eta}, \xi]$. Then the rotation is given by:

$$\begin{pmatrix} \xi & -\bar{\eta} \\ \eta & \xi \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \begin{pmatrix} \xi & -\bar{\eta} \\ \eta & \xi \end{pmatrix}^{-1}$$

since this transformation has the required fixed points, and, with characteristic constant $k = e^{i\theta}$, is elliptic. Denote this Möbius transformation by $R(\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \theta)$.

Boosts: A boost with rapidity ϕ in the z-direction acts by

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \cosh(\phi)ct - \sinh(\phi)z \\ x \\ y \\ \sinh(\phi)ct - \cosh(\phi)z \end{pmatrix}$$

A boost with rapidity ϕ in the z-direction has the following effect:

$$\begin{pmatrix} -1 \\ x \\ y \\ z \end{pmatrix} \mapsto \frac{1}{\cosh(\phi) + z\sinh(\phi)} \begin{pmatrix} -\cosh(\phi) - \sinh(\phi)z \\ x \\ y \\ -\sinh(\phi) - \cosh(\phi)z \end{pmatrix}$$

We can now ask how this acts on the homogeneous coordinates $[\xi, \eta]$, associated to the direction $(x, y, z)^T$.

We note that $[\xi, \eta] = [x + iy, 1 - z]$, and this is sent to $\left[\frac{x+iy}{\cosh(\phi)+z\sinh(\phi)}, 1 - \frac{-\sinh(\phi)-\cosh(\phi)z}{\cosh(\phi)+z\sinh(\phi)}\right]$. On multiplying up by the denominator, and using $\cosh(\phi) - \sinh(\phi) = e^{-\phi}$ we see that the result is:

$$\left[e^{\phi}(x + iy), 1 - z\right]$$

Therefore the Möbius transformation corresponding to a Lorentz boost with rapidity ϕ in the z-direction has representing matrix:

$$\begin{pmatrix} e^{\phi/2} & 0 \\ 0 & e^{-\phi/2} \end{pmatrix}$$

that is, hyperbolic, with characteristic constant $k = e^{\phi}$.

4.4 Consequences

This remarkable connection has some profound consequences.

The first of which, is that the visual appearance of objects is not, as might be first thought upon reading about length contraction, as if they are contracted in length. Indeed, if it were the case that we would visually observe the length contraction of a relativistically moving object, then one might claim that a sphere moving relativistically with respect to an observer would appear elliptical. But the fact that the visual observations of two observers transforms via a Möbius transformation tells us that the circular outline of the sphere, as viewed by an observer at rest relative to it, will remain circular in the second observer's view. Thus a sphere will appear spherical to any observer, and it is clear that one will not visually observe length contraction.

Moreover, there are major computational advantages to working with Möbius transformations over working directly with Lorentz transformations. This is because 2x2 matrices are easy to handle, when compared to the 4x4 matrices corresponding to Lorentz transformations.

This connection is in fact a first step towards using spinors, which are fundamental to physics. An excellent reference for this is Penrose and Rindler [1987].

5 The Program

5.1 Raytracing Relativity

The main thing that we wished to bring out in the program was that circles are mapped to circles in the visual sphere. The standard perspective projection, as is found in most video games, lacks this property. Instead, stereographic projection is more desirable. For the accuracy that we desire, it also seems like raytracing is a good idea.

5.2 The Classical Case

We first deal with the case that the camera is at rest with respect to all objects in the scene.
Then the camera has a position in the scene, along with an orthornormal triple of directions: right, forward, up. We imagine that there is a sphere of sight centred around the camera, and we wish to draw this to the screen. The sphere of sight amounts to a sphere of some fixed radius which is painted in every direction by the colour that the camera would see in that direction. Our problem is therefore reduced to that of drawing this sphere to a flat screen. One way of doing this is to stereographically project from the back of the camera's view to a plane in front of the camera and perpendicular to the camera's forward direction. Note that the back of the camera's view corresponds to the point on the sphere which is in the negative forward direction from the camera's position.
We can now define our screen as some rectangle in this plane, aligned in the obvious way with the camera's right and up vectors.
Now we can raytrace, as follows.

For each pixel in the screen, find the point on the camera's sphere of sight which corresponds to this pixel. Then raytrace in the direction of that point on the sphere of sightas normal; that is, send out a ray, and see what it intersects first. We shall not be worried about light sources in this implementation.

To summarise the situation so far: given a function which will take in a scene and trace a ray in any given direction emanating from any given point, you can perform stereographic raytracing to the screen. This is conformal;

that is, angle-preserving.

5.3 The Relativistic Case

We now move on to the case where the camera is moving with respect to some object. We assume that the velocities and positions of all objects are stored relative to some inertial frame, which we call the scene.

The first thing to notice is that another observer, O' who is at the same instantaneous position as O, the camera, will see precisely the same photons, as was explained before. Moreover, we have already solved the case where the camera is at rest with respect to the object. It thus seems sensible to apply the particular boost to O which will take him into the object's rest frame. Then we can perform classical raytracing.

So this is what we do.

More explicitly, suppose that the object is moving with velocity $\mathbf{u}$ relative to the scene and the observer with velocity $\mathbf{v}$ relative to the scene. Suppose also we know that the ray we are tracing has direction $\mathbf{d}$ in the frame of the camera, and position $\mathbf{p}$ in the frame of the scene.

Let the Lorentz boost by velocity $\mathbf{w}$ be denoted $B(\mathbf{w})$. Then the direction of the ray that we are considering is transformed by $B(-\mathbf{v})B(\mathbf{u})$. As is the case with most relativistic calculations, working with four-vectors proves preferable to almost any other approach.
In this case, our ray is a past pointing light-ray, and so is null. Thus if it has direction given by the unit vector $\mathbf{d} = (d_1, d_2, d_3)$, then our four-vector is given by:

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

It should be noted that, much like classical ray-tracing involving moving objects, it absolutely does matter which direction the light ray is moving in time. We are interested in the direction that an observer in the rest frame of the object O' perceives our given light-ray to travel, and both are past-

pointing.
Also explicitly, our boost, $B(\mathbf{v})$ is given by:

$$\begin{pmatrix} ct' \\ \mathbf{r}' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\boldsymbol{\beta}^T \\ -\gamma\boldsymbol{\beta} & \mathbf{I} + (\gamma - 1)\boldsymbol{\beta}\boldsymbol{\beta}^T/\beta^2 \end{pmatrix} \begin{pmatrix} ct \\ \mathbf{r} \end{pmatrix}$$

where $\boldsymbol{\beta} = \mathbf{v}/c$ is simply the velocity in units where $c = 1$.

So our task is to transform this to a problem about a ray in the rest frame of the object, O'.

A note on specifying orientations

Classically, specifying the orientation of an observer is straightforward, and can be done by specifying their forward, right and up directions, which is an orthonormal triple of vectors; that is, three vectors which are mutually perpendicular and all of length one. Relativistically, this is not so simple, as two observers moving with different speeds will not agree on the matter of whether two vectors are perpendicular.
To get around this, we specify the camera's forward, right and up directions in scene coordinates. This way, for a camera moving with nonzero velocity, the vectors will no longer be perpendicular but, when viewed in a rest frame of the camera, will be.
Then to apply a rotation to the camera, it is sufficient to move to a rest frame of the camera, then apply the rotation, which makes sense in the camera's frame, before returning back to scene coordinates to store the rotated vectors.

With this in mind, we return to the problem of raytracing.

For simplicity, we may assume that the camera is situated at the origin in the scene, by simply translating all relevant spatial vectors by the camera's position.

We then know the position, $\mathbf{c}$, of the object in the scene at some fixed time in the scene.

Now, the camera claims to be looking in a direction $\mathbf{d}$, in his frame, so we first relate this to the scene's frame. This can be done by applying a boost by the camera's negated velocity to the null four-vector $\binom{-1}{\mathbf{d}}$. Given that we have stored the camera's orientation in the scene's frame, there is no more work to be done with orientation, as $\mathbf{d}$ was already specified in terms of the forward, right and up vectors in the camera's frame, which are known in the scene's frame.

Now we are working with a ray emanating from the origin in the frame of the scene, say $\binom{-1}{\mathbf{d}'}$, and a (possibly) moving object in the frame of the scene.

We want to move to a rest frame of this object. This can be done by applying the boost $B(\mathbf{u})$ to both the ray $\binom{-1}{\mathbf{d}'}$ and the object's position in the scene, $\binom{0}{\mathbf{c}}$. Note that it doesn't matter if we choose a position on the object at a different time, since we are transforming into a rest frame of the object, and so we will get the same spatial component regardless.

Our achievement is that we have now transformed the problem to that of raytracing in the direction $\mathbf{d}''$ from the origin with a static object at known position. So we have reduced relativistic raytracing to classical raytracing with a transformation of the directions of the rays.

The whole process, therefore, looks like this:

1. Apply the boost $B(\mathbf{u}) \circ B(-\mathbf{v})$ to the ray $\binom{-1}{\mathbf{d}}$, to get a ray $\binom{-1}{\mathbf{d}''}$.
2. Subtract the camera's position from all spatial vectors in the frame of the scene.
3. Apply the boost $B(\mathbf{u})$ to the four-vector $\binom{0}{\mathbf{c}}$ to get the position, $\mathbf{c}'$, of the object in a rest frame of the object.
4. Perform classical raytracing in direction $\mathbf{d}''$ from the origin, with the static object at position $\mathbf{c}'$.

6 Frequently Asked Questions

When first learning this subject, I had many questions in my mind, often rather easy to ask, but rather difficult to find an answer. Below is a selection of some of the more interesting ones, together with an attempt to answer them.

6.1 List of questions

1. Is there a simple way to visualise the correspondence between Möbius transformations and Lorentz transformations?
2. What does a Möbius transformation look like?
3. What does the outline of a sphere look like under a Lorentz transformation?
4. What does a rod look like under a Lorentz transformation?
5. Other objects?
6. How do colours transform under Lorentz transformations?
7. Will you actually see behind a sphere? Or just a rearrangement of the same coordinates?

Answers

1. Possibly the easiest way to see the correspondence is first to ignore orientation considerations, and just assume that two observers are orientated the same way and at rest at the same position. Then one observer instantaneously changes their velocity to, say $\mathbf{v}$, in the direction that they are looking. The appearance of their sphere of sight is changed by a hyperbolic Möbius transformation. In this simplified case, we are essentially looking just at relativistic aberration. If instead the observers are still looking in the same direction to start with, but their respective right and up vectors are rotated around their shared forward vector relative to one another (in the language of yaw, pitch and roll, this is roll), then the map between their sphere of sights will

be a loxodromic Möbius transformation. The last remaining case is parabolic. If we have a general rotation between their two orientations at rest, and with a boost in some given direction, then almost always the transformation will still be loxodromic, but if the velocity is chosen very carefully, then the two fixed points can be made to coincide, and we get a parabolic transformation. These are rare.

2. A Möbius transformation is characterised by its characteristic constant, k. A non-identity transformation either has two fixed points, when $k^2 \neq 4$, or has a single fixed point when $k^2 = 4$. In the case of two fixed points, then k controls both how much the transformation attracts towards one fixed point and repels away from the other, and also how much the transformation twists around each fixed point. In the case of a single fixed point, then the transformation is by stereographic projection to a plane, followed by translation of the plane, followed by inverse stereographic projection.

3. Möbius transformations on the sphere are conformal, which means that they preserve angles. A direct consequence of this is that circles are mapped to circles on the sphere by Möbius transformations. Since the outline of a stationary sphere on the sphere of sight is always a circle, then under a Möbius transformation, this circle is preserved. It follows that the outline of a relativistically moving sphere is also circular. Hence, *a relativistically moving sphere always appears spherical to an observer*, a fact that was first discovered by Roger Penrose in 1959.

4. It is commonly thought that the appearance of a relativistically moving object would be that it was shorter in the direction of its motion. When applied to a rod, one might assume that if the rod is moving in the direction of its length, then it will appear to be shorter due to the Lorentz contraction. In fact this is not the case, and due to the fact that the projection of a line segment onto a sphere, which is the appearance of a rod in the sphere of sight, is in fact a great circle, then a line segment will appear to be a segment of some circle (the image of a great circle under a Möbius transformation) to a relativistically moving observer.

5. Conformal maps preserve the shape of infinitesimally small objects, but do not in general preserve their size. Consequently, in very small

patches, the object will appear dilated.

6. The Doppler shift affects the frequency by the Doppler factor, $\frac{1+\beta}{1-\beta}$, if the movement is directly away from the observer. In particular, when β is positive, then the observed frequency is higher - so the object is blue shifted, and when β is negative then the observed frequency is lower - so the object is red shifted.

7. An observer at rest relative to an object, situated at your position will see as in the classical case. You, moving relativistically, but instantaneously situated at that position, will see a rearrangement of their sphere of sight by a Möbius transformation. The light that you see has taken some finite time to reach you, and in that sense, you are currently seeing light which from your measurement of the sphere's position at the time that the light reaches you, could have come from obstructed parts of the sphere. But this is just as much a conflation of the terms measurement and visual observation.

7 Appendix

7.1 A Derivation of Lorentz Transformations in Two Dimensions

We first assume that we are looking for some linear transformation to relate the two coordinate systems:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

This makes sense, because space is assumed to be homogeneous; that is, to have a uniform structure throughout. Only a linear transformation will preserve this uniformity.
Now, in the case of an object at rest in the frame of O', say with $x' = 0$, and passing through the point $(0, 0)$ in the frame of O, we must have $x = vt$. Thus by the equation for x' we have $D = -Cv$.
Hence $x' = C(x - vt)$. We now argue by the assumption that there is no preferred frame of reference, that the same argument holds with O in place of O', and thus merely by replacing v with $-v$ we get: $x = C(x' + vt')$.

Under the assumption that the speed of light is constant, it must be that we have: $x = ct$ and $x' = ct'$ for any photon observed by O and O'. Simply substituting these into our equations for x and x' give:

$$x' = C(x - vx/c) \tag{1}$$

$$x = C(x' + vx'/c) \tag{2}$$

and we can multiply these together to get

$$xx' = C^2(1 - v^2/c^2)xx'$$

In general we may divide both sides by xx' and rearrange to find:

$$C = \frac{1}{\sqrt{1 - v^2/c^2}}$$

This is known as the *gamma factor*, $\gamma := \frac{1}{\sqrt{1-v^2/c^2}}$.

It is now straightforward to substitute $x = ct$, $x' = ct'$ into our equations for x, x' in order to get:

$$ct = \gamma(ct' + xv/c) \tag{3}$$
$$ct' = \gamma(ct - xv/c) \tag{4}$$

and thus our full equation is:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v/c \\ -\gamma v/c & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

where it should be noted that we have written ct instead of t for simplicity.

References

R. Penrose and W. Rindler. *Spinors and Space-Time: Volume 1, Two-Spinor Calculus and Relativistic Fields*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1987. ISBN 9780521337076. URL `http://books.google.co.uk/books?id=CzhhKkf1xJUC`.

N.M.J. Woodhouse. *Special Relativity*. Springer, 2003.