Firm Scope and Innovation: The Role of Intangibles*

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Abstract

Horizontal expansion through an increasing product portfolio lies at the core of modern endogenous growth literature. Yet evidence remains limited on how diversification across industries influences a firm's trade-off between generating social surplus and capturing private returns. To investigate this, I categorize intangible assets by their spillovers: transferable intangibles (patents, software) generate social surplus, whereas embedded intangibles (organizational capital, brand value) primarily yield private returns. I document that diversified firms reallocate investment toward embedded intangibles, a strategic shift accompanied by declining markups and productivity, together with reduced innovation by their rivals. Motivated by this evidence, I extend a canonical endogenous-growth framework to endogenize firms' allocation between transferable and embedded intangibles, allowing for both horizontal and vertical expansion. A key prediction of the model is that embedded intangibles are the primary driver of a firm's ability to expand across industries, which also raises entry barriers for competitors and decreases social return rather than promoting long-run growth. Thus, a shift in innovative effort ultimately sacrifices economy-wide growth for firm-level market advantages, and quantitative analysis indicates that size-dependent taxes can substantially improve welfare.

Keywords: Schumpeterian growth, step-by-step innovation, intangibles, firm dynamics, span of control.

JEL Classification: E22, O31, O32, O33, O34.

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1 Introduction

Horizontal and vertical innovation are two essential strategies firms use to expand their size (Klette and Kortum, 2004; Aghion, Harris, Howitt, and Vickers, 2001). The literature mainly assumes that horizontal expansion does not negatively affect the profitability of a firm's existing operations.[1] Consequently, there has been limited attention to how diversification across industries influences the allocation of investment in intangible assets and the associated trade-offs between social and private returns.

Expanding a firm's business across segments[2] requires organizational divisions, which naturally influence managerial attention and innovation strategies. For instance, Colgate-Palmolive operates with a narrow focus on Personal & Home Care and Pet Nutrition, whereas Procter-Gamble manages a wide array of segments, including Beauty, Grooming, Health Care, Fabric & Home Care, and Feminine & Family Care. This divergence raises two critical questions: (i) How does diversification shape a firm's efficiency, market power, and allocation of intangible investment? (ii) What are the corresponding implications for social welfare?

To address these questions, I develop a unified framework to analyze how firm diversification shapes innovation incentives and social welfare. First, I document that firm productivity, markups, and the ratio of R&D to firm-specific intangible investment (net SG&A)[3] vary systematically with the number of firm segments. Guided by this empirical finding, I construct an endogenous-growth model in which firms' expansion decisions determine the strategic allocation of innovative effort.

[1]There are two common approaches to modeling horizontal expansion: one assumes diminishing returns to scale, while the other assumes constant returns to scale and relies on Gibrat's Law, which states that innovation is independent of firm size.

[2]Firm segments, scope, and production lines are used interchangeably to indicate how broadly diversified a firm is across industries. This diversification depends on how the firm defines its own business. It may involve closely related industries or more widely diversified operations.

[3]SG&A also includes R&D expenditures; net SG&A is calculated by subtracting R&D, leaving only expenses for employee compensation, advertising, and other operational costs of the firm. See Section 2.2 for details.

In the model, firms offset the profitability costs of diversification by shifting resources away from R&D and toward firm-specific intangible investments. This reallocation raises entry barriers and tilts innovation incentives toward private returns, at the expense of the broader social gains typically associated with R&D.

To formalize this mechanism, I classify intangible assets according to their transferability between firms. In this framework, transferable intangibles include patents and software, with the associated R&D investments emphasizing their non-rivalrous nature and limited excludability (Romer, 1990; Aghion and Howitt, 1992; Grossman and Helpman, 1991). These investments can be transferred between firms and generate spillover effects, with each successful R&D project building on previous product improvements. Their benefits persist even if the firm exits the market. In contrast, embedded intangibles like brand value and organizational capital are inherently firm-specific and inseparable from the firm that created them. They primarily provide a firm-specific comparative advantage and do not generate spillovers. Consequently, when a firm exits the market, the economic value of embedded intangibles becomes a sunk cost[4].

I further subdivide embedded intangibles into two categories based on their effects on demand and supply (Table 1). Brand value [5] acts as a demand shifter, positively influencing the perceived quality of a firm's output (Cavenaile and Roldan-Blanco, 2021; Cavenaile, Celik, Roldan-Blanco, and Tian, 2025). Evidence also suggests that brand value provides target marketing by increasing consumer awareness, thereby incentivizing substantial firm investment in advertising (Cavenaile, Celik, Perla, and Roldan-Blanco, 2025; Baslandze, Greenwood, Marto, and Moreira, 2023). On the supply side, I conceptualize organizational capital as managerial productivity, including the firm's embodied managerial talent and its contri-

[4]Transferable intangibles investment corresponds directly to R&D investments. By contrast, embedded intangibles include advertising and organizational expenditures, which accumulate as brand value and organizational capital.

[5]Alternatively, Pearce and Wu (2025) suggests that brand value is transferable between firms. However, in the framework of this project, brand value is considered non-transferable, as its only channel of transfer—through mergers and acquisitions (M&A)—lies outside the scope of this paper.

bution to future production profitability (Carlin, Chowdhry, and Garmaise, 2012; Eisfeldt and Papanikolaou, 2013; Prescott and Visscher, 1980)

Table 1. Taxonomy of Intangibles

	Supply Side	**Demand Side**
Embedded	Organizational Capital	Brand Value
Transferable	Software, Patent	

I merge Compustat Fundamentals with Compustat Segment data and document a key empirical finding: as firms diversify across segments, their markups, productivity, growth rates, and the ratio of transferable to embedded intangible investment all decline. To quantify competitive pressures, I use the product market fluidity dataset from Hoberg, Phillips, and Prabhala (2014), which shows that more diversified firms face lower competitive threats and operate in less fluid sectors. I further refine the analysis using granular, within-industry product-scope measures from Hoberg and Phillips (2025) to disentangle sector-level from within-sector expansion effects on firm dynamics. Moreover, I use the dataset from Kogan, Papanikolaou, Seru, and Stoffman (2017) to obtain forward citations and value per patent, which serve as proxies for the social benefits and private value of innovation.

Building on the empirical evidence, I extend the canonical endogenous growth framework along two dimensions. The first dimension introduces vertical and horizontal firm growth, with span-of-control frictions[6] (Lucas, 1978) arising from horizontal expansion. The second dimension endogenizes firms' choices between transferable and embedded intangible investments. The economy consists of a final-good sector and a continuum of intermediate-good sectors, each featuring a single superstar firm alongside a continuum of fringe firms. Superstar firms

[6]See also Jovanovic (2025). Alternatively, Acemoglu, Akcigit, Alp, Bloom, and Kerr (2018) use skilled labor in operational activities as a fixed cost, creating a trade-off between its allocation to R&D and operations. While an increase in firm scale raises operational demands, this does not directly reduce firms' efficiency or pricing power with size.

can invest either in embedded intangibles, which enhance managerial productivity and the perceived quality of their products through brand value and organizational capital, or in transferable intangibles, which improve product quality in two ways: (i) by upgrading existing product lines, and (ii) by enhancing new product lines. Importantly, internal quality improvements do not create span-of-control constraints, whereas expansion into new product lines does. Moreover, brand value and organizational capital transfer freely across a firm's existing lines, while quality improvement requires separate investments for each line. This asymmetry generates increasing returns to scale for embedded investments as firms expand their product portfolios.

Fringe firms cannot invest in embedded intangibles; their only path to becoming a superstar is through radical innovation. When a superstar exits the market, its transferable intangibles in that product line—existing quality—become freely available to fringe firms. Under oligopolistic Bertrand competition, a superstar's markup in each production line is determined endogenously by the levels of transferable and embedded intangibles, as well as by the number of product lines it operates. Moreover, superstars from other industries can enter a sector by improving the quality of a production line and displacing the incumbent. Such entry, however, is only feasible if their brand value and organizational capital are at least as high as the incumbent superstar's. Consequently, substantial investment in embedded intangibles allows a superstar to strengthen its market position and reduce market fluidity.

The model predicts that horizontal expansion decreases managerial productivity and competitiveness due to span-of-control constraints. To offset these inefficiencies, multiproduct firms exploit cross-product synergies and reallocate investment toward embedded intangibles, which deliver increasing returns to scale but simultaneously reduce market fluidity. This strategic shift extends the life cycle of superstar firms in existing markets but generates three adverse aggregate consequences: (i) reduced markups and productivity for multiproduct firms as operational fragmentation intensifies; (ii) a contraction in the innovation possibilities for new entrants; and (iii) depressed long-run quality improvements due to lower investment in transferable intangibles.

I discipline the key parameters of the model using the simulated method of moments (SMM). Based on the calibrated model, I examine how the relative shares of brand value (demand-side) and organizational capital (supply-side) shape firm dynamics. An increase in the share of brand value relative to organizational capital induces firms to concentrate production within a single line, as brand value alone cannot offset the managerial frictions associated with expansion. Consequently, markups decline: the reduction in organizational capital lowers managerial productivity and outweighs the demand-side gains from brand value. This shift also raises entry barriers for potential entrants, thereby reducing market fluidity and aggregate growth. In contrast, when organizational capital constitutes a larger share, firms are more likely to expand, as organizational capital directly enhances managerial efficiency. This fosters higher markups, increases market fluidity, and economic growth.

Next, I run two counterfactuals: I remove the span-of-control constraint and eliminate embedded intangibles to isolate their effects on markups, firm size, market fluidity, and aggregate growth. Removing the span-of-control constraint allows firms to operate more product lines, raises average markups, increases investment in embedded intangibles (thereby reducing market fluidity), and raises aggregate growth. Eliminating embedded intangibles concentrates production on a single line, lowers markups modestly (0.5–1.5%), and raises aggregate growth via a large increase in creative-destruction-driven fluidity.

In the last part of the quantitative analysis, I examine how misallocation operates through two channels: (i) markup dispersion and (ii) entry barriers created by embedded intangibles. Pure markup dispersion accounts for only 0.5% of the output loss. In contrast, removing entry barriers more than triples aggregate output; this is driven primarily by substantial quality-improvement gains, during which the contribution of embedded capital falls slightly. Motivated by these results, I evaluate three tax experiments: a size-dependent profit tax (10% → 12.5%), a flat 11.3% tax on embedded and expansion investment, and a joint application of all three taxes. In consumption-equivalent welfare, the size tax delivers the largest gain (+10.915%); the embedded-investment taxes yield modest gains (+1.745% and +2.176%); and the joint policy produces the most significant complementary benefit (+16.237%).

Related Literature. First, this paper contributes to the growing literature on intangibles and their effects on firm dynamics, which suggests that intangibles increase market concentration, markups, and reduce investment in tangible capital (Chiavari and Goraya, 2025; Crouzet and Eberly, 2019; Weiss, 2020). Building on this, De Ridder (2024) conceptualizes software intangibles as firm-specific fixed costs and shows how incumbents' strategic investments lower marginal production costs, creating asymmetric barriers to innovation that favor incumbents. Similarly, Aghion, Bergeaud, Boppart, Klenow, and Li (2023) distinguishes product and process (firm-specific) innovation and highlights the roles of information and communication technologies in driving concentration. In related work, Cavenaile and Roldan-Blanco (2021) and Cavenaile, Celik, Roldan-Blanco, and Tian (2025) show that advertising can substitute for R&D and dampen innovation intensity, while Pearce and Wu (2025) examines brand-value transfer between firms in the context of market concentration. My paper contributes to the literature by proposing a unified, generalizable taxonomy of intangibles and by modeling their accumulation as an endogenous outcome of firm optimization, rather than treating them solely as expenses or fixed costs. This framework provides the microfoundations for firms' investment and accumulation decisions across different types of intangibles, thereby producing novel insights into how those choices shape markups, firm size, market fluidity, and the nature of firm productivity.

Second, this paper contributes to the literature on horizontal and vertical innovation. Akcigit and Kerr (2018) develop an endogenous growth model in which incumbents engage in internal and external innovation with heterogeneous returns. Garcia-Macia, Hsieh, and Klenow (2019) show that most innovation originates from incumbents improving existing products. In contrast, Berlingieri, De Ridder, Lashkari, and Rigo (2025) document that firms often expand through sequential product diversification rather than by improving existing products. My framework synthesizes and extends these insights by focusing on sector-level horizontal expansion that increases firm scope. This type of expansion is distinct from within similar product lines and highlights its direct implications for the span-of-control constraint. My model thus identifies an additional cost of diversification: beyond diminishing returns, it reduces productivity and markups by straining managerial capacity. This mechanism highlights why strategic innovation portfolios are essential for sustaining firms over the life cycle.

Third, this paper offers a complementary explanation for several documented trends: reduced knowledge spillovers (Akcigit and Ates, 2021; Akcigit and Ates, 2023), declining patent quality (Olmstead-Rumsey, 2019), production lock-in (Casal, 2024), and strategic patenting (Jo and Kim, 2024). The mechanism centers on the strategic reallocation of investment from transferable to embedded intangibles by multiproduct firms. This reallocation depresses spillovers due to the firm-specific nature of embedded intangibles. The shift subsequently replaces economy-wide product quality improvements with firm-specific productivity gains, which in turn reduces market fluidity. This illustrates a mechanism through which firms exercise broader control over their competitive environment, ultimately limiting rivals' innovation potential and the diffusion of knowledge.

Fourth, this paper contributes to the literature on resource misallocation (Hsieh and Klenow, 2009; Restuccia and Rogerson, 2008) by identifying two distinct channels. First, misallocation can arise from markup dispersion (Peters, 2020; Edmond, Midrigan, and Xu, 2023), driven by the accumulation of both transferable and embedded intangibles. Second, embedded intangibles create entry barriers, which amplify distortions and reduce allocative efficiency.

Fifth, the empirical and theoretical literature on the span of control constraint has primarily focused on hierarchical organization, knowledge flow frictions, managerial ability, and associated premiums (Smeets, Waldman, and Warzynski, 2019; Bandiera, Prat, Sadun, and Wulf, 2014; Garicano, 2000; Bloom and Van Reenen, 2007). This paper extends the literature by adopting a macro perspective, examining how span of control constraints shape firm dynamics and growth, and highlighting their broader implications for innovation.

Outline. The remainder of the paper is organized as follows: Section 2 presents the datasets and empirical facts; Section 3 introduces the theoretical model and characterizes its equilibrium; Section 4 discusses the calibration; Section 5 examines counterfactual analysis; Section 6 analyzes misallocation and policy implications; and Section 8 concludes.

2 Datasets and Empirical Facts

In this section, I first describe the data sources and measurement details, and then present empirical evidence on how the investment ratio, productivity, markups, and market fluidity vary with firm multiproductness.

2.1 Data Description

Compustat Fundementals and Segment. Compustat Fundamentals provides comprehensive firm-level financial information for publicly listed companies in North America[7], with extensive longitudinal coverage. It includes detailed balance sheet items, income statement components, cash flow data, and key financial ratios. An additional advantage is that it enables merging with external datasets through a unique firm identifier.

Compustat also offers two distinct segment datasets: (i) the Historical Segment

[7]Foreign firms such as Toyota and Unilever are included in Compustat North America due to their U.S. listings via American Depositary Receipts (ADRs). Although they adhere to home-country governance, these firms comply with SEC reporting requirements. Excluding them does not affect the paper's main conclusions but reduces the sample size by roughly 25%. For this reason, they are retained in the main analysis. See Figure A2

dataset, which contains buyer–supplier relationships and firm segmentation with long-term coverage[8], and (ii) the Compustat Segment dataset, introduced in 2016, which provides more detailed segment-level information but with limited historical depth. [9] To maximize both coverage and detail, I combine these two segment datasets.[10] To estimate markups, productivity, and the investment ratio across production lines, I merge the Compustat Fundamentals and Segment datasets.[11]

Fludity and Firm Scope Dataset. The product market fluidity metric from Hoberg, Phillips, and Prabhala (2014)(HPP) measures the rate at which firms in similar markets change their product or service offerings annually. It is calculated using natural language processing (NLP) on the product descriptions from firms' annual 10-K reports filed with the US Securities and Exchange Commission. This method tracks year-over-year changes in how companies describe their business. A high fluidity score indicates that competitors are adapting rapidly by launching new products, shifting strategies, and entering new markets. Consequently, firms in high-fluidity markets face heightened competitive threats from rivals reconfiguring their offerings and positions.

Hoberg and Phillips (2025)(HP) construct their firm scope dataset using a similar text-based methodology as their fluidity metric. By applying NLP to product descriptions in firms' annual 10-K reports, they calculate pairwise similarity scores between all public firms. This methodology allows them to identify a firm's num-

[8]Under Regulation SFAS No. 131—codified as ASC 280 after 2009—U.S. public firms are required to disclose the identity of aany one customer that accounts for more than 10% of its total revenue, along with the nature of the products or services provided to that customer. These mandated disclosures constitute the foundation of the Historical Segment datasets.

[9]In this project, for the Compustat Historical Segment dataset, geographic and operational segments are excluded; only business segments are retained. For the Compustat Segment dataset, only non-missing entries from the Product–Service (`PD-SRVC`) category are included.

[10]When segment information for a firm is available in the Compustat Segment dataset, I prioritize that source. Otherwise, I use data from the Historical Segment dataset.

[11]The merged Compustat dataset contains fewer firms than the Fundamentals database because segment information is unavailable for some firms. In addition, I restrict the sample to firms with positive R&D and SG&A expenditures. This cleaning and merging process does not affect the representativeness of the merged dataset relative to the full Compustat Fundamentals sample; see Figure A5.

ber of distinct product markets based on the uniqueness of its product descriptions relative to others. The key advantage of this dataset lies in its granular, text-based measurement of firm scope, which offers a more nuanced and dynamic alternative to static industrial classification codes (NAICS)[12]

Forward Citation and Patent Value Dataset. The Kogan, Papanikolaou, Seru, and Stoffman (2017) dataset contains patent-level information, including patent ID, filing and issue dates, firm identifiers, forward citations, and patent value. Forward citations are calculated as the total number of subsequent citations each patent receives, including citations from the patent-owning firm. The private value of each patent is estimated using stock market reactions around the patent grant date, which captures investors' expectations regarding future profits.

2.2 Measurement

Productivity and Markup Estimation. I estimate firm-level total factor productivity using the approach developed by Gandhi, Navarro, and Rivers (2020)[13]. They propose a nonparametric identification strategy that uses a transformation of the first-order condition for intermediate inputs to isolate flexible-input effects and identify the production function and input elasticities without relying solely on proxy inversion. To estimate firm-level markups, I follow the methodology of De Loecker, Eeckhout, and Unger (2020) and define markups as the ratio of sales to the cost of goods sold (cogs) multiplied by the output elasticity of the variable input, which I obtain from the first-stage production function estimation using Levinsohn and Petrin, 2003.

Investment Ratio. Following Peters and Taylor (2017), I measure two categories of intangible investment. I treat total R&D expenditures as investment in transfer-

[12]For a comparative illustration of firm segment classification between the Compustat Segment and HP Firm Scope datasets, see Tables A1 and A2.

[13]The productivity estimation results are robust to alternative production function estimation methods, including those proposed by Ackerberg, Caves, and Frazer (2015) and Levinsohn and Petrin (2003), see Figure A4. For methodological details, see Appendix A.1

able intangibles.[14] On the other hand, 30% of Selling, General, and Administrative (SG&A) expenses, net of R&D, is treated as investment in embedded intangibles. Net SG&A primarily includes employee compensation, advertising, and other expenditures necessary to sustain firm operations. Only a fraction of these expenses are considered investments, as the remainder reflects routine operating costs.

2.3 Empirical Facts

Figure 1(A) and Figure 2(A) document an inverse relationship between a firm's number of production lines and its productivity, productivity growth, and markups[15]. This pattern is consistent with a span of control constraint: as firms expand their scope, managerial attention is weakened per line, reducing efficiency and pricing power.

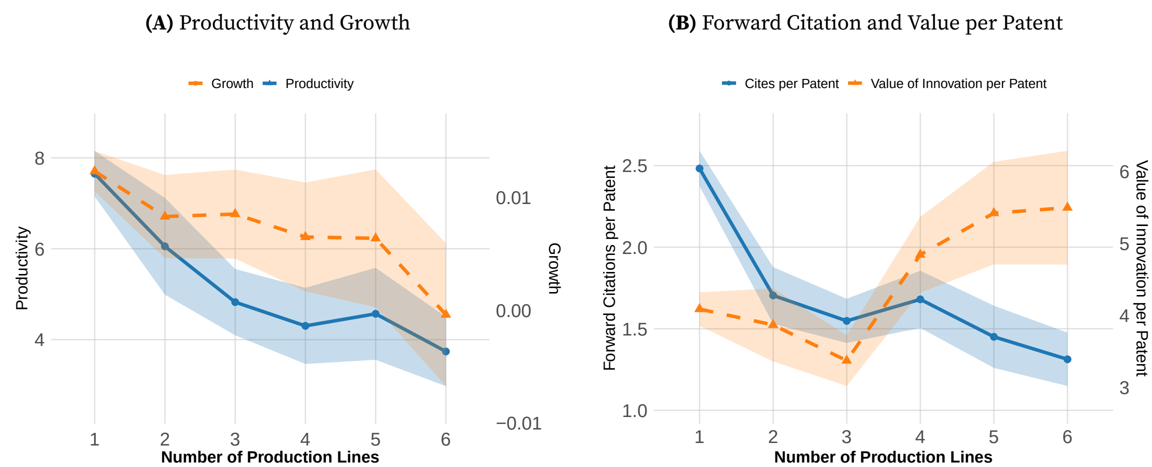

Figure 1. Productivity, Growth & Forward Citation / Value per Patent by Production Line

Note: The sample excludes firms in the utilities and finance sectors, and those with missing or non-positive R&D or SG&A. For forward citations and patent values, observations with missing or non-positive patent values are also excluded. The growth rate is defined as the two-year log change in productivity, $\Delta_2 \ln(\text{prod})_{i,t} = \ln(\text{prod}_{i,t}) - \ln(\text{prod}_{i,t-2})$, averaged over 2005–2019 and winsorized at the 10th and 90th percentiles. Log productivity is measured from the 2019 cross-section and winsorized at the 95th percentile. Forward citations and value per patent are averaged over 2015–2019 and winsorized at the 5th and 95th percentiles.

[14] Peters and Taylor (2017) refer to this as knowledge capital, which is termed transferable intangibles in this paper.

[15] This finding aligns with the pattern in Autor, Dorn, Katz, Patterson, and Van Reenen (2020), where higher-productivity firms charge higher markups.

Figure 1(B) suggest that, on average, forward citations per patent decline as the number of production lines increases. This pattern implies a reduction in the social value of patents as firms diversify. In contrast, the private value of innovation per patent appears to rise with additional production lines, moving in the opposite direction to forward citations. The growing divergence between these two measures highlights a misalignment between the private surplus captured by firms and the social value generated by their patent output. Further, Figure 2(B) indicates that the composition of intangible investments is non-uniform and varies systematically with firm scope. Firms with fewer lines tend to prioritize transferable intangibles, whereas firms that expand into multiple products increasingly allocate investment toward embedded intangibles to reinforce competitive advantages in existing lines. Moreover, Figure 2(C) shows that markets dominated by multiproduct firms are less fluid than those dominated by firms with fewer products.[16] A strong negative correlation between market fluidity and the investment ratio suggests that larger embedded intangible investments strengthen incumbency, making market entry and displacement more difficult for rivals. These relationships are formally confirmed by the regression estimates in appendix Table A4, which control for two-way fixed effects. The results show a significant negative relationship between the number of production lines and both productivity and markups, with even stronger negative coefficients for investment ratios and market fluidity. Crucially, appendix Figure A3 demonstrates that these patterns are not explained solely by firm size or age, underscoring their unique link to a firm's expansion strategy.

To investigate why firm scope affects margins, I compute firm variables using the HP firm scope dataset. In the Appendix, Figure A1 and Table A5 show that these measures increase with firm scope, in contrast to the patterns observed using Compustat Segment data. This stark divergence suggests that the effects of diversification are not uniform, but rather critically depend on the type of expan-

[16] See Table A3 for summary statistics.

sion: scope increases margins when it involves entering closely related markets, but it undermines them when it forces expansion into a distant sector.

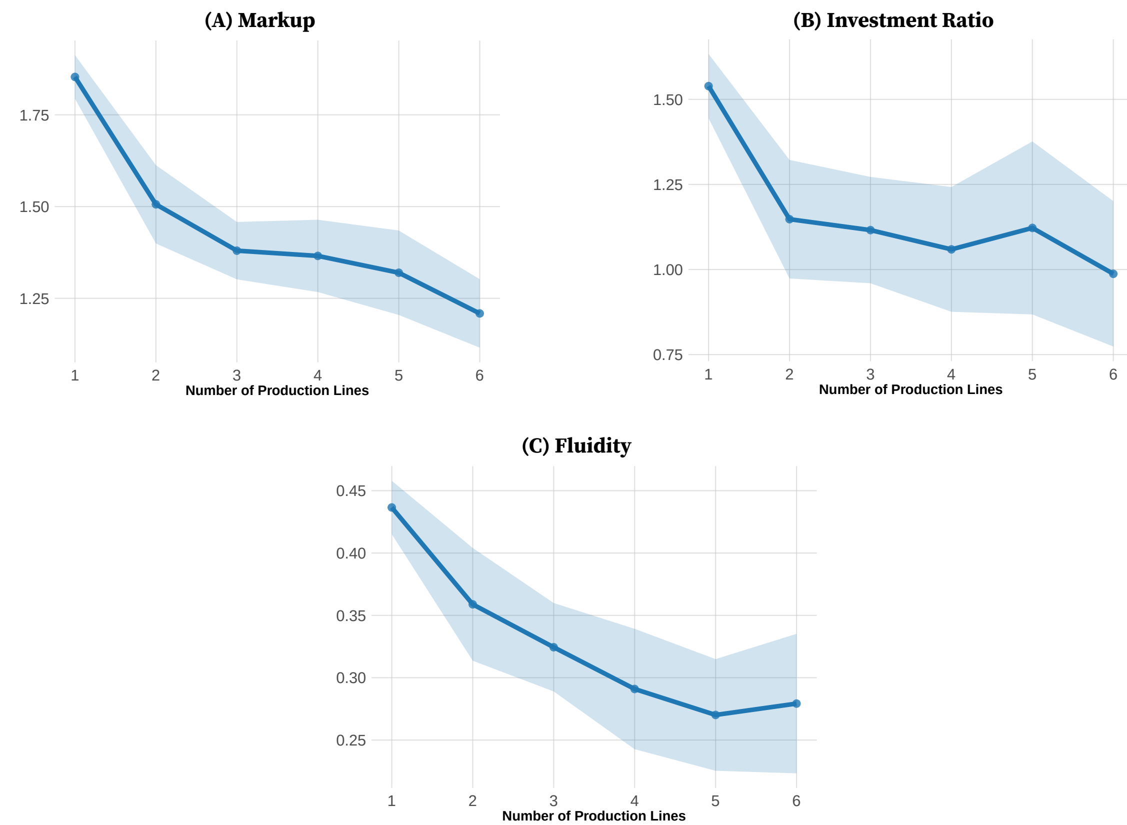

Figure 2. Markup, Investment Ratio, and Fluidity with Production Lines

Note: The sample excludes utilities and finance sectors, and firms with missing or non-positive R&D and SG&A. Markups, the investment ratio, and market fluidity are measured for the 2019 cross-section. The investment ratio is transferable over embedded investment, which is described in Section 2.2. The investment ratio is winsorized at the 95th percentile, the markup at the 90th percentile, and labor market fluidity at the top and bottom 5th percentiles. For calibration purposes, fluidity is normalized using min-max scaling. Each value x was transformed according to $x_{\text{scaled}} = (x - \min(x))/(\max(x) - \min(x))$, mapping all values linearly into the range $[0, 1]$.

3 Theoretical Model

This section develops an endogenous growth model to characterize the equilibrium relationship between innovation, intangible heterogeneity, and firm scope. Unifying the vertical innovation framework of Aghion, Harris, Howitt, and Vickers (2001) with the horizontal expansion mechanism of Klette and Kortum (2004), the model introduces two key elements: heterogeneity in intangible investment and a span of control constraint. These features jointly determine firms' investment decisions, markups, and output based on their competitive positions and the composition of their intangibles.

3.1 Economic Environment

Preferences. In this economy, continuous time is represented by t, and household preferences are described by a logarithmic utility function:

$$\int_0^\infty e^{-\rho t} \ln(C_t)\, dt, \tag{1}$$

where C_t represents household consumption, and $\rho > 0$ denotes the time discount rate. The budget constraint is expressed as

$$\dot{A}_t = r_t\, A_t + w_t - C_t. \tag{2}$$

The term A_t represents the total assets in the economy at time t, and the labor supply is normalized to 1. I normalize the price of the consumption good; therefore, w_t and r_t show the relative prices of wage and the interest rate, respectively. Because households own firms, total assets in the economy can be expressed as the sum of the firm values

$$A_t = \int_0^1 \left(V_{sjt} + V_{fjt}\right) dj,$$

where V_{sjt} and V_{fjt} denote the values of superstar and fringe firms in the intermediate good sector j at time t.

Final Good Technology and Market Structure. The final good sector used for consumption is produced according to the following technology:

$$\ln(Y_t) = \int_0^1 \ln(y_{jt})\, dj. \tag{3}$$

It is produced using a continuum of intermediate varieties $j \in [0, 1]$ in a perfectly competitive market. In each intermediate goods sector, one superstar firm, y_{sjt}, and a continuum of homogeneous small firms, y_{fjt}, compete à la Bertrand to supply the final good producer. Their output is aggregated by a constant elasticity of substitution:

$$y_{jt} = \left(\chi(e_{st})\, y_{sjt}^{\varepsilon} + y_{fjt}^{\varepsilon}\right)^{\frac{1}{\varepsilon}}, \tag{4}$$

where $\varepsilon \in (0, 1)$. In each production line j, a superstar firm s may own multiple lines. It is characterized by the countable set of lines for which it owns the leading technology, $J_s \subseteq [0, 1]$. The number of leading product lines owned by superstar firm s is given by $n_s = |J_s| \in \mathbb{Z}_+$.

Because the superstar firm has a differentiated product, the term $\chi(\xi e_{st})$ is an endogenous and concave demand shifter, defined as $\chi(\xi e_{st}) = (\xi e_{st})^{\beta}$. Here, e_{st} represents firm s's embedded intangibles, while ξe_{st} denotes the portion of embedded intangibles associated with brand value, with $\xi \in (0, 1)$. The parameter $\beta \in (0, 1)$ captures the curvature of the demand shifter. If the relative brand value of a superstar firm increases, the perceived benefit (quality) of its product in the final good sector will be higher than that of fringe firms. For simplicity, the embedded intangible level and brand value of fringe firms are normalized to one. Finally, fringe firms are homogeneous within each intermediate sector and can be represented as:

$$y_{fjt} = \int_0^1 y_{ijt}\, di, \tag{5}$$

where each fringe firm $i \in (0, 1)$.

Superstar Firm Production. The production function for superstar firm s in line j at time t is given by

$$y_{sjt} \;=\; q_{sjt} \cdot \psi(e_{st}, n_{st}) \cdot l_{sjt}, \tag{6}$$

$$\text{with} \quad \psi(e_{st}, n_{st}) \;=\; \frac{\left((1-\xi)\, e_{st}\right)^{\alpha}}{\gamma\, n_{st}^{\alpha_s}}.$$

The term $q_{sjt} \cdot \psi(\cdot)$ represents the total productivity of firm s in production line j, where q_{sjt} denotes product quality and $\psi(\cdot)$ captures the firm's managerial productivity. The input l_{sjt} is the quantity of labor employed by superstar firm s in line j.

The component $(1-\xi)e_{st}$ represents the fraction $(1-\xi)$ of embedded intangibles interpreted as organizational capital, which improves managerial efficiency. The variable n_s denotes the number of product lines owned by firm s. As n_s increases, managerial productivity per line declines due to the span of control—expansion reduces the firm's ability to effectively oversee each individual line[17]. Furthermore, the curvature of organizational capital and the span of control constraint are governed by α and α_s, respectively, while the value of γ determines whether it represents a cost ($\gamma > 1$) or a benefit ($0 < \gamma < 1$) scale for managerial quality.

Fringe Firm Production. Fringe firms produce output according to a linear technology:

$$y_{fjt} = q_{fjt} \cdot l_{fjt}, \tag{7}$$

where q_{fjt} denotes productivity and l_{fjt} is labor input. Unlike superstar firms, a fringe firm operates in only one sector and has a managerial quality normalized to 1. Its productivity is inherited: when a superstar firm exits, its transferable intangible assets, such as patents, become publicly available, allowing a fringe firm to adopt the previous leader's productivity level. Finally, as fringe firms produce a homogeneous good, they are price takers.

Investment Functions and Innovation. A key assumption of the model is that quality improvements in each production line require separate investments, whereas embedded intangibles are freely mobile across a firm's production lines. Thus, a successful investment in embedded intangibles simultaneously improves the brand value and the organizational capital of all production lines. Under this framework, superstar firms face three investment decisions: they can expand their portfolio with new production lines, improve the quality of existing lines through transferable intangibles, or improve brand value and organizational capital of all lines via embedded intangible investments. These investment scenarios are illustrated in Figure 3.

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[17]The span of control constraint imposes a natural upper bound $\bar{n}$ on horizontal firm expansion.

The variables $I^{\text{Emb}}_{s,j,t}$, $I^{\text{Int}}_{s,j,t}$, and $I^{\text{Ex}}_{s,j,t}$ represent superstar s's investment in embedded intangibles, internal transferable investment on its own production lines, and external transferable investment on other production lines, respectively. If a firm is a leader in at least one production line, it can conduct an external innovation. Each unit of investment generates a successful flow rate of innovation on internal $z^{\text{Int}}_{s,j,t}$, external $z^{\text{Ex}}_{s,j,t}$, or on the embedded intangible level $z^{\text{Emb}}_{s,j,t}$, respectively. Investments occur with convex costs and represented by

$$I^{\text{Int}}_{sjt} = \gamma^{Int}\left(z^{Int}_{sjt}\right)^{\vartheta^{Int}} Y_t \quad , \quad I^{\text{Ex}}_{sjt} = \gamma^{Ex}\left(z^{Ex}_{sjt}\right)^{\vartheta^{Ex}} Y_t \tag{8}$$

$$\text{and} \quad I^{\text{Emb}}_{sjt} = \gamma^{Emb}\left(z^{Emb}_{sjt}\right)^{\vartheta^{Emb}} Y_t.$$

In the above expressions, the investment cost function scales with the size of the economy, Y_t. The parameters γ^{Int}, γ^{Ex}, and γ^{Emb} determine the cost scale of the investment functions, whereas ϑ^{Int}, ϑ^{Emb}, and ϑ^{Ex} determine their curvature, respectively. Moreover, the total investment in transferable intangibles by a superstar firm in production line j at time t is equal to

$$I^{T}_{sjt} = I^{\text{Int}}_{sjt} + I^{\text{Ex}}_{sjt}. \tag{9}$$

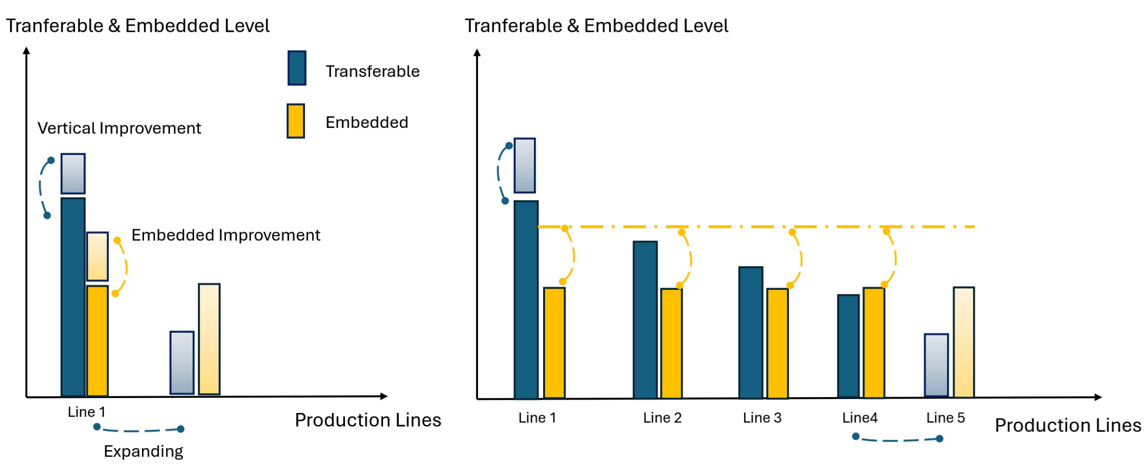


Figure 3. Firm Investment and Innovation Types

Fringe firms invest only in transferable intangibles within their production line, aiming to achieve drastic innovations that could elevate them to become a new superstar. Fringe firms invest in transferable intangibles, followed by

$$I_{sjt}^{\text{Fri}} = \gamma^f \left(z_{sjt}^{Fri}\right)^{\vartheta^f} Y_t. \tag{10}$$

The term γ^f represents the fringe firms' cost of scale, and ϑ^f is the curvature of their investment.

On each production line, firms invest in transferable and embedded intangibles to improve product quality or increase brand value and managerial productivity. The dynamics are given by

$$q_{sjt} = \lambda^{m_{sjt}} q_{sj0}, \quad \text{and} \quad e_{st} = \theta^{k_{st}} e_{j0}, \tag{11}$$

with initial levels $q_{sj0} = 1$ and $e_{j0} = 1$. The variables m_{sjt} and k_{st} denote the cumulative numbers of product-quality and managerial/brand-improving innovations by firm s on line j up to time t, respectively. When a firm successfully innovates between t and $t+\Delta t$, its quality increases by a factor of $\lambda > 1$. Brand value and managerial productivity evolve similarly; however, improvement by a factor of $\theta > 1$ is firm-specific and depends only on firm s.

Upon the exit of the superstar firm, its transferable technology becomes imitable, allowing fringe firms on production line j to adopt it. Consequently, the quality gap between a superstar firm and a fringe firm in line j at time t can be expressed as

$$\frac{q_{sjt}}{q_{fjt}} = \frac{\lambda^{m_{sjt}}}{\lambda^{m_{fjt}}} = \lambda^{m_{sjt}-m_{fjt}} = \lambda^{m_{jt}}, \tag{12}$$

where $m_{jt} \equiv m_{sjt} - m_{fjt}$ denotes the technology gap in transferable intangibles. Because the embedded level of fringe firms on each production line is normalized to one, the gap in brand value and managerial quality is given by the superstar's embedded stock:

$$\frac{e_{sjt}}{e_{fjt}} = \frac{\theta^{k_{sjt}}}{1} = \theta^{k_{st}}, \tag{13}$$

where k_{st} represents the gap in embedded intangibles. To ensure a finite state space, I impose upper bounds $\bar{m}$ and $\bar{k}$ on the transferable gap m_{jt} and the embedded gap k_{st}, respectively.

Creative Destruction. Both superstar and fringe firms engage in creative destruction via product quality innovations, but their takeover dynamics differ. If a superstar firm s successfully makes an external innovation at a flow rate $z^{\text{Ex}}_{i,j,t}$, this innovation is assigned to a randomly chosen production line j'. It establishes firm s as the new superstar on the production line j' if and only if its embedded intangible level is at least as high as the incumbent's. The probability that an external innovation from firm s displaces the incumbent on a randomly chosen line is

$$p^{\text{Ex}}_{s \geq s'} \equiv \mathbb{P}\big(e_{s,t} \geq e_{s',t}\big), \tag{14}$$

where $e_{s,t}$ and $e_{s',t}$ denote the firm-level embedded intangible levels of firm s and the incumbent superstar s' at time t, respectively. Hence, even if firm s achieves creative destruction in line j', this alone does not ensure leadership: the firm's embedded intangible level must also be at least as high as that of the incumbent superstar on the target line.

In contrast, fringe firms already in the market have a baseline level of brand value and managerial productivity normalized to 1. To effectively challenge a superstar firm, fringe firms must undertake drastic innovations that neutralize both the superstar firm's product quality and its embedded intangible level advantages.

3.2 Equilibrium

This section characterizes the general equilibrium of the model, which consists of a static and a dynamic component. The analysis begins with the static equilibrium, determining prices and allocations for a given set of states. Subsequently, I define the Markov Perfect Equilibrium for the dynamic game, outlining the value functions, optimal policy functions, and the evolution of the aggregate state distribution.

Household's Problem. Household maximizes utility with the Euler Equation:

$$\frac{\dot{C}_t}{C_t} = r_t - \rho. \tag{15}$$

Along the balanced growth path, consumption and output grow at the same rate, $g = r - \rho$, and the transversality condition holds.

Final and Intermediate Good Sectors. The final-good producer's demand for the continuum of intermediate goods on line j satisfies

$$p_{jt} \;=\; \frac{Y_t}{y_{jt}}. \tag{16}$$

This implies the demand functions for the superstar and the fringe firms:[18]

$$y_{sjt} = p_{jt}^{\frac{\varepsilon}{1-\varepsilon}} p_{sjt}^{\frac{1}{\varepsilon-1}} Y_t \big(\chi(e_s)\big)^{\frac{1}{1-\varepsilon}} \quad \text{and} \quad y_{fjt} = p_{jt}^{\frac{\varepsilon}{1-\varepsilon}} p_{fjt}^{\frac{1}{\varepsilon-1}} Y_t, \tag{17}$$

where p_{sjt} and p_{fjt} represent the product prices of the superstar and fringe firms, respectively. Furthermore, p_{jt} is the ideal price index for production line j, given by the following equation:

$$p_{jt} = \Big(\big(\chi(e_s)\big)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}}\Big)^{\frac{\varepsilon-1}{\varepsilon}}. \tag{18}$$

Prices and Market Share Function. The Cobb-Douglas production function for the final good indicates equal expenditure shares across all production lines. The market share of superstar firm s in industry j at time t is defined as

$$\frac{p_{sjt}y_{sjt}}{p_{jt}y_{jt}} = \frac{p_{sjt}y_{sjt}}{Y_t} = p_{jt}^{\frac{\varepsilon}{1-\varepsilon}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} \chi(e_s)^{\frac{1}{1-\varepsilon}} \;\equiv\; \phi_{sjt}. \tag{19}$$

Because the sum of market shares equals one, the fringe firms' share is $1 - \phi_{sjt}$.

[18] See Appendix B.2 for the full derivations

The equilibrium price of the superstar firm on production line j at time t under à la Bertrand competition[19] is then given by:[20]

$$p_{sjt} = \frac{1 - \varepsilon\,\phi_{sjt}}{(1 - \phi_{sjt})\,\varepsilon}\,MC_{sjt}, \tag{20}$$

where $MC_{sjt} = w_t \times \dfrac{\gamma_s\, n_{st}^{\alpha_s}}{q_{sjt}\left((1-\xi)e_{st}\right)^{\alpha}}$ denotes the marginal cost of the superstar firm, and the fringe firms' price is equal to their marginal cost[21]. The superstar firm's price equation demonstrates that its price is positively correlated with product quality and embedded intangible level, while it is inversely related to the number of production lines operated. The price ratio of fringe firms to superstar follows

$$\frac{p_{fjt}}{p_{sjt}} = \frac{(1-\phi_{sjt})\varepsilon}{1-\varepsilon\phi_{ijt}} \cdot \lambda^{m_j}\left(\frac{(1-\xi)e_{st}}{n_s}\right)^{\alpha}. \tag{21}$$

Using the definition of market share in equation (19) and substituting the ideal price index from equation (18), the market share of the superstar firm can be expressed in terms of relative prices as

$$\phi_{sjt} \;=\; \frac{1}{1+\left(\frac{1}{(\xi e_{st})^{\frac{\beta}{1-\varepsilon}}}\left(\frac{p_{fjt}}{p_{ijt}}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right)}. \tag{22}$$

Replacing the relative price ratio with equation (21) shows that the superstar firm's market share depends on the quality gap m_j, embedded intangible level e_{st}, and the number of production lines n_s it operates.

Profit, Markup and Labor Demand. The static operational profit of the superstar

[19] For the Cournot Competition version, see Appendix B.4

[20] See Appendix B.3 for the full derivations

[21] Even though fringe firms lack independent pricing power, their presence creates a competitive constraint that disciplines superstar firms. This competitive pressure forces superstars to engage in limit pricing strategies, preventing them from fully exercising their market power and extracting monopolistic rents.

firm is proportional to its market share and the size of the economy:

$$\pi_{sjt} = \frac{(1-\varepsilon)\,\phi_{sjt}}{1-\varepsilon\phi_{sjt}}\,Y_t, \tag{23}$$

with the corresponding markup given by

$$\sigma_{sjt} = \frac{1-\varepsilon\phi_{sjt}}{(1-\phi_{sjt})\,\varepsilon}. \tag{24}$$

Both markup and profit increase with market share. However, because market share on a given line is inversely related to the number of lines a firm operates, a superstar's markup and profit decrease, reflecting the natural consequence of diminishing managerial productivity across production lines. The optimal labor inputs for superstar and fringe firms, respectively, are[22]

$$l_{sjt} = \frac{\phi_{sjt}}{\sigma_{sjt}}\,\omega_t^{-1}, \tag{25}$$

$$l_{fjt} = (1-\phi_{sjt})\,\omega_t^{-1}, \tag{26}$$

where $\omega_t = \frac{w_t}{Y_t}$ denotes the wage share of the economy.

The static equilibrium provides only an implicit solution. Nevertheless, the model yields tractable dynamics because the equilibrium outcome for a superstar firm depends solely on its market share, which is in turn determined by the quality gap, the level of embedded intangibles, and the number of production lines operated. This tractability makes it possible to analyze firms' endogenous investment decisions in different types of intangibles and to study how these choices affect firm dynamics.

Superstar Value Function. The superstar value function $V_t(\mathbf{m}, e_{st}, n_s)$ relevant payoff depends on the quality gap vector $\mathbf{m} = \{m_j\}_{j=1}^{n_s}$, embedded intangible level e_{st}, and the number of production lines the superstar firm has n_s. Superstar firm s

[22] See equation (63) in the Appendix for details.

optimizes the flow rate of innovation of z_{sjt}^{Int}, z_{sjt}^{Ex}, and z_{stj}^{Emb} to maximize the value function given by equation 27.

The left-hand side of the value function shows the return on the value function and its gain over time. The first line on the right-hand side represents the profit of the superstar firm in production line j. The second line's first term shows that with an innovation flow rate of $p_{s\geq s'}^{\text{Ex}} z_{sjt}^{Ex}$, firm s increases its production line from n_s to n_s+1, while the subsequent terms describe how the superstar increases the transferable intangible gap by one rung with an innovation flow rate of z_{sjt}^{Int}. The third line shows that firm s can increase its embedded intangible level one rung across all production lines, and the next term shows that superstars in other industries can innovate and firm s exit production line j. The last three terms represent the cost of investment in improving existing production quality, improving the embedded intangible level, and taking on a new production line, respectively. Cost function details are described in equations (8).

$$
\begin{aligned}
r_t V_t(\mathbf{m}, e_s, n_s) - \dot{V}_t(\mathbf{m}, e_s, n_s) = \max_{z_{sjt}^{Int}, z_{sjt}^{Ex}, z_{sjt}^{Emb}} \sum_{j=1}^{n_s} \Bigg(& \pi_{jt}(m_j, e_s, n_s) \\
+ & \underbrace{p_{s\geq s'}^{\text{Ex}} z_{sjt}^{Ex} \big(V_t((m_j, 1), e_s, n_s+1) - V_t(m_j, e_s, n_s)\big)}_{\text{Expansion with new production line}} + \underbrace{z_{sjt}^{Int} \big(V_t(m_j+1, e_s, n_s) - V_t(m_j, e_s, n_s)\big)}_{\text{Internal innovation}} \\
+ & \underbrace{z_{sjt}^{Emb} \big(V_t(m_j, e_s+1, n_s) - V_t(m_j, e_s, n_s)\big)}_{\text{Innovation on embedded}} + \underbrace{p_{s'\geq s}^{\text{Ex}} Z_{jt}^{Ex} \big[V_t(m_j, e_s, n_s-1) - V_t(m_j, e_s, n_s)\big]}_{\text{Superstars in oher industries innovation}} \\
+ & \underbrace{Z_{jt}^{f} \big(V_t(m_j, e_s, n_s-1) - V_t(m_j, e_s, n_s)\big)}_{\text{Fringe firms' innovation}} - \gamma^{Int} \left(z_{sjt}^{Int}\right)^{\vartheta^{Int}} Y_t - \gamma^{Emb} \left(z_{sjt}^{Emb}\right)^{\vartheta^{Emb}} Y_t \\
- & \gamma^{Ex} \left(z_{sjt}^{Ex}\right)^{\vartheta^{Ex}} Y_t \Bigg)
\end{aligned} \tag{27}
$$

Additionally, Z_{jt}^{Ex} and Z_{jt}^{f} represent, respectively, aggregate external innovation by superstar firms and aggregate innovation by fringe firms:

$$
Z_{jt}^{Ex} = \int_0^1 z_{sjt}^{Ex}\, dj, \qquad Z_{jt}^{f} = \int_0^1 z_{ijt}\, di.
$$

The value function of fringe firms is not explicitly described here. This is primarily because fringe firms may make a drastic innovation that displaces the incumbent superstar in a given production line. Consequently, regardless of the incumbent's quality or embedded intangible gap, a successful innovation by a fringe firm immediately makes it the superstar in that line, with quality and embedded gap set to 1 and operating a single production line, $V(1,1,1)$. Thus, the fringe firm's value function depends solely on the superstar's value at $V(1,1,1)$ and is constant for each rung distance in state space. Section 7 discusses alternative frictions and scenarios involving fringe firms.

In the balanced growth path, aggregate output Y_t, consumption C_t, and the value function $V(\mathbf{m},k,n)$ all grow at the constant rate g. Defining the stationary value function as $v(\mathbf{m},k,n) = V(\mathbf{m},k,n)/Y_t$, the HJB equation for a superstar firm on the balanced growth path is given by:

$$
\begin{aligned}
\rho\, v(\mathbf{m},k,n) = \max_{z_j^{Int}, z_j^{Ex}, z_J^{Emb}} \sum_{j=1}^{n} \Bigg(& \pi_j(m_j,k,n) \\
& + p^{\text{Ex}}_{s\geq s'}\, z_j^{Ex} \big(v((m_j,1),k,n+1) - v(m_j,k,n)\big) + z_j^{Int} \big(v(m_j+1,k,n) - v(m_j,k,n)\big) \\
& + z_j^{Emb} \big(v(m_j,k+1,n) - v(m_j,k,n)\big) + p^{\text{Ex}}_{s'\geq s}\, Z_j^{Ex} \big[v(m_j,k,n-1) - v(m_j,k,n)\big] \\
& + Z_j \big(v(m_j,k,n-1) - v(m_j,k,n)\big) - \gamma^{Int} \left(z_j^{Int}\right)^{\vartheta^{Int}} - \gamma^{Emb} \left(z_j^{Emb}\right)^{\vartheta^{Emb}} \\
& - \gamma^{Ex} \left(z_j^{Ex}\right)^{\vartheta^{Ex}} \Bigg)
\end{aligned} \tag{28}
$$

Innovation Decisions. The first-order conditions of the superstar and fringe firms' value functions determine their optimal innovation intensities. Along the balanced growth path, the superstar firm's optimal rates of internal, embedded, and external innovations are given by

$$z_{sjt}^{\text{Int}} = \left(\frac{v_t(m_j + 1, e_s, n_s) - v_t(m_j, e_s, n_s)}{\gamma^{\text{Int}} \cdot \vartheta^{\text{Int}}} \right)^{\frac{1}{\vartheta^{\text{Int}} - 1}}, \tag{29}$$

$$z_{sjt}^{\text{Emb}} = \left(\frac{v_t(m_j, e_s + 1, n_s) - v_t(m_j, e_s, n_s)}{\gamma^{\text{z}_{\text{E}}\text{mb}} \cdot \vartheta^{\text{Emb}}} \right)^{\frac{1}{\vartheta^{\text{Emb}} - 1}}, \tag{30}$$

$$z_{sjt}^{\text{Ex}} = \left(\frac{v_t(m_j + 1, e_s, n_s + 1) - v_t(m_j, e_s, n_s)}{\gamma^{\text{Ex}} \cdot \vartheta^{\text{Ex}}} \right)^{\frac{1}{\vartheta^{\text{Ex}} - 1}}. \tag{31}$$

For the fringe firm, the optimal flow rate of innovation is

$$z_{fjt} = \left(\frac{v_t(1, 1, 1) - v_t^f}{\gamma^f \cdot \vartheta^f} \right)^{\frac{1}{\vartheta^f - 1}} \tag{32}$$

where v_t^f shows the value function of fringe firms. The above expressions indicate that the optimal investment in each type of innovation depends on the marginal increase in the firm value function relative to the convex cost parameters (γ, ϑ).

Distribution Evolution. For notational simplicity, I suppress explicit indices for firm s, industry j, and describe the evolution of the distribution based on state variables for quality gap (m), embedded intangible level (k), and number of production lines (n).

$$\begin{aligned}
\mu_t(m, k, n) = {} & z_t^{Int}(m-1, k, n) \cdot \mu_t(m-1, k, n) + z_t^{Emb}(m, k-1, n) \cdot \mu_t(m, k-1, n) \\
& + p_{k \geq k'}^{\text{Ex}}\, z_t^{Ex}(m, k, n-1) \cdot \mu_t(m, k, n-1) \\
& + z_t^{Int}(m-1, k-1, n) \cdot z_t^{Emb}(m-1, k-1, n) \cdot \mu_t(m-1, k-1, n) \\
& + z_t^{Int}(m-1, k, n-1) \cdot p_{k \geq k'}^{\text{Ex}}\, z_t^{Ex}(m-1, k, n-1) \cdot \mu_t(m-1, k, n-1) \\
& + z_t^{Emb}(m, k-1, n-1) \cdot p_{k \geq k'}^{\text{Ex}}\, z_t^{Ex}(m, k-1, n-1) \cdot \mu_t(m, k-1, n-1) \\
& + z_t^{Int}(m-1, k-1, n-1) \cdot z_t^{Emb}(m-1, k-1, n-1) \\
& \cdot \quad p_{k \geq k'}^{\text{Ex}}\, z_t^{Ex}(m-1, k-1, n-1) \cdot \mu_t(m-1, k-1, n-1) \\
& - z_t^{Int}(m, k, n) \cdot \mu_t(m, k, n) - z_t^{Emb}(m, k, n) \cdot \mu_t(m, k, n) \\
& - p_{k \geq k'}^{\text{Ex}}\, z_t^{Ex}(m, k, n) \cdot \mu_t(m, k, n) - Z_t^f(m, k, n) \cdot \mu_t(m, k, n) - p_{k' \geq k}^{\text{Ex}}\, Z_t^{Ex} \cdot \mu_t(m, k, n)
\end{aligned} \tag{33}$$

The law of motion of the distribution is driven by net flows into and out of the cohort in state (m, k, n): inflows consist of firms that, following successful innovation, enter (m, k, n); outflows consist of firms that leave (m, k, n) as a result of their own innovation or innovation elsewhere. Concretely, the first three terms on the right-hand side capture inflows from predecessor states that are one step behind in a single dimension. The next three terms capture inflows from states that are one step behind in two dimensions (pairwise lags), while the subsequent term captures inflows from firms that are simultaneously one step behind in all three dimensions. The following group of terms describes outflows from state (m, k, n) owing to the superstar firm's own internal, embedded, or external innovation. Finally, the last two terms account for outflows induced by innovation by fringe firms or by leaders in other industries.

Distribution at Boundaries. The state space is bounded above and below, so specific conditions apply at its edges. If any state attains an upper bound, $m = \bar{m}$, $k = \bar{k}$, or $n = \bar{n}$ (including any joint combination), the outflow resulting from a successful innovation that would advance the firm further in that dimension is zero. This reflects the assumption that a firm's lead in any dimension cannot exceed a technologically feasible maximum. Conversely, if any state attains a lower bound, $m = 1$, $k = 1$, or $n = 1$ (or any joint combination), the outflow to a state with a lower value in that dimension is zero. This condition functions as an absorbing barrier, preventing a firm's position from deteriorating below a fundamental minimum.

Aggregate Variables. The joint distribution of (m, k, n) satisfies

$$\sum_{m=1}^{\bar{m}} \sum_{k=1}^{\bar{k}} \sum_{n=1}^{\bar{n}} \mu_t(m, k, n) = 1. \tag{34}$$

The labor market clears:

$$1 = \int_0^1 \left(l_{sjt} + l_{fjt}\right) dj, \tag{35}$$

and, using (25), (26) and (35), the normalized wage is

$$\omega_t = \sum_{m=1}^{\bar{m}} \sum_{k=1}^{\bar{k}} \sum_{n=1}^{\bar{n}} \left(\frac{\phi_t(m,k,n)}{\sigma_t(m,k,n)} + 1 - \phi_t(m,k,n) \right) \mu_t(m,k,n). \tag{36}$$

Combining intermediate good sectors output (3), (6), (7) with the labor demand of superstar (25) and fringe firms (26) produces aggregate output

$$Y_t = Q_t\, \omega_t^{-1} \exp\left(\sum_{m=1}^{\bar{m}} \sum_{k=1}^{\bar{k}} \sum_{n=1}^{\bar{n}} \underbrace{\ln\left[\left(\xi\theta^{k_t}\right)^{\beta} \left(\frac{((1-\xi)\theta^{k_t})^{\alpha}}{\gamma n_t^{\alpha_s}} \frac{\phi_t(m,k,n)}{\sigma_t(m,k,n)} \right)^{\varepsilon} + \left(\lambda^{-m_t} (1-\phi_t(m,k,n)) \right)^{\varepsilon} \right]^{\frac{1}{\varepsilon}}}_{\equiv R_t(m,k,n)} \mu_t(m,k,n) \right). \tag{37}$$

where

$$Q_t \;=\; \exp\left(\int_0^1 \ln q_{sjt}\, dj \right)$$

The growth rate of the economy is[23]

$$g_t = -g_{\omega,t} + g_{Q,t} + g_{R,t}. \tag{38}$$

In balanced growth, the economy grows at rate g, which is given by

$$g \;=\; \ln \lambda \sum_{m=1}^{\bar{m}} \sum_{k=1}^{\bar{k}} \sum_{n=1}^{\bar{n}} \left(z^{\text{Int}}(m,k,n) + p^{\text{Ex}}_{k \geq k'}\, z^{\text{Ex}}(m,k,n) + Z^{f}(m,k,n) \right) \mu(m,k,n). \tag{39}$$

Finally, the resource constraint satisfies

$$Y_t = C_t + \int_0^1 \left(I_{jt}^{\text{Int}} + I_{jt}^{\text{Emb}} + I_{jt}^{\text{Ex}} \right) dj + \int_0^1 I_{jt}^{f}\, dj, \tag{40}$$

with

$$I_{jt}^{f} = \int_{F_j} I_{ijt}\, di.$$

Equilibrium Definition. The Markov Perfect equilibrium of the economy consists of an allocation $\{C_t, Y_t, y_{sjt}, y_{fjt}\}$, prices $\{r_t, w_t, p_{sjt}, p_{fjt}\}$, and policies $\{z^{Int}, z^{Emb}, z^{Ex},$

[23] For details of the growth-rate calculation, see Appendix B.5.

$z_f, z^{Ent}, l_{sjt}, l_{fjt}\}$, such that the final goods sector maximizes profit given prices. The superstar firm maximizes profit given the quality gap, embedded level gap, and number of production lines it has, $(\mathbf{m}, k, n)$, while the fringe firm maximizes profit given prices. The superstar firm chooses the values of internal z^{Int} (29), external z^{Ex} (31), and embedded innovation z^{Emb} (29) whereas the fringe firm selects its optimal innovation value z_f (32). The real wage clears the labor market, aggregate consumption and output grow at the same rate, equation (39), and the resource constraint satisfies equation (40).

4 Quantitative Analysis

In this section, I present calibration parameters to illustrate how the model responds to variations in the investment ratio of transferable to embedded intangibles, and to changes in productivity, growth rate, and markup. Section 5 then examines counterfactual scenarios, first shutting down the effects of embedded intangibles and then considering the span-of-control problem, to assess their impact on the economy. Section 6 subsequently discusses misallocation and the associated policy implications.

4.1 Empirical counterpart

For each production line n, I aggregate the joint distribution over the quality gap and the embedded-intangible level, described as

$$\mu(n) \;=\; \sum_m \sum_k \mu(m, k, n).$$

Fluidity at production line n is measured as the flow rate of incumbent replacement. It is given by[24]

[24]The first term in the numerator captures external innovations (from higher embedded levels $k' > k$) that displace incumbents in (m, k, n); the second term captures innovation coming from the fringe.

$$\text{Fluidity}(n) = \frac{\sum_m \sum_{k'>k} \left(Z^{\text{Ex}}(m,k',n) \right) \mu(m,k,n) + \sum_m \sum_k Z^f(m,k,n)\, \mu(m,k,n)}{\mu(n)}. \tag{41}$$

The average markup and the investment ratio at production line n are

$$\sigma(n) = \frac{\sum_m \sum_k \frac{1-\varepsilon\,\phi(m,k,n)}{(1-\phi(m,k,n))\,\varepsilon}\, \mu(m,k,n)}{\mu(n)}, \tag{42}$$

$$\frac{I^T}{I^{\text{Emb}}}(n) = \frac{\sum_m \sum_k \frac{I^{\text{Int}}(m,k,n) + I^{\text{Ex}}(m,k,n) + I^f(m,k,n)}{I^{\text{Emb}}(m,k,n)}\, \mu(m,k,n)}{\mu(n)}. \tag{43}$$

The growth and productivity by production line are

$$g(n) = \frac{\ln\lambda \sum_m \sum_k \left(z^{\text{Int}}(m,k,n) + p^{\text{Ex}}_{k\geq k'}\, z^{\text{Ex}}(m,k,n) + Z^f(m,k,n) \right) \mu(m,k,n)}{\mu(n)}, \tag{44}$$

$$Q(n) = e^{g(n)}. \tag{45}$$

4.2 Calibration and Model Performance

The model is disciplined by 36 empirical moments, comprising 18 targeted and 18 untargeted moments. The calibration relies on 17 parameters, of which 4 are set externally. The time-discount rate is fixed at $\rho = 0.05$, and the quality-improvement step size is set to $\lambda = 1.10$ following Akcigit and Ates (2023). The cost scale and curvature of embedded intangibles are taken from Cavenaile, Celik, Roldan-Blanco, and Tian (2025), with $\gamma^{\text{emb}} = 0.0664$ and $\vartheta^{\text{emb}} = 3.3646$. The remaining 13 parameters,

$$\{\varepsilon, \theta, \alpha_s, \alpha, \gamma, \beta, \xi, \gamma_{\text{Int}}, \gamma_{\text{Ex}}, \gamma_f, \vartheta_{\text{Int}}, \vartheta_{\text{Ex}}, \vartheta_f\},$$

are estimated internally. These parameters govern the key structural features of the model. All parameter values are reported in Table 2.[25]

Table 2. Parameter Values

Parameter	Description	Value
	————————*External Calibration*————————	
ρ	Discount rate	0.05
λ	Transferable innovation step size	1.0100
γ^{Emb}	Cost scale of embedded innovation	0.0664
ϑ^{Emb}	Curvature of embedded innovation	3.3646
	————————*Internal Calibration*————————	
ε	CES parameter	0.7747
θ	Embedded innovation step size	1.0100
α_s	Curvature of Span of Control	0.5876
α	Curvature of managerial productivity	0.5980
γ	Scale of managerial productivity	0.4001
β	Curvature of brand value	0.0661
ξ	Share of brand value on embedded intangible	0.4351
γ^{Int}	Cost scale of internal innovation	2.8571
γ^{Ex}	Cost scale of horizontal innovation	0.4001
γ^{f}	Cost scale of fringe	5.1777
ϑ^{Int}	Curvature of internal innovation	15.3154
ϑ^{Ex}	Curvature of horizontal innovation	5.4183
ϑ^{f}	Curvature of fringe	7.9917

Note: The upper limit for the number of production lines $\bar{n}$ is set to 6, and the upper bounds for $\bar{m}$ and $\bar{k}$ are set to 9.

Figure 4 reports the targeted moments used in the simulated method of moments (SMM) estimation, focusing on markup dynamics, the ratio of transferable-to-embedded intangible investment, and the distribution of firms across production lines. Overall, the model reproduces the principal empirical patterns: it captures both the direction and magnitude of the observed trends. Panel (A) shows that the model tracks the decline in markups as the number of production lines increases, although modest deviations remain at the extremes of the distribution—the model understates markups for single-line firms and slightly overstates them for firms operating many lines. Panel (B) illustrates that the transferable-to-embedded investment ratio is well matched across production-line categories, with simulated

[25] Appendix C.2 provides details of the solution algorithm and the simulated method of moments.

moments closely following the empirical shape. Panel (C) demonstrates a strong fit for the firm distribution, in which simulated shares align closely with observed data. Taken together, these results suggest that the model is well disciplined by the targeted moments and captures key margins of firm behavior, with only minor discrepancies concentrated in the tails of the markup profile.

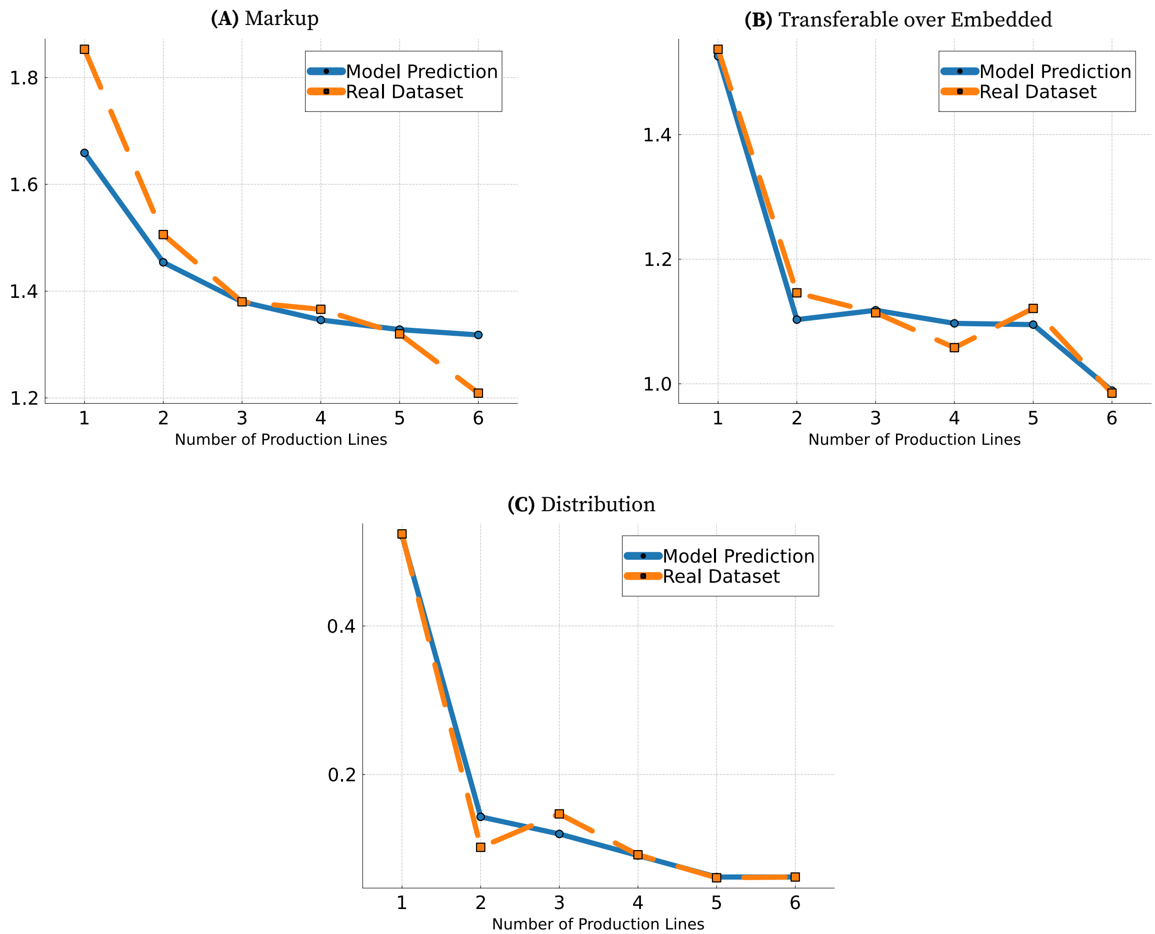


Figure 4. Targeted Moments: Markup, Investment Ratio, and Fluidity by Production Lines

Note: The orange line represents the dataset values, while the blue line shows the model simulation results along the balanced growth path. The horizontal axis corresponds to the production line dimension.

Further, Figure 5 evaluates the model's performance on untargeted moments not used in the SMM, namely productivity, aggregate growth rates, and measures of fluidity across production-line categories. Panel (A) indicates that the model captures the declining pattern of productivity as production-line count rises, with only small departures from the data at intermediate values. Panel (B) reveals systematic differences in the aggregate growth rate: the model tends to underestimate actual growth for firms with one to three production lines and slightly overshoots growth at the upper end of the distribution. These discrepancies imply that, while the model reproduces the overall downward growth trend, it misses some non-monotonic features present in the data. Panel (C) shows that the model generally undershoots empirical fluidity measures across most production-line categories, with the notable exception of the sixth line, where simulated fluidity converges more closely to the observed value.

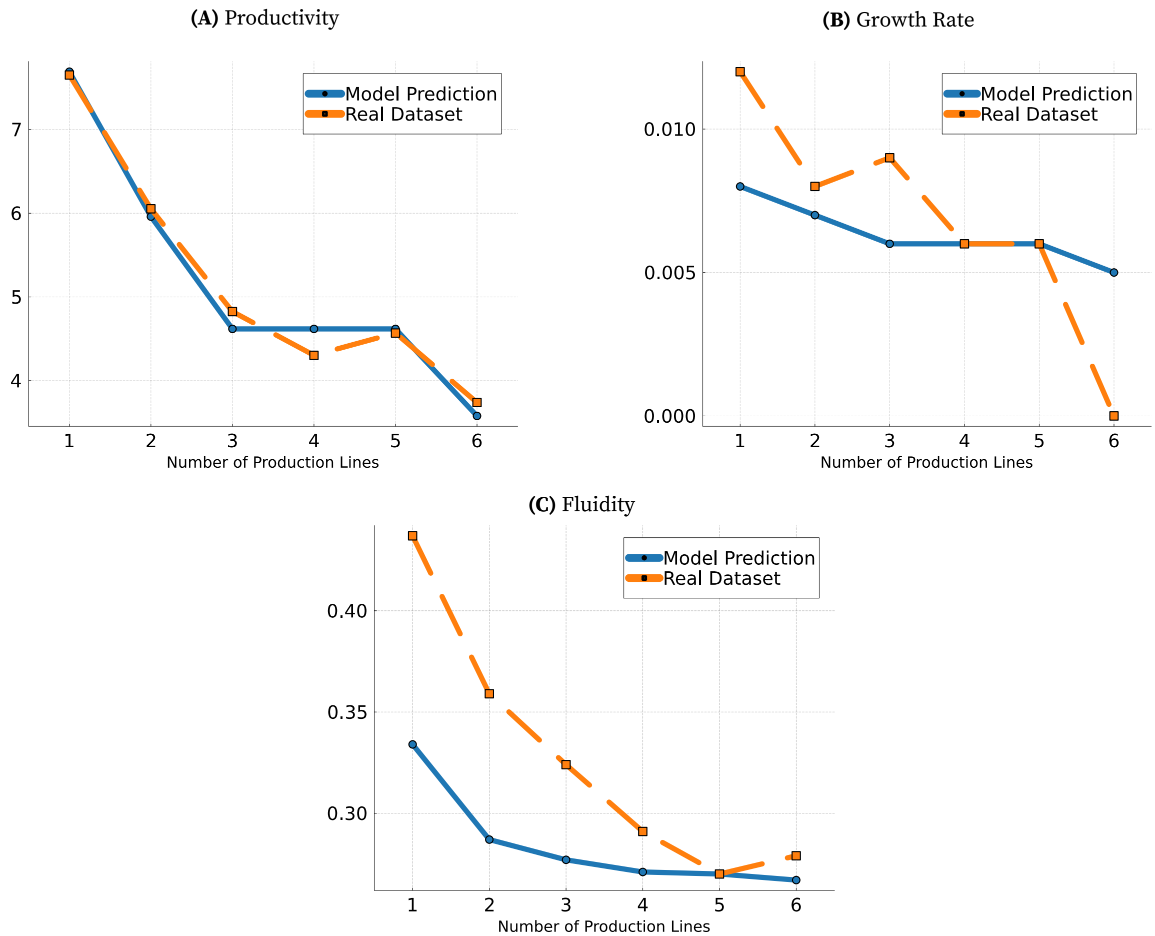


Figure 5. Untargeted Moments: Productivity, Fluidity and Growth Rate by Production Lines

Note: The orange line represents the dataset values, while the blue line shows the model simulation results along the balanced growth path. The horizontal axis corresponds to the production line dimension.

4.3 Demand and Supply Effect of Embedded Intangibles

In the model, parameter ξ governs the share of brand value (demand side), while $1 - \xi$ corresponds to organizational capital (supply side). A higher ξ implies that brand value dominates (green line), whereas a lower ξ (blue line) indicates a greater role for organizational capital. Figure 6 shows that firms derive greater benefits from organizational capital than from brand value when expanding their scope. The primary reason for this is that organizational capital can directly offset some of the managerial difficulties associated with expansion, thereby promoting growth across production lines.

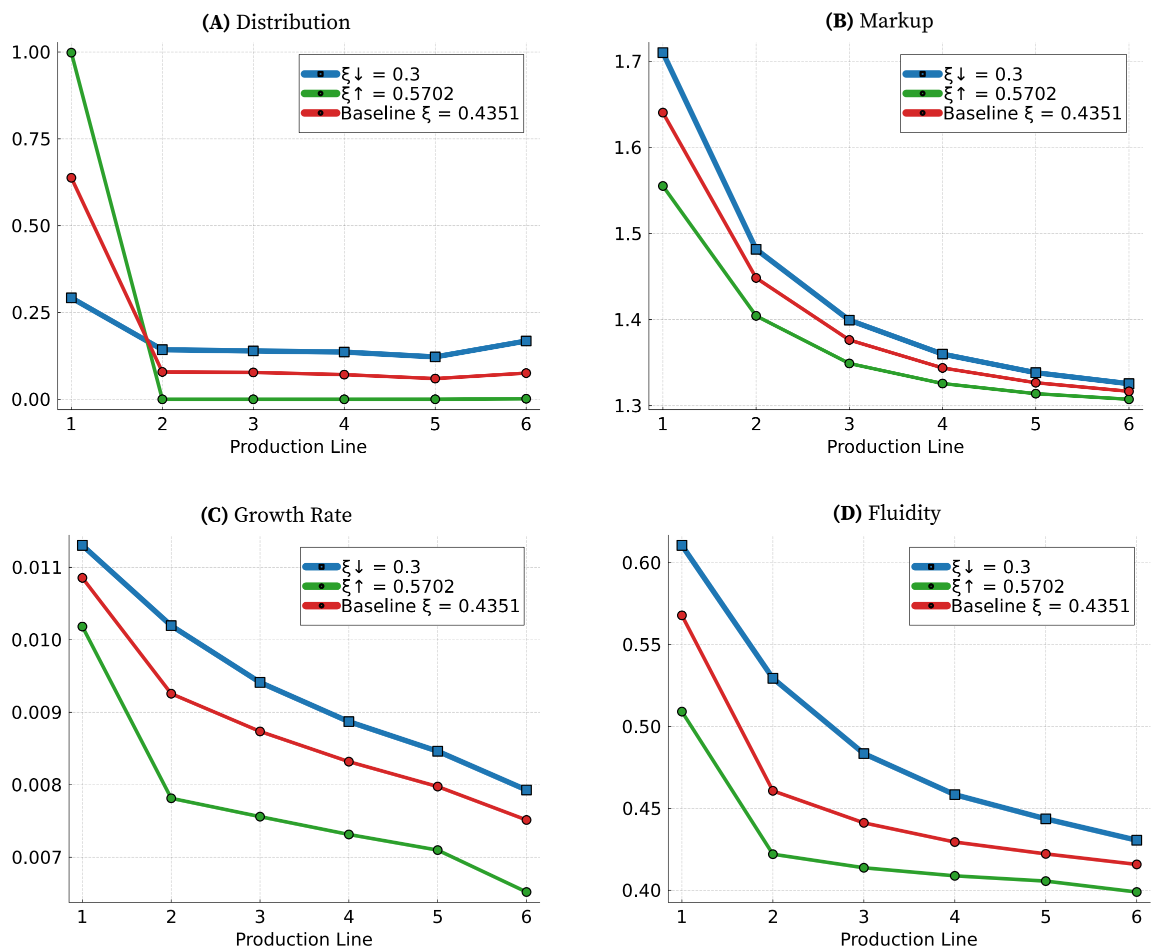


Figure 6. Impact of ξ on Distribution, Markup, Growth, and Fluidity

Note: The green line represents the internally calibrated optimal value of ξ, while the blue line shows an upward shift in ξ and the orange line shows a downward shift. The horizontal axis corresponds to the production line dimension.

In contrast, brand value does not provide this offsetting capability. Consequently, when the share of brand value (ξ) is high, firms tend to stop expanding and predominantly operate a single production line. The associated decrease in markups may seem counterintuitive at first. However, this decline is primarily driven by reductions in organizational capital, which diminishes managerial productivity gains. This negative supply-side effect outweighs any positive demand-side effects from brand value. [26]Moreover, when ξ is high, superstar firms continue to accumulate embedded intangibles, but do so while operating only a single line. This accumulation creates a significant barrier to entry, as new entrants must match this high level of intangibles to compete, even on a single line. As a result, both market fluidity and the aggregate growth rate decrease. All these effects are reversed when parameter ξ is lower. In this case, the larger share of organizational capital provides firms a greater advantage, facilitating expansion and improving the overall dynamics of creative destruction and growth. [27]

5 Counterfactual Analysis

In this section, I conduct a series of counterfactual analyses to isolate the mechanisms driving the results. First, I deactivate the span of control constraint. Second, I shut down the accumulation of embedded intangibles. Finally, I shut down both mechanisms jointly. This sequence allows me to quantify how each feature—and their interaction—affects key outcomes: markups, the firm size distribution, market fluidity, and the aggregate growth rate of the economy.

Shutting Down the Span of Control Constraint. To evaluate the impact of the span-of-control constraint, I conduct a counterfactual analysis by setting the parameter

[26]In the calibration, the curvature of brand value is relatively low compared with that of managerial productivity, which causes supply-side effects to dominate demand-side effects. The parameter γ, which captures the benefit scale of managerial quality, further reinforces this dominance by amplifying supply-side responses relative to demand.

[27]This increase occurs relative to the composition of brand value versus organizational capital. By contrast, Section 6 shows that increasing embedded intangibles leads to substantial inefficiency, regardless of brand value or organizational capital.

α_s to zero[28]. This removes the mechanism that causes managerial productivity to decline with the number of production lines. The results are presented in Figure 7, where the counterfactual scenario is plotted in green and the baseline calibration in red.

The counterfactual generates a rightward shift in the firm size distribution, with firms operating more product lines relative to the baseline. This expansion follows directly from the removal of span-of-control constraints: in the absence of diminishing managerial returns, firms can add product lines without increasing marginal costs. As a result, markups now increase with the number of product lines, reversing the baseline pattern in which diversification reduced markups due to rising marginal costs from span-of-control frictions. Under Bertrand competition, a firm's markup in each product line is constrained by both its own marginal cost and the marginal costs of its competitors. In the baseline, span-of-control limitations increase the firm's marginal costs as it diversifies, reducing markups. Eliminating this channel keeps marginal costs low across all lines, allowing firms to sustain higher markups as they expand.

In contrast, relaxation of managerial constraints reduces market fluidity. As firms grow larger and manage more product lines, they face stronger incentives to invest in embedded intangibles due to increasing returns to scale. The resulting accumulation of intangible capital raises entry barriers and lowers overall market fluidity. Despite the decline in market fluidity, the aggregate growth rate rises. This counterintuitive result stems from a shift in the source of growth: while creative destruction diminishes, innovation by incumbent superstar firms increases. Relaxation of span-of-control constraints directly benefits multiproduct firms, which were previously the most constrained. Consequently, their enhanced ability to expand and innovate boosts firm-level growth rates, increasing aggregate growth relative to the baseline.

[28]The parameter α_s is highly sensitive; setting it directly to zero prevents the model from converging and producing results. Therefore, when $\alpha_s = 0$, $\bar{k}$ is set to 6 rather than 9 in the baseline calibration.

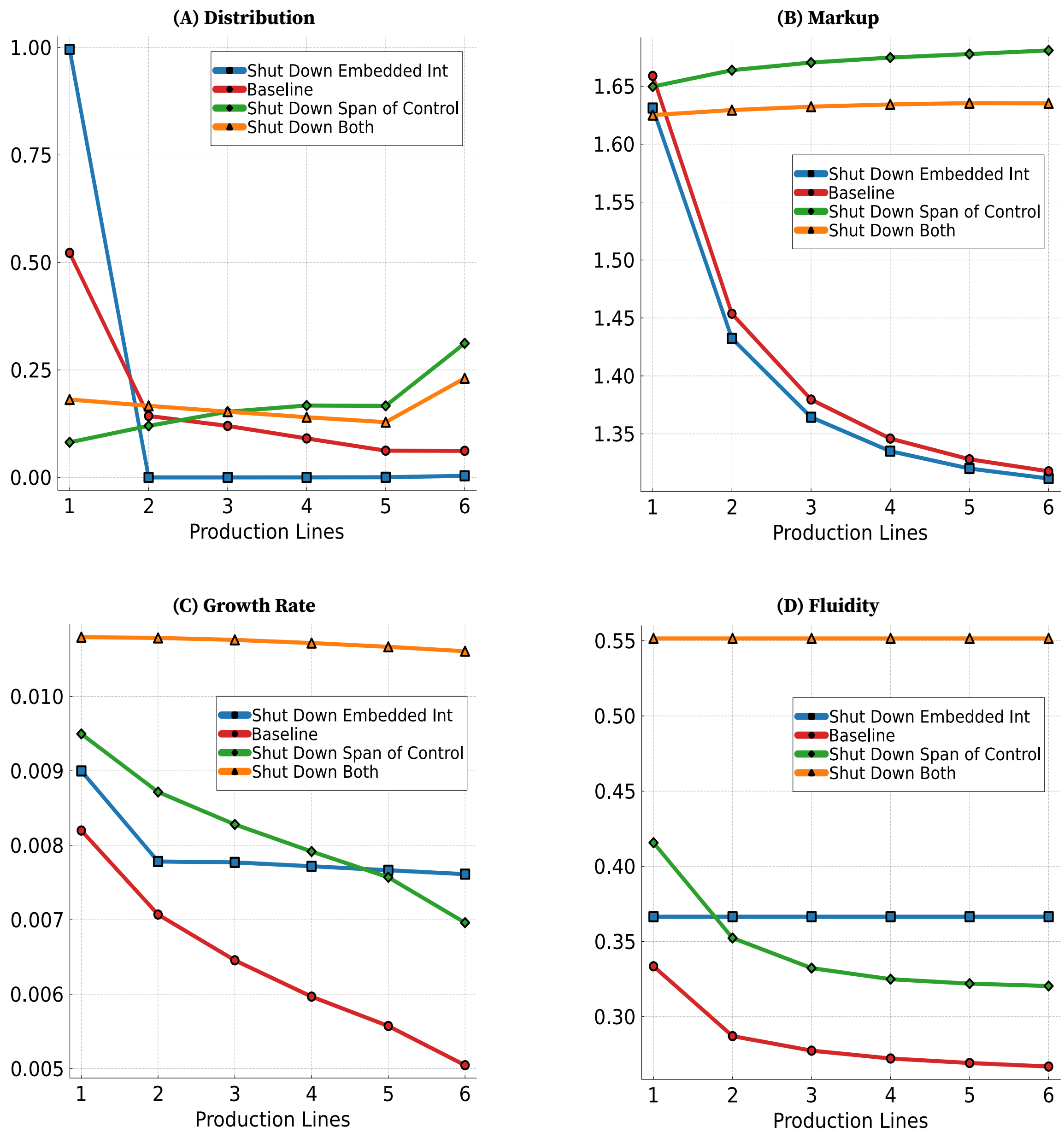


Figure 7. Counterfactual Analysis Under Different Cases

Note: The red line shows the baseline calibration; the green line shows the case with span-of-control constraint shut down ($\alpha_s = 0$); the blue line shows the case with embedded intangibles shut down ($\bar{k} = 1$); and the orange line shows the case with both the span-of-control constraint and embedded intangibles shut down ($\alpha_s = 0$ and $\bar{k} = 1$).

Shutting Down Embedded Intangibles. Next, I examine the effects of shutting down embedded intangibles by setting $\bar{k} = 1$. In this counterfactual, superstar firms can no longer benefit from accumulating brand value or organizational capital. As shown in Figure 7 (blue line), this leads to a leftward shift in the firm size distribution, which becomes concentrated on a single production line. The mechanism driving this shift is that expanding to additional lines remains costly due to the persistent span-of-control constraint, and firms can no longer offset these costs by improving managerial quality through embedded intangibles. Markups also decline, as firms lose the pricing advantage provided by embedded intangibles relative to rivals. However, this reduction is modest, indicating that while embedded intangibles contribute to markup growth, their overall quantitative impact is limited.

The aggregate growth rate rises, primarily driven by a significant increase in market fluidity. Eliminating the accumulation of embedded intangibles removes entry barriers for other superstar firms, allowing them to enter production lines more easily without facing entrenched incumbents. This enhanced entry raises the rate of creative destruction, and the resulting higher market fluidity directly contributes to the increase in aggregate growth. These results highlight the important role of embedded intangibles in shaping market dynamics and economic growth.

Both the removal of span-of-control constraints and the elimination of embedded intangibles raise the aggregate growth rate; however, the sources and magnitudes of these increases differ. In the span-of-control counterfactual, the growth boost is driven primarily by incumbents. In contrast, when embedded intangibles are eliminated, the growth increase is driven by innovations from other superstar firms on incumbent lines.

Shutting Down Both of Mechanisms. Shutting down both mechanisms simultaneously allows multiproduct firms to expand horizontally, which shifts the firm size distribution rightward. However, the magnitude of this shift is significantly limited compared to the scenario in which only the span-of-control constraint is removed. The primary reason is that, in this combined counterfactual, superstar firms can no longer leverage the increasing returns to scale afforded by embedded intangibles to facilitate their expansion. For the same reason, markups increase with the number of production lines, but this increase is more muted than in the counterfactual involving only the span-of-control constraint. Without embedded intangibles, firms lack one of the tools to amplify their pricing power as they grow.

On the other hand, market fluidity and the aggregate growth rate are higher in this combined counterfactual than in either single shutdown case. This elevated growth stems from a dual source: intensified innovation by incumbents and heightened innovation by potential superstar firms in other industries, who find it easier to enter the market and to contest existing production lines due to lower barriers arising from the absence of embedded intangibles.

To summarize, all three counterfactuals demonstrate that both span-of-control constraints and embedded intangibles are necessary to replicate the empirical patterns shown in Section 2.3. Omitting either component results in deviations from the observed facts: markups no longer decrease with scope, firm size distribution becomes concentrated on a single production line, or the declining trends in growth and market fluidity across production lines are not reproduced. Therefore, these two mechanisms together provide the minimal conditions required to capture the key empirical regularities.

6 Policy Implication and Misallocation

This section first quantifies the effects of resource misallocation on aggregate output and economic growth. Based on these insights, I implement policy tools to examine how these frictions affect welfare and how they can be mitigated.

6.1 Misallocation

Misallocation in the model operates through two distinct channels. The first stems from markup dispersion across firms. The second arises because a potential superstar entrant cannot capture a market unless its embedded intangible level is at least as high as the incumbent's, creating a barrier to entry. To quantify the effect of markup dispersion, I adapt the method by Peters (2020) and decompose aggregate output Y into four components: the contribution from quality improvements

(Q), the contribution from embedded intangible capital (E), the misallocation due to markup dispersion (M), and a leftover term (S)[29],

$$Y_t = Q_t \times E_t \times M_t \times S_t \tag{46}$$

$$\text{where} \quad E_t = \exp\left(\sum_m \sum_k \sum_n \frac{1}{\varepsilon} \ln\left[(\xi\theta)^{k\times\beta}\left((1-\xi)\theta)^{k\times\alpha}\right)^{\varepsilon}\right]\mu(m,k,n)\right),$$

$$M_t = \frac{\exp\left(\sum_m \sum_k \sum_n \ln\left[\frac{\phi(m,k,n)}{\sigma(m,k,n)} + (1-\phi(m,k,n)\right]\mu(m,k,n)\right)}{\sum_m \sum_k \sum_n \left(\frac{\phi(m,k,n)}{\sigma(m,k,n)} + (1-\phi(m,k,n)\right)\mu(m,k,n)}, \quad \text{and}$$

$$S_t = \exp\left(\sum_m \sum_k \sum_n \ln\left[\frac{\left(\frac{1}{\gamma n^{\alpha_s}}\frac{\phi(m,k,n)}{\sigma(m,k,n)}\right)^{\varepsilon} + \frac{1}{(\xi\theta^k)^{\beta}}\left(\frac{\lambda^{-m}}{((1-\xi)\theta^k)^{\alpha}}(1-\phi(m,k,n))\right)^{\varepsilon}}{\left(\frac{\phi(m,k,n)}{\sigma(m,k,n)} + (1-\phi(m,k,n))\right)^{\varepsilon}}\right]^{\frac{1}{\varepsilon}}\mu(m,k,n)\right).$$

Defining M as the ratio of the geometric mean to the arithmetic mean, the key misallocation term endogenously evolves with markup dispersion. [30] The results, presented in Table 3, show that although there is substantial markup dispersion across production lines, the aggregate dispersion effect on misallocation is relatively small. As a result, the misallocation arising purely from markup dispersion is minimal, accounting for only about 0.5% of output.

To analyze the second type of misallocation, I conduct a counterfactual experiment that eliminates entry barriers. In this scenario, any superstar firm that successfully makes a horizontal innovation can immediately enter a new production line and become the incumbent without facing the embedded intangible constraint. The results are striking: aggregate output more than triples. This dramatic gain is primarily driven by a large increase in the quality improvement component (Q). The contribution from embedded intangibles (E) slightly decreases, a likely result

[29]For details see Appendix B.6

[30]The markup dispersion term M differs slightly from Peters (2020). Even if markups were constant across superstar firms, dispersion between superstars and fringe firms would still exist, governed by the market share ϕ of superstar firms.

of the policy boosting investment in transferable technology (horizontal innovation) and making embedded investment less critical for market entry. Furthermore, the markup dispersion term (M) approaches one, and the residual term (S) increases. These results suggest that policy attention should focus on lowering entry barriers. Enabling frictionless reallocation following horizontal innovations substantially raises aggregate output, primarily through higher quality growth, and therefore can materially reduce misallocation in the economy.

Table 3. Misallocation

	Y	Q	E	M	S	g
Base Scenario	2.512	1.923	0.691	0.995	1.900	0.007
Shut Down Entry Barrier	7.148	4.588	0.673	1.000	2.314	0.017

6.2 Policy Implication

Based on the frictions discussed in Section 6.1, I analyze three distinct tax regimes. First, I implement a size-dependent tax on profit starting at 10% and gradually increasing to 12.5%. Second, I consider a flat tax of 11.3% specifically on investments in embedded intangibles and horizontal expansion. Third, I examine the joint implementation of both tax policies simultaneously. In all cases, the government collects the tax revenue and redistributes it to households through lump-sum transfers. To evaluate the welfare effects of these policies, I employ a consumption-equivalent measure[31] This metric quantifies the permanent percentage change in consumption that would make households indifferent between the baseline economic path and the policy-induced path. The welfare results show substantial variation across policies. The size-dependent tax generates a welfare increase of 10.915%, accompanied by corresponding rises in consumption and output. In contrast, taxes targeting embedded and expansion investments yield more modest gains of 1.745% and 2.176%, respectively. The size-dependent tax's superior performance operates through two main channels. First, it reduces market segmenta-

[31]See Appendix B.7 for details on the consumption-equivalence welfare calculation.

tion by making firms less exposed to diminishing returns from managerial span of control. Second, it attenuates incentives to over-invest in embedded intangibles by reducing cross-product synergies, ultimately lowering barriers to entry. The joint implementation of all taxes demonstrates significant positive synergies. The total welfare increase of 16.237% exceeds the sum of individual contributions, indicating that the policies work together to mitigate economic distortions and enhance aggregate efficiency relative to the baseline economy.

Table 4. Welfare and Output Effects at Varying Tax Levels

	Tax Type			
	Size-Dependent Tax [0.1, 0.105, 0.11, 0.115, 0.12, 0.125]	Embedded Inv Tax 0.113	Expanding Inv Tax 0.113	All Three Combined
Δ Welfare (%)	10.915	1.745	2.176	16.237
ΔC (%)	10.542	1.602	2.093	15.549
ΔY (%)	8.517	1.566	2.084	13.255

7 Model Extension and Discussion

Fringe Firm Value Function - Extension. There are several ways to introduce frictions into the value function of fringe firms. I consider two examples. (i) The probability of drastic innovation may decrease with the embedded intangible gap: as a superstar firm increases its brand value and organizational capital, it becomes more difficult for fringe firms to innovate successfully. In this case, entry barriers become sharper than in the baseline, intensifying downward pressure on growth and fluidity across production lines. (ii) Alternatively, an additional cost parameter $\eta \in (0, 1)$ can be introduced, such that a larger gap in quality or embedded intangibles reduces the benefit of becoming a superstar. At the same time, when a superstar firm expands its number of production lines, the span-of-control constraint reduces its market share in each line, thereby encouraging fringe entry. Consequently, the second friction has a smoother effect on firm dynamics than the first scenario. The formal expression for the value function of a fringe firm is given by:

$$
\begin{aligned}
r_t V_t^f(m_j, e_s, n_s) - \dot{V}_t^f(m_j, e_s, n_s) = \max_{z_{fjt}} \Bigg\{ & \underbrace{\overbrace{p_s^{Ex}}^{\text{Prob of Dras. Inv.}} z_{fjt}\big[V_t(1,1,1) - V_t^f(m_j, e_s, n_s) - \overbrace{\eta V_t(m,k,n)}^{\text{Diff of Dras. Inv.}}\big]}_{\text{Fringe Firm Innovation}} \\
& + \underbrace{p_{s \geq s'}^{\text{Ex}} z_{sjt}^{Ex}\big[V_t^f(m_j, e_s, n_s+1) - V_t^f(m_j, e_s, n_s)\big]}_{\text{superstar expansion}} + \underbrace{z_{sjt}^{\text{Int}}\big[V_t^f(m_j+1, e_s, n_s) - V_t^f(m_j, e_s, n_s)\big]}_{\text{superstar innovation on transferable}} \\
& + \underbrace{z_{sjt}^{\text{Emb}}\big[V_t^f(m_j, e_s+1, n_s) - V_t(m_j, e_s, n_s)\big]}_{\text{superstar innovation on embedded}} + \underbrace{p_{s' \geq s}^{\text{Ex}} Z_j^{Ex} \mathbb{E}_{s' \geq s}\Big[V_t^f(1, e_{s'}, n_{s'}) - V_t^f(m_j, e_s, n_s)\Big]}_{\text{Superstar in other industries innovation}} \\
& - \gamma^f \left(z_{sjt}^{Fri}\right)^{\vartheta^f} Y_t \Bigg\}.
\end{aligned}
$$

Fringe firms produce at a price equal to marginal cost; therefore, unlike superstar firms, they cannot generate profit. The right-hand side of the equation represents a fringe firm with a flow rate of $p_s^{Ex} z_{fjt}$ that successfully innovates and becomes a new superstar with one gap level. Here, p_s^{Ex} captures the first type of friction: as the embedded intangible level increases, the probability of successful innovation by a fringe firm decreases. By contrast, the red-highlighted additional term represents the second type of friction, in which the drastic innovation cost parameter η reduces the value of fringe firm innovation. The first term in the second line represents the superstar firm expanding its production line, which increases the marginal cost of each existing production line; the fringe firm's value function also increases. The next two terms represent the superstar firm improving the productivity gap on its existing production line and increasing its level of embedded intangible assets. The third line of the second term indicates that superstars in other industries successfully innovate. The last term captures the cost of innovation required to become a new superstar.

Discussion on Size and Age. Firms shifting their investment ratios are associated with declines in productivity, markups, the growth rate, and fluidity, all of which correlate with firm age and size. However, as shown empirically in Figure A3 and within the model, these dynamics cannot be fully explained by age and size alone. From a theoretical side, when a superstar firm expands horizontally, its size increases. Yet horizontal expansion is not the only path to growth; a firm may also expand vertically by improving quality within existing product lines. Similarly, firm age is inversely related to fluidity, as firms with more product lines extend their life cycle while fluidity decreases.

Nevertheless, it is possible to observe superstar firms of the same age with one operating a single product line while another spans multiple lines. Likewise, a vertically improving firm may attain the same size as a horizontally expanding firm. Even if such firms similar age and size, their investment ratios, productivity, markups, and growth dynamics differ fundamentally. The reason is that, although age and size are correlated with these declining variables in the model, they are not the causal drivers. Instead, the model identifies two essential mechanisms: horizontal expansion under span-of-control constraints, and increasing returns to embedded intangibles. It is through these channels that superstar firms become multiproduct producers and reduce fluidity via intangible investment, thereby explaining the stylized facts that age and size alone cannot account for.

8 Conclusion

This paper proposes that, as firms diversify across segments, their markups, productivity, and the ratio of transferable to firm-specific intangible investment decline. To explain this, I develop a unified endogenous growth model in which the strategic allocation of innovative effort emerges directly from firms' expansion decisions. Horizontal expansion introduces span-of-control frictions, which weaken managerial attention and reduce per-segment profitability. In response, multi-segment firms do not simply innovate less; they innovate differently. They strategically reallocate investment away from transferable intangibles, which generate social spillovers and fuel creative destruction, and toward embedded intangibles, which provide private firm-specific advantages. This shift allows firms to exploit cross-product synergies and higher entry barriers, thereby privatizing returns at a significant social cost: it reduces market fluidity and depresses the long-run engine of growth, which is quality-improving innovation.

Quantitative analysis reveals that this mechanism constitutes a major source of economic misallocation. Crucially, I find that the entry barriers created by embedded intangibles are far more consequential for welfare than pure markup dispersion. This finding underscores the need for policies to address this specific distortion. Counterfactual experiments suggest that well-designed interventions, such as a size-dependent tax on profits or a targeted tax on embedded investment, can effectively rebalance private incentives with social goals. By discouraging excessive, friction-inducing diversification and encouraging R&D, such policies can help realign the trade-off between diversification and growth to foster a more dynamic and productive economy.

This framework establishes a foundation for several routes of future research. First, the evolution of firm scope over the lifecycle would permit a richer analysis of dynamic market segmentation strategies and their long-term consequences for the direction and pace of innovation. Second, a formal welfare analysis comparing a decentralized equilibrium to a social planner's solution is a critical next step. Such a comparison would allow for the precise derivation of optimal policy instruments to counteract the inefficiencies and knowledge-spillover frictions engendered by diversification. Third, the model generates testable empirical predictions regarding how firms endogenously respond to span-of-control constraints. Micro-econometric work could investigate the relative efficacy of adaptive strategies such as organizational redesign, investments in information technology, and human capital accumulation. Finally, a critical question is whether embedded intangible capital slows idea diffusion by limiting spillovers. Answering this with richer microdata and dynamic structural methods will be key to guiding policies that balance the private gains from diversification with the broader social returns to innovation and competition.

REFERENCES

Acemoglu, Daron, Ufuk Akcigit, Harun Alp, Nicholas Bloom, and William R. Kerr (2018). "Innovation, Reallocation, and Growth". *American Economic Review*, 108 (11), 3450–3491.

Ackerberg, Daniel A., Kevin Caves, and Garth Frazer (2015). "Identification Properties of Recent Production Function Estimators". *Econometrica*, 83 (6), 2411–2451.

Aghion, Philippe, Antonin Bergeaud, Timo Boppart, Peter J. Klenow, and Huiyu Li (2023). "A Theory of Falling Growth and Rising Rents". *Review of Economic Studies*, 90 (6), 2675–2702.

Aghion, Philippe, Christopher Harris, Peter Howitt, and John Vickers (2001). "Competition, Imitation and Growth with Step-by-Step Innovation". *The Review of Economic Studies*, 68 (3), 467–492.

Aghion, Philippe and Peter Howitt (1992). "A Model of Growth Through Creative Destruction". *Econometrica*, 60 (2), 323–351.

Akcigit, Ufuk and Sina T. Ates (2021). "Ten facts on declining business dynamism and lessons from endogenous growth theory". *American Economic Journal: Macroeconomics*, 13 (1), 257–298.

— (2023). "What happened to US business dynamism?" *Journal of Political Economy*, 131 (8), 2059–2124.

Akcigit, Ufuk and William R. Kerr (2018). "Growth through Heterogeneous Innovations". *Journal of Political Economy*, 126 (4), 1374–1443.

Autor, David, David Dorn, Lawrence F. Katz, Christina Patterson, and John Van Reenen (2020). "The fall of the labor share and the rise of superstar firms". *The Quarterly Journal of Economics*, 135 (2), 645–709.

Bandiera, Oriana, Andrea Prat, Raffaella Sadun, and Julie M. Wulf (2014). "Span of Control and Span of Attention". Harvard Business School Strategy Unit Working Paper No. 12-053; Columbia Business School Research Paper No. 14-22; posted April 30, 2014; originally dated December 2011.

Baslandze, Sukhrob, Jeremy Greenwood, Rodrigo Marto, and Sérgio Moreira (2023). "The Expansion of Varieties in the New Age of Advertising". *Review of Economic Dynamics*, 50, 171–210.

Berlingieri, Giuseppe, Maarten De Ridder, Danial Lashkari, and Davide Rigo (2025). “Creative Destruction through Innovation Bursts”. Working paper, April 2025.

Bloom, Nicholas and John Van Reenen (2007). “Measuring and Explaining Management Practices Across Firms and Countries”. *The Quarterly Journal of Economics*, 122 (4), 1351–1408.

Carlin, Bruce I., Bhashkar Chowdhry, and Mark J. Garmaise (2012). “Investment in Organization Capital”. *Journal of Financial Intermediation*, 21 (2), 268–286.

Casal, Lucía (2024). “Lock-In and Productive Innovations”. Working paper.

Cavenaile, L., M. A. Celik, P. Roldan-Blanco, and X. Tian (2025). “Style over Substance? Advertising, Innovation, and Endogenous Market Structure”. *Journal of Monetary Economics*, 149, 103683.

Cavenaile, Laurent, Murat Alp Celik, Jesse Perla, and Pau Roldan-Blanco (2025). “A Theory of Dynamic Product Awareness and Targeted Advertising”. Working paper.

Cavenaile, Laurent and Pablo Roldan-Blanco (2021). “Advertising, innovation, and economic growth”. *American Economic Journal: Macroeconomics*, 13 (3), 251–303.

Chiavari, Andrea and Sampreet S. Goraya (Apr. 2025). *The Rise of Intangible Capital and the Macroeconomic Implications*. Discussion Paper Series, Working Paper 1078. Working paper; “The Rise of Intangible Capital and the Macroeconomic Implications”, Discussion Paper Series 1078. Department of Economics, University of Oxford.

Crouzet, Nicolas and Janice C. Eberly (2019). *Understanding weak capital investment: The role of market concentration and intangibles*. Tech. rep. w25869. National Bureau of Economic Research.

De Loecker, Jan, Jan Eeckhout, and Gabriel Unger (2020). “The rise of market power and the macroeconomic implications”. *The Quarterly Journal of Economics*, 135 (2), 561–644.

De Ridder, Maarten (2024). “Market power and innovation in the intangible economy”. *American Economic Review*, 114 (1), 199–251.

Edmond, Chris, Virgiliu Midrigan, and Daniel Yi Xu (2023). “How Costly Are Markups?” *Journal of Political Economy*, 131 (7), 1619–1675.

Eisfeldt, Andrea L. and Dimitris Papanikolaou (2013). “Organization capital and the cross-section of expected returns”. *The Journal of Finance*, 68 (4), 1365–1406.

Gandhi, Amit, Salvador Navarro, and David Rivers (2020). "On the Identification of Production Functions: How Heterogeneous is Productivity?" *Journal of Political Economy*, 128 (11), 4135–4182.

Garcia-Macia, Diego, Chang-Tai Hsieh, and Peter J. Klenow (2019). "How Destructive is Innovation?" *Econometrica*, 87 (5), 1507–1541.

Garicano, Luis (2000). "Hierarchies and the Organization of Knowledge in Production". *Journal of Political Economy*, 108 (5), 874–904.

Grossman, Gene M. and Elhanan Helpman (1991). "Trade, knowledge spillovers, and growth". *European Economic Review*, 35 (2–3), 517–526.

Hoberg, Gerard and Gordon M. Phillips (2025). "Scope, Scale, and Concentration: The 21st-Century Firm". *Journal of Finance*, 80 (1), 415–466.

Hoberg, Gerard, Gordon M. Phillips, and Nagpurnanand Prabhala (2014). "Product Market Threats, Payouts, and Financial Flexibility". *Journal of Finance*, 69 (1), 293–324.

Hsieh, Chang-Tai and Peter J. Klenow (2009). "Misallocation and Manufacturing TFP in China and India". *The Quarterly Journal of Economics*, 124 (4), 1403–1448.

Jo, Karam and Seula Kim (July 2024). "Competition, Firm Innovation, and Growth under Imperfect Technology Spillovers". CES Working Paper No. 24-40.

Jovanovic, Boyan (2025). "Robert Lucas's Models of Firm Size Distributions and Investment". *Journal of Political Economy*.

Klette, Tor Jakob and Samuel Kortum (2004). "Innovating Firms and Aggregate Innovation". *Journal of Political Economy*, 112 (5), 986–1018.

Kogan, Leonid, Dimitris Papanikolaou, Amit Seru, and Nathan Stoffman (2017). "Technological innovation, resource allocation, and growth". *The Quarterly Journal of Economics*, 132 (2), 665–712.

Levinsohn, James and Amil Petrin (2003). "Estimating Production Functions Using Inputs to Control for Unobservables". *The Review of Economic Studies*, 70 (2), 317–341.

Lucas, Robert E. (1978). "On the size distribution of business firms". *The Bell Journal of Economics*, 508–523.

Olmstead-Rumsey, Jennifer (2019). "Market Concentration and the Productivity Slowdown". MPRA Paper No. 93260, 2019.

Pearce, Jeremy and Liangjie Wu (July 2025). “Market Concentration and Aggregate Productivity: The Role of Demand”. Federal Reserve Bank of New York, Staff Report No. 1159, July 2025.

Peters, Michael (2020). “Heterogeneous Markups, Growth, and Endogenous Misallocation”. *Econometrica*, 88 (5), 2037–2073.

Peters, Ryan H. and Lucian A. Taylor (2017). “Intangible capital and the investment-q relation”. *Journal of Financial Economics*, 123 (2), 251–272.

Prescott, Edward C. and Mark Visscher (1980). “Organization Capital”. *Journal of Political Economy*, 88 (3), 446–461.

Restuccia, Diego and Richard Rogerson (2008). “Policy Distortions and Aggregate Productivity with Heterogeneous Establishments”. *Review of Economic Dynamics*, 11 (4), 707–720.

Romer, Paul M. (1990). “Endogenous technological change”. *Journal of Political Economy*, 98 (5, Part 2), S71–S102.

Smeets, Valerie, Michael Waldman, and Frederic Warzynski (2019). “Performance, Career Dynamics, and Span of Control”. *Journal of Labor Economics*, 37 (4), 899–932.

Weiss, Joshua (2020). “Intangible Investment and Market Concentration”. Working paper, January 2020.

Appendices

A Empirical Appendix

A.1 Dataset and Measurement Details

Table A1. Example Firms Segment in Compustat Segment Dataset

Company	Segments
TOYOTA MOTOR CORP	Financial Services Automotive All Other
PROCTER & GAMBLE CO	Health Care Grooming Corporate Beauty Baby, Feminine & Family Care Fabric & Home Care
TESLA INC	Energy Generation & Storage Automotive

Table A2. Example Firm Segment in HP Firm Scope Dataset

Company	Segments
TESLA INC	Batteries Automotive (Brakes, Trim, Axle, Engines, Chassis) Automotive Safety (Airbags) Car Dealerships Energy / Cogeneration Utilities / Electric Power Smart Metering / Grid Tech Power Electronics / Voltage Solar / Renewable Energy Hardware & Software Solutions Ticketing / Scanning (Software/Systems)

Note: data source, Hoberg and Phillips, 2025

Measurement Details. The sample covers 1990–2019. The finance and utilities sectors are excluded from all analyses, and estimations are conducted separately for each 2-digit NAICS industry. I employ the approach of Gandhi, Navarro, and Rivers (2020) to estimate firm-level total factor productivity. The gross-output production function is specified as

$$Y_{it} = F(L_{it}, K_{it}, M_{it}) + \omega_{it} + \epsilon_{it}, \tag{47}$$

where Y_{it} is (log) gross output (`sale`), L_{it} is labor (`emp`), K_{it} is capital (`ppegt`), and M_{it} denotes flexible (intermediate) inputs (`cogs`). The term ω_{it} denotes an unobserved firm productivity shock that is observed by firms when choosing inputs, and ϵ_{it} is an iid error term.

A central identification challenge is that the presence of flexible inputs creates a non-identification problem for nonparametric gross-output production functions when standard proxy-variable approaches are applied, because flexible-input choices reflect contemporaneous productivity. They resolve this by exploiting a transformation of the firm's short-run first-order condition for intermediates to obtain cross-equation restrictions that isolate the flexible-input contribution and thus permit nonparametric identification of the production function and input elasticities.

Empirical proxy (material share): Define the intermediate share

$$s_{it} \equiv \frac{\texttt{COGS}_{it}}{\texttt{Sales}_{it}},$$

with both numerator and denominator deflated with `cpi`. GNR show that s_{it} is the empirical moment implied by the transformed FOC and can be used to recover the flexible-input elasticity nonparametrically[32].

First stage (nonparametric share regression): Apply the GNR transformation of the FOC for intermediates and estimate the resulting relation between s_{it} and the observable state variables nonparametrically. This yields an observation-level flexible-input elasticity $\hat{\beta}_{m,it}$

[32] Estimating using methodology by Ackerberg, Caves, and Frazer (2015) and Levinsohn and Petrin (2003) methodology, I replace proxy variable revenue share with capital expenditure (`capx`).

Second stage (fixed-input elasticities and TFP): With $\hat{\beta}_{m,it}$ in hand, identifies the remaining input elasticities $\hat{\beta}_l, \hat{\beta}_k$ using the cross-equation restrictions and standard Markov term $\omega_{it} = g(\omega_{it-1}) + \xi_{it}$. Construct firm TFP as the residual

$$\hat{\omega}_{it} = Y_{it} - \hat{\beta}_l L_{it} - \hat{\beta}_k K_{it} - \hat{\beta}_{m,it} M_{it}.$$

A.2 Additional Figures and Empirical Results

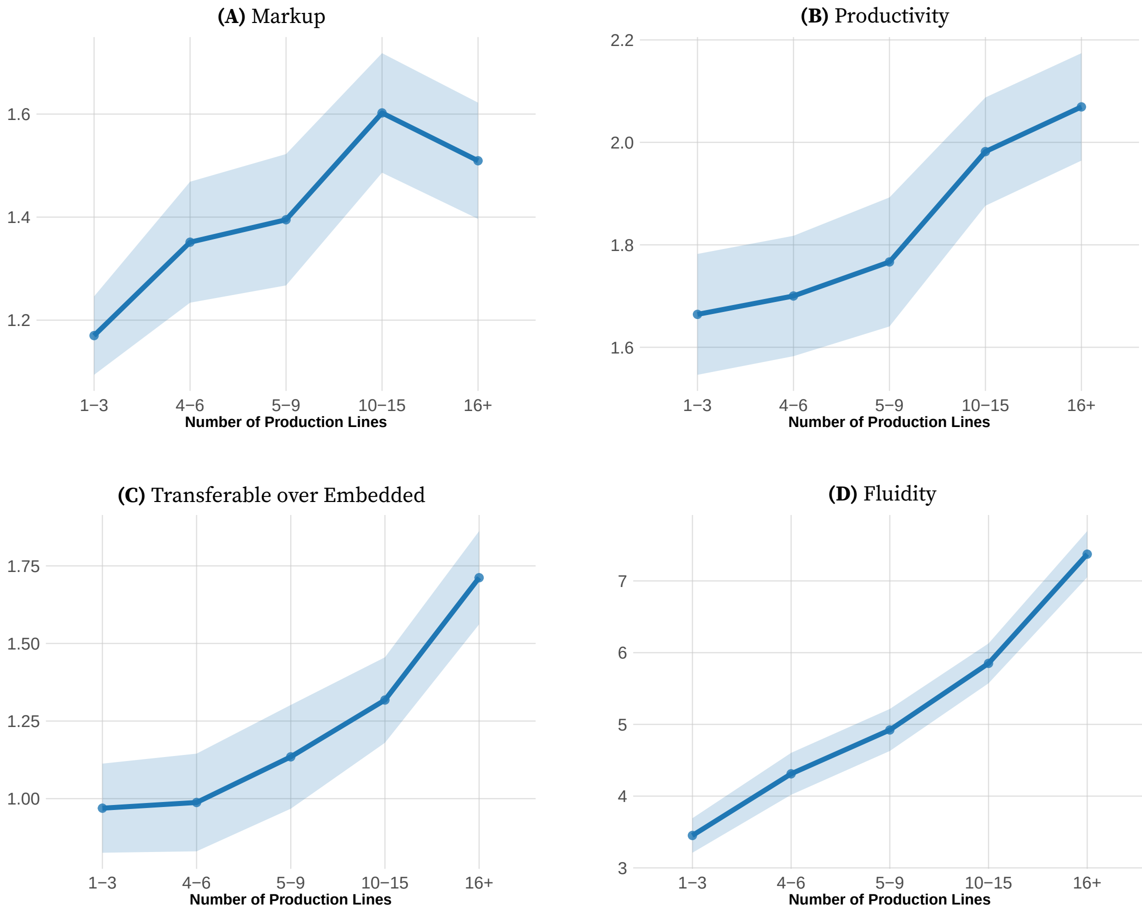


Figure A1. Markup, Productivity, Investment Ratio and Fluidity with HP Firm Scope Dataset

Note: The sample excludes utilities and finance sectors, as well as firms with missing or non-positive R&D and SG&A. Markups, the investment ratio, and market fluidity are measured for the 2019 cross-section. All variables are winsorized at the 95th percentile. Firms are grouped by their number of production lines (1–3, 4–6, 7–9, 10–15, and 16+). The plotted values are the averages within each bin.

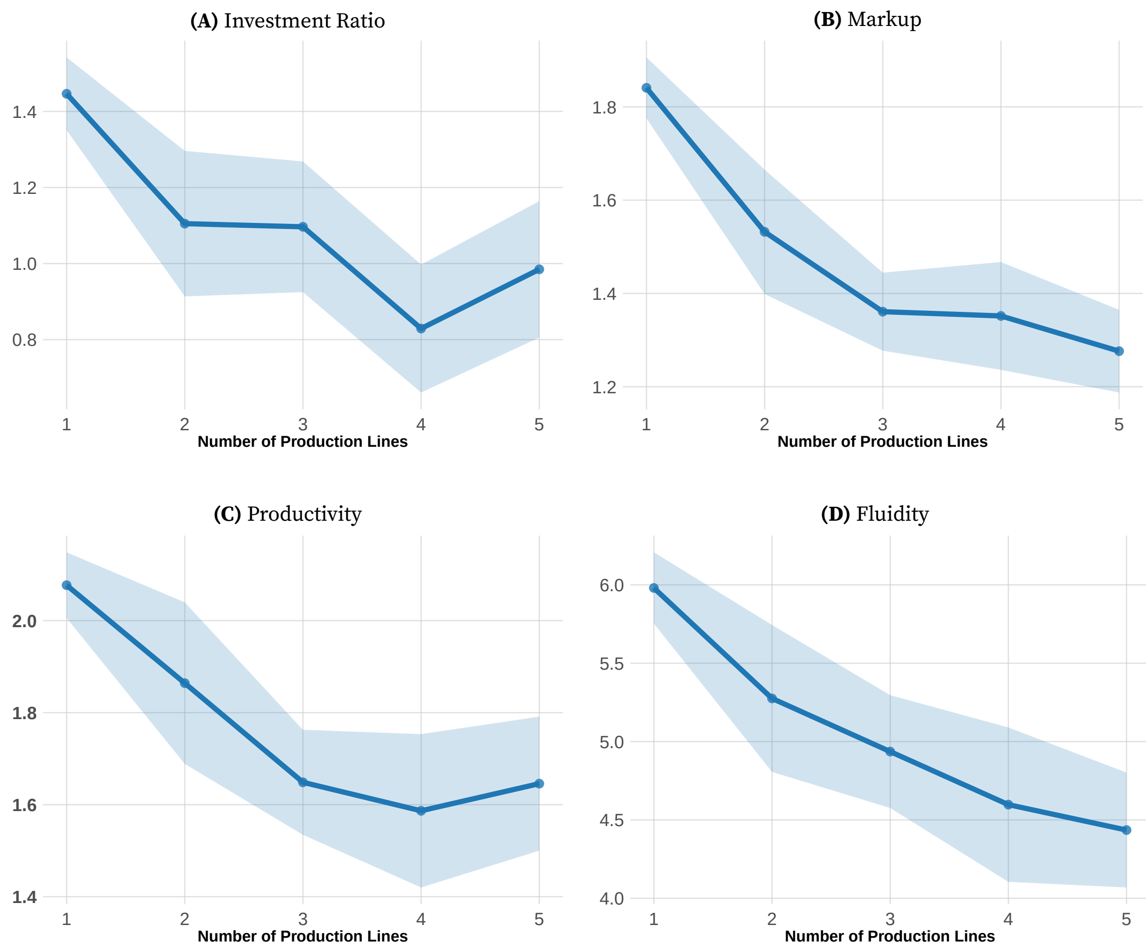
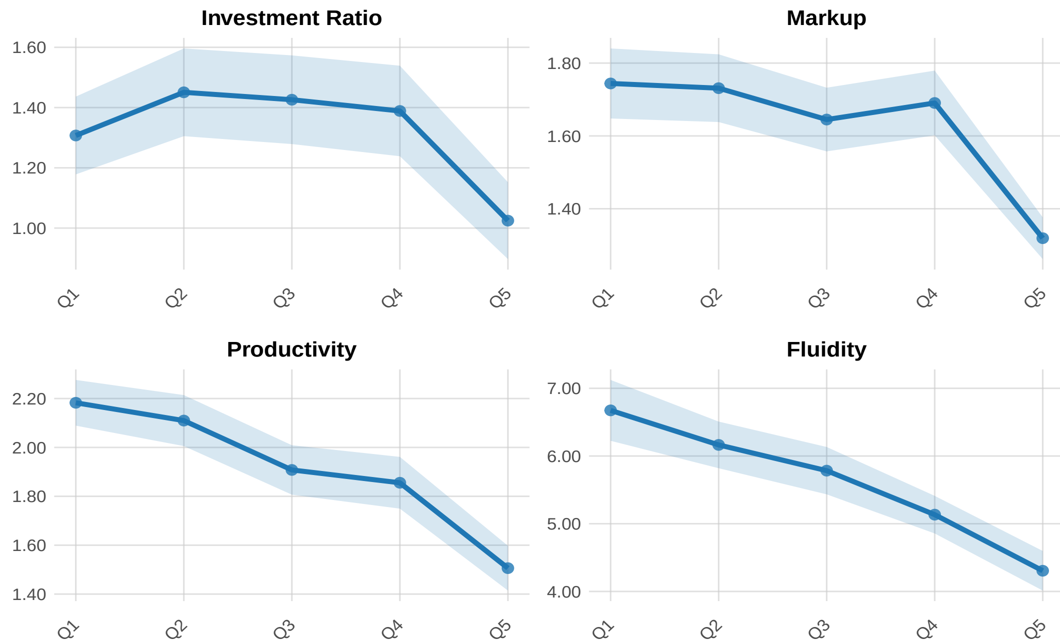


Figure A2. US Firms; Markup, Productivity, Investment Ratio and Fluidity by Production Lines

Note: The sample excludes utilities and finance sectors, as well as firms with missing or non-positive R&D and SG&A. Variables are measured for the 2019 cross-section and winsorized at the 95th percentile.

(A) Firm Age Quintile



(B) Firm Size Quintile

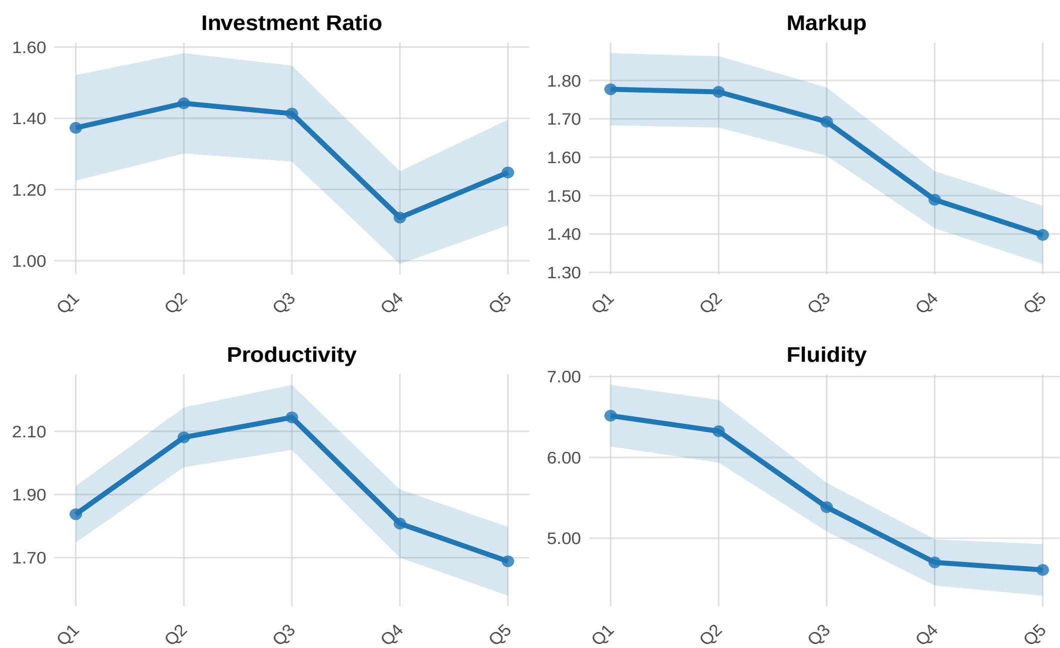


Figure A3. Firm Age and Size

Note: The sample excludes utilities and finance sectors, as well as firms with missing or non-positive R&D and SG&A. Markups, the investment ratio, and market fluidity are measured for the 2019 cross-section. All variables are winsorized at the 95th percentile. Firm size is measured by total employment (EMP), and firm age is measured by years since IPO. Firms are grouped into quantiles; the plotted values represent the average within each quantile.

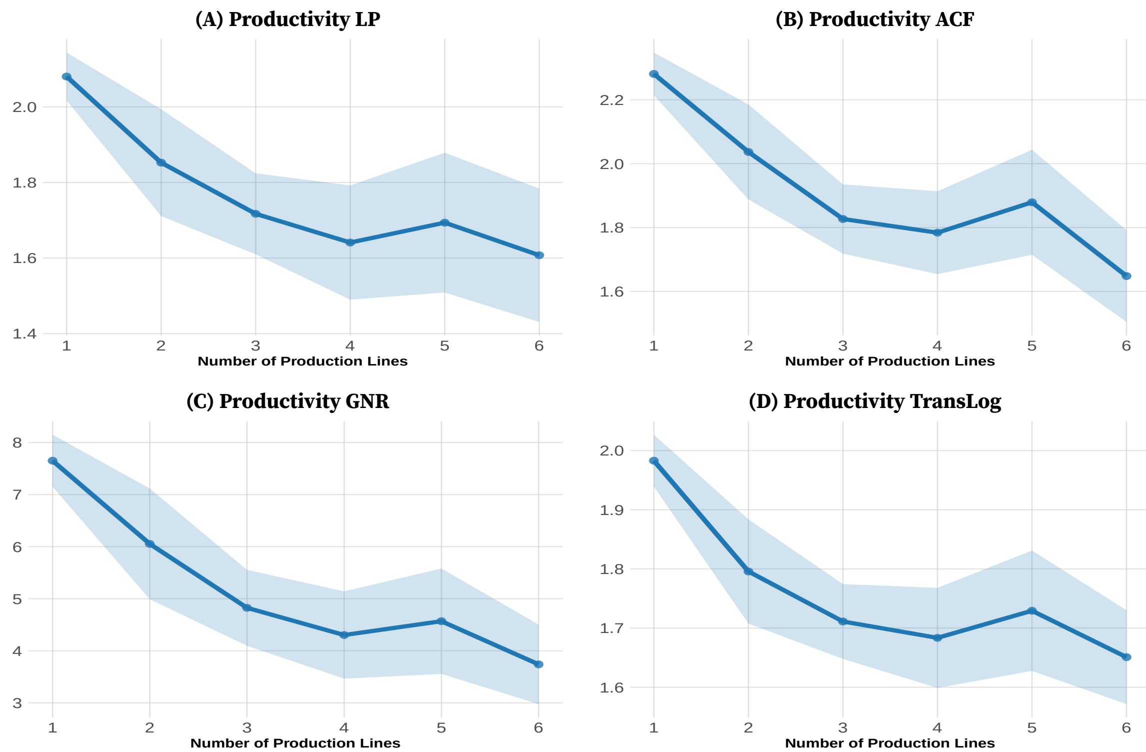


Figure A4. Productivity Measures with Different Methods

Note: The sample excludes utilities and finance sectors, as well as firms with missing or non-positive R&D and SG&A. All variables are winsorized at the 95th percentile. Productivities are measured for the 2019 cross-section. LP: Levinsohn and Petrin (2003); ACF: Ackerberg, Caves, and Frazer (2015); GNR: Gandhi, Navarro, and Rivers (2020)

Table A3. Summary Statistics

Variable	Summary Statistics						
	Mean	SD	Median	P10	P25	P75	P90
Investment ratio	1.300	1.310	0.873	0.129	0.317	1.760	3.290
R&D to Sales ratio	0.143	0.187	0.0715	0.006 93	0.0202	0.185	0.364
Markup	1.630	0.830	1.320	0.870	1.020	1.990	3.430
Productivity	1.910	0.972	1.730	0.754	1.010	2.770	3.300
Log sales	13.100	2.570	13.200	9.840	11.400	14.900	16.400
Log total assets	13.600	2.520	13.600	10.300	11.800	15.300	16.800
Log employees	7.490	2.310	7.530	4.380	5.820	9.130	10.500
Number of unique firms				1,711			

This table reports summary statistics for firm characteristics and the main variables used in the paper. The investment ratio of Transferable over Embedded is defined in Section 2. Investment ratio, R&D-to-sales, and productivity are winsorized at the 95th percentile, while markup is winsorized at the 90th percentile. All other variables are presented in logarithmic form.

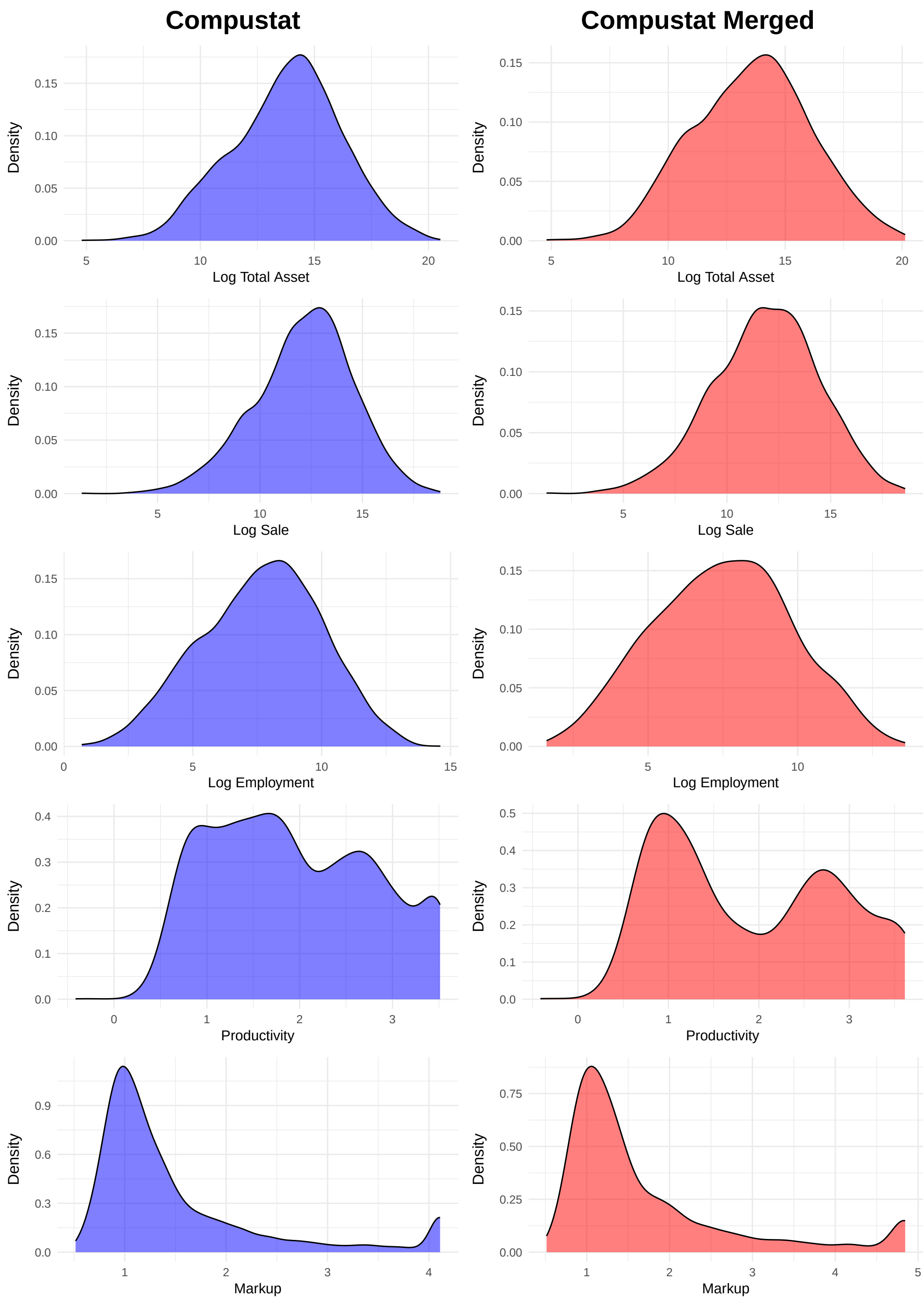


Figure A5. Density Comparison Compustat and Compustat Merged

Note: Both datasets exclude utilities and finance sectors, and all variables are measured for the 2019 cross-section. The Compustat merged dataset further excludes firms with missing or non-positive R&D, SG&A, or segment information.

Table A4. Regression Results: Compustat Dataset

Panel A: *Markup* and *Productivity*

	Pooled OLS		Two-way FE	
	(2) *Markup*	(3) *Productivity*	(6) *Markup*	(7) *Productivity*
Production Lines	-0.099^{***}	-0.056^{***}	-0.111^{***}	-0.032^{***}
	(0.007)	(0.010)	(0.008)	(0.004)
Num. Obs.	40,517	40,517	40,517	40,517
Adj. R^2	0.042	0.025	0.111	0.808
Covariates	Yes	Yes	Yes	Yes
FE: *Year*	No	No	Yes	Yes
FE: *Industry*	No	No	Yes	Yes

Panel B: *Investment Ratio* and *Fluidity*

	Pooled OLS		Two-way FE	
	(1) *Transferable/Embedded*	(4) *Fludity*	(5) *Transferable/Embedded*	(8) *Fludity*
Production Lines	-0.092^{***}	-0.449^{***}	-0.100^{***}	-0.302^{***}
	(0.008)	(0.031)	(0.009)	(0.033)
Num. Obs.	41,071	30,403	41,071	30,403
Adj. R^2	0.085	0.043	0.127	0.169
Covariates	Yes	Yes	Yes	Yes
FE: *Year*	No	No	Yes	Yes
FE: *Industry*	No	No	Yes	Yes

Notes: Each column reports coefficients from a separate regression. Standard errors clustered by *firm id* in parentheses. Significance levels: $^{*}p < 0.05$, $^{**}p < 0.01$, $^{***}p < 0.001$. Columns 1–4: Pooled OLS specifications; Columns 5–8: Two-way fixed effects (year and industry).

Table A5. Regression Results: Hoberg - Phillips Dataset:

Panel A: *Markup* and *Productivity*

	Pooled OLS		Two-way FE	
	(2) *Markup*	(3) *Productivity*	(6) *Markup*	(7) *Productivity*
Production Line	0.030^{***}	0.015^{***}	0.020^{***}	0.009^{***}
	(0.005)	(0.002)	(0.005)	(0.001)
Num. Obs.	55,250	55,250	55,250	55,250
Adj. R^2	0.001	0.029	0.006	0.778
Covariates	Yes	Yes	Yes	Yes
FE: *Year*	No	No	Yes	Yes
FE: *Industy*	No	No	Yes	Yes

Panel B: *Investment Ratio* and *Fluidity*

	Pooled OLS		Two-way FE	
	(1) *Transferable/Embedded*	(4) *Fluidity*	(5) *Transferable/Embedded*	(8) *Fluidity*
Production Line	0.028^{***}	0.201^{***}	0.027^{***}	0.247^{***}
	(0.002)	(0.006)	(0.002)	(0.006)
Num. Obs.	55,514	50,116	55,514	50,116
Adj. R^2	0.101	0.163	0.137	0.344
Covariates	Yes	Yes	Yes	Yes
FE: *Year*	No	No	Yes	Yes
FE: *Industry*	No	No	Yes	Yes

Notes: Each column reports coefficients from a separate regression. Standard errors clustered by *firm id* in parentheses. Significance levels:$^{*}p < 0.05$, $^{**}p < 0.01$, $^{***}p < 0.001$. Columns 1–4: Pooled OLS specifications; Columns 5–8: Two-way fixed effects (year and industry).

B Model Appendix

B.1 Final Good Sector Demand

The final good sector's profit maximization problem is:

$$\max_{y_{jt}} \exp\left(\int_0^1 \ln y_{jt} dj\right) - \int_0^1 p_{jt} y_{jt} dj. \tag{48}$$

The first-order condition yields the inverse demand function for each intermediate good j:

$$p_{jt} = \frac{Y_t}{y_{jt}}. \tag{49}$$

B.2 Intermediate Good Sector Demand

The cost minimization problem for the intermediate sector j is:

$$\min_{y_{sjt}, y_{fjt}} \quad p_{sjt} y_{sjt} + p_{fjt} y_{fjt} \quad s.t. \quad y_{jt} = \left(\chi(e_s) y_{sjt}^{\varepsilon} + y_{fjt}^{\varepsilon}\right)^{\frac{1}{\varepsilon}}. \tag{50}$$

The first-order condition with respect to the superstar firm's output y_{sjt} is:

$$p_{sjt} = \lambda \chi(e_s) y_{jt}^{1-\varepsilon} y_{sjt}^{\varepsilon-1}, \tag{51}$$

where λ is the Lagrange multiplier. Raising both sides to the power $\frac{\varepsilon}{\varepsilon-1}$ and simplifying allows us to solve for the ideal price index p_j for sector j:

$$\lambda = \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{\varepsilon-1}{\varepsilon}} \equiv p_j. \tag{52}$$

Substituting $\lambda = p_j$ back into the first-order condition yields the inverse demand function faced by the superstar firm:

$$p_{sjt} = p_j \chi(e_s) y_{jt}^{1-\varepsilon} y_{sjt}^{\varepsilon-1}. \tag{53}$$

Finally, substituting the final good producer's demand $y_{jt} = Y_t/p_j$ and solving for y_{sjt} provides demand function:

$$y_{sjt} = p_j^{\frac{\varepsilon}{1-\varepsilon}} \chi(e_s)^{\frac{1}{1-\varepsilon}} p_{sjt}^{\frac{1}{\varepsilon-1}} Y_t. \tag{54}$$

B.3 Superstar Firm Maximization Problem: Bertrand Competition

The superstar firm competes à la Bertrand with a continuum of fringe firms. Its profit maximization problem in industry j is:

$$\max_{p_{sjt}} (p_{sjt} - MC_{sjt})\, y_{sjt} \quad s.t. \quad y_{sjt} = p_j^{\frac{\varepsilon}{1-\varepsilon}} \chi(e_s)^{\frac{1}{1-\varepsilon}} p_{sjt}^{\frac{1}{\varepsilon-1}} Y_t. \tag{55}$$

Substituting the expression for p_j into the demand function, the objective function can be expanded as:

$$\max_{p_{sjt}} \left[p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} \chi(e_s)^{\frac{1}{1-\varepsilon}} Y_t \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{-1} - MC_{sjt} \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{-1} p_{sjt}^{\frac{1}{\varepsilon-1}} \chi(e_s)^{\frac{1}{1-\varepsilon}} Y_t \right]. \tag{56}$$

After computing the derivative and factoring common terms, this condition can be expressed as:

$$\begin{aligned} \frac{\partial \pi}{\partial p_{sjt}} = Y_t \chi(e_s)^{\frac{1}{1-\varepsilon}} \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{-1} \times \Bigg\{ \left[\frac{\varepsilon}{\varepsilon-1} p_{sjt}^{\frac{1}{\varepsilon-1}} - \mathrm{MC}_{sjt} \frac{1}{\varepsilon-1} p_{sjt}^{\frac{2-\varepsilon}{\varepsilon-1}} \right] \\ - \left(p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} - \mathrm{MC}_{sjt} p_{sjt}^{\frac{1}{\varepsilon-1}} \right) \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} \frac{\varepsilon}{\varepsilon-1} p_{sjt}^{\frac{1}{\varepsilon-1}} \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{-1} \right) \Bigg\} = 0. \end{aligned} \tag{57}$$

$$\begin{aligned} 0 = & \left[\frac{\varepsilon}{\varepsilon-1} p_{sjt}^{\frac{1}{\varepsilon-1}} - \mathrm{MC}_{sjt} \frac{1}{\varepsilon-1} p_{sjt}^{\frac{2-\varepsilon}{\varepsilon-1}} \right] \\ & - \left(p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} - \mathrm{MC}_{sjt} p_{sjt}^{\frac{1}{\varepsilon-1}} \right) \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} \frac{\varepsilon}{\varepsilon-1} p_{sjt}^{\frac{1}{\varepsilon-1}} \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{-1} \right). \end{aligned} \tag{58}$$

To simplify, multiply both sides by p_{sjt} and substitute the market share definition ϕ_{sjt} from (19), yielding:

$$\left(p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} - \mathrm{MC}_{sjt} p_{sjt}^{\frac{1}{\varepsilon-1}}\right)\left(\phi_{sjt}\frac{\varepsilon}{\varepsilon-1}\right) = \left[\frac{\varepsilon}{\varepsilon-1} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} - \mathrm{MC}_{sjt}\frac{1}{\varepsilon-1} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}}\right]. \tag{59}$$

First, divide both sides by $p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}}$, then by $\frac{\varepsilon}{\varepsilon-1}$. After rearranging terms, the expression simplifies to:

$$\varepsilon(1-\phi_{sjt}) = \frac{\mathrm{MC}_{sjt}}{p_{sjt}}(1-\varepsilon\phi_{sjt}). \tag{60}$$

Solving for the optimal price p_{sjt} gives:

$$p_{sjt} = \frac{1-\varepsilon\phi_{sjt}}{\varepsilon(1-\phi_{sjt})} \cdot \mathrm{MC}_{sjt}. \tag{61}$$

To determine the optimal labor demand, equating output (6) and demand (54) yields

$$q_{sjt}\,\psi(e_s,n_s)\,l_{sjt} = p_j^{\frac{\varepsilon}{1-\varepsilon}} \chi(e_s)^{\frac{1}{1-\varepsilon}} p_{sjt}^{\frac{1}{\varepsilon-1}} Y_t. \tag{62}$$

Multiplying both sides by p_{sjt} and dividing by w_t gives

$$p_{sjt}\underbrace{\frac{q_{sjt}\,\psi(e_s,n_s)}{w_t}}_{\text{inverse MC}_{sjt}} l_{sjt} = \underbrace{p_j^{\frac{\varepsilon}{1-\varepsilon}} \chi(e_s)^{\frac{1}{1-\varepsilon}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}}}_{\phi_{sjt}} \underbrace{\frac{Y_t}{w_t}}_{\omega_t^{-1}}. \tag{63}$$

This expression leads directly to equation (25).

B.4 Superstar Firm Maximization Problem: Cournot Competition

In the à la Cournot setup, a superstar firm and a continuum of fringe firms compete by choosing quantities to sell rather than engaging in price competition as in the à la Bertrand case. Its profit-maximization problem in industry j is

$$\max_{y_{sjt}} \pi_{sjt} = \max_{y_{sjt}} (p_{sjt} - MC_{sjt})\, y_{sjt}, \tag{64}$$

$$s.t. \quad p_{sjt} = \chi(e_s)\, y_{jt}^{-\varepsilon}\, y_{sjt}^{\varepsilon-1}\, Y_t, \quad \text{and} \quad y_{jt} = \left(\chi(e_s)\, y_{sjt}^{\varepsilon} + y_{fjt}^{\varepsilon}\right)^{1/\varepsilon}. \tag{65}$$

Differentiate the profit function with respect to y_{sjt}:

$$\frac{d\pi_{sjt}}{dy_{sjt}} = (p_{sjt} - MC_{sjt}) + y_{sjt}\,\frac{dp_{sjt}}{dy_{sjt}} = 0. \tag{66}$$

Differentiate the inverse demand (65) to obtain

$$\frac{dp_{sjt}}{dy_{sjt}} = p_{sjt}\left(-\varepsilon\frac{1}{y_{jt}}\frac{dy_{jt}}{dy_{sjt}} + \frac{\varepsilon - 1}{y_{sjt}}\right). \tag{67}$$

Differentiate y_{jt} with respect to y_{sjt}:

$$\frac{dy_{jt}}{dy_{sjt}} = \chi(e_s)\, y_{sjt}^{\varepsilon-1}\, y_{jt}^{1-\varepsilon}. \tag{68}$$

Substituting (68) and (65) into (66) yields

$$(p_{sjt} - MC_{sjt}) + y_{sjt}\, p_{sjt}\left(-\varepsilon\chi(e_s)\, y_{sjt}^{\varepsilon-1} y_{jt}^{-\varepsilon} + \frac{\varepsilon - 1}{y_{sjt}}\right) = 0. \tag{69}$$

Using the market-share definition

$$\frac{p_{sjt}\, y_{sjt}}{Y_t} = \chi(e_s)\, y_{sjt}^{\varepsilon}\, y_{jt}^{-\varepsilon},$$

and dividing both sides by p_{sjt}, rearrangement gives the inverse markup condition

$$\frac{MC_{sjt}}{p_{sjt}} = \varepsilon(1 - \phi_{sjt}). \tag{70}$$

The relative price ratio between fringe and superstar firms is therefore

$$\frac{p_{fjt}}{p_{sjt}} = \frac{1 - \phi_{sjt}}{\phi_{sjt}}\,\frac{MC_{fjt}}{MC_{sjt}}. \tag{71}$$

Using the inverse demand expressions leads to

$$\chi(e_s)\left(\frac{y_{fjt}}{y_{sjt}}\right)^{\varepsilon-1} = \frac{1 - \phi_{sjt}}{\phi_{sjt}}\,\frac{MC_{fjt}}{MC_{sjt}}. \tag{72}$$

Substituting the marginal-cost expressions for the fringe and superstar firms yields

$$\left(\frac{y_{fjt}}{y_{sjt}}\right)^{\varepsilon-1} = \frac{1}{\chi(e_s)} \frac{1-\phi_{sjt}}{\phi_{sjt}} \frac{1}{\lambda^{m_j}\,\psi(e_s, n_s)}. \tag{73}$$

Equation (73) shows that the relative output depends on the market share, the quality gap, the embedded intangible level, and the firm's production-line parameter. Rearranging the market-share definition gives

$$\chi(e_s)\, y_{sjt}^{\varepsilon}\, y_{jt}^{-\varepsilon} = \frac{\chi(e_s)\, y_{sjt}^{\varepsilon}}{\chi(e_s)\, y_{sjt}^{\varepsilon} + y_{fjt}^{\varepsilon}} = \frac{1}{1 + \frac{1}{\chi(e_s)}\left(\frac{y_{fjt}}{y_{sjt}}\right)^{\varepsilon}}, \tag{74}$$

which shows that market share depends on the output gap and the embedded intangible level. Therefore, the output gap depends on the quality gap, the embedded intangible level, and the number of production lines associated with superstar s.

B.5 Aggregate Output and Growth Rate

Using the superstar (6) and fringe firm (7) output equations into (3) gives

$$Y_t = \exp\left(\int_0^1 \ln\left[(\xi e_{st})^{\beta}\left(q_{sjt}\,\frac{((1-\xi)e_{st})^{\alpha}}{\gamma n_{st}^{\alpha_s}}\, l_{sjt}\right)^{\varepsilon} + \left(q_{fjt}\, l_{fst}\right)^{\varepsilon}\right]^{1/\varepsilon} dj\right)$$

$$Y_t = \exp\left(\int_0^1 \ln\left[\left(q_{sjt}^{\varepsilon}\left((\xi e_{st})^{\beta}\Big(\frac{((1-\xi)e_{st})^{\alpha}}{\gamma n_{st}^{\alpha_s}}\, l_{sjt}\Big)^{\varepsilon} + \left(\lambda^{-m_{jt}}\, l_{fst}\right)^{\varepsilon}\right)\right)\right]^{1/\varepsilon} dj\right)$$

$$= \underbrace{\exp\left(\int_0^1 \ln q_{sjt} dj\right)}_{Q_t} \exp\left(\int_0^1 \ln\left[((\xi e_{st})^{\beta}\left(\frac{((1-\xi)e_{st})^{\alpha}}{\gamma n_{st}^{\alpha_s}}\, l_{sjt}\right)^{\varepsilon} + \left(\lambda^{-m_{jt}}\, l_{fst}\right)^{\varepsilon}\right]^{\frac{1}{\varepsilon}} dj\right), \tag{75}$$

and , along with the labor demands in (25) and (26), and factoring out ω_t^{-1}, aggregate output can be expressed as

$$Y_t = Q_t\,\omega_t^{-1}\,\exp\left(\int_0^1 \ln\left[(\xi e_{st})^{\beta}\left(\frac{((1-\xi)e_{st})^{\alpha}}{\gamma n_{st}^{\alpha_s}}\frac{\phi_{sjt}}{\sigma_{sjt}}\right)^{\varepsilon} + \left(\lambda^{-m_{jt}}(1-\phi_{sjt})\right)^{\varepsilon}\right]^{\frac{1}{\varepsilon}} dj\right). \tag{76}$$

Since everything in the integrand depends only on the gaps (m, k, n), the expression can be written in discrete state space as

$$Y_t = Q_t\,\omega_t^{-1}\,\exp\left(\sum_{m=1}^{\bar{m}}\sum_{k=1}^{\bar{k}}\sum_{n=1}^{\bar{n}} \underbrace{\ln\left[(\xi\theta^{k})^{\beta}\left(\frac{((1-\xi)\theta^{kt})^{\alpha}}{\gamma n^{\alpha_s}}\frac{\phi_t(m,k,n)}{\sigma_t(m,k,n)}\right)^{\varepsilon} + \left(\lambda^{-mt}(1-\phi_t(m,k,n))\right)^{\varepsilon}\right]^{\frac{1}{\varepsilon}}}_{\equiv R_t(m,k,n)} \mu_t(m,k,n)\right). \tag{77}$$

$$\begin{aligned}\ln Y_{t+\Delta t} - \ln Y_t = {}& \big(\ln Q_{t+\Delta t} - \ln Q_t\big) + \ln\omega_t - \ln\omega_{t+\Delta t} \\ &+ \sum_{m=1}^{\bar{m}}\sum_{k=1}^{\bar{k}}\sum_{n=1}^{\bar{n}}\Big(R_{t+\Delta t}(m,k,n) - R_t(m,k,n)\Big)\Big(\mu_{t+\Delta t}(m,k,n) - \mu_t(m,k,n)\Big) + o(\Delta t).\end{aligned} \tag{78}$$

where

$$\begin{aligned}\ln Q_{t+\Delta t} - \ln Q_t = \ln\lambda\left[\sum_{m=1}^{\bar{m}}\sum_{k=1}^{\bar{k}}\sum_{n=1}^{\bar{n}}\Big(z_t^{Int}(m,k,n) + p^{\mathrm{Ex}}_{s\geq s'}\,z_t^{Ex}(m,k,n) + Z_t^{f}(m,k,n)\Big)\mu_t(m,k,n)\right]\Delta t \\ + o(\Delta t).\end{aligned} \tag{79}$$

Dividing by Δt and taking the limit $\Delta t \to 0$, the growth rate of the economy is

$$g_t = -g_{\omega,t} + g_{Q,t} + g_{R,t}. \tag{80}$$

In the steady state, the distribution $\mu_t(m, k, n)$ is constant, implying that R_t is constant. Moreover, wages grow at the same rate as output, so the real wage remains constant. Therefore, in steady state the growth rate of the economy is determined solely by quality improvements:

$$g = g_Q = \ln \lambda \left[\sum_{m=1}^{\bar{m}} \sum_{k=1}^{\bar{k}} \sum_{n=1}^{\bar{n}} \Big(z^{\mathrm{Int}}(m,k,n) + p^{\mathrm{Ex}}_{s \geq s'}\, z^{\mathrm{Ex}}(m,k,n) + Z^f(m,k,n) \Big) \mu(m,k,n) \right]. \tag{81}$$

B.6 Decomposition of Output

Starting from (76) and factoring out the term $(\xi e_{st})^{\beta}\left(\frac{((1-\xi)e_{st})^{\alpha}}{\gamma\, n_{st}^{\alpha_s}}\right)$, aggregate output can be written as

$$Y_t = Q_t\, \omega_t^{-1}\, \exp\left(\int_0^1 \ln \left[\left(\frac{\phi_{sjt}}{\sigma_{sjt}}\right)^{\varepsilon} + \frac{1}{(\xi e_{st})^{\beta}} \left(\frac{\gamma\, n_{st}^{\alpha_s}}{((1-\xi)e_{st})^{\alpha}}\, \lambda^{-m_{jt}} (1-\phi_{sjt}) \right)^{\varepsilon} \right]^{1/\varepsilon} dj \right). \tag{82}$$

Define the multiplicative factor that collects the factored-out terms as

$$E_t \;=\; \exp\left(\int_0^1 \ln \left[(\xi e_{st})^{\beta} \left(\frac{((1-\xi)e_{st})^{\alpha}}{\gamma\, n_{st}^{\alpha_s}} \right)^{\varepsilon} \right]^{1/\varepsilon} dj \right). \tag{83}$$

Next multiply and divide the integrand by the linear weight $\dfrac{\phi_{sjt}}{\sigma_{sjt}} + (1-\phi_{sjt})$. After this algebraic step I obtain a decomposition that isolates a simple mean term and a residual term:

$$\begin{aligned} Y_t = Q_t\, E_t\, \omega_t^{-1}\, &\exp\left(\int_0^1 \ln \left[\frac{\phi_{sjt}}{\sigma_{sjt}} + (1-\phi_{sjt}) \right] dj \right) \\ &\times \exp\left(\int_0^1 \ln \left[\frac{ \left(\frac{1}{\gamma n_{st}^{\alpha_s}} \frac{\phi_{sjt}}{\sigma_{sjt}} \right)^{\varepsilon} + \frac{1}{(\xi e_{st})^{\beta}} \left(\frac{1}{((1-\xi)e_{st})^{\alpha}} \lambda^{-m_{jt}} (1-\phi_{sjt}) \right)^{\varepsilon} }{ \frac{\phi_{sjt}}{\sigma_{sjt}} + (1-\phi_{sjt}) } \right]^{1/\varepsilon} dj \right). \end{aligned} \tag{84}$$

Finally, using (36) and defining the multiplicative mean term

$$M_t \;=\; \frac{ \exp\Big(\int_0^1 \ln \left[\frac{\phi_{sjt}}{\sigma_{sjt}} + (1-\phi_{sjt}) \right] dj \Big) }{ \int_0^1 \left[\frac{\phi_{sjt}}{\sigma_{sjt}} + (1-\phi_{sjt}) \right] dj }, \tag{85}$$

the output decomposition can be written as

$$Y_t \;=\; Q_t \times E_t \times M_t \times S_t,$$

where

$$S_t = \exp\left(\int_0^1 \ln\left[\frac{\left(\frac{1}{\gamma\, n_{st}^{\alpha_s}}\frac{\phi_{sjt}}{\sigma_{sjt}}\right)^{\varepsilon} + \frac{1}{(\xi e_{st})^{\beta}}\left(\frac{1}{((1-\xi)e_{st})^{\alpha}}\,\lambda^{-m_{jt}}(1-\phi_{sjt})\right)^{\varepsilon}}{\frac{\phi_{sjt}}{\sigma_{sjt}} + (1-\phi_{sjt})}\right]^{1/\varepsilon} dj\right). \tag{86}$$

B.7 Consumption Equivalence Welfare Measure

On the balanced growth path, consumption grows at rate g, so that $C(t) = C_0 \exp(gt)$.[33] Defining welfare Ω as the present value of lifetime utility from consumption yields:

$$\Omega = \int_0^{\infty} e^{-\rho t} \ln(C(t))\mathrm{d}t \tag{88}$$

$$= \ln(C_0)\int_0^{\infty} e^{-\rho t}\mathrm{d}t + g\int_0^{\infty} te^{-\rho t}\mathrm{d}t. \tag{89}$$

Solving these integrals gives:

$$\Omega = \frac{\ln(C_0)}{\rho} + \frac{g}{\rho^2} \tag{90}$$

$$= \frac{1}{\rho}\left(\ln C_0 + \frac{g}{\rho}\right). \tag{91}$$

Equivalent Welfare Changes Between Economies To compare welfare between two economies—a calibrated benchmark economy (Cal) and a taxed economy (Tax) on their respective balanced growth paths—I compute the percentage change δ in

[33] The C_0 consumption level is given by:

$$C_0 = Y_0 - \int_0^1 \left(I_0^{\mathrm{Int}} + I_0^{\mathrm{Emb}} + I_0^{\mathrm{Ex}} + I_0^{f}\right) dj + G_0. \tag{87}$$

In this equation, the subscript 0 represents calibrated optimum values on the balanced growth path, and G is the lump-sum transfer of government taxes.

lifetime consumption that would make households indifferent between the two. The required compensation δ satisfies:

$$\Omega^{\text{Tax}} = \frac{1}{\rho}\left(\ln\left(C_0^{\text{Cal}}(1+\delta)\right) + \frac{g^{\text{Cal}}}{\rho}\right). \tag{92}$$

Solving equation (92) for δ:

$$\frac{\ln C_0^{\text{Tax}}}{\rho} + \frac{g^{\text{Tax}}}{\rho^2} = \frac{\ln[C_0^{\text{Cal}}(1+\delta)]}{\rho} + \frac{g^{\text{Cal}}}{\rho^2} \tag{93}$$

$$\ln\left(\frac{C_0^{\text{Tax}}}{C_0^{\text{Cal}}}\right) + \frac{g^{\text{Tax}} - g^{\text{Cal}}}{\rho} = \ln(1+\delta) \tag{94}$$

$$\delta = \frac{C_0^{\text{Tax}}}{C_0^{\text{Cal}}}\exp\left(\frac{g^{\text{Tax}} - g^{\text{Cal}}}{\rho}\right) - 1. \tag{95}$$

If $\delta > 0$: households require compensation to remain in the benchmark economy (Cal).

If $\delta < 0$: households would pay to move to the taxed economy (Tax).

C Numerical Appendix

C.1 Additional Numerical Results

Table C1. Sensitivity Matrix

Parameter	Markup	Growth Rate	Investment Ratio	Innovation Rate
ε	-1.052%	-2.113%	-25.815%	-3.763%
α	-0.010%	-0.324%	-3.736%	-0.521%
α_s	0.000%	-0.027%	-0.043%	-0.062%
β	-0.002%	-0.064%	-0.862%	-0.106%
γ	-0.013%	-0.398%	-4.342%	-0.639%
ξ	-0.004%	-0.129%	-1.484%	-0.210%

Note: Each row reports the percentage change in variables resulting from a 1% change in the parameter value.

C.2 Solution Algorithm

This algorithm computes the balanced growth path with a three dimensional state space (m, k, n). The solution involves finding the value functions $v_s(m, k, n)$ and $v_f(m, k, n)$, the innovation rates $z^{\text{Int}}, z^{\text{Emb}}, z^{\text{Ex}}, z^f$, and the stationary distribution $\mu(m, k, n)$ that jointly satisfy the model's equilibrium conditions.

BGP Equilibrium Solution:

1. **Compute static values:** Calculate static market shares and profit values using equations (23) and (19).
2. **Initialization:** Initialize the value functions $v_s(m, k, n)$ and $v_f(m, k, n)$, and the stationary distribution $\mu(m, k, n)$.
3. **Step 1: Solve HJB Equations (Backward Iteration)**
   (a) Set $v^{\text{old}}(m, k, n)$.
   (b) Repeat until $\max |v^{\text{new}} - v^{\text{old}}| <$ tolerance:
      - Compute policy functions $x(m, k, n)$ from the FOCs using v^{old}.
      - Solve the discretized HJB equations for $v^{\text{new}}(m, k, n)$.
      - Update $v^{\text{old}} \leftarrow v^{\text{new}}$.
4. **Step 2: Solve the Kolmogorov Forward Equation (KFE)**
   (a) Set $\mu^{\text{old}}(m, k, n)$.
   (b) Repeat until $\max |\mu^{\text{new}} - \mu^{\text{old}}| <$ tolerance:
      - Solve the discretized KFE for $\mu^{\text{new}}(m, k, n)$ using the policy functions $z^{\text{Int}}, z^{\text{Emb}}, z^{\text{Ex}}, z^f$.
      - Update $\mu^{\text{old}} \leftarrow \mu^{\text{new}}$.
5. **Step 3: Repeat Steps Until Value Functions and Distribution Converge**

Finally, to determine the optimal parameter values, search over the parameter space to minimize the objective function,

$$\text{Minimize}(z) = \sum_{z=1}^{Z} \frac{\left|\text{model}(z) - \text{data}(z)\right|}{\frac{1}{2}\left|\text{model}(z)\right| + \frac{1}{2}\left|\text{data}(z)\right|}.$$