1. **Proposition.** Let $a, b, c \in \mathbb{N}$. If $a + b = a + c$, then $b = c$.

   Proof. Assume $a, b, c \in \mathbb{N}$. We will show that if $a + b = a + c$, then $b = c$ for any $a \in \mathbb{N}$. Consider the base case where $a = 0$. Thus we have that

   $$b + 0 = c + 0.$$

   Because $b + 0 = b$ and $c + 0 = c$, then it follows that $b = c$. Thus, the base case $a = 0$ is true. For the inductive step, assume that if $a + b = a + c$, then $b = c$ for some $a \in \mathbb{N}$. We will show that this is also true for the successor of a given by $S(a)$ so that if

   $$b + S(a) = c + S(a),$$

   then b and c are equal. We begin by using the axiomatic arithmetic definition on the left-hand side of the above equation to write that

   $$b + S(a) = S(b + a).$$

   Having already assumed that $b + a = c + a$, we can substitute to write that

   $$b + S(a) = S(c + a).$$

   Once again applying the axiomatic arithmetic definition, we then write that

   $$b + S(a) = c + S(a).$$

   Thus, we have that the statement also holds true for successors of a. Therefore, we have shown via induction that if $a + b = a + c$, then $b = c$ when a is a natural number greater than or equal to the base case of $a = 0$, as desired.

   $\square$

2. Let $a \in \mathbb{N}$.

   (a) **Proposition.** For some natural number a, $a + a = 2 \cdot a$.

   Proof. Let $a \in \mathbb{N}$. We know that $a \cdot 1 = a$, so we can substitute to write that

   $$a + a = a + a \cdot 1.$$

   The axiomatic definition of multiplication states that for $a, b \in \mathbb{N}$ it is true that $a \cdot S(b) = a + (a \cdot b)$. Then, we can write that

   $$a + a = a \cdot S(1).$$

   In the set of natural numbers $\mathbb{N}$, $S(1)$ is 2, so we can substitute to write that

   $$a + a = a \cdot 2.$$

   The commutativity theorem states that for all $a, b \in \mathbb{N}$, it follows that $a \cdot b = b \cdot a$. Thus,

   $$a + a = 2 \cdot a,$$

   for some $a \in \mathbb{N}$, as desired.

□

(b) **Proposition.** The n-fold sum $a + \ldots + a = n \cdot a$.

Proof. Let $a \in \mathbb{N}$ and let statement $A(n)$ be given by n-fold sum $a+\ldots+a = n\cdot a$. We will prove that $A(n)$ is true using proof by induction. We will use $A(0)$ as the base case. For $n = 0$, we have that the 0-fold sum on a is equivalent to 0; we also have that $0 \cdot a$ is equal to 0 by the axiomatic multiplication definition. By axiom 1, which states that for $x \in \mathbb{N}$, then $x = x$, we can write that $0 = 0$. Thus, the base case $A(0)$ is true.

Now we will show that the same is true when $n \geq 0$ by showing that the statement of equivalence given by $A(n + 1)$ is also true. In other words, we must show that the $(n+1)$-fold sum $a+\ldots+a$ is equal to $(n+1)\cdot a$. The statement $A(n)$ will serve as the inductive assumption. Beginning with the left-hand side of the statement A(n+1), we have an incrementation of one a to the n-fold sum on a. Thus, we write that

$$a + (a + \ldots + a) = a + n \cdot a.$$

Remember that $a \cdot 1 = a$. Thus we substitute on the right-hand side of the equation to reveal that

$$a + (a + \ldots + a) = a \cdot 1 + n \cdot a.$$

The theorem of distributivity states that for $a, b, c \in \mathbb{N}$, it is true that $a \cdot (b + c) = a \cdot b + a \cdot c$. Therefore, we can write that

$$a + (a + \ldots + a) = (n + 1) \cdot a.$$

Thus, we have arrived at the statement $A(n + 1)$: the $(n + 1)$-fold sum $a + \ldots + a$ is equal to $(n + 1) \cdot a$. Therefore, by inductive assumption, we can assume that the statement given by $A(n)$ is true for values of n greater than the base case of $n = 0$, as desired.

□

3. Let $a, b \in \mathbb{N}$. Define $a \leq b$ if and only if there exists some $c \in \mathbb{N}$ such that $a + c = b$.

(a) **Proposition.** Let $a, b, c \in \mathbb{N}$ such that $c \neq 0$ and $a = b \cdot c$. It follows that $b \leq a$.

Proof. Axiom 9 states every nonzero number is a successor to some other number. Since we have assumed that $c \neq 0$, then there exists some natural number n such that $S(n) = c$. Thus, $a = b \cdot c$ is the same as $a = b \cdot S(n)$. Then, using the definition of axiomatic arithmetic, we write that $a = b + b \cdot n$. Since b and n are natural numbers, they must be non-negative. In the case that $b \cdot n$ is 0, because we know that $b + 0 = b$, then $a = b$. By the theorem of commutativity, it follows that $b = a$. Thus, the case that $b \cdot n$ is 0 satisfies $b \leq a$ because the $\leq$ symbol asks if b is less than *or* equal to a. In the case that $b \cdot n$ is positive, then we have that a is less than b, because $b \cdot n$ must be added to b to make it equivalent to a. This case also satisfies $b \leq a$. Thus, we have shown that if $c \neq 0$ and $a = b \cdot c$, then it follows that $b \leq a$, as desired.

□

(b) **Proposition.** Let $a \in \mathbb{N}$. Then $a \leq a$. This is known as the *reflexive* property.

Proof. Axiom 1 states that for $x \in \mathbb{N}$, then $x = x$. Therefore, since $a \in \mathbb{N}$, we know that $a = a$. The $\leq$ symbol asks if a is less than *or* equal to a. Since we have already noted that a is equal to a, we satisfy the statement $a \leq a$, as desired.

□

(c) **Proposition.** Let $a, b, c \in \mathbb{N}$. If $a \leq b$ and $b \leq c$, then $a \leq c$. This is known as the *transitive* property.

Proof. Assume that $a \leq b \leq c \in \mathbb{N}$. Then, assume a natural number x such that $a + x = b$ and a natural number y such that $b + y = c$. By substituting for b, we can combines the equations to read that $(a + x) + y = c$. By the theorem of associativity, we can write that $a + (x + y) = c$. Thus, in the same vein as the steps in (a), because x and y are contained in the set of natural numbers $\mathbb{N}$, we know that they are non-negative integers. Therefore, their sum is either 0 or positive. In both cases $a \leq c$, as desired.

□

(d) **Proposition.** Let $a, b \in \mathbb{N}$. If $a \leq b$ and $b \leq a$, then $a = b$. This property is called *antisymmetry*.

Proof. We have that $a \leq b$ and $b \leq a$. Assume some $x, y \in \mathbb{N}$ so that $a + x = b$ and $b + y = a$. We will show that both x and y are 0, to the effect that $a = b$. First substitute the equation $a + x = b$ into the second equation, writing that $(a + x) + y = a$. By the theorem of associativity we have $a + (x + y) = a$. We already know that for $j \in \mathbb{N}$, $j + 0 = j$. Moreover, by subtracting a from both sides of the equation $a + (x + y) = a$, we reveal that $x + y = 0$. Thus, the sum of x and y is 0. Since x and y are contained in the set of natural numbers, they are both non-negative. We can further assume that they are both equal to 0 because it is impossible to write that $x + y = 0$ when either x or y are natural numbers not equal to 0.

In a brief proof by contradiction, we will assume to the contrary that $x > 0$. Thus, since $x \neq 0$, there exists some $n \in \mathbb{N}$ such that $S(n) = x$. Substituting, we write that $S(n) + y = 0$. Using the axiomatic definition of arithmetic, we can further state that $S(n + y) = 0$. However, since 0 cannot be the successor of any number (given by axiom 7), we have arrived at a contradiction. We are forced to assume that $x = 0$. (Since x and y are used similarly in the statement $x + y = 0$, we can assume that the same will be true for y without loss by generalization. However, the reader is invited to note that because $x = 0$ and $0 + y = y$, that y also is 0.)

Therefore, for the equations $a + x = b$ and $b + y = a$, we in fact have that $a + 0 = b$ and $b + 0 = a$. Once again, since $a + 0 = a$ and $b + 0 = b$, then $a = b$ and $b = a$. Thus, we have shown that if $a \leq b$ and $b \leq a$, then $a = b$, as desired.

□