1. **Proposition.** Let $r \neq 1$ be a real number. Using mathematical induction,

$$\sum_{j=0}^{n} r^j = \frac{1 - r^{n+1}}{1 - r}.$$

Proof. Assume $r \neq 1$ is contained in the set of real numbers $\mathbb{R}$. Let the statement $A(n)$ be given by

$$\sum_{j=0}^{n} r^j = \frac{1 - r^{n+1}}{1 - r}.$$

We will use mathematical induction to show that $A(n)$ is true for all $n \geq 0$. First we confirm that the base case A(0) is true. We have that

$$\sum_{j=0}^{0} r^j = r^0 = 1.$$

Moreover, since $r \neq 1$, we can say that

$$\frac{1 - r^{0+1}}{1 - r} = \frac{1 - r}{1 - r} = 1.$$

Thus A(0) is true because both sides of the equation are equal to one; thus, they are equivalent. Next, we perform the inductive step. We assume that $A(n)$ is true. In other words, we assume that

$$\sum_{j=0}^{n} r^j = \frac{1 - r^{n+1}}{1 - r}$$

for some $n \geq 0$. We will use this assumption to prove that $A(n+1)$ is true by showing that

$$\sum_{j=0}^{n+1} r^j = \frac{1 - r^{(n+1)+1}}{1 - r}.$$

Beginning with the left-hand side of the $A(n+1)$ statement and using our inductive assumption, we have that

$$\sum_{j=0}^{n+1} r^j = \left[\sum_{j=0}^{n} r^j\right] + r^{n+1}.$$

Substituting $\sum_{j=0}^{n} r^j$ for $\frac{1-r^{n+1}}{1-r}$ (as they are equivalent via the inductive assumption), we then write that

$$\sum_{j=0}^{n+1} r^j = \frac{1 - r^{n+1}}{1 - r} + r^{n+1}.$$

Using basic algebra, we can manipulate the right-hand side of the equation to write that

$$\sum_{j=0}^{n+1} r^j = \frac{1 - r^{n+1} + (1 - r)r^{n+1}}{1 - r}.$$

Then, foiling and simplifying, we discover that

$$\sum_{j=0}^{n+1} r^j = \frac{1 - r^{n+1} + r^{n+1} - r \cdot r^{n+1}}{1-r}.$$

Canceling the addition and subtraction of the value r^{n+1}, we then write that:

$$\sum_{j=0}^{n+1} r^j = \frac{1 - r \cdot r^{n+1}}{1-r};$$
$$\sum_{j=0}^{n+1} r^j = \frac{1 - r^{(n+1)+1}}{1-r}.$$

Thus we have shown that $A(n+1)$ is true. By induction, we can now conclude that $A(n)$ given by

$$\sum_{j=0}^{n} r^j = \frac{1 - r^{n+1}}{1-r}$$

where $r \neq 1$ is a real number, is true for all n greater than the base case $n = 0$, as desired.

$\square$

2. Consider the function $f(x) = \dfrac{1}{1-x}$.

   (a) Compute the first several derivatives of f and conjecture a pattern for $f^{(n)}(x)$.

   First, to ease integration we will rewrite $f(x)$ as

$$f(x) = \frac{1}{1-x} = \frac{-1}{-(1-x)} = \frac{-1}{x-1} = (-1)(x-1)^{-1}.$$

   The first several derivatives of the function f follow:

$$\begin{aligned} f(x) &= (-1)(x-1)^{-1} \\ f'(x) &= (1)(x-1)^{-2} \\ f''(x) &= (-2)(x-1)^{-3} \\ f^{(3)}(x) &= (6)(x-1)^{-4} \\ f^{(4)}(x) &= (-24)(x-1)^{-5}. \end{aligned}$$

   The conjectured pattern is

$$f^{(n)}(x) = (-1)^{n+1}(n!)(x-1)^{-(n+1)}.$$

(b) **Proposition.** The conjectured pattern is true for $f^{(n)}(x)$.

Proof. Let $A(n)$ be the statement

$$f^{(n)}(x) = (-1)^{n+1}(n!)(x-1)^{-(n+1)}.$$

We wish to show that $A(n)$ is true for all integers $n \geq 0$. We must verify that the base case $A(0)$ is true, which says that

$$f^{(0)}(x) = (-1)^{0+1}(0!)(x-1)^{-(0+1)}.$$

Since $f^{(0)}$ is simply the original function, and

$$f^{(0)}(x) = (-1)^1(1)(x-1)^{-1} = (-1)(x-1)^{-1} = \frac{-1}{x-1} = \frac{1}{1-x},$$

which is also the original function, then the base case $A(0)$ is true. We will assume for some $n \geq 0$ that

$$f^{(n)}(x) = (-1)^{n+1}(n!)(x-1)^{-(n+1)}$$

is true. This is the inductive assumption. We will then show $A(n+1)$ is also true by showing

$$f^{(n+1)}(x) = (-1)^{(n+1)+1}((n+1)!)(x-1)^{-((n+1)+1)}.$$

We will do so by writing the $(n+1)$st derivative as the derivative of the n-th derivative of function f. We will use basic techniques of derivation. To start,

$$\begin{aligned} f^{(n+1)}(x) &= \frac{d}{dx} f^{(n)}(x) \\ &= \frac{d}{dx}\Big[(-1)^{n+1}(n!)(x-1)^{-(n+1)}\Big] \\ &= (-1)^{n+1}(n!) \cdot \frac{d}{dx}\Big[(x-1)^{-(n+1)}\Big]. \end{aligned}$$

Then, we take the derivative of $(x-1)^{-(n+1)}$ for the statement

$$f^{(n+1)}(x) = (-1)^{n+1}(n!) \cdot \Big[-(n+1)(x-1)^{-(n+1)-1}\Big].$$

Simplifying using basic algebra, we write

$$\begin{aligned} f^{(n+1)}(x) &= (-1)^{n+1}(n!) \cdot \Big[-(n+1)(x-1)^{-(n+1)-1}\Big]. \\ &= (-1)^{n+1}(n!)(-1)(n+1)(x-1)^{-((n+1)+1)}. \end{aligned}$$

Because $(-1)^{n+1}$ multiplied by -1 is equal to $(-1)^{n+2}$ we write that

$$f^{(n+1)}(x) = (-1)^{n+2}(n!)(n+1)(x-1)^{-((n+1)+1)}.$$

Then, because $n!$ multiplied by $n+1$ can be written as $(n+1)!$ we write that

$$f^{(n+1)}(x) = (-1)^{n+2}(n+1)!(x-1)^{-((n+1)+1)}.$$

We have therefore shown our statement $A(n+1)$ to be true, and our inductive step is complete. By induction, we know that the statement $A(n)$ given by

$$f^{(n)}(x) = (-1)^{n+1}(n!)(x-1)^{-(n+1)}$$

is true for all $n \geq 0$, as desired.

$\square$

3. **Proposition.** Let $x > -1$. The statement

$$(1+x)^n \geq 1 + nx$$

is true for all integers $n \geq 1$.

Proof. Assume an x greater than -1. We will use proof by induction to show the statement $A(n)$ given by $(1+x)^n \geq 1+nx$ is true for all integers $n \geq 1$. Thus we have our base case $n = 1$. The statement $A(1)$ reads that

$$\begin{aligned}(1+x)^1 &\geq 1 + 1x;\\ 1+x &\geq 1+x.\end{aligned}$$

The above is a true statement because $1+x = 1+x$. Thus $A(1)$ is true. Next we perform the inductive step. Thus we assume $A(n)$ is true for some n greater than the base case of $n = 1$. In other words, we assume

$$(1+x)^n \geq 1 + nx$$

for $n \geq 1$. We wish to show $A(n+1)$, that

$$(1+x)^{n+1} \geq 1 + (n+1)x.$$

We will begin on the left-hand side with

$$(1+x)^{n+1} = (1+x)^n \cdot (1+x).$$

Because we have assumed that x is greater than -1, it follow that $1+x$ is greater than 0. Thus we can multiply both sides of the initial statement A by the value $1+x$ to reveal that

$$(1+x)^n \cdot (1+x) \geq (1+nx) \cdot (1+x).$$

By combining the previous two statements, we then write that

$$(1+x)^{n+1} = (1+x)^n \cdot (1+x) \geq (1+nx) \cdot (1+x).$$

In foiling the right-hand side of this inequality, we have that

$$\begin{aligned}(1+x)^{n+1} &\geq (1+nx) \cdot (1+x);\\ (1+x)^{n+1} &\geq 1 + nx + x + nx^2;\\ (1+x)^{n+1} &\geq 1 + (n+1)x + nx^2.\end{aligned}$$

Because n and x^2 are both greater than or equal to 0, it follows that $nx^2 \geq 0$ as well. Thus, we can write that

$$1 + (n+1)x + nx^2 \geq 1 + (n+1)x.$$

Therefore, by combination of the previous two inequalities, we can write

$$(1+x)^{n+1} \geq 1 + (n+1)x + nx^2 \geq 1 + (n+1)x.$$

Elimination of the middle value reveals the original $A(n+1)$ statement:

$$(1+x)^{n+1} \geq 1 + (n+1)x.$$

Thus, we have that $A(n+1)$ is a true statement. By induction, we can now conclude that the statement $A(n)$ given by $(1+x)^n \geq 1 + nx$ is true for all integers $n \geq 1$, as desired.

$\square$