1. (a) Attempt 2 seems to be the best. It starts with a true statement and ends with the proposition.

   □

   (b) Attempt 1 starts by assuming that what he/she is attempting to prove is true. Attempt 3 seems to use unnecessary geometry, when they could simply write something along the lines of $(\sqrt{a} - \sqrt{b})^2 \geq 0$, a la Attempt 2.

   □

2. **Proposition.** A number is divisible by 3 if and only if it is the sum of three consecutive whole numbers.

   **Discussion.** Let the number $n$ be some integer in $\mathbb{Z}$. Here we have a biconditional statement $p \Leftrightarrow q$, where $p$ is that $n$ is divisible by 3 (i.e., $3 \mid n$), while $q$ is that $n$ is the sum of three consecutive whole numbers. Thus, we want to prove the following two conditional statements:

   - $p \Rightarrow q$: "If the number $n$ is divisible by 3, then it is the sum of three consecutive whole numbers."
   - $q \Rightarrow p$: "If the number $n$ is the sum of three consecutive whole numbers, then it is divisible by 3."

   We will assume that "whole numbers" implies numbers contained in the set of natural numbers $\mathbb{N}$, which includes the element 0. Therefore, any negative number, even if contained in the set of integers $\mathbb{Z}$, is not a whole number. We will show that the proposition holds when $n$ is greater than 0, but fails when $n$ is less than or equal to 0. Note that when we show the conditional statement $q \Rightarrow p$, because we have assumed that $n$ is the sum of three consecutive whole numbers, we are not concerned with whether or not $n > 0$.

   **Proof.** Let the number $n$ be an integer contained in set $\mathbb{Z}$. We will prove our biconditional statement by showing two conditional statements: "If the number $n$ is divisible by 3, then it is the sum of three consecutive whole numbers," and "If the number $n$ is the sum of three consecutive whole numbers, then it is divisible by 3."

   For the first conditional statement, assume that 3 divides $n$. We wish to show that there exists some $j \in \mathbb{N}$ such that $n = (j+1) + j + (j-1)$ where the values of $j+1$, $j$, and $j-1$ represent three consecutive whole numbers.

   By the definition of of divisibility, there exists some $k \in \mathbb{Z}$ such that $n = 3k$. By the axiomatic definition of arithmetic, we know that $3k + 0 = 3k$. Thus, we may write $n = 3k + 0$. Further consider that $0 = 1 - 1$. We may substitute in for 0, to the effect that $n = 3k+(1-1)$. Moreover, because $3k = k+k+k$ by the definition of multiplication, we can write that $n = k+k+k+(1-1)$. By the theorem of associativity, we can further manipulate this equation to write that $n = k+1+k+k-1$, and $n = (k+1)+k+(k-1)$. Note that this is equivalent to the sum of three consecutive *integers*, not necessarily whole numbers. This is because the set of whole numbers is contained in the set of natural numbers $\mathbb{N}$, and while $\mathbb{N}$ is a subset of the set of integers $\mathbb{Z}$, it does not follow that $\mathbb{Z} \in \mathbb{N}$.

Here, we will show that the conditional statement $p \Rightarrow q$ is true when $n > 0$ but is not true when $n \leq 0$. First, assume that $n > 0$. Thus, $n$ is positive. The least value of $n$ greater than 0 that is divisible by 3 is 3. Since $3 \mid 3$, we can write that $n = 3k$. If $n = 3$, then by dividing both sides of $3 = 3k$ by three, we reveal that $k$ is 1. Therefore, in the equation $n = (k+1) + k + (k-1)$, we have the consecutive integers $n = 2 + 1 + 0$. Because the values 2, 1, and 0 are contained in the set of whole numbers, we have that $p \Rightarrow q$ is true when $n = 3$. As $n$ becomes increasingly positive, so does $k$; this is shown in the relationship of $n = 3k$. By dividing by three, we reveal that $k = \frac{n}{3}$. Thus, as $n$ increases, so must $k$. Because there are no values of $n$ divisible by 3 that are less than 3 but greater than 0, we may write that the statement $p \Rightarrow q$ holds when $n > 0$.

However, the statement does not hold when $n \leq 0$. In the case that $n \leq 0$, then by the definition of divisibility, $n = 3k$ for some $k \in \mathbb{Z}$, we have that $k = \frac{n}{3}$. Since the denominator of 3 is positive, when $n < 0$, we know that $k < 0$. Since the set of whole numbers does not include negative numbers, we know that $k$ cannot be less than 0. But what about the case that $k$ is equal to 0? By writing that $n = (k+1) + k + (k-1)$, however, we have accounted for some number such that $k$ is its successor. We cannot have $k = 0$, because by axiom 7, there are no natural numbers such that 0 is a successor. Therefore, we cannot have $k = 0$. Because $n = 3k$, if $k = 0$, then $n = 0$. Therefore, by extension, we are forced to conclude that $p \Rightarrow q$ is not true for $n \leq 0$, but *is* true when $n > 0$.

Now we will turn our attention to the statement given by $q \Rightarrow p$: "If the number $n$ is the sum of three consecutive whole numbers, then it is divisible by 3." Thus, we have some whole number $w$ such that $w + (w+1) + (w+2) = n$. This is true because the value $w$ has a successor of $w+1$, which in turn has a successor of $w+2$. Thus, they are three consecutive whole numbers. By basic algebra and the theorem of associativity, we may write that:

$$\begin{aligned} w + (w+1) + (w+2) &= n; \\ w + w + w + 3 &= n; \\ 3w + 3 &= n; \\ 3(w+1) &= n. \end{aligned}$$

Because $w$ is some whole number, it is contained in the set of integers $\mathbb{Z}$. Thus, by the definition of divisibility, $n$ is divisible by 3 because there exists an integer $w$ such that $n = 3w$. Therefore, we may conclude that if the number $n$ is the sum of three consecutive whole numbers, then it is divisible by 3, as desired.

We have now shown that $p \Rightarrow q$ is true when $n > 0$, and we have shown $q \Rightarrow p$. Therefore, we may conclude that $p \Leftrightarrow q$ is true when $n$ is some positive number greater than zero. We then have a partial proof for the statement "A number $n$ is divisible by 3 if and only if it is the sum of three consecutive whole numbers," on the further condition that $n$ is greater than 0.

$\square$

3. **Proposition.** A general pattern for the n-th derivative of $f(x) = xe^x$ is given by

$$f^{(n)}(x) = (x+n)e^x.$$

**Discussion.** The first several derivatives of the function $f$ are as follows:

$$\begin{aligned}
f(x) &= xe^x && = (x+0)e^x;\\
f'(x) &= xe^x + e^x && = (x+1)e^x;\\
f''(x) &= xe^x + e^x + e^x && = (x+2)e^x;\\
f^{(3)}(x) &= xe^x + e^x + e^x + e^x && = (x+3)e^x;\\
f^{(4)}(x) &= xe^x + e^x + e^x + e^x + e^x && = (x+4)e^x.
\end{aligned}$$

By the above patterns, we may conjecture that the $n$-th derivative of function $f$ may be given by

$$f^{(n)}(x) = (x+n)e^x.$$

**Proof.** We will show this is the case by using a proof by induction. Let $A(n)$ be the statement

$$f^{(n)}(x) = (x+n)e^x.$$

We wish to show that $A(n)$ is true for all integers $n \geq 0$. We must verify that $A(0)$ is true, which says that

$$f^{(0)}(x) = (x+0)e^x.$$

Since the 0-th derivative of a function is just the function itself, this is true because the above simplifies to $f(x) = xe^x$. For the inductive step, we will assume that, for some $k \geq 0$ that

$$f^{(k)}(x) = (x+k)e^x.$$

We must show that $A(k+1)$ is also true by showing that

$$f^{(k+1)}(x) = (x+(k+1))e^x.$$

We will do so by writing the $(k+1)$-st derivative as the derivative of the $k$-th derivative and by using basic calculus and factorial rules. We will begin with the left-hand side of our desired equation, writing that:

$$\begin{aligned}
f^{(k+1)}(x) &= \frac{d}{dx} f^{(k)}(x); \\
f^{(k+1)}(x) &= \frac{d}{dx}\Big[(x+k)e^x\Big]; \\
f^{(k+1)}(x) &= \frac{d}{dx}\Big[xe^x + ke^x\Big]; \\
f^{(k+1)}(x) &= \frac{d}{dx} xe^x + \frac{d}{dx} ke^x.
\end{aligned}$$

Then, employing the product rule followed by simple algebra, we can write:

$$\begin{aligned}
f^{(k+1)}(x) &= xe^x + e^x + ke^x; \\
f^{(k+1)}(x) &= xe^x + (k+1)e^x.
\end{aligned}$$

Finally, we isolate $e^x$ to conclude that

$$f^{(k+1)}(x) = (x + (k+1))e^x.$$

Thus, we have shown our statement $A(k+1)$ to be true and thus our inductive step is complete. By induction, we know that the statement $A(n)$ given by $f^{(n)}(x) = (x+n)e^x$ is true for all $n \geq 0$.

$\square$