1. **Proposition.** Let $a, b \in \mathbb{Z}$. $4 \mid a^2 - b^2$ if and only if a and b are of the same parity.

   Discussion. This proposition is a conditional statement $p \Leftrightarrow q$ with p being that $4 \mid a^2 - b^2$ and q being that a and b are of the same parity. We will show individually that $p \Rightarrow q$ and that $q \Rightarrow p$. The statement $p \Rightarrow q$ is that if $4 \mid a^2 - b^2$, then a and b are of the same parity. Similarly, the statement $q \Rightarrow p$ is that if a and b are of the same parity, then $4 \mid a^2 - b^2$. Notice that the statement $p \Rightarrow q$ relies on information regarding $a^2 - b^2$. To simplify, we will use proof by contrapositive for the statement $p \Rightarrow q$, where the contrapositive is $\neg q \Rightarrow \neg p$, or, if a and b are not of the same parity, then $4 \nmid a^2 - b^2$.

   Proof. This proposition is a conditional statement $p \Leftrightarrow q$ with p being $4 \mid a^2 - b^2$ and q being that a and b are of the same parity. We will show individually that $p \Rightarrow q$ and that $q \Rightarrow p$.

   We will begin by demonstrating $p \Rightarrow q$ using proof by contrapositive. The contrapositive of the statement $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$. In other words, if a and b are not of the same parity, then $4 \nmid a^2 - b^2$. So, we will assume some $a, b \in \mathbb{Z}$ that are not of the same parity. We know that one variable of a and b is odd, while the other is even. We will check the cases that a is even while b is odd, and that a is odd while b is even.

   To start, we will make a the even integer, and b the odd integer. So $a = 2k$ for some $k \in \mathbb{Z}$ and $b = 2j + 1$ for some $j \in \mathbb{Z}$. We must now demonstrate that $4 \nmid a^2 - b^2$. We will substitute a and b for

   $$4 \nmid (2k)^2 - (2j + 1)^2.$$

   Now, we can expand the equations for

   $$4 \nmid 4k^2 - (4j^2 + 4j + 1).$$

   Distribute the negative for

   $$4 \nmid 4k^2 - 4j^2 - 4j - 1.$$

   Next, we will isolate 4 on the right-hand side to see that

   $$4 \nmid 4(k^2 - j^2 - j) - 1.$$

   We know that $4(k^2 - j^2 - j)$ is some integer multiplied by four because $\mathbb{Z}$ is closed under multiplication and addition, and $k, j \in \mathbb{Z}$. Then, note that this is equivalent to the subtraction of one from an integer $4(k^2 - j^2 - j)$ that is otherwise divisible by four. Therefore $4 \nmid 4(k^2 - j^2 - j) - 1$ is true. Thus $4 \nmid (2k)^2 - (2j + 1)^2$. and we have found that $4 \nmid a^2 - b^2$ when a is even and b is odd.

   Now, let us make a odd and b even. Therefore, $a = 2k + 1$ for some $k \in \mathbb{Z}$ and $b = 2j$ for some $j \in \mathbb{Z}$. We must demonstrate that $4 \nmid a^2 - b^2$. Once again, we will substitute a and b for

   $$4 \nmid (2k + 1)^2 - (2j)^2.$$

   Now, we can expand the equations for

   $$4 \nmid 4k^2 + 4k + 1 - 4j^2.$$

Next, we will isolate 4 on the right-hand side to see that

$$4 \nmid 4(k^2 + k - j^2) + 1.$$

We know that $4(k^2 + k - j^2)$ is some integer multiplied by four because $\mathbb{Z}$ is closed under multiplication and addition, and $k, j \in \mathbb{Z}$. Then, note that this is equivalent to the addition of one to an integer $4(k^2 + k - j)$ that is otherwise be divisible by four. Therefore, $4 \nmid 4(k^2 + k - j^2) + 1$ is true. Thus $4 \nmid (2k + 1)^2 - (2j)^2$ and we have found that $4 \nmid a^2 - b^2$ when a is odd and b is even.

Thus, we have proven the contrapositive of the initial statement, that if a and b are not of the same parity, then $4 \nmid a^2 - b^2$. By proving the contrapositive $\neg q \Rightarrow \neg p$, we have proved $p \Rightarrow q$.

Now we will show that $q \Rightarrow p$. In other words, we must demonstrate that if a and b are of the same parity, then $4 \mid a^2 - b^2$. There are two cases: when a and b are both even, and when a and b are both odd. We will explore this statement for both cases.

Let us begin by assuming that a and b are both even. Thus, $a = 2k$ for some $k \in \mathbb{Z}$ and $b = 2j$ for some $j \in \mathbb{Z}$. We must show that $4 \mid a^2 - b^2$. We will begin by substituting a and b to see that

$$4 \mid (2k)^2 - (2j)^2.$$

Next, we expand the equation for

$$4 \mid 4k^2 - 4j^2.$$

Finally, isolate the number four to see that

$$4 \mid 4(k^2 - j^2).$$

We know that $k^2 - j^2$ is an integer because $\mathbb{Z}$ is closed under multiplication and addition, and $k, j \in \mathbb{Z}$. Therefore, $4(k^2 - j^2)$ is equivalent to multiplying some integer $k^2 - j^2$ by four, and therefore it is true that $4 \mid 4(k^2 - j^2)$. Thus $4 \mid a^2 - b^2$ when a and b are both even.

Now we will explore the case that both a and b are odd. Thus, $a = 2k + 1$ for some $k \in \mathbb{Z}$ and $b = 2j + 1$ for some $j \in \mathbb{Z}$. We must show that $4 \mid a^2 - b^2$. We will begin by substituting a and b to see that

$$4 \mid (2k + 1)^2 - (2j + 1)^2.$$

Next, we expand the equation for

$$4 \mid 4k^2 + 4k + 1 - (4j^2 + 4j + 1).$$

Distribute the negative for the equation

$$4 \mid 4k^2 + 4k + 1 - 4j^2 - 4j - 1.$$

Next, we will isolate the number four to see that

$$4 \mid 4(k^2 + k - j^2 - j) + 1 - 1,$$

which is equivalent to $4 \mid 4(k^2+k-j^2-j)$.We know that k^2+k-j^2-j is an integer because $\mathbb{Z}$ is closed under multiplication and addition, and $k, j \in \mathbb{Z}$. Therefore, $4(k^2 + k - j^2 - j)$ is equivalent to multiplying some integer $k^2 + k - j^2 - j$ by four, and therefore it is true that $4 \mid 4(k^2 + k - j^2 - j)$. Thus $4 \mid a^2 - b^2$ when a and b are both odd.

We have now shown that if a and b are of the same parity, then $4 \mid a^2 - b^2$. Thus $q \Rightarrow p$.

Since we have shown that $p \Rightarrow q$ and $q \Rightarrow p$, it is true that $p \Leftrightarrow q$. Therefore, we have demonstrated that $4 \mid a^2 - b^2$ if and only if a and b are of the same parity, as desired.

□

2. (a) **Proposition.** Let $a \in \mathbb{Z}$. $3 \mid a$ if and only if $3 \mid a^2$.

Discussion. This proposition is a conditional statement $p \Leftrightarrow q$ with p being $3 \mid a$ and q being $3 \mid a^2$. We will show individually that $p \Rightarrow q$ and that $q \Rightarrow p$. The statement $p \Rightarrow q$ is that if $3 \mid a$, then $3 \mid a^2$. Similarly, the statement $q \Rightarrow p$ is that if $3 \mid a^2$, then $3 \mid a$. Notice that the statement $q \Rightarrow p$ relies on information regarding a^2. To simplify, we will use proof by contrapositive for the statement $q \Rightarrow p$, where the contrapositive is $\neg p \Rightarrow \neg q$, or, if $3 \nmid a$, then $3 \nmid a^2$.

Proof. This proposition is a conditional statement $p \Leftrightarrow q$ with p being $3 \mid a$ and q being $3 \mid a^2$. We will show individually that $p \Rightarrow q$ and that $q \Rightarrow p$. The statement $p \Rightarrow q$ is that if $3 \mid a$, then $3 \mid a^2$. Similarly, the statement $q \Rightarrow p$ is that if $3 \mid a^2$, then $3 \mid a$.

We will start with proof that $p \Rightarrow q$. In other words, we must demonstrate that if $3 \mid a$, then $3 \mid a^2$. Assume some $a \in \mathbb{Z}$ such that $3 \mid a^2$. In other words, there exists some $k \in \mathbb{Z}$ so that $a = 3k$. We must show that $3 \mid a^2$. To do so, we will substitute a for $3k$ so that $3 \mid (3k)^2$. Simplifying, we see that $3 \mid 9k^2$. We know that k^2 is an integer because $\mathbb{Z}$ is closed under multiplication, and $k \in \mathbb{Z}$. Therefore, $9k^2$ is equivalent to multiplying some integer k^2 by nine (or rather, multiplying some integer k^2 by three, twice), and therefore it is true that $3 \mid 9k^2$. Thus if $3 \mid a$, then $3 \mid a^2$. We have shown that $p \Rightarrow q$, as desired.

Now we will show that $q \Rightarrow p$. In other words, we must demonstrate that if $3 \mid a^2$, then $3 \mid a$. Notice that the statement $q \Rightarrow p$ relies on information regarding a^2. To simplify, we will use proof by contrapositive for the statement $q \Rightarrow p$, where the contrapositive is $\neg p \Rightarrow \neg q$, or, if $3 \nmid a$, then $3 \nmid a^2$. Assume some $a \in \mathbb{Z}$ such that $3 \nmid a$. Then, there are two cases regarding $3 \nmid a$ for some $k \in \mathbb{Z}$. First, that $a = 3k + 1$; second, that $a = 3k + 2$. Notice that replacing the 1 or 2 in the above statements with another number not divisible by 3 will reduce to one of the above two cases. We must show that $3 \nmid a^2$ for $a = 3k + 1$ and for $a = 3k + 2$.

Let us begin with the case that $a = 3k+1$ for some $k \in \mathbb{Z}$. We will show that $3 \nmid a^2$. Begin by substituting a for $3k+1$ to see that $3 \nmid (3k+1)^2$. Expand the equation for $3 \nmid 9k^2 + 6k + 1$. Isolate 3 to show that $3 \nmid 3(3k^2 + 2k) + 1$. We know that $3k^2 + 2k$ is an integer because $\mathbb{Z}$ is closed under multiplication and addition, and $k \in \mathbb{Z}$. Therefore, $3(3k^2 + 2k)$ is equivalent to multiplying some integer $3k^2 + 2k$ by three. Then, note that $3(3k^2 + 2k) + 1$ is equivalent to the addition of one to an integer $3(3k^2 + 2k)$ that is otherwise be divisible by three. Therefore, $3 \nmid 3(3k^2 + 2k) + 1$ is true and $3 \nmid a^2$ when $a = 3k + 1$ for some $k \in \mathbb{Z}$.

Now we will consider the case that $a = 3k + 2$ for some $k \in \mathbb{Z}$. We will show that $3 \nmid a^2$. Begin by substituting a for $3k + 2$ to see that $3 \nmid (3k + 2)^2$. Expand the equation for $3 \nmid 9k^2 + 12k + 4$. Isolate 3 to see that $3 \nmid 3(3k^2 + 4k + 1) + 1$. We know that $3k^2 + 4k + 1$ is an integer because $\mathbb{Z}$ is closed under multiplication and addition, and $k \in \mathbb{Z}$. Therefore, $3(3k^2 + 4k + 1)$ is equivalent to multiplying some integer $3k^2 + 4k + 1$ by three. Then, note that $3(3k^2 + 4k + 1) + 1$ is equivalent to the addition of one to an integer $3(3k^2 + 4k + 1)$ that is otherwise be divisible by three. Therefore, $3 \nmid 3(3k^2 + 4k + 1) + 1$ is true and $3 \nmid a^2$ when $a = 3k + 2$ for some $k \in \mathbb{Z}$.

Since we have shown that $3 \nmid a^2$ for $a = 3k + 1$ and for $a = 3k + 2$, it is true that if $3 \nmid a$, then $3 \nmid a^2$. Thus we have proved the contrapositive of the initial statement. Therefore we have shown the initial statement, if $3 \mid a^2$, then $3 \mid a$, as well. Thus, $q \Rightarrow p$ is true.

Since we have shown that $p \Rightarrow q$ and $q \Rightarrow p$, it is true that $p \Leftrightarrow q$. Therefore, we have demonstrated that $3 \mid a$ if and only if $3 \mid a^2$, as desired.

(b) **Proposition.** $\sqrt{3}$ is irrational.

Discussion. We wish to show that $\sqrt{3}$ is an irrational number. To do so, we will use a proof by contradiction and assume that $\sqrt{3}$ is rational. By assuming that $\sqrt{3}$ is rational, we assume that we can write $\sqrt{3}$ as a fraction $\frac{p}{q} \in \mathbb{Q}$ in lowest terms. Eventually, we will contradict the fact that $\frac{p}{q}$ is in lowest terms by showing that they share a common divisor. In doing so, we can use the previously demonstrated theorem that $3 \mid a$ if and only if $3 \mid a^2$.

Proof. Assume, to the contrary, that $\sqrt{3}$ is rational. Thus, we may write

$$\sqrt{3} = \frac{p}{q},$$

where p and q have no common divisors. Squaring both sides, we obtain

$$3 = \frac{p^2}{q^2},$$

which is equivalent to $3q^2 = p^2$. Now, p^2 must be divisible by 3. Using the previously demonstrated theorem that $3 \mid a$ if and only if $3 \mid a^2$, we can conclude that p is also divisible by 3. Thus, we can write that $p = 3k$ for some $k \in \mathbb{Z}$. Substituting this new equation in, we arrive at $3q^2 = (3k)^2$, or $3q^2 = 9k^2$. Thus, dividing by three, we have that $q^2 = 3k^2$, which implies that q^2 is also divisible by three. Thus both p^2 and q^2 are divisible by three. Once again applying the previously discussed theorem that $3 \mid a$ if and only if $3 \mid a^2$, it becomes apparent that p and q are both divisible by 3. We arrive at a contradiction. We are forced to conclude that $\sqrt{3} = \frac{p}{q}$ cannot be written in lowest terms, and $\sqrt{3}$ is irrational, as desired.

$\square$

3. **Proposition.** Let $a, b \in \mathbb{R}$. Show that if $a + b$ is rational, then a is irrational or b is rational.

   Discussion. This statement can be represented as $p \Rightarrow q$, where p is that $a + b$ is rational and q is that a is irrational or b is rational. We will use proof by contrapositive $\neg q \Rightarrow \neg p$. Using DeMorgan's Logic Laws, we find the negation of q to be that a is rational and b is irrational. In full, the contrapositive of the initial statement is: if a is rational and b is irrational, then $a + b$ is irrational. Furthermore, we will use proof by contradiction in assuming that $a + b$ is rational. By arriving at a contradiction, we will be forced to conclude that $a+b$ is indeed irrational. Thus, we can conclude that the contrapositive $\neg q \Rightarrow \neg p$ is true. By extension, the initial statement $p \Rightarrow q$ is true, as desired.

   Proof. We will be employing proof by contrapositive. The initial statement can be represented as $p \Rightarrow q$, where p is that $a + b$ is rational and q is that a is irrational or b is rational. In full, the contrapositive $\neg q \Rightarrow \neg p$ of the initial statement is: if a is rational and b is irrational, then $a + b$ is irrational.

   Assume, to the contrary, that $a + b$ is rational. Thus, we have that $a \in \mathbb{Q}$, $b \notin \mathbb{Q}$, and $a + b \in \mathbb{Q}$. Remember that $\mathbb{Q}$ is closed under addition. Thus, when we subtract $a \in \mathbb{Q}$ from $a + b \in \mathbb{Q}$, we find that $(a + b) - a \in \mathbb{Q}$. This is equivalent to $b \in \mathbb{Q}$. However, we have already stated that $b \notin \mathbb{Q}$. Thus, we have arrived at a contradiction. We are forced to conclude that $a + b \notin \mathbb{Q}$. Therefore, we have shown the contrapositive: if a is rational and b is irrational, then $a + b$ is irrational.

   We conclude that the contrapositive $\neg q \Rightarrow \neg p$ is true. By extension, the initial statement $p \Rightarrow q$ is true. Thus if $a + b$ is rational, then a is irrational or b is rational, as desired.

$\square$