1. **Proposition.** The two angle-sum formulae hold:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha;$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Proof. Euler's equation states that $e^{i\theta} = \cos\theta + i\sin(\theta)$. Consider the value $e^{i(\alpha+\beta)}$. From Euler's equation we know that $e^{i(\alpha+\beta)} = \cos(\alpha+\beta)+i\sin(\alpha+\beta)$. Because complex exponents follow the same rules as real exponents, we also know that $e^{i(\alpha+\beta)} = e^{i\alpha}\cdot e^{i\beta}$. Further expansion via Euler's equation reveals that $e^{i\alpha}\cdot e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$. Expanding this result shows

$$e^{i\alpha}\cdot e^{i\beta} = \cos\alpha\cos\beta + i(\sin\alpha\cos\beta) + i(\sin\beta\cos\alpha) - \sin\alpha\sin\beta.$$

By rearranging this equivalence to separate the real and imaginary parts, we can see that

$$e^{i\alpha}\cdot e^{i\beta} = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \sin\beta\cos\alpha).$$

Thus, by equating the Euler expansions on the equivalent values $e^{i(\alpha+\beta)}$ and $e^{i\alpha}\cdot e^{i\beta}$, we have that

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \sin\beta\cos\alpha).$$

For two complex numbers $x, y \in \mathbb{C}$ to be equivalent, their real parts and their imaginary parts must be equal. In other words, $\mathrm{Re}(x) = \mathrm{Re}(y)$ and $\mathrm{Im}(x) = \mathrm{Im}(y)$ must be true. Therefore, because we have equated the Euler expansions as above, we have that

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha.$$

These are the double-angle formulae, as desired.

$\square$

2. (a) **Proposition.** Show $|z| = \mathrm{Re}(z)$ if and only if z is a non-negative real number.

Proof. Here we have a biconditional statement $p \Leftrightarrow q$, where p is that $|z| = \mathrm{Re}(z)$ and q is that z is a non-negative real number. We will show $p \Rightarrow q$ as well as $q \Rightarrow p$. First we will demonstrate $p \Rightarrow q$ using proof by contrapositive $\neg q \Rightarrow \neg p$. The contrapositive is that if z is negative or z is not a real number, then $|z| \neq \mathrm{Re}(z)$. Thus $\neg q$ has two cases: when z is negative, and when z is not a real number (i.e., has an imaginary component).

For the first case, let us consider a $z < 0$ contained in $\mathbb{R}$. Thus, in the Cartesian representation $z = a + bi$, we assume a is some negative real number and $b = 0$. Then we can write that

$$|z| = |a + bi|.$$

Using the definition of modulus, we have that

$$|a+bi| = \sqrt{a^2+b^2} = \sqrt{a^2+0^2} = \sqrt{a^2} = |a|.$$

Additionally, we know that $\mathrm{Re}(z) = a$, where a is some negative number less than 0. However, $|a|$ must be some nonnegative number by the properties of absolute values. Thus, $|a| \neq a$. By extension, $|z| \neq \mathrm{Re}(z)$ in the case that z is some negative real number.

Now, consider the second case that z has some imaginary part not equal to 0. Then in Cartesian representation, $z = a+bi$, we know that $b \neq 0$. Thus, by the definition of modulus, $|z| = |a+bi| = \sqrt{a^2+b^2}$. We know that $\mathrm{Re}(z) = a$. However, in the case that $b \neq 0$, then $a \neq \sqrt{a^2+b^2}$. Thus, when z has some imaginary part, $|z| \neq \mathrm{Re}(z)$.

Having shown the contrapositive $\neg q \Rightarrow \neg p$ is true for both cases, we conclude that $p \Rightarrow q$ is true as well.

Now we must show $q \Rightarrow p$, which states that if z is a non-negative real number, then $|z| = \mathrm{Re}(z)$. Assume some real non-negative z. Using Cartesian form, $z = a+bi$, we know that $\mathrm{Re}(z) = a$ and $a \geq 0$. Furthermore, because $z \in \mathbb{R}$, then $b = 0$.

Consider that the modulus $|z|$. We write that

$$|z| = |a+bi| = \sqrt{a^2+b^2} = \sqrt{a^2+0^2} = \sqrt{a^2} = |a|.$$

Because $a \geq 0$, the real part $\mathrm{Re}(z) = a$ is non-negative, as well. Therefore, $|a| = a$. Thus, since $|z| = a$ and $\mathrm{Re}(z) = a$, we can conclude that $|z| = \mathrm{Re}(z)$ when z is a non-negative real number. We have thus shown that $q \Rightarrow p$, in addition to previously showing $p \Rightarrow q$. Therefore, $p \Leftrightarrow q$ is true. We have shown that $|z| = \mathrm{Re}(z)$ if and only if z is a non-negative real number, as desired.

$\square$

(b) **Proposition.** Show that $(\overline{z})^2 = z^2$ if and only if z is purely real or purely imaginary (i.e., its real part is 0).

Proof. Here we have a biconditional statement $p \Leftrightarrow q$, where p is that $(\overline{z})^2 = z^2$ and q is that z is purely real or purely imaginary. For q to be true, either $\mathrm{Re}(z) = 0$ or $\mathrm{Im}(z) = 0$. We will show $p \Rightarrow q$ and $q \Rightarrow p$ separately.

Let us begin with $p \Rightarrow q$ by proof of contrapositive $\neg q \Rightarrow \neg p$. Written, the contrapositive states that if $\mathrm{Im}(z) \neq 0$ and $\mathrm{Re}(z) \neq 0$, then $(\overline{z})^2 \neq z^2$. As such, consider a $z \in \mathbb{C}$. We will represent z in Cartesian form as $z = a+bi$ where $\mathrm{Im}(z) \neq 0$ and $\mathrm{Re}(z) \neq 0$. Thus $a \neq 0$ and $b \neq 0$. Then, we have that

$$z^2 = z \cdot z = (a+bi)(a+bi).$$

Foiling, we find

$$z^2 = a^2 + i(2ab) - b^2.$$

Now, consider the conjugate of z, given by $\overline{z} = a - bi$. Squaring, we find

$$(\overline{z})^2 = \overline{z} \cdot \overline{z} = (a - bi)(a - bi).$$

Foiling the result, we see that

$$(\overline{z})^2 = a^2 - i(2ab) - b^2.$$

When $a \neq 0$ and $b \neq 0$, as is the case, then

$$\begin{aligned} z^2 &= a^2 + i(2ab) + b^2; \\ (\overline{z})^2 &= a^2 - i(2ab) - b^2, \end{aligned}$$

are not always equivalent. Therefore, we have shown the contrapositive $\neg q \Rightarrow \neg p$: if $\text{Im}(z) \neq 0$ and $\text{Re}(z) \neq 0$, then $(\overline{z})^2 \neq z^2$. Therefore, we conclude that $p \Rightarrow q$, as desired.

Now, we will show $q \Rightarrow p$. Thus, if $\text{Im}(z) = 0$ or $\text{Re}(z) = 0$, then $(\overline{z})^2 = z^2$. In the statement q, we have two cases: when $\text{Im}(z) = 0$, and when $\text{Re}(z) = 0$.

Let us consider the case when $\text{Im}(z) = 0$. Thus, in Cartesian representation $z = a + bi$, we assume that b is zero. Therefore, $z = a + bi = a + 0i = a$. Now, consider the conjugate $\overline{z} = a - bi$. Again, since $\text{Im}(z) = 0$, we know that $b = 0$ and $\overline{z} = a - bi = a - 0i = a$. Thus, $\overline{z} = a = z$ and $\overline{z} = z$. Squaring both sides, we see that $(\overline{z})^2 = z^2$. Therefore, when $\text{Im}(z) = 0$, it is true that $(\overline{z})^2 = z^2$.

Now, let us consider the second case when $\text{Re}(z) = 0$. Then, in Cartesian representation $z = a + bi$, we assume that a is zero. Therefore, $z = a + bi = 0 + bi = bi$. Squaring both sides,

$$z^2 = (bi)^2 = b^2 \cdot i^2 = b^2 \cdot -1 = -b^2.$$

In short, $z^2 = -b^2$. Now consider the conjugate $\overline{z} = a - bi$. Once again, since Re(z) $= 0$, we know that $a = 0$. Then, $\overline{z} = a - bi = 0 - bi = -bi$. Once again squaring both sides of the equation, we get that

$$(\overline{z})^2 = (-bi)^2 = (-b)^2 \cdot i^2 = b^2 \cdot -1 = -b^2.$$

Therefore, $(\overline{z})^2$ is also equivalent to $-b^2$. We now see that in the case when $\text{Re}(z) = 0$, it is true that $(\overline{z})^2 = z^2$, because they are both equivalent to $-b^2$.

Therefore, we have shown both cases: if $\text{Re}(z) = 0$ or $\text{Im}(z) = 0$, then $(\overline{z})^2 = z^2$. Thus, we have proven that $q \Rightarrow p$. Having already shown that $p \Rightarrow q$, we can now conclude that $p \Leftrightarrow q$. In other words, $(\overline{z})^2 = z^2$ if and only if z is purely real or purely imaginary, as desired.

$\square$

3. (a) **Proposition.** If $z, w \in \mathbb{C}$, then

$$|z \cdot w| = |z| \cdot |w|$$

using the Cartesian form $z = a+bi$ and $w = c+di$ for the complex numbers z and w.

Proof. Assume a $z, w \in \mathbb{C}$. Using the Cartesian representation $z = a + bi$ and $w = c + di$, we will show that $|z \cdot w| = |z| \cdot |w|$ because they are both equivalent to $\sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}$.
First, consider $|z \cdot w|$. Substitute the above Cartesian forms to see that $|z \cdot w|$ can be represented as $|(a + bi)(c + di)|$. Foiling, we reveal that

$$|z \cdot w| = |(a + bi)(c + di)| = |ac - bd + i(bc + ad)|$$

where $ac - bd$ is the real part and $bc + ad$ is the imaginary part. By the definition of modulus, we can take the square root of the sum of the squared real part and the squared imaginary part to write that

$$|z \cdot w| = |ac - bd + i(bc + ad)| = \sqrt{(ac - bd)^2 + (bc + ad)^2}.$$

Foiling once again, we can show that

$$|z \cdot w| = \sqrt{(ac - bd)^2 + (bc + ad)^2} = \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}.$$

Now, consider $|z| \cdot |w|$. In Cartesian form, we represent this value as $|a+bi| \cdot |c+di|$. Thus, employing the definition of modulus, we reveal that

$$|z| \cdot |w| = |a + bi| \cdot |c + di| = \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2}.$$

Using basic algebraic laws, we can further rewrite this equation to show that

$$|z| \cdot |w| = \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2} = \sqrt{(a^2 + b^2)(c^2 + d^2)}.$$

Foiling this equivalence reveals to us that

$$|z| \cdot |w| = \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}.$$

Therefore, we have shown that $|z \cdot w|$ and $|z| \cdot |w|$ are both equivalent to the value $\sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}$. Thus, they must be also equivalent. In other words, $|z \cdot w| = |z| \cdot |w|$ for complex numbers z and w, as desired.

□

(b) **Proposition.** If $z, w \in \mathbb{C}$, then

$$|z \cdot w| = |z| \cdot |w|$$

using the polar form $z = r_1e^{i\theta_1}$ and $w = r_2e^{i\theta_2}$ for the complex numbers z and w.

Proof. Now we will demonstrate the same phenomena $|z \cdot w| = |z| \cdot |w|$ using complex polar representation such that $z = r_1e^{i\theta_1}$ and $w = r_2e^{i\theta_2}$. We will show that $|z \cdot w|$ and $|z| \cdot |w|$ are equivalent because they are both equal to the value r_1r_2.

First, consider $|z \cdot w| = |r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}|$. Using basic algebraic laws, we can rewrite this equivalence as $|z \cdot w| = |r_1 r_2 e^{i(\theta_1+\theta_2)}|$. Since any complex number $n = re^{i\theta}$ can be understood as $n = r\cos\theta + ir\sin\theta$, we may say that

$$|z \cdot w| = |r_1 r_2 e^{i(\theta_1+\theta_2)}| = |r_1 r_2 \cos(\theta_1+\theta_2) + ir_1 r_2 \sin(\theta_1+\theta_2)|.$$

Then, using the definition of modulus, we reveal that that

$$|z \cdot w| = \sqrt{(r_1 r_2 \cos(\theta_1+\theta_2))^2 + (r_1 r_2 \sin(\theta_1+\theta_2))^2}.$$

Once again employing basic algebraic laws, we can rewrite the equivalence statement as

$$|z \cdot w| = r_1 r_2 \sqrt{\cos^2(\theta_1+\theta_2) + \sin^2(\theta_1+\theta_2)}.$$

Next, we use the Pythagorean identity that $\cos^2\theta + \sin^2\theta = 1$ for all values of θ. Thus, we have that

$$|z \cdot w| = r_1 r_2 \sqrt{1} = r_1 r_2.$$

We will show that the same is true for $|z| \cdot |w|$. Substituting for polar notation, we write that $|z| \cdot |w| = |r_1 e^{i\theta_1}| \cdot |r_2 e^{i\theta_2}|$. Then,

$$|z| \cdot |w| = |r_1 e^{i\theta_1}| \cdot |r_2 e^{i\theta_2}| = |r_1 \cos\theta_1 + ir_1 \sin\theta_1| \cdot |r_2 \cos\theta_2 + ir_2 \sin\theta_2|.$$

Employing the definition of modulus, we see that

$$|z| \cdot |w| = \sqrt{(r_1 \cos\theta_1)^2 + (r_1 \sin\theta_1)^2} \cdot \sqrt{(r_2 \cos\theta_2)^2 + (r_2 \sin\theta_2)^2}.$$

Isolating r_1 and r_2 using algebraic laws, we see that

$$|z| \cdot |w| = r_1 \sqrt{\cos^2\theta_1 + \sin^2\theta_1} \cdot r_2 \sqrt{\cos^2\theta_2 + \sin^2\theta_2}.$$

Finally, employing the Phythagorean identity, we see that

$$|z| \cdot |w| = r_1 \sqrt{1} \cdot r_2 \sqrt{1} = r_1 r_2.$$

Thus, because they are both equivalent to $r_1 r_2$, we have that $|z \cdot w| = |z| \cdot |w|$ for complex numbers z and w, as desired.

□

4. For the parts below, let $z = a + bi$ and $w = c + di$ be complex numbers.

   (a) **Proposition.** $\overline{z+w} = \overline{z} + \overline{w}$.

   Proof. We will show that $\overline{z+w} = \overline{z} + \overline{w}$ by demonstrating that both $\overline{z+w}$ and $\overline{z} + \overline{w}$ are equal to $a + c - i(b+d)$.
   First, consider

$$\overline{z+w} = \overline{(a+bi)+(c+di)} = \overline{a+c+i(b+d)}.$$

By the definition of a conjugate, $\overline{z+w}$ is equivalent to the real part minus the imaginary part of $z+w$. In other words, $\overline{z+w} = a+c-i(b+d)$. Now, consider $\overline{z}+\overline{w}$. Substituting, we have

$$\overline{z}+\overline{w} = \overline{a+bi}+\overline{c+di}.$$

By the definition of a conjugate, we can write that $\overline{z}+\overline{w} = (a-bi)+(c-di)$. Rearranging to separate the real and imaginary parts, we have that

$$\overline{z}+\overline{w} = a+c-i(b+d).$$

Thus, because they are both equivalent to $a+c-i(b+d)$, we have shown that $\overline{z+w} = \overline{z}+\overline{w}$, as desired.

$\square$

(b) **Proposition.** $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$.

Proof. We will show that $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$ by demonstrating that both $\overline{z \cdot w}$ and $\overline{z} \cdot \overline{w}$ are equal to $ac-bd-i(bc+ad)$. First, consider $\overline{z \cdot w}$. Substituting for Cartesian form, we have

$$\overline{z \cdot w} = \overline{(a+bi)\cdot(c+di)}.$$

Foiling, we find that

$$\overline{z \cdot w} = \overline{ac+ibc+iad-bd};$$
$$\overline{z \cdot w} = \overline{ac-bd+i(bc+ad)}.$$

Then by definition of the complex conjugate, we know that

$$\overline{z \cdot w} = ac-bd-i(bc+ad).$$

Now, consider $\overline{z} \cdot \overline{w}$. Thus, we have,

$$\overline{z} \cdot \overline{w} = \overline{a+bi} \cdot \overline{c+di}.$$

Using the the definition of a complex conjugate and foiling, we know that this is also equivalent to

$$\overline{z} \cdot \overline{w} = (a-bi)\cdot(c-di) = ac+ibc+iad-bd.$$

Then, with rearrangement of real and imaginary parts, we can conclude that

$$\overline{z} \cdot \overline{w} = ac-bd-i(bc+ad).$$

Since both $\overline{z \cdot w}$ and $\overline{z} \cdot \overline{w}$ are equivalent to $ac-bd-i(bc+ad)$, we have that $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$, as desired.

$\square$

(c) **Proposition.** $\overline{z^n} = (\overline{z})^n$ for any natural number $n \in \mathbb{N}$.

Proof. Consider a natural number $n \in N$. We will show that $\overline{z^n} = (\overline{z})^n$ where $n \geq 0$. Recall the theorem proved in (b) above, where $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$ for complex numbers $z, w \in \mathbb{C}$. If $n = 0$, then it is true that $\overline{z^0} = (\overline{z})^0$ because a complex number raised to the 0th power is 1, and $\overline{1} = 1$ because 1 is a real number (i.e., its imaginary part is 0).
Having shown that the statement $\overline{z^n} = (\overline{z})^n$ is true for the base case $n = 0$, we will now use an inductive assumption. Assume that $\overline{z^n} = (\overline{z})^n$ is true for some n. We will use proof by induction to show that this is also true for $n+1$ so that

$$\overline{z^{n+1}} = (\overline{z})^{n+1}.$$

Having assumed $\overline{z^n} = (\overline{z})^n$, then let us consider the case for $n+1$, where

$$\overline{z^{n+1}} = \overline{z^n \cdot z}.$$

Using the property found in (b) above, we can write that $\overline{z^n \cdot z} = \overline{z^n} \cdot \overline{z}$. We can then continue this process, separating z to find the following pattern:

$$\begin{aligned}
\overline{z^{n+1}} &= \overline{z^n} \cdot \overline{z} \\
&= \overline{z^{n-1}} \cdot \overline{z} \cdot \overline{z} \\
&= \overline{z^{n-2}} \cdot \overline{z} \cdot \overline{z} \cdot \overline{z} \\
&= \overline{z^{n-3}} \cdot \overline{z} \cdot \overline{z} \cdot \overline{z} \cdot \overline{z},
\end{aligned}$$

until we have reached

$$\begin{aligned}
\overline{z^{n+1}} &= \overline{z^0} \cdot (\overline{z})^{n+1} \\
&= \overline{1} \cdot (\overline{z})^{n+1} \\
&= 1 \cdot (\overline{z})^{n+1}.
\end{aligned}$$

Thus, we have shown that $\overline{z^{n+1}} = (\overline{z})^{n+1}$. By inductive assumption, we may now conclude that $\overline{z^n} = (\overline{z})^n$ for any natural number $n \in \mathbb{N}$, as desired.

$\square$

(d) **Proposition.** Consider the following polynomial $p(z)$ with *real coefficients*:

$$p(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0,$$

where each α_i is a real number. If a complex number w is a root to the above polynomial with real coefficients, then its conjugate $\overline{w}$ is also a root to the same polynomial. That is, if $p(w) = 0$, then $p(\overline{w}) = 0$.

Proof. We will show that if $p(w) = 0$, then $p(\overline{w}) = 0$. Using sigma notation, we can write that

$$p(z) = \sum_{k=0}^{n} a_k z^k.$$

Thus, having assumed a $w \in \mathbb{C}$ such that $p(w) = 0$, we substitute for w to write that

$$p(w) = \sum_{k=0}^{n} a_k w^k = 0.$$

Note that with the above assumption, we can also write that

$$\overline{\sum_{k=0}^{n} a_k w^k} = \overline{0}.$$

Now, consider $p(\overline{w})$ such that

$$p(\overline{w}) = \sum_{k=0}^{n} a_k \overline{w}^k.$$

Using the previously shown theorems of (a) - (c), we will show that $p(\overline{w}) = 0$. First, consider the concept shown in (c), in which $\overline{z^n} = (\overline{z})^n$ for natural number n and complex number z. Applying this, we see that

$$p(\overline{w}) = \sum_{k=0}^{n} a_k \overline{w}^k = \sum_{k=0}^{n} a_k \overline{w^k}.$$

Now, recall that when z is a real number, $\overline{z} = z$ because the imaginary part is equal to 0. Therefore, since each a_i is a real number, $a_i = \overline{a_i}$. Following, it is true that

$$p(\overline{w}) = \sum_{k=0}^{n} a_k \overline{w^k} = \sum_{k=0}^{n} \overline{a_k} \cdot \overline{w^k},$$

and we can apply the theorem in (b) where $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$ for $z, w \in \mathbb{C}$. Thus we may write that

$$p(\overline{w}) = \sum_{k=0}^{n} \overline{a_k} \cdot \overline{w^k} = \sum_{k=0}^{n} \overline{a_k w^k}.$$

Finally, consider proof (a), which states that $\overline{z + w} = \overline{z} + \overline{w}$ for $z, w \in \mathbb{C}$. Because sigma notation is representation for repeated additions, we may write that

$$p(\overline{w}) = \sum_{k=0}^{n} \overline{a_k w^k} = \overline{\sum_{k=0}^{n} a_k w^k}.$$

Since we already know that

$$\overline{\sum_{k=0}^{n} a_k w^k} = \overline{0},$$

and it is true that $0 = \overline{0}$ because $0 \in \mathbb{R}$, then we have that

$$p(\overline{w}) = \overline{\sum_{k=0}^{n} a_k w^k} = 0.$$

In other words, $\overline{w}$ is a root of polynomial p. Therefore, if a complex number w is a root to the polynomial p with real coefficients, then its conjugate $\overline{w}$ is also a root to p, as desired.

$\square$