1. Let $m \neq 0$ and b be real numbers and consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = mx + b$.

   (a) **Proposition.** The function f is a bijection.

   Discussion. A function $f : S \to T$ is a bijection if it is both an injection and a surjection. In a bijection, we have that for every $t \in T$, there is exactly one pre-image. To prove that f is a bijection, we will individually prove that it is a surjection and that it is an injection.

   A function is surjective when the image of the function is the entire co-domain. To prove that f is a surjection, we must show that for every $t \in T$, there exists some $s \in S$ such that $f(s) = t$. To do so, we will assume a $y \in \mathbb{R}$ and show that it has a corresponding pre-image in $\mathbb{R}$.

   A function is injective if it has the property that, if s_1 and s_2 are distinct elements, then their outputs $f(s_1)$ and $f(s_2)$ are also distinct elements. To prove that f is an injection, we must show that whenever $f(s_1) = f(s_2)$, then $s_1 = s_2$.

   Proof. A function f is a bijection if it is both an injection and a surjection. In a bijection, we have that for every element in the output set, there is exactly one pre-image. We will first show that f is a surjection, and then that it is an *injection*.

   To show that f is surjective, assume some $y \in \mathbb{R}$, the output set. We will show that all $y \in \mathbb{R}$ have a corresponding pre-image in input set $\mathbb{R}$. We will show that $y = f(x)$, or $y = mx + b$. Let us solve for x by subtracting b, then dividing by m, for the equation $\frac{y-b}{m} = x$. Because y, b, and m are all real numbers, x is a real number, too. Therefore $x \in \mathbb{R}$ and the function f is surjective.

   To show that f is injective, assume two real numbers s_1 and s_2 such that $f(s_1) = f(s_2)$. We will show that if $f(s_1) = f(s_2)$, then $s_1 = s_2$. Note that $f(s_1) = ms_1 + b$ and that $f(s_2) = ms_2 + b$. Since we have assumed that $f(s_1) = f(s_2)$, then $ms_1 + b = ms_2 + b$. First, subtract both sides of the equation by b for $ms_1 = ms_2$. Next, divide by m to show that $s_1 = s_2$, therefore f is injective.

   Since we have shown that f is both surjective and injective, f is a bijection, as desired.

   □

   (b) **Proposition.** Since f is a bijection, it is invertible. Show that f^{-1} is an inverse by demonstrating that

   $$f^{-1}(f(x)) = x$$

   Discussion. Since $f : \mathbb{R} \to \mathbb{R}$ is a bijection, it is invertible. The inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ is given by the function $f^{-1}(x) = \frac{x-b}{m}$. We must show that $f^{-1}(f(x)) = x$.

Proof. Assume that the inverse of the function f is given by $f^{-1}(x) = \frac{x-b}{m}$. We will show that $f^{-1}(f(x)) = x$. One can see that this is indeed the inverse since

$$f^{-1}(f(x)) = f^{-1}(mx + b) = \frac{(mx + b) - b}{m} = \frac{mx}{m} = x$$

Therefore, since it is true that $f^{-1}(f(x)) = x$, then $f^{-1}(x) = \frac{x-b}{m}$ is the inverse of the bijection $f = mx + b$.

$\square$

2. **Proposition.** Let γ, $\rho \in \mathbb{R}$ be real numbers such that $\gamma \cdot \rho \neq 1$. Let $\mathbb{R} - \{\gamma\}$ and $\mathbb{R} - \{-\rho\}$ be the set of all real numbers $\mathbb{R}$ except for γ and $-\rho$, respectively. Consider the function $f : \mathbb{R} - \{-\rho\} \to \mathbb{R} - \{\gamma\}$ given by

$$f(x) = \frac{\gamma x + 1}{x + \rho}$$

The function $f(x)$ is a bijection.

Discussion. A function $f : S \to T$ is a bijection if it is both an injection and a surjection. In a bijection, we have that for every $t \in T$, there is exactly one pre-image. To prove that f is a bijection, we will individually prove that it is a surjection and that it is an injection.

A function is surjective when the image of the function is the entire co-domain. To prove that f is a surjection, we must show that for every $t \in T$, there exists some $s \in S$ such that $f(s) = t$. To do so, we will assume a $y \in \mathbb{R} - \{\gamma\}$ and show that it has a corresponding pre-image in $\mathbb{R} - \{-\rho\}$.

A function is injective if it has the property that, if s_1 and s_2 are distinct elements, then their outputs $f(s_1)$ and $f(s_2)$ are also distinct elements. To prove that f is an injection, we must show that whenever $f(s_1) = f(s_2)$, then $s_1 = s_2$.

The described function f is interesting in that the exact natures of the input and output sets are unknown. Because $\gamma \cdot \rho \neq 1$ has been declared, we must note that γ and ρ can be both any any real number with the exception that they cannot both simultaneously be equal to 1 and cannot both simultaneously be equal to -1. Thus, having an input set $\mathbb{R} - \{-\rho\}$ and an output set $\mathbb{R} - \{\gamma\}$, there are two cases involving the nature of these sets. One, that the input set is all real numbers except $\{1\}$ while the output set is all real numbers except $\{-1\}$. Two, that the input set is all real numbers except $\{-1\}$ while the output set is all real numbers except $\{1\}$. We will consider both of these cases in the proof.

Proof. A function $f : S \to T$ is a bijection if it is both an injection and a surjection. In a bijection, we have that for every $t \in T$, there is exactly one pre-image. To prove that f is a bijection, we will individually prove that it is a surjection and that it is an injection.

To prove that f is surjective, assume some $y \in \mathbb{R} - \{\gamma\}$, the output set. Note that we have declared that $\gamma \cdot \rho \neq 1$. Therefore, we cannot have that both $\gamma = 1$ and $\rho = 1$, and we cannot have that both $\gamma = -1$ and $\rho = -1$, because in these two cases, $\gamma \cdot \rho = 1$. In both cases, we will consider the $y \in \mathbb{R} - \{\gamma\}$ to show that all y have a corresponding pre-image in input set $\mathbb{R} - \{-\rho\}$.

First, we will set $y = f(x)$. This is equal to $y = \frac{\gamma x+1}{x+\rho}$. Now, we will solve for x. First, multiply both sides by $x + \rho$ for

$$y(x + \rho) = \gamma x + 1.$$

Next, expand the equation for

$$xy + \rho y = \gamma x + 1.$$

Now subtract $1 + xy$ from the equation for

$$y\rho - 1 = \gamma x - xy.$$

Isolate x, then divide both sides by $\gamma - y$ for

$$y\rho - 1 = x(\gamma - y)$$
$$\frac{y\rho - 1}{\gamma - y} = x.$$

Now we know that $x = \frac{y\rho-1}{\gamma-y}$, where $\gamma \cdot \rho \neq 1$. Because y, γ, and ρ are all real numbers, x is a real number, as well. Moreover, remember that $\gamma \cdot \rho \neq 1$ and $y \in \mathbb{R} - \{\gamma\}$. Therefore $y \neq \gamma$, so x will never be equal to $-\rho$. Therefore $x \in \mathbb{R} - \{-\rho\}$ and the function f is surjective.

To show that f is injective, assume two real numbers s_1 and s_2 such that $f(s_1) = f(s_2)$. We will show that if $f(s_1) = f(s_2)$, then $s_1 = s_2$. We will begin with the assumption that $f(s_1) = f(s_2)$. Note that

$$f(s_1) = \frac{\gamma s_1 + 1}{s_1 + \rho}$$
$$f(s_2) = \frac{\gamma s_2 + 1}{s_2 + \rho}.$$

Now, since we know that $f(s_1) = f(s_2)$, we can set the two equations equal to each other, like so

$$\frac{\gamma s_1 + 1}{s_1 + \rho} = \frac{\gamma s_2 + 1}{s_2 + \rho}.$$

Next, we cross-multiply first by $s_1 + \rho$ and then by $s_2 + \rho$ for

$$(\gamma s_1 + 1)(s_2 + \rho) = (\gamma s_2 + 1)(s_1 + \rho).$$

Next, we will expand the equation for

$$s_1\gamma s_2 + s_1\gamma\rho + s_2 + \rho = \gamma s_2 s_1 + \gamma s_2\rho + s_1 + \rho.$$

Next, rearrange the equation for readability, to

$$\rho + s_2 + s_1\gamma\rho + s_1\gamma s_2 = \rho + \gamma\rho s_2 + s_1 + s_1\gamma s_2.$$

Now we will isolate s_1 to find that

$$\rho + s_2 + s_1(\gamma\rho + \gamma s_2) = \rho + \gamma\rho s_2 + s_1(1 + \gamma s_2).$$

We continue the process of isolating s_1 by subtracting ρ, s_2, and $s_1(1 + \gamma s_2)$ from both sides of the equation. First, we subtract ρ for

$$s_2 + s_1(\gamma\rho + \gamma s_2) = \gamma\rho s_2 + s_1(1 + \gamma s_2).$$

Next, subtract s_2 for

$$s_1(\gamma\rho + \gamma s_2) = \gamma\rho s_2 + s_1(1 + \gamma s_2) - s_2.$$

Finally, subtract $s_1(1 + \gamma s_2)$ for

$$s_1(\gamma\rho + \gamma s_2) - s_1(1 + \gamma s_2) = \gamma\rho s_2 - s_2.$$

Now we can simplify the left side of this equation by removing the addition and subsequent subtraction of $s_1\gamma s_2$ for the equation

$$\begin{aligned} s_1(\gamma\rho) - s_1(1) &= \gamma\rho s_2 - s_2 \\ s_1(\gamma\rho - 1) &= \gamma\rho s_2 - s_2. \end{aligned}$$

Isolate the variable s_2 to achieve

$$s_1(\gamma\rho - 1) = s_2(\gamma\rho - 1)$$

Lastly, divide both sides by $\gamma\rho - 1$ for

$$s_1 = s_2$$

Because we have shown that if $f(s_1) = f(s_2)$, then it is known that $s_1 = s_2$, we have proved that the function f is an injection.

Since we have shown that f is both surjective and injective, f is a bijection, as desired.

$\square$

3. **Proposition.** Let S, T, and R be sets, and let $f : S \to T$ and $g : T \to R$ be functions. Show that if $g \circ f$ is injective, then f is injective.

   Discussion. By the definition of an injective function, if s_1 and s_2 are distinct elements, then their outputs are also distinct elements. In other words, if $g \circ f$ is injective, then when $s_1 \neq s_2$, it is true that $g \circ f(s_1) \neq g \circ f(s_2)$. Note that $g \circ f$ is a composition function so that $g \circ f : S \to R$ given by

   $$g \circ f(s) = g(f(s)).$$

   It is also worthy to note that injective function property as noted above can be manipulated using DeMorgan's Logic Law: $\neg(p \vee q) \equiv \neg p \wedge \neg q$. Thus, it is also the case that whenever $g \circ f(s_1) = g \circ f(s_2)$, then $s_1 = s_2$.

   We will use proof by contrapositive to show that f is injective; therefore, we will assume that f is not injective. Next, we will show that because f is not injective, the composition function $g \circ f$ cannot be injective either. We arrive at a contradiction that shows that $g \circ f$ cannot be injective, although it indeed has been declared injective. Thus we will prove that because the function $g \circ f$ is injective, f must be injective, as well.

   Proof. Assume distinct elements $s_1, s_2 \in S$ such that $s_1 \neq s_2$. Since $g \circ f$ is injective, then when $s_1 \neq s_2$, it is true that $g \circ f(s_1) \neq g \circ f(s_2)$ by the property of injective functions. Furthermore, assume that the function f is not injective. Thus, there are such $s_1 \neq s_2$ so that $f(s_1) = f(s_2)$. Now, we will input these elements into our composition function $g \circ f$. Since $g \circ f$ is injective, we can see that when $f(s_1) = f(s_2)$, it is implied that $g(f(s_1)) = g(f(s_2))$ by the property of injective functions. In other notation, $g \circ f(s_1) = g \circ (s_2)$.

   However, from the very beginning, we concluded that because $s_1 \neq s_2$, it must be true that $g \circ f(s_1) \neq g \circ f(s_2)$. Thus, when we assume that f is not injective, we arrive at a contradiction and are unable to show that $g \circ f$ is injective. Therefore, for $g \circ f$ to be injective, as declared, f must be injective too, as desired.

   $\square$

4. Let $C([0,1])$ be the set of all real, continuous functions on the interval [0,1]. That is,

   $$C([0,1]) = \{f | f : [0,1] \to \mathbb{R} \text{ is a continuous function}\}.$$

   Thus, an element of the set $C([0,1])$ is simply a function $f(x)$ that is continuous on [0,1]. Furthermore, consider the function $\varphi : C([0,1]) \to \mathbb{R}$ given by

   $$\varphi(f) = \int_0^1 f(x)dx.$$

   (a) **Proposition.** The function φ is surjective because for every $a \in \mathbb{R}$, there exists a pre-image $f \in C([0,1])$ such that $\varphi(f) = a$.

Discussion. A function is surjective when the image of the function is the entire co-domain. To prove that the function $\varphi(f) = \int_0^1 f(x)dx$ is surjective, we will show that for every $a \in \mathbb{R}$, the output set, there exists a pre-image $f \in C([0,1])$ such that $\varphi(f) = a$. In other words, for every $a \in \mathbb{R}$, there is a real, continuous function that defines an area of a under itself and above the x-axis over the closed interval [0,1].

Proof. Consider the function $f(x) = n$ where $n \in \mathbb{R}$. The function f is any function that is equal to some real number n at any point x. Therefore, f is a real, continuous function over [0,1], and is thus contained in set C. We will apply $f(x) = n$ to the function φ to see that

$$\varphi(f(x)) = \int_0^1 f(x)dx = \int_0^1 ndx = nx\Big|_0^1 = n(1) - n(0) = n.$$

We can see that for some function $f(x) = n$ in φ, there exists some output $n \in \mathbb{R}$. Having already stated that n can be any number contained in $\mathbb{R}$, it becomes apparent that there is indeed some function f for every $a \in \mathbb{R}$. Therefore, the function φ is surjective, as desired.

$\square$

(b) **Proposition.** The function φ is not injective because there exist two distinct functions $f, g \in C([0,1])$ such that $\varphi(f) = \varphi(g)$.

Discussion. An injective function has the property that, for two distinct inputs, there are two distinct outputs. So, to prove that the function φ is not injective, we will simply procure two distinct example functions $f, g \in C([0,1])$ and demonstrate that $\varphi(f) = \varphi(g)$. Therefore, the function φ is not injective.

Proof. Consider a function $f(x) = 1$ and a function $g(x) = 2x$. These two functions are contained in the set C because they are real, continuous functions over the closed interval [0,1].

If it were the case that the function φ is injective, it would be true that for two distinct inputs, such as the functions f and g defined, then $\varphi(f) \neq \varphi(g)$. However, this is not the case.

$$\varphi(f(x)) = \int_0^1 f(x)dx = \int_0^1 (1)dx = x\Big|_0^1 = 1 - 0 = 1$$
$$\varphi(g(x)) = \int_0^1 g(x)dx = \int_0^1 (2x)dx = x^2\Big|_0^1 = 1^2 - 0^2 = 1$$

Thus, because $\varphi(f) = 1 = \varphi(g)$ despite having two distinct inputs f and g, the function φ violates the properties of an injection. Therefore, the φ function is not injective, as desired.

$\square$