Quantum Field Theory for Mathematicians

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Lecture 1

1.1 History

Date	People	What	Why	Techniques
1969	Faddeev and Popov	Gauge fixing (adding ghosts)	Quantize Yang-Mills	Berezinian integration
1973	't Hooft and Veltman	Quantized Yang-Mills	Quantize Yang-Mills	Feynman diagrams
1975	Becchi, Rouet, Stora, Tyutin (BRST)	Cohomological theory to quantize Yang-Mills	Understanding 't Hooft and Veltman	Derived invariants (Lie algebra cohomology)
1981	Batallin and Vilkovisky (BV)	Quantize systems with complicated gauge symmetries	Supergravity	Derived intersections (Koszul complexes)
1992	Henneaux	Quantize Yang-Mills using BV	Analyze Yang-Mills using BV	Derived intersections
2007	Costello	Combine BV with effective field theory	Make BV quantization rigorous	Derived everything, analysis, and homotopy theory

1.2 References

The main references for this seminar will be:

- Costello - Renormalization and Effective Field Theory [Cos11];
- Elliot, Williams, Yoo - Asymptotic Freedom in the BV Formalism [**elliottAsymptotic2018**];
- Gwilliam - Factorization algebras and free field theories [**gwilliamFactorization**].

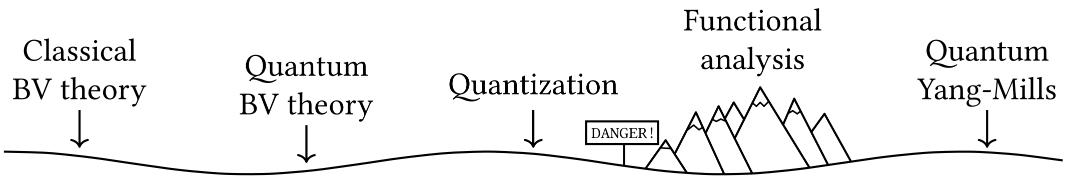

Figure 1.1: Roadmap to BV quantization.

1.3 Roadmap to BV Quantization

The space of fields $\mathscr{E}^\bullet$ is a cochain complex

$$\dots \longrightarrow \mathscr{E}^{-1} \xrightarrow{Q} \mathscr{E}^0 \xrightarrow{Q} \mathscr{E}^1 \xrightarrow{Q} \mathscr{E}^2 \longrightarrow \dots$$

equipped with a differential Q such that $Q^2 = 0$. Moreover, $\mathscr{E}$ admits a -1-shifted symplectic structure, that is, there exists a non degenerate pairing of degree -1

$$\langle \cdot, \cdot \rangle : \mathscr{E} \otimes \mathscr{E} \longrightarrow \mathbb{R}[-1]$$

such that $\langle x, y \rangle = -(-1)^{(|x|+1)(|y|+1)} \langle y, x \rangle$. This structure defines a $+1$-shifted Poisson bracket

$$\{\cdot, \cdot\} : \mathscr{O}(\mathscr{E}) \otimes \mathscr{O}(\mathscr{E}) \longrightarrow \mathscr{O}(\mathscr{E})$$

where $\mathscr{O}(\mathscr{E}) \cong \mathrm{Sym}^\bullet(\mathscr{E}^\vee)$ is the (graded) commutative algebra of polynomial functions on the dual complex $\mathscr{E}^\vee$. Pick $S \in \mathscr{O}(\mathscr{E})$ obeying the **classical master equation** (CME)

$$\{S, S\} = 0. \tag{1.1}$$

The data $(\mathscr{E}, \langle \cdot, \cdot \rangle, S)$ defines a **classical BV theory**. The CME says $\{S, \cdot\}$ is a differential which makes $(\mathscr{O}(\mathscr{E}), \{S, \cdot\})$ into a cochain complex such that

$$\mathrm{H}^0 \mathscr{O}(\mathscr{E}) \cong \mathscr{O}(\mathrm{Crit}(S)),$$

where $\mathrm{Crit}(S)$ denotes the critical locus of S. We will restrict to S of the form

$$S(e) = \underbrace{\langle e, Qe \rangle}_{\substack{\text{free part} \\ \text{(kinetic +} \\ \text{mass terms)}}} + \underbrace{I(e)}_{\substack{\text{interaction} \\ \text{part (cubic} \\ \text{or higher)}}} .$$

Example. Why are the cubic and higher order terms called interaction terms? For electromagnetism on a manifold M we have a space of fields $\mathscr{F} = \Omega^1(M) \oplus \Omega^0(M, S)$ in degree 0. Let $F = \mathrm{d}A$ and define

$$S(A, \psi) = \int_M \underbrace{F \wedge \star F + \langle \psi, \not{d} \psi \rangle \,\mathrm{dvol}}_{\text{quadratic terms}} + \underbrace{\langle \psi, \not{A} \psi \rangle \,\mathrm{dvol}}_{\text{interaction term}} .$$

Computing the Euler-Lagrange equations we obtain the system of differential equations

$$\begin{cases} \star \mathrm{d} \star F = \bar{\psi} \gamma^\mu \psi \,\mathrm{d}x_\mu \\ \not{d}_A \psi = 0 \end{cases}$$

which is coupled because of the interaction term.

1.4 Quantization in the BV formalism

The slogan of quantization in the BV formalism is to *deform the differential.* In the perturbative context we work in formal power series in $\hbar$, for example, over the ring $\mathbb{R}[[\hbar]]$. Quantization results in a cochain complex $(\mathcal{O}(\mathscr{E})[[\hbar]], \{S^q, \cdot\} + \hbar\Delta)$, where Δ is called the BV Laplacian, and $S^q \in \mathcal{O}(\mathscr{E})[[\hbar]]$ satisfies the **quantum master equation** (QME)

$$(\{S^q, \cdot\} + \hbar\Delta)^2 = 0 \tag{1.2}$$

Example. In finite dimensions ($\mathscr{F} \cong \mathbb{R}^n$) the BV fields are $\mathscr{E} = \mathbb{R}^n \longrightarrow \mathbb{R}^n$ therefore

$$\mathcal{O}(\mathscr{E}) \cong \mathbb{R}[x^1, \dots, x^n, \xi^1, \dots, \xi^n]$$

and the BV Laplacian takes the form

$$\Delta = \sum_{\mu=1}^{n} \frac{\partial}{\partial \xi^\mu} \frac{\partial}{\partial x^\mu}.$$

In this form, it becomes clear that Δ is a differential operator of degree 1 such that $\Delta^2 = 0$.

$$S^q(e) = \langle e, Qe \rangle + I^q(e)$$

where $I^q \in \mathcal{O}(\mathscr{E})[[\hbar]]$ is cubic mod $\hbar$ and satisfies the QME

$$QI^q + \frac{1}{2}\{I^q, I^q\} + \hbar\Delta I^q = 0$$

which resembles, in this form, the **Maurer-Cartan** (MC) **equation**. In infinite dimensions, some problems arise:

1. there may be no solution to this equation. In this case we say that quantization is obstructed (there is an anomaly);

2. the QME in infinite dimensions is ill-defined. Some functional analysis is needed to make sense of this problem.

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Lecture 2

In this lecture we consider a naive example that aims to exemplify how the Euler-Lagrange equations lead us to classical BV theories.

Example. Let $\mathscr{F}$ be a finite-dimensional vector space encoding the naive space of fields and consider an action

$$S : \mathscr{F} \longrightarrow \mathbb{R}.$$

We say that S is a naive action because it might be necessary to add additional terms to S to guarantee that it satisfies the CME. The solutions to the Euler-Lagrange equation are fields $f \in \mathscr{F}$ such that $\mathrm{d}S_f = 0$. Restricting to the case $\mathscr{F} = M$ for some finite-dimensional manifold M, we say that critical points of the action form the **critical locus** of S

$$\mathrm{Crit}(S) = \left\{ p \in M \,\middle|\, \mathrm{d}S_p = 0 \right\}.$$

Alternatively, we can characterize the critical locus of S as an intersection in T^*M

$$\mathrm{Crit}(S) = \mathrm{Graph}(\mathrm{d}S) \cap \mathrm{Graph}(M)$$

where we identify M with the zero section. It follows that

$$\mathscr{O}(\mathrm{Crit}(S)) = \mathscr{O}(\mathrm{Graph}(\mathrm{d}S)) \otimes_{\mathscr{O}(T^*M)} \mathscr{O}(M).$$

We are going to consider a derived version of this construction, where the tensor product $\otimes$ is replaced by a derived tensor product $\otimes^{\mathbb{L}}$. This raises the obvious questions:

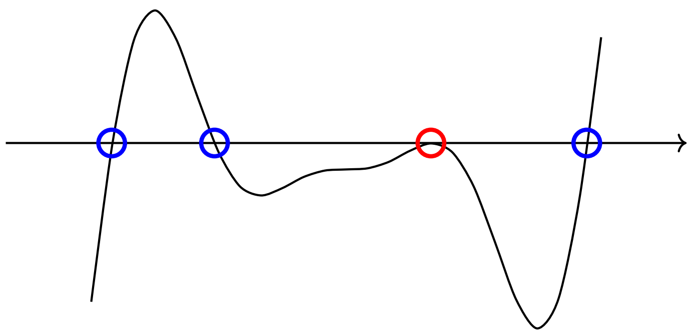

Figure 2.1: Well-behaved (in red) and badly-behaved (in red) points of an intersection.

- **Why?** This intersection might not be well behaved, in the sense that dS and the zero section might not intersect transversally at every point, as illustrated in figure 2.1. The derived approach allows us to study these badly-behaved points using Serre's intersection formula.

- **How?** We replace $\mathcal{O}(\mathrm{Graph}(\mathrm{d}S)) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M)$ with a dg commutative algebra A such that
$$\mathrm{H}^0 A = \mathcal{O}(\mathrm{Graph}(\mathrm{d}S)) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M).$$
To compute the derived tensor product $\otimes^{\mathbb{L}}$ we need to resolve either $\mathcal{O}(M)$ or $\mathcal{O}(\mathrm{Graph}(\mathrm{d}S))$ in $\mathcal{O}(T^*M)$-modules. Let us make use of Darboux coordinates to resolve
$$\begin{aligned}\mathcal{O}(\mathrm{Graph}(\mathrm{d}S)) &= \mathcal{O}(T^*M)\Big/\big(f|_{\mathrm{Graph}(\mathrm{d}S)} = 0\big)\\ &= \mathcal{O}(T^*M)\Big/\big(p_\mu - \partial_\mu S\big).\end{aligned}$$
Consider the resolution
$$\dots \longrightarrow \mathcal{O}(T^*M)(\xi_1, \dots, \xi_n) \xrightarrow{\xi_\mu \mapsto p_\mu - \partial_\mu S} \mathcal{O}(T^*M) \longrightarrow \mathcal{O}(\mathrm{Graph}(\mathrm{d}S))$$
which we extend to the left as a Koszul complex $K^{-p} = \bigwedge^p_{\mathcal{O}(T^*M)}(\xi_1, \dots, \xi_n)$ with differential
$$\mathrm{d} = \sum_\mu (p_\mu - \partial_\mu S)\frac{\partial}{\partial \xi_\mu}.$$
This complex freely resolves $\mathcal{O}(\mathrm{Graph}(\mathrm{d}S))$. Alternatively, $(K^\bullet, \mathrm{d})$ admits a coordinate free description where
$$K^{-p} = \mathcal{O}(T^*M) \otimes_{\mathcal{O}(M)} \mathfrak{X}^p(M).$$
A model for $\mathcal{O}(\mathrm{Graph}(\mathrm{d}S)) \otimes^{\mathbb{L}}_{\mathcal{O}(T^*M)} \mathcal{O}(M)$ is given by
$$\mathcal{O}(\mathrm{dCrit}(S)) = K^{-\bullet} \otimes_{T^*M} \mathcal{O}(M)$$
which we call the **derived critial locus**. But notice that
$$\mathcal{O}(T^*M) \otimes_{\mathcal{O}(M)} \mathfrak{X}^\bullet(M) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M) \cong \mathfrak{X}^\bullet(M)$$
where $\mathfrak{X}^\bullet(M)$ denotes the complex of polyvector fields on M. The differential is given by contracting with dS, so we write
$$\mathcal{O}(\mathrm{dCrit}(S)) = (\mathfrak{X}^\bullet(M), \iota_{\mathrm{d}S}).$$

Bibliography

[Cos11] Kevin Costello. *Renormalization and Effective Field Theory*. Mathematical Surveys and Monographs 170. American Mathematical Society, 2011. ISBN: 978-0-8218-5288-0.