

Quantum Field Theory for Mathematicians

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Lecture 1

1.1 History

| Date | People | What | Why | Techniques |
|------|-------------------------------------|--|------------------------------------|---|
| 1969 | Faddeev and Popov | Gauge fixing (adding ghosts) | Quantize Yang-Mills | Berezinian integration |
| 1973 | 't Hooft and Veltman | Quantized Yang-Mills | Quantize Yang-Mills | Feynman diagrams |
| 1975 | Becchi, Rouet, Stora, Tyutin (BRST) | Cohomological theory to quantize Yang-Mills | Understanding 't Hooft and Veltman | Derived invariants (Lie algebra cohomology) |
| 1981 | Batallin and Vilkovisky (BV) | Quantize systems with complicated gauge symmetries | Supergravity | Derived intersections (Koszul complexes) |
| 1992 | Henneaux | Quantize Yang-Mills using BV | Analyze Yang-Mills using BV | Derived intersections |
| 2007 | Costello | Combine BV with effective field theory | Make BV quantization rigorous | Derived everything, analysis, and homotopy theory |

1.2 References

The main references for this seminar will be:

- Costello - Renormalization and Effective Field Theory [Cos11];
- Elliot, Williams, Yoo - Asymptotic Freedom in the BV Formalism [EWY18];
- Gwilliam - Factorization algebras and free field theories [Gwi].

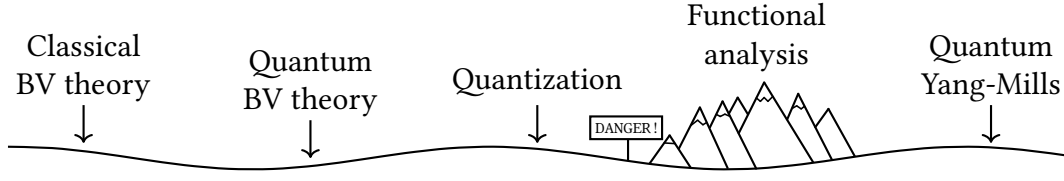


Figure 1.1: Roadmap to BV quantization.

1.3 Roadmap to BV Quantization

The space of fields \mathcal{E}^\bullet is a cochain complex

$$\dots \longrightarrow \mathcal{E}^{-1} \xrightarrow{Q} \mathcal{E}^0 \xrightarrow{Q} \mathcal{E}^1 \xrightarrow{Q} \mathcal{E}^2 \longrightarrow \dots$$

equipped with a differential Q such that $Q^2 = 0$. Moreover, \mathcal{E} admits a -1 -shifted symplectic structure, that is, there exists a non degenerate pairing of degree -1

$$\langle \cdot, \cdot \rangle : \mathcal{E} \otimes \mathcal{E} \longrightarrow \mathbb{R}[-1]$$

such that $\langle x, y \rangle = -(-1)^{(|x|+1)(|y|+1)} \langle y, x \rangle$. This structure defines a $+1$ -shifted Poisson bracket

$$\{ \cdot, \cdot \} : \mathcal{O}(\mathcal{E}) \otimes \mathcal{O}(\mathcal{E}) \longrightarrow \mathcal{O}(\mathcal{E})$$

where $\mathcal{O}(\mathcal{E}) \cong \text{Sym}^\bullet(\mathcal{E}^\vee)$ is the (graded) commutative algebra of polynomial functions on the dual complex \mathcal{E}^\vee . Pick $S \in \mathcal{O}(\mathcal{E})$ obeying the **classical master equation** (CME)

$$\{S, S\} = 0. \quad (1.1)$$

The data $(\mathcal{E}, \langle \cdot, \cdot \rangle, S)$ defines a **classical BV theory**. The CME says $\{S, \cdot\}$ is a differential which makes $(\mathcal{O}(\mathcal{E}), \{S, \cdot\})$ into a cochain complex such that

$$H^0 \mathcal{O}(\mathcal{E}) \cong \mathcal{O}(\text{Crit}(S)),$$

where $\text{Crit}(S)$ denotes the critical locus of S . We will restrict to S of the form

$$S(e) = \underbrace{\langle e, Qe \rangle}_{\substack{\text{free part} \\ \text{(kinetic +} \\ \text{mass terms)}}} + \underbrace{I(e)}_{\substack{\text{interaction} \\ \text{part (cubic} \\ \text{or higher)}}}.$$

Example. Why are the cubic and higher order terms called interaction terms? For electromagnetism on a manifold M we have a space of fields $\mathcal{F} = \Omega^1(M) \oplus \Omega^0(M, S)$ in degree 0. Let $F = dA$ and define

$$S(A, \psi) = \int_M \underbrace{F \wedge \star F + \langle \psi, \not{d}\psi \rangle \text{ dvol}}_{\text{quadratic terms}} + \underbrace{\langle \psi, \not{A}\psi \rangle \text{ dvol}}_{\text{interaction term}}.$$

Computing the Euler-Lagrange equations we obtain the system of differential equations

$$\begin{cases} \star d\star F = \bar{\psi} \gamma^\mu \psi dx_\mu \\ \not{d}_A \psi = 0 \end{cases}$$

which is coupled because of the interaction term.

1.4 Quantization in the BV formalism

The slogan of quantization in the BV formalism is to *deform the differential*. In the perturbative context we work in formal power series in \hbar , for example, over the ring $\mathbb{R}[[\hbar]]$. Quantization results in a cochain complex $(\mathcal{O}(\mathcal{E})[[\hbar]], \{S^q, \cdot\} + \hbar\Delta)$, where Δ is called the BV Laplacian, and $S^q \in \mathcal{O}(\mathcal{E})[[\hbar]]$ satisfies the **quantum master equation** (QME)

$$(\{S^q, \cdot\} + \hbar\Delta)^2 = 0 \quad (1.2)$$

Example. In finite dimensions ($\mathcal{F} \cong \mathbb{R}^n$) the BV fields are $\mathcal{E} = \mathbb{R}^n \longrightarrow \mathbb{R}^n$ therefore

$$\mathcal{O}(\mathcal{E}) \cong \mathbb{R}[x^1, \dots, x^n, \xi^1, \dots, \xi^n]$$

and the BV Laplacian takes the form

$$\Delta = \sum_{\mu=1}^n \frac{\partial}{\partial \xi^\mu} \frac{\partial}{\partial x^\mu}.$$

In this form, it becomes clear that Δ is a differential operator of degree 1 such that $\Delta^2 = 0$.

$$S^q(e) = \langle e, Qe \rangle + I^q(e)$$

where $I^q \in \mathcal{O}(\mathcal{E})[[\hbar]]$ is cubic mod \hbar and satisfies the QME

$$QI^q + \frac{1}{2}\{I^q, I^q\} + \hbar\Delta I^q = 0$$

which resembles, in this form, the **Maurer-Cartan (MC) equation**. In infinite dimensions, some problems arise:

1. there may be no solution to this equation. In this case we say that quantization is obstructed (there is an anomaly);
2. the QME in infinite dimensions is ill-defined. Some functional analysis is needed to make sense of this problem.

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Lecture 2

In this lecture we consider a naive example that aims to exemplify how the Euler-Lagrange equations lead us to classical BV theories.

Example. Let \mathcal{F} be a finite-dimensional vector space encoding the naive space of fields and consider an action

$$S : \mathcal{F} \longrightarrow \mathbb{R}.$$

We say that S is a naive action because it might be necessary to add additional terms to S to guarantee that it satisfies the CME. The solutions to the Euler-Lagrange equation are fields $f \in \mathcal{F}$ such that $dS_f = 0$. Restricting to the case $\mathcal{F} = M$ for some finite-dimensional manifold M , we say that critical points of the action form the **critical locus** of S

$$\text{Crit}(S) = \{p \in M \mid dS_p = 0\}.$$

Alternatively, we can characterize the critical locus of S as an intersection in T^*M

$$\text{Crit}(S) = \text{Graph}(dS) \cap \text{Graph}(M)$$

where we identify M with the zero section. It follows that

$$\mathcal{O}(\text{Crit}(S)) = \mathcal{O}(\text{Graph}(dS)) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M).$$

We are going to consider a derived version of this construction, where the tensor product \otimes is replaced by a derived tensor product $\otimes^{\mathbb{L}}$. This raises the obvious questions:

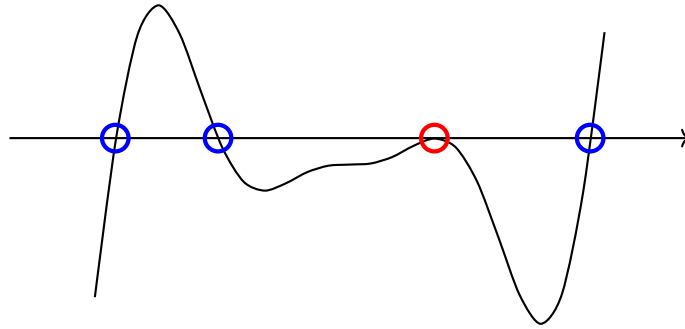


Figure 2.1: Well-behaved (in red) and badly-behaved (in red) points of an intersection.

- **Why?** This intersection might not be well behaved, in the sense that dS and the zero section might not intersect transversally at every point, as illustrated in figure 2.1. The derived approach allows us to study these badly-behaved points using Serre's intersection formula.
- **How?** We replace $\mathcal{O}(\text{Graph}(dS)) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M)$ with a dg commutative algebra A such that

$$H^0 A = \mathcal{O}(\text{Graph}(dS)) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M).$$

To compute the derived tensor product $\otimes^{\mathbb{L}}$ we need to resolve either $\mathcal{O}(M)$ or $\mathcal{O}(\text{Graph}(dS))$ in $\mathcal{O}(T^*M)$ -modules. Let us make use of Darboux coordinates to resolve

$$\begin{aligned} \mathcal{O}(\text{Graph}(dS)) &= \mathcal{O}(T^*M) / (f|_{\text{Graph}(dS)} = 0) \\ &= \mathcal{O}(T^*M) / (p_\mu - \partial_\mu S). \end{aligned}$$

Consider the resolution

$$\dots \longrightarrow \mathcal{O}(T^*M)(\xi_1, \dots, \xi_n) \xrightarrow{\xi_\mu \mapsto p_\mu - \partial_\mu S} \mathcal{O}(T^*M) \longrightarrow \mathcal{O}(\text{Graph}(dS))$$

which we extend to the left as a Koszul complex $K^{-p} = \bigwedge_{\mathcal{O}(T^*M)}^p (\xi_1, \dots, \xi_n)$ with differential

$$d = \sum_{\mu} (p_\mu - \partial_\mu S) \frac{\partial}{\partial \xi_\mu}.$$

This complex freely resolves $\mathcal{O}(\text{Graph}(dS))$. Alternatively, (K^\bullet, d) admits a coordinate free description where

$$K^{-p} = \mathcal{O}(T^*M) \otimes_{\mathcal{O}(M)} \mathfrak{X}^p(M).$$

A model for $\mathcal{O}(\text{Graph}(dS)) \otimes_{\mathcal{O}(T^*M)}^{\mathbb{L}} \mathcal{O}(M)$ is given by

$$\mathcal{O}(\text{dCrit}(S)) = K^{-\bullet} \otimes_{T^*M} \mathcal{O}(M)$$

which we call the **derived critical locus**. But notice that

$$\mathcal{O}(T^*M) \otimes_{\mathcal{O}(M)} \mathfrak{X}^\bullet(M) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M) \cong \mathfrak{X}^\bullet(M)$$

where $\mathfrak{X}^\bullet(M)$ denotes the complex of polyvector fields on M . The differential is given by contracting with dS , so we write

$$\mathcal{O}(\text{dCrit}(S)) = (\mathfrak{X}^\bullet(M), \iota_{dS}).$$

Bibliography

- [Cos11] Kevin Costello. *Renormalization and Effective Field Theory*. 1st ed. Vol. 170. Mathematical Surveys and Monographs 81. 2011. ISBN: 978-1-4704-7008-1.
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- [Gwi] Owen Gwilliam. “Factorization Algebras and Free Field Theories”.