Quantum Field Theory for Mathematicians

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Lecture 1

1.1 History

Date	People	What	Why	Techniques
1969	Faddeev and Popov	Gauge fixing (adding ghosts)	Quantize Yang-Mills	Berezinian integration
1973	’t Hooft and Veltman	Quantized Yang-Mills	Quantize Yang-Mills	Feynman diagrams
1975	Becchi, Rouet, Stora, Tyutin (BRST)	Cohomological theory to quantize Yang-Mills	Understanding ’t Hooft and Veltman	Derived invariants (Lie algebra cohomology)
1981	Batallin and Vilkovisky (BV)	Quantize systems with complicated gauge symmetries	Supergravity	Derived intersections (Koszul complexes)
1992	Henneaux	Quantize Yang-Mills using BV	Analyze Yang-Mills using BV	Derived intersections
2007	Costello	Combine BV with effective field theory	Make BV quantization rigorous	Derived everything, analysis, and homotopy theory

1.2 References

The main references for this seminar will be:

- Costello - Renormalization and Effective Field Theory [**CosRenormalization11**];
- Elliot, Williams, Yoo - Asymptotic Freedom in the BV Formalism [**EWYAsymptotic18**];
- Gwilliam - Factorization algebras and free field theories [**GwiFactorization**].

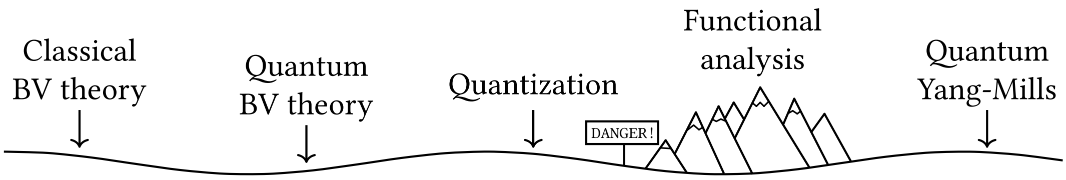

Figure 1.1: Roadmap to BV quantization.

1.3 Roadmap to BV Quantization

The space of fields $\mathscr{E}^\bullet$ is a cochain complex

$$\dots \longrightarrow \mathscr{E}^{-1} \xrightarrow{Q} \mathscr{E}^0 \xrightarrow{Q} \mathscr{E}^1 \xrightarrow{Q} \mathscr{E}^2 \longrightarrow \dots$$

equipped with a differential Q such that $Q^2 = 0$. Moreover, $\mathscr{E}$ admits a -1-shifted symplectic structure, that is, there exists a non degenerate pairing of degree -1

$$\langle \cdot , \cdot \rangle : \mathscr{E} \otimes \mathscr{E} \longrightarrow \mathbb{R}[-1]$$

such that $\langle x, y\rangle = -(-1)^{(|x|+1)(|y|+1)}\langle y, x\rangle$. This structure defines a $+1$-shifted Poisson bracket

$$\{\cdot , \cdot\} : \mathscr{O}(\mathscr{E}) \otimes \mathscr{O}(\mathscr{E}) \longrightarrow \mathscr{O}(\mathscr{E})$$

where $\mathscr{O}(\mathscr{E}) \cong \mathrm{Sym}^\bullet(\mathscr{E}^\vee)$ is the (graded) commutative algebra of polynomial functions on the dual complex $\mathscr{E}^\vee$. Pick $S \in \mathscr{O}(\mathscr{E})$ obeying the **classical master equation** (CME)

$$\{S, S\} = 0. \tag{1.1}$$

The data $(\mathscr{E}, \langle \cdot , \cdot \rangle, S)$ defines a **classical BV theory**. The CME says $\{S, \cdot\}$ is a differential which makes $(\mathscr{O}(\mathscr{E}), \{S, \cdot\})$ into a cochain complex such that

$$\mathrm{H}^0\mathscr{O}(\mathscr{E}) \cong \mathscr{O}(\mathrm{Crit}(S)),$$

where $\mathrm{Crit}(S)$ denotes the critical locus of S. We will restrict to S of the form

$$S(e) = \underbrace{\langle e, Qe\rangle}_{\substack{\text{free part}\\ \text{(kinetic +}\\ \text{mass terms)}}} + \underbrace{I(e)}_{\substack{\text{interaction}\\ \text{part (cubic}\\ \text{or higher)}}}.$$

Example. Why are the cubic and higher order terms called interaction terms? For electromagnetism on a manifold M we have a space of fields $\mathscr{F} = \Omega^1(M) \oplus \Omega^0(M, S)$ in degree 0. Let $F = \mathrm{d}A$ and define

$$S(A, \psi) = \int_M \underbrace{F \wedge \star F + \langle \psi, \not{\mathrm{d}}\psi\rangle\, \mathrm{dvol}}_{\text{quadratic terms}} + \underbrace{\langle \psi, \not{A}\psi\rangle\, \mathrm{dvol}}_{\text{interaction term}}.$$

Computing the Euler-Lagrange equations we obtain the system of differential equations

$$\begin{cases} \star\mathrm{d}\star F = \bar{\psi}\gamma^\mu\psi\,\mathrm{d}x_\mu \\ \not{\mathrm{d}}_A\psi = 0 \end{cases}$$

which is coupled because of the interaction term.

1.4 Quantization in the BV formalism

The slogan of quantization in the BV formalism is to *deform the differential.* In the perturbative context we work in formal power series in $\hbar$, for example, over the ring $\mathbb{R}[[\hbar]]$. Quantization results in a cochain complex $(\mathscr{O}(\mathscr{E})[[\hbar]], \{S^q, \cdot\} + \hbar\Delta)$, where Δ is called the BV Laplacian, and $S^q \in \mathscr{O}(\mathscr{E})[[\hbar]]$ satisfies the **quantum master equation** (QME)

$$(\{S^q, \cdot\} + \hbar\Delta)^2 = 0 \tag{1.2}$$

Example. In finite dimensions ($\mathscr{F} \cong \mathbb{R}^n$) the BV fields are $\mathscr{E} = \mathbb{R}^n \longrightarrow \mathbb{R}^n$ therefore

$$\mathscr{O}(\mathscr{E}) \cong \mathbb{R}[x^1, \dots, x^n, \xi^1, \dots, \xi^n]$$

and the BV Laplacian takes the form

$$\Delta = \sum_{\mu=1}^{n} \frac{\partial}{\partial \xi^\mu} \frac{\partial}{\partial x^\mu}.$$

In this form, it becomes clear that Δ is a differential operator of degree 1 such that $\Delta^2 = 0$.

$$S^q(e) = \langle e, Qe \rangle + I^q(e)$$

where $I^q \in \mathscr{O}(\mathscr{E})[[\hbar]]$ is cubic mod $\hbar$ and satisfies the QME

$$QI^q + \frac{1}{2}\{I^q, I^q\} + \hbar\Delta I^q = 0$$

which resembles, in this form, the **Maurer-Cartan** (MC) **equation**. In infinite dimensions, some problems arise:

1. there may be no solution to this equation. In this case we say that quantization is obstructed (there is an anomaly);

2. the QME in infinite dimensions is ill-defined. Some functional analysis is needed to make sense of this problem.

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Lecture 2

In this lecture we consider a naive example that aims to exemplify how the Euler-Lagrange equations lead us to classical BV theories.

Example. Let $\mathscr{F}$ be a finite-dimensional vector space encoding the naive space of fields and consider an action

$$S : \mathscr{F} \longrightarrow \mathbb{R}.$$

We say that S is a naive action because it might be necessary to add additional terms to S to guarantee that it satisfies the CME. The solutions to the Euler-Lagrange equation are fields $f \in \mathscr{F}$ such that $\mathrm{d}S_f = 0$. Restricting to the case $\mathscr{F} = M$ for some finite-dimensional manifold M, we say that critical points of the action form the **critical locus** of S

$$\mathrm{Crit}(S) = \left\{p \in M \,\middle|\, \mathrm{d}S_p = 0\right\}.$$

Alternatively, we can characterize the critical locus of S as an intersection in T^*M

$$\mathrm{Crit}(S) = \mathrm{Graph}(\mathrm{d}S) \cap \mathrm{Graph}(M)$$

where we identify M with the zero section. It follows that

$$\mathscr{O}(\mathrm{Crit}(S)) = \mathscr{O}(\mathrm{Graph}(\mathrm{d}S)) \otimes_{\mathscr{O}(T^*M)} \mathscr{O}(M).$$

We are going to consider a derived version of this construction, where the tensor product $\otimes$ is replaced by a derived tensor product $\otimes^{\mathbb{L}}$. This raises the obvious questions:

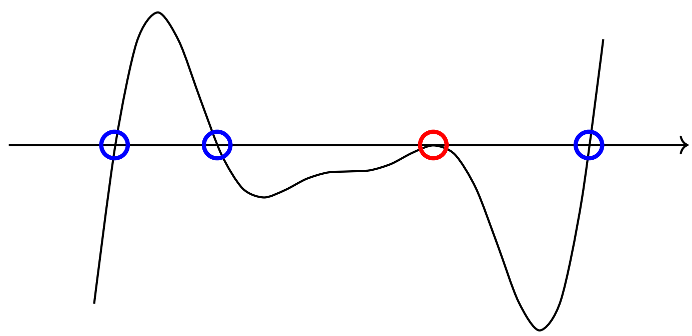

Figure 2.1: Well-behaved (in red) and badly-behaved (in red) points of an intersection.

- **Why?** This intersection might not be well behaved, in the sense that $\mathrm{d}S$ and the zero section might not intersect transversally at every point, as illustrated in figure 2.1. The derived approach allows us to study these badly-behaved points using Serre's intersection formula.

- **How?** We replace $\mathcal{O}(\mathrm{Graph}(\mathrm{d}S)) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M)$ with a dg commutative algebra A such that
$$\mathrm{H}^0 A = \mathcal{O}(\mathrm{Graph}(\mathrm{d}S)) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M).$$
To compute the derived tensor product $\otimes^{\mathbb{L}}$ we need to resolve either $\mathcal{O}(M)$ or $\mathcal{O}(\mathrm{Graph}(\mathrm{d}S))$ in $\mathcal{O}(T^*M)$-modules. Let us make use of Darboux coordinates to resolve
$$\begin{aligned}\mathcal{O}(\mathrm{Graph}(\mathrm{d}S)) &= \mathcal{O}(T^*M)\Big/\big(f|_{\mathrm{Graph}(\mathrm{d}S)} = 0\big)\\ &= \mathcal{O}(T^*M)\Big/\big(p_\mu - \partial_\mu S\big).\end{aligned}$$
Consider the resolution
$$\cdots \longrightarrow \mathcal{O}(T^*M)(\xi_1, \dots, \xi_n) \xrightarrow{\xi_\mu \mapsto p_\mu - \partial_\mu S} \mathcal{O}(T^*M) \longrightarrow \mathcal{O}(\mathrm{Graph}(\mathrm{d}S))$$
which we extend to the left as a Koszul complex $K^{-p} = \bigwedge^p_{\mathcal{O}(T^*M)}(\xi_1, \dots, \xi_n)$ with differential
$$\mathrm{d} = \sum_\mu (p_\mu - \partial_\mu S)\frac{\partial}{\partial \xi_\mu}.$$
This complex freely resolves $\mathcal{O}(\mathrm{Graph}(\mathrm{d}S))$. Alternatively, $(K^\bullet, \mathrm{d})$ admits a coordinate free description where
$$K^{-p} = \mathcal{O}(T^*M) \otimes_{\mathcal{O}(M)} \mathfrak{X}^p(M).$$
A model for $\mathcal{O}(\mathrm{Graph}(\mathrm{d}S)) \otimes^{\mathbb{L}}_{\mathcal{O}(T^*M)} \mathcal{O}(M)$ is given by
$$\mathcal{O}(\mathrm{dCrit}(S)) = K^{-\bullet} \otimes_{T^*M} \mathcal{O}(M)$$
which we call the **derived critical locus**. But notice that
$$\mathcal{O}(T^*M) \otimes_{\mathcal{O}(M)} \mathrm{PV}^\bullet(M) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M) \cong \mathrm{PV}^\bullet(M)$$
where $\mathrm{PV}^\bullet(M)$ denotes the complex of polyvector fields on M. The differential is given by contracting with $\mathrm{d}S$, so we write
$$\mathcal{O}(\mathrm{dCrit}(S)) = (\mathrm{PV}^\bullet(M), -\iota_{\mathrm{d}S}).$$

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Lecture 3

We want to sketch how to go from the Yang-Mills action

$$S^{\text{naive}}(A) = \int_{M^n} \operatorname{tr}(F_A \wedge \star F_A)$$

to the Yang-Mills classical BV theory

$$\underbrace{\overset{-1}{\Omega^0(M,\mathfrak{g})}}_{\text{ghosts}} \xrightarrow{\quad \mathrm{d} \quad} \underbrace{\overset{0}{\Omega^1(M,\mathfrak{g})}}_{\text{fields}} \xrightarrow{\quad \mathrm{d}\star\mathrm{d} \quad} \underbrace{\overset{1}{\Omega^{n-1}(M,\mathfrak{g})}}_{\text{antifields}} \xrightarrow{\quad \mathrm{d} \quad} \underbrace{\overset{2}{\Omega^n(M,\mathfrak{g})}}_{\text{antighosts}}$$

with BV action

$$S^{\text{BV}}(e) = \langle e, Qe \rangle + I(e)$$

where

$$\langle e, f \rangle = \int_{M^n} \operatorname{tr}(e \wedge f)$$

is the -1-shifted symplectic structure. There are some points to motivate:

i) **fields $\rightsquigarrow$ fields and antifields:** coming from the derived critical locus $\mathrm{dCrit}(S)$;

ii) **ghosts:** coming from taking the derived coinvariants of $\mathfrak{g} \curvearrowright V$.

For Yang-Mills the manifold M^n is spacetime and $\Omega^1(M,\mathfrak{g})$ is the space of fields. In what follows, let M be the space of fields. Recall that

$$\begin{aligned} \mathrm{Crit}(S) &= \{p \in M \mid \mathrm{d}S_p = 0\} \\ &= \mathrm{Graph}(\mathrm{d}S) \cap M \end{aligned}$$

in T^*M. Dually

$$\mathcal{O}\big(\mathrm{Crit}(S)\big) = \mathcal{O}\big(\mathrm{Graph}(\mathrm{d}S)\big) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M).$$

By homological yoga, taking the derived intersection means that we replace the tensor product $\otimes$ with the derived tensor product $\otimes^{\mathbb{L}}$. To find $\mathrm{dCrit}(S)$ we resolve either $\mathcal{O}\big(\mathrm{Graph}(\mathrm{d}S)\big)$ or $\mathcal{O}(M)$ as $\mathcal{O}(T^*M)$-modules. Last time we wrote the Koszul complex

$$K^{-p} = \mathrm{PV}^p(M) \otimes_{\mathcal{O}(M)} \mathcal{O}(T^*M)$$

where $\mathrm{PV}^p = \bigwedge^p \mathfrak{X}(M)$ and differential

$$Q : v_1 \wedge \cdots \wedge v_k \otimes 1 \longmapsto \sum_{i=1}^{k} (-1)^{i+1} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k \otimes \big(p(v_i) - \mathrm{d}S(v_i)\big)$$

Exercise. Check that $\mathrm{H}^0(K^\bullet, Q) \cong \mathcal{O}\big(\mathrm{Graph}(\mathrm{d}S)\big)$, so

$$\mathrm{dCrit}(S) \cong K^\bullet \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M) \cong \mathrm{PV}^\bullet(M)$$

and thus $\mathcal{O}\big(\mathrm{dCrit}(S)\big) \simeq (\mathrm{PV}^\bullet, -\iota_{\mathrm{d}S})$.

Exercise. Show that $\mathrm{H}^0\mathcal{O}(\mathrm{dCrit}) \cong \mathcal{O}(\mathrm{Crit})$.

We can enhance $\mathcal{O}\big(\mathrm{dCrit}(S)\big)$ to a sheaf on M. Following Grothendieck

$$\mathrm{dCrit}(S) = \big(M, \mathrm{PV}^\bullet_M, -\iota_{\mathrm{d}S}\big)$$

is an example of a **dg manifold**.

Definition 1. A dg manifold is a smooth manifold M with a sheaf $\mathcal{O}_M$ of **dg commutative algebras** (DGCA) locally isomorphic to $\mathcal{O}_M(U) \cong \bigwedge \mathcal{E}(U)$ where $\mathcal{E}$ are the smooth sections of $E \to M$.

Ignoring the differential, we get a sheaf $(M, \mathrm{PV}^\bullet_M)$ on M such that

$$\mathrm{PV}_M = \bigwedge \mathfrak{X}_M \cong \mathrm{Sym}\, \mathfrak{X}[1].$$

The underlying graded manifold of $\mathrm{dCrit}(S)$ is

$$T^*[-1]M = \big(M, \mathrm{Sym}\, \mathfrak{X}[1]\big)$$

displaying the following properties:

i) the graded manifold $T^*[-1]M$ is a -1-shifted symplectic graded manifold just as T^*M is a 0-shifted symplectic manifold;

ii) Induced from the -1-shifted symplectic structure we get a 1-shifted Poisson bracket on $\mathcal{O}\big(T^*[-1]M\big) = \mathrm{PV}(M)$ known as the **Schouten bracket**

$$\begin{aligned} \{f, g\} &= 0, \\ \{v, f\} &= vf, \\ \{v, w\} &= [v, w], \\ \{u, v \wedge w\} &= \{u, v\} \wedge w + v \wedge \{u, w\} \end{aligned}$$

for $f, g \in \mathcal{O}(M)$ and $u, v, w \in \mathrm{PV}^{-1}(M)$.

Exercise. Show that $-\iota_{\mathrm{d}S} = \{S, \cdot\}$.

Definition 2. A $\mathbb{P}_0$ algebra $\big(A, \mathrm{d}, \{\cdot,\cdot\}\big)$ is a DGCA (A, d) with a -1-shifted Poisson bracket $\{\cdot,\cdot\} : A \otimes A \to A$ obeying:

i) **graded skew-symmetry:**

$$\{x, y\} = -(-1)^{(|x|+1)(|y|+1)}\{y, x\};$$

ii) **graded Poisson identity**:

$$\{x, yz\} = \{x, y\}z + (-1)^{(|x|+1)|y|}y\{x, z\}$$

so $\{x, \cdot\}$ is a degree $|x| + 1$ derivation;

iii) **graded Jacobi identity**:

$$\{x, \{y, z\}\} = \{\{x, y\}, z\} + (-1)^{(|x|+1)(|y|+1)}\{y, \{x, z\}\};$$

iv) **compatibility with differential:**

$$\mathrm{d}\{x, y\} = \{\mathrm{d}x, y\} + (-1)^{|x|+1}\{x, \mathrm{d}y\}.$$

Exercise. Check that the Schouten bracket defines a $\mathbb{P}_0$ algebra on $\mathcal{O}(T^*[-1]M)$.

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Lecture 20240429

(John Huerta)

4.1 Ultimate Goal

Define and use the Feynman "path" integral

$$\int_{\phi \in \mathscr{F}} e^{-\frac{S(\phi)}{\hbar}}$$

(Euclidean field theory)

In the constructive track: see Gonçalo on how to do this. In the BV track: we will produce a formal power series in $\hbar$.

4.2 Recall

- From now on: We work *perturbatively*, i.e., *formally* (in Algebraic Geometry speak), i.e., in *formal power series*, i.e., *infinitesimally*.
- Now M is going to be a finite dimensional manifold, denoting space-time. E.g.,

  $$M = \mathbb{R}^d$$

  or

  $$M = \mathrm{pt}$$

- $\mathscr{F}$ always denotes the naive fields, a sheaf of vector spaces on M; specifically, sections of some vector bundle $F \longrightarrow M$.

  Example: Yang-Mills fields for a trivial G-bundle $M \times G \longrightarrow M$, then $\mathscr{F}(M) = \Omega^1(M, \mathfrak{g})$, where $\mathfrak{g} = \mathrm{Lie}(G)$.

- $\mathscr{E}$ ("extended"), the space of BV-fields, a sheaf of *graded* vector spaces over M, sections of a graded vector bundle $E \longrightarrow M$. $\mathscr{E}^0(M) = \mathscr{F}(M)$.

  In the Yang-Mills example, where $d = \dim M$

$$\mathscr{E}(M) = \underbrace{\overset{-1}{\Omega^0(M,\mathfrak{g})}}_{\text{"ghosts"}} \oplus \underbrace{\overset{0}{\Omega^1(M,\mathfrak{g})}}_{\text{"fields"}} \oplus \underbrace{\overset{1}{\Omega^{d-1}(M,\mathfrak{g})}}_{\text{"anti-fields"}} \oplus \underbrace{\overset{2}{\Omega^{d}(M,\mathfrak{g})}}_{\text{"anti-ghosts"}}$$

4.3 BV Formulation of Gauge Theory

Input: Naive gauge theory

$$\underbrace{\mathscr{L}}_{\substack{\text{Lie algebra of}\\ \text{infinitesimal gauge}\\ \text{transformations}}} \curvearrowright \underbrace{\mathscr{F}}_{\text{space of naive fields}} .$$

The action may be non-linear. In the Young-Mills example it is an affine action

$$\Omega^0(M\mathfrak{g}) \curvearrowright \Omega^1(M\mathfrak{g}) .$$

There is a two-step process to writing down the gauge theory:

1. Take the *"stacky quotient"*

$$\mathscr{F} \rightsquigarrow \mathscr{F} /\!/ \mathscr{L} \qquad \text{(this lecture)}$$

2. Take the derived critical locus of S_{gauge}:

$$T^*[-1]\,(\mathscr{F} /\!/ \mathscr{L}) \,. \qquad \text{(already done)}$$

4.4 Lightning Fast Introduction to Derived Invariants

$\mathfrak{g}$ a finite dimensional Lie algebra, R a finite dimensional representation of $\mathfrak{g}$

$$\mathfrak{g} \to \mathfrak{gl}(R)$$

over some field $\boldsymbol{k} \in \{\mathbb{R}, \mathbb{C}\}$. Observe that

$$\begin{aligned} R^{\mathfrak{g}} &= \{v \in \mathbb{R} \mid Xv = 0 \text{ for all } X \in \mathfrak{g}\} \\ &= \mathrm{Hom}_{\mathfrak{g}}(\boldsymbol{k}, R) \end{aligned}$$

Derived version Hom $\rightsquigarrow$ $\mathbb{R}$Hom.

Try $R^{\mathrm{h}\mathfrak{g}} = \mathbb{R}\mathrm{Hom}_{U\mathfrak{g}}(\boldsymbol{k}, R)$, where U is the eveloping algebra. I.e.,

$$U\mathfrak{g} = \frac{T\mathfrak{g}}{x \otimes y - y \otimes x - [x,y]}$$

where $T\mathfrak{g}$ is the tensor algebra.

Fact. $\mathrm{Rep}_{\mathfrak{g}} \simeq U\mathfrak{g}\text{–mod}$.

To compute $\mathbb{R}\mathrm{Hom}_{U\mathfrak{g}}(\boldsymbol{k}, R)$ we need to resolve $\boldsymbol{k}$ or R as $U\mathfrak{g}$ modules. Similar to the Koszul complex

$$\cdots \longrightarrow \overset{-k}{\Lambda^k \mathfrak{g} \otimes U\mathfrak{g}} \longrightarrow \cdots \longrightarrow \overset{-1}{\mathfrak{g} \otimes U\mathfrak{g}} \longrightarrow \overset{0}{U\mathfrak{g}}$$

with differential

$$\begin{aligned} \Lambda^{k+1}\mathfrak{g}\otimes &\longrightarrow \Lambda^k \otimes U\mathfrak{g} \\ x_0 \wedge \cdots \wedge x_k \otimes y &\longmapsto \sum_{i=0}^{k} (-1)^i x_0 \wedge \cdots \widehat{x_i} \cdots \wedge x_k \otimes x_i y \\ &\quad + \sum_{i<j} (-1)^{i+j} [x_i, x_j] \wedge x_0 \wedge \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots \wedge x_k \otimes y. \end{aligned}$$

With this differential

$$\begin{aligned} H^0\left(\Lambda^\bullet \mathfrak{g} \otimes U\mathfrak{g}\right) &\simeq \boldsymbol{k} \\ H^k\left(\Lambda^\bullet \mathfrak{g} \otimes U\mathfrak{g}\right) &= 0 \qquad \text{for } k < 0 \end{aligned}$$

Hence

$$\begin{aligned} R^{\mathrm{h}\mathfrak{g}} &= \mathbb{R}\mathrm{Hom}_{U\mathfrak{g}}(\boldsymbol{k}, R) \\ &= \mathrm{Hom}_{U\mathfrak{g}}(\Lambda^\bullet \mathfrak{g} \otimes U\mathfrak{g}, R) \\ &\simeq \mathrm{Hom}_{\boldsymbol{k}}(\Lambda^\bullet \mathfrak{g}, R) \end{aligned}$$

because $\Lambda^\bullet \mathfrak{g} \otimes U\mathfrak{g}$ is free.

Definition 3. For $\mathfrak{g}$ a Lie algebra, R a representation of $\mathfrak{g}$, the *Chevalley-Eilenberg complex* $C^\bullet(\mathfrak{g}, R)$ is defined as

$$C^k(\mathfrak{g}, R) = \mathrm{Hom}(\Lambda^k \mathfrak{g}, R)$$

with

$$\begin{aligned} \mathrm{d}\omega(x_0, \dots, x_k) = &\sum_{i=0}^{k} (-1)^i x_i \cdot \omega(x_0, \dots, \widehat{x_i}, \dots, x_k) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([x_i, x_j], x_0, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_k) \end{aligned}$$

Conclusion. Back to $R = \mathcal{O}(V)$, then

$$\begin{aligned} R^{\mathrm{h}\mathfrak{g}} &= \mathcal{O}(V)^{\mathrm{h}\mathfrak{g}} \\ &= \mathrm{Hom}_{\boldsymbol{k}}(\Lambda^\bullet \mathfrak{g}, \mathcal{O}(V)) \\ &\simeq \Lambda^\bullet \mathfrak{g}^* \otimes \mathcal{O}(V) \\ &\simeq \mathrm{Sym}(\mathfrak{g}^*[-1]) \otimes \mathrm{Sym}(V^*) \\ &\simeq \mathrm{Sym}(\overset{0}{V^*} \oplus \overset{1}{\mathfrak{g}^*[-1]}) \\ &\simeq \mathcal{O}(\overset{-1}{\mathfrak{g}[1]} \oplus \overset{0}{V}) \\ &=: \mathcal{O}(V /\!\!/ \mathfrak{g}). \end{aligned}$$

Hence

Definition 4. $V /\!/ \mathfrak{g} := \mathfrak{g}[1] \oplus V$

Puzzle: what happened to the differential d. It becomes a vector field on $\mathfrak{g} \oplus V$!.

4.5 Back to Yang-Mills

$$
\begin{aligned}
V &\rightsquigarrow \Omega^1(M, \mathfrak{g}) \\
\mathfrak{g} &\rightsquigarrow \Omega^0(M, \mathfrak{g})
\end{aligned}
$$

Step 1: $\Omega^1(M, \mathfrak{g}) /\!/ \Omega^0(M, \mathfrak{g}) := \Omega^0(M, \mathfrak{g})[1] \oplus \Omega^1(M, \mathfrak{g})$
Step 2: $\mathscr{E}$ for Yang-Mills

$$
\begin{aligned}
T^*[-1](\overset{-1}{\Omega^0(M, \mathfrak{g})[1]} \oplus \overset{0}{\Omega^1(M, \mathfrak{g})}) &\simeq \Omega^0(M, \mathfrak{g})[1] \oplus \Omega^1(M, \mathfrak{g}) \\
&\quad \oplus (\Omega^0(M, \mathfrak{g})[1] \oplus \Omega^1(M, \mathfrak{g}))^*[-1] \\
&\simeq \Omega^0(M, \mathfrak{g})[1] \oplus \Omega^1(M, \mathfrak{g}) \\
&\quad \oplus \Omega^{d-1}(M, \mathfrak{g})[-1] \oplus \Omega^d(M, \mathfrak{g})[-2]
\end{aligned}
$$