Quantum Field Theory for Mathematicians

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Lecture 1

1.1 History

Date	People	What	Why	Techniques
1969	Faddeev and Popov	Gauge fixing (adding ghosts)	Quantize Yang-Mills	Berezinian integration
1973	't Hooft and Veltman	Quantized Yang-Mills	Quantize Yang-Mills	Feynman diagrams
1975	Becchi, Rouet, Stora, Tyutin (BRST)	Cohomological theory to quantize Yang-Mills	Understanding 't Hooft and Veltman	Derived invariants (Lie algebra cohomology)
1981	Batallin and Vilkovisky (BV)	Quantize systems with complicated gauge symmetries	Supergravity	Derived intersections (Koszul complexes)
1992	Henneaux	Quantize Yang-Mills using BV	Analyze Yang-Mills using BV	Derived intersections
2007	Costello	Combine BV with effective field theory	Make BV quantization rigorous	Derived everything, analysis, and homotopy theory

1.2 References

The main references for this seminar will be:

- Costello - Renormalization and Effective Field Theory [Cos11];
- Elliot, Williams, Yoo - Asymptotic Freedom in the BV Formalism [EWY18];
- Gwilliam - Factorization algebras and free field theories [Gwi].

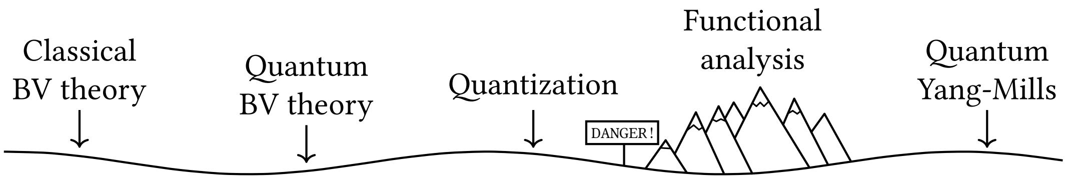

Figure 1.1: Roadmap to BV quantization.

1.3 Roadmap to BV Quantization

The space of fields $\mathscr{E}^\bullet$ is a cochain complex

$$\dots \longrightarrow \mathscr{E}^{-1} \xrightarrow{Q} \mathscr{E}^0 \xrightarrow{Q} \mathscr{E}^1 \xrightarrow{Q} \mathscr{E}^2 \longrightarrow \dots$$

equipped with a differential Q such that $Q^2 = 0$. Moreover, $\mathscr{E}$ admits a -1-shifted symplectic structure, that is, there exists a non degenerate pairing of degree -1

$$\langle \cdot, \cdot \rangle : \mathscr{E} \otimes \mathscr{E} \longrightarrow \mathbb{R}[-1]$$

such that $\langle x, y \rangle = -(-1)^{(|x|+1)(|y|+1)} \langle y, x \rangle$. This structure defines a $+1$-shifted Poisson bracket

$$\{\cdot, \cdot\} : \mathscr{O}(\mathscr{E}) \otimes \mathscr{O}(\mathscr{E}) \longrightarrow \mathscr{O}(\mathscr{E})$$

where $\mathscr{O}(\mathscr{E}) \cong \mathrm{Sym}^\bullet(\mathscr{E}^\vee)$ is the (graded) commutative algebra of polynomial functions on the dual complex $\mathscr{E}^\vee$. Pick $S \in \mathscr{O}(\mathscr{E})$ obeying the **classical master equation** (CME)

$$\{S, S\} = 0. \tag{1.1}$$

The data $(\mathscr{E}, \langle \cdot, \cdot \rangle, S)$ defines a **classical BV theory**. The CME says $\{S, \cdot\}$ is a differential which makes $(\mathscr{O}(\mathscr{E}), \{S, \cdot\})$ into a cochain complex such that

$$\mathrm{H}^0\, \mathscr{O}(\mathscr{E}) \cong \mathscr{O}(\mathrm{Crit}(S)),$$

where $\mathrm{Crit}(S)$ denotes the critical locus of S. We will restrict to S of the form

$$S(e) = \underbrace{\langle e, Qe \rangle}_{\substack{\text{free part} \\ \text{(kinetic +} \\ \text{mass terms)}}} + \underbrace{I(e)}_{\substack{\text{interaction} \\ \text{part (cubic} \\ \text{or higher)}}} .$$

Example. Why are the cubic and higher order terms called interaction terms? For electromagnetism on a manifold M we have a space of fields $\mathscr{F} = \Omega^1(M) \oplus \Omega^0(M, S)$ in degree 0. Let $F = \mathrm{d}A$ and define

$$S(A, \psi) = \int_M \underbrace{F \wedge \star F + \langle \psi, \not{d}\psi \rangle\, \mathrm{dvol}}_{\text{quadratic terms}} + \underbrace{\langle \psi, \not{A}\psi \rangle\, \mathrm{dvol}}_{\text{interaction term}} .$$

Computing the Euler-Lagrange equations we obtain the system of differential equations

$$\begin{cases} \star \mathrm{d} \star F = \bar{\psi} \gamma^\mu \psi\, \mathrm{d}x_\mu \\ \not{d}_A \psi = 0 \end{cases}$$

which is coupled because of the interaction term.

1.4 Quantization in the BV formalism

The slogan of quantization in the BV formalism is to *deform the differential.* In the perturbative context we work in formal power series in $\hbar$, for example, over the ring $\mathbb{R}[[\hbar]]$. Quantization results in a cochain complex $(\mathcal{O}(\mathscr{E})[[\hbar]], \{S^q, \cdot\} + \hbar\Delta)$, where Δ is called the BV Laplacian, and $S^q \in \mathcal{O}(\mathscr{E})[[\hbar]]$ satisfies the **quantum master equation** (QME)

$$(\{S^q, \cdot\} + \hbar\Delta)^2 = 0 \tag{1.2}$$

Example. In finite dimensions ($\mathscr{F} \cong \mathbb{R}^n$) the BV fields are $\mathscr{E} = \mathbb{R}^n \longrightarrow \mathbb{R}^n$ therefore

$$\mathcal{O}(\mathscr{E}) \cong \mathbb{R}[x^1, \dots, x^n, \xi^1, \dots, \xi^n]$$

and the BV Laplacian takes the form

$$\Delta = \sum_{\mu=1}^{n} \frac{\partial}{\partial\theta^\mu} \frac{\partial}{\partial x^\mu}.$$

In this form, it becomes clear that Δ is a differential operator of degree 1 such that $\Delta^2 = 0$.

BV quantization is the choice of an action

$$S^q(e) = \langle e, Qe\rangle + I^q(e)$$

where $I^q \in \mathcal{O}(\mathscr{E})[[\hbar]]$ is cubic mod $\hbar$ and satisfies the QME

$$QI^q + \frac{1}{2}\{I^q, I^q\} + \hbar\Delta I^q = 0$$

which resembles, in this form, the **Maurer-Cartan** (MC) **equation**. In infinite dimensions, some problems arise:

1. there may be no solution to this equation. In this case we say that quantization is obstructed (there is an anomaly);
2. the QME in infinite dimensions is ill-defined. Some functional analysis is needed to make sense of this problem.

Bibliography

[Cos11] Kevin Costello. *Renormalization and Effective Field Theory*. 1st ed. Vol. 170. Mathematical Surveys and Monographs 81. 2011. ISBN: 978-1-4704-7008-1.

[EWY18] Chris Elliott, Brian Williams, and Philsang Yoo. "Asymptotic Freedom in the BV Formalism". In: *Journal of Geometry and Physics* 123 (Jan. 2018), pp. 246–283. ISSN: 03930440. DOI: 10.1016/j.geomphys.2017.08.009. arXiv: 1702.05973 [hep-th, physics:math-ph].

[Gwi] Owen Gwilliam. "Factorization Algebras and Free Field Theories".