Almost Equilateral Heronian Triangles

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A *Heronian triangle* is one whose sides (a, b, c) and area K are integers. An *almost equilateral Heronian triangle* is a Heronian triangle whose sides are consecutive integers such as (3, 4, 5), with area 6, and (13, 14, 15), with area 84 [1]. Are there others? If so, how many?

To answer, we first recall that the area K of an arbitrary triangle with sides a, b, c is given by Heron's formula:

$$K = \sqrt{s(s-a)(s-b)(s-c)},$$

where s denotes the *semiperimeter* $s = (a+b+c)/2$. Letting $(b-1, b, b+1)$ denote the sides of an almost equilateral Heronian triangle yields $s = 3b/2$ and

$$K^2 = \left(\frac{3b}{2}\right)\left(\frac{b+2}{2}\right)\left(\frac{b}{2}\right)\left(\frac{b-2}{2}\right)$$

or

$$16K^2 = 3b^2(b^2-4).$$

Hence, b must be even. If h denotes the altitude to the even side b, then we also have $K = bh/2$. Thus,

$$3b^2(b^2-4) = 4b^2h^2,$$

or

$$3b^2 - (2h)^2 = 12.$$

We now prove the following theorem:

Theorem 1. *Infinitely many almost equilateral Heronian triangles exist.*

Proof. In Figure 1, we have an equilateral triangle with side $3b+4h$, where b and h are integers. The triangle is divided into three equilateral triangles each in two shades of gray with side $2b+2h$, overlapping pairwise in three equilateral triangles each in dark gray with side b, and a white equilateral triangle with side $2h$ in the center.

Using the inclusion-exclusion principle to count the small $\triangle$ and $\triangledown$ in the figure (noting that the total number of $\triangle$ and $\triangledown$ triangles in a larger equilateral triangle is the square of its side length) yields

$$(3b+4h)^2 = 3(2b+2h)^2 - 3b^2 + (2h)^2,$$

or equivalently,

$$3b^2 - (2h)^2 = 3(2b+2h)^2 - (3b+4h)^2.$$

Thus, if

$$3b^2 - (2h)^2 = 12,$$

Math. Mag. **93** (2020) 378–379. doi:10.1080/0025570X.2020.1817708 © Mathematical Association of America
MSC: 51M04, 97C40

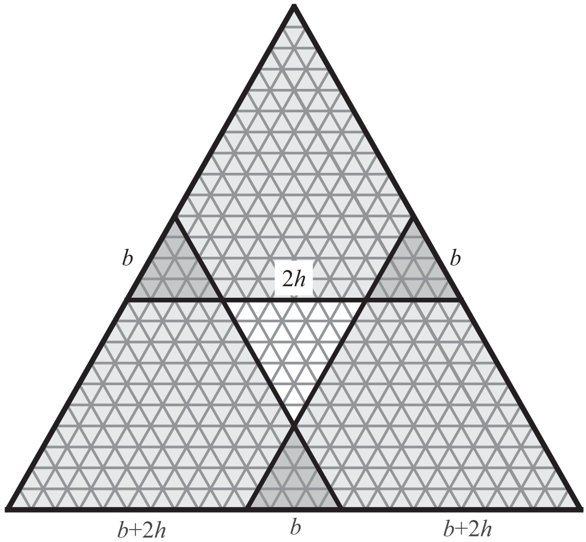

Figure 1 Dividing up a large equilateral triangle, of side length $3b + 4h$.

then

$$3(2b + 2h)^2 - (3b + 4h)^2 = 12.$$

Now let b denote the even side, and h the altitude to the even side, of an almost equilateral triangle satisfying

$$3b^2 - (2h)^2 = 12.$$

The above recursion then produces a larger almost equilateral Heronian triangle with even side $2b + 2h$ and altitude to the even side $(3b + 4h)/2$.

When $(b, h) = (4, 3)$, as in the $(3, 4, 5)$ right triangle, the recursion yields the following infinite sequence of almost equilateral Heronian triangles:

$$(3, 4, 5) \to (13, 14, 15) \to (51, 52, 53)$$
$$\to (193, 194, 195) \to (723, 724, 725) \ldots.$$

■

Acknowledgments The author wishes to thank an anonymous referee for helpful suggestions on an earlier draft of this note.

REFERENCE

[1] Beauregard, R. A., Suryanarayan, E. R. (1998). The Brahmagupta triangles. *College Math. J.* 29(1): 13–17. doi.org/10.1080/07468342.1998.11973907

Summary. We use Heron's formula and the inclusion–exclusion principle to show that infinitely many almost equilateral Heronian triangles exist.

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