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Author(s): ROGER B. NELSEN

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# Almost Equilateral Heronian Triangles

ROGER B. NELSEN

Lewis & Clark College

Portland, OR 97219

[nelsen@lclark.edu](mailto:nelsen@lclark.edu)

A *Heronian triangle* is one whose sides  $(a, b, c)$  and area  $K$  are integers. An *almost equilateral Heronian triangle* is a Heronian triangle whose sides are consecutive integers such as  $(3, 4, 5)$ , with area 6, and  $(13, 14, 15)$ , with area 84 [1]. Are there others? If so, how many?

To answer, we first recall that the area  $K$  of an arbitrary triangle with sides  $a, b, c$  is given by Heron's formula:

$$K = \sqrt{s(s-a)(s-b)(s-c)},$$

where  $s$  denotes the *semiperimeter*  $s = (a + b + c)/2$ . Letting  $(b-1, b, b+1)$  denote the sides of an almost equilateral Heronian triangle yields  $s = 3b/2$  and

$$K^2 = \left(\frac{3b}{2}\right) \left(\frac{b+2}{2}\right) \left(\frac{b}{2}\right) \left(\frac{b-2}{2}\right)$$

or

$$16K^2 = 3b^2(b^2 - 4).$$

Hence,  $b$  must be even. If  $h$  denotes the altitude to the even side  $b$ , then we also have  $K = bh/2$ . Thus,

$$3b^2(b^2 - 4) = 4b^2h^2,$$

or

$$3b^2 - (2h)^2 = 12.$$

We now prove the following theorem:

**Theorem 1.** *Infinitely many almost equilateral Heronian triangles exist.*

*Proof.* In Figure 1, we have an equilateral triangle with side  $3b + 4h$ , where  $b$  and  $h$  are integers. The triangle is divided into three equilateral triangles each in two shades of gray with side  $2b + 2h$ , overlapping pairwise in three equilateral triangles each in dark gray with side  $b$ , and a white equilateral triangle with side  $2h$  in the center.

Using the inclusion-exclusion principle to count the small  $\triangle$  and  $\nabla$  in the figure (noting that the total number of  $\triangle$  and  $\nabla$  triangles in a larger equilateral triangle is the square of its side length) yields

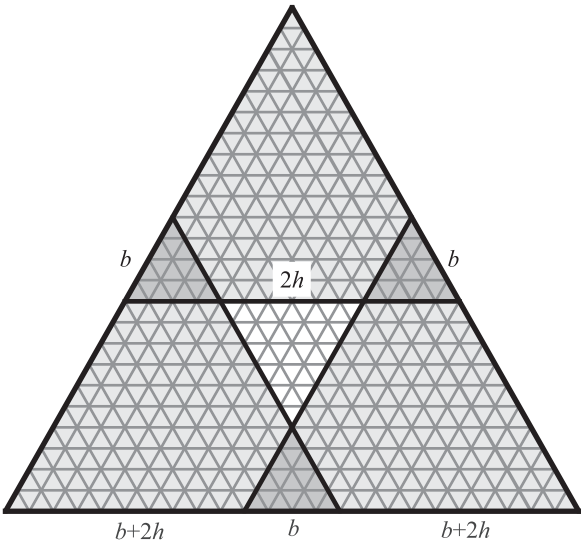
$$(3b + 4h)^2 = 3(2b + 2h)^2 - 3b^2 + (2h)^2,$$

or equivalently,

$$3b^2 - (2h)^2 = 3(2b + 2h)^2 - (3b + 4h)^2.$$

Thus, if

$$3b^2 - (2h)^2 = 12,$$



**Figure 1** Dividing up a large equilateral triangle, of side length  $3b + 4h$ .

then

$$3(2b + 2h)^2 - (3b + 4h)^2 = 12.$$

Now let  $b$  denote the even side, and  $h$  the altitude to the even side, of an almost equilateral triangle satisfying

$$3b^2 - (2h)^2 = 12.$$

The above recursion then produces a larger almost equilateral Heronian triangle with even side  $2b + 2h$  and altitude to the even side  $(3b + 4h)/2$ .

When  $(b, h) = (4, 3)$ , as in the  $(3, 4, 5)$  right triangle, the recursion yields the following infinite sequence of almost equilateral Heronian triangles:

$$\begin{aligned} (3, 4, 5) &\rightarrow (13, 14, 15) \rightarrow (51, 52, 53) \\ &\rightarrow (193, 194, 195) \rightarrow (723, 724, 725) \dots \end{aligned}$$

■

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[1] Beauregard, R. A., Suryanarayan, E. R. (1998). The Brahmagupta triangles. *College Math. J.* 29(1): 13–17. [doi.org/10.1080/07468342.1998.11973907](https://doi.org/10.1080/07468342.1998.11973907)

**Summary.** We use Heron’s formula and the inclusion–exclusion principle to show that infinitely many almost equilateral Heronian triangles exist.

**ROGER B. NELSEN** (MR Author ID [237909](#)) is a professor emeritus at Lewis & Clark College, where he taught mathematics and statistics for 40 years.