A Probabilistic Derivation of the Formula for the Sum of a Geometric Series

JEREMY F. ALM
Lamar University
Beaumont, TX 10009
jalm@lamar.edu

The usual derivation for the sum of a geometric series involves taking limits of both sides of the algebraic identity

$$1 + x + x^2 + \cdots + x^n = (1 - x^{n+1})/(1 - x).$$

In this note, we provide a derivation that uses almost no algebra and avoids explicit limits.

Suppose Alice and Bob play a rock-paper-scissors-like game in which, in any given round, Alice can win, Bob can win, or they can draw. The probability that Alice wins any given round is p, the probability that Bob wins is q, and the probability of a draw is r, where $p + q + r = 1$. Any two distinct rounds are independent. The first player to win a round wins the game. What is the probability that Alice wins the game?

On the one hand, there are infinitely many ways for Alice to win, as there can be any nonnegative integer number of draws before Alice wins a round. The probability that Alice wins the game in the first round is p, the probability that she wins in the second round is pr, the probability that she wins in the third round is pr^2, and so on. Since these events are mutually exclusive, we may sum them to get

$$p + pr + pr^2 + pr^3 + \cdots = \sum_{k=0}^{\infty} p \cdot r^k$$

On the other hand, we may compute directly the probability that Alice wins by observing that the number of draws before Alice wins a round is irrelevant! So suppose someone wins in the kth round. Then we know that in *this* round, there was no draw, so the probability that it was Alice who won is $p/(p + q)$. Hence,

$$\sum_{k=0}^{\infty} p \cdot r^k = \frac{p}{p+q} = \frac{p}{1-r},$$

as desired. This derivation is easily modified for the case of a truncated series, which is left as an exercise for the reader.

Summary. We derive the formula for the sum of a geometric series using elementary probability theory. Limits and algebra are avoided.

JEREMY F. ALM (MR Author ID 707759) is Professor and Chair of Mathematics and Director of the Quality Enhancement Plan at Lamar University. He occasionally gets to do mathematics. In order to remain sane, he pretends to be a musician; he plays electric guitar, harmonica, and ukulele rather poorly but with great enthusiasm. He hails from southern Wisconsin, but loves the seafood in his new home on the Gulf Coast.

Math. Mag. **93** (2020) 381. doi:10.1080/0025570X.2020.1818525
MSC: 00A08