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Source: *Mathematics Magazine*, December 2020, Vol. 93, No. 5 (December 2020), p. 381

Published by: Taylor & Francis, Ltd. on behalf of the Mathematical Association of America

Stable URL: <https://www.jstor.org/stable/10.2307/48676014>

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A Probabilistic Derivation of the Formula for the Sum of a Geometric Series

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The usual derivation for the sum of a geometric series involves taking limits of both sides of the algebraic identity

$$1 + x + x^2 + \cdots + x^n = (1 - x^{n+1})/(1 - x).$$

In this note, we provide a derivation that uses almost no algebra and avoids explicit limits.

Suppose Alice and Bob play a rock-paper-scissors-like game in which, in any given round, Alice can win, Bob can win, or they can draw. The probability that Alice wins any given round is p , the probability that Bob wins is q , and the probability of a draw is r , where $p + q + r = 1$. Any two distinct rounds are independent. The first player to win a round wins the game. What is the probability that Alice wins the game?

On the one hand, there are infinitely many ways for Alice to win, as there can be any nonnegative integer number of draws before Alice wins a round. The probability that Alice wins the game in the first round is p , the probability that she wins in the second round is pr , the probability that she wins in the third round is pr^2 , and so on. Since these events are mutually exclusive, we may sum them to get

$$p + pr + pr^2 + pr^3 + \cdots = \sum_{k=0}^{\infty} p \cdot r^k$$

On the other hand, we may compute directly the probability that Alice wins by observing that the number of draws before Alice wins a round is irrelevant! So suppose someone wins in the k th round. Then we know that in *this* round, there was no draw, so the probability that it was Alice who won is $p/(p + q)$. Hence,

$$\sum_{k=0}^{\infty} p \cdot r^k = \frac{p}{p + q} = \frac{p}{1 - r},$$

as desired. This derivation is easily modified for the case of a truncated series, which is left as an exercise for the reader.

Summary. We derive the formula for the sum of a geometric series using elementary probability theory. Limits and algebra are avoided.

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Math. Mag. **93** (2020) 381. doi:10.1080/0025570X.2020.1818525 © Mathematical Association of America
MSC: 00A08