The Pythagorean Theorem Via Equilateral Triangles

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Throughout history, there have been many different proofs of the Pythagorean theorem. Here we propose another one, which is nonstandard in that we use neither squares nor similarity of polygons. To proceed, consider the right triangle in Figure 1, with equilateral triangles drawn on each of its sides.

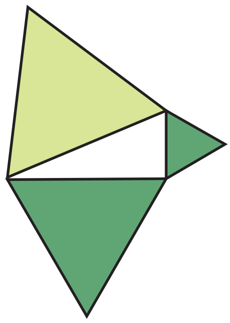

Figure 1 A right triangle with equilateral triangles drawn on its sides.

We will prove the following theorem:

Theorem 1. In Figure 1, the areas of the small equilateral triangles sum up to the area of the larger one.

We prove this by the classical method of rearrangement. Let A, B, C be the vertices of the triangle, and let a, b, c be the lengths of the corresponding opposite sides. Rotate the triangle $\triangle ABC$ counter-clockwise by 60^{o} at the vertex A, and clockwise by 60^{o} at B. Let C_1, B_1 and C_2, A_2 be the images of the vertices under these transformations, as shown in Figure 2. Notice that

$$|BA_2| = c = |AB_1|, \qquad \text{and} \qquad \angle ABA_2 = \angle BAB_1 = 60^{\mathrm{o}}.$$

Hence, A_2 and B_1 coincide. Denoting this point by D, we have that the vertices A, B and D determine an equilateral triangle of side length c.

Also, notice that $\triangle BCC_2$ and $\triangle ACC_1$ are equilateral triangles of side length a and b, respectively. Moreover, triangles $\triangle BC_2D$ and $\triangle AC_1D$ are both congruent to triangle $\triangle ABC$, as shown in Figure 3.

Now, referring to the areas, we have

$$\begin{aligned} ABC_2DC_1 &= ABD + BC_2D + AC_1D \\ &= ACC_1 + BCC_2 + ABC + C_2DC_1C. \end{aligned}$$

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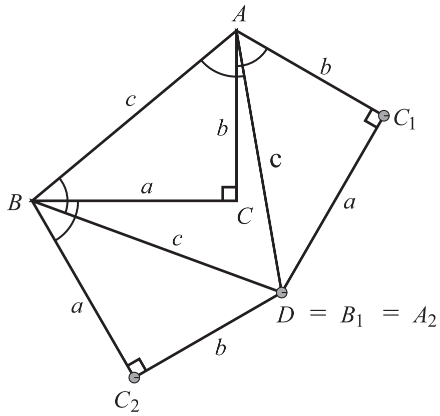


Figure 2 The result of rotating $\triangle ABC$ counter-clockwise 60° about A, and 60° clockwise about B.

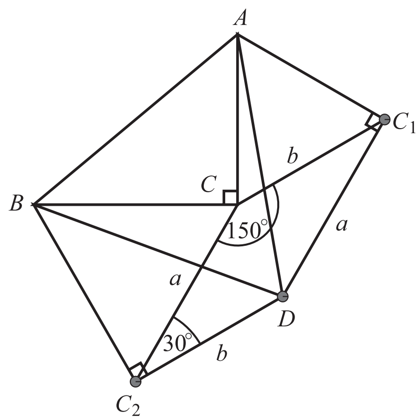
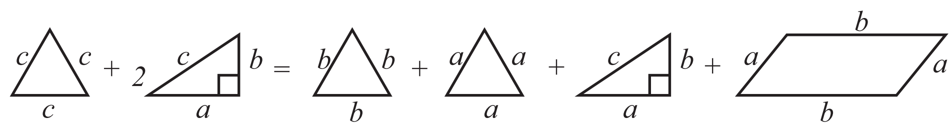
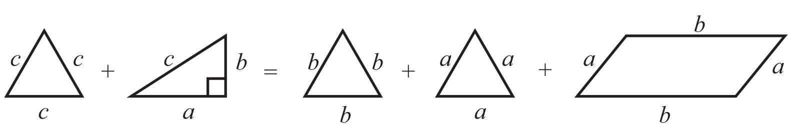


Figure 3 Our construction implies that $\triangle BCC_2$ and $\triangle ACC_1$ are both equilateral. Moreover, $\triangle BC_2D$ and $\triangle AC_1D$ are both congruent to $\triangle ABC$.

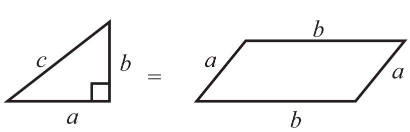
In pictures,



which yields



Hence, we are left to prove that



To do this, notice that C_2DC_1C is a parallelogram of side lengths a and b. Also, since $\angle BCA = 90^{\mathrm{o}}$ and

$$\angle ACC_1 = \angle BCC_2 = 60^{\mathrm{o}},$$

we have that $\angle C_1CC_2$ must equal 150^{o}. Moreover,

$$\angle CC_2D = \angle BC_2D - \angle BC_2C = 30^{\mathrm{o}},$$

and similarly $\angle DC_1C = 30^{\mathrm{o}}$. Thus,

$$\mathrm{area}(C_1CC_2D) = ab\sin(30^{\mathrm{o}}) = \frac{1}{2}ab = \mathrm{area}(ABC),$$

as desired.

In the last step above, one can certainly avoid the use of trigonometry just by looking at Figure 4. Indeed, the height of a parallelogram with angles 30° and 150° equals half the length of the side that is not being considered as the base. This is simply because a triangle of angles 30°, 60°, and 90° is half of an equilateral triangle. Hence, the length of its smallest side is half of the length of the largest one.

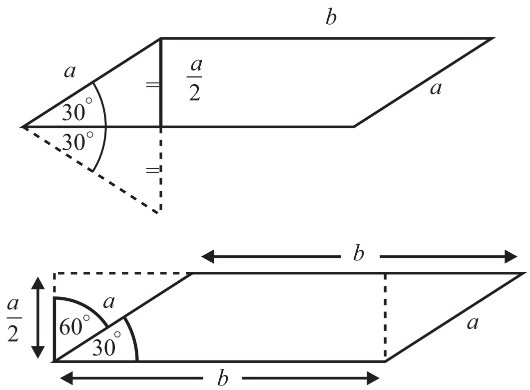


Figure 4 Another approach to finding the area of parallelogram C_1CC_2D.

It is straightforward to show that Theorem 1 implies the Pythagorean theorem.

Since the area of an equilateral triangle of side length ℓ equals $\ell^2\sqrt{3}/4$, the equality proven above may be rewritten as

$$\frac{a^2\sqrt{3}}{4} + \frac{b^2\sqrt{3}}{4} = \frac{c^2\sqrt{3}}{4}.$$

This implies the Pythagorean equality $a^2 + b^2 = c^2$ just by canceling the common factor $\sqrt{3}/4$.

Actually, if we start with the Pythagorean identity and multiply each term by $\sqrt{3}/4$, we obtain the equality between the area of the equilateral triangle built on the hypotenuse and the sum of those built on the legs. The two claims are therefore equivalent.

Finally, the Pythagorean identity $a^2 + b^2 = c^2$ also implies that the area of a regular n-gon of side length c is the sum of the areas of two regular n-gons of side lengths a and b, respectively (see Figure 5). To see this, as above, it suffices to multiply each term of this identity by an appropriate constant (namely, the area of the regular n-gon

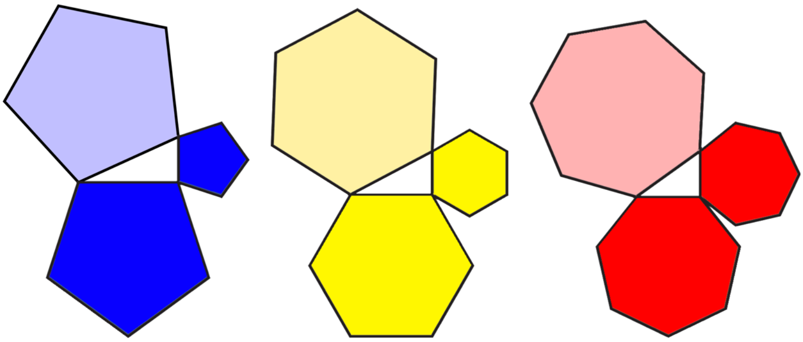


Figure 5 The area of a regular n-gon of side length c is equal to the sum of the areas of two regular n-gons of side lengths a and b, respectively.

of side length 1). Actually, this statement, for any given $n \geq 3$, also implies (and hence it is equivalent to) the Pythagorean theorem, just by reversing this argument.

Certainly, a more geometric proof for regular polygons different from equilateral triangles or squares would be desirable. Of course, in this framework, the Wallace–Bolyai–Gerwien decomposition theorem applies, hence there should be a decomposition of the two small n-gons into polygonal pieces that, after rearrangement, yield the larger one. Nevertheless, the number of required pieces seems to be quite large in general.

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Summary. We prove the Pythagorean theorem by constructing equilateral triangles on each side of a right triangle. Our approach is unusual in that it uses neither squares nor similarity of polygons.

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