HUDM6052 Psychometric II Homework_02

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Contents

Q1

Find the maximum discrimination for 1PL, 2PL, and 3PL logistic models...

My Solution:
2PL MODEL

I begin with a 2PL model since 1PL is a special case of 2PL.

For the 2PL model,

$$P(\theta) = \frac{1}{1 + e^{-\alpha(\theta-\beta)}},$$

let $Z = \alpha(\theta - \beta)$. Using the chain rule and take the partial derivative on α, one can have

$$\frac{\partial P}{\partial \theta} = \frac{\partial P}{\partial Z}\frac{\partial Z}{\partial \theta}.$$

Then, expand the formula above, one can have

$$\frac{\partial P}{\partial \alpha} = P(1 - P)\alpha.$$

Note, this is the slope function of a 2PL model, one still need to take the first derivative (i.e., the second derivative of the original 2PL model) and let it equal to 0 to get the local minimum or maximum. For simplification, I use $S(\theta)$ to represent the slope function of a 2PL model. Then,

$$S(\theta) = \alpha P(1 - P).$$

Take the partial derivative of the slope function, one can have

$$\frac{\partial S}{\partial \theta} = \frac{\partial S}{\partial P}\frac{\partial P}{\partial \theta} = (\alpha - 2\alpha P)[\alpha P(1 - P)] = 0.$$

Solve the equation above, one can find that the extreme values of slope occurs at $P = 0, P = 1, P = \frac{1}{2}$. Rigorously, we still need to take the second partial derivative of this slope function to determine the whether it is the local minimum or maximum. But, from the ICC one can easily find that slope will have maximum at $P = \frac{1}{2}$. I skipped this rigorous math proof here. Finally, solve the equation of $P(\theta) = \frac{1}{2}$, we can have the solution $\theta = \beta$.

Therefore, the maximum value of discrimination of a 2PL model is at the point $\theta = \beta$.

1PL MODEL

As for the 1PL model, plug the $\alpha = 1$ into the partial derivative of slope function $\frac{\partial S}{\partial \theta}$ above. One can easily have same conclusion that the maximum of slope of 1PL model is at the point $\theta = \beta$.

3PL MODEL

Q2

Let the discrimination, difficulty and guessing parameters of five items be...

My Solution:
First, I write a 3PL model function:

```
> # items' discrimination
> a_ <- c(0.5,1,1.5,2.5,1)
> # items' difficulty levels
> b_ <- c(-1,-0.5,0,0.5,1)
> # items's guessing parameters
> c_ <- c(0,0.1,0.15,0.05,0.32)
> # trait vector
> theta_ <- c(-2.0, -1.0, 0,1,2)
> # write a 3PL model
> irt_3pl <- function(theta,a,b,c){
+   z <- 1.702*a*(theta - b)
+   output <- c + (1-c)/(1+exp(-z))
+   return(output)
+ }
```

Next, for each test-taker (i.e., each trait), get the required values iteratively.

```
> # using a for-loop to get the values
> for (j in 1:5) {
+   #print(paste0("-------------For the theta =", theta_[j]," : -------------"))
+   exp_cor <- 0
+   for (i in 1:5) {
+     #print(paste0("-----For the item i=", i," : -----"))
+     P <- irt_3pl(theta= theta_[j],
+                  a=a_[i], b=b_[i], c=c_[i])
+     Q <- 1-P
+     # get the odds
+     odds <- round(P/Q,3)
+     # get the logit
+     logit <- round(log(odds),3)
+     # if the odds greater than 1, we can expected this student j may
+     # get this item correctly
+     if(odds >=1){
```

```r
+         exp_cor <- exp_cor + 1
+       }
+       # print the results for this student
+       #print(paste0("Odds: ", odds," ."))
+       #print(paste0("Logit:", logit," ."))
+     }
+     # get the expected proportion of correct
+     prop <- round(exp_cor/5,3)
+     #print(paste0("Expected Correct #:", exp_cor," ."))
+     #print(paste0("Expected Correct proportion:", prop," ."))
+ }
```

To make the layout in a good-looking manner, I loaded the results from above to a table as followed. Please remove the comment mark # before `print()` function to see the returned results.

Theta	theta = -2.0		theta = -1.0		theta = -0.0		theta = 1.0		theta = 2.0	
Item	**odds**	**logit**	**odds**	**logit**	**odds**	**logit**	**odds**	**logit**	**odds**	**logit**
1	0.427	-0.861	1	0	2.342	0.851	5.485	1.702	12.846	2.553
2	0.198	-1.619	0.586	-0.534	2.713	0.998	14.384	2.666	78.396	4.362
3	0.184	-1.693	0.268	-1.317	1.353	0.302	15.289	2.727	194.305	5.269
4	0.053	-2.937	0.054	-2.919	0.178	-1.726	8.888	2.185	622.584	6.434
5	0.48	-0.734	0.519	-0.656	0.739	-0.302	1.941	0.663	8.537	2.144
Expected Correct	0		1		3		5		5	
Expected Propor.	0		0.2		0.6		1		1	

Q3-Part a

Find the maximum likelihood estimates of the item parameters under...

My Solution

```r
> # load the given values
> ## the assumed theta_j
> theta_set <- seq(-3,3,by =0.5)
> ## number of correct responses for each theta_j
> r_set <- c(0,0,1,2,3,4,5,6,4,4,4,2,1)
> ## number of respondents for each theta_j
> f_set <- c(1,2,4,7,8,9,10,8,6,5,5,2,1)
>
> # get the proportion of correct responses for each theta_j
> p_set <- r_set/f_set
>
> # to count the iteration times
> iter_time <- 0
> deltas <- t(t(c(10,10)))
> zeta <- 0
> lambda <- 1.0
>
> while((abs(deltas[1,1])>0.005) & (abs(deltas[2,1])>0.005)) {
+   # set the initial value
+   iter_time <- iter_time+1
+   # define the 2PL model:
+   # note, both the input and output are vectors rather than scalars
+   P <- 1/(1+exp(-(zeta+lambda*theta_set)))
+   Q <- 1-P
```

```
+
+   # define the weight in case of using normal ogive in the future
+   W <- P*Q
+
+   # Define the L1, L11, L2, L22, L12
+   # Note, to make computation efficient, I use matrix operation
+   vec_tempt_1 <- (p_set -P)/(P*Q)
+   vec_tempt_2 <- (p_set -P)*theta_set/(P*Q)
+   theta_sq_vec <- theta_set^2
+   L1 <- t(f_set)%*%diag(W)%*%t(t(vec_tempt_1))
+   L2 <- t(f_set)%*%diag(W)%*%t(t(vec_tempt_2))
+   L11 <- -t(f_set)%*%t(t(W))
+   L22 <- -t(f_set)%*%diag(W)%*%t(t(theta_sq_vec))
+   L12 <- -t(f_set)%*%diag(W)%*%t(t(theta_set))
+
+   # make them into matrix form
+   matrix_L <- matrix(c(L11,L12,L12,L22),2,2)
+   vector_L <- t(t(c(L1, L2)))
+
+   # get the delta zeta and delta lambda
+   deltas <- -solve(matrix_L)%*%vector_L
+
+   # update the zeta and lambda
+   # note here is to add the deltas not to minus!!!!
+   updated_parameters <- t(t(c(zeta, lambda))) + deltas
+   zeta <-updated_parameters[1,1]
+   lambda <- updated_parameters[2,1]
+ }
> print(deltas[1])
[1] -0.0001296827
> print(deltas[2])
[1] 0.000476612
> print(paste0("The iteration time is: ", iter_time," ."))
[1] "The iteration time is: 3 ."
> print(paste0("The estimated zeta is: ", round(zeta,4)," ."))
[1] "The estimated zeta is: 0.2016 ."
> print(paste0("The estimated lambda is: ", round(lambda,4)," ."))
[1] "The estimated lambda is: 0.8236 ."
```

The results show that the $\hat{\zeta} = .202$ and $\hat{\lambda} = .824$ after 4 iterations. Convert the value into $\hat{\alpha} = \hat{\lambda} = .824$ and $\hat{\beta} = -\frac{\hat{\zeta}}{\hat{\lambda}} = -0.245$. Therefore, the estimated logistic model is

$$P(\theta) = \frac{1}{1 + e^{-.824(\theta + .245)}}.$$

Notes about coding: The way my code to get the `L1`,`L11`, etc., is kinda wordy. R actually provides more efficient way. For example, one can just type `L1 <- sum(f_set*W*vec_tempt_1)` to get the same result. Here, I use the matrix alegra style to refresh my memory about the past math knowledge.

Q3-Part b

Calculate the standard errors of the estimates

My Solution
From the equations in our lecture notes, one can have the following code

```
> alpha <- 0.824
> beta <- -0.245
> var_alpha <- 1/sum(f_set*W*(theta_set-mean(theta_set))^2)
> var_zeta <- 1/sum(f_set*W) + var_alpha*mean(theta_set)^2
> var_beta <- (sum(f_set*W)^(-1)+var_alpha*(beta-mean(theta_set))^2)*alpha^(-2)
> (se_alpha <-sqrt(var_alpha))
[1] 0.2393551
> (se_zeta <- sqrt(var_zeta))
[1] 0.2732285
> (se_beta <- sqrt(var_beta))
[1] 0.3391392
```

The results show that standard error (SE) for α or λ is 0.239; SE for ζ is 0.273; SE for β is 0.339.

Q3-Part c

Find the minimum logit...

My Solution:
Refer to the equation (2.35) and (2.36) from Baker & Kim(2004), one can have

```
> # re-define the p_set and get the log ratios
> q_set <- 1-p_set
> l_set <- log(p_set[3:11]/q_set[3:11])
> # use the revised p to replace the 0 or 1
> l_set <- c(c(0.5,0.25),l_set,c(0.75,0.5))
> l_set
 [1]  0.5000000  0.2500000 -1.0986123 -0.9162907 -0.5108256 -0.2231436
 [7]  0.0000000  1.0986123  0.6931472  1.3862944  1.3862944  0.7500000
[13]  0.5000000
> # prepare some values
> l_mean <- mean(l_set)
> theta_mean <- mean(theta_set)
>
> # using the equation (2.35)
> zeta_2_nom <- l_mean*sum(f_set*W*theta_sq_vec)-theta_mean*sum(f_set*W*l_set*theta_set)
> zeta_2_denom <- sum(f_set*W*theta_sq_vec)- (sum(f_set*W*theta_set)^2)/sum(f_set*W)
> zeta_2 <- zeta_2_nom/zeta_2_denom
>
> # using the equation (2.36)
> lambda_2_nom <- sum(f_set*W*l_set*theta_set)-sum(f_set*W*theta_set)*sum(f_set*W*l_set)/sum(f_set*W)
> lambda_2_denom <- sum(f_set*W*theta_sq_vec)-sum(f_set*W*theta_set)^2/sum(f_set*W)
> lambda_2 <- lambda_2_nom/lambda_2_denom
> print(paste0("The estimated zeta is ", round(zeta_2,3)))
[1] "The estimated zeta is 0.299"
> print(paste0("The estimated lambda is ", round(lambda_2,3)))
[1] "The estimated lambda is 0.573"
```

Comparing the results from two method,ζ vs. ζ_2 is .202 vs. .299; and the λ vs. λ_2 is .824 vs. .573. The two results are not very close. The posssible reason is due to the sample size will given relatively large estimation error. Especially, at two extreme ends, very few responses make our estimation unstable. Comparing them by using the 95% CI of each statistic may be a good choice.

Q4-Part a

For the five items given in the Table…

My Solution:
I use P_i to represent the probability of success for a test-taker on the $item_i$. Since the response vector of this given examinee is $X = [0, 0, 0, 1, 1]$, the likelihood function is

$$L(x) = (1 - P_1)(1 - P_2)(1 - P_3)P_4P_5$$

.

Here are some assumptions:
- A1: The binary-scored-item parameters are known (already given by the question).
- A2: The examinee are independent (it is satisfied since only one examinee is given).
- A3: All items in this test are modeled by the same IRT model (here, 3PL).
- A4: All items are independent on a given trait (Local Independent).
- A5: All items measure one type of trait (Unidimensionality).

Q4-Part b

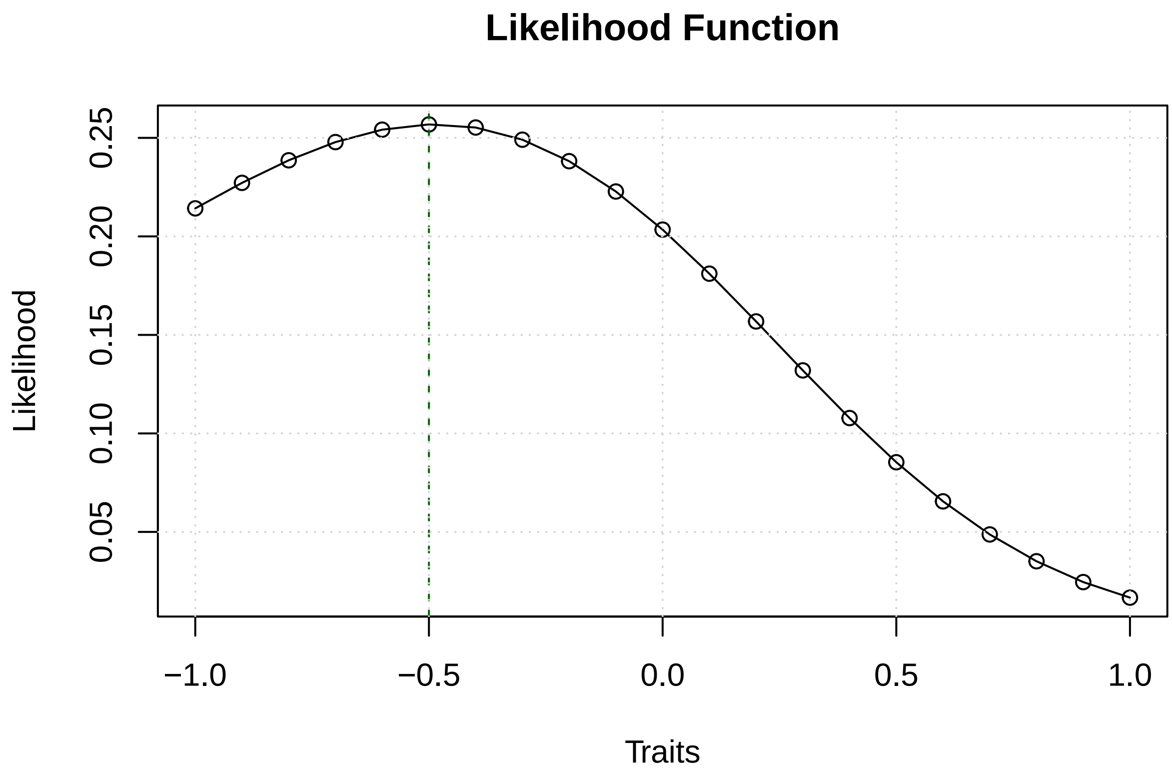
Plot the likelihood function at values from…

My Solution:
Expanded the likelihood function $L(x)$ above, one can have the following code in R:

```
> # write a 3PL model
> L_X <- function(theta){
+   a_set <- c(1.25,1.35,1.15,1,0.75)
+   b_set <- c(1.20,0.60,0.15,-0.6,-2)
+   c_set <- c(0.1,0.15,0.15,0.20,0.10)
+   P1 <- c_set[1] + (1-c_set[1])/(1+exp(-(1.702*a_set[1]*(theta - b_set[1]))))
+   Q1 <- 1-P1
+   P2 <- c_set[2] + (1-c_set[2])/(1+exp(-(1.702*a_set[2]*(theta - b_set[2]))))
+   Q2 <- 1-P2
+   P3 <- c_set[3] + (1-c_set[3])/(1+exp(-(1.702*a_set[3]*(theta - b_set[3]))))
+   Q3 <- 1-P3
+   P4 <- c_set[4] + (1-c_set[4])/(1+exp(-(1.702*a_set[4]*(theta - b_set[4]))))
+   P5 <- c_set[5] + (1-c_set[5])/(1+exp(-(1.702*a_set[5]*(theta - b_set[5]))))
+   # write the likelihood function
+   L_x <- Q1*Q2*Q3*P4*P5
+   return(L_x)
+ }
```

Based on the likelihood function above, plug the trait vector and plot the results:

```
> theta_set <- seq(-1,1,by=0.1)
> L_set <- L_X(theta_set)
> plot(theta_set,L_set,type='o',
+      xlab="Traits",ylab="Likelihood",
+      main="Likelihood Function")
> abline(v=-0.5, col="darkgreen",lty=2)
> grid()
```



The plot shows that the maximum likelihood occurs at around $trait = -0.5$. The maximum likelihood is 0.26.