BONUS PROBLEMS

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. My primary resources were my notes from lectures , lab example problems and pauls notes. I combined these resources with AI so I could import these resources into chatGPT and ask specific questions about the parts I did not understand, or where I made an error in a step when I got stuck.

1. Problem 1

Let $\vec{F}(x,y,z) = \langle sin(x^2), xz, z^2 \rangle$ Evaluate $\int_C \vec{F} \cdot d\vec{r}$ around the curve C of the intersection of the cylinder $x^2 + y^2 = 4$ with the surface $z = x^2$ in the counter-clockwise direction as viewed from the top of the z-axis.

Use Stokes Theorem:
Which applies when:
-C is closed, simple and piecewise smooth
-Does not intersect
-Oriented counter clockwise when viewed from above.

Therefore for this problem we can say: $\int_C \vec{F} \cdot d\vec{r} = \iint_D \operatorname{curl} \vec{F} \cdot d\vec{s}$

$$\text{Curl}(\vec{F}) = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ F_x & F_y & F_z \\ sin(x^2) & xz & z^2 \end{bmatrix}$$

this yields: $\iint_D \langle -x, 0, z \rangle \cdot d\vec{s}$
$d\vec{s}$ can be defined as $r_u \times r_v$
Our surface is given by the intersection between this cylinder and parabolic sheet.

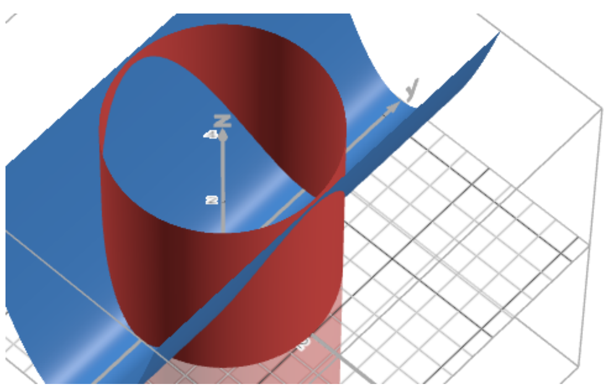

Figure 1. Intersection between cylinder and parabolic sheet

We can parameterize this surface to get our surface s: Let x = u , y = v and because $z = x^2 \quad z = u^2$
now we have $r(u,v) = \langle u, v, u^2 \rangle$ for u and v $\in x^2 + y^2 \leq 4$

Date: May 15, 2025.

$r_u = \langle 1, 0, 2u\rangle \quad r_v = \langle 0, 1, 0\rangle$

$$r_u \times r_v = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 0 \end{bmatrix}$$

This gives our normal vector $\vec{n} = \langle -2u, 0, 1\rangle$
to get the integrand: $\langle -u, 0, u^2\rangle \cdot \langle -2u, 0, 1\rangle$
$\iint_D 2u^2 + 2u \cdot d\vec{s}$
$\iint_D 3u^2 \cdot d\vec{s}$
let $\quad u = r\,cos\,(\theta), v = r\,sin\,(\theta)$
we now have:
$\int_0^{2\pi} \int_0^0 3r^2 cos^2(\theta)\, dr\, d\theta$
$\int_0^{2\pi} [\frac{3}{4} r^4 cos^2(\theta)]\big|_0^2\, d\theta$
$\int_0^{2\pi} 12cos^2(\theta)\, d\theta$
The double angle formula is given by: $cos^2(\theta) = \frac{1+cos(2\theta)}{2}$
$12 \int_0^{2\pi} \frac{1+cos(2\theta)}{2}$
$\int_0^{2\pi} 6 + 6cos(2\theta)$
$6 \int_0^{2\pi} 1 + cos(2\theta)$
$6[2\pi + \frac{1}{2} sin(2\theta)]$
$12\pi + 3sin(4\pi)$
Sin over any even number of pi over 1 period $= 0$
$12\pi + 0 = 12\pi$

2. Problem 2

Let E be the solid region between the plane $z = 4$ and the parabolid $z = x^2 + y^2$ Let $\vec{F}(x, y, z) = \langle \frac{-1}{3}x^3 + e^{z^2}, \frac{-1}{3}y^3 + xsin(z), 4z\rangle$ Let S be the surface that encloses E. Find $\int\int_S \vec{F} \cdot d\vec{s}$

We can apply divergence theorem as E is an enclosed region between our paraboloid and our plane. Our parabolid is not closed but we can artificially close it through our limits of integration.

Divergence theorem is given by $\iint_D \vec{F} \cdot d\vec{s} = \iiint_V \text{div}(\tilde{\text{F}}) dV$

Our region is bounded by $z = 4, z = x^2 + y^2$ as shown in figure 2.

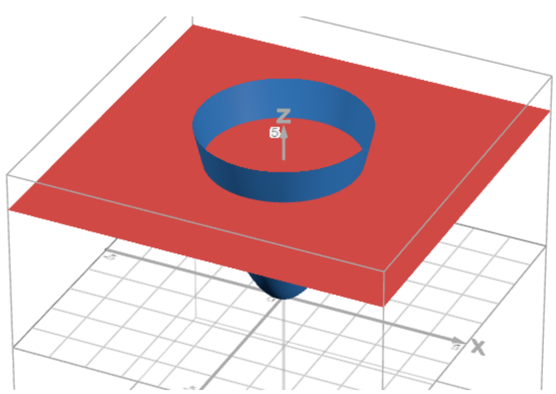

Figure 2. Intersection between paraboloid and plane

$\text{div}(\vec{F}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

now that the problem is contextualized we break down the integral:
$\frac{\partial F_x}{\partial x} = -x^2 \quad \frac{\partial F_y}{\partial y} = -y^2 \quad \frac{\partial F_z}{\partial z} = 4$
$\iiint_V -x^2 - y^2 + 4\,dV$

The paraboloid itself that encloses the region has an open top but we can artificially enclose the paraboloid by limiting the upper bound to $z = x^2 + y^2$

let
$x = rcos\theta$
$y = rsin\theta$
$z = x^2 + y^2$
$z = r^2cos^2\theta + r^2sin^2\theta$
$z = r^2$
because $z = x^2 + y^2 \quad r^2 = x^2 + y^2 \quad r = 2$

set up the bounds of integration: $\int_0^{2\pi}\int_0^2\int_4^{r^2} -r^2cos^2\theta - r^2sin^2\theta + 4r\,dz\,dr\,d\theta$
$\int_0^{2\pi}\int_0^2\int_4^{r^2} -r^3cos^2\theta - r^3sin^2\theta + 4r\,dz\,dr\,d\theta$
$\int_0^{2\pi}\int_0^2 -zr^3cos^2\theta - zr^3sin^2\theta + 4zr\,\Big|_4^{r^2}\,dr\,d\theta$
$\int_0^{2\pi}\int_0^2 -r^5cos^2\theta - r^5sin^2\theta + 4r^3 + 4r^3cos^2\theta + 4r^3sin^2\theta - 16r\,dr\,d\theta$
$4r^3cos^2\theta + 4r^3sin^2\theta = 4r^3$
$\int_0^{2\pi}\int_0^2 -r^5cos^2\theta - r^5sin^2\theta + 8r^3 - 16r\,dr\,d\theta$
$\int_0^{2\pi} -\frac{1}{6}r^6cos^2\theta - \frac{1}{6}r^6sin^2\theta + 2r^4 - 8r^2\,\Big|_0^2\,d\theta$
$\int_0^{2\pi} -\frac{64}{6}cos^2\theta - \frac{64}{6}sin^2\theta\,d\theta$
$\frac{64}{6}\int_0^{2\pi} -cos^2\theta - sin^2\theta\,d\theta$
Use trig substitutions:
$cos^2\theta = \frac{1+cos(2\theta)}{2} sin^2\theta = \frac{1-cos(2\theta)}{2}$
$\frac{64}{6}\int_0^{2\pi} \frac{-1-cos(2\theta)}{2} - \frac{1-cos(2\theta)}{2}\,d\theta$
$\frac{64}{6}\int_0^{2\pi} -1\,d\theta$
$\frac{64}{6}(-2\pi)$
$\frac{-128\pi}{6}$
$\frac{-64\pi}{3}$

Answer represents a volume so should be positive but is negative because I put r in the upper limit of integration instead of lower.