Cross Product Notes

Christopher Greene

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The unit vector cross products:

- $\vec{i} \times \vec{j} = \vec{k}$
- $\vec{j} \times \vec{i} = -\vec{k}$
- $\vec{j} \times \vec{k} = \vec{i}$
- $\vec{k} \times \vec{j} = -\vec{i}$
- $\vec{k} \times \vec{i} = \vec{j}$
- $\vec{i} \times \vec{k} = -\vec{j}$
- $\vec{i} \times \vec{i} = \vec{0}$ (self cross product is zero)

General Cross Product Definition

For vectors:

$$\begin{aligned} \vec{u} &= u_1\vec{i} + u_2\vec{j} + u_3\vec{k} \\ \vec{v} &= v_1\vec{i} + v_2\vec{j} + v_3\vec{k} \end{aligned}$$

The cross product expands as:

$$\begin{aligned} \vec{u} \times \vec{v} &= (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \times (v_1\vec{i} + v_2\vec{j} + v_3\vec{k}) \\ &= u_1v_1(\cancel{\vec{i} \times \vec{i}})0 + u_1v_2(\cancel{\vec{i} \times \vec{j}})k + u_1v_3(\cancel{\vec{i} \times \vec{k}}) - j \\ &\quad + u_2v_1(\vec{j} \times \vec{i}) + u_2v_2(\vec{j} \times \vec{j}) + u_2v_3(\vec{j} \times \vec{k}) \\ &\quad + u_3v_1(\vec{k} \times \vec{i}) + u_3v_2(\vec{k} \times \vec{j}) + u_3v_3(\vec{k} \times \vec{k}) \end{aligned}$$

Using the self-cross product property $\vec{a} \times \vec{a} = \vec{0}$, this simplifies to:

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$$

$$\begin{aligned} \text{2D matrices:} &\quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\ \text{This yields:} &\quad ad - bc \quad \text{(The result of this is called the determinant).} \end{aligned}$$

$$\begin{aligned} \text{3D matrices:} &\quad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ \text{This yields:} &\quad ((a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}) \\ \text{this can be broken down into:} &\quad \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\vec{k} \end{aligned}$$

Important takeaways:

Let w equal the resultant vector from the cross product of u and v:

$$\vec{u} \times \vec{v} = \vec{w}$$

The resultant vector will be orthogonal to the plane which u and v are on.

this vector w must be orthogonal to both u and v at any point

this can be verified by using the dot poduct:

$$(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$$

This must equal zero

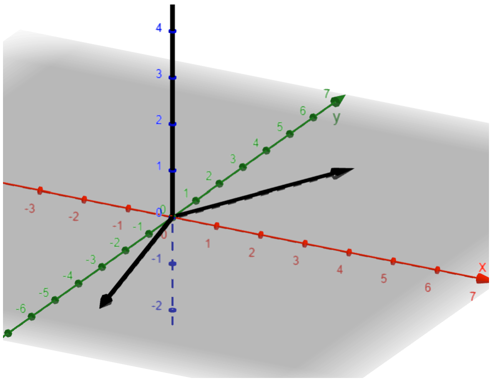

Figure 1: This shows the vector generated by the cross product

Important derivations:

- $\vec{u} \times \vec{v} = |\vec{u}||\vec{v}| \cos$
- $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin$

Examples:

Example 1: Find the volume of the parallelepiped created by vectors u,v,w

$$\vec{u} = \langle 1, -1, 1 \rangle \quad \vec{v} = \langle 1, 1, 2 \rangle \quad \vec{w} = \langle -1, -1, 1 \rangle$$

recall that volume = (basearea) x (height)

base area = $|\vec{u} \times \vec{v}|$

height = $|\vec{w}|$

step 1. find$(\vec{u} \times \vec{v})$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \vec{i}(-1(2) - 1(1)) - \vec{j}(1(2) - 1(1)) + \vec{k}(1(1) - (-1)(1))$$

$$= -3\vec{i} - \vec{j} + 2\vec{k}$$

volume = $\langle -3, -1, 2 \rangle \times$ height

step 2. find height

height = $\vec{w}$

$$\langle -3, -1, 2 \rangle \cdot \langle -1, -1, 1 \rangle$$

volume = $((-3)(-1) + (-1)(-1) + (2)(1))$

$= (3 + 1 + 2)$

volume = 6

Example 2: Find the area of 2d triangle created by points P,Q,rangle

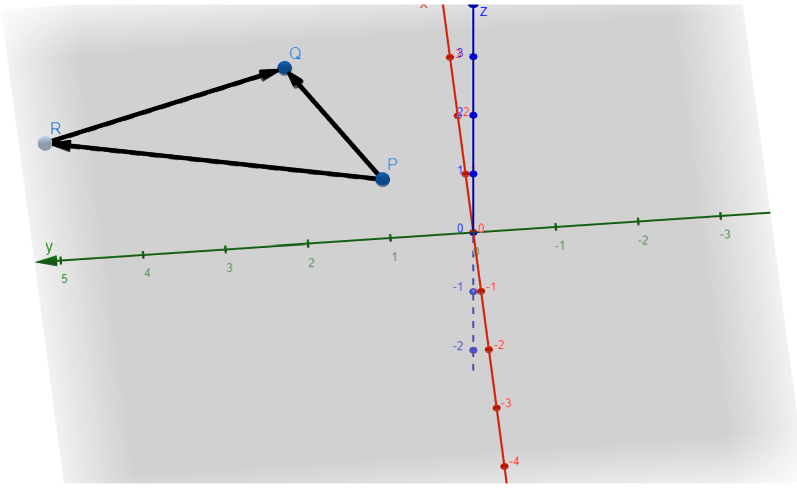

Figure 2: This shows the triangle created by the points

a parallelogram is created by 2 triangles.
Therefore to get the area of the triangle we just need to find the area of the parallelogram and divide by 2.

$$\vec{PQ} = \langle 1, 1, 0 \rangle \quad \vec{PR} = \langle 2, 0, 1 \rangle$$

$$\text{area of parallelogram} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 1 & 4 & 0 \end{vmatrix}$$

$$= 0mathvfi - 0mathbfj + (8-1)mathbfk$$

$$= 7$$

$$\text{divide by 2 for area of triangle}$$

$$a = 3.5$$