Math 132A Homework Three

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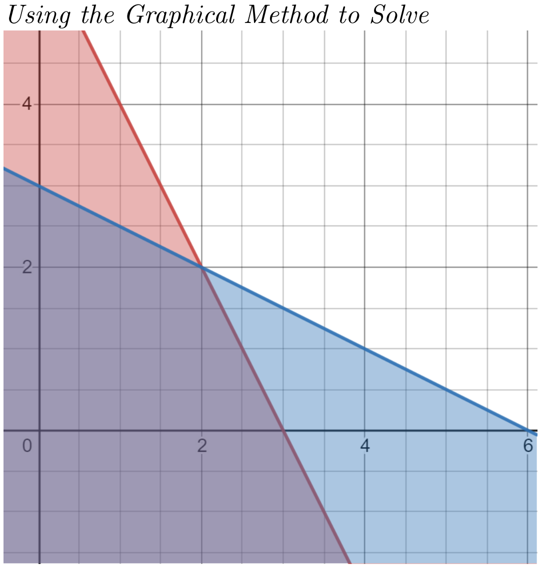
1 Part One

1.1 Exercise One

max $3x_1 + 2x_2$ subject to

$$2x_1 + x_2 \leq 6$$
$$x_1 + 2x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

Using the Graphical Method to Solve



Note the four extreme points: (0,0), (0,3), (2,2), (3,0) with the four corresponding optimal values: 0, 6, 10, 9. Thus (2,2) is the optimal solution and 10 is the optimal value.

Using the Simplex Method to Solve

First, we rewrite our model in standard form:

$$\text{MAX } z(x_1, x_2, s_1, s_2) = 3x_1 + 2x_2 + 0s_1 + 0s_2 \text{ SUBJECT TO}$$

$$2x_1 + x_2 + s_1 = 6$$
$$x_1 + 2x_2 + s_2 = 6$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Thus, the feasible region of this problem is

$$P = \{x \in \mathbb{R}^4 : Ax = b, x \geq 0\}$$

where

A = $\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

Let $(0,0,6,6)$ be our initial BFS where $x_N = \{x_1, x_2\}$ and $x_B = \{s_1, s_2\}$. Thus our objective function is $z(x_1, x_2, s_1, s_2) = 0 + 3x_1 + 2x_2$ subject to

$$s_1 = 6 - 2x_1 - x_2$$
$$s_2 = 6 - x_1 - 2x_2$$

Thus, the reduced cost coefficients are $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ so we choose to go in the direction of x_1 and make it a basic variable.

$$s_1 = 6 - 2x_1$$
$$s_2 = 6 - x_1$$

Thus $x_1 \leq 3$ in order to maintain the non-negativity constraint, so we let $x_1 = 3$. Then our BFS becomes (3, 0, 0, 3), $x_N = \{x_2, s_1\}$, $x_B = \{x_1, s_2\}$, and the optimal value is 9. So, our model becomes

$$\text{MAX } 3(\frac{6 - s_1}{2}) + 2x_2 = \frac{-3}{2}s_1 + 2x_2 - 9$$

subject to

$$x_1 = \frac{6 - x_2}{2}$$
$$s_2 = 6 - 2x_2$$

Thus, we should move in the x_2 direction to see if there is a better optimal solution. So, we should keep $s_1 = 0$.

$$x_1 = \frac{6 - x_2}{2}$$
$$s_2 = 6 - x_1 - 2x_2 = 6 - \frac{6 - x_2}{2} - 2x_2 = 3 - \frac{3}{2}x_2$$

To maintain non-negativity, $x_2 \leq 2$ so $x_2 = 2$ to yield a BFS of (2, 2, 0, 0) where $x_B = \{x_1, x_2\}$, $x_N = \{s_1, s_2\}$. This yields an optimal value of 10. Rewriting our objective function in terms of non basic variables, we get

$$\text{MAX } \frac{-3}{2}s_1 + 2x_2 - 9 = \frac{-3}{2}s_1 + 2(\frac{-2}{3}s_2 + 2) - 9 = \frac{-3}{2}s_1 + \frac{-4}{3}s_2 - 5$$

Thus, we should not move further from this point since the reduced cost coefficients are $\begin{bmatrix} \frac{-3}{2} \\ \frac{-4}{3} \end{bmatrix}$ Therefore $(2, 2, 0, 0)$ is our optimal solution.

Relating Simplex and Graphical Solutions
Essentially, the simplex method starts at the origin (or another simple BFS) and then decides which direction to go around the boundary of the feasible region to find a better optimal value until you cannot move any further. In our case, we start at the extreme point (0,0) and go along the x-axis to (3,0) first because it increases quickly and then once we cannot move further, we go along the y axis and checking for the best optimal solution until getting to (2,2).

1.2 Exercise Two

max $4x + 3y + 6z$ subject to

$$3x + y + 3z \leq 30$$

$$2x + 2y + 3z \leq 40$$

$$x, y, z \geq 0$$

First, we want to convert this problem into standard form, so the constraints are now

$$3x + y + 3z + s_1 = 30$$

$$2x + 2y + 3z + s_2 = 40$$

$$x, y, z, s_1, s_2 \geq 0$$

Let (0,0,0,30,40) be our initial BFS, which yields an optimal value of 0 with $x_B = \{s_1, s_2\}$, $x_N = \{x, y, z\}$. Note the RCC = $\begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}$ Choosing to increase in the z direction and making it a basic variable, we observe (x,z=0):

$$s_1 = 30 - 3z$$

$$s_2 = 40 - 3z$$

With the MRT, we see z should equal 10. Then, s_1 becomes a non-basic variable, so our new BFS is (0, 0, 10, 0, 10) with an optimal value of 60 and $x_B = \{z, s_2\}$, $x_N = \{x, y, s_1\}$. We adjust our model as follows:

$$\text{MAX } 4x+3y+6\left(\frac{30 - 3x - y - s_1}{3}\right) = 4x+3y+60-6x-2y-2s_1 = -2x+y-2s_1+60$$

Thus, our RCC are $\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$ so we should choose to move further in the y direction $(x, s_1 = 0)$:

$$z = 10 - \frac{1}{3}y$$

$$s_2 = 40 - 2y - 3(10 - \frac{1}{3}y) = 40 - 2y - 30 + y = 10 - y$$

Using the MRT, we see y should equal 10. Then, s_2 becomes a non-basic variable, so our new BFS is $(0, 10, \frac{20}{3}, 0, 0)$ with an optimal value of 70 and $x_B = \{y, z\}$, $x_N = \{x, s_1, s_2\}$. We adjust our model:

$$\text{MAX } -2x + (10 - s_2) - 2s_1 + 60 = -2x - 2s_1 - s_2 + 70$$

Because our RCC are $\begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix}$ this tells us this our unique optimal solution.

2 Part Two

2.1 Exercise One

Consider the model

$$\text{MAX } x + y \text{ SUBJECT TO}$$

$$x \leq 2$$

$$y \leq 2$$

$$x, y \geq 0$$

So, our feasible region is a square with four extreme points. Using the simplex method, we'll see this has a unique optimal solution.
First, we convert to standard form.

$$x + s_1 = 2$$

$$y + s_2 = 2$$

$$x, y, s_1, s_2 \geq 0$$

Since the RCC are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we should move further in the x or y direction, lets say x. Then,

$$s_1 = 2 - x$$

$$s_2 = 2$$

with MRT, we see x should be 2. This leads us to our new BFS (2,0,0,2) and an optimal value of 2 with $x_B = \{x, s_2\}$, $x_N = \{y, s_1\}$. We adjust our model

$$\text{MAX } x + y = y - s_1 + 2$$

Since RCC = $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ we should move further in the y direction ($s_1 = 0$). Thus,

$$x = 2$$

$$s_2 = 2 - y$$

with MRT, y should equal 2, giving us a new BFS of (2,2,0,0) and an optimal value of 4 with $x_B = \{x, y\}$, $x_N = \{s_1, s_2\}$. Then,

$$\text{MAX } y - s_1 + 2 = 2 - s_2 - s_1 + 2 = 4 - s_1 - s_2$$

making the RCC $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ so this is our unique optimal solution.

2.2 Exercise Two

Consider the model

$$\text{MAX } x + y \text{ SUBJECT TO}$$

$$2x + 2y \leq 4$$

$$y \leq 1$$

$$x, y \geq 0$$

We will show this has many optimal solutions using the simplex method. First, lets convert to standard form

$$2x + 2y + s_1 = 4$$

$$y + s_2 = 1$$

$$x, y, s_1, s_2 \geq 0$$

Since the RCC are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we should increase in the x direction (y=0). So,

$$s_1 = 4 - 2x$$

$$s_2 = 1$$

with MRT, x should equal 2. Thus, we get a new BFS (2,0,0,1) and an optimal value of 2 with $x_B = \{x, s_2\}$, $x_N = \{y, s_1\}$. We adjust our model with the new non-basic variable:

$$\text{MAX } x + y = (2 - y - \frac{1}{2}s_1) + y = 2 - \frac{1}{2}s_1$$

Which makes our new RCC $\begin{bmatrix} 0 \\ \frac{-1}{2} \end{bmatrix}$ Since the MRT=2 $\neq 0$, there are many optimal solutions along the line segment $y = 2 - x$, $x \leq 2$.

2.3 Exercise Three

Consider the problem

$$\text{MAX } x + y \text{ SUBJECT TO}$$

$$x + y \geq 1$$

$$x, y \geq 0$$

Notice this problem is unbounded (but has at least two extreme points) so its reasonable to anticipate that there are no optimal solutions, which we will show using the simplex method. Converting to standard form, we get

$$x + y - s_1 = 1$$

$$x, y, s_1 \geq 0$$

Choosing an initial BFS of (0,0,1) yields an optimal value of 0 with $x_B = \{s_1\}$, $x_N = \{x, y\}$. Since the RCC are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we should move in the x direction (y=0). Then,

$$s_1 = x + 1$$

so the MRT=ϕ since x can be increased as much as we want and this value will remain more than 0. Thus, this problem has no optimal solutions.