前言

大地電磁法(MT)最常用的資料交換格式是 EDI 格式(副檔名是.edi)，是目前多數大地電磁分析軟體採用的格式標準之一。EDI 格式標準的定義是來自「Society of Exploration Geophysicists MT/EMAP Data Interchange Standard Revision 1.0 December 14, 1987」所制定的 SEG v1.0 的 EDI 格式標準。另外，1991 年有將 1987 年這份文件追加說明並重新發表，內容與原本的基本相同，但提供許多註解與範例，稍微有比較好懂一點點。

但很悲傷的是，並沒有免費且開源的軟體提供讀寫及分析 EDI 檔案的軟體，也很少提供資料預處理軟體的計算如何搭配 EDI 檔案使用。多數文獻專注於改善資料預處理的方案，但常常因各種理由使用自訂的格式而非 EDI 格式，自訂格式也都不會像 EDI 格式這麼完整規範，結果造成普通的使用者還是較依賴支援 EDI 格式的商用軟體。商用軟體最大的弊病就是隱藏了許多關鍵的計算技術，這保護了商業利益，但也造成許多使用者在錯誤的觀念下進行不正確的分析，也一定程度阻礙了分析方法的進步。

我所知道的商用軟體包含 SSMT2000、MTEditor、WinGLink、EMPower、3D-Grid 等軟體都支援 EDI 格式。一個免費開源的程式 mtpy 也號稱有支援 edi 格式(目前 2023/08/30，穩定版為 v1.0，預發佈版本為 v2.0)，但是他讀取 EDI 後輸出的資料與商用軟體還是有點不一樣。

美國地質學會的學者「Anna Kelbert」在 2020 年發表了「EMTF XML:

New data interchange format and conversion tools for electromagnetic transfer functions」一文，文章中提出了新一代的大地電磁法建議通用交換格式(副檔名.xml)。XML(可延伸標記式語言)是一種成熟且易用的語言，廣泛用來作為跨平台之間互動數據的形式，例如，常用在詮釋資料(metadata)。該文章同樣描述了如何進行一些基礎的 EDI 格式讀寫、資料預處理的分析、以及提供明確的新交換格式的規範，使得資料更容易的被歸檔及資料庫搜索。另外，作者提供了免費開源的工具進行 EDI 檔、J 檔、Z 檔等多種不同分析軟體的格式轉換至 XML 的工具「EMTF FCU」(目前 2023/08/30，程式碼標註版本為 v4.1，說明標註版本為 v4.0)。若有其他必要的詮釋資料可以自行填充至 XML 中。在部分狀況下也支援逆向轉換，但因為只有 XML 擁有最完整的詮釋資料，轉換為其他格式將會損失資訊。但這程式是用 fortran 90 寫的，用起來有點不太友善。

感謝「Anna Kelbert」的文章及提供的開源程式碼，我就利用這程式碼回頭破解一下 EDI 格式與預處理的關係是怎麼回事。

方程式符號慣例

大寫粗體英文字: 矩陣

小寫粗體英文字: 矩陣

斜體英數字: 變數

斜體小寫 i 要留給複數使用。

正體英數字: 常數

複數運算與程式中定義

共軛複數

$$A^* = (a + bi)^* = a - bi$$

矩陣定義與運算

　　身為一個軟體工程師，早就忘記數學是怎麼一回事了，回頭整理一下相關的數學工具。

<1x1>矩陣範例: $\boldsymbol{A}_{\langle 1\mathrm{x}1\rangle} = [A(1,1)]$
<2x1>矩陣範例: $\boldsymbol{A}_{\langle 2\mathrm{x}1\rangle} = \begin{bmatrix} A(1,1) \\ A(2,1) \end{bmatrix}$
<2x2>矩陣範例: $\boldsymbol{A}_{\langle 2\mathrm{x}2\rangle} = \begin{bmatrix} A(1,1) & A(1,2) \\ A(2,1) & A(2,2) \end{bmatrix}$
<3x2>矩陣範例: $\boldsymbol{A}_{\langle 3\mathrm{x}2\rangle} = \begin{bmatrix} A(1,1) & A(1,2) \\ A(2,1) & A(2,2) \\ A(3,1) & A(3,2) \end{bmatrix}$
<2x1><1x2>矩陣運算-矩陣乘法: $\begin{bmatrix} a \\ b \end{bmatrix}_{\langle 2\mathrm{x}1\rangle} \begin{bmatrix} c & d \end{bmatrix}_{\langle 1\mathrm{x}2\rangle} = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}_{\langle 2\mathrm{x}2\rangle}$
<2x2><2x1>矩陣運算-矩陣乘法: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{\langle 2\mathrm{x}2\rangle} \begin{bmatrix} e \\ f \end{bmatrix}_{\langle 2\mathrm{x}1\rangle} = \begin{bmatrix} ae + bf \\ ce + df \end{bmatrix}_{\langle 2\mathrm{x}1\rangle}$
<2x2><2x2>矩陣運算-矩陣乘法: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{\langle 2\mathrm{x}2\rangle} \begin{bmatrix} e & f \\ g & h \end{bmatrix}_{\langle 2\mathrm{x}2\rangle} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}_{\langle 2\mathrm{x}2\rangle}$
<2x2>矩陣運算-反矩陣:

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{\langle 2\text{x}2\rangle}\right)^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}_{\langle 2\text{x}2\rangle} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}_{\langle 2\text{x}2\rangle}$$

矩陣運算-複共軛轉置(complex conjugate transpose)或稱埃爾米特轉置(Hermitian transpose):

$$\left(\boldsymbol{A}_{\langle\mathrm{mxn}\rangle}\boldsymbol{B}_{\langle\mathrm{nxp}\rangle}\right)^{\dagger} = \left(\boldsymbol{B}_{\langle\mathrm{nxp}\rangle}\right)^{\dagger}\left(\boldsymbol{A}_{\langle\mathrm{mxn}\rangle}\right)^{\dagger}$$

<1x2>矩陣運算-複共軛轉置(complex conjugate transpose)或稱埃爾米特轉置(Hermitian transpose):

$$\left(\begin{bmatrix} A & B \end{bmatrix}_{\langle 1\mathrm{x}2\rangle}\right)^{\dagger} = \left(\begin{bmatrix} a+bi & c+di \end{bmatrix}_{\langle 1\mathrm{x}2\rangle}\right)^{\dagger} = \begin{bmatrix} a-bi \\ c-di \end{bmatrix}_{\langle 2\mathrm{x}1\rangle} = \begin{bmatrix} A^* \\ B^* \end{bmatrix}_{\langle 2\mathrm{x}1\rangle}$$

<2x1>矩陣運算-複共軛轉置(complex conjugate transpose)或稱埃爾米特轉置(Hermitian transpose):

$$\left(\begin{bmatrix} A \\ B \end{bmatrix}_{\langle 2\mathrm{x}1\rangle}\right)^{\dagger} = \left(\begin{bmatrix} a+bi \\ c+di \end{bmatrix}_{\langle 2\mathrm{x}1\rangle}\right)^{\dagger} = \begin{bmatrix} a-bi & c-di \end{bmatrix}_{\langle 1\mathrm{x}2\rangle} = \begin{bmatrix} A^* & B^* \end{bmatrix}_{\langle 1\mathrm{x}2\rangle}$$

<2x2>矩陣運算-複共軛轉置(complex conjugate transpose)或稱埃爾米特轉置(Hermitian transpose):

$$\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\langle 2\mathrm{x}2\rangle}\right)^{\dagger} = \left(\begin{bmatrix} a+bi & c+di \\ e+fi & g+hi \end{bmatrix}_{\langle 2\mathrm{x}2\rangle}\right)^{\dagger} = \begin{bmatrix} a-bi & e-fi \\ c-di & g-hi \end{bmatrix}_{\langle 2\mathrm{x}2\rangle} = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}_{\langle 2\mathrm{x}2\rangle}$$

單位矩陣乘法:

$$\boldsymbol{I}_{\langle 2\mathrm{x}2\rangle} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{\langle 2\mathrm{x}2\rangle}$$

$$\boldsymbol{A}_{\langle 1\mathrm{x}2\rangle} = \begin{bmatrix} A(1,1) & A(1,2) \end{bmatrix}_{\langle 1\mathrm{x}2\rangle}$$

$$\boldsymbol{A}_{\langle 1\mathrm{x}2\rangle}\boldsymbol{I}_{\langle 2\mathrm{x}2\rangle} = \begin{bmatrix} A(1,1) & A(1,2) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A(1,1) & A(1,2) \end{bmatrix} = \boldsymbol{A}_{\langle 1\mathrm{x}2\rangle}$$

簡化描述下，單位矩陣自己會配合相乘的矩陣，所以:

$$\boldsymbol{A}_{\langle\mathrm{mxn}\rangle}\boldsymbol{I}_{\langle\mathrm{nxn}\rangle} = \boldsymbol{A}_{\langle\mathrm{mxn}\rangle}$$

簡化常看到的寫法:

$$\boldsymbol{AI} = \boldsymbol{A}$$

線性回歸問題

根據 Anna Kelbert(2020)，整理一下線性回歸問題的相關公式。

常用簡單線性問題的符號表示下，線性回歸問題: $\boldsymbol{Y}=\boldsymbol{X}\boldsymbol{b}+\boldsymbol{\varepsilon}$ 已知: $\boldsymbol{Y}$、$\boldsymbol{X}$ 未知: $\boldsymbol{b}$ 誤差: $\boldsymbol{\varepsilon}$ 公式解: $\widehat{\boldsymbol{b}}=(\boldsymbol{X}^{\dagger}\boldsymbol{X})^{-1}(\boldsymbol{X}^{\dagger}\boldsymbol{Y})$ 逆訊號功率矩陣(inverse signal power matrix): $\boldsymbol{S}=(\boldsymbol{X}^{\dagger}\boldsymbol{X})^{-1}$ P 矩陣: $\boldsymbol{P}=\boldsymbol{X}(\boldsymbol{X}^{\dagger}\boldsymbol{X})^{-1}\boldsymbol{X}^{\dagger}$ 此時，$(\boldsymbol{I}-\boldsymbol{P})$為投影矩陣，重要性質: $(\boldsymbol{I}-\boldsymbol{P})^{\dagger}=(\boldsymbol{I}-\boldsymbol{P})$及$(\boldsymbol{I}-\boldsymbol{P})^{2}=(\boldsymbol{I}-\boldsymbol{P})$ 殘差值的共變異數矩陣(residual covariance matrix): $\boldsymbol{N}=\hat{\sigma}_{v}^{2}\big((\boldsymbol{I}-\boldsymbol{P})\boldsymbol{Y}\big)^{\dagger}\big((\boldsymbol{I}-\boldsymbol{P})\boldsymbol{Y}\big)$ 殘差值的共變異數矩陣(residual covariance matrix)整理: $\boldsymbol{N}=\hat{\sigma}_{v}^{2}\big((\boldsymbol{I}-\boldsymbol{P})\boldsymbol{Y}\big)^{\dagger}\big((\boldsymbol{I}-\boldsymbol{P})\boldsymbol{Y}\big)=\hat{\sigma}_{v}^{2}\boldsymbol{Y}^{\dagger}(\boldsymbol{I}-\boldsymbol{P})^{\dagger}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{Y}$ $=\hat{\sigma}_{v}^{2}\boldsymbol{Y}^{\dagger}(\boldsymbol{I}-\boldsymbol{P})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{Y}=\hat{\sigma}_{v}^{2}\boldsymbol{Y}^{\dagger}(\boldsymbol{I}-\boldsymbol{P})^{2}\boldsymbol{Y}=\hat{\sigma}_{v}^{2}\boldsymbol{Y}^{\dagger}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{Y}$
以 MT 的符號表示下，單站估算法問題: $\boldsymbol{E}=\boldsymbol{H}\boldsymbol{Z}+\boldsymbol{\varepsilon}$ 已知: $\boldsymbol{E}$、$\boldsymbol{H}$ 未知: $\boldsymbol{Z}$ 誤差: $\boldsymbol{\varepsilon}$ 公式解: $\widehat{\boldsymbol{Z}}=(\boldsymbol{H}^{\dagger}\boldsymbol{H})^{-1}(\boldsymbol{H}^{\dagger}\boldsymbol{E})$ 逆訊號功率矩陣(inverse signal power matrix): $\boldsymbol{S}=(\boldsymbol{H}^{\dagger}\boldsymbol{H})^{-1}$ P 矩陣: $\boldsymbol{P}=\boldsymbol{H}(\boldsymbol{H}^{\dagger}\boldsymbol{H})^{-1}\boldsymbol{H}^{\dagger}$ 此時，$(\boldsymbol{I}-\boldsymbol{P})$為投影矩陣，重要性質: $(\boldsymbol{I}-\boldsymbol{P})^{\dagger}=(\boldsymbol{I}-\boldsymbol{P})$及$(\boldsymbol{I}-\boldsymbol{P})^{2}=(\boldsymbol{I}-\boldsymbol{P})$ 殘差值的共變異數矩陣(residual covariance matrix): $\boldsymbol{N}=\hat{\sigma}_{v}^{2}\big((\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}\big)^{\dagger}\big((\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}\big)$ 殘差值的共變異數矩陣(residual covariance matrix)整理: $\boldsymbol{N}=\hat{\sigma}_{v}^{2}\big((\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}\big)^{\dagger}\big((\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}\big)=\hat{\sigma}_{v}^{2}\boldsymbol{E}^{\dagger}(\boldsymbol{I}-\boldsymbol{P})^{\dagger}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}$ 利用$(\boldsymbol{I}-\boldsymbol{P})$為投影矩陣改寫: $\boldsymbol{N}=\hat{\sigma}_{v}^{2}\boldsymbol{E}^{\dagger}(\boldsymbol{I}-\boldsymbol{P})^{\dagger}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}=\hat{\sigma}_{v}^{2}\boldsymbol{E}^{\dagger}(\boldsymbol{I}-\boldsymbol{P})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}=\hat{\sigma}_{v}^{2}\boldsymbol{E}^{\dagger}(\boldsymbol{I}-\boldsymbol{P})^{2}\boldsymbol{E}=\hat{\sigma}_{v}^{2}\boldsymbol{E}^{\dagger}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}$ 把 P 矩陣換掉: $=\hat{\sigma}_{v}^{2}\left(\boldsymbol{E}^{\dagger}(\boldsymbol{I}-\boldsymbol{P})\right)\boldsymbol{E}=\hat{\sigma}_{v}^{2}(\boldsymbol{E}^{\dagger}\boldsymbol{I}-\boldsymbol{E}^{\dagger}\boldsymbol{P})\boldsymbol{E}=\hat{\sigma}_{v}^{2}(\boldsymbol{E}^{\dagger}-\boldsymbol{E}^{\dagger}\boldsymbol{P})\boldsymbol{E}$ $=\hat{\sigma}_{v}^{2}(\boldsymbol{E}^{\dagger}\boldsymbol{E}-\boldsymbol{E}^{\dagger}\boldsymbol{P}\boldsymbol{E})=\hat{\sigma}_{v}^{2}(\boldsymbol{E}^{\dagger}\boldsymbol{E}-\boldsymbol{E}^{\dagger}(\boldsymbol{H}(\boldsymbol{H}^{\dagger}\boldsymbol{H})^{-1}\boldsymbol{H}^{\dagger})\boldsymbol{E})$ $=\hat{\sigma}_{v}^{2}\left((\boldsymbol{E}^{\dagger}\boldsymbol{E})-(\boldsymbol{E}^{\dagger}\boldsymbol{H})(\boldsymbol{H}^{\dagger}\boldsymbol{H})^{-1}(\boldsymbol{H}^{\dagger}\boldsymbol{E})\right)$ 注意: 與文章的公式 28 不同，我覺得文章寫錯。

以 MT 的符號表示下，遠端站參考估算法問題: $\boldsymbol{E} = \boldsymbol{H}\boldsymbol{Z} + \boldsymbol{\varepsilon}$

已知: $\boldsymbol{E}$、$\boldsymbol{H}$、$\boldsymbol{R}$

未知: $\boldsymbol{Z}$

誤差: $\boldsymbol{\varepsilon}$

公式解: $\widehat{\boldsymbol{Z}} = (\boldsymbol{R}^{\dagger}\boldsymbol{H})^{-1}(\boldsymbol{R}^{\dagger}\boldsymbol{E})$

逆訊號功率矩陣(inverse signal power matrix): $\boldsymbol{S} = (\boldsymbol{R}^{\dagger}\boldsymbol{H})^{-1}(\boldsymbol{R}^{\dagger}\boldsymbol{R})(\boldsymbol{H}^{\dagger}\boldsymbol{R})^{-1}$

P 矩陣: $\boldsymbol{P} = \boldsymbol{H}(\boldsymbol{R}^{\dagger}\boldsymbol{H})^{-1}\boldsymbol{R}^{\dagger}$

此時，$(\boldsymbol{I}-\boldsymbol{P})$非投影矩陣，不過$\boldsymbol{H}\widehat{\boldsymbol{Z}} = (\boldsymbol{R}^{\dagger}\boldsymbol{H})^{-1}(\boldsymbol{R}^{\dagger}\boldsymbol{E}) = \boldsymbol{P}\boldsymbol{E}$

殘差值的共變異數矩陣(residual covariance matrix): $\boldsymbol{N} = \hat{\sigma}_v^2\left((\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}\right)^{\dagger}\left((\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}\right)$

殘差值的共變異數矩陣(residual covariance matrix)整理:

$$
\begin{aligned}
\boldsymbol{N} &= \hat{\sigma}_v^2\left((\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}\right)^{\dagger}\left((\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}\right) = \hat{\sigma}_v^2\boldsymbol{E}^{\dagger}(\boldsymbol{I}-\boldsymbol{P})^{\dagger}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}\\
&= \hat{\sigma}_v^2\boldsymbol{E}^{\dagger}(\boldsymbol{I}^{\dagger}-\boldsymbol{P}^{\dagger})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E} = \hat{\sigma}_v^2\boldsymbol{E}^{\dagger}(\boldsymbol{I}-\boldsymbol{P}^{\dagger})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}\\
&= \hat{\sigma}_v^2\left(\boldsymbol{E}^{\dagger}(\boldsymbol{I}-\boldsymbol{P}^{\dagger})\right)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E} = \hat{\sigma}_v^2(\boldsymbol{E}^{\dagger}\boldsymbol{I}-\boldsymbol{E}^{\dagger}\boldsymbol{P}^{\dagger})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}\\
&= \hat{\sigma}_v^2(\boldsymbol{E}^{\dagger}-\boldsymbol{E}^{\dagger}\boldsymbol{P}^{\dagger})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E} = \hat{\sigma}_v^2\left(\boldsymbol{E}^{\dagger}-(\boldsymbol{E}^{\dagger}\boldsymbol{P}^{\dagger})\right)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E}\\
&= \hat{\sigma}_v^2(\boldsymbol{E}^{\dagger}-(\boldsymbol{P}\boldsymbol{E})^{\dagger})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{E} = \hat{\sigma}_v^2\left((\boldsymbol{E}^{\dagger}-(\boldsymbol{P}\boldsymbol{E})^{\dagger})(\boldsymbol{I}-\boldsymbol{P})\right)\boldsymbol{E}\\
&= \hat{\sigma}_v^2(\boldsymbol{E}^{\dagger}\boldsymbol{I}-(\boldsymbol{P}\boldsymbol{E})^{\dagger}\boldsymbol{I}-\boldsymbol{E}^{\dagger}\boldsymbol{P}+(\boldsymbol{P}\boldsymbol{E})^{\dagger}\boldsymbol{P})\boldsymbol{E}\\
&= \hat{\sigma}_v^2(\boldsymbol{E}^{\dagger}-(\boldsymbol{P}\boldsymbol{E})^{\dagger}-\boldsymbol{E}^{\dagger}\boldsymbol{P}+(\boldsymbol{P}\boldsymbol{E})^{\dagger}\boldsymbol{P})\boldsymbol{E} = \hat{\sigma}_v^2\left((\boldsymbol{E}^{\dagger}-(\boldsymbol{P}\boldsymbol{E})^{\dagger}-\boldsymbol{E}^{\dagger}\boldsymbol{P}+(\boldsymbol{P}\boldsymbol{E})^{\dagger}\boldsymbol{P})\boldsymbol{E}\right)\\
&= \hat{\sigma}_v^2(\boldsymbol{E}^{\dagger}\boldsymbol{E}-(\boldsymbol{P}\boldsymbol{E})^{\dagger}\boldsymbol{E}-\boldsymbol{E}^{\dagger}\boldsymbol{P}\boldsymbol{E}+(\boldsymbol{P}\boldsymbol{E})^{\dagger}\boldsymbol{P}\boldsymbol{E})
\end{aligned}
$$

把 P 矩陣換掉:

$$
\begin{aligned}
\boldsymbol{N} &= \hat{\sigma}_v^2(\boldsymbol{E}^{\dagger}\boldsymbol{E}-(\boldsymbol{P}\boldsymbol{E})^{\dagger}\boldsymbol{E}-\boldsymbol{E}^{\dagger}\boldsymbol{P}\boldsymbol{E}+(\boldsymbol{P}\boldsymbol{E})^{\dagger}\boldsymbol{P}\boldsymbol{E})\\
&= \hat{\sigma}_v^2\left(\boldsymbol{E}^{\dagger}\boldsymbol{E}-\left(\boldsymbol{H}\widehat{\boldsymbol{Z}}\right)^{\dagger}\boldsymbol{E}-\boldsymbol{E}^{\dagger}\boldsymbol{H}\widehat{\boldsymbol{Z}}+\left(\boldsymbol{H}\widehat{\boldsymbol{Z}}\right)^{\dagger}\boldsymbol{H}\widehat{\boldsymbol{Z}}\right)\\
&= \hat{\sigma}_v^2\left(\boldsymbol{E}^{\dagger}\boldsymbol{E}-\widehat{\boldsymbol{Z}}^{\dagger}\boldsymbol{H}^{\dagger}\boldsymbol{E}-\boldsymbol{E}^{\dagger}\boldsymbol{H}\widehat{\boldsymbol{Z}}+\widehat{\boldsymbol{Z}}^{\dagger}\boldsymbol{H}^{\dagger}\boldsymbol{H}\widehat{\boldsymbol{Z}}\right)\\
&= \hat{\sigma}_v^2\left((\boldsymbol{E}^{\dagger}\boldsymbol{E})-\left(\widehat{\boldsymbol{Z}}^{\dagger}\right)(\boldsymbol{H}^{\dagger}\boldsymbol{E})-(\boldsymbol{E}^{\dagger}\boldsymbol{H})\left(\widehat{\boldsymbol{Z}}\right)+\left(\widehat{\boldsymbol{Z}}^{\dagger}\right)(\boldsymbol{H}^{\dagger}\boldsymbol{H})\left(\widehat{\boldsymbol{Z}}\right)\right)
\end{aligned}
$$

注意: 與文章的公式 31 相同。故意括號起來是因為這些內容可以從 EDI 檔案中取得。

注意: 文章建議遠端參考估算法只要把$\boldsymbol{R}$用$\boldsymbol{H}$取代，也等同是處理單站估算法。為了能同時處理兩個方法，我們可以只將遠端參考站的公式轉換為程式碼，缺點是將會浪費些許對現代個人電腦來說微不足道的計算資源。

實際搭配 SpectraEdi 運算

大地電磁法主要的三個關係式:

$E_x = Z_{xx}H_x + Z_{xy}H_y$ ...(1)

$E_y = Z_{yx}H_x + Z_{yy}H_y$ ...(2)

$H_z = T_xH_x + T_yH_y$ ...(3)

一次只看一個關係式，這裡取用關係式:

$E_x = Z_{xx}H_x + Z_{xy}H_y$

n 次獨立觀測可產生下列關係式:

$E_{x,\#1} = Z_{xx}H_{x,\#1} + Z_{xy}H_{y,\#1} + \varepsilon_{x,\#1}$ ... (1)

$E_{x,\#2} = Z_{xx}H_{x,\#2} + Z_{xy}H_{y,\#2} + \varepsilon_{x,\#2}$ ... (2)

$E_{x,\#3} = Z_{xx}H_{x,\#3} + Z_{xy}H_{y,\#3} + \varepsilon_{x,\#3}$ ... (3)

...

$E_{x,\#n} = Z_{xx}H_{x,\#n} + Z_{xy}H_{y,\#n} + \varepsilon_{x,\#n}$ ... (n)

整理成矩陣形式<nx1>=<nx2><2x1>+<nx1>:

$$\begin{bmatrix} E_{x,\#1} \\ E_{x,\#2} \\ E_{x,\#3} \\ \vdots \\ E_{x,\#n} \end{bmatrix}_{\langle \mathrm{nx1} \rangle} = \begin{bmatrix} H_{x,\#1} & H_{y,\#1} \\ H_{x,\#2} & H_{y,\#2} \\ H_{x,\#3} & H_{y,\#3} \\ \vdots & \vdots \\ H_{x,\#n} & H_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle} \begin{bmatrix} Z_{xx} \\ Z_{xy} \end{bmatrix}_{\langle \mathrm{2x1} \rangle} + \begin{bmatrix} \varepsilon_{x,\#1} \\ \varepsilon_{x,\#2} \\ \varepsilon_{x,\#3} \\ \vdots \\ \varepsilon_{x,\#n} \end{bmatrix}_{\langle \mathrm{nx1} \rangle}$$

套用到遠端站估算法問題:

線性回歸模型: $\boldsymbol{E} = \boldsymbol{HZ} + \boldsymbol{\varepsilon}$

已知: $\boldsymbol{E} = \begin{bmatrix} E_{x,\#1} \\ E_{x,\#2} \\ E_{x,\#3} \\ \vdots \\ E_{x,\#n} \end{bmatrix}_{\langle \mathrm{nx1} \rangle}$、$\boldsymbol{H} = \begin{bmatrix} H_{x,\#1} & H_{y,\#1} \\ H_{x,\#2} & H_{y,\#2} \\ H_{x,\#3} & H_{y,\#3} \\ \vdots & \vdots \\ H_{x,\#n} & H_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle}$、$\boldsymbol{R} = \begin{bmatrix} R_{x,\#1} & R_{y,\#1} \\ R_{x,\#2} & R_{y,\#2} \\ R_{x,\#3} & R_{y,\#3} \\ \vdots & \vdots \\ R_{x,\#n} & R_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle}$

未知: $\boldsymbol{Z} = \begin{bmatrix} Z_{xx} \\ Z_{xy} \end{bmatrix}_{\langle \mathrm{2x1} \rangle}$

誤差: $\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{x,\#1} \\ \varepsilon_{x,\#2} \\ \varepsilon_{x,\#3} \\ \vdots \\ \varepsilon_{x,\#n} \end{bmatrix}_{\langle \mathrm{nx1} \rangle}$

求未知: $\hat{\boldsymbol{Z}} = \begin{bmatrix} \hat{Z}_{xx} \\ \hat{Z}_{xy} \end{bmatrix}_{\langle \mathrm{2x1} \rangle}$

公式解: $\hat{\boldsymbol{Z}} = (\boldsymbol{R}^{\dagger}\boldsymbol{H})^{-1}(\boldsymbol{R}^{\dagger}\boldsymbol{E})$

逆訊號功率矩陣(inverse signal power matrix): $\boldsymbol{S}=(\boldsymbol{R}^{\dagger}\boldsymbol{H})^{-1}(\boldsymbol{R}^{\dagger}\boldsymbol{R})(\boldsymbol{H}^{\dagger}\boldsymbol{R})^{-1}$

殘差值的共變異數矩陣(residual covariance matrix):

$$\boldsymbol{N}=\hat{\sigma}_v^2\left((\boldsymbol{E}^{\dagger}\boldsymbol{E})-(\widehat{\boldsymbol{Z}}^{\dagger})(\boldsymbol{H}^{\dagger}\boldsymbol{E})-(\boldsymbol{E}^{\dagger}\boldsymbol{H})(\widehat{\boldsymbol{Z}})+(\widehat{\boldsymbol{Z}}^{\dagger})(\boldsymbol{H}^{\dagger}\boldsymbol{H})(\widehat{\boldsymbol{Z}})\right)$$

探討自功率譜矩陣資料與互功率譜矩陣資料:

計算要用到的 8 個矩陣: $\boldsymbol{R}^{\dagger}\boldsymbol{H}$、$\boldsymbol{R}^{\dagger}\boldsymbol{R}$、$\boldsymbol{H}^{\dagger}\boldsymbol{R}$、$\boldsymbol{R}^{\dagger}\boldsymbol{E}$、$\boldsymbol{E}^{\dagger}\boldsymbol{E}$、$\boldsymbol{H}^{\dagger}\boldsymbol{E}$、$\boldsymbol{E}^{\dagger}\boldsymbol{H}$、$\boldsymbol{H}^{\dagger}\boldsymbol{H}$，數量不算太多，逐個探討。

第 1 個矩陣:

$$\boldsymbol{R}^{\dagger}\boldsymbol{H}=\left(\begin{bmatrix}R_{x,\#1} & R_{y,\#1}\\ R_{x,\#2} & R_{y,\#2}\\ R_{x,\#3} & R_{y,\#3}\\ \vdots & \vdots\\ R_{x,\#n} & R_{y,\#n}\end{bmatrix}_{\langle\mathrm{nx2}\rangle}\right)^{\dagger}\begin{bmatrix}H_{x,\#1} & H_{y,\#1}\\ H_{x,\#2} & H_{y,\#2}\\ H_{x,\#3} & H_{y,\#3}\\ \vdots & \vdots\\ H_{x,\#n} & H_{y,\#n}\end{bmatrix}_{\langle\mathrm{nx2}\rangle}$$

$$=\begin{bmatrix}R^*_{x,\#1} & R^*_{x,\#2} & R^*_{x,\#3} & \cdots & R^*_{x,\#n}\\ R^*_{y,\#1} & R^*_{y,\#2} & R^*_{y,\#3} & \cdots & R^*_{y,\#n}\end{bmatrix}_{\langle\mathrm{2xn}\rangle}\begin{bmatrix}H_{x,\#1} & H_{y,\#1}\\ H_{x,\#2} & H_{y,\#2}\\ H_{x,\#3} & H_{y,\#3}\\ \vdots & \vdots\\ H_{x,\#n} & H_{y,\#n}\end{bmatrix}_{\langle\mathrm{nx2}\rangle}$$

$$=\begin{bmatrix}\sum_{k=1}^{n}R^*_{x,\#k}H_{x,\#k} & \sum_{k=1}^{n}R^*_{x,\#k}H_{y,\#k}\\ \sum_{k=1}^{n}R^*_{y,\#k}H_{x,\#k} & \sum_{k=1}^{n}R^*_{y,\#k}H_{y,\#k}\end{bmatrix}_{\langle\mathrm{2x2}\rangle}$$

定義比較好讀的中括號符號:

$\langle R_x^*H_x\rangle=\sum_{k=1}^{n}R^*_{x,\#k}H_{x,\#k}$

$\langle R_x^*H_y\rangle=\sum_{k=1}^{n}R^*_{x,\#k}H_{y,\#k}$

$\langle R_y^*H_x\rangle=\sum_{k=1}^{n}R^*_{y,\#k}H_{x,\#k}$

$\langle R_y^*H_y\rangle=\sum_{k=1}^{n}R^*_{y,\#k}H_{y,\#k}$

重新整理第 1 個矩陣:

$$\boldsymbol{R}^{\dagger}\boldsymbol{H}=\begin{bmatrix}\langle R_x^*H_x\rangle & \langle R_x^*H_y\rangle\\ \langle R_y^*H_x\rangle & \langle R_y^*H_y\rangle\end{bmatrix}_{\langle\mathrm{2x2}\rangle}$$

第 2 個矩陣:

$$\boldsymbol{R}^{\dagger}\boldsymbol{R}=\left(\begin{bmatrix}R_{x,\#1} & R_{y,\#1}\\ R_{x,\#2} & R_{y,\#2}\\ R_{x,\#3} & R_{y,\#3}\\ \vdots & \vdots\\ R_{x,\#n} & R_{y,\#n}\end{bmatrix}_{\langle\mathrm{nx2}\rangle}\right)^{\dagger}\begin{bmatrix}R_{x,\#1} & R_{y,\#1}\\ R_{x,\#2} & R_{y,\#2}\\ R_{x,\#3} & R_{y,\#3}\\ \vdots & \vdots\\ R_{x,\#n} & R_{y,\#n}\end{bmatrix}_{\langle\mathrm{nx2}\rangle}$$

$$= \begin{bmatrix} R_{x,\#1}^* & R_{x,\#2}^* & R_{x,\#3}^* & \cdots & R_{x,\#n}^* \\ R_{y,\#1}^* & R_{y,\#2}^* & R_{y,\#3}^* & \cdots & R_{y,\#n}^* \end{bmatrix}_{\langle 2\mathrm{xn} \rangle} \begin{bmatrix} R_{x,\#1} & R_{y,\#1} \\ R_{x,\#2} & R_{y,\#2} \\ R_{x,\#3} & R_{y,\#3} \\ \vdots & \vdots \\ R_{x,\#n} & R_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx}2 \rangle}$$

$$= \begin{bmatrix} \sum_{k=1}^{n} R_{x,\#k}^* R_{x,\#k} & \sum_{k=1}^{n} R_{x,\#k}^* R_{y,\#k} \\ \sum_{k=1}^{n} R_{y,\#k}^* R_{x,\#k} & \sum_{k=1}^{n} R_{y,\#k}^* R_{y,\#k} \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle}$$

定義比較好讀的中括號符號:

$\langle R_x^* R_x \rangle = \sum_{k=1}^{n} R_{x,\#k}^* R_{x,\#k}$

$\langle R_x^* R_y \rangle = \sum_{k=1}^{n} R_{x,\#k}^* R_{y,\#k}$

$\langle R_y^* R_x \rangle = \sum_{k=1}^{n} R_{y,\#k}^* R_{x,\#k}$

$\langle R_y^* R_y \rangle = \sum_{k=1}^{n} R_{y,\#k}^* R_{y,\#k}$

重新整理第 2 個矩陣:

$$\boldsymbol{R}^\dagger \boldsymbol{R} = \begin{bmatrix} \langle R_x^* R_x \rangle & \langle R_x^* R_y \rangle \\ \langle R_y^* R_x \rangle & \langle R_y^* R_y \rangle \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle}$$

第 3 個矩陣:

$$\boldsymbol{H}^\dagger \boldsymbol{R} = \left(\begin{bmatrix} H_{x,\#1} & H_{y,\#1} \\ H_{x,\#2} & H_{y,\#2} \\ H_{x,\#3} & H_{y,\#3} \\ \vdots & \vdots \\ H_{x,\#n} & H_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx}2 \rangle} \right)^\dagger \begin{bmatrix} R_{x,\#1} & R_{y,\#1} \\ R_{x,\#2} & R_{y,\#2} \\ R_{x,\#3} & R_{y,\#3} \\ \vdots & \vdots \\ R_{x,\#n} & R_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx}2 \rangle}$$

$$= \begin{bmatrix} H_{x,\#1}^* & H_{x,\#2}^* & H_{x,\#3}^* & \cdots & H_{x,\#n}^* \\ H_{y,\#1}^* & H_{y,\#2}^* & H_{y,\#3}^* & \cdots & H_{y,\#n}^* \end{bmatrix}_{\langle 2\mathrm{xn} \rangle} \begin{bmatrix} R_{x,\#1} & R_{y,\#1} \\ R_{x,\#2} & R_{y,\#2} \\ R_{x,\#3} & R_{y,\#3} \\ \vdots & \vdots \\ R_{x,\#n} & R_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx}2 \rangle}$$

$$= \begin{bmatrix} \sum_{k=1}^{n} H_{x,\#k}^* R_{x,\#k} & \sum_{k=1}^{n} H_{x,\#k}^* R_{y,\#k} \\ \sum_{k=1}^{n} H_{y,\#k}^* R_{x,\#k} & \sum_{k=1}^{n} H_{y,\#k}^* R_{y,\#k} \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle}$$

定義比較好讀的中括號符號:

$\langle H_x^* R_x \rangle = \sum_{k=1}^{n} H_{x,\#k}^* R_{x,\#k}$

$\langle H_x^* R_y \rangle = \sum_{k=1}^{n} H_{x,\#k}^* R_{y,\#k}$

$\langle H_y^* R_x \rangle = \sum_{k=1}^{n} H_{y,\#k}^* R_{x,\#k}$

$\langle H_y^* R_y \rangle = \sum_{k=1}^{n} H_{y,\#k}^* R_{y,\#k}$

重新整理第 3 個矩陣:

$$\boldsymbol{H}^\dagger \boldsymbol{R} = \begin{bmatrix} \langle H_x^* R_x \rangle & \langle H_x^* R_y \rangle \\ \langle H_y^* R_x \rangle & \langle H_y^* R_y \rangle \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle}$$

第 4 個矩陣:

$$\boldsymbol{R}^{\dagger}\boldsymbol{E}=\left(\begin{bmatrix}R_{x,\#1} & R_{y,\#1}\\ R_{x,\#2} & R_{y,\#2}\\ R_{x,\#3} & R_{y,\#3}\\ \vdots & \vdots\\ R_{x,\#n} & R_{y,\#n}\end{bmatrix}_{\langle \mathrm{nx2}\rangle}\right)^{\dagger}\begin{bmatrix}E_{x,\#1}\\ E_{x,\#2}\\ E_{x,\#3}\\ \vdots\\ E_{x,\#n}\end{bmatrix}_{\langle \mathrm{nx1}\rangle}$$

$$=\begin{bmatrix}R_{x,\#1}^{*} & R_{x,\#2}^{*} & R_{x,\#3}^{*} & \cdots & R_{x,\#n}^{*}\\ R_{y,\#1}^{*} & R_{y,\#2}^{*} & R_{y,\#3}^{*} & \cdots & R_{y,\#n}^{*}\end{bmatrix}_{\langle \mathrm{2xn}\rangle}\begin{bmatrix}E_{x,\#1}\\ E_{x,\#2}\\ E_{x,\#3}\\ \vdots\\ E_{x,\#n}\end{bmatrix}_{\langle \mathrm{nx1}\rangle}$$

$$=\begin{bmatrix}\sum_{k=1}^{n}R_{x,\#k}^{*}E_{x,\#k}\\ \sum_{k=1}^{n}R_{y,\#k}^{*}E_{x,\#k}\end{bmatrix}_{\langle \mathrm{2x1}\rangle}$$

定義比較好讀的中括號符號:

$$\langle R_x^{*}E_x\rangle=\sum_{k=1}^{n}R_{x,\#k}^{*}E_{x,\#k}$$

$$\langle R_y^{*}E_x\rangle=\sum_{k=1}^{n}R_{y,\#k}^{*}E_{x,\#k}$$

重新整理第 4 個矩陣:

$$\boldsymbol{R}^{\dagger}\boldsymbol{E}=\begin{bmatrix}\langle R_x^{*}E_x\rangle\\ \langle R_y^{*}E_x\rangle\end{bmatrix}_{\langle \mathrm{2x1}\rangle}$$

第 5 個矩陣:

$$\boldsymbol{E}^{\dagger}\boldsymbol{E}=\left(\begin{bmatrix}E_{x,\#1}\\ E_{x,\#2}\\ E_{x,\#3}\\ \vdots\\ E_{x,\#n}\end{bmatrix}_{\langle \mathrm{nx1}\rangle}\right)^{\dagger}\begin{bmatrix}E_{x,\#1}\\ E_{x,\#2}\\ E_{x,\#3}\\ \vdots\\ E_{x,\#n}\end{bmatrix}_{\langle \mathrm{nx1}\rangle}$$

$$=\begin{bmatrix}E_{x,\#1}^{*} & E_{x,\#2}^{*} & E_{x,\#3}^{*} & \cdots & E_{x,\#n}^{*}\end{bmatrix}_{\langle \mathrm{1xn}\rangle}\begin{bmatrix}E_{x,\#1}\\ E_{x,\#2}\\ E_{x,\#3}\\ \vdots\\ E_{x,\#n}\end{bmatrix}_{\langle \mathrm{nx1}\rangle}$$

$$=\left[\sum_{k=1}^{n}E_{x,\#k}^{*}E_{x,\#k}\right]_{\langle \mathrm{1x1}\rangle}$$

定義比較好讀的中括號符號:

$$\langle E_x^{*}E_x\rangle=\sum_{k=1}^{n}E_{x,\#k}^{*}E_{x,\#k}$$

重新整理第 5 個矩陣:

$$\boldsymbol{E}^{\dagger}\boldsymbol{E}=[\langle E_x^{*}E_x\rangle]_{\langle \mathrm{1x1}\rangle}$$

第 6 個矩陣:

$$\boldsymbol{H}^{\dagger}\boldsymbol{E}=\left(\begin{bmatrix} H_{x,\#1} & H_{y,\#1} \\ H_{x,\#2} & H_{y,\#2} \\ H_{x,\#3} & H_{y,\#3} \\ \vdots & \vdots \\ H_{x,\#n} & H_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle}\right)^{\dagger}\begin{bmatrix} E_{x,\#1} \\ E_{x,\#2} \\ E_{x,\#3} \\ \vdots \\ E_{x,\#n} \end{bmatrix}_{\langle \mathrm{nx1} \rangle}$$

$$=\begin{bmatrix} H_{x,\#1}^{*} & H_{x,\#2}^{*} & H_{x,\#3}^{*} & \cdots & H_{x,\#n}^{*} \\ H_{y,\#1}^{*} & H_{y,\#2}^{*} & H_{y,\#3}^{*} & \cdots & H_{y,\#n}^{*} \end{bmatrix}_{\langle \mathrm{2xn} \rangle}\begin{bmatrix} E_{x,\#1} \\ E_{x,\#2} \\ E_{x,\#3} \\ \vdots \\ E_{x,\#n} \end{bmatrix}_{\langle \mathrm{nx1} \rangle}$$

$$=\begin{bmatrix} \sum_{k=1}^{n} H_{x,\#k}^{*} E_{x,\#k} \\ \sum_{k=1}^{n} H_{y,\#k}^{*} E_{x,\#k} \end{bmatrix}_{\langle \mathrm{2x1} \rangle}$$

定義比較好讀的中括號符號:

$$\langle H_x^{*} E_x \rangle = \sum_{k=1}^{n} H_{x,\#k}^{*} E_{x,\#k}$$

$$\langle H_y^{*} E_x \rangle = \sum_{k=1}^{n} H_{y,\#k}^{*} E_{x,\#k}$$

重新整理第 6 個矩陣:

$$\boldsymbol{H}^{\dagger}\boldsymbol{E}=\begin{bmatrix} \langle H_x^{*} E_x \rangle \\ \langle H_y^{*} E_x \rangle \end{bmatrix}_{\langle \mathrm{2x1} \rangle}$$

第 7 個矩陣:

$$\boldsymbol{E}^{\dagger}\boldsymbol{H}=\left(\begin{bmatrix} E_{x,\#1} \\ E_{x,\#2} \\ E_{x,\#3} \\ \vdots \\ E_{x,\#n} \end{bmatrix}_{\langle \mathrm{nx1} \rangle}\right)^{\dagger}\begin{bmatrix} H_{x,\#1} & H_{y,\#1} \\ H_{x,\#2} & H_{y,\#2} \\ H_{x,\#3} & H_{y,\#3} \\ \vdots & \vdots \\ H_{x,\#n} & H_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle}$$

$$=\begin{bmatrix} E_{x,\#1}^{*} & E_{x,\#2}^{*} & E_{x,\#3}^{*} & \cdots & E_{x,\#n}^{*} \end{bmatrix}_{\langle \mathrm{1xn} \rangle}\begin{bmatrix} H_{x,\#1} & H_{y,\#1} \\ H_{x,\#2} & H_{y,\#2} \\ H_{x,\#3} & H_{y,\#3} \\ \vdots & \vdots \\ H_{x,\#n} & H_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle}$$

$$=\begin{bmatrix} \sum_{k=1}^{n} E_{x,\#k}^{*} H_{x,\#k} & \sum_{k=1}^{n} E_{x,\#k}^{*} H_{y,\#k} \end{bmatrix}_{\langle \mathrm{1x2} \rangle}$$

定義比較好讀的中括號符號:

$$\langle E_x^{*} H_x \rangle = \sum_{k=1}^{n} E_{x,\#k}^{*} H_{x,\#k}$$

$$\langle E_x^{*} H_y \rangle = \sum_{k=1}^{n} E_{x,\#k}^{*} H_{y,\#k}$$

重新整理第 7 個矩陣:

$$\boldsymbol{E}^{\dagger}\boldsymbol{H}=\begin{bmatrix} \langle E_x^{*} H_x \rangle & \langle E_x^{*} H_y \rangle \end{bmatrix}_{\langle \mathrm{1x2} \rangle}$$

第 8 個矩陣:

$$\boldsymbol{H}^\dagger\boldsymbol{H} = \left(\begin{bmatrix} H_{x,\#1} & H_{y,\#1} \\ H_{x,\#2} & H_{y,\#2} \\ H_{x,\#3} & H_{y,\#3} \\ \vdots & \vdots \\ H_{x,\#n} & H_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle}\right)^\dagger \begin{bmatrix} H_{x,\#1} & H_{y,\#1} \\ H_{x,\#2} & H_{y,\#2} \\ H_{x,\#3} & H_{y,\#3} \\ \vdots & \vdots \\ H_{x,\#n} & H_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle}$$

$$= \begin{bmatrix} H^*_{x,\#1} & H^*_{x,\#2} & H^*_{x,\#3} & \cdots & H^*_{x,\#n} \\ H^*_{y,\#1} & H^*_{y,\#2} & H^*_{y,\#3} & \cdots & H^*_{y,\#n} \end{bmatrix}_{\langle \mathrm{2xn} \rangle} \begin{bmatrix} H_{x,\#1} & H_{y,\#1} \\ H_{x,\#2} & H_{y,\#2} \\ H_{x,\#3} & H_{y,\#3} \\ \vdots & \vdots \\ H_{x,\#n} & H_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle}$$

$$= \begin{bmatrix} \sum_{k=1}^n H^*_{x,\#k} H_{x,\#k} & \sum_{k=1}^n H^*_{x,\#k} H_{y,\#k} \\ \sum_{k=1}^n H^*_{y,\#k} H_{x,\#k} & \sum_{k=1}^n H^*_{y,\#k} H_{y,\#k} \end{bmatrix}_{\langle \mathrm{2x2} \rangle}$$

定義比較好讀的中括號符號:

$$\langle H_x^* H_x \rangle = \sum_{k=1}^n H^*_{x,\#k} H_{x,\#k}$$

$$\langle H_x^* H_y \rangle = \sum_{k=1}^n H^*_{x,\#k} H_{y,\#k}$$

$$\langle H_y^* H_x \rangle = \sum_{k=1}^n H^*_{y,\#k} H_{x,\#k}$$

$$\langle H_y^* H_y \rangle = \sum_{k=1}^n H^*_{y,\#k} H_{y,\#k}$$

重新整理第 8 個矩陣:

$$\boldsymbol{H}^\dagger\boldsymbol{H} = \begin{bmatrix} \langle H_x^* H_x \rangle & \langle H_x^* H_y \rangle \\ \langle H_y^* H_x \rangle & \langle H_y^* H_y \rangle \end{bmatrix}_{\langle \mathrm{2x2} \rangle}$$

將 8 個矩陣都以中括號整理:

#1: $\boldsymbol{R}^\dagger\boldsymbol{H} = \begin{bmatrix} \langle R_x^* H_x \rangle & \langle R_x^* H_y \rangle \\ \langle R_y^* H_x \rangle & \langle R_y^* H_y \rangle \end{bmatrix}_{\langle \mathrm{2x2} \rangle}$

#2: $\boldsymbol{R}^\dagger\boldsymbol{R} = \begin{bmatrix} \langle R_x^* R_x \rangle & \langle R_x^* R_y \rangle \\ \langle R_y^* R_x \rangle & \langle R_y^* R_y \rangle \end{bmatrix}_{\langle \mathrm{2x2} \rangle}$

#3: $\boldsymbol{H}^\dagger\boldsymbol{R} = \begin{bmatrix} \langle H_x^* R_x \rangle & \langle H_x^* R_y \rangle \\ \langle H_y^* R_x \rangle & \langle H_y^* R_y \rangle \end{bmatrix}_{\langle \mathrm{2x2} \rangle}$

#4: $\boldsymbol{R}^\dagger\boldsymbol{E} = \begin{bmatrix} \langle R_x^* E_x \rangle \\ \langle R_y^* E_x \rangle \end{bmatrix}_{\langle \mathrm{2x1} \rangle}$

#5: $\boldsymbol{E}^\dagger\boldsymbol{E} = [\langle E_x^* E_x \rangle]_{\langle \mathrm{1x1} \rangle}$

#6: $\boldsymbol{H}^\dagger\boldsymbol{E} = \begin{bmatrix} \langle H_x^* E_x \rangle \\ \langle H_y^* E_x \rangle \end{bmatrix}_{\langle \mathrm{2x1} \rangle}$

#7: $\boldsymbol{E}^\dagger\boldsymbol{H} = [\langle E_x^* H_x \rangle \quad \langle E_x^* H_y \rangle]_{\langle \mathrm{1x2} \rangle}$

#8: $\boldsymbol{H}^\dagger\boldsymbol{H} = \begin{bmatrix} \langle H_x^* H_x \rangle & \langle H_x^* H_y \rangle \\ \langle H_y^* H_x \rangle & \langle H_y^* H_y \rangle \end{bmatrix}_{\langle \mathrm{2x2} \rangle}$

用 8 個矩陣資料進行計算:

求未知: $\widehat{\boldsymbol{Z}} = \begin{bmatrix} \hat{Z}_{xx} \\ \hat{Z}_{xy} \end{bmatrix}_{\langle 2\text{x}1 \rangle}$

$$\widehat{\boldsymbol{Z}} = (\boldsymbol{R}^{\dagger}\boldsymbol{H})^{-1}(\boldsymbol{R}^{\dagger}\boldsymbol{E})$$

$$= \left(\begin{bmatrix} \langle R_x^* H_x \rangle & \langle R_x^* H_y \rangle \\ \langle R_y^* H_x \rangle & \langle R_y^* H_y \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle} \right)^{-1} \left(\begin{bmatrix} \langle R_x^* E_x \rangle \\ \langle R_y^* E_x \rangle \end{bmatrix}_{\langle 2\text{x}1 \rangle} \right)$$

$$= \frac{1}{\langle R_x^* H_x \rangle \langle R_y^* H_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* H_x \rangle} \begin{bmatrix} \langle R_y^* H_y \rangle & -\langle R_x^* H_y \rangle \\ -\langle R_y^* H_x \rangle & \langle R_x^* H_x \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle} \begin{bmatrix} \langle R_x^* E_x \rangle \\ \langle R_y^* E_x \rangle \end{bmatrix}_{\langle 2\text{x}1 \rangle}$$

$$= \frac{\begin{bmatrix} \langle R_y^* H_y \rangle & -\langle R_x^* H_y \rangle \\ -\langle R_y^* H_x \rangle & \langle R_x^* H_x \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle} \begin{bmatrix} \langle R_x^* E_x \rangle \\ \langle R_y^* E_x \rangle \end{bmatrix}_{\langle 2\text{x}1 \rangle}}{\langle R_x^* H_x \rangle \langle R_y^* H_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* H_x \rangle} = \frac{\begin{bmatrix} \langle R_y^* H_y \rangle \langle R_x^* E_x \rangle - \langle R_x^* H_y \rangle \langle R_y^* E_x \rangle \\ -\langle R_y^* H_x \rangle \langle R_x^* E_x \rangle + \langle R_x^* H_x \rangle \langle R_y^* E_x \rangle \end{bmatrix}_{\langle 2\text{x}1 \rangle}}{\langle R_x^* H_x \rangle \langle R_y^* H_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* H_x \rangle}$$

$$= \begin{bmatrix} \dfrac{\langle R_y^* H_y \rangle \langle R_x^* E_x \rangle - \langle R_x^* H_y \rangle \langle R_y^* E_x \rangle}{\langle R_x^* H_x \rangle \langle R_y^* H_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* H_x \rangle} \\ \dfrac{-\langle R_y^* H_x \rangle \langle R_x^* E_x \rangle + \langle R_x^* H_x \rangle \langle R_y^* E_x \rangle}{\langle R_x^* H_x \rangle \langle R_y^* H_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* H_x \rangle} \end{bmatrix}_{\langle 2\text{x}1 \rangle}$$

最終元素計算:

$$\widehat{\boldsymbol{Z}} = \begin{bmatrix} \hat{Z}_{xx} \\ \hat{Z}_{xy} \end{bmatrix}_{\langle 2\text{x}1 \rangle} = \begin{bmatrix} \dfrac{\langle R_y^* H_y \rangle \langle R_x^* E_x \rangle - \langle R_x^* H_y \rangle \langle R_y^* E_x \rangle}{\langle R_x^* H_x \rangle \langle R_y^* H_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* H_x \rangle} \\ \dfrac{-\langle R_y^* H_x \rangle \langle R_x^* E_x \rangle + \langle R_x^* H_x \rangle \langle R_y^* E_x \rangle}{\langle R_x^* H_x \rangle \langle R_y^* H_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* H_x \rangle} \end{bmatrix}_{\langle 2\text{x}1 \rangle}$$

順便算一下另一個後面需要的矩陣:

$$\widehat{\boldsymbol{Z}}^{\dagger} = \left(\begin{bmatrix} \hat{Z}_{xx} \\ \hat{Z}_{xy} \end{bmatrix}_{\langle 2\text{x}1 \rangle} \right)^{\dagger} = \begin{bmatrix} \hat{Z}_{xx}^* & \hat{Z}_{xy}^* \end{bmatrix}_{\langle 1\text{x}2 \rangle}$$

逆訊號功率矩陣(inverse signal power matrix):

$$\boldsymbol{S} = (\boldsymbol{R}^{\dagger}\boldsymbol{H})^{-1}(\boldsymbol{R}^{\dagger}\boldsymbol{R})(\boldsymbol{H}^{\dagger}\boldsymbol{R})^{-1}$$

$$= \left(\begin{bmatrix} \langle R_x^* H_x \rangle & \langle R_x^* H_y \rangle \\ \langle R_y^* H_x \rangle & \langle R_y^* H_y \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle} \right)^{-1} \left(\begin{bmatrix} \langle R_x^* R_x \rangle & \langle R_x^* R_y \rangle \\ \langle R_y^* R_x \rangle & \langle R_y^* R_y \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle} \right) \left(\begin{bmatrix} \langle H_x^* R_x \rangle & \langle H_x^* R_y \rangle \\ \langle H_y^* R_x \rangle & \langle H_y^* R_y \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle} \right)^{-1}$$

$$= \frac{\begin{bmatrix} \langle R_y^* H_y \rangle & -\langle R_x^* H_y \rangle \\ -\langle R_y^* H_x \rangle & \langle R_x^* H_x \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle}}{\langle R_x^* H_x \rangle \langle R_y^* H_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* H_x \rangle} \left(\begin{bmatrix} \langle R_x^* R_x \rangle & \langle R_x^* R_y \rangle \\ \langle R_y^* R_x \rangle & \langle R_y^* R_y \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle} \right) \frac{\begin{bmatrix} \langle H_y^* R_y \rangle & -\langle H_x^* R_y \rangle \\ -\langle H_y^* R_x \rangle & \langle H_x^* R_x \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle}}{\langle H_x^* R_x \rangle \langle H_y^* R_y \rangle - \langle H_x^* R_y \rangle \langle H_y^* R_x \rangle}$$

$$= \left(\frac{\begin{bmatrix} \langle R_y^* H_y \rangle & -\langle R_x^* H_y \rangle \\ -\langle R_y^* H_x \rangle & \langle R_x^* H_x \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle} \begin{bmatrix} \langle R_x^* R_x \rangle & \langle R_x^* R_y \rangle \\ \langle R_y^* R_x \rangle & \langle R_y^* R_y \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle}}{\langle R_x^* H_x \rangle \langle R_y^* H_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* H_x \rangle} \right) \frac{\begin{bmatrix} \langle H_y^* R_y \rangle & -\langle H_x^* R_y \rangle \\ -\langle H_y^* R_x \rangle & \langle H_x^* R_x \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle}}{\langle H_x^* R_x \rangle \langle H_y^* R_y \rangle - \langle H_x^* R_y \rangle \langle H_y^* R_x \rangle}$$

$$= \left(\frac{\begin{bmatrix} \langle R_y^* H_y \rangle \langle R_x^* R_x \rangle - \langle R_x^* H_y \rangle \langle R_y^* R_x \rangle & \langle R_y^* H_y \rangle \langle R_x^* R_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* R_y \rangle \\ -\langle R_y^* H_x \rangle \langle R_x^* R_x \rangle + \langle R_x^* H_x \rangle \langle R_y^* R_x \rangle & -\langle R_y^* H_x \rangle \langle R_x^* R_y \rangle + \langle R_x^* H_x \rangle \langle R_y^* R_y \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle}}{\langle R_x^* H_x \rangle \langle R_y^* H_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* H_x \rangle} \right) \frac{\begin{bmatrix} \langle H_y^* R_y \rangle & -\langle H_x^* R_y \rangle \\ -\langle H_y^* R_x \rangle & \langle H_x^* R_x \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle}}{\langle H_x^* R_x \rangle \langle H_y^* R_y \rangle - \langle H_x^* R_y \rangle \langle H_y^* R_x \rangle}$$

$$= \frac{\begin{bmatrix} \langle R_y^* H_y \rangle \langle R_x^* R_x \rangle - \langle R_x^* H_y \rangle \langle R_y^* R_x \rangle & \langle R_y^* H_y \rangle \langle R_x^* R_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* R_y \rangle \\ -\langle R_y^* H_x \rangle \langle R_x^* R_x \rangle + \langle R_x^* H_x \rangle \langle R_y^* R_x \rangle & -\langle R_y^* H_x \rangle \langle R_x^* R_y \rangle + \langle R_x^* H_x \rangle \langle R_y^* R_y \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle} \begin{bmatrix} \langle H_y^* R_y \rangle & -\langle H_x^* R_y \rangle \\ -\langle H_y^* R_x \rangle & \langle H_x^* R_x \rangle \end{bmatrix}_{\langle 2\text{x}2 \rangle}}{\left(\langle R_x^* H_x \rangle \langle R_y^* H_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* H_x \rangle \right) \left(\langle H_x^* R_x \rangle \langle H_y^* R_y \rangle - \langle H_x^* R_y \rangle \langle H_y^* R_x \rangle \right)}$$

$$= \frac{\begin{bmatrix} \left(\langle R_y^* H_y \rangle \langle R_x^* R_x \rangle - \langle R_x^* H_y \rangle \langle R_y^* R_x \rangle \right) \left(\langle H_y^* R_y \rangle \right) + \left(\langle R_y^* H_y \rangle \langle R_x^* R_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* R_y \rangle \right) \left(-\langle H_y^* R_x \rangle \right) & \left(\langle R_y^* H_y \rangle \langle R_x^* R_x \rangle - \langle R_x^* H_y \rangle \langle R_y^* R_x \rangle \right) \left(-\langle H_x^* R_y \rangle \right) + \left(\langle R_y^* H_y \rangle \langle R_x^* R_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* R_y \rangle \right) \left(\langle H_x^* R_x \rangle \right) \\ \left(-\langle R_y^* H_x \rangle \langle R_x^* R_x \rangle + \langle R_x^* H_x \rangle \langle R_y^* R_x \rangle \right) \left(\langle H_y^* R_y \rangle \right) + \left(-\langle R_y^* H_x \rangle \langle R_x^* R_y \rangle + \langle R_x^* H_x \rangle \langle R_y^* R_y \rangle \right) \left(-\langle H_y^* R_x \rangle \right) & \left(-\langle R_y^* H_x \rangle \langle R_x^* R_x \rangle + \langle R_x^* H_x \rangle \langle R_y^* R_x \rangle \right) \left(-\langle H_x^* R_y \rangle \right) + \left(-\langle R_y^* H_x \rangle \langle R_x^* R_y \rangle + \langle R_x^* H_x \rangle \langle R_y^* R_y \rangle \right) \left(\langle H_x^* R_x \rangle \right) \end{bmatrix}_{\langle 2\text{x}2 \rangle}}{\left(\langle R_x^* H_x \rangle \langle R_y^* H_y \rangle - \langle R_x^* H_y \rangle \langle R_y^* H_x \rangle \right) \left(\langle H_x^* R_x \rangle \langle H_y^* R_y \rangle - \langle H_x^* R_y \rangle \langle H_y^* R_x \rangle \right)}$$

$$
=\begin{bmatrix}
\frac{\left(\langle R_y^*H_y\rangle\langle R_x^*R_x\rangle-\langle R_x^*H_y\rangle\langle R_y^*R_x\rangle\right)\left(\langle H_y^*R_y\rangle\right)+\left(\langle R_y^*H_y\rangle\langle R_x^*R_y\rangle-\langle R_x^*H_y\rangle\langle R_y^*R_y\rangle\right)\left(-\langle H_y^*R_x\rangle\right)}{\left(\langle R_x^*H_x\rangle\langle R_y^*H_y\rangle-\langle R_x^*H_y\rangle\langle R_y^*H_x\rangle\right)\left(\langle H_x^*R_x\rangle\langle H_y^*R_y\rangle-\langle H_x^*R_y\rangle\langle H_y^*R_x\rangle\right)} & \frac{\left(\langle R_y^*H_y\rangle\langle R_x^*R_x\rangle-\langle R_x^*H_y\rangle\langle R_y^*R_x\rangle\right)\left(-\langle H_x^*R_y\rangle\right)+\left(\langle R_y^*H_y\rangle\langle R_x^*R_y\rangle-\langle R_x^*H_y\rangle\langle R_y^*R_y\rangle\right)\left(\langle H_x^*R_x\rangle\right)}{\left(\langle R_x^*H_x\rangle\langle R_y^*H_y\rangle-\langle R_x^*H_y\rangle\langle R_y^*H_x\rangle\right)\left(\langle H_x^*R_x\rangle\langle H_y^*R_y\rangle-\langle H_x^*R_y\rangle\langle H_y^*R_x\rangle\right)} \\
\frac{\left(-\langle R_y^*H_x\rangle\langle R_x^*R_x\rangle+\langle R_x^*H_x\rangle\langle R_y^*R_x\rangle\right)\left(\langle H_y^*R_y\rangle\right)+\left(-\langle R_y^*H_x\rangle\langle R_x^*R_y\rangle+\langle R_x^*H_x\rangle\langle R_y^*R_y\rangle\right)\left(-\langle H_y^*R_x\rangle\right)}{\left(\langle R_x^*H_x\rangle\langle R_y^*H_y\rangle-\langle R_x^*H_y\rangle\langle R_y^*H_x\rangle\right)\left(\langle H_x^*R_x\rangle\langle H_y^*R_y\rangle-\langle H_x^*R_y\rangle\langle H_y^*R_x\rangle\right)} & \frac{\left(-\langle R_y^*H_x\rangle\langle R_x^*R_x\rangle+\langle R_x^*H_x\rangle\langle R_y^*R_x\rangle\right)\left(-\langle H_x^*R_y\rangle\right)+\left(-\langle R_y^*H_x\rangle\langle R_x^*R_y\rangle+\langle R_x^*H_x\rangle\langle R_y^*R_y\rangle\right)\left(\langle H_x^*R_x\rangle\right)}{\left(\langle R_x^*H_x\rangle\langle R_y^*H_y\rangle-\langle R_x^*H_y\rangle\langle R_y^*H_x\rangle\right)\left(\langle H_x^*R_x\rangle\langle H_y^*R_y\rangle-\langle H_x^*R_y\rangle\langle H_y^*R_x\rangle\right)}
\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}
$$

最終元素計算:

$$
\boldsymbol{S}=\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}_{\langle 2\mathrm{x}2\rangle}
$$

$$
=\begin{bmatrix}
\frac{\left(\langle R_y^*H_y\rangle\langle R_x^*R_x\rangle-\langle R_x^*H_y\rangle\langle R_y^*R_x\rangle\right)\left(\langle H_y^*R_y\rangle\right)+\left(\langle R_y^*H_y\rangle\langle R_x^*R_y\rangle-\langle R_x^*H_y\rangle\langle R_y^*R_y\rangle\right)\left(-\langle H_y^*R_x\rangle\right)}{\left(\langle R_x^*H_x\rangle\langle R_y^*H_y\rangle-\langle R_x^*H_y\rangle\langle R_y^*H_x\rangle\right)\left(\langle H_x^*R_x\rangle\langle H_y^*R_y\rangle-\langle H_x^*R_y\rangle\langle H_y^*R_x\rangle\right)} & \frac{\left(\langle R_y^*H_y\rangle\langle R_x^*R_x\rangle-\langle R_x^*H_y\rangle\langle R_y^*R_x\rangle\right)\left(-\langle H_x^*R_y\rangle\right)+\left(\langle R_y^*H_y\rangle\langle R_x^*R_y\rangle-\langle R_x^*H_y\rangle\langle R_y^*R_y\rangle\right)\left(\langle H_x^*R_x\rangle\right)}{\left(\langle R_x^*H_x\rangle\langle R_y^*H_y\rangle-\langle R_x^*H_y\rangle\langle R_y^*H_x\rangle\right)\left(\langle H_x^*R_x\rangle\langle H_y^*R_y\rangle-\langle H_x^*R_y\rangle\langle H_y^*R_x\rangle\right)} \\
\frac{\left(-\langle R_y^*H_x\rangle\langle R_x^*R_x\rangle+\langle R_x^*H_x\rangle\langle R_y^*R_x\rangle\right)\left(\langle H_y^*R_y\rangle\right)+\left(-\langle R_y^*H_x\rangle\langle R_x^*R_y\rangle+\langle R_x^*H_x\rangle\langle R_y^*R_y\rangle\right)\left(-\langle H_y^*R_x\rangle\right)}{\left(\langle R_x^*H_x\rangle\langle R_y^*H_y\rangle-\langle R_x^*H_y\rangle\langle R_y^*H_x\rangle\right)\left(\langle H_x^*R_x\rangle\langle H_y^*R_y\rangle-\langle H_x^*R_y\rangle\langle H_y^*R_x\rangle\right)} & \frac{\left(-\langle R_y^*H_x\rangle\langle R_x^*R_x\rangle+\langle R_x^*H_x\rangle\langle R_y^*R_x\rangle\right)\left(-\langle H_x^*R_y\rangle\right)+\left(-\langle R_y^*H_x\rangle\langle R_x^*R_y\rangle+\langle R_x^*H_x\rangle\langle R_y^*R_y\rangle\right)\left(\langle H_x^*R_x\rangle\right)}{\left(\langle R_x^*H_x\rangle\langle R_y^*H_y\rangle-\langle R_x^*H_y\rangle\langle R_y^*H_x\rangle\right)\left(\langle H_x^*R_x\rangle\langle H_y^*R_y\rangle-\langle H_x^*R_y\rangle\langle H_y^*R_x\rangle\right)}
\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}
$$

殘差值的共變異數矩陣(residual covariance matrix):

$$
\boldsymbol{N}=\hat{\sigma}_v^2\left((\boldsymbol{E}^\dagger\boldsymbol{E})-(\widehat{\boldsymbol{Z}}^\dagger)(\boldsymbol{H}^\dagger\boldsymbol{E})-(\boldsymbol{E}^\dagger\boldsymbol{H})(\widehat{\boldsymbol{Z}})+(\widehat{\boldsymbol{Z}}^\dagger)(\boldsymbol{H}^\dagger\boldsymbol{H})(\widehat{\boldsymbol{Z}})\right)
$$

$$
=\hat{\sigma}_v^2\left([\langle E_x^*E_x\rangle]_{\langle 1\mathrm{x}1\rangle}-\left(\begin{bmatrix}\hat{Z}_{xx}^* & \hat{Z}_{xy}^*\end{bmatrix}_{\langle 1\mathrm{x}2\rangle}\begin{bmatrix}\langle H_x^*E_x\rangle \\ \langle H_y^*E_x\rangle\end{bmatrix}_{\langle 2\mathrm{x}1\rangle}\right)-\left([\langle E_x^*H_x\rangle \quad \langle E_x^*H_y\rangle]_{\langle 1\mathrm{x}2\rangle}\begin{bmatrix}\hat{Z}_{xx} \\ \hat{Z}_{xy}\end{bmatrix}_{\langle 2\mathrm{x}1\rangle}\right)+\left(\begin{bmatrix}\hat{Z}_{xx}^* & \hat{Z}_{xy}^*\end{bmatrix}_{\langle 1\mathrm{x}2\rangle}\begin{bmatrix}\langle H_x^*H_x\rangle & \langle H_x^*H_y\rangle \\ \langle H_y^*H_x\rangle & \langle H_y^*H_y\rangle\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}\begin{bmatrix}\hat{Z}_{xx} \\ \hat{Z}_{xy}\end{bmatrix}_{\langle 2\mathrm{x}1\rangle}\right)\right)
$$

$$
=\hat{\sigma}_v^2\left([\langle E_x^*E_x\rangle]_{\langle 1\mathrm{x}1\rangle}-\left[\hat{Z}_{xx}^*\langle H_x^*E_x\rangle+\hat{Z}_{xy}^*\langle H_y^*E_x\rangle\right]_{\langle 1\mathrm{x}1\rangle}-\left([\langle E_x^*H_x\rangle \quad \langle E_x^*H_y\rangle]_{\langle 1\mathrm{x}2\rangle}\begin{bmatrix}\hat{Z}_{xx} \\ \hat{Z}_{xy}\end{bmatrix}_{\langle 2\mathrm{x}1\rangle}\right)+\left(\begin{bmatrix}\hat{Z}_{xx}^* & \hat{Z}_{xy}^*\end{bmatrix}_{\langle 1\mathrm{x}2\rangle}\begin{bmatrix}\langle H_x^*H_x\rangle & \langle H_x^*H_y\rangle \\ \langle H_y^*H_x\rangle & \langle H_y^*H_y\rangle\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}\begin{bmatrix}\hat{Z}_{xx} \\ \hat{Z}_{xy}\end{bmatrix}_{\langle 2\mathrm{x}1\rangle}\right)\right)
$$

$$
=\hat{\sigma}_v^2\left(\left[\langle E_x^*E_x\rangle-\hat{Z}_{xx}^*\langle H_x^*E_x\rangle-\hat{Z}_{xy}^*\langle H_y^*E_x\rangle\right]_{\langle 1\mathrm{x}1\rangle}-\left([\langle E_x^*H_x\rangle \quad \langle E_x^*H_y\rangle]_{\langle 1\mathrm{x}2\rangle}\begin{bmatrix}\hat{Z}_{xx} \\ \hat{Z}_{xy}\end{bmatrix}_{\langle 2\mathrm{x}1\rangle}\right)+\left(\begin{bmatrix}\hat{Z}_{xx}^* & \hat{Z}_{xy}^*\end{bmatrix}_{\langle 1\mathrm{x}2\rangle}\begin{bmatrix}\langle H_x^*H_x\rangle & \langle H_x^*H_y\rangle \\ \langle H_y^*H_x\rangle & \langle H_y^*H_y\rangle\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}\begin{bmatrix}\hat{Z}_{xx} \\ \hat{Z}_{xy}\end{bmatrix}_{\langle 2\mathrm{x}1\rangle}\right)\right)
$$

$$
=\hat{\sigma}_v^2\left(\left[\langle E_x^*E_x\rangle-\hat{Z}_{xx}^*\langle H_x^*E_x\rangle-\hat{Z}_{xy}^*\langle H_y^*E_x\rangle\right]_{\langle 1\mathrm{x}1\rangle}-\left[\langle E_x^*H_x\rangle\hat{Z}_{xx}+\langle E_x^*H_y\rangle\hat{Z}_{xy}\right]_{\langle 1\mathrm{x}1\rangle}+\left(\begin{bmatrix}\hat{Z}_{xx}^* & \hat{Z}_{xy}^*\end{bmatrix}_{\langle 1\mathrm{x}2\rangle}\begin{bmatrix}\langle H_x^*H_x\rangle & \langle H_x^*H_y\rangle \\ \langle H_y^*H_x\rangle & \langle H_y^*H_y\rangle\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}\begin{bmatrix}\hat{Z}_{xx} \\ \hat{Z}_{xy}\end{bmatrix}_{\langle 2\mathrm{x}1\rangle}\right)\right)
$$

$$
=\hat{\sigma}_v^2\left(\left[\langle E_x^*E_x\rangle-\hat{Z}_{xx}^*\langle H_x^*E_x\rangle-\hat{Z}_{xy}^*\langle H_y^*E_x\rangle-\langle E_x^*H_x\rangle\hat{Z}_{xx}-\langle E_x^*H_y\rangle\hat{Z}_{xy}\right]_{\langle 1\mathrm{x}1\rangle}+\left(\begin{bmatrix}\hat{Z}_{xx}^* & \hat{Z}_{xy}^*\end{bmatrix}_{\langle 1\mathrm{x}2\rangle}\begin{bmatrix}\langle H_x^*H_x\rangle & \langle H_x^*H_y\rangle \\ \langle H_y^*H_x\rangle & \langle H_y^*H_y\rangle\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}\begin{bmatrix}\hat{Z}_{xx} \\ \hat{Z}_{xy}\end{bmatrix}_{\langle 2\mathrm{x}1\rangle}\right)\right)
$$

$$
=\hat{\sigma}_v^2\left(\left[\langle E_x^*E_x\rangle-\hat{Z}_{xx}^*\langle H_x^*E_x\rangle-\hat{Z}_{xy}^*\langle H_y^*E_x\rangle-\langle E_x^*H_x\rangle\hat{Z}_{xx}-\langle E_x^*H_y\rangle\hat{Z}_{xy}\right]_{\langle 1\mathrm{x}1\rangle}+\left(\begin{bmatrix}\hat{Z}_{xx}^* & \hat{Z}_{xy}^*\end{bmatrix}_{\langle 1\mathrm{x}2\rangle}\begin{bmatrix}\langle H_x^*H_x\rangle\hat{Z}_{xx}+\langle H_x^*H_y\rangle\hat{Z}_{xy} \\ \langle H_y^*H_x\rangle\hat{Z}_{xx}+\langle H_y^*H_y\rangle\hat{Z}_{xy}\end{bmatrix}_{\langle 2\mathrm{x}1\rangle}\right)\right)
$$

$$
=\hat{\sigma}_v^2\left(\left[\langle E_x^*E_x\rangle-\hat{Z}_{xx}^*\langle H_x^*E_x\rangle-\hat{Z}_{xy}^*\langle H_y^*E_x\rangle-\langle E_x^*H_x\rangle\hat{Z}_{xx}-\langle E_x^*H_y\rangle\hat{Z}_{xy}\right]_{\langle 1\mathrm{x}1\rangle}+\left[\hat{Z}_{xx}^*\left(\langle H_x^*H_x\rangle\hat{Z}_{xx}+\langle H_x^*H_y\rangle\hat{Z}_{xy}\right)+\hat{Z}_{xy}^*\left(\langle H_y^*H_x\rangle\hat{Z}_{xx}+\langle H_y^*H_y\rangle\hat{Z}_{xy}\right)\right]_{\langle 1\mathrm{x}1\rangle}\right)
$$

$$
=\hat{\sigma}_v^2\left(\left[\langle E_x^*E_x\rangle-\hat{Z}_{xx}^*\langle H_x^*E_x\rangle-\hat{Z}_{xy}^*\langle H_y^*E_x\rangle-\langle E_x^*H_x\rangle\hat{Z}_{xx}-\langle E_x^*H_y\rangle\hat{Z}_{xy}+\hat{Z}_{xx}^*\left(\langle H_x^*H_x\rangle\hat{Z}_{xx}+\langle H_x^*H_y\rangle\hat{Z}_{xy}\right)+\hat{Z}_{xy}^*\left(\langle H_y^*H_x\rangle\hat{Z}_{xx}+\langle H_y^*H_y\rangle\hat{Z}_{xy}\right)\right]_{\langle 1\mathrm{x}1\rangle}\right)
$$

$$
=\left[\hat{\sigma}_v^2\left(\langle E_x^*E_x\rangle-\hat{Z}_{xx}^*\langle H_x^*E_x\rangle-\hat{Z}_{xy}^*\langle H_y^*E_x\rangle-\langle E_x^*H_x\rangle\hat{Z}_{xx}-\langle E_x^*H_y\rangle\hat{Z}_{xy}+\hat{Z}_{xx}^*\left(\langle H_x^*H_x\rangle\hat{Z}_{xx}+\langle H_x^*H_y\rangle\hat{Z}_{xy}\right)+\hat{Z}_{xy}^*\left(\langle H_y^*H_x\rangle\hat{Z}_{xx}+\langle H_y^*H_y\rangle\hat{Z}_{xy}\right)\right)\right]_{\langle 1\mathrm{x}1\rangle}
$$

最終元素計算:

$$
\boldsymbol{N}=[N_{11}]_{\langle 1\mathrm{x}1\rangle}
$$

$$
=\left[\hat{\sigma}_v^2\left(\langle E_x^*E_x\rangle-\hat{Z}_{xx}^*\langle H_x^*E_x\rangle-\hat{Z}_{xy}^*\langle H_y^*E_x\rangle-\langle E_x^*H_x\rangle\hat{Z}_{xx}-\langle E_x^*H_y\rangle\hat{Z}_{xy}+\hat{Z}_{xx}^*\left(\langle H_x^*H_x\rangle\hat{Z}_{xx}+\langle H_x^*H_y\rangle\hat{Z}_{xy}\right)+\hat{Z}_{xy}^*\left(\langle H_y^*H_x\rangle\hat{Z}_{xx}+\langle H_y^*H_y\rangle\hat{Z}_{xy}\right)\right)\right]_{\langle 1\mathrm{x}1\rangle}
$$

計算對應 EDI 檔案中的 ZXXR、ZXXI、ZXX.VAR、ZXYR、ZXYI、ZXY.VAR

已求得: $\widehat{\boldsymbol{Z}}=\begin{bmatrix}\hat{Z}_{xx} \\ \hat{Z}_{xy}\end{bmatrix}_{\langle 2\mathrm{x}1\rangle}$、$\boldsymbol{S}=\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}_{\langle 2\mathrm{x}2\rangle}$、$\boldsymbol{N}=[N_{11}]_{\langle 1\mathrm{x}1\rangle}$

從程式碼中推測:

$$\mathrm{ZXXR}=\Re\mathfrak{e}\left(\hat{Z}_{xx}\right)$$

$$\mathrm{ZXXI}=\Im\mathfrak{m}\left(\hat{Z}_{xx}\right)$$

$$\mathrm{ZXX.VAR}=\Re\mathfrak{e}(N_{11}S_{11})$$

$$\mathrm{ZXYR} = \Re\mathrm{e}(\hat{Z}_{xy})$$
$$\mathrm{ZXYI} = \Im\mathrm{m}(\hat{Z}_{xy})$$
$$\mathrm{ZXY.VAR} = \Re\mathrm{e}(N_{11}S_{22})$$

大地電磁法主要的三個關係式:

$E_x = Z_{xx}H_x + Z_{xy}H_y$ ...(1)

$E_y = Z_{yx}H_x + Z_{yy}H_y$ ...(2)

$H_z = T_xH_x + T_yH_y$ ...(3)

一次只看前兩個關係式，這裡取用關係式:

$$E_x = Z_{xx}H_x + Z_{xy}H_y$$
$$E_y = Z_{yx}H_x + Z_{yy}H_y$$

n 次獨立觀測可產生下列關係式:

$$\begin{cases} E_{x,\#1} = Z_{xx}H_{x,\#1} + Z_{xy}H_{y,\#1} + \varepsilon_{x,\#1} \\ E_{y,\#1} = Z_{yx}H_{x,\#1} + Z_{yy}H_{y,\#1} + \varepsilon_{y,\#1} \end{cases} \quad \text{... (1)}$$

$$\begin{cases} E_{x,\#2} = Z_{xx}H_{x,\#2} + Z_{xy}H_{y,\#2} + \varepsilon_{x,\#2} \\ E_{y,\#2} = Z_{yx}H_{x,\#2} + Z_{yy}H_{y,\#2} + \varepsilon_{y,\#2} \end{cases} \quad \text{... (2)}$$

$$\begin{cases} E_{x,\#3} = Z_{xx}H_{x,\#3} + Z_{xy}H_{y,\#3} + \varepsilon_{x,\#3} \\ E_{y,\#3} = Z_{yx}H_{x,\#3} + Z_{yy}H_{y,\#3} + \varepsilon_{y,\#3} \end{cases} \quad \text{... (3)}$$

...

$$\begin{cases} E_{x,\#n} = Z_{xx}H_{x,\#n} + Z_{xy}H_{y,\#n} + \varepsilon_{x,\#n} \\ E_{y,\#n} = Z_{yx}H_{x,\#n} + Z_{yy}H_{y,\#n} + \varepsilon_{y,\#n} \end{cases} \quad \text{... (n)}$$

整理成矩陣形式<nx2>=<nx2><2x2>+<nx2>:

$$\begin{bmatrix} E_{x,\#1} & E_{y,\#1} \\ E_{x,\#2} & E_{y,\#2} \\ E_{x,\#3} & E_{y,\#3} \\ \vdots & \vdots \\ E_{x,\#n} & E_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle} = \begin{bmatrix} H_{x,\#1} & H_{y,\#1} \\ H_{x,\#2} & H_{y,\#2} \\ H_{x,\#3} & H_{y,\#3} \\ \vdots & \vdots \\ H_{x,\#n} & H_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle} \begin{bmatrix} Z_{xx} & Z_{yx} \\ Z_{xy} & Z_{yy} \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle} + \begin{bmatrix} \varepsilon_{x,\#1} \\ \varepsilon_{x,\#2} \\ \varepsilon_{x,\#3} \\ \vdots \\ \varepsilon_{x,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle}$$

套用到遠端站估算法問題:

線性回歸模型: $\boldsymbol{E} = \boldsymbol{HZ} + \boldsymbol{\varepsilon}$

已知: $\boldsymbol{E} = \begin{bmatrix} E_{x,\#1} & E_{y,\#1} \\ E_{x,\#2} & E_{y,\#2} \\ E_{x,\#3} & E_{y,\#3} \\ \vdots & \vdots \\ E_{x,\#n} & E_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle}$ 、$\boldsymbol{H} = \begin{bmatrix} H_{x,\#1} & H_{y,\#1} \\ H_{x,\#2} & H_{y,\#2} \\ H_{x,\#3} & H_{y,\#3} \\ \vdots & \vdots \\ H_{x,\#n} & H_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle}$ 、$\boldsymbol{R} = \begin{bmatrix} R_{x,\#1} & R_{y,\#1} \\ R_{x,\#2} & R_{y,\#2} \\ R_{x,\#3} & R_{y,\#3} \\ \vdots & \vdots \\ R_{x,\#n} & R_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle}$

未知: $\boldsymbol{Z} = \begin{bmatrix} Z_{xx} & Z_{yx} \\ Z_{xy} & Z_{yy} \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle}$

誤差: $\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{x,\#1} \\ \varepsilon_{x,\#2} \\ \varepsilon_{x,\#3} \\ \vdots \\ \varepsilon_{x,\#n} \end{bmatrix}_{\langle \mathrm{nx1} \rangle}$

求未知: $\widehat{\boldsymbol{Z}} = \begin{bmatrix} \hat{Z}_{xx} & \hat{Z}_{yx} \\ \hat{Z}_{xy} & \hat{Z}_{yy} \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle}$

公式解: $\widehat{\boldsymbol{Z}} = (\boldsymbol{R}^\dagger \boldsymbol{H})^{-1}(\boldsymbol{R}^\dagger \boldsymbol{E})$

逆訊號功率矩陣(inverse signal power matrix): $\boldsymbol{S} = (\boldsymbol{R}^\dagger \boldsymbol{H})^{-1}(\boldsymbol{R}^\dagger \boldsymbol{R})(\boldsymbol{H}^\dagger \boldsymbol{R})^{-1}$

殘差值的共變異數矩陣(residual covariance matrix):

$$\boldsymbol{N} = \hat{\sigma}_v^2 \left((\boldsymbol{E}^\dagger \boldsymbol{E}) - (\widehat{\boldsymbol{Z}}^\dagger)(\boldsymbol{H}^\dagger \boldsymbol{E}) - (\boldsymbol{E}^\dagger \boldsymbol{H})(\widehat{\boldsymbol{Z}}) + (\widehat{\boldsymbol{Z}}^\dagger)(\boldsymbol{H}^\dagger \boldsymbol{H})(\widehat{\boldsymbol{Z}}) \right)$$

探討自功率譜矩陣資料與互功率譜矩陣資料:

計算要用到的 8 個矩陣: $\boldsymbol{R}^\dagger \boldsymbol{H}$、$\boldsymbol{R}^\dagger \boldsymbol{R}$、$\boldsymbol{H}^\dagger \boldsymbol{R}$、$\boldsymbol{R}^\dagger \boldsymbol{E}$、$\boldsymbol{E}^\dagger \boldsymbol{E}$、$\boldsymbol{H}^\dagger \boldsymbol{E}$、$\boldsymbol{E}^\dagger \boldsymbol{H}$、$\boldsymbol{H}^\dagger \boldsymbol{H}$

將 8 個矩陣都以中括號整理:

#1: $\boldsymbol{R}^\dagger \boldsymbol{H} = \begin{bmatrix} \langle R_x^* H_x \rangle & \langle R_x^* H_y \rangle \\ \langle R_y^* H_x \rangle & \langle R_y^* H_y \rangle \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle}$

#2: $\boldsymbol{R}^\dagger \boldsymbol{R} = \begin{bmatrix} \langle R_x^* R_x \rangle & \langle R_x^* R_y \rangle \\ \langle R_y^* R_x \rangle & \langle R_y^* R_y \rangle \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle}$

#3: $\boldsymbol{H}^\dagger \boldsymbol{R} = \begin{bmatrix} \langle H_x^* R_x \rangle & \langle H_x^* R_y \rangle \\ \langle H_y^* R_x \rangle & \langle H_y^* R_y \rangle \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle}$

#4: $\boldsymbol{R}^\dagger \boldsymbol{E} = \begin{bmatrix} \langle R_x^* E_x \rangle & \langle R_x^* E_y \rangle \\ \langle R_y^* E_x \rangle & \langle R_y^* E_y \rangle \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle}$

#5: $\boldsymbol{E}^\dagger \boldsymbol{E} = \begin{bmatrix} \langle E_x^* E_x \rangle & \langle E_x^* E_y \rangle \\ \langle E_y^* E_x \rangle & \langle E_y^* E_y \rangle \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle}$

#6: $\boldsymbol{H}^\dagger \boldsymbol{E} = \begin{bmatrix} \langle H_x^* E_x \rangle & \langle H_x^* E_y \rangle \\ \langle H_y^* E_x \rangle & \langle H_y^* E_y \rangle \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle}$

#7: $\boldsymbol{E}^\dagger \boldsymbol{H} = \begin{bmatrix} \langle E_x^* H_x \rangle & \langle E_x^* H_y \rangle \\ \langle E_y^* H_x \rangle & \langle E_y^* H_y \rangle \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle}$

#8: $\boldsymbol{H}^\dagger \boldsymbol{H} = \begin{bmatrix} \langle H_x^* H_x \rangle & \langle H_x^* H_y \rangle \\ \langle H_y^* H_x \rangle & \langle H_y^* H_y \rangle \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle}$

用 8 個矩陣資料進行計算:

求未知: $\widehat{\boldsymbol{Z}} = \begin{bmatrix} \hat{Z}_{xx} & \hat{Z}_{yx} \\ \hat{Z}_{xy} & \hat{Z}_{yy} \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle}$

$$\widehat{\boldsymbol{Z}} = (\boldsymbol{R}^\dagger \boldsymbol{H})^{-1}(\boldsymbol{R}^\dagger \boldsymbol{E})$$

$$= \begin{bmatrix} \hat{Z}_{xx} & \hat{Z}_{yx} \\ \hat{Z}_{xy} & \hat{Z}_{yy} \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle} = \left(\begin{bmatrix} \langle R_x^* H_x \rangle & \langle R_x^* H_y \rangle \\ \langle R_y^* H_x \rangle & \langle R_y^* H_y \rangle \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle} \right)^{-1} \left(\begin{bmatrix} \langle R_x^* E_x \rangle & \langle R_x^* E_y \rangle \\ \langle R_y^* E_x \rangle & \langle R_y^* E_y \rangle \end{bmatrix}_{\langle 2\mathrm{x}2 \rangle} \right)$$

順便算一下另一個後面需要的矩陣:

$$\widehat{\boldsymbol{Z}}^{\dagger}=\left(\begin{bmatrix}\hat{Z}_{xx} & \hat{Z}_{yx}\\ \hat{Z}_{xy} & \hat{Z}_{yy}\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}\right)^{\dagger}=\begin{bmatrix}\hat{Z}_{xx}^{*} & \hat{Z}_{yx}^{*}\\ \hat{Z}_{xy}^{*} & \hat{Z}_{yy}^{*}\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}$$

逆訊號功率矩陣(inverse signal power matrix):

$$\boldsymbol{S}=(\boldsymbol{R}^{\dagger}\boldsymbol{H})^{-1}(\boldsymbol{R}^{\dagger}\boldsymbol{R})(\boldsymbol{H}^{\dagger}\boldsymbol{R})^{-1}$$

$$=\begin{bmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}=\left(\begin{bmatrix}\langle R_x^{*}H_x\rangle & \langle R_x^{*}H_y\rangle\\ \langle R_y^{*}H_x\rangle & \langle R_y^{*}H_y\rangle\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}\right)^{-1}\left(\begin{bmatrix}\langle R_x^{*}R_x\rangle & \langle R_x^{*}R_y\rangle\\ \langle R_y^{*}R_x\rangle & \langle R_y^{*}R_y\rangle\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}\right)\left(\begin{bmatrix}\langle H_x^{*}R_x\rangle & \langle H_x^{*}R_y\rangle\\ \langle H_y^{*}R_x\rangle & \langle H_y^{*}R_y\rangle\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}\right)^{-1}$$

殘差值的共變異數矩陣(residual covariance matrix):

$$\boldsymbol{N}=\hat{\sigma}_{v}^{2}\left((\boldsymbol{E}^{\dagger}\boldsymbol{E})-(\widehat{\boldsymbol{Z}}^{\dagger})(\boldsymbol{H}^{\dagger}\boldsymbol{E})-(\boldsymbol{E}^{\dagger}\boldsymbol{H})(\widehat{\boldsymbol{Z}})+(\widehat{\boldsymbol{Z}}^{\dagger})(\boldsymbol{H}^{\dagger}\boldsymbol{H})(\widehat{\boldsymbol{Z}})\right)$$

$$=\begin{bmatrix}N_{11} & N_{12}\\ N_{21} & N_{22}\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}$$

計算對應 EDI 檔案中的 ZXXR、ZXXI、ZXX.VAR、ZXYR、ZXYI、ZXY.VAR、ZYXR、ZYXI、ZYX.VAR、ZYYR、ZYYI、ZYY.VAR

已求得: $\widehat{\boldsymbol{Z}}=\begin{bmatrix}\hat{Z}_{xx} & \hat{Z}_{yx}\\ \hat{Z}_{xy} & \hat{Z}_{yy}\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}$、$\boldsymbol{S}=\begin{bmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}$、$\boldsymbol{N}=\begin{bmatrix}N_{11} & N_{12}\\ N_{21} & N_{22}\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}$

從程式碼中推測:

$\mathrm{ZXXR}=\Re\mathrm{e}(\hat{Z}_{xx})$

$\mathrm{ZXXI}=\Im\mathrm{m}(\hat{Z}_{xx})$

$\mathrm{ZXX.VAR}=\Re\mathrm{e}(N_{11}S_{11})$

$\mathrm{ZXYR}=\Re\mathrm{e}(\hat{Z}_{xy})$

$\mathrm{ZXYI}=\Im\mathrm{m}(\hat{Z}_{xy})$

$\mathrm{ZXY.VAR}=\Re\mathrm{e}(N_{11}S_{22})$

$\mathrm{ZYXR}=\Re\mathrm{e}(\hat{Z}_{yx})$

$\mathrm{ZYXI}=\Im\mathrm{m}(\hat{Z}_{yx})$

$\mathrm{ZYX.VAR}=\Re\mathrm{e}(N_{22}S_{11})$

$\mathrm{ZYYR}=\Re\mathrm{e}(\hat{Z}_{yx})$

$\mathrm{ZYYI}=\Im\mathrm{m}(\hat{Z}_{yx})$

$\mathrm{ZYY.VAR}=\Re\mathrm{e}(N_{22}S_{22})$

大地電磁法主要的三個關係式:

$E_x=Z_{xx}H_x+Z_{xy}H_y$ ...(1)

$E_y = Z_{yx}H_x + Z_{yy}H_y$ ...(2)

$H_z = T_xH_x + T_yH_y$ ...(3)

一次看三個關係式，這裡取用關係式:

$E_x = Z_{xx}H_x + Z_{xy}H_y$

$E_y = Z_{yx}H_x + Z_{yy}H_y$

$H_z = T_xH_x + T_yH_y$

n 次獨立觀測可產生下列關係式:

$$\begin{cases} E_{x,\#1} = Z_{xx}H_{x,\#1} + Z_{xy}H_{y,\#1} + \varepsilon_{x,\#1} \\ E_{y,\#1} = Z_{yx}H_{x,\#1} + Z_{yy}H_{y,\#1} + \varepsilon_{y,\#1} \\ H_{z,\#1} = T_xH_{x,\#1} + T_yH_{y,\#1} + \varepsilon_{z,\#1} \end{cases} \quad \ldots (1)$$

$$\begin{cases} E_{x,\#2} = Z_{xx}H_{x,\#2} + Z_{xy}H_{y,\#2} + \varepsilon_{x,\#2} \\ E_{y,\#2} = Z_{yx}H_{x,\#2} + Z_{yy}H_{y,\#2} + \varepsilon_{y,\#2} \\ H_{z,\#2} = T_xH_{x,\#2} + T_yH_{y,\#2} + \varepsilon_{z,\#2} \end{cases} \quad \ldots (2)$$

$$\begin{cases} E_{x,\#3} = Z_{xx}H_{x,\#3} + Z_{xy}H_{y,\#3} + \varepsilon_{x,\#3} \\ E_{y,\#3} = Z_{yx}H_{x,\#3} + Z_{yy}H_{y,\#3} + \varepsilon_{y,\#3} \\ H_{z,\#3} = T_xH_{x,\#3} + T_yH_{y,\#3} + \varepsilon_{z,\#3} \end{cases} \quad \ldots (3)$$

...

$$\begin{cases} E_{x,\#n} = Z_{xx}H_{x,\#n} + Z_{xy}H_{y,\#n} + \varepsilon_{x,\#n} \\ E_{y,\#n} = Z_{yx}H_{x,\#n} + Z_{yy}H_{y,\#n} + \varepsilon_{y,\#n} \\ H_{z,\#n} = T_xH_{x,\#n} + T_yH_{y,\#n} + \varepsilon_{z,\#n} \end{cases} \quad \ldots (n)$$

整理成矩陣形式<nx3>=<nx2><2x1>+<nx3>:

$$\begin{bmatrix} E_{x,\#1} & E_{y,\#1} & H_{z,\#1} \\ E_{x,\#2} & E_{y,\#2} & H_{z,\#1} \\ E_{x,\#3} & E_{y,\#3} & H_{z,\#1} \\ \vdots & \vdots & \vdots \\ E_{x,\#n} & E_{y,\#n} & H_{z,\#1} \end{bmatrix}_{\langle \mathrm{nx3} \rangle} = \begin{bmatrix} H_{x,\#1} & H_{y,\#1} \\ H_{x,\#2} & H_{y,\#2} \\ H_{x,\#3} & H_{y,\#3} \\ \vdots & \vdots \\ H_{x,\#n} & H_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle} \begin{bmatrix} Z_{xx} & Z_{yx} & T_x \\ Z_{xy} & Z_{yy} & T_y \end{bmatrix}_{\langle \mathrm{2x3} \rangle} + \begin{bmatrix} \varepsilon_{x,\#1} & \varepsilon_{y,\#1} & \varepsilon_{z,\#1} \\ \varepsilon_{x,\#2} & \varepsilon_{y,\#2} & \varepsilon_{z,\#2} \\ \varepsilon_{x,\#3} & \varepsilon_{y,\#3} & \varepsilon_{z,\#3} \\ \vdots & \vdots & \vdots \\ \varepsilon_{x,\#n} & \varepsilon_{y,\#n} & \varepsilon_{z,\#n} \end{bmatrix}_{\langle \mathrm{nx3} \rangle}$$

套用到遠端站估算法問題:

線性回歸模型: $\boldsymbol{E} = \boldsymbol{HZ} + \boldsymbol{\varepsilon}$

已知: $\boldsymbol{E} = \begin{bmatrix} E_{x,\#1} & E_{y,\#1} & H_{z,\#1} \\ E_{x,\#2} & E_{y,\#2} & H_{z,\#1} \\ E_{x,\#3} & E_{y,\#3} & H_{z,\#1} \\ \vdots & \vdots & \vdots \\ E_{x,\#n} & E_{y,\#n} & H_{z,\#1} \end{bmatrix}_{\langle \mathrm{nx3} \rangle}$ 、$\boldsymbol{H} = \begin{bmatrix} H_{x,\#1} & H_{y,\#1} \\ H_{x,\#2} & H_{y,\#2} \\ H_{x,\#3} & H_{y,\#3} \\ \vdots & \vdots \\ H_{x,\#n} & H_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle}$ 、$\boldsymbol{R} = \begin{bmatrix} R_{x,\#1} & R_{y,\#1} \\ R_{x,\#2} & R_{y,\#2} \\ R_{x,\#3} & R_{y,\#3} \\ \vdots & \vdots \\ R_{x,\#n} & R_{y,\#n} \end{bmatrix}_{\langle \mathrm{nx2} \rangle}$

未知: $\boldsymbol{Z} = \begin{bmatrix} Z_{xx} & Z_{yx} & T_x \\ Z_{xy} & Z_{yy} & T_y \end{bmatrix}_{\langle \mathrm{2x3} \rangle}$

誤差: $\boldsymbol{\varepsilon}=\begin{bmatrix}\varepsilon_{x,\#1} & \varepsilon_{y,\#1} & \varepsilon_{z,\#1}\\ \varepsilon_{x,\#2} & \varepsilon_{y,\#2} & \varepsilon_{z,\#2}\\ \varepsilon_{x,\#3} & \varepsilon_{y,\#3} & \varepsilon_{z,\#3}\\ \vdots & \vdots & \vdots\\ \varepsilon_{x,\#n} & \varepsilon_{y,\#n} & \varepsilon_{z,\#n}\end{bmatrix}_{\langle \mathrm{nx3}\rangle}$

求未知: $\widehat{\boldsymbol{Z}}=\begin{bmatrix}\hat{Z}_{xx} & \hat{Z}_{yx} & \hat{T}_x\\ \hat{Z}_{xy} & \hat{Z}_{yy} & \hat{T}_y\end{bmatrix}_{\langle 2\mathrm{x}3\rangle}$

公式解: $\widehat{\boldsymbol{Z}}=(\boldsymbol{R}^{\dagger}\boldsymbol{H})^{-1}(\boldsymbol{R}^{\dagger}\boldsymbol{E})$

逆訊號功率矩陣(inverse signal power matrix): $\boldsymbol{S}=(\boldsymbol{R}^{\dagger}\boldsymbol{H})^{-1}(\boldsymbol{R}^{\dagger}\boldsymbol{R})(\boldsymbol{H}^{\dagger}\boldsymbol{R})^{-1}$

殘差值的共變異數矩陣(residual covariance matrix):

$$\boldsymbol{N}=\hat{\sigma}_{v}^{2}\left((\boldsymbol{E}^{\dagger}\boldsymbol{E})-\left(\widehat{\boldsymbol{Z}}^{\dagger}\right)(\boldsymbol{H}^{\dagger}\boldsymbol{E})-(\boldsymbol{E}^{\dagger}\boldsymbol{H})\left(\widehat{\boldsymbol{Z}}\right)+\left(\widehat{\boldsymbol{Z}}^{\dagger}\right)(\boldsymbol{H}^{\dagger}\boldsymbol{H})\left(\widehat{\boldsymbol{Z}}\right)\right)$$

探討自功率譜矩陣資料與互功率譜矩陣資料:

計算要用到的 8 個矩陣: $\boldsymbol{R}^{\dagger}\boldsymbol{H}$、$\boldsymbol{R}^{\dagger}\boldsymbol{R}$、$\boldsymbol{H}^{\dagger}\boldsymbol{R}$、$\boldsymbol{R}^{\dagger}\boldsymbol{E}$、$\boldsymbol{E}^{\dagger}\boldsymbol{E}$、$\boldsymbol{H}^{\dagger}\boldsymbol{E}$、$\boldsymbol{E}^{\dagger}\boldsymbol{H}$、$\boldsymbol{H}^{\dagger}\boldsymbol{H}$

將 8 個矩陣都以中括號整理:

#1: $\boldsymbol{R}^{\dagger}\boldsymbol{H}=\begin{bmatrix}\langle R_x^*H_x\rangle & \langle R_x^*H_y\rangle\\ \langle R_y^*H_x\rangle & \langle R_y^*H_y\rangle\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}$

#2: $\boldsymbol{R}^{\dagger}\boldsymbol{R}=\begin{bmatrix}\langle R_x^*R_x\rangle & \langle R_x^*R_y\rangle\\ \langle R_y^*R_x\rangle & \langle R_y^*R_y\rangle\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}$

#3: $\boldsymbol{H}^{\dagger}\boldsymbol{R}=\begin{bmatrix}\langle H_x^*R_x\rangle & \langle H_x^*R_y\rangle\\ \langle H_y^*R_x\rangle & \langle H_y^*R_y\rangle\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}$

#4: $\boldsymbol{R}^{\dagger}\boldsymbol{E}=\begin{bmatrix}\langle R_x^*E_x\rangle & \langle R_x^*E_y\rangle & \langle R_x^*H_z\rangle\\ \langle R_y^*E_x\rangle & \langle R_y^*E_y\rangle & \langle R_y^*H_z\rangle\end{bmatrix}_{\langle 2\mathrm{x}3\rangle}$

#5: $\boldsymbol{E}^{\dagger}\boldsymbol{E}=\begin{bmatrix}\langle E_x^*E_x\rangle & \langle E_x^*E_y\rangle & \langle E_x^*H_z\rangle\\ \langle E_y^*E_x\rangle & \langle E_y^*E_y\rangle & \langle E_y^*H_z\rangle\\ \langle H_z^*E_x\rangle & \langle H_z^*E_y\rangle & \langle H_z^*H_z\rangle\end{bmatrix}_{\langle 3\mathrm{x}3\rangle}$

#6: $\boldsymbol{H}^{\dagger}\boldsymbol{E}=\begin{bmatrix}\langle H_x^*E_x\rangle & \langle H_x^*E_y\rangle & \langle H_x^*H_z\rangle\\ \langle H_y^*E_x\rangle & \langle H_y^*E_y\rangle & \langle H_y^*H_z\rangle\end{bmatrix}_{\langle 2\mathrm{x}3\rangle}$

#7: $\boldsymbol{E}^{\dagger}\boldsymbol{H}=\begin{bmatrix}\langle E_x^*H_x\rangle & \langle E_x^*H_y\rangle\\ \langle E_y^*H_x\rangle & \langle E_y^*H_y\rangle\\ \langle H_z^*H_x\rangle & \langle H_z^*H_y\rangle\end{bmatrix}_{\langle 3\mathrm{x}2\rangle}$

#8: $\boldsymbol{H}^{\dagger}\boldsymbol{H}=\begin{bmatrix}\langle H_x^*H_x\rangle & \langle H_x^*H_y\rangle\\ \langle H_y^*H_x\rangle & \langle H_y^*H_y\rangle\end{bmatrix}_{\langle 2\mathrm{x}2\rangle}$

用 8 個矩陣資料進行計算:

求未知: $\widehat{\boldsymbol{Z}}=\begin{bmatrix}\hat{Z}_{xx} & \hat{Z}_{yx} & \hat{T}_x\\ \hat{Z}_{xy} & \hat{Z}_{yy} & \hat{T}_y\end{bmatrix}_{\langle 2\mathrm{x}3\rangle}$

$\widehat{\boldsymbol{Z}}=(\boldsymbol{R}^{\dagger}\boldsymbol{H})^{-1}(\boldsymbol{R}^{\dagger}\boldsymbol{E})$

$$= \begin{bmatrix} \hat{Z}_{xx} & \hat{Z}_{yx} & \hat{T}_x \\ \hat{Z}_{xy} & \hat{Z}_{yy} & \hat{T}_y \end{bmatrix}_{\langle 2x3 \rangle} = \left(\begin{bmatrix} \langle R_x^* H_x \rangle & \langle R_x^* H_y \rangle \\ \langle R_y^* H_x \rangle & \langle R_y^* H_y \rangle \end{bmatrix}_{\langle 2x2 \rangle} \right)^{-1} \left(\begin{bmatrix} \langle R_x^* E_x \rangle & \langle R_x^* E_y \rangle & \langle R_x^* H_z \rangle \\ \langle R_y^* E_x \rangle & \langle R_y^* E_y \rangle & \langle R_y^* H_z \rangle \end{bmatrix}_{\langle 2x3 \rangle} \right)$$

順便算一下另一個後面需要的矩陣:

$$\hat{\boldsymbol{Z}}^{\dagger} = \left(\begin{bmatrix} \hat{Z}_{xx} & \hat{Z}_{yx} & \hat{T}_x \\ \hat{Z}_{xy} & \hat{Z}_{yy} & \hat{T}_y \end{bmatrix}_{\langle 2x3 \rangle} \right)^{\dagger} = \begin{bmatrix} \hat{Z}_{xx}^* & \hat{Z}_{xy}^* \\ \hat{Z}_{yx}^* & \hat{Z}_{yy}^* \\ \hat{T}_x^* & \hat{T}_y^* \end{bmatrix}_{\langle 3x2 \rangle}$$

逆訊號功率矩陣(inverse signal power matrix):

$$\boldsymbol{S} = (\boldsymbol{R}^{\dagger}\boldsymbol{H})^{-1}(\boldsymbol{R}^{\dagger}\boldsymbol{R})(\boldsymbol{H}^{\dagger}\boldsymbol{R})^{-1}$$

$$= \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}_{\langle 2x2 \rangle} = \left(\begin{bmatrix} \langle R_x^* H_x \rangle & \langle R_x^* H_y \rangle \\ \langle R_y^* H_x \rangle & \langle R_y^* H_y \rangle \end{bmatrix}_{\langle 2x2 \rangle} \right)^{-1} \left(\begin{bmatrix} \langle R_x^* R_x \rangle & \langle R_x^* R_y \rangle \\ \langle R_y^* R_x \rangle & \langle R_y^* R_y \rangle \end{bmatrix}_{\langle 2x2 \rangle} \right) \left(\begin{bmatrix} \langle H_x^* R_x \rangle & \langle H_x^* R_y \rangle \\ \langle H_y^* R_x \rangle & \langle H_y^* R_y \rangle \end{bmatrix}_{\langle 2x2 \rangle} \right)^{-1}$$

殘差值的共變異數矩陣(residual covariance matrix):

$$\boldsymbol{N} = \hat{\sigma}_v^2 \left((\boldsymbol{E}^{\dagger}\boldsymbol{E}) - (\hat{\boldsymbol{Z}}^{\dagger})(\boldsymbol{H}^{\dagger}\boldsymbol{E}) - (\boldsymbol{E}^{\dagger}\boldsymbol{H})(\hat{\boldsymbol{Z}}) + (\hat{\boldsymbol{Z}}^{\dagger})(\boldsymbol{H}^{\dagger}\boldsymbol{H})(\hat{\boldsymbol{Z}}) \right)$$

$$= \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix}_{\langle 3x3 \rangle}$$

計算對應 EDI 檔案中的 ZXXR、ZXXI、ZXX.VAR、ZXYR、ZXYI、ZXY.VAR、ZYXR、ZYXI、ZYX.VAR、ZYYR、ZYYI、ZYY.VAR

已求得: $\hat{\boldsymbol{Z}} = \begin{bmatrix} \hat{Z}_{xx} & \hat{Z}_{yx} & \hat{T}_x \\ \hat{Z}_{xy} & \hat{Z}_{yy} & \hat{T}_y \end{bmatrix}_{\langle 2x3 \rangle}$ 、$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}_{\langle 2x2 \rangle}$ 、$\boldsymbol{N} = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix}_{\langle 3x3 \rangle}$

從程式碼中推測:

$\mathrm{ZXXR} = \Re\mathrm{e}(\hat{Z}_{xx})$

$\mathrm{ZXXI} = \Im\mathrm{m}(\hat{Z}_{xx})$

$\mathrm{ZXX.VAR} = \Re\mathrm{e}(N_{11}S_{11})$

$\mathrm{ZXYR} = \Re\mathrm{e}(\hat{Z}_{xy})$

$\mathrm{ZXYI} = \Im\mathrm{m}(\hat{Z}_{xy})$

$\mathrm{ZXY.VAR} = \Re\mathrm{e}(N_{11}S_{22})$

$\mathrm{ZYXR} = \Re\mathrm{e}(\hat{Z}_{yx})$

$\mathrm{ZYXI} = \Im\mathrm{m}(\hat{Z}_{yx})$

$\mathrm{ZYX.VAR} = \Re\mathrm{e}(N_{22}S_{11})$

$\mathrm{ZYYR} = \Re\mathrm{e}(\hat{Z}_{yx})$

$\mathrm{ZYYI} = \Im\mathrm{m}(\hat{Z}_{yx})$

$\mathrm{ZYY.VAR} = \Re\mathrm{e}(N_{22}S_{22})$

$$\mathrm{TFVar} = \begin{bmatrix} \mathrm{TFVar}(1,1) & \mathrm{TFVar}(1,2) \\ \mathrm{TFVar}(2,1) & \mathrm{TFVar}(2,2) \end{bmatrix} = \begin{bmatrix} \mathrm{Var}(\mathrm{Z}_{xx}) & \mathrm{Var}(\mathrm{Z}_{xy}) \\ \mathrm{Var}(\mathrm{Z}_{yx}) & \mathrm{Var}(\mathrm{Z}_{yy}) \end{bmatrix}$$

$$\begin{aligned}
\mathrm{Var}(\mathrm{Z}_{xx}) &= \mathrm{TFVar}(1,1) = \mathrm{ResidCov}(1,1) * InvSigCov(1,1) \\
\mathrm{Var}(\mathrm{Z}_{xy}) &= \mathrm{TFVar}(1,2) = \mathrm{ResidCov}(1,1) * InvSigCov(2,2) \\
\mathrm{Var}(\mathrm{Z}_{yx}) &= \mathrm{TFVar}(2,1) = \mathrm{ResidCov}(2,2) * InvSigCov(1,1) \\
\mathrm{Var}(\mathrm{Z}_{yy}) &= \mathrm{TFVar}(2,2) = \mathrm{ResidCov}(2,2) * InvSigCov(2,2)
\end{aligned}$$

$$= \begin{bmatrix} E^*_{x,\#1} & E^*_{x,\#2} & E^*_{x,\#3} & \cdots & E^*_{x,\#n} \\ E^*_{y,\#1} & E^*_{y,\#2} & E^*_{y,\#3} & \cdots & E^*_{y,\#n} \\ H^*_{z,\#1} & H^*_{z,\#2} & H^*_{z,\#3} & \cdots & H^*_{z,\#n} \end{bmatrix}_{\langle 3\mathrm{xn}\rangle} \begin{bmatrix} E_{x,\#1} & E_{y,\#1} & H_{z,\#1} \\ E_{x,\#2} & E_{y,\#2} & H_{z,\#1} \\ E_{x,\#3} & E_{y,\#3} & H_{z,\#1} \\ \vdots & \vdots & \vdots \\ E_{x,\#n} & E_{y,\#n} & H_{z,\#1} \end{bmatrix}_{\langle \mathrm{nx}3\rangle}$$

$$= \begin{bmatrix} \langle E^*_x E_x \rangle & \langle E^*_x E_y \rangle & \langle E^*_x H_z \rangle \\ \langle E^*_y E_x \rangle & \langle E^*_y E_y \rangle & \langle E^*_y H_z \rangle \\ \langle H^*_z E_x \rangle & \langle H^*_z E_y \rangle & \langle H^*_z H_z \rangle \end{bmatrix}$$