states, $a_{X_1Y_2}$, $a_{X_2Y_3}$, $a_{X_3Y_1}$; and twelve probabilities of transitions to the *end* state, a_{X_10}, a_{X_20}, a_{X_30}.

Note that some transitions between hidden states are impossible:

$$a_{X_1Y_1} = a_{X_2Y_2} = a_{X_3Y_3} = a_{X_1Y_3} = a_{X_2Y_1} = a_{X_3Y_2} = 0.$$

These zero value parameters are not included in the above counts. □

Problem 3.13 **Prove that $P(x) = \prod_{j=1}^{L} s_j$ if scaling variables s_j are defined by the following equations:**

$$\tilde{f}_l(i) = \frac{f_l(i)}{\prod_{j=1}^{i} s_j}, \sum_l \tilde{f}_l(i) = 1.$$

Solution With the scaling variables s_i, $i = 1, \ldots, L$, chosen to satisfy equations $\sum_l \tilde{f}_l(i) = 1$ for any i, the forward algorithm equation for the probability $P(x)$ of sequence $x = (x_1, \ldots, x_L)$ becomes

$$P(x) = \sum_k f_k(L) a_{k0} = \left(\sum_k \tilde{f}_k(L) a_{k0}\right) \prod_{j=1}^{L} s_j = \prod_{j=1}^{L} s_j.$$

Here the probabilities of transition from any state k to the end state, a_{k0}, are equal to 1 at position L, thus all sequences have a fixed length L.

For such a choice of s_i, the following recurrent equations hold:

$$\tilde{f}_l(i+1) = \frac{1}{s_{i+1}} e_l(x_{i+1}) \sum_k \tilde{f}_k(i) a_{kl},$$

$$s_{i+1} = \sum_l e_l(x_{i+1}) \sum_k \tilde{f}_k(i) a_{kl}.$$

For the backward variables rescaled with the same set of s_j, the recurrence equation becomes

$$\tilde{b}_k(i) = \frac{1}{s_i} \sum_l a_{kl} \tilde{b}_l(i+1) e_l(x_{i+1}).$$

Use of the rescaled variables $\tilde{f}_l(i)$ ($\tilde{b}_k(i)$) allows us to avoid an underflow error when running the forward (the backward) algorithm for long sequences. □