Figure 1: The Gauss Digitization $\mathtt{D}_h(X)$ of and Euclidean shape $X$ (in gray) is the set of black points that lies within $X$. The set of surfels of the Gauss Digitization is pictured in red.
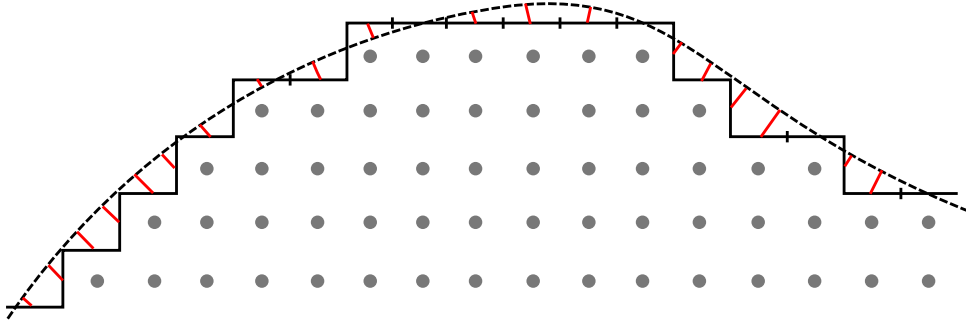


Figure 2: For a given linel $l$, the projection map $\xi$ associates the centroid of $l$ to the closest point on $X$.

# TP7: Curvature estimation of digital curves

We consider a compact shape $X \in \mathbb{R}^2$. The Gauss Digitization of $X$, namely $\mathtt{D}_h(X)$, at a grid step $h$ is:

$$\mathtt{D}_h(X) := X \cap (h\mathbb{Z})^2,$$

which is simply the set of points of the infinite regular grid of size $h$ that are inside $X$ (see Fig.1). The discrete border of $\mathtt{D}_h(X)$ is roughly defined as the border of the shifted $h + h/2$-grid. In 2D, elements of dimensions 1 are called linels and elements of dimension 0 are called pointels (they correspond to the center of the original $h$-grid). The set of linels is denoted $\mathbb{E}^1$ and the set of pointels $\mathbb{E}^0$ for convenience.

For each elements of $l \in \mathbb{E}^1$ we call the projection of $l$ the closest point of the centroid of $l$ on $X$ (see Fig.2). The associated map is denoted $\xi$ and is called the projection map between $X$ and $\mathtt{D}_h(X)$.
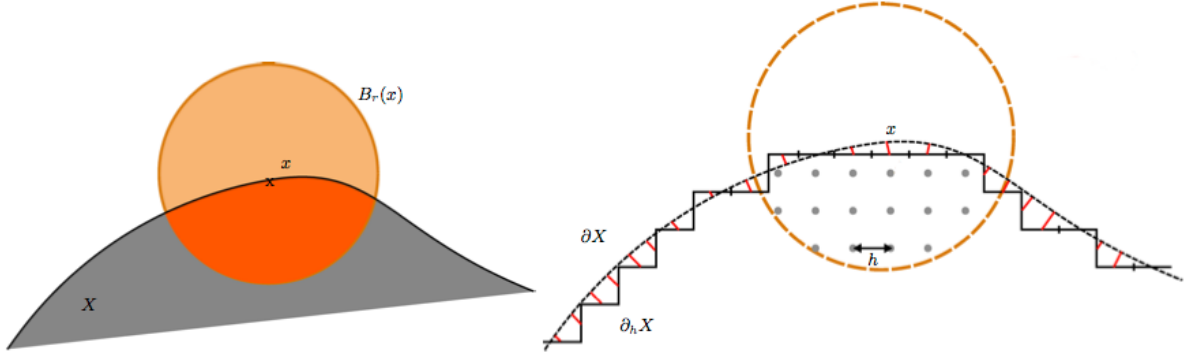
Figure 3: The volumetric integral (on the left) $V_R(x)$ is the area of the intersection between an euclidean ball $B_r(x)$ centered in $x$ and the shape $X$.

**Goal**  Compute a convergent estimator $\tilde{\kappa}$ of the real curvature $\kappa$ at a point $x \in X$ using only the discrete structure. More precisely we want that $\forall \mathbf{x} \in \mathbb{E}^{\mathbf{1}}$

$$||\kappa(\xi(\dot{\mathbf{x}})) - \tilde{\kappa}(\dot{\mathbf{x}})||_\infty \leq \sigma(h)$$

where the limit of $\sigma$ is zero as $h$ tends to zero.

## Integral Invariant Estimator

Given an euclidean ball $B_r(x)$ of radius $r$ centered in $x$, we define the volumetric integral $V_r(x)$ (see Fig.3) as

$$V_r(x) := \int_{B_r(x)} \mathcal{X}(p) dp$$

where $\mathcal{X}$ is the characteristic function of $X$ (it is equal to one if $x$ is in $X$, zero otherwise).

We have

$$\kappa(x) \approx \frac{3\pi}{2r} - \frac{3V_r(x)}{r^3}$$

that links the curvature $\kappa$ and the volumetric integral $V_r$ at a point $x$. If we are able to compute the volumetric integral at a point $x$, we can estimate the curvature at the same point.

An existing result for digital area approximation is

$$\text{Area}(\mathtt{D}_h(X)) := h^2 \text{Card}(\mathtt{D}_h(X)) = \text{Area}(X) + O(h)$$

where Card is simply the number of digital points within $\mathtt{D}_h(X)$. Therefore, to estimate $V_r(x)$ we can count the number of digital points in the intersection between $B_r(x)$ and $\mathtt{D}_h(X)$.

## Assignment

You have to implement the integral invariant curvature estimator $\tilde{\kappa}$ on 2D digital curves. To do so, have a look at the file generate_digital_contour.cpp in DGtalSkel. This program computes the Gauss Digitization $\mathtt{D}_h(X)$ (where $X$ can be here either a Ball, a Flower or and Accelerated Flower), extracts its digital border and finally gives the real curvature by projecting digital points onto $X$.

**Question** Given a surfel $\mathbf{s}$, implement the estimator of the volumetric integral (it must be parametrized by the ball radius $r$)

**Question** Implement the curvature estimator $\tilde{\kappa}$. For each linels $l$, compute the mean of the volumetric integral estimator of the two adjacent surfels.

**Question** Plot (using gnuplot for example) the maximum error between the real curvature $\kappa$ and the estimated one $\tilde{\kappa}$ with a decreasing grid-step $h$.