Introduction - Differential invariance

This work deals with various aspects of the computation of differential invariants on manifolds. The work will take the form of a cumulative thesis meaning that chapter 2, chapter 3 and chapter 4 are published papers with references collected at the end. We dedicate this introduction to establishing the common ground for these articles as well as provide context and motivation for the research and further investigation.
Firstly chapter 2 is devoted to the systematic study of invariant connections on isotropy irreducible homogeneous manifolds and the last two chapters will deal with normal forms of Lie algebra actions on contact and Euclidean space. In the introduction we begin by stating what these subjects have in common. Then we present the reader with the background needed to understand the material. In particular for chapter 3 and chapter 4 we present the application of these results to the geometric theory of differential equations.

First section 1.1 will be a philosophical introduction to the notion of invariants in geometry and will bridge the gap between the results found in the following chapters.

Next section 1.2 will present elementary notions in the classification and study of Lie algebras and their representation. This material will be crucial to chapter 2 and will be the foundation to the authors contribution to the article. These notions however also tie in to chapter 3 and chapter 4 as the fundamental object of study is a Lie algebra representation.

Then in section 1.3 we introduce the notion of a connection, namely the fundamental object of study in chapter 2. Thus we present the Levi-Civita connection as the unique torsion-free connection preserving the metric. Afterwards we focus in on homogeneous manifolds equipped with a metric and define the notion of the canonical connection on such spaces in order to motivate investigation into connections with torsion.

Next we switch our focus to the geometric theory of differential equations in section 1.4. Here we prove the Lie-Bianchi theorem which gives conditions on a symmetry algebra of a differential equation to produce a quadrature solution. Not only will this theorem provide us with an algebraic condition of solubility but also an algorithmic procedure to provide the solution. This will be the key motivation of chapter 3 and chapter 4. Next we will take a result obtained in chapter 3 and show how one computes the differential equations invariant under this action and how one constructs a quadrature from the provided symmetry algebra.

Finally in section 1.5 we present the background need in chapter 4 and give some context to the notion of normal forms. We begin by proving the so-called resonance conditions which provide necessary and sufficient conditions for the linearization of a euclidean vector field which has a non-zero first jet. Next we provide the reader with a classical result in the theory of normal forms in symplectic geometry namely the so-called Birkhoff normal forms. This will require some background in construction of Hamiltonian vector fields. However it will provide some context in terms of Hamiltonian mechanics and perturbation theory. Finally we introduce the odd dimensional sibling of symplectic geometry namely contact geometry. We give an introduction to contact geometry needed in chapter 3 in terms of the contact Hamiltonian, the Reeb vector field and the Lagrange bracket.

1.1 Introduction

Informally speaking given a mathematical object X equipped with the action of say a group G, we say that $x \in X$ is invariant under G if it remains unchanged under the action of G. For example the rigid motions of the plane maps the set of lines to itself.
The study of invariant objects has initiated multiple branches of mathematical disciplines, for instance homology, cohomology and Cartan geometry.

Invariants together with symmetries provide the building blocks for solving equivalence problems. Given objects $\mathcal{E}_1$ and $\mathcal{E}_2$ they can only be equivalent if their invariants are equal. Thus providing necessary conditions for the solubility given equivalence problem. For a through introduction on the topic of symmetries, equivalences and invariants we refer the reader to [Ol95].

We begin by introducing the notion of a scalar invariant. In this case our set is manifold M and our group G is a Lie group. Consider now the following case:
Let G be a Lie group acting on $\mathbb{R}^2$, then locally G acts on the space of graphs of the form $y = f(x)$. Then a k-nth order differential invariant I is a function

$$I\left(x, y, \frac{dy}{dx}, \dots, \frac{d^k y}{dx^k}\right)$$

invariant under the action of the group G.
Utilizing the language of jets one may formulate this as follows:
Let G be a Lie group. A differential invariant for G is a differential function

$$I : J^n \to \mathbb{R},$$

which satisfies

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)}),$$

for all $g \in G$ and all $(x, u^{(n)}) \in J^n$ where $g^{(n)} \cdot (x, u^{(n)})$ is defined.
It is easy to see how this construction generalizes to several variables.

It is well known that for ordinary and differential invariants that the following holds:
Given a set of (differential) invariants $I_1, \dots, I_n$ then for any smooth function H we have that $H(I_1, \dots, I_n)$ will also be invariant. This gives rise to the notion of a functionally independent invariant. That is an invariant which cannot be expressed as the function of another invariant. Key result in the construction of higher order functional invariants from lower order invariants was developed by Tresse in [Tr1], [Tr2].

Historically the modern notions of differential invariants dates back to the work of Sophus Lie. Lie was motivated by the study of differential equations and their intregrability, and published various works relating to transformation groups [LE88] [LE90] [LE93], differential invariants [LE76] and contact geometry [LS96]. How these notions tie together to form a theory of integrability of differential equations will be presented in a later section. However the ideas developed by Lie are fundamental to modern geometry and invariant theory.
The notion of invariance can be naturally extended to other geometric objects. For instance a vectorfield $X \in \Gamma(TM)$ is said to be invariant with respect to the action of a group $G : M \to M$ if

$$g_* X = X,$$

for every $g \in G$.
The infinitesmal equivalent of this statment can be expressed through the Lie algebra $\mathfrak{g} = T_e G$ of G as:

$$[X, Y] = 0$$

for every $Y \in \mathfrak{g}$.

In Riemannian geometry, given two manifolds $(M_1, g_1), (M_2, g_2)$ it is natural question to pose the following question:
Does there exist a map

$$\phi : M_1 \to M_2$$

such that $\phi^* g_1 = g_2$?
In dimension two this was well known by the work of Gauss, however the higher dimensional case was treated by Christoffel and some of these ideas will be presented later.
Further Cartan developed the so called *Cartan equivalence method*, which generalizes the answer to this question to several geometric structures and is considered one of Cartans major contributions to mathematics. However in order solve these types problems one needs to study the $G-$structure of the manifolds in question. That is given a manifold M with structure group G a G-structure is a G-sub-bundle of the frame bundle $FM \to M$. Cartans equivalence method can be thus considered as a method of moving frames. It is important to keep the notions of principle bundles and G-structures separated even though they are closely related.

In chapter 2 we work with the notion of a invariant affine connection ∇. The authours contribution is essentially a direct application of the following theorem(See [Wa]):

Theorem 1 (Wang). *Given a bundle $p : E \to B$, with structure group S and denote by J the isotropy subgroup. If G act transitively on the fibers of E, then there is a bijective correspondence between the G-invariant connection over E and linear maps $\psi : \mathfrak{g} \to \mathfrak{s}$ satisfying:*

$$\psi \circ \operatorname{Ad} j = \operatorname{Ad} \psi(j) \circ \phi, \quad j \in J,$$
$$\psi(\bar{j}) = \phi(\bar{j}), \quad \bar{j} \in \mathfrak{j},$$

where ϕ denotes the natural homomorphism $\phi : J \to S$ as well as the induced homomorphism of Lie algebras $\phi : \mathfrak{j} \to \mathfrak{s}$.

This reduces the problem of finding G-invariant connections to a problem in representation theory. Thus we will go through the necessary theory of the representation of Lie algebras. Furthermore we will recall some basics on the theory of homogeneous Riemannian manifold.
In chapter 3 and chapter 4 we find the normal forms of Lie algebras of vector fields on contact and Euclidean spaces respectively. We will present the theory needed in contact geometry and in the so called resonance conditions for linearization of a singular vector field as well as some results regarding normal forms in Symplectic geometry. Finally we motivate further development into this area by directly applying the Lie-Bianchi theorem to give a solution to certain classes of second order differential equations.

1.2 The representations of Lie algebras

Our goal in this section is to acquaint the reader with the following topics

- Classification of semi-simple Lie algebras,
- Representations of Lie algebras,
- The Weyl dimensional formula,
- Real structures in representation theory.

We assume the reader is familiar with basic notions in the theory of Lie groups and Lie algebras. A vast amount of literature exists on the topic presented here. See for instance [Ki17] [ČS] [Hal]. The material presented here was primarily gathered from [ČS].

1.2.1 Basic structure theory

A representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a homomorphism:

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V),$$

of Lie algebras.
Equivalently one can view a representation as a billinear map:

$$\rho : \mathfrak{g} \times V \to V$$

such that

$$\rho([X,Y],v) = \rho(X,\rho(Y,v)) - \rho(Y,\rho(X,v)).$$

From now on we will denote a representation as $X \cdot v$ or Xv instead of $\rho(X)(v)$.
Given a Lie algebra $\mathfrak{g}$ then by a complex representation, one simply means a homomorphism from $\mathfrak{g}$ to the Lie algebra of complex linear maps.
Consider now two representations

$$\rho_1, \rho_2 : \mathfrak{g} \to \mathfrak{gl}(V).$$

Then a morphism from ρ_1 to ρ_2 is a linear map $\phi : V \to V'$ which is compatible with $\mathfrak{g}$. This is expressed explicitly as

$$\phi(\rho_1(X)(v)) = \rho_2(X)(\phi(v)),$$
$$\phi(Xv) = X \cdot \phi(v).$$

Such morphisms are referred to as intertwining operators or equivariant maps.

An isomorphism of representations is a bijective morphism between two representations.

Naturally if $\phi : V \to W$ is an isomorphism then $\phi^{-1} : W \to V$ is an isomorphism also.
We now introduce a very important represntation, namely the adjoint representation:

$$\mathrm{ad}(X,Y) := [X,Y].$$

The Jacobi identity states that this is a representation of $\mathfrak{g}$.

1.2.2 The Killing form

Given a finite dimensional representation of a Lie algebra $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. We define a billinear form

$$\begin{gathered} B_\rho = B_V : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K} \\ B_\rho(X,Y) := tr(\rho(X) \circ \rho(Y)) \end{gathered} \tag{1.2.1}$$

Proposition 1. *This form is symmetric and invariant, and we call it the trace form.*

When we let ρ be the adjoint representation we call this construction the Killing form. In this case we have the following relation for $\phi : \mathfrak{g} \to \mathfrak{g}$

$$B(\phi(X), \phi(Y)) = B(X,Y).$$

This means that the killing form is invariant under automorphisms of $\mathfrak{g}$. We now state the following fundamental lemma

Lemma 1. *Let V be a finite-dimensional complex vector space and $X \in \mathfrak{gl}(V)$ be a linear mapping with Jordan decomposition*

$$X = X_s \oplus X_n.$$

Then the Jordan decomposition of

$$\mathrm{ad}(X) : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$$

is given by $\mathrm{ad}(X) = \mathrm{ad}(X_s) + \mathrm{ad}(X_n)$.

Theorem 1.

- *Let V be a vector space and $\mathfrak{g} \subset \mathfrak{gl}(V)$ a Lie subalgebra. If $B_V = 0$ on $\mathfrak{g}$, then $\mathfrak{g}$ is solvable.*
- *A Lie algebra $\mathfrak{g}$ is solvable if and only if its Killing for has the property that $B(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$*
- *A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is non-degenerate.*

From this we state the following corollary:

Corollary 1.

- *If $\mathfrak{g}$ is a semisimple Lie algebra, then there are simple ideals $\mathfrak{g}_1, \dots \mathfrak{g}_k \in \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$ as a Lie algebra. Moreover, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and any ideal in $\mathfrak{g}$ as well as any homomorphic image of $\mathfrak{g}$ is semisimple*

- *A real Lie algebra $\mathfrak{g}$ is semisimple if and only if its complexification $\mathfrak{g}_{\mathbb{C}}$ is semisimple.*

- *If $\mathfrak{g}$ is a complex simple Lie algebra and $\Phi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a $\mathfrak{g}$-invariant complex billinear form, then Φ is a multiple of the Killing form. In particular, if Φ is nonzero, then it is automatically symmetric and non-degenerate.*

The center $\mathfrak{z}$ of a Lie algebra $\mathfrak{g}$ is defined as:

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} | [X, Y] = 0 \; \forall \; Y \in \mathfrak{g}\}.$$

A derivation of a Lie algebra $\mathfrak{g}$ is a linear map $D : \mathfrak{g} \to \mathfrak{g}$

$$D([X, Y]) = [D(X), Y] + [X, D(Y)],$$

for every $X, Y \in \mathfrak{g}$.

The Jacobi identity tells the following: Given $X \in \mathfrak{g}$ the map $\mathrm{ad}(X)$ is a derivation of $\mathfrak{g}$. We call derevations of this form the inner derivations of $\mathfrak{g}$.
Having two derivations, their bracket is also a derivation. So $\mathfrak{der}(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$. Moreover we have that the inner derivations form an ideal $\mathrm{ad}(\mathfrak{g})$ in $\mathfrak{der}(\mathfrak{g})$

Corollary 2. *For a Lie algebra $\mathfrak{g}$, the following conditions are equivalent*

- *$\mathfrak{g}$ is reductive,*

- *$[\mathfrak{g}, \mathfrak{g}]$ is semisimple and $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ as a Lie algebra,*

- *the adjoint representation of $\mathfrak{g}$ is semisimple.*

For a semisimple Lie algebra $\mathfrak{g}$ the adjoint action is an isomorphism of Lie algebras from $\mathfrak{g}$ onto $\mathfrak{der}(\mathfrak{g})$. In particular any derivation of a semisimple Lie algebra is inner.

1.2.3 Jordan decomposition and Cartan subalgebra

From earlier we have seen the Jordan decomposition $f = f_{ss} + f_n$ of an endomorphism of a complex vector space. Now we present the following lemma

Lemma 2. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra*

- *Suppose that $\mathfrak{g}$ is a complex Lie subalgebra of $\mathfrak{gl}(V)$ for a finite-dimensional complex vector space V. For $X \in \mathfrak{g}$ let $X = S + N$ be the Jordan decomposition of $X : V \to V$. Then $S, N \in \mathfrak{g}$*

- *For any $\mathfrak{g}$ there are unique elements $X_s, X_n \in \mathfrak{g}$ such that $\mathrm{ad}(X) = \mathrm{ad}(X_s) + \mathrm{ad}(X_n)$ is the Jordan decomposition of $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$*

- *If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite-dimensional complex representation, then for any element $X \in \mathfrak{g}$ the Jordan decomposition of $\rho(X) : V \to V$ is given by $\rho(X) = \rho(X_s) + \rho(X_n)$*

This notion is fundamental to understand representations of semisimple Lie algebras. Moreover we call an element $X \in \mathfrak{g}$ semisimple if the linear map $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable.
Let $X \in \mathfrak{g}$ be semisimple and $\mathfrak{g} = \bigoplus_\lambda \mathfrak{g}_\lambda(X)$ be the decomposition into the eigenspaces for $\mathrm{ad}(X)$. By the Jacobi identity this defines a subalgebra, moreover for any finite-dimensional complex representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ the map $\rho(X) : V \to V$ is diagonalizable. Thus we have a decomposition

$$V = \bigoplus_\mu V_\mu$$

for the linear map $\rho(X)$.

Utilizing this and the basic fact from linear algebra that two commuting diagonalizable matrices are simultaneously diagonalizable we motivate the following definition:
A Cartan subalgebra of a complex semisimple Lie algebra is a maximal commutative subalgebra which consists of semisimple elements.
In order to state the fundamental theorem related to the Cartan subalgebra we define the notion of a regular element.
Given an element $H \in \mathfrak{g}$ we define the centralizer of H in $\mathfrak{g}$ by:

$$\mathfrak{c}(H) := \{X \in \mathfrak{g} | [X, H] = 0\}.$$

It is evident that this is a Lie subalgebra of $\mathfrak{g}$. The minimal dimension of $\mathfrak{c}(H)$ as H varies of all the semi simple elements is called the rank of $\mathfrak{g}$. The semisimple elements for which this minimal value is attained are called regular. We now provide the following theorem;

Theorem 2. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra.*

- *If $H \in \mathfrak{g}$ is a regular semisimple element, then $\mathfrak{c}(H) \leq \mathfrak{g}$ is a Cartan subalgebra,*
- *Any two Cartan subalgebras in $\mathfrak{g}$ are conjugate by an inner automorphisms of $\mathfrak{g}$.*

Thus for a fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and a complex representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, we consider a linear functional:

$$\lambda : \mathfrak{h} \to \mathbb{C}.$$

The functionals correspondingsponding to nontrivial eigenspaces are called weights of the representation V. A functional $\lambda : \mathfrak{h} \to \mathbb{C}$ of V is a weight if and only if there is a nonzero vector $v \in V$ such that

$$\rho(H)(v) = \lambda(H)v,$$

for every $H \in \mathfrak{h}$.
If λ is a weight, then the weight space V_λ corresponding to λ is defined by:

$$V_\lambda := \{v \in V | \rho(H)(v) = \lambda(H)v \ \forall H \in \mathfrak{h}\}.$$

We focus on the case of the adjoint representation. The non-zero weights of the adjoint representation are called the roots of the Lie algebra $\mathfrak{g}$. As above the weight space corresponding to a root is called the root space.
We will denote the set of all roots of $\mathfrak{g}$ by Δ. The weight space corresponding to the weight zero is exactly the Cartan subalgebra so we obtain the root decomposition.

$$\mathfrak{g} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha. \tag{1.2.2}$$

The Jacobi identity gives us that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta},$$

for $\alpha, \beta \in \Delta$.
We study the roots of general complex semisimple Lie algebras, however to truly understand this we must study the irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$. For this Lie algebra we can write out the root decomposition and define the so called standard generators:

$$E := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ F := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \ H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The investigation of this can be summarized in the following proposition:

Proposition 2. *For any $n \in \mathbb{N}$, there is a unique(up to isomorphism) irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ of dimension $n + 1$. The weights of this representation are $\{n, n-2, \dots, -n+2, -n\}$ and all weight spaces are one-dimensional. More precisely there is a basis $\{v_0, \dots, v_n\}$ of V such that*

$$\begin{aligned} Hv_j &= (n - 2j)v_j \\ Fv_j &= v_{j+1} \ \text{for } j < n \\ Ev_j &= j(n - j + 1)v_{j-1} \ \forall j. \end{aligned}$$

1.2.4 The root system of a semisimple Lie algebra

Consider now a general semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ and the corresponding root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

Consider now the Killing form B from (1.2.1), then we have that $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\beta = -\alpha$. Thus non-degeneracy of the Killing form implies that if $\alpha \in \Delta$ then $-\alpha \in \Delta$, and the Killing form induces a duality between the root spaces. Moreover the restriction of the Killing form to $\mathfrak{h}$ is non degenerate.
In particular for each linear functional $\lambda \in \mathfrak{h}^*$ there exists a unique element $H_\lambda \in \mathfrak{h}$ such that

$$\lambda(H) = B(H_\lambda, H),$$

for every $H \in \mathfrak{h}$.
One can also use the Killing form to define a non-degenerate complex billinear form on $\mathfrak{h}^*$ by:

$$\langle \lambda, \mu\rangle := B(H_\lambda, H_\mu).$$

Summarizing we have the following proposition.

Proposition 3. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \leq \mathfrak{g}$ a Cartan subalgebra, $\Delta \subset \mathfrak{h}^*$ the corresponding set of roots, and $\langle , \rangle$ the complex billinear form on $\mathfrak{h}^*$ induced by the Killing form, then we have:*

- *For any $\alpha \in \Delta$, we have that $-\alpha \in \Delta$ and these are the only complex multiples of α which are roots.*
- *For any $\alpha \in \Delta$, the root space $\mathfrak{g}_\alpha$ is one dimensional and the subspace $\mathfrak{s}$ of $\mathfrak{g}$ spanned by $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ is a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.*
- *For $\alpha, \beta \in \Delta$ where $\beta \neq \alpha$ we have that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta \in \Delta$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ otherwise.*
- *For $\alpha, \beta \in \Delta$ where $\beta \neq \pm\alpha$ and $z \in \mathbb{C}$ a functional of the form $\beta + z\alpha$ can only be a root if $x \in \mathbb{Z}$. The roots of this form are an unbroken string:*

  $$\beta - p\alpha, \beta - (p-1)\alpha, \ldots, \beta + (q-1)\alpha, \beta + q\alpha,$$

  where $p, q \geq 0$ and

  $$p - q = \frac{2\langle\beta, \alpha\rangle}{\langle\alpha, \alpha\rangle}.$$

Let now $V \subset \mathfrak{h}^*$ be the real span of the set Δ of roots and let $\mathfrak{h}_0 \subset \mathfrak{h}$ be the real subspace spanned by the elements H_α for $\alpha \in \Delta$. Necessarily one can restrict the Killing form B to $\mathfrak{h}_0$ and the complex billinear form $\langle , \rangle$ to V. We claim that the $\mathfrak{h}_0$ is a real form on $\mathfrak{h}$, its dual is exactly V, and the restriction of the forms are positive definite.
By definition:

$$\langle\alpha, \alpha\rangle = B(H_\alpha, H_\alpha) = tr(ad(H_\alpha) \circ ad(H_\alpha)).$$

Now $\mathrm{ad}(H_\alpha)$ acts trivially on $\mathfrak{h}$ and by multiplication of $\beta(H_\alpha)$ on $\mathfrak{g}_\beta$. Therefore

$$\langle\alpha, \alpha\rangle = \sum_{\beta\in\Delta} \beta(H_\alpha)^2.$$

By inserting $\beta(H_\alpha) = \langle\alpha, \beta\rangle$ we obtain

$$\langle\alpha, \alpha\rangle = 2\langle\alpha, \alpha\rangle^2 + \sum_{\beta\in\Delta:\beta\neq\pm\alpha} \langle\beta, \alpha\rangle^2 \tag{1.2.3}$$

Dividing (1.2.3) by $\langle\alpha, \alpha\rangle^2$ we conclude that

$$\langle\alpha, \alpha\rangle = \alpha(H_\alpha) \in \mathbb{Q}.$$

This in turn implies that $\beta(H_\alpha) \in \mathbb{Q}$. Again using (1.2.3) shows that $\langle \alpha, \alpha \rangle > 0$ for all α. Thus the restriction of $\langle , \rangle$ to the subspace $\mathfrak{h}_0 \subset \mathfrak{h}$ spanned by H_α is positive definite. Moreover we see that the restriction of all roots to $\mathfrak{h}_0$ are real valued. Hence the real dimension of $\mathfrak{h}_0$ can be at most the complex dimension of $\mathfrak{h}$, but since the roots span $\mathfrak{h}^*$ it cannot be smaller than that either.
The final idea in this context is to consider the reflections of real Euclidean spaces $\mathfrak{h}_0^*$ in the hyper planes orthogonal to roots. For $\alpha \in \Delta$ the root reflection $s_\alpha : \mathfrak{h}_0^* \to \mathfrak{h}_0^*$ on the hyper plane $\alpha^\perp$ is given by $s_\alpha(\phi) = \phi - \frac{2\langle \phi, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$. As a reflection on a hyper plane this is an orthogonal mapping of determinant -1, and $s_\alpha(\alpha) = -\alpha \in \Delta$. For a root $\beta \neq \pm\alpha$ we have that $s_\alpha(\beta) = \beta - (p-q)\alpha$, where the α-string through β has the form

$$\mathfrak{g}_{\beta - p\alpha} \oplus \ldots \oplus \mathfrak{g}_{\beta + q\alpha}$$

By the previous proposition. Since $-p \leq q - p \leq q$, we conclude $s_\alpha(\beta) \in \Delta$. Summarizing we have the theorem.

Theorem 3. *Let $\mathfrak{g}$ be a semisimple Lie algebra of rank r, let $\mathfrak{h} \leq \mathfrak{g}$ be a Cartan subalgebra, $\Delta \subset \mathfrak{h}^*$ be the corresponding set of roots, $\mathfrak{h}_0^* \subset \mathfrak{h}^*$ the real span of Δ and let $\langle , \rangle$ be the complex billinear form on $\mathfrak{h}^*$ induced by the Killing form.*
Then

- *$\mathfrak{h}_0$ has real dimension r and the restriction of $\langle , \rangle$ to $\mathfrak{h}_0^*$ is positive definite,*
- *The subset $\Delta \subset \mathfrak{h}_0^*$ spans the space $\mathfrak{h}_0^*$, any root reflection s_α for $\alpha \in \Delta$ maps Δ to itself for $\alpha, \beta \in \Delta$ we have $\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$,*
- *For $\alpha \in \Delta$ we have that $2\alpha \notin \Delta$.*

A finite subset Δ in a Euclidean vector space which satisfies the second condition of the above theorem is called an abstract root system If it satisfies the final condition we say the root system is reduced. Finally we mention that there is an obvious notion of direct sums of abstract root system. By taking the orthogonal direct sum of the surrounding Euclidean spaces and the union of the abstract root systems. An abstract root system is called irreducible if and only if it does not decompose as a direct sum in a nontrivial way. It can be shown that $\mathfrak{g}$ is simple if and only if its root system is irreducible.

1.2.5 Dynkin diagrams

In this section we disregard Lie algebra structure for a while and keep in mind our goal to the description of abstract root systems, namely Dynkin diagrams. This will mainly revolve around notions in Euclidean geometry.
Since an abstract root system Δ on a Euclidean vector space V is symmetric under root reflection, it follows that $\alpha \in \Delta$ implies $-\alpha \in \Delta$. Thus a natural idea is to split the set of roots into positive and negative parts. We choose a stronger concept namely the notion of positivity on our space i.e. a subset:

$$V^+ \subset V/\{0\},$$

such that V is a disjoint union of $V^+, \{0\}$ and $-V^+$. Moreover we require that V^+ is stable under addition and multiplication by positive scalars.
Having chosen such a decomposition, we get a total ordering on V defined by $v \leq w$ if and only if $v = w$ or $w - v \in V^+$.
The simplest way to get such an ordering is to choose a basis $\phi_1, \ldots, \phi_n$ for the dual space V^* and define $v \in V^+$ if and only if there is an index j such that $\phi_i(v) = 0$ for $i < j$ and $\phi_j(v) > 0$.
Having chosen V^+ we define the set of positive roots as $\Delta^+ \subset \Delta$ by $\Delta^+ = \Delta \cap V^+$. Thus we can write $\alpha > 0$ to indicate that the root α is positive. Furthermore we can define $\Delta^0 \subset \Delta^+$ of simple roots as the set of those positive roots, which cannot be written as the sum of two positive roots. It can be shown that the simple roots form a basis of V, and that any root can be written as an integral linear combination of simple roots, in which the coefficients are either all positive or all negative. In particular defining the root lattice Λ_R to be the set of all integral linear combinations of roots, we see that $\Lambda_R \cong \mathbb{Z}^{\dim(V)}$.
Given $\alpha, \beta \in \Delta$, the Cauchy-Schwartz inequality shows that

$$\left| \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \frac{2\langle \beta, \alpha \rangle}{\langle \beta, \beta \rangle} \right| \leq 4,$$

and equality is achieved when α, β are proportional.
Hence, each of the integers $\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ lies between $-4, 4$. and for $\alpha \neq \beta$ and reduced root systems ± 4 cannot occur. Moreover either $\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ or $\frac{\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle}$ must be 0 or ± 1. For $\alpha, \beta \in \Delta^0$ we clearly have that $\alpha - \beta \notin \Delta$ and thus the inner product $\langle\alpha, \beta\rangle \leq 0$.
Finally one can show also that in the case of reduced abstract root systems that for $\alpha, \beta \in \Delta$ with $\beta \neq \pm\alpha$ the elements of Δ of the form $\beta + n\alpha$ form an unbroken string of the form $\beta - p\alpha, \ldots, \beta + q\alpha$ with $p - q = \frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}$. Replacing β with $\beta - p\alpha$ we conclude that such string must have length at most four.
Fix now an abstract root system Δ on V and a subset $V^+ \subset V$ and let $\Delta^0 = \{\alpha_1, \ldots, \alpha_l\}$ be the corresponding set of simple roots. Then the Cartan matrix of Δ is the $l \times l$-matrix $A = (a_{ij})$ defined by $a_{ij} = \frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_i,\alpha_i\rangle}$.

Obviously this matrix depends on the ordering of the simple roots, so it at best unique up to conjugation and permutation matrices. We already know that all a_{ij} are integers, the diagonal entries are 2, while the off-diagonal ones are greater than zero. Obviously we also have that if $a_{ij} = 0$ then $a_{ji} = 0$. Finally one shows that A is symmetrizable i.e. there is a diagonal matrix D with positive entries such that DAD^{-1} is symmetric and positive definite. Matrices with this property is called abstract Cartan matrices.
The final step now is to associate to every Cartan matrix a type of graph called its Dynkin Diagram by taking a vertex for its simple root, and then joins the i-th and j-th vertex by $a_{ij}a_{ji}$ many lines. Otherwise put the number of edges joining the vertex corresponding to two simple roots equal to 4 times the square of the cosine of the angle between the roots. If two roots are of different length then in addition one orients the edge by an arrow pointing from the longer to the shorter root.

1.2.6 The Weyl group

We now discuss a method of reducing the classification of finite-dimensional simple Lie algebra to the classification of Dynkin diagrams. Up until now we have shown how to associate to a semisimple Lie algebra a Dynkin diagram by making two choices. Firstly we have chosen a Cartan subalgebra $\mathfrak{h} \leq \mathfrak{g}$, and secondly we choose $\Delta^+ \subset \Delta$ of positive roots.

The independence of choice of Cartan subalgebra follows from the fact that any two Cartan subalgebras are conjugate under inner automorphisms. To show Independence's on the ordering on the roots we introduce the notion of a Weyl group W of an abstract root system.

Given a complex semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$, the Weyl group of the corresponding abstract root system is given by $W = W(\mathfrak{g}, \mathfrak{h})$.

If $\Delta \subset V$ is an abstract root system , then for any $\alpha \in \Delta$ we have the root reflection $s_\alpha : V \to V$, and we know from earlier that $s_\alpha(\Delta) \subset \Delta$. Thus we may now define the Weyl group $W = W(\Delta)$ of Δ to be the subgroup of the orthogonal group $O(V)$ generated by all the reflections s_α.

Then any element $w \in W$ maps the root system Δ to itself. Moreover if $w \in W$ fixes all elements of Δ, then w must be the identity map, since Δ spans V. We define the sign of an element $w \in W$ as the determinant of w, viewed as a linear automorphisms of V. Since w is an orthogonal map, the sign is either 1 or -1, depending on whether w is a product of even or odd number of reflections.

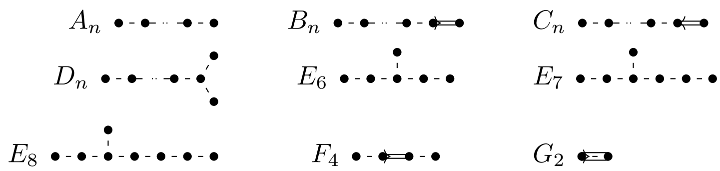
Proposition 4. *Let Δ be an abstract root system and let $\Delta^0 = \{\alpha_1, \ldots, \alpha_n\}$ be a simple subsystem.*

- *The Weyl group W of Δ is generated by the reflections s_{α_i} corresponding to simple roots,*
- *Mapping w to $w(\Delta^0)$ induces a bijection between W and the family of all simple subsystems of Δ.*

This implies that the Cartan matrix and hence the Dynkin diagram is independent of order. We skip the investigation of dominant elements and Weyl chambers and give the main result regarding the classification of Dynkin diagrams:

Theorem 4. *Any irreducible root system is isomorphic to exactly one of the systems:* $A_n(n \geq 1), B_n(n \geq 2), C_n(n \geq 3), D_n(n \geq 4), E_6, E_7, E_8, F_4$ *or* G_2,

Here we give the Dykin diagrams of semi-simple Lie algebras:



Thus we have show how to pass from a Lie algebra to a root system and from a root system further to a Dynkin diagram. Moreover we have seen that the diagram does not depend on the choice of a Cartan subalgebra and a set of positive roots. Thus any isomorphic Lie algebra leads to the same Dynkin diagram. The only question remaining is weather there exists simple Lie algebras corresponding to the root systems of type G_2, F_4 and the family E. The answer to this question is given by the so called Serre relations which is treated in detail in [ČS].

1.2.7 Finite-dimensional representations

Given a fixed complex simple Lie algebra $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h} \leq \mathfrak{g}$ and let Δ be the corresponding set of roots. Fix now and order on $\mathfrak{h}^*$ and let Δ^+ and Δ^0 be the corresponding set of positive and simple roots respectively. By $\mathfrak{h}_0^* \subset \mathfrak{h}^*$ we denote the real span of the roots. Let $\langle , \rangle$ be the complex billinear form on $\mathfrak{h}^*$ induced by the Killing form. From earlier we have that the restriction of $\langle , \rangle$ to $\mathfrak{h}_0^*$ is positive definite. Moreover we know that any finite-dimensional representation admits a weight decomposition into the direct sum of joint eigenspaces for the actions of the elements of $\mathfrak{h}$. These are the weight spaces and the eigenvalues are called weights and are linear functionals on $\mathfrak{h}$.
Here we introduce a few notions on the weights of finite dimensional representations. A weight $\lambda \in \mathfrak{h}^*$ is called real if it lies in the subspace $\mathfrak{h}_0^*$ or equivalently if $\langle \lambda, \alpha \rangle \in \mathbb{R}$ for all $\alpha \in \Delta$, moreover a real weight called algebraically integral if

$$\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z},$$

for every $\alpha \in \Delta$.
Recall that simple roots form a complex basis for $\mathfrak{h}^*$ and a real basis for $\mathfrak{h}_0^*$. Writing $\Delta^0 = \{\alpha_1, \ldots, \alpha_n\}$ we define the elements $\omega_1, \ldots, \omega \in \mathfrak{h}_0^*$ by the relation:

$$\frac{2\langle \omega_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}.$$

These are called the fundamental weights corresponding to the simple system Δ^0. Moreover we call an elements $\lambda \in \mathfrak{h}_0^*$ dominant if $\langle \lambda, \alpha \rangle \geq 0$ for every j.
We are now ready to formulate the following lemma

Lemma 3.

- *A weight* $\lambda \in \mathfrak{h}^*$ *is algebraically integral if and only if*

$$\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z},$$

  holds for all simple roots $\alpha \in \Delta^0$

- *The fundamental weights* $\omega_1, \ldots \omega_n$ *form a real basis for* $\mathfrak{h}_0^*$ *and a complex basis for* $\mathfrak{h}^*$. *Given* $\lambda \in \mathfrak{h}^*$ *let* $\lambda = \sum_i \lambda_i \omega_i$ *be the expansion in this basis. Then* λ *is real, algebraically integral and dominant if and only if all* $\lambda_i \in \mathbb{R}$, $\lambda_i \in \mathbb{Z}$, $\lambda_i \geq 0$ *respectively for every* i.

Consider now the set Λ_W of all algebraically integral elements of $\mathfrak{h}_0^*$ forms a lattice in this space. This is called the weight lattice of $\mathfrak{g}$. Let now ρ be a complex representation of $\mathfrak{g}$ on a finite-dimensional vector space V. Let $\mathrm{wt}(V) \subset \mathfrak{h}^*$ be the set of weights and

$$V = \bigoplus_{\lambda \in \mathrm{wt}(V)} V_\lambda,$$

be the weight decomposition. The dimension of the weight space V_λ is called the multiplicity of the weight λ in V. We are now in position to formulate the following proposition

Proposition 5. *Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra and $\mathfrak{h} \leq \mathfrak{g}$ a Cartan subalgebra and Δ the corresponding set of roots. Fix a choice of positive roots Δ^+ with simple roots Δ^0, and let W be the Weyl group. Let V be a finite dimensional complex representation of $\mathfrak{g}$ with weight decomposition*

$$V = \bigoplus_{\lambda \in \mathrm{wt}(V)} V_\lambda.$$

Then we have

- *Any weight λ of V is algebraically integral and at least one weight of V is dominant*
- *For any weight $\lambda \in \mathrm{wt}(V)$ and any $w \in W$, then $w(\lambda) \in \mathrm{wt}(V)$ and the two weights have the same multiplicity in V*
- *Suppose that for each $\lambda \in \mathrm{wt}(V)$ and $\alpha \in \Delta$ the integer*

$$k := \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle},$$

  is positive. Then for every $l \in \{1, \ldots, k\}$ we have that $\lambda - l\alpha \in \mathrm{wt}(V)$ with at least the same multiplicity as λ

Consider now a representation V of $\mathfrak{g}$ and let $\mathfrak{h}$ be the Cartan subalgebra. Let the action of $\mathfrak{g}$ on V be such that any action of $\mathfrak{h}$ is diagonalizable. Thus V admits a decomposition into weight spaces. A highest weight vector in V is a weight vector $v \in V$ such that $X \cdot v = 0$ for any element X lying in a root space $\mathfrak{g}_\alpha$ with $\alpha \in \Delta^+$. Fix now the set $\{E_i, F_i, H_i : i = 1, \ldots, n\}$ of standard generators. Then $v \in V$ is a highest weight vector if and only if $E_i \cdot v = 0$ for every $i = 1, \ldots, n$

Theorem 5. *Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra. Let $\mathfrak{h} \leq \mathfrak{g}$ be a fixed Cartan subalgebra and let Δ be the corresponding set of roots. Fix and order on the real span $\mathfrak{h}_0^*$ of Δ, let Δ^+ and Δ^0 be the sets of positive and simple roots respectively and let $\{E_i, F_i, H_i : i = 1, \ldots, n\}$ be a standard set of generators. Let V be a representation of $\mathfrak{g}$ such that the elements of $\mathfrak{h}$ act simultaneously diagonalizable.*

- *For any highest weight vector $v \in V$ the elements of the form $F_{i_1} \cdots F_{i_l} \cdot v$ span an non decompose-able sub representation V' of V. Denoting by λ the weight of v, all weights occurring in V' have the form*

$$\lambda - \sum n_i \alpha_i,$$

  for $\alpha_i \in \Delta^0$ and non negative integers n_i and the weight space V'_λ is one-dimensional
- *Suppose further that V is finite dimensional. Then the weight of any highest vector is dominant and algebraically integral, and there exists at least one highest weight vector. In this case the sub module $V' \subset V$ from above is irreducible.*

We now state the main theorem regarding highest weights.

Theorem 6 (Theorem of the highest weights)**.** *If $\mathfrak{g}$ is a finite dimensional complex semisimple Lie algebra, then for any dominant algebraically integral weight $\lambda \in \mathfrak{h}_0^*$ there is a up to isomorphism unique finite-dimensional irreducible representation with highest weight λ.*

1.2.8 Formulas for multiplicities, character and dimensions

In this subsection we discuss various methods for extracting additional information about a irreducible representation of a complex semisimple Lie algebra $\mathfrak{g}$, as this will be of high importance in a later sections.

A first example of a formula for multiplicity of a weight in a irreducible representation is the Freudenthal multiplicity formula. It expresses the multiplicity $n_\mu(\Gamma_\lambda)$ in terms of the multiplicities of weights which are higher than μ. By the symmetry of the weights under the Weyl group, it suffices to determine the multiplicities of dominant weights, so one may use the formula to recursively compute all multiplicities. Explicitly we have:

$$(2\langle\lambda-\mu,\mu+\delta\rangle+||\lambda-\mu||^2)n_\mu(\Gamma_\lambda)=2\sum_{\alpha\in\Delta^+}\sum_{k\geq 1}\langle\mu+k\alpha,\alpha\rangle n_{\mu+k\alpha}(\Gamma_\lambda),$$

where δ is the lowest form of the Lie algebra, namely the half sum of all positive roots. If $\mu\neq\lambda$ is a dominant weight of Γ_λ, then $\lambda-\mu$ is a linear combination of simple roots with non negative integral coefficients. This implies that $\langle\lambda-\mu,\mu+\delta\rangle>0$ so the numerical values on the left this equation is positive.
Its possible to compute the multiplicities of the weights Γ_λ into an expression called the Weyl character formula. Fix a Cartan subalgebra $\mathfrak{h}$. Consider a representation V of $\mathfrak{g}$ on which the elements of $\mathfrak{h}$ act simultaneously diagonalizable. Then V splits into the direct sum of weight spaces V_λ. If all of these spaces are finite dimensional we say that V has character. In this case one defines the character by

$$\mathrm{char}(V):\mathfrak{h}^*\to\mathbb{Z},$$

by $\mathrm{char}(V)(\lambda)=\dim(V_\lambda)$.

If V is finite dimensional then of course V has a character and since $\mathrm{char}(V)$ has finite support, one may view it as an element on the group ring $\mathbb{Z}[\mathfrak{h}^*]$ of the abelian group $\mathfrak{h}^*$. The multiplication on $\mathbb{Z}[\mathfrak{h}^*]$ is given by the convolution of functions. Denoting by $e^\phi\in\mathbb{Z}[\mathfrak{h}^*]$ the function which is one on ϕ and zero on all other elements of $\mathfrak{h}^*$, thus one gets $e^\phi e^\psi=e^{\phi+\psi}$. Thus by construction these elements form a basis of $\mathbb{Z}[\mathfrak{h}^*]$, so we may write elements of this set as $f=\sum_{\phi\in\mathfrak{h}^*}a_\phi e^\phi$ where $a_\phi\in\mathbb{Z}$.
We are now in position to formulate the first version of Weyl character formula by first defining some central concepts. For each weight $\lambda\in\mathfrak{h}_0^*$ define an element $A_\lambda\in\mathbb{Z}[\mathfrak{h}^*]$ by the formula

$$A_\lambda:=\sum_{w\in W}\mathrm{sgn}(w)e^{w(\lambda)},$$

where W is the Weyl group of $\mathfrak{g}$. Denote by δ the lowest form on $\mathfrak{g}$, the Weyl character formula states

$$A_\delta\,\mathrm{char}(\Gamma_\lambda)=A_{\lambda+\delta}$$

We now assume some knowledge of Verma modules in order to derive the dimension formula.
Consider a Verma module $M_\mathfrak{b}(\lambda)$ for $\lambda\in\mathfrak{h}^*$. Putting $\Delta^+=\{\alpha_1,\dots,\alpha_l\}$, we know that the monomials $F_{\alpha_1}^{i_1}\cdots F_{\alpha}^{i_l}\otimes 1$ form a linear basis of $M_\mathfrak{b}(\lambda)$ and that this element is a weight vector of weight $\lambda-\sum_j i_j\alpha_j$. Thus $M_\mathfrak{b}(\lambda)$ has character and that this lies in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$, where this denotes the set of roots with non negative integral coefficients. In order to write this character explicitly we define the Konstant partiaion function $\mathcal{P}:\mathfrak{h}^*\to\mathbb{N}$ by setting the number of all tuples $(a_\alpha)_{\alpha\in\Delta^+}$ with each $a_\alpha\in\mathbb{N}$ such that $\phi=\sum_{\alpha\in\Delta^+}a_\alpha\alpha$. Thus, $\mathcal{P}(\phi)$ is the number of different expressions of ϕ as a linear combination of positive roots with non negative integral coefficients. Thus we introduce the convention $\mathcal{P}(0)=1$ and naturally $\mathcal{P}(\phi)=0$ unless ϕ can be expressed as a linear combination of positive roots with non-negative integral coefficients denoted Q^+. Then one can show that:

$$\mathrm{char}(M_\mathfrak{b}(\lambda))=\sum_{\mu\in\mathfrak{h}^*}\mathcal{P}(\lambda-\mu)e^\mu=\sum_{\phi\in Q^+}\mathcal{P}(\phi)e^{\lambda-\phi}=e^\lambda\sum_{\phi\in Q^+}\mathcal{P}(\phi)e^{-\phi}.$$

Now the Weyl denominator A_δ can be written as

$$A_\delta=e^\delta\prod_{\alpha\in\Delta^+}(1-e^{-\alpha}).$$

Using this one can verify that in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$ one has $Ke^{-\delta}A_\delta=1$, where $K=\sum_{\phi\in Q^+}\mathcal{P}(\phi)e^\phi$. Thus we have that A_δ is invertible in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$ and the inverse is exactly the character of the Verma module $M_\mathfrak{b}(-\delta)$. Summarizing we can now express the character formula

$$\mathrm{char}(\Gamma_\lambda)=e^{-\delta}\left(\sum_{\phi\in Q^+}\mathcal{P}e^{-\phi}\right)\left(\sum_{w\in W}\mathrm{sgn}(w)e^{w(\lambda+\delta)}\right).$$

From this one can extract the Weyl dimension formula which computes the dimension of Γ_λ. To get this one applies the homomorphism $\mathbb{Z}[\mathfrak{h}^*] \to \mathbb{Z}$ that sends any e^ϕ to one to the character formula. After some computation one obtains:

$$\dim(\Gamma_\lambda) = \prod_{\alpha\in\Delta^+} \frac{\langle\lambda+\delta,\alpha\rangle}{\langle\delta,\alpha\rangle}.$$

In our application of Wang's theorem we use the dimensional formula to find representations of desired dimension.

1.2.9 Real semisimple Lie algebras and their representation

Representations arising from real structures will play an important role in chapter 2, we therefore introduce the theory of real Lie algebras and their representations. It is easy to see that the complexification of a semisimple Lie algebra is also semisimple. Thus real semisimple Lie algebras can be viewed as real forms of complex semisimple Lie algebras.

More precisely we introduce the complexification $\mathfrak{g}_\mathbb{C} = \mathfrak{g}\otimes\mathbb{C}$ of a real Lie algebra $\mathfrak{g}$, and with it the corresponding involutive automorphisms σ of $\mathfrak{g}_\mathbb{C}$ such that

$$\mathfrak{g}_\mathbb{C}^\sigma = \{A\in\mathfrak{g}_\mathbb{C} : \sigma(A) = A\} = \mathfrak{g},$$

thus $\mathfrak{g}$ is a real from of $\mathfrak{g}_\mathbb{C}$. Moreover any real form can be described as $\mathfrak{g}_\mathbb{C}^\sigma$ for a conjugate linear involutive automorphisms σ. Moreover the Killing form of $\mathfrak{g}_\mathbb{C}$ is the complex billinear extension of the Killing form of $\mathfrak{g}$ thus $\mathfrak{g}_\mathbb{C}$ is semisimple if and only if $\mathfrak{g}$ is semisimple.

We begin with a digression on compact Lie algebras. Suppose that G is a compact Lie group with Lie algebra $\mathfrak{g}$. Then there is an inner product on $\mathfrak{g}$ such that for each $g\in G$ the map $\mathrm{Ad}(g):\mathfrak{g}\to\mathfrak{g}$ is orthogonal. Similarly for any element $A\in\mathfrak{g}$ the map $\mathrm{ad}(A):\mathfrak{g}\to\mathfrak{g}$ is skew symmetric. Therefore the acting on the complexification $\mathrm{ad}(A)$ is skew Hermintian and thus diagonalizable with purely imaginary eigenvalues. Thus $\mathrm{ad}(A)^2$ is diagonalizable with real non-positive eigenvalues, this implies that the Killing form B of $\mathfrak{g}$ is negative semi definite. Utilizing the fact that the maps $\mathrm{ad}(A)$ are all skew symmetric and the orthogonal compliment of any ideal in $\mathfrak{g}$ is also an ideal. We see that $\mathfrak{g}$ is reductive and $[\mathfrak{g},\mathfrak{g}]$ is semisimple and $\mathfrak{g} = \mathfrak{z}(\mathfrak{g})\oplus[\mathfrak{g},\mathfrak{g}]$, thus the Killing form must be negative definite.
This shows us that any connected Lie group with Lie algebra $\mathfrak{g}$ is compact, and this property is equivalent to the fact that the Killing form is negative definite. A semisimple Lie algebra which satisfies these equivalent conditions is called compact. There is a opposite extremal class of real forms, which is in a sense maximally non-compact. Such a real form is called a split real form and is characterized by the following property. For each real Lie algebra $\mathfrak{g}$ with complexification $\mathfrak{g}_\mathbb{C}$ we define a Cartan subalgebra in $\mathfrak{g}$ as a subalgebra $\mathfrak{h}\leq\mathfrak{g}$ whose complexification $\mathfrak{h}_\mathbb{C}$ is a Cartan subalgebra in $\mathfrak{g}_\mathbb{C}$. Given a Cartan subalgebra $\mathfrak{h}\leq\mathfrak{g}$ one can then look at the roots of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{h}_\mathbb{C}$ and restrict these linear functionals to $\mathfrak{h}$. We call $\mathfrak{g}$ a split real form of $\mathfrak{g}_\mathbb{C}$ if there exists a Cartan subalgebra $\mathfrak{h}\leq\mathfrak{g}$ for which all these restrictions are real values.

Proposition 6. *Any complex semisimple Lie algebra $\mathfrak{g}$ admits at least one split real form an at least one compact real form*

Having the two extremal real forms of complex semisimple Lie algebras, we initiate a study of generalized real forms. The first step is to construct a vector space decomposition of a real semisimple Lie algebra into maximal compact subalgebra and a complementary subspace. Let $\mathfrak{g}$ be a real semisimple Lie algebra. A Cartan involution for $\mathfrak{g}$ is an involutive automorphisms $\theta:\mathfrak{g}\to\mathfrak{g}$ such that the billinear form $B_\theta(X,Y) := -B(X,\theta Y)$ is positive definite. A Cartan decomposition of $\mathfrak{g}$ is a vector space decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ such that $\mathfrak{k}$ is a subalgebra, $[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$. Such that $\mathfrak{k}\oplus\mathfrak{p}$ is the $\pm$-eigenspaces of θ. Moreover every Cartan decompositions of $\mathfrak{g}$ are related by inner automorphisms. A Cartan subalgebra $\mathfrak{h}$ of a real-semisimple Lie algebra $\mathfrak{g}$ is the pre-image of the Cartan subalgebra $\mathfrak{h}^\mathbb{C}$ of $\mathfrak{g}^\mathbb{C}$. Real Cartan subalgebras are characterized by being maximal self-normalizing abelian subalgebras of $\mathfrak{g}$. We choose a Cartan involution θ, $\mathfrak{h}$ is called θ-stable if $\theta(\mathfrak{h})\subset\mathfrak{h}$. In this case we have that

$$\mathfrak{h} = (\mathfrak{h}\cap\mathfrak{k})\oplus(\mathfrak{h}\cap\mathfrak{p}).$$

This defines the notion of a compact and non-compact components of $\mathfrak{h}$. Any Cartan subalgebra is conjugate to a θ stable Cartan subalgebra by an inner automorphisms.
A Cartan subalgebra is called maximally non-compact if the dimension of $\mathfrak{p}$-component is maximal possible among the Cartan subalgebras of $\mathfrak{g}$. In a similar fashion one defines the notion of a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$.

1.2.10 Satake diagrams

Let $\mathfrak{g}$ be a real simple Lie algebra. Let $\mathfrak{h}$ be a maximally non compact Cartan subalgebra The Satake diagram of $\mathfrak{g}$ is constructed as follows: Take the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$ wrt. $\mathfrak{h}^{\mathbb{C}}$. The roots will be complex valued when restricted to $\mathfrak{h}$. Color the roots that vanish on $\mathfrak{h} \cap \mathfrak{p}$ black, and the rest white. Complex conjugation will preserve the set of roots. The roots that are not mapped to themselves under conjugation indicate its image by an arrow.
Thus the Sakate diagram of a split $\mathfrak{g}$ is white, while the Sakate diagram of a compact $\mathfrak{g}$ is black. We now state the real equivalent of the main theorem regarding Dynkin diagrams:

Theorem 7. *The real simple Lie algebras are uniquely defined by their Satake diagram, thus two real Lie algebras with isomorphic Satake diagrams are isomorphic.*

Shur's lemma in the setting of real Lie algebras tells us the following:

Theorem 8 (Shur's Lemma)**.**

- *If V_1, V_2 are real representations, then any morphism $\phi : V_1 \to V_2$ is either 0 or invertible*
- *If V is a real irreducible representation of $\mathfrak{g}$, then the ring $End_{\mathfrak{g}}(V)$ of linear operators commuting with the $\mathfrak{g}$-action is a real division algebra. That is it must be isomorphic to $\mathbb{R}, \mathbb{C}$ or the quaternions $\mathbb{H}$.*

As with complex representation all representations of real simple Lie algebras can be decomposed into irreducibles.
In order to describe real irreducible representations from complex ones, we introduce the following notions:

- A real structure on a complex irrep $\mathfrak{m}$ of a real $\mathfrak{g}$ is a $\mathbb{C}$-antilinear $\mathfrak{g}$-equivariant endomorphism θ s.t. $\theta^2 = 1$.
- A quaternionic structure on a complex irrep $\mathfrak{m}$ of real $\mathfrak{g}$ is a $\mathbb{C}$-anti-linear $\mathfrak{g}$-equivariant endomorphism σ s.t. $\sigma^2 = 1$.

Both real and quaternionic complex irreps m satisfy $\mathfrak{m} \simeq \overline{\mathfrak{m}}$ as complex representations. There is a third class of complex irrep, defined by the absence of this isomorphism: $\mathfrak{m} \not\simeq \overline{\mathfrak{m}}$ as complex reps. This means that there is an involution ξ on the set of dominant weights, $\lambda \to \overline{\lambda}$. When restricted to fundamental weights, ξ is always a Satake-automorphisms. The Satake automorphisms are either unique or non-existent, so finding all true complex fundamental irreps is quite easy, it is enough to find one example.

We now describe all the real irreps of $\mathfrak{g}$.

- A complex irrep with real structure is the complexification of a unique real irrep.
- A true complex or quaternionic irrep is irreducible over the reals.
- Two conjugate true complex reps are equivalent over reals.

1.3 Connections on homogenous spaces

A connection ∇ defines the notion of parallel transport of geometric data along a family of curves. The study of connections play a fundamental role in modern geometry, and have broad applications in mathematics and physics. There are several equivalent definitions of general connections over a vector bundle. We will however focus on the more algebraic of them. But its worth noting that a connection ∇ can be equivalently defined as a splitting of the tangent bundle into a horizontal and vectical part of the total space of a fibration.

Let $p: E \to M$ be a vector bundle over M and let $\Gamma(E)$ denote the vector space of sections of this bundle. A linear connection or a co-variant derivative on E is a map

$$\begin{aligned} \nabla : TM \times \Gamma(E) &\to \Gamma(E) \\ (X, s) &\to \nabla_X s \end{aligned}$$

satisfing:

$$\begin{aligned} \nabla_{fX+hY} s &= f\nabla_X s + h\nabla_Y s, \\ \nabla_X(s+t) &= \nabla_X s + \nabla_x t, \\ \nabla_X(fs) &= X(f)s + f\nabla_X s \end{aligned}$$

for any $X, Y \in TM$, any sections $s, t \in \Gamma(E)$ and for any $f, h \in C^\infty(M)$
It is worth noting that for any $s \in \Gamma(E)$ the map $\nabla s : TM \to \Gamma(E)$ is $C^\infty(M)$-linear. Thus we may define ∇s is a 1-form on M with values in E.
Given two connections ∇ and ∇' of the same bundle E, the difference

$$A(X, s) = \nabla'_X s - \nabla_x s$$

is $C^\infty(M)$-linear, thus defines a section of $\bigwedge^1 M \otimes E^* \otimes E$. From this one may deduce the following: Given any connection on a vector bundle E over B, let (U, ϕ) be a local trivialization of E i.e. U is an open subset of B and ϕ a $C^\infty(M)$-fibred isomorphism

$$p^{-1}(U) \to U \times F.$$

Then $U \times F$ admits the trivial connection ∇^F, namely the unique connection such that constant sections satisfy $\nabla_X s = 0 \ \forall \ X$. We may consider the connection ∇^ϕ on $p^{-1}(U)$ such that ϕ interchanges ∇^ϕ and ∇^F. The difference Ξ between a given connection ∇ and ∇^ϕ is a section of $\bigwedge^1 U \otimes E^* \otimes E$. This construction is called the Christoffel tensor of ∇ with respect to the trivialization ϕ.

Given a chart U of M with coordinates (x_i) and identify the fiber F with $\mathbb{R}^p$, then the components $\Xi^\alpha_{i\beta}$ of Ξ in the canonical basis $\left(\frac{\partial}{\partial x_i}\right)$ of TU and (ϵ_α) of $\mathbb{R}^p$ are the so called Christoffel symbols of ∇.

We now present the two main geometric invariants one may derive from a given connection.

- The curvature R^∇ of a linear connection ∇ on a vector bundle E is a 2-form on M with values in $E^* \otimes E$ defined by

  $$R^\nabla_{X,Y} s = \nabla_{[X,Y]} s - [\nabla_X, \nabla_Y] s.$$

  When we consider a linear connection on a manifold, then this will define a tensor field.

- The torsion T of a linear connection ∇ on a manifold M is the $(2,1)$-tensor field defined by:

  $$T_{X,Y} = \nabla_X Y - \nabla_Y X - [X, Y]$$

We now fix the notion of a linear connection ∇ on a manifold M as a linear connection on the tangent bundle TM of M.
The following notions are fundamental to the theory of connection:
A connection ∇^g is called the Levi-Civita connection on a Riemannian manifold (M, g) if:

- $\nabla^g g = 0$ i.e. it preserves the metric,

- The connection ∇^g has zero torsion.

Theorem 9 (Levi-Cevita)**.** *If the Levi Civita connection exists it is unique.*

We now present the some basic theory on the geometry of homogeneous Riemannian manifolds.
A Riemannian manifold (M, g) is sad to be homogenous if the group of isometries $I(M, g)$ acts transetively. That is $\forall x, y \in M$ $\exists$ an isometry f such that $f(x) = y$.
Using some familiar notions from the geometry of homogenous spaces we say that a Riemannian manifold (M, g) is G-homogeneous if G is a closed subgroup of (I, g) which acts transetively on M
Given such a G-homogeneous manfiold we define the subgroup $K \subset G$ as the isotropy subgroup given by

$$K = \{f \in G | f(x) = x\}.$$

A given isometry f is determined by its image $f(x)$ at $x.$ and the corresponding tangent map $T_x f$ defines the linear isotropy representation $\chi(f) = T_x f$ of $K \subset \mathrm{GL}(T_x M)$. It is worth noting that this representation will always be faithful.

This construction will pay a key role in chapter 2 where the isotropy representation will be utilized in order to describe invariant tensors on a reductive homogenous manifold. Moreover the spaces we will be working with will be isotropy irreducible meaning that the isotropy representation is an irreducible representation.

An important concept in chapter 2, is the notion of the canonical connections on a homogenous manfiold $M = G/K$, thus we state its definition here:

Let M be a homogeneous space, that is a smooth manifold on which a Lie group G acts transitively by diffeomorphisms. Then $M \cong G/H$ where H is the isotropy group of some base point $p \in M$, and the action map $\pi : G \to M, g \to gp$ becomes the canonical projection $G \to G/H$ which is a principle bundle with structure group H. Let $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ denote the adjoint representation of G. Its restriction $\mathrm{Ad}(H)$ keeps $\mathfrak{h}$ invariant. We will assume that the homogeneous space $M = G/H$ is reductive. That is there is a vectorspace compliment $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ which is invariant under $\mathrm{Ad}(H)$. Note that if H is compact, then the space is reductive. since $d\pi_e : \mathfrak{g} \to T_p M$ has kernel $\mathfrak{h}$ it is an isomorphism on the compliment $\mathfrak{m}$ and carries the representation $\mathrm{Ad}(H)|_{\mathfrak{m}}$ into the isotropy representation of H on $T_p M$. Via $d\pi_e$ we identify $\mathfrak{m}$ with $T_p M$.
Using the left translation $L_g, g \in G$, $\mathfrak{m}$ defines a distribution $\mathcal{H}$ which is complementary to the vertical distiribution $\mathcal{V}_g = dL_g \mathfrak{h}$ and which is invariant under the right translations of H if $\mathfrak{m}$ is a reductive compliment. Thus this defines a connection on the H-principle bundle $G \to G/H$ called the canonical connection.

1.4 The geometry of second order differential equations

In this section we present the geometric analysis of an ordinary differential equation of second order. Our approach will be based on the material presented in [CtB]. In practice one is presented a differential equation, and in order to compute its solution in quadratures one finds a solvable Lie algebra of sufficient dimension to apply the Lie-Bianchi theorem.

The approach presented here however will be different. We will begin with an Lie algebra of shuffling symmetries and compute which Ode's are invariant with respect to this action. Since the Lie algebras we are working with are solvable, the Lie-Bianchi theorem gives us an algorithmic method to give the solution in quadratures.

This analysis will build upon the results found in chapter 3 which generalize Lie's theorem given in chapter 4 for the contact case. However the general scheme outlined here applies to all the results given in chapter 4 and chapter 3. This motivates further investigation into representations of singular contact vector fields, which will be given in a later publication. For literature regarding the geometry of differential equation, we refer the reader to [CtB], [KrVi]. For a more focused book on jets and partial differential equations we refer the reader to [KLV]. We now give a historical exposition on the Lie-Bianchi theorem.
Inspired by Abel's work on the solubility of fifth-order polynomial equations, Lie set out to prove a similar results regarding differential equation. Roughly sketched the solubility of a polynomial equation is determined by the existence of a chain of subgroups given by the automorphisms on the set of solutions. That is given a polynomial $P(x)$, and denote by G the set of automorphisms

of the solutions of $P(x)$. Then the polynomial is solvable by radicals if there is a derived chain of groups factored by the commutator subgroup:

$$\{1\} \subset \mathbb{Z}_2 \ldots \subset G_1 \subset G$$

terminates. For an exposition on the topic of solubility of polynomial equations we refer the reader to [HaEd].
Lie however was not able to find such a theorem, however his work on transformation groups, invariance and contact geometry are quintessential to modern geometry.
Bianchi however found the solution. Not only will it give us a condition on the solubility of a differential equation but also a algorithmic method to give its solution by quadratures that is as an integral of a closed differential form. Remarkably this will depend on the existance of a solvable Lie algebra of suffling symmetries. That is a derived chain of the so called derived subalgebras:

$$\{0\} \subset \mathfrak{g}^k \ldots \subset \mathfrak{g}^{(1)} \subset \mathfrak{g}$$

1.4.1 Differential equations as geometric objects

We begin our analysis by studying the simplest differential equation, namely a first order ODE. Assume that it can be solved for the derivative i.e.

$$y' = f(x, y) \tag{1.4.1}$$

Naturally this equation can be interpreted as a vector field on the (x, y)-plane. Namely to each point (x_0, y_0), one has to consider the vector $(1, f(x_0, y_0))$ and the operator $\frac{\partial}{\partial x} + f(x_0, y_0)\frac{\partial}{\partial y}$ of derivations in the direction of this vector. The trajectories of the field $\frac{\partial}{\partial x} + f(x, y)\frac{\partial}{\partial y}$ are called the integral curves of (1.4.1).
The generalization of this procedure to an arbitrary first order ODE will be initiated as follows: Consider the equation:

$$F(x, y, y') = 0.$$

We study the space $\mathbb{R}^3$ with coordinates (x, y, y_1). Then for every solution $u = f(x)$ there corresponds a curve on the surface given by the equation

$$y = f(x), \; y_1 = f'(x).$$

Since $y_0 = f'(y_0)$ the tangent vector to this curve at the point $a = (x_0, y_0, y_1)$ will be given as:

$$\frac{\partial}{\partial x} + y_0\frac{\partial}{\partial y} + f''(x_0)\frac{\partial}{\partial y_1}.$$

Thus this vector lies in the plane given by the equation

$$u - y_0 = y_0(x - x_0),$$

equivalently it belongs to the kernel of the 1-form

$$\omega = dy - y_1 dx$$

For the case of a first order ODE we define the distribution $\ker(\omega) = \mathcal{C}$. The distribution $\mathcal{C}$ is called the Cartan distribution. We describe $\mathcal{C}$ for a general n-th order ODE:

$$\overline{F}(x, y, y', \ldots, y^{(n)}) = 0.$$

Assume we can algebraically express this equation as $y^{(n)} = F(x, y, y^{(1)}, \ldots, y^{(n-1)})$ and let $M = \mathbb{R}^n$. Denote the coordinates in M by $(x, y_0, y_1, \ldots, y_n)$ and consider the following differential 1-forms:

$$\begin{aligned}
\omega_0 &= dy - y_1 dx, \\
\omega_1 &= dy_1 - y_2 dx, \\
&\vdots \\
\omega_{n-2} &= dy_{n-2} - y_{n-1} dx, \\
\omega_{n-1} &= dy_{n-1} - F(x, y, y_1, \ldots, y_{n-1}) dx.
\end{aligned}$$

The distribution $\mathcal{D} = \ker(\omega_0, \ldots, \omega_{n-1})$ is a one-dimensional distribution and is called the Cartan distribution and is of co-dimension 1. This distribution can be equivalently be defined through a single vector field

$$\mathcal{D} = \frac{\partial}{\partial x} + y_2 \frac{\partial}{\partial y_1} + \ldots + y_{k-1} \frac{\partial}{\partial y_{k-2}} + F(x, y, y_1, \ldots, y_{k-1}) \frac{\partial}{\partial y_{k-1}}$$

It is worth noting that for $M = J^1(\mathbb{R}, \mathbb{R})$ which can be described as $\mathbb{R}^3(x, y, y_1)$, equipped with the 1-form $\omega = dy - y_1 dx$ defines a contact manifold.

Moreover these notions can be extended in a similar fashion to partial differential equations however we will not cover this here and refer the reader to [CtB], [KLV]. Summarizing we define a ODE as a manifold $\mathcal{E}$ equipped with a Cartan distribution $\mathcal{D}$.

1.4.2 Symmetries and distributions

Consider now a manifold M equipped with a distribution P and denote the set $D(P)$ as the set of all vector fields that belong to P.
We say that a submanifold $N \subset M$ is integral for P if

$$T_a N \subset P_a$$

moreover we say that N is a maximal integral manifold if for any $a \in N$ one can find a neighborhood $\mathcal{O}$ of a such that there exists no integral manifold N' such that

$$N \cap \mathcal{O} \subset N'.$$

Moreover a distribution which has integral manifolds of dimensions equal to the dimension of the distributions are said to be completely integrable.

Our aim is to construct solutions to an ODE given its symmetry, in order to accomplish this we introduce fundamental notions in the theory of symmetries of distributions.

We say that a map

$$\phi : M \to M$$

is a symmetry of the distribution P if

$$\phi_*(P) = P_{\phi(a)},$$

for every $a \in M$.

We now present the infinitesimal equivalent of this construction. That is we say that a vector field $X \in \mathfrak{X}(M)$ is a symmetry of P if the flow A_t of X is a one parameter group of symmetries. We denote the set of all (infinitesimal) symmetries of P as $\mathrm{Sym}(P)$.

Theorem 10. *The following conditions are equivalent:*

1. $X \in \mathrm{Sym}(P)$
2. $\forall Y \in D(P) \implies [X, Y] \in D(P)$
3. $\forall \omega \in \ker D(P) \implies L_X(\omega) \in \ker P$

We immediately have that:

Corollary 1. $\mathrm{Sym}(P)$ *is a Lie algebra with respect to the commutator of vector fields.*

Consider now a symmetry X of the distribution P and let A_t denote the flow of X. Given a integral submanifold N of P we have that $A_t(N)$ is an integral manifold of P. This illustrates an important property of symmetries.
We now describe a very distinguished class of symmetries that transforms each maximal integral manifold onto itself. We call these for characteristic symmetries, and they are given as:

$$\mathrm{Char}(P) := \mathrm{Sym}(P) \cap D(P).$$

We now present a key theorem regarding characteristic symmetries:

Theorem 11.

1. *$\mathrm{Char}(P)$ is an ideal of the Lie algebra $\mathrm{Sym}(P)$. That is, $\mathrm{Char}(P)$ is a linear subspace of $\mathrm{Sym}(P)$ and*
$$X \in \mathrm{Sym}(P),\ Y \in \mathrm{Char}(P) \implies [X, Y] \in \mathrm{Char}(P)$$

2. *$\mathrm{Char}(P)$ is $C^\infty(M)$-module:*
$$f \in C^\infty(M),\ Y \in \mathrm{Char}(P) \implies fY \in \mathrm{Char}(P)$$

Since $\mathrm{Char}(P)$ is an ideal of the Lie algebra $\mathrm{Sym}(P)$, we define the quotient Lie algebra

$$\mathrm{Shuf}(P) := \mathrm{Sym}(P)/\,\mathrm{Char}(P).$$

An element $X \in \mathrm{Shuf}(P)$ is called a shuffling symmetry of P. The geometric interpretation of this algebra is as follows;
Denote by $\mathrm{Sol}(P)$ the set of all maximal integral manifolds of P. Then any symmetry $X \in \mathrm{Sym}(P)$ generates a flow on $\mathrm{Sol}(P)$, and the characteristic symmetries generate trivial flows. Moreover given two symmetries $X, Y \in \mathrm{Sym}(P)$ if $X - Y \in \mathrm{Char}(P)$ then the corresponding flows on $\mathrm{Sol}(P)$ are the same.
Thus symmetries of the form $X \mod \mathrm{Char}(P)$ shuffle the set of maximal integral manifolds like a deck of cards.
Assume now that we have a k-dimensional distribution P on a manifold M of dimension m. It is well known that P can be defined as the kernel of $(m-k)$ linearly independent 1-forms ω_i.
If for any pair of vector fields $X, Y \in \Gamma(P)$ we have that $[X, Y] \in \Gamma(P)$ we say our distribution is involutive.
We now state a well known theorem:

Theorem 12 (Frobenius)**.** *The following conditions are equivalent:*

1. *P is completely integrable*

2. *P is involutive*

3. *$\forall x \in M$ $\exists$ neighborhood $x_0 \in U \subset M$ and 1-forms θ_{ij} defined on U such that*
$$d\omega_i = \sum_{j=1}^{m-k} \theta_{ij} \wedge \omega_j,$$
for $1 \le i \le m-k$

4. *for all indices $1 \le m-k$ the following exterior product vanishes*
$$d\omega_i \wedge (\omega_1, \ldots, \omega_{m-k}) = 0$$

Proof. For a detailed and complete proof see [AgFr] □

1.4.3 The Lie-Bianchi theorem

Let P be n-co-dimensional completely integrable distribution expressed as the kernel of the differential 1-forms $\omega_1, \ldots, \omega_n$ and let $\mathfrak{g}$ be a Lie subalgebra of $\mathrm{Shuf}(P)$.

We say that $\mathfrak{g}$ is transversal to P if $\dim \mathfrak{g} = n$, and values of symmetries generate the factor $TM_a/P_a\ \forall\, a \in M$.

Denote by $\overline{X} \in \mathrm{Sym}(P)$ a representative of a shuffling symmetry $X \in \mathrm{Shuf}(P)$. It is easy to see that $\theta(X) := \theta(\overline{X})$ is correctly defined for any $\theta \in \ker(P)$.

Let $X_1, \ldots, X_n$ be a basis of the Lie algebra $\mathfrak{g}$. Then the transversality condition for the algebra $\mathfrak{g}$ means that the matrix:
$$W := |w_i(X_j)|_{i,j=1,\ldots,n},$$

is non-degenerate at any point.
We choose now another basis $\overline{\omega}_1, \ldots, \overline{\omega}_n$ of the module $\ker(P)$ such that

$$\overline{\omega_i}(\overline{X}) = \delta_{ij} \ (i, j = 1, \ldots, n) \tag{1.4.2}$$

Suppose now that for the forms $\omega_1, \ldots, \omega_n$ that this condition holds, and let $c_{ij}^l \in \mathbb{R}$ be the structure constants of $\mathfrak{g}$ s.t.

$$[X_i, X_j] = \sum_{l=1}^{n} c_{ij}^l X_l \tag{1.4.3}$$

Then we have:

Theorem 13. *Let P be a completely integrable distribution defined by the 1-forms $\omega_1, \ldots, \omega_n$ satisfying* (1.4.2) *and* (1.4.3)*. Then the we have the following:*

$$d\omega_l + \sum_{i<j} c_{ij}^l \omega_l \wedge \omega_j = 0$$

for all l.

We call this the Maurer-Cartan equation which is analogous to the Mauer-Cartan equation in Lie group theory, see [ČS].
Denote $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ the commutator ideal of $\mathfrak{g}$. Assume that $\mathfrak{g}$ is not simple i.e. $\mathfrak{g}^{(1)} \neq \mathfrak{g}$ and set $l = \operatorname{codim}_{\mathfrak{g}} \mathfrak{g}^{(1)}$.
Choose a basis $X_1, \ldots, X_l, \ldots, X_n$ of $\mathfrak{g}$ such that

$$X_1, \ldots, X_l \notin \mathfrak{g}^{(1)} \qquad X_{l+1}, \ldots, X_m \in \mathfrak{g}^{(1)}.$$

Then the Mauer-Cartan equations imply that

$$d\overline{\omega_i} = 0, \qquad i = 1, \ldots, l.$$

Denote by

$$H_i = \int_L \omega_i$$

Then the sub-manifolds

$$M_c = \{H_1 = c_1, \ldots, H_l = c_l\},$$

where $(c_1, \ldots, c_l) \in \mathbb{R}^l$ are invariant with respect to the commutator $\mathfrak{g}^{(1)}$ since

$$X_i(H_j) = dH_j(X_i) = \omega(X_j) = 0$$

for $i = l+1, \ldots, n$ and $j = 1, \ldots, l$
Denote by P_c the restriction of P on M_c. Since H_i are integrals, P_c is a completely integrable distribution of dimension n.
Thus $\mathfrak{g}^{(1)}$ is a transversal symmetry algebra for P_c for any c.
Apply now the same procedure for

$$\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}].$$

Recall that a Lie algebra $\mathfrak{g}$ is said to be solvable if there is a number r such that

$$\mathfrak{g}^{(r)} = 0.$$

For any solvable Lie algebra one had that the descending sequence of Lie subalgebras

$$0 = \mathfrak{g}^{(r)} \subset \mathfrak{g}^{(r-1)} \subset \ldots \subset \mathfrak{g}^{(1)} \subset \mathfrak{g}^{(0)=\mathfrak{g}}.$$

This procedure shows that we can find the complete sequence of first integrals by integration of closed 1-forms also called quadratures. Summarizing we have that:

Theorem 14. *Let P be a completely integrable distribution and let $\mathfrak{g} \subset \operatorname{Shuf}(P)$ be a solvable symmetry Lie algebra transversal to P, $\dim \mathfrak{g} = \operatorname{codim}(P)$. Then P is integrable by quadratures.*

1.4.4 Application to a class of second order Ode's

Assume we can bring our differential equation to the form

$$y'' = F(x, y, y').$$

In this section we illustrate and apply the theory presented in the previous section in order to solve a class of second order ODE's, with a fixed 2-dimensional non-abelian symmetry algebra. More precisely we seek equations $\mathcal{E}$ with a distribution $P \subset T(\mathcal{E})$ such that

$$L_X(\mathcal{E}) = \mathcal{E}.$$

for every $X \in D(P)$.
Moreover we require that $\mathrm{Shuf}(P) = \mathfrak{g}$ where $\mathfrak{g}$ is spanned by:

$$X = \frac{\partial}{\partial x}$$
$$Y = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

First recall the Cartan distribution of a second order ODE $y'' = F(x, y, y_1)$ is given by:

$$\omega_1 = dy - y_1 dx$$
$$\omega_2 = dy_1 - F(x, y, y_1)dx$$

or equivalently as a vector field

$$\mathcal{D} = \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + F\frac{\partial}{\partial y_1} \tag{1.4.4}$$

Define now:

$$\phi_i = \omega_1(X_i) \tag{1.4.5}$$

as the so called generating functions.

From [CtB] we have the following relations:

$$\mathcal{D}^{k+1}(\phi) - \sum_{i=0}^{k} \frac{\partial F}{\partial p_i}\mathcal{D}^i(\phi) = 0, \tag{1.4.6}$$
$$\mathcal{D}^i = \mathcal{D}(\mathcal{D}^{i-1}), \tag{1.4.7}$$
$$\mathcal{D}^0 = \phi \tag{1.4.8}$$

Next we must compute a basis as described in (1.4.2). In the case of a second order ODE we have:

$$\overline{\omega_1} = \frac{\mathcal{D}(\phi_2)\omega_1 - \phi_2\omega_2}{\phi_1\mathcal{D}(\phi_2) - \phi_2\mathcal{D}(\phi_1)} \tag{1.4.9}$$
$$\overline{\omega_2} = \frac{\mathcal{D}(\phi_1)\omega_1 - \phi_1\omega_2}{\phi_1\mathcal{D}(\phi_2) - \phi_2\mathcal{D}(\phi_1)} \tag{1.4.10}$$

Summarizing we arrive at the following partial differential equation

$$2F = F_y + y_1 F_{y_1} \tag{1.4.11}$$

Thus we find its solution and we have that the following ode is invariant with respect to $\mathfrak{g}$:

$$y'' = \lambda\,(y_1 e^{-y})e^{2y}). \tag{1.4.12}$$

For some arbitrary smooth function λ.
Compute now which of the forms (1.4.9), (1.4.10) is exact and compute its integral

$$H_{c_1} := \int \overline{\omega}_1 = c_1.$$

And to find the solution we compute the integral of the form:

$$y(x) = \int \pi^* \overline{\omega}_2$$

where π is the projection down to the submanifold H_{c_1}.

A more elegant way to arrive at (1.4.11) is using the language of jets and prolongation. We present this theory for second order ODE's but these notions generalize easily to pde's and higher order differential equation.

Given a local diffeomorphism ϕ of $\mathbb{R}^2$, then there exists a unique local contact diffeomorphism $\phi^{(1)} : J^1(\mathbb{R}) \to J^1(\mathbb{R})$ such that the following diagram commutes The transformation $\phi^{(1)}$ is called

$$\begin{tikzcd}
J^1(\mathbb{R}^2) \arrow[r, dashed, "\phi^{(1)}"] \arrow[d, "\pi"] & J^1(\mathbb{R}^2) \arrow[d, "\pi"] \\
\mathbb{R}^2 \arrow[r, "\phi"] & \mathbb{R}^2
\end{tikzcd}$$

the first prolongation of ϕ. Higher order prolongations are obtained in the same way, in addition one defines the first prolongation of a vector field $X^{(1)}$ on $J^1(\mathbb{R})$ such that $\phi_*(X^{(1)}) = X$.
We note that the first prolongation of a symmetry algebra gives an algebra of contact vector field. This is a very important remark since the classification of Lie algebras of contact vector fields generalize the classification of Lie algebras of Euclidean vector fields. Thus it is natural to think to truly find all symmetry algebras of a given second order equation one must classify all Lie algebras actions on the space $\mathbb{R}(x, y, y_1, y_2)$. This however turns out not to be the case from [KLV] we have the following theorem:

Theorem 15. *Any symmetry of a differential equation X of the jet manifold $J^k(\pi), k \geq 1$ is described as follows*

1. *For $\dim \pi = 1$, the transformation X is the $(k-1)$-st lift of some contact transformation of the space $J^1(\pi)$*

2. *For $m = \dim \pi \geq 1$, the transformation X is the k-th lifting of some diffeomorphism of the space J^0 of dependent and independent variables.*

Informally stated this means that any symmetry of a differential equation $\mathcal{E}$ is either the prolongation of a point transformation or contact transformation. This holds true also for vector fields.

1.5 Normal forms

In this section we present the reader with known theory on the normal forms in Euclidean and Symplectic geometry as well as some introductory theory behind contact vector fields. Essentially providing theoretical background for the reader to understand the results given in chapter 3 and chapter 4.

1.5.1 Resonance conditions

Let X be a vector field of the form

$$X = \sum_i X_i(x_1, \ldots, x_i) \frac{\partial}{\partial x_i}$$

Where $X_i(0, \ldots, 0) = 0$ for all i
Let λ_i be the eigenvalues of the first jet of X_i at zero, i.e. the matrix $||\frac{\partial X_i}{\partial x_j}(0)||$.
We say our system has resonance, if there exist an eigenvalues λ_i and non-negative integers m_{ij} such that:

$$\lambda_i = \sum_j m_{ij}\lambda_j,$$

where $\sum_j m_{ij} \geq 2$.
We now give a description as do how the following statement is proved.

Theorem 16 (Poincare). *If the first jet of the vector field is non-vanishing and has no resonance, then the vector field is linearizable. That is let λ_i denote eigenvalues then there exists coordinates such that*

$$\sum_i \lambda_i x_i \frac{\partial}{\partial x_i}$$

We now present the proof of this statement: Consider the autonomous system of ordinary differential equations

$$\frac{dx_i}{dt} = Ax + \mathcal{O}(|x|^2),\ x \in \mathbb{R}^n, \tag{1.5.1}$$

where $A \in GL(n)$.
Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of A. We say that the set of eigenvalues of the operator A is called *resonant* if a relation of the form

$$\lambda_s = m_1\lambda_1 + m_2\lambda_2 + \ldots + m_n\lambda_n,$$

is satisfied for some set of non-negative integers $m_1, m_2, \ldots, m_n$ where $\sum_{i=1}^n m_i \geq 2$. The value $\sum_{i=1}^n m_i \geq 2$ is called the *order of resonance*. The classical theorem of Poincare is then:
If an eigenvalue of an equilibrium of (1.5.1) do not satisfy any resonance up to order N. Then there exist a smooth change of coordinates

$$x = y + \mathcal{O}(|y^2|),$$

such that the system is reducible to the form

$$\frac{dy_i}{dt} = Ay + \mathcal{O}(|y|^{N+1}).$$

Write the system as:

$$\frac{dx}{dt} = Ax + V(x)$$
$$V(x) = v_2(x) + v_3(x) + \ldots + v_N(x) + \mathcal{O}(|x|^{N+1}),$$

where $v_i(x)$ is a homogeneous polynomial of degree i.
Thus we are looking for a change of variables $x \to y$ of the form:

$$x = y + h(y),$$
$$h(y) = h_2(y) + h_3(y) + \ldots + h_N(y),$$

where h_i is a homogeneous polynomial of degree i. This transforms the system to

$$\frac{dy}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} = A(y + h(y)) + V(y + h(y)) + \mathcal{O}(|y|^{N+1}).$$

Assume now this transform brings the system to the desired form. Then we equate the terms of a fixed order, say r and we arrive at the *homological equations*.

$$\frac{\partial h}{\partial y} Ay - Ah_r(y) = V_r(y), \tag{1.5.2}$$

where V_r is the homogeneous polynomial of degree r whose coefficients are expressed through coefficients of v_i, h_i. Assume for simplicity that $\lambda_i \neq \lambda_j$ for $i \neq j$ and let $e_1, e_2, \ldots, e_n$ be the eigenvalues of the operator A corresponding to the eigenvalues $\lambda_i \in \mathbb{C}$. Let y_i be the coordinates of y in this basis. Define now the formally the functions

$$U = \sum_{\substack{s=1,\ldots,n \\ |m|=r}} U_{s,m} y_m e_s,$$

$$h = \sum_{\substack{s=1,\ldots,n \\ |m|=r}} h_{s,m} y_m e_s,$$

where $m = (m_1, \ldots, m_m) \in \mathbb{Z}^n$, $m_i \geq 0$, $|m| = \sum m_i$ and $y_m = y_1^{m_1} y_2^{m_2}, \ldots, y_n^{m_n}$.
Finally from (1.5.2) one arrives at the equation

$$(m_1\lambda_1 + m_2\lambda_2 + \ldots + m_n\lambda_n - \lambda_s)h_{s,m} = U_{s,m}.$$

Induction on the level r completes the proof.
This however holds true however for vector fields of finite jets. Using topological arguments in [SsV], Sternberg showed that this classification gives a smooth classification of the action.

1.5.2 Symplectic normal forms and the harmonic oscillator

Consider now a particular Hamiltonian system, and investigate the integrabilty of this system, moreover we consider a perturbation of this system and its normalization properties. This will result in what is call the Birkhoff normal forms.

Let $M = \mathbb{R}^{2n}(q_i, p_i)$ where $i = 1, \ldots, n$ and consider the Hamiltonian system associated to

$$H_0(q,p) = \sum_{i=1}^{n} \omega_i \frac{p_i^2 + q_i^2}{2},$$

where $\omega_i \in \mathbb{R}$.
We call this system the *harmonic oscillator* and its integrals of motion is the set

$$I_j = \frac{q_j^2 + p_j^2}{2},\ j = 1, \ldots, n$$

Consider now a perturbation of the harmonic oscillator, that is: Let H be a Hamiltonian function on M^{2n} having an isolated and elliptic equilibrium at the origin (equivalently at any point).
This means that in some local coordinates on M^{2n}, H takes the form

$$H = H_0 + P,$$

where P is some real valued smooth function function which has a zero at the origin of at least cubic order.
We say that the system is in *normal form* if

$$\{H_0, P\} = 0.$$

We now pose the question: Does there exist a canonical transformation that brings every perturbation of the harmonic oscillator to a normal form? The answer is given by the following theorem due to Birkhoff.

Theorem 17. *Consider the Hamiltonian system $H = H_0 + P$, where H_0 is the harmonic oscillator and P is a function which has a zero at the origin of fixed order $r \geq 3$. Then there exist a smooth canonical transformation*

$$\tau : \mathcal{O} \to \tilde{\mathcal{O}},$$

where $\mathcal{O}, \tilde{\mathcal{O}}$ are neighborhoods of the origin which brings H to a normal form up to order r. That is

$$H \circ \tau = H_0 + Z + R,$$

such that

1. *Z is a polynomial of order r and $\{Z, H_0\} = 0$*
2. *R is a smooth function vanishing of order $l \geq r + 1$*
3. *τ is a perturbation of the identity: $\tau = Id + \dots$*

Proof. We will do the work in the complex coordinates

$$\xi_j = \frac{1}{\sqrt{2}}(q_i + ip_i), \qquad \eta_j = \frac{1}{\sqrt{2}}(q_i - ip_i).$$

In these coordinates we have that $H_0 = \sum_{j=1}^{n} \omega_j \xi_j \eta_j$ and the Poisson bracket is given by:

$$\{F, G\} = i \sum_{j=1}^{n} \frac{\partial F}{\partial \xi_j} \frac{\partial G}{\partial \eta_j} - \frac{\partial F}{\partial \eta_j} \frac{\partial G}{\partial \xi_j}$$

The idea of the proof is to iteratively construct a canonical transformation τ_k for $k = 1 \dots, r$ such that

$$H_K = H \circ \tau_k = H_0 + Z_k + P_{k+1} + R_{k+2} \tag{1.5.3}$$

where Z_k is a polynomial of degree k such that $\{Z_k, H_0\} = 0$, P_{k+1} is a homogeneous polynomial of degree $k+1$ and R_{k+2} has a zero of order $k+2$ at the origin. Thus proving the theorem for $Z = Z_k$ and $R = P_{k+1} + R_{k+2}$.

The first observation we make is that for $k = 2, \tau_2 = Id, Z_2 = 0, P_3$ is the Taylor polynomial of order 3 and $R = P - P_3$ satisfies our conditions on (1.5.3), and we now construct $\tau_{k+1} = \tau_k \circ \phi_{k+1}$. Let $X_{\gamma_{k+1}}$ be a Hamiltonian vector field corresponding to ϕ_{k+1} in the following way:
Let ψ_t be a one parameter subgroup of canonical transformations generated by the flow of the Hamiltonian vector field X_γ. This is canonically defined in a sufficiently small neighborhood of the point under consideration. By re-parameterization ψ_1 can be chosen to be well defined. Moreover the Baker-Campbell-Hausdorff formula provides its action on functions through Poisson brackets:

$$F \circ \psi_1 = F + \{F, \gamma\} + \frac{1}{2}\{\{F, \gamma\}, \gamma\} + \dots$$

We now state two basic facts about this construction and the Poisson bracket:

- If F_1, F_2 are two homogeneous polynomials of degree d_1, d_2 respectively. Then $\{F_1, F_2\}$ is a homogeneous polynomial of degree $d_1 + d_2 - 2$
- If γ_{k+1} is a homogeneous polynomial of degree $k+1$, then the canonical transformation ϕ_{k+1} associated to $X_{\gamma_{k+1}}$ is of the form : $\phi_{k+1} = (\xi, \eta) + \mathcal{O}(||(\xi, \eta)||)^k$

Back to the original problem, we search for γ_{k+1} a homogeneous polynomials of degree $k+1$ and decompose the action as follows:

$$\begin{aligned} H_k \circ \gamma_{k+1} = H_0 &+ Z_k + \{H_0, \gamma_{k+1}\} + P_{k+1} \\ &+ R_{k+2} \circ \phi_{k+1} + H_0 \circ \phi_{k+1} - H_0 - \{H_0, \gamma_{k+1}\} \\ &+ Z_k \circ \phi_{k+1} - Z_k + P_{k+1} \circ \phi_{k+1} - P_{k+1} \end{aligned}$$

Using the last two facts stated above, we see that the last two lines of this equation must have zero of order $q \geq k+1$. Thus all that remains is showing that γ_{k+1} can be chosen in such a way that:

$$Z_{k+1} = Z_k + \{H_0, \gamma_{k+1}\} + P_{k+1},$$

and commutes with the harmonic oscillator.
Summarizing we arrive that the following homological equation

$$\{H_0, \gamma\} + Q = Z,$$

where H_0 is the harmonic oscillator in ξ, η coordinates, Q is a given k-th order homogeneous polynomial and γ, Z are k-th order homogeneous polynomials we are free to choose. In addition we require that

$$\{Z, H_0\} = 0.$$

It turns out that this system is always solvable. Due to our construction of ϕ_{k+1} through the one-parameter groups of transformation associated to $X_{\gamma_{k+1}}$, it will always be a perturbation of the identity. □

1.5.3 Contact geometry

Consider a $2n+1-$dimensional manifold M, a *contact structure* ξ is a $2n$-dimensional distribution which is completely non-integrable.
Locally we may express any co-dimension 1 distribution as the kernel of a differential 1-from α called the contact form

$$\alpha = \ker \xi.$$

We define a strict contact manifold as the pair (M, α) when $\xi = \ker \alpha$ is contact.

The non-integrabilty of the distribution can be expressed through the contact form by the Forbenious condition

$$\alpha \wedge (d\alpha)^n \neq 0.$$

Let M be a strict contact manifold, then for every $a \in M$ we have the following decomposition:

$$T_a M = \ker \alpha_a \oplus \ker d_a \alpha,$$

where $d_a \alpha$ defines a symplectic structure on ξ_a.

A smooth map ϕ between contact manifolds (M_1, ξ_1) and (M_2, ξ_2) is said to be contact if

$$\phi_*(\xi_1) = \xi_2.$$

Then if ϕ is a diffeomorphism which is also contact, we say that ϕ is a contact diffeomorphism, similarly one defines a contact transformation when $M_1 = M_2$.

The dual version of this statement is expressed the following way:

$$\phi^*(\alpha_2) = \lambda \alpha_1,$$

where α_1, α_2 correspond to ξ_1, ξ_2 respectively, and $\lambda \in C^\infty(M_1)$ which we call the *conformal factor*.
A vector field $X \in \Gamma(TM)$ is called a contact vector field, if the flow of X consists of one-parameter groups of contact transformations. This condition is expressed through the Lie derivative:

$$L_X(\alpha) = \lambda_X \alpha, \tag{1.5.4}$$

where L_X denotes the Lie derivative along X, and λ_X is some smooth function.
Equivalently

$$L_X(\alpha) \wedge \alpha = 0.$$

Moreover the pairing $\alpha(X_f) = f$ defines the generating function of a contact vector field.
This definition is justified by the following proposition:

Proposition 7. *Let (M, α) be a strict contact manifold. Then for any function $f \in C^\infty(M)$ there exist a unique contact vector field X_f such that:*

$$\begin{aligned} X_f &= f, \\ i_{X_f} d\alpha &= X_1 \alpha - df, \\ L_{X_f} &= X_1(f)\alpha. \end{aligned}$$

where X_1 the contact vector field with generating function 1, also called the Reeb vector field.

These properties define the bijection

$$\begin{aligned} \Delta : C^\infty(M) &\to \mathfrak{ct}(M), \\ f &\to X_f, \end{aligned}$$

where Δ is a linear differential operator and $\mathfrak{ct}(M)$ is the Lie algebra of contact vector fields.

This bijection Δ endows $C^\infty(M)$ with a Lie algebra structure by

$$\Delta([f, g]) = [\Delta(f), \Delta(g)].$$

This bracket is called the Lagrange bracket and denote it by $\{f, g\}$.

From the proposition above, we have that

$$\{f, g\} = d\alpha(X_g, X_f) = X_f(g) - X_1(f)g. \tag{1.5.5}$$

The Darboux theorem states that, two contact manifolds of the same dimension are locally contact equivalent.

This means that locally every contact structure looks like the standard contact structure on $\mathbb{R}^{2n+1}$ with coordinates (q_i, u, p_i) such that the contact form is given by:

$$\alpha = du - \sum_i p_i dq_i.$$

In these (canonical) coordinates we have

$$X_f = \sum_i -f_{p_i} \frac{\partial}{\partial q_i} + \left(\sum_i f - p_i f_{p_i} \right) \frac{\partial}{\partial u} + \sum_i (f_{q_i} + p_i f_u) \frac{\partial}{\partial p_i}. \tag{1.5.6}$$

The Reeb vector field in this case has the form

$$X_1 = \frac{\partial}{\partial u}.$$

Invariant connections and ∇-Einstein structures on isotropy irreducible spaces

Ioannis Chrysikos, Christian O'Cadiz Gustad and Henrik Winther

Abstract

This paper is devoted to a systematic study and classification of invariant affine or metric connections on certain classes of naturally reductive spaces. For any non-symmetric, effective, strongly isotropy irreducible homogeneous Riemannian manifold $(M = G/K, g)$, we compute the dimensions of the spaces of G-invariant affine and metric connections. For such manifolds we also describe the space of invariant metric connections with skew-torsion. For the compact Lie group U_n we classify all bi-invariant metric connections, by introducing a new family of bi-invariant connections whose torsion is of vectorial type. Next we present applications related with the notion of ∇-Einstein manifolds with skew-torsion. In particular, we classify all such invariant structures on any non-symmetric strongly isotropy irreducible homogeneous space.

Table 2.1: Invariant connections on compact irreducible symmetric spaces due to [L2, L3]

Type I	$M = G/K$	invariant connections $\mathcal{A}ff_G(F(M))$
AI	$\mathrm{SU}_n / \mathrm{SO}_n$ $(n \geq 3)$	1-dimensional family
AII	$\mathrm{SU}_{2n} / \mathrm{Sp}_n$ $(n \geq 3)$	1-dimensional family
EIV	$\mathrm{E}_6 / \mathrm{F}_4$	1-dimensional family
	all the other cases	canonical connection $\equiv$ Levi-Civita connection
Type II	$M = (G \times G)/\Delta G$	bi-invariant connections $\mathcal{A}ff_{G\times G}(F(M))$
	SU_n $(n \geq 3)$	2-dimensional family
	all the other simple Lie groups	1-dimensional family (inducing the flat ± 1-connections)

Introduction

Motivation. Given a homogeneous space $M = G/K$ with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, a G-invariant affine connection ∇ is nothing but a connection in the frame bundle $F(M) = G \times_K \mathrm{GL}(\mathfrak{m})$ of M which is also G-invariant. The first studies of invariant connections were perfomed by Nomizu [N], Wang [W] and Kostant [K] during the fifties. After that, homogeneous connections on principal bundles have attracted the interest of both mathematicians and physicists, with several different perspectives; for example Cartan connections and parabolic geometries [ČS], Lie triple systems and Yamaguti-Lie algebras [EℓM, BEM], Yang–Mills and gauge theories [I, L1], etc. From another point of view, invariant connections are crucial in the holonomy theory of naturally reductive spaces and Dirac operators, mainly due to the special properties of the *canonical connection* (or the characteristic connection in terms of special structures, see [KN, OR1, OR2, CℓS, A, AFr1, AFH]).
According to [W], given a G-homogeneous principal bundle $P \to G/K$ with structure group U, there is a bijective correspondence between G-invariant connections on P and certain linear maps $\Lambda : \mathfrak{g} \to \mathfrak{u}$, where $\mathfrak{g}, \mathfrak{u}$ are the Lie algebras of G and U, respectively. Wang's correspondence was successfully used by Laquer [L2, L3] during the nineties to describe the set of invariant affine connections, denoted by $\mathcal{A}ff_G(F(G/K))$, on compact irreducible Riemannian symmetric spaces $M = G/K$. For most cases, Laquer proved that $\mathcal{A}ff_G(F(G/K))$ consists of the canonical connection (simple Lie groups admit a line of canonical connections), except for a few cases where new 1-parameter families arise, see Table 2.1. By contrast, much less is known about invariant connections on *non-symmetric* homogeneous spaces, even in the isotropy irreducible case. For example, the first author in [C1], considered invariant connections on manifolds G/K diffeomorphic to a symmetric space, which however do not induce a symmetric pair (G, K), e.g. $\mathrm{G}_2 / \mathrm{SU}_3 \cong \mathrm{S}^6$ and $\mathrm{Spin}_7 / \mathrm{G}_2 \cong \mathrm{S}^7$. There, it was shown that the space of G_2-invariant affine or metric connections on the sphere $\mathrm{S}^6 = \mathrm{G}_2 / \mathrm{SU}(3)$ is 2-dimensional, while the space of Spin_7-invariant affine or metric connections on the 7-sphere $\mathrm{S}^7 = \mathrm{Spin}_7 / \mathrm{G}_2$ is 1-dimensional. This is a remarkable result, since the only SO_7- (resp. SO_8)-invariant affine (or metric) connection on the symmetric space $\mathrm{S}^6 = \mathrm{SO}_7 / \mathrm{SO}_6$ (resp. $\mathrm{S}^7 = \mathrm{SO}_8 / \mathrm{SO}_7$) is the canonical connection.
Motivated by this simple result, in this article we classify invariant affine connections on *(compact) non-symmetric strongly isotropy irreducible homogeneous Riemannian manifolds.* A connected effective homogeneous space G/K is called isotropy irreducible if K acts irreducibly on $T_o(G/K)$ via the isotropy representation. If the identity component K_0 of K also acts irreducibly on $T_o(G/K)$, then G/K is called *strongly isotropy irreducible*. Obviously, any strongly isotropy irreducible space (SII space for short) is also isotropy irreducible but the converse is false, see [B]. Non-symmetric strongly isotropy irreducible homogeneous spaces were originally classified by Manturov (see for example [B]) and were later studied by Wolf [Wo1] and others. Any SII space admits a unique invariant Einstein metric, the so-called Killing metric and in the non-compact case such a manifold is a symmetric space of non-compact type. In fact, SII spaces share many properties with symmetric spaces and indeed, any irreducible (as Riemannian manifold) symmetric space is strongly isotropy irreducible. A conceptual relationship between symmetric spaces and SII spaces was explained in [WZ]. More recently, isotropy irreducible homogeneous spaces endowed with their canonical connection ∇^c was shown to have a special relationship with geometric structures with torsion (see [CℓS, Cℓ]).

Outline and classification results. After recalling preliminaries in Section 2.1 about (invariant) metric connections and their torsion types, in Section 2.2 we fix a reductive homogeneous space

$(M = G/K, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$ and introduce the notion of generalized derivations of a tensor $F : \otimes^p \mathfrak{m} \to \mathfrak{m}$. When F is $\mathrm{Ad}(K)$-invariant and $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ is a K-intertwining map, we prove that μ induces a generalized derivation of F if and only if F is ∇^μ-parallel, where ∇^μ is the invariant connection on M associated to μ (Theorem 19). Moreover, we conclude that for an invariant tensor field F the operation induced by a generalized derivation $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, coincides with the covariant derivative $\nabla^\mu F$ (Corollary 1). Then we consider derivations on $\mathfrak{m}$ and provide necessary and sufficient conditions for their existence (Theorem 20), generalizing results from [C1].
Next, in Section 2.3 we present a series of new results related to invariant connections and their torsion type, on compact, effective, naturally reductive Riemannian manifolds. In particular, we examine both the symmetric and non-symmetric case and we develop some theory available for the classification of all G-invariant metric connections, with respect to a naturally reductive metric (see Lemma 10, Lemma 12, Theorem 22). In fact, in this way we correct some wrong conclusions given in [AFH, C1]. For example, for the compact Lie group U_n $(n \geq 3)$ endowed with a bi-invariant metric we present a class of bi-invariant metric connections whose torsion is not a 3-form, but of vectorial type (Theorem 23, Proposition 11).
In Section 2.4 we focus on (compact) non-symmetric, strongly isotropy irreducible homogeneous Riemannian manifolds $(M = G/K, g = -B|_{\mathfrak{m}})$ with aim the classification of all G-invariant affine or metric connections. We always work with an effective G-action, and based on our previous results on effective naturally reductive spaces we first prove that a G-invariant metric connection on $(M = G/K, g = -B|_{\mathfrak{m}})$ cannot admit a component of vectorial type (Proposition 13). Then, in the spin case we describe an application about the formal self-adjointness of Dirac operators associated to invariant metric connections on such types of homogeneous spaces (Corollary 6).
Notice now that any (effective) non-symmetric SII space $M = G/K$ admits a family of invariant metric connections induced by the $\mathrm{Ad}(K)$-invariant bilinear map $\eta^\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ with $\eta^\alpha(X, Y) := \frac{1-\alpha}{2}[X, Y]_{\mathfrak{m}}$. In full details, this family, which we call the *Lie bracket family*, has the form

$$\nabla^\alpha_X Y = \nabla^c_X Y + \eta^\alpha(X, Y) = \nabla^g_X Y + \frac{\alpha}{2} T^c(X, Y),$$

where ∇^c denotes the canonical connection associated to $\mathfrak{m}$ and ∇^g the Levi-Civita connection of the Killing metric.[1] Hence, its torsion is an invariant 3-form on M, given by $T^\alpha = \alpha \cdot T^c$, where T^c is the torsion of ∇^c (see [A, C1]). However, we will show that in general the family ∇^α does not exhaust all G-invariant metric connections, even with skew-torsion. In particular, for the classification of invariant connections on $M = G/K$ one needs to decompose the modules $\Lambda^2(\mathfrak{m})$ and $\mathrm{Sym}^2(\mathfrak{m})$ into irreducible submodules. For such a procedure we mainly use the LiE program[2], but also provide examples of how such spaces can be treated only by pure representation theory arguments, without the aid of a computer (see paragraph 2.4.5). As a result, for any effective non-symmetric (compact) SII homogeneous Riemannian manifold $(M = G/K, g = -B|_{\mathfrak{m}})$ we state the dimension of the space $\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ (see Theorem 24, Tables 2.4, 2.5). In addition to this, for any such homogeneous space we present the space of G-invariant torsion-free connections and classify the dimension of the space of G-invariant metric connections. Moreover, we state the multiplicity of the (real) trivial representation inside the space $\Lambda^3(\mathfrak{m})$ of 3-forms. This last step yields finally the presentation of the subclass of G-invariant metric connections with skew-torsion. Note that all these desired multiplicities were also obtained in [Cℓ], up to some errors/omissions, see Remark 13 and Table 2.4 for corrections. We summarize our results as follows:

Theorem A.1. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective non-symmetric SII space. Then:*
(i) The family $\{\nabla^\alpha : \alpha \in \mathbb{R}\}$ exhausts all G-invariant affine or metric connections on $M = G/K$, if and only if $G = \mathrm{Sp}_n$, or $M = G/K$ is one of the manifolds

$$\begin{array}{lllll} \mathrm{SO}_{14}/\mathrm{Sp}_3, & \mathrm{SO}_{4n}/\mathrm{Sp}_n \times \mathrm{Sp}_1\ (n \geq 2), & \mathrm{SO}_7/\mathrm{G}_2, & \mathrm{SO}_{16}/\mathrm{Spin}_9, & \mathrm{G}_2/\mathrm{SO}_3, \\ \mathrm{F}_4/(\mathrm{G}_2 \times \mathrm{SU}_2), & \mathrm{E}_7/(\mathrm{G}_2 \times \mathrm{Sp}_3), & \mathrm{E}_7/(\mathrm{F}_4 \times \mathrm{SU}_2), & \mathrm{E}_8/(\mathrm{F}_4 \times \mathrm{G}_2). & \end{array}$$

The same family exhausts also all SU_{2q}-invariant metric connections on the homogeneous space $\mathrm{SU}_{2q}/\mathrm{SU}_2 \times \mathrm{SU}_q$ $(q \geq 3)$, but not all the SU_{2q}-invariant affine connections.
(ii) The family $\{\nabla^\alpha : \alpha \in \mathbb{C}\}$ exhausts all G-invariant affine or metric connections on $M = G/K$, if and only if $M = G/K$ is one of the manifolds

$$\begin{array}{llll} \mathrm{SO}_8/\mathrm{SU}_3, & \mathrm{G}_2/\mathrm{SU}_3, & \mathrm{F}_4/(\mathrm{SU}_3 \times \mathrm{SU}_3), & \mathrm{E}_6/(\mathrm{SU}_3 \times \mathrm{SU}_3 \times \mathrm{SU}_3), \\ \mathrm{E}_7/(\mathrm{SU}_3 \times \mathrm{SU}_6), & \mathrm{E}_8/\mathrm{SU}_9, & \mathrm{E}_8/(\mathrm{E}_6 \times \mathrm{SU}_3). & \end{array}$$

[1] The parameter α can be a real or complex number, depending on the type of the isotropy representation $\mathfrak{m}$.
[2] `http://wwwmathlabo.univ-poitiers.fr/~maavl/LiE/`

For invariant metric connections different from ∇^α, we prove that

Theorem A.2. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective non-symmetric SII space which admits at least one invariant metric connection, different from the Lie bracket family. Then, $M = G/K$ is isometric to a space given in Table 2.2. In this table we present the dimensions of the spaces $\mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m})$ and $(\Lambda^3\mathfrak{m})^K$, which respectively parametrize the space of invariant metric connections and the space of invariant metric connections with totally skew-symmetric torsion. In particular:*
(i) Any homogeneous space in Table 2.2 whose isotropy representation is of real type and which is not isometric to $\mathrm{SO}_{10} / \mathrm{Sp}_2$, admits a 2-dimensional space of G-invariant metric connections with skew-torsion. For $\mathrm{SO}_{10} / \mathrm{Sp}_2$, the unique family of SO_{10}-invariant metric connections with skew-torsion is given by ∇^α ($\alpha \in \mathbb{R}$). However, the space of all SO_{10}-invariant metric connections is 2-dimensional.
(ii) Any homogeneous space in Table 2.2 whose isotropy representation is of complex type, admits a 6-dimensional space of G-invariant metric connections and a 4-dimensional subspace of G-invariant metric connections with skew-torsion.

For invariant connections induced by some $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$, we prove the following:

Theorem B. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective non-symmetric SII space, which admits at least one invariant affine connection ∇^μ, induced by some $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$. Then:*
(i) If the associated isotropy representation is of real type, then $M = G/K$ is isometric to a manifold in Table 2.3. In this table, $\mathbf{s}$ states for the dimension of the module $\mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$, which parametrizes the space of such invariant connections.
(ii) If the associated isotropy representation is of complex type, then $M = G/K$ is isometric to one of the manifolds $\mathrm{SO}_{n^2-1} / \mathrm{SU}_n$ ($n \geq 4$) or $\mathrm{E}_6 / \mathrm{SU}_3$, where the dimension of the space of such invariant connections is 2 and 4, respectively.
(iii) The G-invariant connection ∇^μ does not preserve the Killing metric $g = -B|_{\mathfrak{m}}$. Thus, ∇^μ is not metric with respect to any G-invariant metric.

Now, a small combination of Theorems A.1, A.2 and B yields the desired dimension of the space of all *G-invariant affine connections* for any non-symmetric (compact) SII space $M = G/K$,

$$\mathcal{N} := \dim_{\mathbb{R}} \mathcal{A}ff_G(F(G/K)) = \dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}).$$

We refer to the Tables 2.4 and 2.5, where the number $\mathcal{N}$ is explicitly indicated. Note that for SII homogeneous spaces $M = G/K$ of the Lie group $G = \mathrm{SU}_n$, we can describe explicitly some of the SU_n-invariant affine connections induced by a symmetric K-intertwining map $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$ (and in a few cases all such connections, see Corollary 7). We also conclude that the space of invariant *torsion-free connections* on a non-symmetric SII space $M = G/K$, denoted by $\mathcal{A}ff_G^0(F(G/K))$, is parametrized by an affine subspace of $\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, which is modelled on $\mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$ and contains the Levi-Civita connection, see Lemma 5 and Remark 1. In particular, for any G-invariant affine connection ∇^μ induced by $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$, the invariant connection $\nabla := \nabla^\mu - \frac{1}{2}T^\mu$ is torsion-free. Thus, the following is a direct consequence of Theorem B.

Corollary of Theorem B. *The classification of non-symmetric SII spaces which admit new invariant torsion-free connections, in addition to the Levi-Civita connection, is given by the manifolds of Theorem B. In particular, for a space in Table 2.3 we have $\dim_{\mathbb{R}} \mathcal{A}ff_G^0(F(G/K)) = \mathbf{s}$, and for the almost complex homogeneous spaces in Theorem B it is $\dim_{\mathbb{R}} \mathcal{A}ff_G^0(F(G/K)) = 2$ or 4, respectively.*

Classification of ∇-Einstein structures with skew-torsion. After obtaining Theorems A.1, A.2 and B, in the final Section 2.5 we turn our attention to more geometric problems. We use our classification results of Table 2 to examine ∇-Einstein structures with skew-torsion. Roughly speaking, such a structure consists of a n-dimensional connected Riemannian manifold (M, g) endowed with a metric connection ∇ which has non-trivial skew-torsion $0 \neq T \in \Lambda^3(T^*M)$ and whose Ricci tensor has symmetric part a multiple of the metric tensor, i.e. (see [FrIv, AFr2, AF, DGP, C1, C2])

$$\mathrm{Ric}_S^\nabla = \frac{\mathrm{Scal}^\nabla}{n} g.$$

For $T = 0$ the whole notion reduces to the original Einstein metrics. In fact, like Einstein metrics on compact Riemannian manifolds, in [AF] it was shown that ∇-Einstein structures can be

Table 2.2: Non-symmetric SII spaces carrying new G-invariant metric connections and the dimension of the space of global G-invariant 3-forms

Real type			
$M = G/K$ (families)	$\dim_{\mathbb{R}} M$	Invariant metric connections $\dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m})$	skew-torsion $\dim_{\mathbb{R}}(\Lambda^3\mathfrak{m})^K$
$\mathrm{SU}_{pq} / \mathrm{SU}_p \times \mathrm{SU}_q$ $(p, q \geq 3)$	$(p^2-1)(q^2-1)$	2	2
$\mathrm{SO}_{\frac{n(n-1)}{2}} / \mathrm{SO}_n$ $(n \geq 9)$	$\frac{1}{8}(6n - 5n^2 - 2n^3 + n^4)$	3	2
$\mathrm{SO}_{\frac{(n-1)(n+2)}{2}} / \mathrm{SO}_n$ $(n \geq 7)$	$\frac{1}{8}(8 - 2n - 9n^2 + 2n^3 + n^4)$	3	2
$\mathrm{SO}_{(n-1)(2n+1)} / \mathrm{Sp}_n$ $(n \geq 4)$	$\frac{1}{2}(2 + n - 9n^2 - 4n^3 + 4n^4)$	3	2
$\mathrm{SO}_{n(2n+1)} / \mathrm{Sp}_n$ $(n \geq 3)$	$\frac{1}{2}(-3n - 5n^2 + 4n^3 + 4n^4)$	3	2
(low-dim cases)			
$\mathrm{SO}_{21} / \mathrm{SO}_7$	189	3	2
$\mathrm{SO}_{28} / \mathrm{SO}_8$	350	4	2
$\mathrm{SO}_{14} / \mathrm{SO}_5$	81	3	2
$\mathrm{SO}_{20} / \mathrm{SO}_6$	175	3	2
$\mathrm{SO}_{10} / \mathrm{Sp}_2$	35	2	1
(exceptions)			
$\mathrm{SO}_{14} / \mathrm{G}_2$	70	2	2
$\mathrm{SO}_{26} / \mathrm{F}_4$	273	2	2
$\mathrm{SO}_{42} / \mathrm{Sp}_4$	825	2	2
$\mathrm{SO}_{52} / \mathrm{F}_4$	1274	2	2
$\mathrm{SO}_{70} / \mathrm{SU}_8$	2352	2	2
$\mathrm{SO}_{248} / \mathrm{E}_8$	30380	2	2
$\mathrm{SO}_{78} / \mathrm{E}_6$	2925	2	2
$\mathrm{SO}_{128} / \mathrm{Spin}_{16}$	8008	2	2
$\mathrm{SO}_{133} / \mathrm{E}_7$	8645	2	2
$\mathrm{E}_7 / \mathrm{SU}_3$	125	2	2
Complex type			
$M = G/K$	$\dim_{\mathbb{R}} M$	Invariant metric connections $\dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2(\mathfrak{m}))$	skew-torsion $\dim_{\mathbb{R}}(\Lambda^3\mathfrak{m})^K$
$\mathrm{SO}_{n^2-1} / \mathrm{SU}_n$ $(n \geq 4)$	$\frac{1}{2}(4 - 5n^2 + n^4)$	6	4
$\mathrm{E}_6 / \mathrm{SU}_3$	70	6	4

Table 2.3: Non-symmetric SII spaces of real type carrying G-invariant affine connections induced by $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$

Real type

$\mathbf{s} = 1$	$\mathbf{s} = 2$	$\mathbf{s} = 3$
$\mathrm{SU}_{10} / \mathrm{SU}_5$	$\mathrm{SU}_{\frac{n(n-1)}{2}} / \mathrm{SU}_n$ $(n \geq 6)$	$\mathrm{SO}_{28} / \mathrm{SO}_8$
$\mathrm{SO}_{\frac{n(n-1)}{2}} / \mathrm{SO}_n$ $(n \geq 9)$	$\mathrm{SU}_{\frac{n(n+1)}{2}} / \mathrm{SU}_n$ $(n \geq 3)$	$\mathrm{E}_7 / \mathrm{SU}_3$
$\mathrm{SO}_{\frac{(n-1)(n+2)}{2}} / \mathrm{SO}_n$ $(n \geq 7)$	$\mathrm{SO}_{20} / \mathrm{SO}_6$	
$\mathrm{SO}_{21} / \mathrm{SO}_7$	$\mathrm{SU}_{27} / \mathrm{E}_6$	
$\mathrm{SO}_{14} / \mathrm{SO}_5$	$\mathrm{SU}_{pq} / \mathrm{SU}_p \times \mathrm{SU}_q$ $(p, q \geq 3)$	
$\mathrm{SO}_{(n-1)(2n+1)} / \mathrm{Sp}_n$ $(n \geq 4)$		
$\mathrm{SO}_{n(2n+1)} / \mathrm{Sp}_n$ $(n \geq 3)$		
$\mathrm{SU}_{2q} / \mathrm{SU}_2 \times \mathrm{SU}_q$ $(q \geq 3)$		
$\mathrm{SO}_{10} / \mathrm{Sp}_2$		
$\mathrm{SU}_{16} / \mathrm{Spin}_{10}$		
$\mathrm{SO}_{70} / \mathrm{SU}_8$		
$\mathrm{E}_6 / \mathrm{G}_2$		
$\mathrm{E}_6 / (\mathrm{G}_2 \times \mathrm{SU}_3)$		

characterized variationally. On the other hand, the classification of ∇-Einstein structures with skew-torsion on a fixed Riemannian manifold (M, g), is initially based on the classification of all metric connections on M whose torsion is a non-trivial 3-form. For example, for odd dimensional spheres $\mathrm{S}^{2n+1} \cong \mathrm{SU}_{n+1} / \mathrm{SU}_n$ endowed with their Sasakian structure, a classification of SU_{n+1}-*invariant* ∇*-Einstein structures with skew-torsion* has been very recently given in [DGP], and it follows only after the classification of SU_{n+1}-invariant metric connections (with skew-torsion) and their description in terms of tensor fields related to the special structure (see also [AF]).
As far as we know, most well-understood examples of ∇-Einstein manifolds appear in the context of non-integrable geometries, where a metric connection with skew-torsion $0 \neq T$ is adapted to the geometry under consideration, the so-called *characteristic connection* ∇^c (see [FrIv]). This connection, which in the homogeneous case coincides with the canonical connection, plays a crucial role in the theory of special geometries and nowadays is a traditional approach to describing the associated non-integrable structure in terms of ∇^c (or the very closely related *intrinsic torsion*). Moreover, the articles [FrIv, AFr2, AF] provide some nice classes of ∇^c-Einstein structures, e.g. nearly Kähler manifolds in dimension 6, nearly-parallel G_2-manifolds in dimension 7, or 7-dimensional 3-Sasakian manifolds. Notice that these special structures admit (∇^c-parallel) real Killing spinors and hence, in some cases one can describe a deeper relation between the ∇-Einstein condition and a class of spinor fields, known as *Killing spinors with torsion*. These are natural generalizations of the original Killing spinor fields, satisfying the Killing spinor equation with respect to a metric connection with skew-torsion. Their existence is known for several types of special geometries (see [ABK, BB, C2]). For example, on 6-dimensional nearly Kähler manifolds, 7-dimensional nearly parallel G_2-manifolds, or even on $\mathrm{S}^3 \cong \mathrm{SU}_2$, such spinors are induced by the associated ∇^c-parallel spinors and their description is given in terms of whole 1-parameter families $\{\nabla^s : s \in \mathbb{R}\}$ of metric connections with skew-torsion. Moreover, their existence imposes the following strong geometric constraint: $\mathrm{Ric}^s = \frac{1}{n} \mathrm{Scal}^s g$ for any $s \in \mathbb{R}$ [C2] (although in general this is not the case, see [BB]). The special value $s = 1/4$ corresponds to the characteristic connection (which has parallel torsion T), while the parameter $s = 0$ induces the original Einstein metric related with the existent real Killing spinor.
Beside these classes of ∇-Einstein manifolds, the first author in [C1] studies homogeneous ∇-Einstein structures for more general manifolds, e.g. on compact isotropy irreducible spaces and a class of normal homogeneous manifolds with two isotropy summands. An important result for us from [C1], is that any effective compact isotropy irreducible homogeneous space $M = G/K$ which is not a symmetric space of Type I, is a ∇^α-Einstein manifold for any parameter $\alpha \neq 0$, where ∇^α is the Lie bracket family. As a consequence of the results in Section 2.4, we conclude that any (effective) non-symmetric SII homogeneous space $(M = G/K, g = -B|_{\mathfrak{m}})$ is a ∇^α-Einstein manifold for any parameter $\alpha \neq 0$. Moreover, our Lemma 12 in combination with Schur's lemma, yield a natural parameterization of the set of G-invariant ∇-Einstein structures with skew-torsion, by the space of invariant metric connections with non-trivial skew-torsion, or equivalently of the vector

space of (global) invariant 3-forms. Hence, in this case the space of all homogeneous ∇-Einstein structures with skew-torsion on $(M = G/K, g = -B|_{\mathfrak{m}})$, denoted by $\mathcal{E}^{sk}_G(\mathrm{SO}(G/K, -B|_{\mathfrak{m}}))$, can be viewed as an affine subspace of the space of all G-invariant metric connections. Combining with our classification results on G-invariant metric connections with skew-torsion (see Theorems A.1, A.2, Table 2), we finally deduce that

Theorem C. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective non-symmetric SII space and assume that the family ∇^α exhausts all G-invariant metric connections. Then, the associated space of G-invariant ∇-Einstein structures with skew-torsion has dimension either*

$$\dim_{\mathbb{R}} \mathcal{E}^{sk}_G(\mathrm{SO}(G/K, -B|_{\mathfrak{m}})) = 1, \quad \textit{or} \quad \dim_{\mathbb{R}} \mathcal{E}^{sk}_G(\mathrm{SO}(G/K, -B|_{\mathfrak{m}})) = 2,$$

for spaces with isotropy representation of real or complex type, respectively, and the manifold is one of the manifolds of Theorem A.1 or $\mathrm{SO}_{10} / \mathrm{Sp}_2$.

For the new invariant metric connections with skew-torsion, different from the family ∇^α, an explicit description seems difficult (for dimensional reasons, see Table 2.2). However, we prove that

Theorem D. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective non-symmetric SII space of Table 2.2, whose isotropy representation χ is of real type and assume that M is not isometric to $\mathrm{SO}_{10} / \mathrm{Sp}_2$. Then, the Ricci tensor associated to the 1-parameter family of invariant metric connections with skew-torsion, orthogonal to the Lie bracket family ∇^α, is also symmetric. Moreover,*

$$\dim_{\mathbb{R}} \mathcal{E}^{sk}_G(\mathrm{SO}(G/K, -B|_{\mathfrak{m}})) = 2.$$

This result is based on Theorem A.2 (Table 2.2) and the fact $(\Lambda^2\mathfrak{m})^K = 0$ for real representations of real type. This means that the space of skew-symmetric 2-forms $\Lambda^2\mathfrak{m}$ associated to a space $M = G/K$ of Theorem D (or even for the space $\mathrm{SO}_{10} / \mathrm{Sp}_2$), does not contain the trivial representation, hence there do not exist G-invariant 2-forms. Consequently, the co-differential of the torsion associated to any existent G-invariant affine metric connection on M must vanish and our assertion follows by Schur's lemma in combination with the expression of the Ricci tensor for a metric connection with skew-torsion.
Theorems C and D give the complete classification of all existent G-homogeneous ∇-Einstein structures on any effective, non-symmetric, SII space $M = G/K$, except of the quotients $\mathrm{SO}_{n^2-1} / \mathrm{SU}_n$ $(n \geq 4)$ and $\mathrm{E}_6 / \mathrm{SU}_3$. These are privileged manifolds with respect to Theorem A.2; the associated space of G-invariant metric connections with skew-torsion is 4-dimensional. Moreover, they both admit an invariant almost complex structure and hence $\Lambda^2(\mathfrak{m})$ contains a copy of the trivial representation $\mathbb{R}$ (Lemma 15), i.e. there exist G-invariant (global) 2-forms. However, since we are interested only on the symmetric part of Ric^∇ and the isotropy representation is (strongly) irreducible, again a combination of the results of Theorem A.2 with Schur's lemma, yields that

Theorem E. Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be one of the manifolds $\mathrm{SO}_{n^2-1} / \mathrm{SU}_n$ $(n \geq 4)$ or $\mathrm{E}_6 / \mathrm{SU}_3$. Then, the space of G-homogeneous ∇-Einstein structures with skew-torsion has dimension

$$\dim_{\mathbb{R}} \mathcal{E}^{sk}_G(\mathrm{SO}(G/K, -B|_{\mathfrak{m}})) = 4.$$

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2.1 Preliminaries

2.1.1 Metric connections and their types

Consider a connected, oriented Riemannian manifold (M^n, g) and identify the tangent and cotangent bundle $TM \cong T^*M$ via the bundle isomorphism provided by the metric tensor g. Any metric connection $\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM) \cong \Gamma(TM \otimes TM)$ on M can be written as $\nabla_X Y = \nabla^g_X Y + A(X, Y)$ for any $X, Y \in \Gamma(TM)$, for some tensor $A \in TM \otimes \Lambda^2(TM)$, where ∇^g is the Levi-Civita connection. Let us denote by $A(X, Y, Z) := g(A(X, Y), Z)$ the induced tensor obtained by contraction with g. The affine connections on M which are compatible with g, form an affine space modelled on the sections of the tensor bundle

$$\mathcal{A} := \{A \in \otimes^3 TM : A(X,Y,Z) + A(X,Z,Y) = 0\} \cong TM \otimes \Lambda^2(TM),$$

which has fibre dimension $n^2(n-1)/2$. It is well-known that $\mathcal{A}$ coincides with the space of torsion tensors

$$\mathcal{T} = \{A \in \otimes^3 TM : A(X,Y,Z) + A(X,Y,Z) = 0\} \cong \Lambda^2(TM) \otimes TM.$$

Moreover, under the action of the structure group SO_n it decomposes into three irreducible representations $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$, defined by

$$\begin{aligned}
\mathcal{A}_1 &:= \{A \in \mathcal{A} : A(X,Y,Z) = g(X,Y)\varphi(Z) - g(X,Z)\varphi(Y),\ \varphi \in \Gamma(T^*M)\} \cong TM,\\
\mathcal{A}_2 &:= \{A \in \mathcal{A} : \mathfrak{S}^{X,Y,Z} A(X,Y,Z) = 0, \Phi(A) = 0\},\\
\mathcal{A}_3 &:= \{A \in \mathcal{A} : A(X,Y,Z) + A(Y,X,Z) = 0\} \cong \Lambda^3 TM.
\end{aligned}$$

Here, the map $\Phi : \mathcal{A} \to T^*M$ is given by $\Phi(A)(Z) := \mathrm{tr} A_Z := \sum_i A(e_i, e_i, Z)$, for a vector field $Z \in \Gamma(TM)$ and a (local) orthonormal frame $\{e_i\}$ of M. The torsion $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ of ∇ satisfies the relation $T(X,Y) = A(X,Y) - A(Y,X)$ and conversely, A is expressed in terms of T by the condition

$$2A(X,Y,Z) = T(X,Y,Z) - T(Y,Z,X) + T(Z,X,Y), \quad \forall X,Y,Z \in \Gamma(TM). \tag{2.1.1}$$

We say that ∇ is of *vectorial type* (and the same for its torsion) if $A \in \mathcal{A}_1 \cong TM$, of *Cartan type, or traceless cyclic* if $A \in \mathcal{A}_2$ and finally (totally) *skew-symmetric* (or, of skew-torsion) if $A \in \mathcal{A}_3 \cong \Lambda^3 TM$. Notice that for $n = 2$, $\mathcal{A} \cong \mathbb{R}^2$ is irreducible. For $n \geq 3$, the mixed types occur by taking the direct sums of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$:

$$\begin{aligned}
\mathcal{A}_1 \oplus \mathcal{A}_2 &= \{A \in \mathcal{A} : \mathfrak{S}^{X,Y,Z} A(X,Y,Z) = 0\},\\
\mathcal{A}_2 \oplus \mathcal{A}_3 &= \{A \in \mathcal{A} : \Phi(A) = 0\},\\
\mathcal{A}_1 \oplus \mathcal{A}_3 &= \{A \in \mathcal{A} : A(X,Y,Z) + A(Y,X,Z) = 2g(X,Y)\varphi(Z) - g(X,Z)\varphi(Y)\\
&\qquad -g(Y,Z)\varphi(X),\ \varphi \in \Gamma(T^*M)\}.
\end{aligned}$$

Usually, connections of type $\mathcal{A}_1 \oplus \mathcal{A}_2$ are called *cyclic* and connections of type $\mathcal{A}_2 \oplus \mathcal{A}_3$ are known as *traceless* connections.
Let us finally recall that a tensor field $A \in \mathcal{A}$ satisfying $\nabla A = 0 = \nabla R$, where R denotes the curvature of the metric connection $\nabla = \nabla^g + A$ is called a *homogeneous structure.* The existence of a metric connection with these properties implies that (M,g) is locally homogeneous and if in addition (M,g) is complete, then it is locally isometric to a homogeneous Riemannian manifold. In particular, a complete, connected and simply-connected Riemannian manifold (M,g) endowed with a metric connection ∇ solving the equations $\nabla A = 0 = \nabla R$ is a homogeneous Riemannian manifold, see [TrV] for more details and proofs.

2.1.2 Connections with skew-torsion and ∇-Einstein manifolds

Let (M^n, g) be a connected Riemannian manifold carrying a metric connection ∇ with skew-torsion $0 \neq T \in \Lambda^3(TM)$, i.e.

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} T(X,Y,Z).$$

We normalize the length of T such that $\|T\|^2 := (1/6)\sum_{i,j} g(T(e_i,e_j), T(e_i,e_j))$ and we denote by $\delta^\nabla T = -\sum_{i=1}^n e_i \lrcorner \nabla_{e_i} T$ the co-differential of T. It is easy to check that $\delta^g T = \delta^\nabla T$. It is also known that (see for example [IvP, DI, FrIv])

Lemma 4. *The Ricci tensor associated to ∇ is given by*

$$\mathrm{Ric}^\nabla(X,Y) \equiv \mathrm{Ric}(X,Y) = \mathrm{Ric}^g(X,Y) - \frac{1}{4}\sum_{i=1}^n g(T(e_i,X), T(e_i,Y)) - \frac{1}{2}(\delta^g T)(X,Y).$$

Thus, in contrast to the Riemannian Ricci tensor Ric^g, the Ricci tensor of ∇ is not symmetric; it decomposes into a symmetric and antisymmetric part $\mathrm{Ric} = \mathrm{Ric}_S + \mathrm{Ric}_A$, given by

$$\mathrm{Ric}_S(X,Y) := \mathrm{Ric}^g(X,Y) - \frac{1}{4}S(X,Y), \quad \mathrm{Ric}_A(X,Y) := -\frac{1}{2}(\delta^g T)(X,Y),$$

respectively, where S is the symmetric tensor defined by $S(X,Y) = \sum_{i=1}^n g(T(e_i,X),T(e_i,Y))$.

Definition 1. ([AF]) *A triple (M,g,T) is called a ∇-Einstein manifold with non-trivial skew-torsion $0 \neq T \in \Lambda^3(TM)$, or for short, a ∇-Einstein manifold, if the symmetric part Ric_S of the Ricci tensor associated to the metric connection $\nabla = \nabla^g + \frac{1}{2}T$ satisfies the equation*

$$\mathrm{Ric}_S = \frac{\mathrm{Scal}}{n} g, \tag{2.1.2}$$

where $\mathrm{Scal} \equiv \mathrm{Scal}^\nabla$ is the scalar curvature associated to ∇ and $n = \dim_{\mathbb{R}} M$. If $\nabla T = 0$, then (M,g,T) is called a ∇-Einstein manifold with parallel skew-torsion.

Notice that in contrast to the Riemannian case, for a ∇-Einstein manifold the scalar curvature $\mathrm{Scal}^\nabla \equiv \mathrm{Scal} = \mathrm{Scal}^g - \frac{3}{2}\|T\|^2$ is not necessarily constant (for details see [AF]). For parallel torsion, i.e. $\nabla T = 0$, one has $\delta^\nabla T = 0$ and the Ricci tensor becomes symmetric $\mathrm{Ric} = \mathrm{Ric}_S$. If in addition $\delta \,\mathrm{Ric}^g = 0$, then the scalar curvature is constant, similarly to an Einstein manifold. This is the case for any ∇-Einstein manifold (M,g,∇,T) with *parallel skew-torsion* [AF, Prop. 2.7].

2.1.3 Invariant connections

Consider a Lie group G acting transitively on a smooth manifold M and let us denote by $\pi : P \to M$ a G-homogeneous principal bundle over M with structure group U. Let K be the isotropy subgroup at the point $o = \pi(p_0) \in M$ with $p_0 \in P$ (this is a closed subgroup $K \subset G$). Then, there is a Lie group homomorphism $\lambda : K \to U$ and hence an action of K on U, given by $ku = \lambda(k)u$. This induces a G-homogeneous principal U-bundle $P_\lambda \to M = G/K$, defined by $P_\lambda := G \times_K U = G \times_\lambda U = G \times U/\sim$, where $(g,u) \sim (gk, \lambda(k^{-1})u)$ for any $g \in G, u \in U, k \in K$. Because the left action of G on P restricts to a left action of K on the fiber P_o of P over a base point $o = eK \in G/K$, for the original bundle P we have $P \cong G \times_K P_o$. But fixing a point $u_0 \in P_o$ we see that the map $U \to P_o$, $u \mapsto u_0 u$ is a diffeomorphism and hence we identify $P \cong G \times_K P_o = G \times_K U = P_\lambda$, see also [ČS, L2].
For G-homogeneous principal U-bundles $P \cong P_\lambda \to G/K$, it makes sense to speak about G-invariant connections, i.e. connections for which the horizontal subspaces $\mathcal{H}_p$ are also invariant by the left G-action, $(L_g)_*\mathcal{H}_p = \mathcal{H}_{gp}$ for any $g \in G$ and $p \in P$. In other words, a connection in P_λ is G-invariant if and only if the associated connection form $Z \in \Omega^1(P,\mathfrak{u})$ is such that $(\tau'_g)^*Z = Z$, for all $g \in G$, where $\tau'_g : P \to P$ is the (right) U-equivariant bundle map.

Theorem 18. ([W]) *Let $P \cong P_\lambda \to G/K$ be a G-homogeneous principal U-bundle associated to a homomorphism $\lambda : K \to U$, as above. Then, G-invariant connections on P_λ are in a bijective correspondence with linear mappings $\Lambda : \mathfrak{g} \to \mathfrak{u}$ satisfying the following conditions:*
(a) $\Lambda(X) = \lambda_*(X)$, for all $X \in \mathfrak{k} = T_eK$, where $\lambda_* : \mathfrak{k} \to \mathfrak{u}$ is the differential of λ,
(b) $\Lambda(\mathrm{Ad}(k)X) = \mathrm{Ad}(\lambda(k))\Lambda(X)$, for all $X \in \mathfrak{g} = T_eG$, $k \in K$.

2.1.4 Reductive homogeneous spaces

Consider now a reductive homogeneous space $M = G/K$, i.e. we assume that there is an orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of $\mathfrak{g} = T_eG$ with $\mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$. Then we may identify $\mathfrak{m} = T_oM$ at $o = eK \in M$ and the isotropy representation $\chi : K \to \mathrm{Aut}(\mathfrak{m})$ of K with the restriction of the adjoint representation $\mathrm{Ad}|_K$ on $\mathfrak{m}$. Therefore, there is a direct sum decomposition $\mathrm{Ad}\big|_K = \mathrm{Ad}_K \oplus \chi$ where Ad_K is the adjoint representation of K. As a further consequence, we identify the tangent bundle TM and the frame bundle $F(M)$ of $M = G/K$ with the homogeneous vector bundle $G \times_K \mathfrak{m}$ and the homogeneous principal bundle $G \times_K \mathrm{GL}(\mathfrak{m})$, respectively, the latter with structure group $\mathrm{GL}(\mathfrak{m}) = \mathrm{GL}_n \mathbb{R}$ ($n = \dim_{\mathbb{R}} \mathfrak{m} = \dim_{\mathbb{R}} M$).
An invariant affine connection on $M = G/K$ is a principal connection on $F(G/K)$ that is G-invariant. By Theorem 18 such an affine connection is described by a $\mathbb{R}$-linear map $\Lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ which is equivariant under the isotropy representation, i.e. $\Lambda(\mathrm{Ad}(k)X) = \mathrm{Ad}(k)\Lambda(X)\,\mathrm{Ad}(k)^{-1}$

for any $X \in \mathfrak{m}$ and $k \in K$. Let us denote by $\mathrm{Hom}_K(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ the set of such linear maps. The assignment $\Lambda(X)Y = \eta(X, Y)$ provides an identification of $\mathrm{Hom}_K(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ (and hence of the space of G-invariant affine connections on $M = G/K$) with the set of all $\mathrm{Ad}(K)$-equivariant bilinear maps $\eta : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$, i.e.

$$\eta(\mathrm{Ad}(k)X, \mathrm{Ad}(k)Y) = \mathrm{Ad}(k)\eta(X, Y), \tag{2.1.3}$$

for any $X, Y \in \mathfrak{m}$ and $k \in K$. Moreover, since any such map $\eta : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ induces a unique linear map $\tilde{\eta} : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ with $\tilde{\eta}(X \otimes Y) = \eta(X, Y)$, one may further identify (see [L2, Thm. 5.1])

$$\mathcal{A}ff_G\big(F(G/K)\big) \cong \mathrm{Hom}_K(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m})) \cong \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}),$$

where in general $\mathcal{A}ff_G(P)$ denotes the affine space of G-invariant affine connections on a homogeneous principal bundle $P \to G/K$ over $M = G/K$ and $\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ is the space of K-intertwining maps $\mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$. Usually we shall work with K connected and in this case we may identify $\mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m}, \mathfrak{m}) = \mathrm{Hom}_{\mathfrak{k}}(\mathfrak{m}\otimes\mathfrak{m}, \mathfrak{m})$. Due to the orthogonal splitting $\mathfrak{m}\otimes\mathfrak{m} = \Lambda^2\mathfrak{m}\oplus\mathrm{Sym}^2\,\mathfrak{m}$ we also remark that

$$\mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}) = \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m}) \oplus \mathrm{Hom}_K(\mathrm{Sym}^2, \mathfrak{m}). \tag{2.1.4}$$

The linear map $\Lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ is usually called *Nomizu map* or *connection map* (for details see [AVL, KN]) and it satisfies the relation $\Lambda(X) = -(\nabla_X - L_X)_o$, where L_X is the Lie derivative with respect to X. Hence it encodes most of the properties of ∇; for example, the torsion $T \in \Lambda^2(\mathfrak{m})\otimes\mathfrak{m}$ and curvature $R \in \Lambda^2(\mathfrak{m}) \otimes \mathfrak{k}$ of ∇ are given by:

$$T(X, Y)_o = \Lambda(X)Y - \Lambda(Y)X - [X, Y]_{\mathfrak{m}}, \quad R(X, Y)_o = [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]_{\mathfrak{m}}) - \mathrm{ad}([X, Y]_{\mathfrak{k}}).$$

Lemma 5. *Let $M = G/K$ be a homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Let $\Lambda, \Lambda' \in \mathrm{Hom}_{\mathfrak{k}}(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ be two connection maps and let $\nabla, \nabla' \in \mathcal{A}ff_G\big(F(G/K)\big)$ be the associated G-invariant affine connections. Set $\eta := \Lambda - \Lambda'$. Then*
(i) ∇ and ∇' have the same geodesics if and only if $\eta \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$.
(ii) ∇ and ∇' have the same torsion if and only if $\eta \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$.

Consider now a homogeneous Riemannian manifold $(M = G/K, g)$. In this case G can be considered as a closed subgroup of the full isometry group $\mathrm{Iso}(M, g)$, which implies that K and the Lie subgroup $\mathrm{Ad}(K) \subset \mathrm{Ad}(G)$ are compact subgroups. Hence, there is always a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with respect to some $\mathrm{Ad}(K)$-invariant inner product in the Lie algebra $\mathfrak{g}$. We shall denote by $\langle\ ,\ \rangle$ the $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$ induced by g. We equivariantly identify the K-modules $\mathfrak{so}(\mathfrak{m}, g) \equiv \mathfrak{so}(\mathfrak{m}) = \Lambda^2(\mathfrak{m})$ via the isomorphism $X \wedge Y \mapsto \langle X, \cdot\rangle Y - \langle Y, \cdot\rangle X$, for any $X, Y \in \mathfrak{m}$. Consider the $\mathrm{SO}(\mathfrak{m})$-principal bundle $\mathrm{SO}(G/K) \to G/K$ of $\langle\ ,\ \rangle$-orthonormal frames. This is a homogeneous principal bundle and an invariant metric connection on $M = G/K$ is a principal connection on $\mathrm{SO}(G/K)$ that is G-invariant. It follows that

Lemma 6. *A G-invariant affine connection ∇ on $(M = G/K, g)$ preserves the G-invariant Riemannian metric g if and only if the associated Nomizu map satisfies $\Lambda(X) \in \mathfrak{so}(\mathfrak{m}, g)$ for any $X \in \mathfrak{m}$.*

Notice that the existence of an invariant metric means that the isotropy representation of $M = G/K$ is self-dual, $\mathfrak{m} \simeq \mathfrak{m}^*$. Thus we may equivariantly identify

$$\mathfrak{gl}(\mathfrak{m}) \simeq \mathrm{End}(\mathfrak{m}) \simeq \mathfrak{m} \otimes \mathfrak{m}, \quad \mathrm{Hom}_K(\mathfrak{m}, \mathrm{End}(\mathfrak{m})) = (\mathfrak{m}^* \otimes \mathfrak{m}^* \otimes \mathfrak{m})^K \simeq (\otimes^3\mathfrak{m})^K.$$

In the last case, a K-equivariant map Λ on the left hand side is equivalent to a K-invariant tensor on the right hand side: $\mathrm{Hom}_K(\mathfrak{m}, \mathrm{End}(\mathfrak{m})) = (\otimes^3\mathfrak{m})^K$. The latter space has the following obvious K-submodules: $\Lambda^2\mathfrak{m} \otimes \mathfrak{m}$, $\mathrm{Sym}^2\,\mathfrak{m} \otimes \mathfrak{m}$, $\mathfrak{m} \otimes \mathrm{Sym}^2\,\mathfrak{m}$ and $\mathfrak{m} \otimes \Lambda^2\mathfrak{m}$. Of these, the last space corresponds to the $\mathfrak{so}(\mathfrak{m})$-valued Nomizu maps, i.e. the space of homogeneous metric connections which we denote by $\mathcal{M}_G(\mathrm{SO}(G/K))$. In particular, there is an equivariant isomorphism

$$\mathcal{M}_G(\mathrm{SO}(G/K, g)) \cong \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m}).$$

Remark 1. The other submodules have different interpretations. For example, $\mathrm{Sym}^2\,\mathfrak{m} \otimes \mathfrak{m}$ is the vector space on which the affine space of invariant torsion-free connections $\mathcal{A}ff^0_G(F(G/K))$ is modelled, and $\Lambda^2\mathfrak{m}\otimes\mathfrak{m}$ is the vector space on which the affine space of possible invariant torsion tensors is modelled. In fact, since the rearrangement of indices is equivariant (even with respect to the bigger algebra $\mathfrak{gl}(\mathfrak{m})$), one has the following isomorphisms: $\Lambda^2\mathfrak{m}\otimes\mathfrak{m}\simeq\mathfrak{m}\otimes\Lambda^2\mathfrak{m}$ and $\mathrm{Sym}^2\,\mathfrak{m}\otimes\mathfrak{m}\simeq\mathfrak{m}\otimes\mathrm{Sym}^2\,\mathfrak{m}$. Let us now relate this to the question of multiplicities of $\mathfrak{m}$ inside $\otimes^2\mathfrak{m}=\mathrm{End}(\mathfrak{m})$. Suppose we have a copy of $\mathfrak{m}$ inside the invariant decomposition of $\Lambda^2\mathfrak{m}$ (or respectively, in $\mathrm{Sym}^2\,\mathfrak{m}$). This is equivalent to a map $\theta:\mathfrak{m}\to\Lambda^2\mathfrak{m}$ (respectively $\mathfrak{m}\to\mathrm{Sym}^2\,\mathfrak{m}$). We may then raise all indices of θ to produce a K-invariant element of $\otimes^3\mathfrak{m}$. However through our freedom to rearrange indices, we may change to which of our four submodules this tensor belongs. For example, one may interpret the tensor corresponding to the instance of $\mathfrak{m}$ in $\Lambda^2\mathfrak{m}$ either as a metric connection in $\mathfrak{m}\otimes\Lambda^2\mathfrak{m}$, or a potentially non-metric connection in $\Lambda^2\mathfrak{m}\otimes\mathfrak{m}$. These coincide up to a scalar when $\theta\in\Lambda^3\mathfrak{m}$.

On a homogeneous Riemannian manifold $(M=G/K,g)$ the Levi-Civita connection ∇^g is the unique G-invariant metric connection determined by (cf. [B, NRS])

$$\langle\nabla^g_XY,Z\rangle=-\frac{1}{2}\big[\langle[X,Y]_{\mathfrak{m}},Z\rangle+\langle[Y,Z]_{\mathfrak{m}},X\rangle-\langle[Z,X]_{\mathfrak{m}},Y\rangle\big],\quad\forall\ X,Y,Z\in\mathfrak{m}.\quad(\star)$$

On the other hand, the *canonical connection* on $M=G/K$ is induced by the principal K-bundle $G\to G/K$ and depends on the choice of the reductive complement $\mathfrak{m}$. It is defined by the horizontal distribution $\{\mathcal{H}_g:=d\ell_g(\mathfrak{m}):g\in G\}$, where ℓ_g denotes the left translation on G and its Nomizu map is given by $\Lambda^c:\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}\xrightarrow{\mathrm{pr}_{\mathfrak{k}}}\mathfrak{k}\xrightarrow{\chi_*}\mathfrak{so}(\mathfrak{m})$, i.e. $\Lambda^c=\chi_*\circ\mathrm{pr}_{\mathfrak{k}}$. Thus, $\Lambda^c(X)=0$ for any $X\in\mathfrak{m}$ (cf. [KN, AVL]). Both the torsion $T^c(X,Y)=-[X,Y]_{\mathfrak{m}}$ and the curvature $R^c(X,Y)=-\operatorname{ad}([X,Y]_{\mathfrak{k}})$ of ∇^c are parallel objects, in particular any G-invariant tensor field on $M=G/K$ is ∇^c-parallel (cf. [N, KN]). Hence, any homogeneous Riemannian manifold $(M=G/K,g)$ admits a homogeneous structure $A^c\in\mathfrak{m}\otimes\Lambda^2\mathfrak{m}\cong\mathcal{A}$ induced by the canonical connection ∇^c associated to the reductive decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$. In the following, we shall refer to this homogeneous structure as the *canonical homogeneous structure*, adapted to $\mathfrak{m}$ and G. Using $(\star)$ it is easy to see that $A^c:=\nabla^c-\nabla^g$ satisfies the relation

$$A^c(X,Y,Z)=\frac{1}{2}T^c(X,Y,Z)-\langle U(X,Y),Z\rangle,\quad\forall\ X,Y,Z\in\mathfrak{m},\tag{2.1.5}$$

where $U:\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ is the symmetric bilinear mapping defined by

$$2\langle U(X,Y),Z\rangle=\langle[Z,X]_{\mathfrak{m}},Y\rangle+\langle X,[Z,Y]_{\mathfrak{m}}\rangle.\tag{2.1.6}$$

2.2 Invariant connections and derivations

Given a reductive homogeneous space $M=G/K$ endowed with a G-invariant affine connection ∇, in the following we examine $\mathrm{Ad}(K)$-equivariant derivations on $\mathfrak{m}$ induced by ∇ in terms of Nomizu maps. For the case of a compact Lie group G, this problem has been analyzed in [C1].

2.2.1 Derivations and generalized derivations

For the following of this section let us fix a (connected) homogeneous manifold $M=G/K$ with a reductive decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$. For simplicity we assume that the transitive G-action is effective. We consider a bilinear mapping $\mu:\mathfrak{m}\otimes\mathfrak{m}\to\mathfrak{m}$ and denote by $\Lambda:\mathfrak{m}\to\mathfrak{gl}(\mathfrak{m})$ the adjoint map, defined by $\Lambda(X)Y=\mu(X,Y)$.

Definition 2. The endomorphism $\Lambda(Z):\mathfrak{m}\to\mathfrak{m}$ ($Z\in\mathfrak{m}$) is called a *derivation* of $\mathfrak{m}$, with respect to the Lie bracket operation $\mathrm{ad}_{\mathfrak{m}}:=[\ ,\ ]_{\mathfrak{m}}:\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$, $\mathrm{ad}_{\mathfrak{m}}(X,Y):=[X,Y]_{\mathfrak{m}}$, if and only if $\mathfrak{Der}^{\mu}(X,Y;Z)=0$ identically, where for any $X,Y,Z\in\mathfrak{m}$ we set

$$\begin{aligned}\mathfrak{Der}^{\mu}(X,Y;Z)\ &:=\ \Lambda(Z)[X,Y]_{\mathfrak{m}}-[\Lambda(Z)X,Y]_{\mathfrak{m}}-[X,\Lambda(Z)Y]_{\mathfrak{m}}\\ &=\ \mu(Z,[X,Y]_{\mathfrak{m}})-[\mu(Z,X),Y]_{\mathfrak{m}}-[X,\mu(Z,Y)]_{\mathfrak{m}}.\end{aligned}$$

From now on, let us denote by $\mathrm{Der}(\mathrm{ad}_{\mathfrak{m}};\mathfrak{m})\equiv\mathrm{Der}(\mathfrak{m})$ the vector space of all derivations on $\mathfrak{m}$. We mention that given a bilinear map $\mu:\mathfrak{m}\otimes\mathfrak{m}\to\mathfrak{m}$, the condition $\mu\in\mathrm{Der}(\mathfrak{m})$ is equivalent to say that the associated connection map Λ is valued in $\mathrm{Der}(\mathfrak{m})$, i.e. $\Lambda\in\mathrm{Hom}(\mathfrak{m},\mathrm{Der}(\mathfrak{m}))$. Restricting on K-intertwining maps $\mu\in\mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m},\mathfrak{m})$ the vector space $\mathrm{Der}(\mathfrak{m})$ becomes a K-module, denoted by $\mathrm{Der}_K(\mathfrak{m})$. In fact, in this case we shall speak about $\mathrm{Ad}(K)$-equivariant derivations on $\mathfrak{m}$. So, let us focus on $\mathrm{Ad}(K)$-equivariant derivations induced by invariant connections on $M=G/K$.

Proposition 8. *Let $\nabla \equiv \nabla^{\mu}$ be a G-invariant connection on $M = G/K$ corresponding to $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$. Then, ∇^{μ} induces a $\mathrm{Ad}(K)$-equivariant derivation $\mu \in \mathrm{Der}_K(\mathfrak{m})$, if and only if $\mathrm{ad}_{\mathfrak{m}} := [\ ,\ ]_{\mathfrak{m}}$ is ∇^{μ}-parallel, i.e. $\nabla^{\mu}\,\mathrm{ad}_{\mathfrak{m}} = 0$ (which is equivalent to say that the torsion T^c of the canonical connection ∇^c associated to the reductive complement $\mathfrak{m}$ is ∇^{μ}-parallel, i.e. $\nabla^{\mu} T^c = 0$).*

Proof. The equivalence $\mu \in \mathrm{Der}(\mathrm{ad}_{\mathfrak{m}}; \mathfrak{m}) \equiv \mathrm{Der}(\mathfrak{m}) \ \Leftrightarrow\ \nabla^{\mu}\,\mathrm{ad}_{\mathfrak{m}} \equiv 0$ is an immediate consequence of the identity

$$\mathfrak{der}^{\mu}(X, Y; Z) = (\nabla^{\mu}_{Z}\,\mathrm{ad}_{\mathfrak{m}})(X, Y) = -(\nabla^{\mu}_{Z} T^c)(X, Y), \quad \forall\ X, Y, Z \in \mathfrak{m}. \tag{2.2.1}$$

The proof of (2.2.1) relies on the fact that G-invariant tensor fields are ∇^c-parallel, where ∇^c is the canonical connection associated to $\mathfrak{m}$. In particular, since ∇ is a G-invariant connection we write $\nabla^{\mu}_{Z} = \nabla^{c}_{Z} + \Lambda(Z)$, for any $Z \in \mathfrak{m}$, where $\Lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ is the associated Nomizu map. Then, for any $X, Y, Z \in \mathfrak{m}$ we obtain that

$$\begin{aligned}
(\nabla^{\mu}_{Z}\,\mathrm{ad}_{\mathfrak{m}})(X, Y) &= \nabla^{\mu}_{Z}\,\mathrm{ad}_{\mathfrak{m}}(X, Y) - \mathrm{ad}_{\mathfrak{m}}(\nabla^{\mu}_{Z} X, Y) - \mathrm{ad}_{\mathfrak{m}}(X, \nabla^{\mu}_{Z} Y) \\
&= \big[\nabla^{c}_{Z}\,\mathrm{ad}_{\mathfrak{m}}(X, Y) - \mathrm{ad}_{\mathfrak{m}}(\nabla^{c}_{Z} X, Y) - \mathrm{ad}_{\mathfrak{m}}(X, \nabla^{c}_{Z} Y)\big] \\
&\quad + \big[\Lambda(Z)\,\mathrm{ad}_{\mathfrak{m}}(X, Y) - \mathrm{ad}_{\mathfrak{m}}(\Lambda(Z) X, Y) - \mathrm{ad}_{\mathfrak{m}}(X, \Lambda(Z) Y)\big] \\
&= (\nabla^{c}_{Z}\,\mathrm{ad}_{\mathfrak{m}})(X, Y) + \mathfrak{der}^{\mu}(X, Y; Z) = \mathfrak{der}^{\mu}(X, Y; Z),
\end{aligned}$$

where the last equality follows since $\nabla^c\,\mathrm{ad}_{\mathfrak{m}} \equiv 0$. Similarly for the second equality in (2.2.1). $\square$

Example 1. The canonical connection ∇^c associated to the reductive complement $\mathfrak{m}$ induces a derivation on $\mathfrak{m}$ (the zero one, corresponding to $0 \in \mathrm{Der}_K(\mathfrak{m})$), since $\nabla^c T^c = 0$, or in other words since T^c is ∇^{μ}-parallel, where $\mu = 0 \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$.

Let us now generalize the notion of derivations on $\mathfrak{m}$, as follows:

Definition 3. *Consider a tensor $F : \otimes^p \mathfrak{m} \to \mathfrak{m}$. Then, a bilinear mapping $\mu : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ is said to be a generalized derivation of F on $\mathfrak{m}$, if and only if μ satisfies the relation*

$$\begin{aligned}
\mu(Z, F(X_1, \dots, X_p)) &= F(\mu(Z, X_1), X_2, \dots, X_p) + \cdots + F(X_1, \dots, X_{p-1}, \mu(Z, X_p)) \ \Leftrightarrow \\
\Lambda(Z) F(X_1, \dots, X_p) &= F(\Lambda(Z) X_1, X_2, \dots, X_p) + \cdots + F(X_1, \dots, X_{p-1}, \Lambda(Z) X_p),
\end{aligned}$$

for any $Z, X_1, \dots, X_p \in \mathfrak{m}$, where $\Lambda \in \mathrm{Hom}(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m}))$ is the adjoint map induced by μ.

For a tensor $F : \otimes^p \mathfrak{m} \to \mathfrak{m}$, the definition of a generalized derivation implies that if $\mu_1, \mu_2 : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ are two such bilinear mappings, then the linear combination $a\mu_1 + b\mu_2$ is also a generalized derivation of F on $\mathfrak{m}$. Hence, the set $\mathrm{Der}(F; \mathfrak{m})$ of all generalized derivations of F on $\mathfrak{m}$ is a vector space. Obviously, for $F = \mathrm{ad}_{\mathfrak{m}}$, a generalized derivation is just a classical derivation on $\mathfrak{m}$. Notice however that F can be much more general than the Lie bracket restriction, e.g. the torsion, or the curvature of a G-invariant connection ∇ on $M = G/K$ induced by some $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, or even μ itself. In particular, one may restrict Definition 3 on K-intertwining maps $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$; then, the space $\mathrm{Der}(F; \mathfrak{m})$ becomes a K-module, which we shall denote by $\mathrm{Der}_K(F; \mathfrak{m})$. If moreover we focus on G-invariant tensor fields, then similarly to Proposition 8 we conclude that

Theorem 19. *Let $(M = G/K, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$ be a reductive homogeneous space endowed with an $\mathrm{Ad}(K)$-invariant tensor $F : \otimes^p \mathfrak{m} \to \mathfrak{m}$. Consider a K-intertwining map $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ and let us denote by ∇^{μ} the associated G-invariant affine connection. Then, μ is an $\mathrm{Ad}(K)$-equivariant generalized derivation of F if and only if F is ∇^{μ}-parallel, i.e. $\mu \in \mathrm{Der}_K(F; \mathfrak{m}) \ \Leftrightarrow\ \nabla^{\mu} F \equiv 0$.*

Proof. A direct computation shows that the evaluation of the covariant differentiation ∇F at the point $o = eK \in G/K$ gives rise to the following $\mathrm{Ad}(K)$-invariant tensor on $\mathfrak{m}$:

$$\begin{aligned}
(\nabla_Z F)(X_1, \ldots, X_p) &= \nabla_Z F(X_1, \ldots, X_p) - \sum_{i=1}^{p} F(X_1, \ldots, \nabla_Z X_i, \ldots, X_p) \\
&= \nabla^c_Z F(X_1, \ldots, X_p) + \Lambda(Z) F(X_1, \ldots, X_p) - \sum_{i=1}^{p} F(X_1, \ldots, \nabla^c_Z X_i, \ldots, X_p) \\
&\quad - \sum_{i=1}^{p} F(X_1, \ldots, \Lambda(Z) X_i, \ldots, X_p) \\
&= \nabla^c_Z F(X_1, \ldots, X_p) - \sum_{i=1}^{p} F(X_1, \ldots, \nabla^c_Z X_i, \ldots, X_p) \\
&\quad + \Lambda(Z) F(X_1, \ldots, X_p) - \sum_{i=1}^{p} F(X_1, \ldots, \Lambda(Z) X_i, \ldots, X_p) \\
&= (\nabla^c_Z F)(X_1, \ldots, X_p) + (\mathcal{D}^\mu_Z F)(X_1, \ldots, X_p),
\end{aligned}$$

where we set $(\mathcal{D}^\mu_Z F)(X_1, \ldots, X_p) := \Lambda(Z) F(X_1, \ldots, X_p) - \sum_{i=1}^{p} F(X_1, \ldots, \Lambda(Z) X_i, \ldots, X_p)$. However, F is by assumption G-invariant, hence $\nabla^c F = 0$ and our claim immediatelly follows. $\square$

Moreover, we conclude that

Corollary 1. *On a reductive homogeneous space $(M = G/K, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$, given an $\mathrm{Ad}(K)$-invariant tensor $F : \otimes^p \mathfrak{m} \to \mathfrak{m}$ and some K-intertwining map $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, the operation*

$$(\mathcal{D}^\mu_Z F)(X_1, \ldots, X_p) := \Lambda(Z) F(X_1, \ldots, X_p) - \sum_{i=1}^{p} F(X_1, \ldots, \Lambda(Z) X_i, \ldots, X_p)$$

coincides with the covariant differentiation of F with respect to the connection $\nabla = \nabla^\mu$ induced on $M = G/K$ by μ, i.e. $(\nabla^\mu_Z F)(X_1, \ldots, X_p) = (\mathcal{D}^\mu_Z F)(X_1, \ldots, X_p)$ for any $X_1, \ldots, X_p, Z \in \mathfrak{m}$.

For a bilinear mapping $\mu : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ let us now introduce the tensor $\mathcal{C}^\mu$, defined by

$$\mathcal{C}^\mu(X, Y; Z) := (\nabla^\mu_Z \mu)(X, Y) - (\nabla^\mu_Z \mu)(Y, X),$$

for any $X, Y, Z \in \mathfrak{m}$. If $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, then we get the further identification $\mathcal{C}^\mu(X, Y; Z) := (\mathcal{D}^\mu_Z \mu)(X, Y) - (\mathcal{D}^\mu_Z \mu)(Y, X)$. In terms of $\mathcal{C}^\mu$ we obtain that

Proposition 9. *Let $\nabla = \nabla^\mu$ be a G-invariant affine connection on a reductive homogeneous space $(M = G/K, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$, corresponding to some $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$. Then, $\mu \in \mathrm{Der}_K(\mathfrak{m})$, if and only if*

$$(\nabla_Z T)(X, Y) \equiv (\mathcal{D}^\mu_Z T)(X, Y) = \mathcal{C}^\mu(X, Y; Z), \quad \forall\ X, Y, Z \in \mathfrak{m},$$

where $T = T^\mu$ is the torsion associated to ∇^μ.

Proof. As in the proof of Theorem 19, we easily get that

$$(\nabla_Z T)(X, Y) = (\mathcal{D}^\mu_Z T)(X, Y) = \mu(Z, T(X, Y)) - T(\mu(Z, X), Y) - T(X, \mu(Z, Y)) \tag{2.2.2}$$

for any $X, Y, Z \in \mathfrak{m}$. We will show now that the left hand side reduces to $(\mathcal{D}^\mu_Z T^c)(X, Y) + \mathcal{C}^\mu(X, Y; Z)$. For this, notice first that

$$\begin{aligned}
(\nabla_Z T)(X, Y) &= \mu(Z, \mu(X, Y)) - \mu(Z, \mu(Y, X)) - \mu(\mu(Z, X), Y) + \mu(Y, \mu(Z, X)) \\
&\quad - \mu(X, \mu(Z, Y)) + \mu(\mu(Z, Y), X) - \mathfrak{der}_{\mathfrak{m}}(X, Y; Z).
\end{aligned}$$

An easy computation also gives that

$$\begin{aligned}
(\mathcal{D}^\mu_Z \mu)(X, Y) - (\mathcal{D}^\mu_Z \mu)(Y, X) &= \mu(Z, \mu(X, Y)) - \mu(Z, \mu(Y, X)) - \mu(\mu(Z, X), Y) + \mu(Y, \mu(Z, X)) \\
&\quad - \mu(X, \mu(Z, Y)) + \mu(\mu(Z, Y), X).
\end{aligned}$$

Hence $(\mathcal{D}^\mu_Z T)(X, Y) = (\mathcal{D}^\mu_Z T^c)(X, Y) + \mathcal{C}^\mu(X, Y; Z)$ and in combination with (2.2.1) one can easily finish the proof. $\square$

Consequently, for some $\mu \in \mathrm{Hom}_K(\mathfrak{m}\otimes\mathfrak{m},\mathfrak{m})$ the condition $\mu \in \mathrm{Der}_K(\mathfrak{m})$ can also be read in terms of the $\mathrm{Ad}(K)$-invariant tensor $\mathcal{C}^{\mu}$, which geometrically, represents the difference

$$(\nabla_Z T)(X,Y) - (\nabla_Z T^c)(X,Y) \equiv (\mathcal{D}^{\mu}_Z T)(X,Y) - (\mathcal{D}^{\mu}_Z T^c)(X,Y),$$

for any $X,Y,Z \in \mathfrak{m}$. In particular, a combination of Proposition 9 and identity (2.1.4), yields that

Theorem 20. *Let $M = G/K$ an effective homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$. Then the following hold:*
(1) A G-invariant affine connection $\nabla = \nabla^{\mu}$ on $M = G/K$ corresponding to $\mu \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m},\mathfrak{m})$, induces an $\mathrm{Ad}(K)$-equivariant derivation $\mu \in \mathrm{Der}_K(\mathfrak{m})$, if and only if

$$(\nabla^{\mu}_Z T)(X,Y) \equiv (\mathcal{D}^{\mu}_Z T)(X,Y) = 2\mathfrak{S}_{X,Y,Z}\,\mu(X,\mu(Y,Z)), \tag{2.2.3}$$

for any $X,Y,Z \in \mathfrak{m}$. This is equivalent to say that

$$(\nabla^{\mu}_Z T)(X,Y) \equiv (\mathcal{D}^{\mu}_Z T)(X,Y) = 2\big\{R(Z,X)Y + \Lambda(Y)(\Lambda(Z)X - [Z,X]_{\mathfrak{m}}) + \mathrm{ad}([Z,X]_{\mathfrak{k}})Y\big\}, \tag{2.2.4}$$

where R is the curvature tensor associated to ∇.
(2) A G-invariant affine connection $\nabla = \nabla^{\mu}$ on $M = G/K$ corresponding to $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\mathfrak{m},\mathfrak{m})$, induces an $\mathrm{Ad}(K)$-equivariant derivation $\mu \in \mathrm{Der}_K(\mathfrak{m})$ on $\mathfrak{m}$ if and only if the torsion T^{μ} associated to ∇^{μ} is ∇^{μ}-parallel.
(3) Let $\mu \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m},\mathfrak{m})$. Then μ is an $\mathrm{Ad}(K)$-equivariant generalized derivation of itself, i.e. $\mu \in \mathrm{Der}_K(\mu;\mathfrak{m})$ if and only if $\mathcal{C}^{\mu} = 0$ identically.

Proof. For a skew-symmetric mapping $\mu \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m},\mathfrak{m})$ a simple computation gives that

$$\mathcal{C}^{\mu}(X,Y;Z) = 2\mathfrak{S}_{X,Y,Z}\,\mu(X,\mu(Y,Z)).$$

Hence, (2.2.3) is an immediate consequence of Proposition 9. For the second relation (2.2.4), using the definition of the curvature tensor R and (2.2.2), for some $\mu \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m},\mathfrak{m})$ we get that

$$\begin{aligned}(\nabla_Z T)(X,Y) = (\mathcal{D}^{\mu}_Z T)(X,Y) \;&=\; 2R(Z,X)Y + 2\Lambda(Y)(\Lambda(Z)X - [Z,X]_{\mathfrak{m}})\\ &\quad +2\,\mathrm{ad}([Z,X]_{\mathfrak{k}})Y - \mathfrak{der}^{\mu}(X,Y;Z),\end{aligned} \tag{2.2.5}$$

for any $X,Y,Z \in \mathfrak{m}$ and our claim immediately follows.
For the second statement and for a symmetric map $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\mathfrak{m},\mathfrak{m})$ it is easy to see that $\mathcal{C}^{\mu} = 0$. Therefore, our assertion is a direct consequence of Proposition 9.
Let us finally prove (3). By definition, it is $\mu \in \mathrm{Der}_K(\mu;\mathfrak{m})$ or equivalent $\Lambda \in \mathrm{Hom}_K(\mathfrak{m},\mathrm{Der}_K(\mu;\mathfrak{m}))$, if and only if

$$\mu(Z,\mu(X,Y)) = \mu(\mu(Z,X),Y) + \mu(X,\mu(Z,Y))$$

for any $X,Y,Z \in \mathfrak{m}$, which is equivalent to say that $\mathfrak{S}_{X,Y,Z}\,\mu(X,\mu(Y,Z)) = 0$ identically. But since $\mathcal{C}^{\mu}(X,Y;Z) = 2\mathfrak{S}_{X,Y,Z}\,\mu(X,\mu(Y,Z))$, we conclude. $\square$

Remark 2. For a compact connected Lie group $G \cong (G\times G)/\Delta G$ endowed with a bi-invariant affine connection ∇ corresponding to a skew-symmetric mapping $\mu \in \mathrm{Hom}_G(\Lambda^2\mathfrak{g},\mathfrak{g})$, formula (2.2.4) has been described in [C1, Prop. 2.4]. In particular, in this case relation (2.2.5) reduces to

$$(\nabla_Z T)(X,Y) = 2R(Z,X)Y + 2\Lambda(Y)(\Lambda(Z)X - [Z,X]_{\mathfrak{m}}) - \mathfrak{der}_{\mathfrak{g}}(X,Y;Z),$$

for any $X,Y,Z \in \mathfrak{g} = T_eG$, see also [C1, Prop. 2.4].

Example 2. Let $M = G/K$ an effective homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$. We consider the restricted Lie bracket $\mathrm{ad}_{\mathfrak{m}} := [-,-]_{\mathfrak{m}} : \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ and denote the associated Nomizu map just by $\Lambda_{\mathfrak{m}}$. Obviously, $\mathrm{ad}_{\mathfrak{m}}$ *induces a derivation on* $\mathfrak{m}$ *if and only if* $\mathrm{Jac}_{\mathfrak{m}} \equiv 0$, where $\mathrm{Jac}_{\mathfrak{m}} : \mathfrak{m}\times\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ is the trilinear map defined by

$$\mathrm{Jac}_{\mathfrak{m}}(X,Y,Z) := \mathfrak{S}_{X,Y,Z}[X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} = [X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} + [Y,[Z,X]_{\mathfrak{m}}]_{\mathfrak{m}} + [Z,[X,Y]_{\mathfrak{m}}]_{\mathfrak{m}},$$

for any $X,Y,Z \in \mathfrak{m}$. The same conclusion follows from Theorem 20. Indeed, let us denote by $\nabla^{\mathfrak{m}}$ the G-invariant connection associated to $\mathrm{ad}_{\mathfrak{m}}$ and by $T^{\mathfrak{m}}$ and $R^{\mathfrak{m}}$ its torsion and curvature, respectively. It is $T^{\mathfrak{m}}(X,Y) = [X,Y]_{\mathfrak{m}}$ and

$$(\nabla^{\mathfrak{m}}_Z T^{\mathfrak{m}})(X,Y) = (\mathcal{D}^{\alpha_{\mathfrak{m}}}_Z T^{\mathfrak{m}})(X,Y) = \mathrm{Jac}_{\mathfrak{m}}(X,Y,Z),$$

for any $X, Y, Z \in \mathfrak{m}$. Moreover, $R^{\mathfrak{m}}(Z, X)Y = \mathrm{Jac}_{\mathfrak{m}}(X, Y, Z) - [[Z, X]_{\mathfrak{k}}, Y]$ and since $\Lambda_{\mathfrak{m}}(Z)X = [Z, X]_{\mathfrak{m}}$, an application of Theorem 20, (1), shows that $\Lambda_{\mathfrak{m}} \in \mathrm{Hom}_K(\mathfrak{m}, \mathrm{Der}(\mathfrak{m}))$ if and only if $\mathrm{Jac}_{\mathfrak{m}}(X, Y, Z) = 0$ for any $X, Y, Z \in \mathfrak{m}$. In fact, for $\mu = \mathrm{ad}_{\mathfrak{m}}$ it is $\mathcal{C}^{\mathrm{ad}_{\mathfrak{m}}}(X, Y; Z) = 2\,\mathrm{Jac}_{\mathfrak{m}}(X, Y, Z)$, hence the same results follows by relation (2.2.3). Finally, for the same assertion one can even apply Theorem 20, (3) for $\mu = \mathrm{ad}_{\mathfrak{m}}$.
Note that if $M = G/K$ is an effective symmetric space, then $\mathrm{Jac}_{\mathfrak{m}}$ is identically zero and $\mathrm{ad}_{\mathfrak{m}}$ is a derivation trivially. For example, any compact connected Lie group $M = G$ with a bi-invariant metric can be viewed as a symmetric space of the form $(G \times G)/\Delta G$. The Cartan decomposition is given by $\mathfrak{g} \oplus \mathfrak{g} = \Delta\mathfrak{g} \oplus \mathfrak{m}$, where $\Delta\mathfrak{g} := \{(X, X) \in \mathfrak{g} \oplus \mathfrak{g} : X \in \mathfrak{g}\} \cong \mathfrak{g}$ and $\mathfrak{m} := \{(X, -X) \in \mathfrak{g} \oplus \mathfrak{g} : X \in \mathfrak{g}\} \cong \mathfrak{g}$, respectively. Obviously, in this case the condition $\mathrm{Jac}_{\mathfrak{m}} \equiv 0$ is the Jacobi identity which leads to the well-known result that the adjoint representation $\Lambda_{\mathfrak{g}} = \mathrm{ad}_{\mathfrak{g}}$ is a derivation of $\mathfrak{g}$. In the following section we examine the condition $\mathrm{Jac}_{\mathfrak{m}} \equiv 0$ also on non-symmetric, compact, effective and simply-connected naturally reductive manifolds, see Corollary 2.

2.3 Invariant connections on naturally reductive manifolds

Next we describe a series of new results related to invariant connections (and their torsion type) on effective naturally reductive spaces. Note that all these results can be applied on an effective, non-symmetric (compact) SII homogeneous Riemannian manifold.

2.3.1 Naturally reductive spaces

A Riemannian manifold (M, g) is called *naturally reductive* if there exists a closed subgroup G of the isometry group $\mathrm{Iso}(M, g)$ which acts transitively on M with isotropy group K and which induces a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that the torsion of the canonical connection ∇^c associated to $\mathfrak{m}$, is a 3-form $T^c \in \Lambda^3(\mathfrak{m})$. This is equivalent to say that $U \equiv 0$ identically, where $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is the bilinear map defined by (2.1.6). Thus, an alternative way to define naturally reductive manifolds is as follows:

Definition 4. *A naturally reductive manifold is a homogeneous Riemannian manifold $(M = G/K, g)$ with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that canonical homogeneous structure $A^c \in \mathfrak{m} \otimes \Lambda^2\mathfrak{m}$ adapted to $\mathfrak{m}$ and G, is totally skew-symmetric, i.e. $2A^c(X, Y, Z) = T^c(X, Y, Z)$ for any $X, Y, Z \in \mathfrak{m}$.*

A special subclass of naturally reductive manifolds $M = G/K$ consists of the so-called *normal* homogeneous Riemannian manifolds. In this case there is an $\mathrm{Ad}(G)$-invariant inner product Q on $\mathfrak{g}$ such that $Q(\mathfrak{k}, \mathfrak{m}) = 0$, i.e. $\mathfrak{m} = \mathfrak{k}^{\perp}$ and $Q|_{\mathfrak{m}} = \langle\ ,\ \rangle$. Thus, a normal metric is defined by a positive definite bilinear form Q. Notice however that Q can be more general, see [K, B]. If $Q = -B$, where B denotes the Killing form of $\mathfrak{g}$, then the normal metric is called the *Killing* (or *standard*) metric; this is the case if the Lie group G is compact and semi-simple. We mention that in this paper whenever we refer to a naturally reductive space $(M = G/K, g, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$ we shall always assume that G acts effectively on M and that $\mathfrak{g}$ coincides with with the ideal $\tilde{\mathfrak{g}} := \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$. On the level of Lie groups this condition means that G coincides with the transvection group of the associated canonical connection ∇^c. Note that any compact normal homogeneous space satisfies this condition, see [R].

2.3.2 Properties of invariant connections in the naturally reductive setting

We start with the following simple observation.

Lemma 7. *Let $(M = G/K, g)$ be an effective compact homogeneous Riemannian manifold with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ which is not a symmetric space of Type I. Then, there is always an instance of $\mathfrak{m}$ inside $\Lambda^2(\mathfrak{m})$, associated to the restriction of the Lie bracket operation on the reductive complement $\mathfrak{m}$. In particular, this specific copy gives rise to a G-invariant metric connection on M if and only if g is naturally reductive with respect to G and $\mathfrak{m}$.*

Proof. Since $M = G/K$ is not isometric to a symmetric space of Type I, the canonical connection ∇^c has non-trivial torsion $T^c(X,Y) = -[X,Y]_{\mathfrak{m}}$, which gives rise to a non-trivial $\mathrm{Ad}(K)$-equivariant skew-symmetric bilinear mapping $\mathrm{ad}_{\mathfrak{m}} : \Lambda^2\mathfrak{m} \to \mathfrak{m}$. The second statement is apparent due to the naturally reductive property. □

Remark 3. If $(M = G/K, g)$ is a Riemannian symmetric space of Type I, then G is a compact simple Lie group and its Killing form B gives rise to a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. Moreover, the restriction $\langle\ ,\ \rangle = -B|_{\mathfrak{m}}$ induces a G-invariant metric which is naturally reductive with respect to $\mathfrak{m}$. However, the K-module $\Lambda^2(\mathfrak{m})$ never contains a copy of $\mathfrak{m}$, see also [L3]. This is in contrast to a Riemannian symmetric space $(M = G, g = \rho)$ of Type II endowed with a bi-invariant metric ρ, where one can always find a copy of $\mathfrak{g}$ inside $\Lambda^2(\mathfrak{g})$, see also Remark 6. Geometrically, this copy represents the existence of 1-parameter family of canonical connections on any compact simple Lie group G (cf. [KN, AFH, C1]). The same is true in the more general compact case (cf. [OR1]).

Lemma 8. ([A, C1]) *Let $(M = G/K, g)$ be an effective naturally reductive manifold with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that $\mathfrak{g} = \tilde{\mathfrak{g}}$. Then,*
(i) A G-invariant metric connection ∇ on $(M = G/K, g)$ has totally skew-symmetric torsion $T \in \Lambda^3(\mathfrak{m})$ if and only if $\Lambda(Z)Z = 0$, for any $Z \in \mathfrak{m}$, where Λ is the associated Nomizu map.
(ii) There is a bijective correspondence between $\mathrm{Ad}(K)$-equivariant maps $\Lambda^\alpha : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$, defined by $\Lambda^\alpha(X)Y = \frac{1-\alpha}{2}[X,Y]_{\mathfrak{m}} = (1-\alpha)\Lambda^g(X)Y$ for any $X, Y \in \mathfrak{m}$, and G-invariant metric connections ∇^α with totally skew-symmetric torsion $T^\alpha \in \Lambda^3(\mathfrak{m})$, such that $T^\alpha = \alpha \cdot T^c$ for some parameter α, where T^c is the torsion of the canonical connection ∇^c associated to $\mathfrak{m}$ and $\Lambda^g : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ the Nomizu map associated to the Levi-Civita connection ∇^g.

Let us finally recall the following fundamental result by Olmos and Reggiani.

Theorem 21. ([OR1, Thm. 1.2], [OR2, Thm. 2.1]) *Let (M^n, g) be a simply-connected and irreducible Riemannian manifold that is not isometric to a sphere, nor to a projective space, nor to a compact simple Lie group with a bi-invariant metric or its symmetric dual. Then the canonical connection is unique, i.e. (M^n, g) admits at most one naturally reductive homogeneous structure.*

Combining the observations in Example 2 and the results of Lemma 8 and Theorem 21, in the compact and simply-connected case we obtain the following conclusion about derivations on $\mathfrak{m}$:

Corollary 2. *Let $(M = G/K, g)$ be an effective, compact and simply-connected naturally reductive manifold, irreducible as Riemannian manifold, endowed with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that $\mathfrak{g} = \tilde{\mathfrak{g}}$. Assume that $M = G/K$ is not isometric to a symmetric space of Type I, neither to a sphere or to a real projective space. Then, the bilinear mapping $\mathrm{ad}_{\mathfrak{m}} := [\ ,\ ]_{\mathfrak{m}}$ gives rise to $\mathrm{Ad}(K)$-equivariant derivation on $\mathfrak{m}$, if and only if M is isometric to a compact simple Lie group G endowed with a bi-invariant metric.*

Proof. A special version of Corollary 2 has been proved in [C1, Lem. 4.5]. Here we improve this result. We omit some details and only present the main idea. Assume that $\mathrm{ad}_{\mathfrak{m}} \in \mathrm{Der}_K(\mathfrak{m})$, i.e. $\mathrm{Jac}_{\mathfrak{m}}(X,Y,Z) = 0$ for any $X, Y, Z \in \mathfrak{m}$ (see Example 2). Then, for the family ∇^α of Lemma 8, a small computation shows that $\nabla^\alpha T^\alpha = 0$ for any α, see for example [A]. On the other hand, one can easily see that $[X, [Y,Z]_{\mathfrak{m}}]_{\mathfrak{k}} + [Y, [Z,X]_{\mathfrak{m}}]_{\mathfrak{k}} + [Z, [X,Y]_{\mathfrak{m}}]]_{\mathfrak{k}} = 0$, for any $X, Y, Z \in \mathfrak{m}$. Combining this identity with $\mathrm{Jac}_{\mathfrak{m}}(X,Y,Z) = 0$, a long computation certifies that $\nabla^\alpha R^\alpha = 0$ for any α, as well. But then, ∇^α is a 1-parameter family of canonical connections on $M = G/K$ (in the sense of the Ambrose-Singer Theorem) and Theorem 21 yields the result. □

Notation: Let $(M = G/K, g)$ be an effective compact naturally reductive Riemannian manifold with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} = \tilde{\mathfrak{g}}$. If $\chi : K \to \mathrm{Aut}(\mathfrak{m})$ is of real type, we shall denote by $\mathbf{s}$ and $\mathbf{a}$ the multiplicity of $\mathfrak{m}$ inside $\mathrm{Sym}^2(\mathfrak{m})$ and $\Lambda^2(\mathfrak{m})$, respectively (or twice the multiplicity of $\mathfrak{m}$ inside $\mathrm{Sym}^2(\mathfrak{m})$ and $\Lambda^2(\mathfrak{m})$, respectively, if $\chi : K \to \mathrm{Aut}(\mathfrak{m})$ is of complex type). We also set

$$\begin{aligned} \mathcal{N} = \mathbf{s} + \mathbf{a} \quad &:= \quad \dim_{\mathbb{R}} \mathcal{A}ff_G(F(G/K)) = \dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}) \\ \mathcal{N}_{\mathrm{mtr}} \quad &:= \quad \dim_{\mathbb{R}} \mathcal{M}_G(\mathrm{SO}(G/K)) = \dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2(\mathfrak{m})) \leq \mathcal{N}. \end{aligned}$$

Since K is compact, and we treat finite dimensional K-representations, we conclude that

Lemma 9. *The dimensions of modules* $\mathrm{Hom}_K(\Lambda^2\mathfrak{m},\mathfrak{m})$ *and* $\mathrm{Hom}_K(\mathfrak{m},\Lambda^2\mathfrak{m})$ *coincide,*

$$\dim_{\mathbb{R}}\mathrm{Hom}_K(\Lambda^2\mathfrak{m},\mathfrak{m})=\dim_{\mathbb{R}}\mathrm{Hom}_K(\mathfrak{m},\Lambda^2\mathfrak{m}),$$

or in other words $\mathbf{a}=\mathcal{N}_{\mathrm{mtr}}$.

Remark 4. Note that there exists compact Lie groups, e.g. $G=\mathrm{U}_n$, admitting skew-symmetric $\mathrm{Ad}(G)$-equivariant maps $\Lambda^2(\mathfrak{g})\to\mathfrak{g}$ which do not induce bi-invariant metric connections with respect to the bi-invariant inner product $\langle X,Y\rangle=-\operatorname{tr}XY$ (see also the proof of Theorem 23). In fact, below will show that Lemma 9 implies that [AFH, Lem. 3.1] or [C1, Corol. 2.3, Thm. 2.9] are in general false. In particular, the corresponding statements hold only for compact simple Lie groups, but fail for general compact Lie groups.

Next, our aim is to clarify Remark 4. For simplicity, given an $\mathrm{Ad}(K)$-equivariant bilinear mapping $\mu:\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ associated to a G-invariant connection ∇ on $(M=G/K,g)$ we shall use the same notation for the corresponding K-intertwining map $\mu:\mathfrak{m}\otimes\mathfrak{m}\to\mathfrak{m}$ (and we shall identify them) and denote by $\hat{\mu}$ the contraction of μ with the $\mathrm{Ad}(K)$-invariant inner product $\langle\ ,\ \rangle:\mathfrak{m}\times\mathfrak{m}\to\mathbb{R}$ associated to g, i.e. $\hat{\mu}(X,Y,Z):=\langle\mu(X,Y),Z\rangle$, for any $X,Y,Z\in\mathfrak{m}$. Notice that $\hat{\mu}$ is an $\mathrm{Ad}(K)$-invariant tensor on $\mathfrak{m}$. Initially, it is useful to examine invariant connections related to some $\mu\in\mathrm{Hom}_K(\mathrm{Sym}^2\mathfrak{m},\mathfrak{m})$. Then, the induced tensor $\hat{\mu}=\hat{\mu}^{\mathbf{s}}$ is such that $\hat{\mu}(X,Y,Z)=\hat{\mu}(Y,X,Z)$ for any $X,Y,Z\in\mathfrak{m}\cong T_oG/K$ and the corresponding Nomizu map $\Lambda:=\Lambda^{\mathbf{s}}:\mathfrak{m}\to\mathrm{Sym}^2(\mathfrak{m})$ is also symmetric in the sense that $\Lambda(X)Y=\Lambda(Y)X$ (since $\mu(X,Y)=\mu(Y,X)$ for any $X,Y\in\mathfrak{m}$). Next we prove that when $0\neq\mu\in\mathrm{Hom}_K(\mathrm{Sym}^2\mathfrak{m},\mathfrak{m})$ is non-trivial, then the induced connection cannot preserve the naturally reductive metric $\langle\ ,\ \rangle$. We start with the non-symmetric case.

Lemma 10. *Let* $(M=G/K,g)$ *be a connected, compact, non-symmetric, naturally reductive space of a compact Lie group* G *modulo a compact subgroup* K*. Assume that the transitive* G*-action is effective and let us denote by* $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}=\tilde{\mathfrak{g}}$ *the associated reductive decomposition. If* ∇ *is an invariant metric connection induced by some* $\mu\in\mathrm{Hom}_K(\mathrm{Sym}^2\mathfrak{m},\mathfrak{m})$*, then* $\mu=0$ *and* ∇ *coincides with the canonical connection* ∇^c *associated to* $\mathfrak{m}$*.*

Proof. Assume that such an invariant connection $\nabla=\nabla^{\mathbf{s}}$ exists, i.e.

$$\langle\Lambda^{\mathbf{s}}(X)Y,Z\rangle+\langle\Lambda^{\mathbf{s}}(X)Z,Y\rangle=0,$$

which is equivalent to say that $\hat{\mu}\in\mathfrak{m}\otimes\Lambda^2(\mathfrak{m})$, i.e. $\hat{\mu}(X,Y,Z)+\hat{\mu}(X,Z,Y)=0$, for any $X,Y,Z\in\mathfrak{m}$. Then, since $\Lambda^{\mathbf{s}}(X)Y=\Lambda^{\mathbf{s}}(Y)X$, its torsion coincides with the torsion of the canonical connection, $T^{\mathbf{s}}(X,Y)=-[X,Y]_{\mathfrak{m}}=T^c(X,Y)$. Since metric connections are determined by their torsion tensor, we conclude that $\nabla^s=\nabla^c$ and $\mu=0$. ☐

In fact, the non-existence of an invariant metric connection $\nabla^{\mathbf{s}}$ corresponding to a non-trivial element $0\neq\mu\in\mathrm{Hom}_K(\mathrm{Sym}^2\mathfrak{m},\mathfrak{m})$ can be proved also as follows. By the condition $T^{\mathbf{s}}(X,Y)=-[X,Y]_{\mathfrak{m}}$ and since $\langle\ ,\ \rangle$ is naturally reductive with respect to G, one concludes that $\nabla^{\mathbf{s}}$ is an invariant connection with skew-torsion, which according to Lemma 8 is equivalent to say that $\Lambda^{\mathbf{s}}(X)X=0$ for any $X\in\mathfrak{m}$. But then, it is also $\Lambda^{\mathbf{s}}(X+Y)(X+Y)=0$ for any $X,Y\in\mathfrak{m}$, i.e. $\Lambda^s(X)Y=-\Lambda^s(Y)X$, which gives rise to a contradiction (since $\mu\neq0$). Let us now explain also the compact symmetric case.

Lemma 11. *Given a connected Riemannian symmetric space* $(M=G/K,g)$ *of Type I (resp.* $(M=(G\times G)/\Delta(G),\rho)$ *of Type II, for some compact, connected, simple Lie group* G *with a bi-invariant metric* ρ*), then the unique* G*-invariant (resp. bi-invariant) metric connection which is induced by a symmetric* $\mathrm{Ad}(K)$*-equivariant mapping on* $\mathfrak{m}$ *(resp, symmetric* $\mathrm{Ad}(G)$*-equivariant mapping on* $\mathfrak{g}$*), is the torsion-free Levi-Civita connection.*

Proof. Consider first a symmetric space $(M=G/K,g)$ of Type I, endowed with a G-invariant affine connection ∇^{μ} associated to an element $\mu\in\mathrm{Hom}_K(\mathrm{Sym}^2\mathfrak{m},\mathfrak{m})$. Then, $T^{\mu}(X,Y)=0$ for any $X,Y\in\mathfrak{m}$, i.e. $\hat{\mu}\in\mathrm{Sym}^2\mathfrak{m}\otimes\mathfrak{m}$. Hence, assuming in addition that ∇ is metric with respect to g, the fundamental theorem in Riemannian geometry implies the identification of ∇ with the unique torsion-free metric connection on $(M=G/K,g)$, i.e. the Levi-Civita connection, or the canonical connection associated to $\mathfrak{m}$. In particular, $\mu=0$ is trivial. The same conclusions, related this time to bi-invariant metric connections corresponding to maps $\mu\in\mathrm{Hom}_G(\mathrm{Sym}^2\mathfrak{g},\mathfrak{g})$, hold for a compact, connected, simple Lie group $G\cong(G\times G)/\Delta G$, endowed with a bi-invariant metric ρ. ☐

Remark 5. Laquer proved in [L3] the existence of (irreducible) compact symmetric spaces $(M = G/K, g)$ which admit invariant affine connections induced by non-trivial elements $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$. And indeed, by [AFH] we know that these G-invariant connections are not metric with respect to $g = -B|_{\mathfrak{m}}$, as it should be according to Lemma 11. The same is true for compact simple Lie groups, such as SU_n, see [AFH, L2].

Let us now consider invariant connections whose torsion is a 3-form. We show that on an effective, non-symmetric, compact, naturally reductive space $(M = G/K, g)$ the G-invariant metric connections whose torsion is a 3-form necessarily correspond to instances of the trivial representation inside the space $\Lambda^3\mathfrak{m}$, and conversely. In particular, the torsion form is a G-invariant 3-form. Let us denote by ℓ the multiplicity of the (real) trivial representation inside $\Lambda^3\mathfrak{m}$.

Lemma 12. *Let $(M = G/K, g)$ be a naturally reductive manifold as in Lemma 10. Then, the dimension of the affine space of G-invariant metric connections on M which have the same geodesics with the Levi-Civita connection ∇^g, i.e. $\Lambda(X)X = 0$, or equivalent whose torsion form T is a non-trivial G-invariant 3-form, is equal to ℓ. In particular,*

$$1 \leq \ell \leq \mathcal{N}_{\mathrm{mtr}} = \mathbf{a} \leq \mathcal{N}.$$

Proof. First notice that $1 \leq \ell \leq \mathcal{N}_{\mathrm{mtr}}$. This follows since the induced $\mathrm{Ad}(K)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ satisfies the naturally reductive property and hence the torsion of the canonical connection $T^c(X, Y, Z) = -\langle [X, Y]_{\mathfrak{m}}, Z\rangle \neq 0$ is a non-trivial G-invariant 3-form. Then, according to Lemma 8, the family $\nabla^\alpha = \nabla^c + \Lambda^\alpha$ induces a 1-parameter family of metric connections with skew-torsion $T^\alpha := \alpha T^c \neq 0$. Now, any instance of the trivial representation inside $\Lambda^3(\mathfrak{m})$ induces a G-invariant (global) 3-form on $M = G/K$, say $0 \neq T \in \Lambda^3(\mathfrak{m})^K$. If $\ell \geq 2$, then we can also assume that $T \neq T^\alpha$. But then, one can define a 1-parameter family of metric connections with skew-torsion, say $2sT$, given by $\nabla^s = \nabla^g + sT$. Obviously, this family is G-invariant and preserves the metric. On the other hand, if $M = G/K$ admits a G-invariant metric connection ∇ with skew-torsion T such that $T \neq T^\alpha$, then T must be a global G-invariant 3-form and hence it corresponds to a new copy of the trivial representation inside $\Lambda^3\mathfrak{m}$. ☐

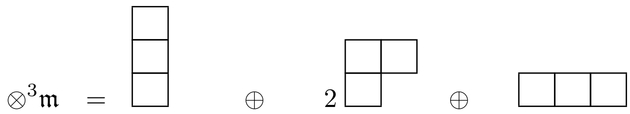
For a complete description of all G-invariant metric connections on $(M = G/K, g)$, one has to encode the "defect"

$$\epsilon := \mathcal{N}_{\mathrm{mtr}} - \ell \geq 0.$$

For this, it is useful to consider the tensor product

$$\otimes^3\mathfrak{m} = \mathfrak{m}\otimes\mathfrak{m}\otimes\mathfrak{m} \cong (\Lambda^2\mathfrak{m} \oplus \mathrm{Sym}^2\,\mathfrak{m})\otimes\mathfrak{m} \cong \big(\Lambda^2\mathfrak{m}\otimes\mathfrak{m}\big) \oplus \big(\mathrm{Sym}^2\,\mathfrak{m}\otimes\mathfrak{m}\big)$$

and its decomposition in terms of Young diagrams:



$$= \mathrm{Sym}^3\,\mathfrak{m} \oplus \mathcal{L}(\mathfrak{m}) \oplus \Lambda^3(\mathfrak{m}),$$

where $\mathcal{L}(\mathfrak{m}) := \ker(P_{\mathbf{sym}})\cap\ker(P_{\mathbf{skew}})$ is the section of the the kernels of the equivariant projections

$$P_{\mathbf{sym}} : \otimes^3\mathfrak{m} \to \mathrm{Sym}^3(\mathfrak{m}), \quad P_{\mathbf{skew}} : \otimes^3\mathfrak{m} \to \Lambda^3(\mathfrak{m}).$$

Notice that the kernel of the natural maps $\mathrm{Sym}^2\,\mathfrak{m}\otimes\mathfrak{m} \to \mathrm{Sym}^3\,\mathfrak{m}$ and $\Lambda^2(\mathfrak{m})\otimes\mathfrak{m} \to \Lambda^3(\mathfrak{m})$ are isomorphic irreducible $\mathrm{GL}(\mathfrak{m})$-modules of real dimension $n(n-1)(n+1)/3$, where $n := \dim_{\mathbb{R}}\mathfrak{m} = \dim_{\mathbb{R}} M$ (for an example see [Sim, p. 246]). Moreover, there is an equivariant isomorphism

$$\mathcal{L}(\mathfrak{m}) \cong \oplus^2 \ker(\mathfrak{m}\otimes\Lambda^2(\mathfrak{m}) \to \Lambda^3(\mathfrak{m})).$$

The intersection of $\mathcal{L}(\mathfrak{m})$ with the K-module $\mathfrak{m}\otimes\Lambda^2\mathfrak{m}$ consists of metric connections and is isomorphic to the so-called (2,1)-plethysm of the K-representation $\mathfrak{m}$:

$$P_{\mathfrak{m}}(2,1) := \mathcal{L}(\mathfrak{m}) \cap \big(\mathfrak{m}\otimes\Lambda^2\mathfrak{m}\big).$$

Theorem 22. *Let $(M^n = G/K, g = \langle\ ,\ \rangle, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$ as in Lemma 10. The existence of the trivial representation inside the $n(n-1)(n+1)/3$-dimensional (2, 1)-plethysm $P_{\mathfrak{m}}(2,1)$ of $\mathfrak{m}$, gives rise to a G-invariant metric connection $\nabla = \nabla^{\mu}$ on $M = G/K$ corresponding to a K-intertwining bilinear mapping $\mu : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ which has both non-trivial symmetric and skew-symmetric part, i.e. $\mu = \mu^{\mathrm{skew}} + \mu^{\mathrm{sym}}$, with $0 \neq \mu^{\mathrm{skew}} \in \mathrm{Hom}_K(\Lambda^2\mathfrak{m}, \mathfrak{m})$ and $0 \neq \mu^{\mathrm{sym}} \in \mathrm{Hom}_K(\mathrm{Sym}^2\,\mathfrak{m}, \mathfrak{m})$, respectively. In particular, the torsion of ∇^{μ} is not totally skew-symmetric and the defect $\epsilon := \mathcal{N}_{\mathrm{mtr}} - \ell \geq 0$ coincides with the multiplicity of the trivial representation inside $P_{\mathfrak{m}}(2,1)$.*

Proof. The trivial representation inside $P_{\mathfrak{m}}(2,1)$ induces an $\mathrm{Ad}(K)$-equivariant 3-tensor $\hat{\mu}$ on $\mathfrak{m}$ which is skew-symmetric with respect the last two indices, i.e. $\hat{\mu} \in \mathfrak{m} \otimes \Lambda^2(\mathfrak{m})$. Since the K-module $\mathfrak{m} \otimes \Lambda^2\mathfrak{m}$ corresponds to the set of $\mathfrak{so}(\mathfrak{m})$-valued Nomizu maps on $M = G/K$ with respect to $\langle\ ,\ \rangle$, the induced invariant connection $\nabla = \nabla^{\mu}$ is necessarily metric. In order to prove that its torsion is not a 3-form we rely on the definition of $P_{\mathfrak{m}}(2,1)$ and the orthogonal decomposition

$$\otimes^3\,\mathfrak{m} = \mathrm{Sym}^3\,\mathfrak{m} \oplus \mathcal{L}(\mathfrak{m}) \oplus \Lambda^3(\mathfrak{m}). \tag{2.3.1}$$

Indeed, since the 3-tensor $\hat{\mu}(X,Y,Z) = \langle \mu(X,Y), Z\rangle$ is induced by the trivial representation inside $P_{\mathfrak{m}}(2,1) := \mathcal{L}(\mathfrak{m}) \cap \big(\mathfrak{m} \otimes \Lambda^2\mathfrak{m}\big)$, the direct sum decomposition (2.3.1) together with Lemma 12, shows that the torsion T^{μ} of ∇^{μ} cannot be totally skew-symmetric. We use now (2.1.4) and write $\mu = \mu^{\mathrm{skew}} + \mu^{\mathrm{sym}}$ for the corresponding K-intertwining bilinear mapping $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$. Since T^{μ} is not a 3-form, μ^{sym} cannot be trivial, $\mu^{\mathrm{sym}} \neq 0$. Indeed, if $\mu^{\mathrm{sym}} = 0$, then $\mu = \mu^{\mathrm{skew}}$ and hence $\mu(X,X) = 0$ for any $X \in \mathfrak{m}$. But then, using Lemma 8, (i) we get a contradiction. Assume now that μ is given by a (non-trivial) symmetric K-intertwining bilinear mapping, i.e $\mu^{\mathrm{skew}} = 0$ and $\mu = \mu^{\mathrm{sym}}$ where $0 \neq \mu^{\mathrm{sym}} : \mathrm{Sym}^2\,\mathfrak{m} \to \mathfrak{m}$. Then, according to Lemma 10 our connection ∇^{μ} cannot be metric with respect to $\langle\ ,\ \rangle$, which contradicts to $\hat{\mu} \in \mathfrak{m} \otimes \Lambda^2(\mathfrak{m})$. This shows that $\mu^{\mathrm{skew}} \neq 0$, as well. Now, the identification of the defect $\epsilon := \mathcal{N}_{\mathrm{mtr}} - \ell \geq 0$ with the multiplicity of the trivial representation in $P_{\mathfrak{m}}(2,1)$ is a direct consequence of (2.3.1) and Lemmas 9, 12. $\square$

We mention that one cannot drop the naturally reductive assumption in Theorem 22, due to the fact that the proof relies on Lemmas 10 and 12.

Remark 6. On a compact simple Lie group G, bi-invariant connections which are compatible with the Killing form are induced by the copy of $\mathfrak{g}$ inside $\Lambda^2(\mathfrak{g})$. Indeed, recall that

$$\mathfrak{so}(\mathfrak{g}) \cong \Lambda^2(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^{\perp}, \quad \mathfrak{g}^{\perp} := \ker \delta_{\mathfrak{g}},$$

where $\delta_{\mathfrak{g}} : \Lambda^2(\mathfrak{g}) \to \mathfrak{g}$ is given by $\delta_{\mathfrak{g}}(X \wedge Y) := [X,Y]$. Since $\delta_{\mathfrak{g}}$ is surjective, $\mathfrak{g}$ always lies inside $\Lambda^2(\mathfrak{g})$. However, the module $P_{\mathfrak{g}}(2,1) := \mathcal{L}(\mathfrak{g}) \cap \big(\mathfrak{g} \otimes \Lambda^2\mathfrak{g}\big)$, where $\mathcal{L}(\mathfrak{g})$ is similarly defined by $\mathcal{L}(\mathfrak{g}) := \oplus^2 \ker\big(\mathfrak{g} \otimes \Lambda^2(\mathfrak{g}) \to \Lambda^3(\mathfrak{g})\big)$, never contains the trivial summand. In contrast, as we noticed in Remark 4, for a compact Lie group the case can be different. Let us focus for example on $G = \mathrm{U}_n$ $(n \geq 3)$.

2.3.3 Bi-invariant metric connections on the compact Lie group U_n

According to Laquer [L2], for $n \geq 3$ the space of bi-invariant affine connections on U_n is 6-dimensional. In particular, the following $\mathrm{Ad}(G)$-equivariant bilinear mappings form a basis of $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ for $\mathfrak{g} = \mathfrak{u}(n)$:

$$\left.\begin{array}{lll} \mu_1(X,Y) = [X,Y], & \mu_2(X,Y) = i(XY + YX), & \mu_3(X,Y) = i\,\mathrm{tr}(X)Y \\ \mu_4(X,Y) = i\,\mathrm{tr}\,(Y)X, & \mu_5(X,Y) = i\,\mathrm{tr}\,(XY)\,\mathrm{Id}, & \mu_6(X,Y) = i\,\mathrm{tr}\,(X)\,\mathrm{tr}\,(Y)\,\mathrm{Id} \end{array}\right\}, \tag{2.3.2}$$

where XY denotes multiplication of matrices and Id is the identity matrix. We also consider the linear combinations

$$\begin{aligned} \nu(X,Y) &:= \mu_3(X,Y) - \mu_4(X,Y) = i(\mathrm{tr}(X)Y - \mathrm{tr}\,(Y)X) \in \mathrm{Hom}_G(\Lambda^2\mathfrak{g}, \mathfrak{g}), \\ \vartheta(X,Y) &:= \mu_3(X,Y) + \mu_4(X,Y) = i(\mathrm{tr}(X)Y + \mathrm{tr}\,(Y)X) \in \mathrm{Hom}_G(\mathrm{Sym}^2\,\mathfrak{g}, \mathfrak{g}). \end{aligned}$$

Theorem 23. *(1) The connection induced by the $\mathrm{Ad}(\mathfrak{u}(n))$-equivariant bilinear mapping $\mu = \mu_4 - \mu_5$, i.e. $\mu(X,Y) := i(\mathrm{tr}\,(Y)X - \mathrm{tr}\,(XY)\,\mathrm{Id})$ for any $X, Y \in \mathfrak{u}(n)$, is a bi-invariant metric connection on U_n $(n \geq 3)$ with respect to the bi-invariant metric induced by $\langle\ ,\ \rangle = -\mathrm{tr}(XY)$. The symmetric and skew-symmetric part of $\mu = \mu^{\mathrm{skew}} + \mu^{\mathrm{sym}}$ are given by*

$$\mu^{\mathrm{skew}}(X,Y) = -(1/2)\nu(X,Y), \quad \textit{and} \quad \mu^{\mathrm{sym}}(X,Y) = (1/2)\vartheta(X,Y) + i\langle X,Y\rangle\,\mathrm{Id},$$

respectively, and its torsion has the form $T^{\mu}(X,Y) = -\nu(X,Y) + T^{c}(X,Y)$. In particular, the induced tensor $T^{\mu}(X,Y,Z) = \langle T^{\mu}(X,Y), Z\rangle$ is not totally skew-symmetric.
(2) Consequently, for $n \geq 3$ the Lie group U_n carries a 2-dimensional space of bi-invariant metric connections, i.e. $\mathcal{N}_{\mathrm{mtr}} = \epsilon + \ell = 2$.

Proof. (1) The module $\mathcal{L}(\mathfrak{g})$ associated to the adjoint representation of $\mathfrak{g} = \mathfrak{u}(n) = \mathbb{R} \oplus \mathfrak{su}(n)$ contains the trivial representation twice. The one copy corresponds to the invariant 3-tensor $\hat{\nu}(X,Y,Z) = \langle \nu(X,Y), Z\rangle$ which is skew-symmetric only with respect to the first two indices, i.e. $\hat{\nu} \in \mathcal{L}(\mathfrak{g}) \cap (\Lambda^2\mathfrak{g} \otimes \mathfrak{g})$ and thus $\nu = \mu_3 - \mu_4$ fails to induce a bi-invariant connection on U_n, preserving $\langle\ ,\ \rangle$. The second copy corresponds to the invariant 3-tensor $\hat{\mu}(X,Y,Z) = \langle \mu(X,Y), Z\rangle$, where $\mu : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is given by $\mu = \mu_4 - \mu_5$. We will show that $\hat{\mu}$ is indeed inside the (2, 1) plethysm $P_{\mathfrak{g}}(2,1) = \mathcal{L}(\mathfrak{g}) \cap (\mathfrak{g} \otimes \Lambda^2\mathfrak{g})$, i.e. $\epsilon = 1$, and hence the associated connection ∇^{μ} gives rise to 1-dimensional family of bi-invariant metric connections on U_n. For simplicity, we set $\mathcal{O}(X,Y,Z) := \langle \mu(X,Y), Z\rangle + \langle Y, \mu(X,Z)\rangle$, for any $X,Y,Z \in \mathfrak{u}(n)$. Then we get that

$$\begin{aligned}
\mathcal{O}(X,Y,Z) &= \langle i(\mathrm{tr}\,(Y)X - \mathrm{tr}\,(XY)\,\mathrm{Id}), Z\rangle + \langle Y, i(\mathrm{tr}\,(Z)X - \mathrm{tr}\,(XZ)\,\mathrm{Id})\rangle \\
&= i\big(\,\mathrm{tr}(Y)\langle X, Z\rangle - \mathrm{tr}(XY)\langle \mathrm{Id}, Z\rangle + \mathrm{tr}(Z)\langle Y, X\rangle - \mathrm{tr}(XZ)\langle Y, \mathrm{Id}\rangle\big) \\
&= i\big(-\mathrm{tr}(Y)\,\mathrm{tr}(XZ) + \mathrm{tr}(XY)\,\mathrm{tr}(Z) - \mathrm{tr}(Z)\,\mathrm{tr}(XY) + \mathrm{tr}(XZ)\,\mathrm{tr}(Y)\big) = 0,
\end{aligned}$$

for any $X,Y,Z \in \mathfrak{u}(n)$ and this proves our assertion. Now, according to Theorem 22, μ has both non-trivial symmetric and skew-symmetric part, namely $\mu^{\mathrm{sym}}(X,Y) = \frac{1}{2}[\mu(X,Y) + \mu(Y,X)]$ and $\mu^{\mathrm{skew}}(X,Y) = \frac{1}{2}[\mu(X,Y) - \mu(Y,X)]$, respectively, and a small computation completes the proof.
(2) For the second statement, the mapping $a\mu_1(X,Y)$ ($a \in \mathbb{R}$) induces a 1-parameter family of bi-invariant metric connections on U_n with skew-torsion and this is the unique family of bi-invariant metric connections with skew-torsion (since the multiplicity of the trivial represenatation inside $\Lambda^3\mathfrak{g}$ is one, i.e. $\ell = 1$, see also the remark below). Then, according to Theorem 22 it must be $\mathcal{N}_{\mathrm{mtr}} = \epsilon + \ell = 1 + 1 = 2$, which also fits with the conclusion that μ is a new bi-invariant metric connection on U_n, and finally also with Lemma 9. This induces a 1-parameter family of bi-invariant metric connections ∇^b ($b \in \mathbb{R}$), corresponding to the bilinear mapping $\mu^b(X,Y) := b[i(\mathrm{tr}\,(Y)X - \mathrm{tr}\,(XY)\,\mathrm{Id})] = b\mu(X,Y)$, with $X,Y \in \mathfrak{u}(n)$. The torsion $T^b \in \Lambda^2\mathfrak{u}(n) \otimes \mathfrak{u}(n)$ is not totally skew-symmetric. Indeed, the torsion of the mapping $\mu = \mu_4 - \mu_5$ is given by $T^{\mu}(X,Y) = -\nu(X,Y) + T^{c}(X,Y)$. It is not totally skew-symmetric since for example $\mu(X,X) \neq 0$ and $\langle\ ,\ \rangle$ is a naturally reductive metric. Similarly for μ^b. This finishes the proof. □

Remark 7. For a verification of the fact $\ell = 1$ for U_n, one can use the LiE program (and stability arguments), or even apply the following. First, for dimensional reasons notice that

$$\Lambda^3(\mathfrak{g}) = \Lambda^3(\mathfrak{u}(n)) = \Lambda^3(\mathbb{R} \oplus \mathfrak{su}(n)) = \Lambda^3(\mathfrak{su}(n)) \oplus \big(\mathbb{R} \otimes \Lambda^2\mathfrak{su}(n)\big). \quad (*)$$

Using (2.4.3) we also see that $\mathbb{R} \otimes \Lambda^2\mathfrak{su}(n)$ doesn't contain the trivial representation. For the decomposition of $\Lambda^3(\mathfrak{su}(n))$, recall first that any compact simple lie group $\hat{G}$ admits a non-trivial global $\hat{G}$-invariant 3-from, the so-called Cartan 3-form $\omega_{\hat{\mathfrak{g}}}(X,Y,Z) = B([X,Y],Z)$, where B denotes the Killing form on the Lie algebra $\hat{\mathfrak{g}}$. On the other hand, the $\mathrm{Ad}(\hat{G})$-equivariant differential $d_{\hat{\mathfrak{g}}} : \Lambda^k(\hat{\mathfrak{g}}) \to \Lambda^{k+1}(\hat{\mathfrak{g}})$ on $\hat{\mathfrak{g}}$ is defined by $d_{\hat{\mathfrak{g}}}(\psi \wedge \varphi) = d_{\hat{\mathfrak{g}}}(\psi) \wedge \varphi + (-1)^{\deg\psi}\psi \wedge d_{\hat{\mathfrak{g}}}(\varphi)$ with $d_{\hat{\mathfrak{g}}}(\varphi) = \sum_i (Z_i \lrcorner \omega_{\hat{\mathfrak{g}}}) \wedge (Z_i \lrcorner \varphi)$ for some $(-B)$-orthonormal basis $\{Z_i\}$ of $\hat{\mathfrak{g}}$. In these terms, in [Le] it was shown that the splitting $\Lambda^3(\hat{\mathfrak{g}}) = \mathrm{span}_{\mathbb{R}}\{\omega_{\hat{\mathfrak{g}}}\} \oplus \delta_{\hat{\mathfrak{g}}}(\Lambda^4(\mathfrak{g})) \oplus d_{\hat{\mathfrak{g}}}(\hat{\mathfrak{g}}^{\perp})$ defines an equivariant orthogonal decomposition of $\Lambda^3(\hat{\mathfrak{g}})$, where $\delta_{\hat{\mathfrak{g}}}$ is the adjoint operator of $d_{\hat{\mathfrak{g}}}$ with respect to $-B$ (see also Remark 6). From this decomposition, one deduces that $\ell = 1$ for any compact simple Lie group $\hat{G}$, and since $\mathfrak{u}(n) = \mathbb{R} \oplus \hat{\mathfrak{g}}$ with $\hat{\mathfrak{g}} = \mathfrak{su}(n)$, by $(*)$ we conclude the same for U_n.

We finally observe that $\mu := \mu_4 - \mu_5$ does not induces a derivation on $\mathfrak{m}$ (apply for example Proposition 8 or Theorem 20). In particular, [C1, Thm. 2.9] holds only for G compact and simple (the direct claim is true even in the compact case, but the converse direction fails for non-simple Lie groups, since [AFH, Lem. 3.1], or [C1, Thm. 2.1], is valid only for a compact simple Lie group).

2.3.4 Characterization of the types of invariant metric connections

Given an effective naturally reductive Riemannian manifold $(M = G/K, g)$, our aim now is to characterize the possible invariant connections with respect to their torsion type (for skew-torsion, see [A] or Lemma 8). We remark that next is not necessary to assume the compactness of $M = G/K$.

Proposition 10. *Let $(M^n = G/K, g)$ be a homogeneous Riemannian manifold which is naturally reductive with respect to a closed subgroup $G \subseteq \mathrm{Iso}(M, g)$ of the isometry group and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the associated reductive decomposition. Assume that the transitive G-action is effective, $\mathfrak{g} = \tilde{\mathfrak{g}}$ and denote by $\nabla \equiv \nabla^{\mu}$ a G-invariant metric connection corresponding to $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$. Set $\hat{\mu}(X, Y, Z) = \langle \mu(X, Y), Z \rangle$, $A(X, Y) = \nabla_X Y - \nabla^g_X Y$ and $A(X, Y, Z) = \langle A(X, Y), Z \rangle$ for any $X, Y, Z \in \mathfrak{m}$, where ∇^g is the Levi-Civita connection. Then, the following hold:*
(1) ∇ is of vectorial type, i.e. $A \in \mathcal{A}_1$, if and only if there is a global G-invariant 1-form φ on M such that

$$\hat{\mu}(X, Y, Z) = \frac{1}{2}\langle [X, Y]_{\mathfrak{m}}, Z \rangle + \langle X, Y \rangle \varphi(Z) - \langle X, Z \rangle \varphi(Y), \quad \forall\ X, Y, Z \in \mathfrak{m}.$$

(2) ∇ is of Cartan type or traceless cyclic, i.e. $A \in \mathcal{A}_2$, if any only if the following two conditions are simultaneously satisfied:

$$\begin{array}{ll} (\alpha) & \mathfrak{S}_{X,Y,Z}\hat{\mu}(X, Y, Z) = \frac{3}{2}\langle [X, Y]_{\mathfrak{m}}, Z \rangle, \quad \forall\ X, Y, Z \in \mathfrak{m}, \\ (\beta) & \sum_i \mu(Z_i, Z_i) = 0, \end{array}$$

where $Z_1, \ldots, Z_n$ is an arbitrary $\langle\ ,\ \rangle$-orthonormal basis of $\mathfrak{m}$.
(3) ∇ is cyclic, i.e. $A \in \mathcal{A}_1 \oplus \mathcal{A}_2$, if and only if $\mathfrak{S}_{X,Y,Z}\hat{\mu}(X, Y, Z) = \frac{3}{2}\langle [X, Y]_{\mathfrak{m}}, Z \rangle$, $\forall\ X, Y, Z \in \mathfrak{m}$.
(4) ∇ is traceless, i.e. $A \in \mathcal{A}_2 \oplus \mathcal{A}_3$, if and only if $\sum_i \mu(Z_i, Z_i) = 0$.

Remark 8. Before proceed with the proof, let us first describe a useful formula. Recall that the torsion of ∇ is given by $T(X, Y) = \mu(X, Y) - \mu(Y, X) - [X, Y]_{\mathfrak{m}}$, or in other words $T(X, Y, Z) = \hat{\mu}(X, Y, Z) - \hat{\mu}(Y, X, Z) - \langle [X, Y]_{\mathfrak{m}}, Z \rangle$ for any $X, Y, Z \in \mathfrak{m}$. Therefore, a short application of (2.1.1) gives rise to

$$\begin{aligned} 2A(X, Y, Z) &= T(X, Y, Z) - T(Y, Z, X) + T(Z, X, Y) \\ &= \hat{\mu}(X, Y, Z) - \hat{\mu}(Y, X, Z) - \langle [X, Y]_{\mathfrak{m}}, Z \rangle - \hat{\mu}(Y, Z, Z) + \hat{\mu}(Z, Y, X) + \langle [Y, Z]_{\mathfrak{m}}, X \rangle \\ &\quad + \hat{\mu}(Z, X, Y) - \hat{\mu}(X, Z, Y) - \langle [Z, X]_{\mathfrak{m}}, Y \rangle \\ &= 2\hat{\mu}(X, Y, Z) - \langle [X, Y]_{\mathfrak{m}}, Z \rangle, \end{aligned}$$

since $\hat{\mu}(X, Y, Z) + \hat{\mu}(X, Z, Y) = 0$ for any $X, Y, Z \in \mathfrak{m}$ and $\langle\ ,\ \rangle$ is naturally reductive. Thus

$$A(X, Y, Z) = \hat{\mu}(X, Y, Z) - \frac{1}{2}\langle [X, Y]_{\mathfrak{m}}, Z \rangle, \quad \forall\ X, Y, Z \in \mathfrak{m}. \tag{2.3.3}$$

Proof. (1) Assume that $M = G/K$ carries a G-invariant metric connection ∇ whose torsion is of vectorial type. Next we shall identify $\mathfrak{m} \cong T_o M$ and for any $X \in \mathfrak{m} \subset \mathfrak{g}$ we shall write X^* for the (Killing) vector field on M induced by $\exp(-tX)$. Recall that $[X^*, Y^*]_o = -[X, Y]^*_o = -[X, Y]_{\mathfrak{m}}$. Since ∇ is a G-invariant connection, identifying $(\nabla_{X^*} Y^*)_o = \nabla_X Y$, we can write

$$\begin{aligned} \langle \nabla_X Y, Z \rangle &= \langle \nabla^c_X Y, Z \rangle + \langle \mu(X, Y), Z \rangle = \langle \nabla^c_X Y, Z \rangle + \langle \Lambda^{\mu}(X) Y, Z \rangle \\ &= -\langle [X, Y]_{\mathfrak{m}}, Z \rangle + \hat{\mu}(X, Y, Z), \quad (*) \end{aligned}$$

where $(\nabla^c_{X^*} Y^*)_o = \nabla^c_X Y = -[X, Y]_{\mathfrak{m}} = [X^*, Y^*]_o$ is the canonical connection with respect to $\mathfrak{m}$ (cf. [OR1, R]). However, ∇ is of vectorial type, hence there is a 1-form φ on $M = G/K$ such that

$$\mathfrak{m} \otimes \Lambda^2 \mathfrak{m} \cong \mathcal{A} \ni A(X, Y, Z) = \langle X, Y \rangle \varphi(Z) - \langle X, Z \rangle \varphi(Y),$$

for any $X, Y, Z \in \mathfrak{m}$. Using that $\langle\ ,\ \rangle$ is naturally reductive with respect to G and $\mathfrak{m}$, we compute $(\nabla^g_{X^*} Y^*)_o = \frac{1}{2}[X^*, Y^*]_o = -\frac{1}{2}[X, Y]_{\mathfrak{m}}$ and

$$\langle \nabla_X Y, Z \rangle = \langle \nabla^g_X Y, Z \rangle + A(X, Y, Z) = -\frac{1}{2}\langle [X, Y]_{\mathfrak{m}}, Z \rangle + \langle X, Y \rangle \varphi(Z) - \langle X, Z \rangle \varphi(Y).$$

Hence, a small combination with $(*)$ gives rise to

$$\hat{\mu}(X, Y, Z) = \frac{1}{2}\langle [X, Y]_{\mathfrak{m}}, Z \rangle + \langle X, Y \rangle \varphi(Z) - \langle X, Z \rangle \varphi(Y). \quad (**)$$

However, $\hat{\mu}$ is an $\mathrm{Ad}(K)$-invariant tensor (or in other words, it corresponds to a G-invariant tensor field on $M = G/K$), and hence by $(**)$ we conclude that φ must be a (global) G-invariant 1-form on

M. This proves the one direction. Assume now that $(M = G/K, g)$ is endowed with a G-invariant tensor $\hat{\mu} \in \mathfrak{m} \otimes \Lambda^2\mathfrak{m}$ satisfying $(**)$ for some G-invariant 1-form φ on M and let us denote by ∇ the associated G-invariant metric connection. Then, a combination of (2.3.3) and $(**)$ yields that $A \in \mathcal{A}_1$, which completes the proof of (1).
(2) Assume that $M = G/K$ carries a G-invariant metric connection ∇ which is traceless cyclic. This means that the invariant tensor $A(X,Y,Z)$ must satisfy the conditions

$$\mathfrak{S}_{X,Y,Z} A(X,Y,Z) = 0 \ \text{ and } \ \sum_i A(Z_i, Z_i, Z) = 0, \quad (\dagger)$$

where $\{Z_i\}$ is an orthonormal basis of $\mathfrak{m}$ with respect to $\langle \ , \ \rangle$. By (2.3.3) we see that

$$\sum_i A(Z_i, Z_i, Z) = 0 \ \Leftrightarrow \ \sum_i \hat{\mu}(Z_i, Z_i, Z) = 0.$$

However, $\sum_i \hat{\mu}(Z_i, Z_i, Z) = \sum_i \langle \mu(Z_i, Z_i), Z\rangle = \langle \sum_i \mu(Z_i, Z_i), Z\rangle = \langle \sum_i \Lambda(Z_i)Z_i, Z\rangle$, where $\Lambda \equiv \Lambda^{\mu} : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ is the associated connection map. Thus, the traceless condition in $(\dagger)$ holds if and only if $\sum_i \mu(Z_i, Z_i) = \sum_i \Lambda(Z_i)Z_i = 0$. Now, for the cyclic condition in $(\dagger)$, using (2.3.3) we obtain the relation

$$\mathfrak{S}_{X,Y,Z} A(X,Y,Z) = \mathfrak{S}_{X,Y,Z} \hat{\mu}(X,Y,Z) - \frac{3}{2}\langle [X,Y]_{\mathfrak{m}}, Z\rangle$$

and in this way we conclude the second stated relation. In fact, this follows also by the cyclic sum $\mathfrak{S}_{X,Y,Z} T(X,Y,Z) = 0$, where T is the torsion of ∇.
(3) Parts (3) and (4) are immediate due to the description given in (2) and the definition of the classes $\mathcal{A}_1 \oplus \mathcal{A}_2$, and $\mathcal{A}_2 \oplus \mathcal{A}_3$. $\square$

Remark 9. If $(M = G/K, g)$ is an effective Riemannian symmetric space endowed with a G-invariant metric connection $\nabla \equiv \nabla^{\mu}$ corresponding to some $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, then the conclusions in Proposition 10 are simplified, i.e. for the tensor $A = \nabla^{\mu} - \nabla^{g}$ we deduce that

- $A \in \mathcal{A}_1$, i.e. ∇ is vectorial, if and only if $\exists$ a global G-invariant 1-form φ on M such that

$$\hat{\mu}(X,Y,Z) = \langle X, Y\rangle \varphi(Z) - \langle X, Z\rangle \varphi(Y), \quad \forall \ X, Y, Z \in \mathfrak{m}.$$

- $A \in \mathcal{A}_2$, i.e. ∇ is traceless cyclic, if any only if $\mathfrak{S}_{X,Y,Z} \hat{\mu}(X,Y,Z) = 0$ and $\sum_i \Lambda(Z_i)Z_i = 0$.
- $A \in \mathcal{A}_1 \oplus \mathcal{A}_2$, i.e. ∇ is cyclic, if and only if $\mathfrak{S}_{X,Y,Z} \hat{\mu}(X,Y,Z) = 0$ for any $X, Y, Z \in \mathfrak{m}$.

Because on a compact Riemannian symmetric space $(M = G/K, g)$ of Type I, the G-invariant metric connections are exhausted by the torsion-free canonical connection $\nabla^c = \nabla^g$ associated to $\mathfrak{m}$, in the compact case the above conditions are of particular interest for compact connected (non-simple) Lie groups endowed with a bi-invariant metric, where $A = \nabla^{\mu} - \nabla^g$ can be non-trivial. For example, below we apply these considerations for the Lie group U_n. Finally notice that considering a naturally reductive space as in Proposition 10 (or even a symmetric space as above), it is easy to certify that any G-invariant metric connection of type $\mathcal{A}_3$ is also of type $\mathcal{A}_2 \oplus \mathcal{A}_3$, any G-invariant metric connection of type $\mathcal{A}_1$ it is also of type $\mathcal{A}_1 \oplus \mathcal{A}_2$, etc.

Proposition 11. *For $n \geq 3$, the bi-invariant metric connection ∇^{μ} on $(\mathrm{U}_n, \langle \ , \ \rangle)$ induced by the map $\mu := \mu_4 - \mu_5$ of Theorem 23 has torsion of vectorial type.*

Proof. The Lie group U_n has 1-dimensional center Z; hence the quotient $(\mathrm{U}_n \times \mathrm{U}_n)/\Delta\,\mathrm{U}_n$ is not yet effective, but the expression $(\mathrm{U}_n/Z)/(\Delta\,\mathrm{U}_n/\Delta Z)$ satisfies this condition. From now on we shall identify $\mathrm{U}_n \cong (\mathrm{U}_n \times \mathrm{U}_n)/\Delta\,\mathrm{U}_n \cong (\mathrm{U}_n/Z)/(\Delta\,\mathrm{U}_n/\Delta Z)$ and write $\mathfrak{u}(n) \oplus \mathfrak{u}(n) = \Delta\mathfrak{u}(n) \oplus \mathfrak{m}$ for the associated symmetric reductive decomposition, where

$$\Delta\mathfrak{u}(n) := \{(X, X) \in \mathfrak{u}(n) \oplus \mathfrak{u}(n) : X \in \mathfrak{u}(n)\}, \quad \mathfrak{m} := \{(X, -X) \in \mathfrak{u}(n) \oplus \mathfrak{u}(n) : X \in \mathfrak{u}(n)\}$$

are both isomorphic to $\mathfrak{u}(n)$ as U_n-modules. Because any compact connected Lie group G endowed with a bi-invariant metric is a compact normal homogeneous space and moreover a compact symmetric space, the condition $\mathfrak{g} = \tilde{\mathfrak{g}}$ of Proposition 10 is satisfied and we can apply the considerations of Remark 9. Consider the Lie algebra $\mathfrak{u}(n)$ endowed with the bilinear mapping

$\mu(X,Y) = i(\mathrm{tr}(Y)X - \mathrm{tr}(XY)\,\mathrm{Id})$, given in Theorem 23. Since $\langle X,Y\rangle = -\mathrm{tr}(XY)$ we conclude that

$$\begin{aligned}\hat{\mu}(X,Y,Z) := \langle \mu(X,Y), Z\rangle &= i\,\mathrm{tr}(Y)\langle X,Z\rangle - i\,\mathrm{tr}(XY)\langle \mathrm{Id}, Z\rangle \\ &= i\,\mathrm{tr}(Y)\langle X,Z\rangle + i\langle X,Y\rangle\langle \mathrm{Id}, Z\rangle \\ &= i\,\mathrm{tr}(Y)\langle X,Z\rangle - i\,\mathrm{tr}(Z)\langle X,Y\rangle, \end{aligned} \tag{2.3.4}$$

for any $X,Y,Z \in \mathfrak{u}(n)$. Consider now the 1-form $\varphi : \mathfrak{u}(n) \to \mathbb{R}$, $Y \mapsto \varphi(Y) := -i\,\mathrm{tr}(Y)$. It is easy to see that φ is a U_n-invariant 1-form with kernel $\mathfrak{su}(n)$. But then, based on (2.3.4) we obtain that

$$\hat{\mu}(X,Y,Z) = -\langle X,Z\rangle\varphi(Y) + \langle X,Y\rangle\varphi(Z),$$

for any $X,Y,Z \in \mathfrak{u}(n)$ and using Remark 9 we conclude that $\mathcal{A}^{\mu} := \nabla^{\mu} - \nabla^{g} \in \mathcal{A}_1$. □

Remark 10. By Theorem 23, the group U_n $(n \geq 3)$ is equipped with a two dimensional space of bi-invariant metric connections ∇^f, given by the bilinear map $f := a\mu_1 + b\mu$ $(a,b \in \mathbb{R})$ where μ_1 and μ are given by (2.3.2) and Theorem 23, respectively. In general, ∇^f is of mixed type $\mathcal{A}_1 \oplus \mathcal{A}_3$, but the conditions that the type of ∇^f is either purely $\mathcal{A}_3$ or purely $\mathcal{A}_1$, naturally defines the one dimensional subfamilies $a\mu_1$ and $b\mu$, respectively. Thus, we can express the space of bilinear mappings inducing bi-invariant metric connections on U_n as a direct sum of these families.

2.3.5 The curvature tensor and the Ricci tensor

Let us now examine the curvature tensor.

Proposition 12. *Let $(M = G/K, g)$ be a naturally reductive Riemannian manifold as in Proposition 10. Then, the curvature tensor $R^{\nabla^{\mu}} \equiv R^{\nabla}$ associated to a G-invariant metric connection $\nabla \equiv \nabla^{\mu}$ on $M = G/K$, induced by some $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, satisfies the following relation*

$$\begin{aligned} R^{\nabla}(X,Y)Z &= R^{g}(X,Y)Z + A(X,\mu(Y,Z)) - A(Y,\mu(X,Z)) - A([X,Y]_{\mathfrak{m}},Z) \\ &\quad + \frac{1}{2}\Big([X,A(Y,Z)]_{\mathfrak{m}} - [Y,A(X,Z)]_{\mathfrak{m}}\Big), \end{aligned}$$

for any $X,Y,Z \in \mathfrak{m}$, where the tensor A is defined by the difference $A = \nabla - \nabla^{g}$ and R^{g} is the Riemannian curvature tensor. If $(\mathfrak{g},\mathfrak{k})$ is a symmetric pair, then the last three terms in the previous relation are canceled.

Proof. The proof relies on a straightforward computation using the formulas

$$R^{\nabla}(X,Y)Z = \mu(X,\mu(Y,Z)) - \mu(Y,\mu(X,Z)) - \mu([X,Y]_{\mathfrak{m}})Z - [[X,Y]_{\mathfrak{k}}, Z],$$

and $A(X,Y) = \mu(X,Y) - \mu^{g}(X,Y) = \Lambda(X)Y - \Lambda^{g}(X)Y$ where $\mu^{g}(X,Y) = \Lambda^{g}(X)Y = \frac{1}{2}[X,Y]_{\mathfrak{m}}$ is the bilinear map associated to the Levi-Civita connection on $M = G/K$, see also (2.3.3). The last conclusion relies on the symmetric reductive decomposition, in particular (2.3.3) reduces to $A(X,Y,Z) = \hat{\mu}(X,Y,Z)$ for any $X,Y,Z \in \mathfrak{m}$. □

Consider now a G-invariant metric connection ∇ of vectorial type. Let us denote by φ the associated $\mathrm{Ad}(K)$-invariant 1-form on $\mathfrak{m}$ and by $\xi \in \mathfrak{m}$ the dual vector with respect to $\langle\ ,\ \rangle$. If $\|\xi\|^2 \neq 0$, then ∇ is called a *G-invariant connection of non-degenerate vectorial type.* In this case, by applying [AKr, Corol. 3.1] or by a direct calculation based on Proposition 12, we get that

Corollary 3. *Let $(M = G/K, g)$ be a naturally reductive manifold as in Proposition 10, endowed with a G-invariant metric connection $\nabla \equiv \nabla^{\mu}$ of non-degenerate vectorial type. Then*
(1) For any $X,Y \in \mathfrak{m}$, the Ricci tensor Ric^{∇} associated to ∇ satisfies the relation

$$\mathrm{Ric}^{\nabla}(X,Y) = \mathrm{Ric}^{g}(X,Y) + (n-2)\langle X,\xi\rangle\langle Y,\xi\rangle + (2-n)\|\xi\|^2\langle X,Y\rangle + \frac{2-n}{2}\langle [X,Y]_{\mathfrak{m}},\xi\rangle. \tag{2.3.5}$$

(2) Ric^{∇} is symmetric if and only if $(\mathfrak{g},\mathfrak{k})$ is a symmetric pair and this is equivalent to say that φ is a closed invariant 1-form.

Proof. We prove only the second claim. By (2.3.5) it follows that

$$\mathrm{Ric}^{\nabla}(X,Y)-\mathrm{Ric}^{\nabla}(Y,X)=(n-2)\langle[X,Y]_{\mathfrak{m}},\xi\rangle,\quad \forall\ X,Y\in\mathfrak{m}.$$

Hence, Ric^{∇} is symmetric if and only if $\langle[X,Y]_{\mathfrak{m}},\xi\rangle=0$. But since $\xi\neq 0$, this is equivalent to say that $(\mathfrak{g},\mathfrak{k})$ is a symmetric pair, i.e. $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{k}$. By the definition of the differential of an invariant form (cf. [Wo2, pp.248-250]), or by [AKr, Prop. 3.2] we get the last correspondence. □

Specializing to the Lie group U_n we conclude that

Corollary 4. *Consider the Lie group* U_n $(n\geq 3)$ *endowed with the bi-invariant metric connection* ∇^{μ} *induced by the map* $\mu=\mu_4-\mu_5$*, as described in Theorem 23. Then, the Ricci tensor* Ric^{μ} *associated to* ∇^{μ} *is given by the following symmetric invariant bilinear form on* $\mathfrak{u}(n)$*:*

$$\mathrm{Ric}^{\mu}(X,Y)=\frac{1}{2}\Big\{(n-4)\,\mathrm{tr}\,XY+(5-2n)\,\mathrm{tr}\,X\,\mathrm{tr}\,Y\Big\}=-\frac{(n-4)}{2}\langle X,Y\rangle+\frac{(5-2n)}{2}\beta(X,Y)$$

for any $X,Y\in\mathfrak{m}\cong\mathfrak{u}(n)$*, where* $\beta(X,Y):=\mathrm{tr}\,X\,\mathrm{tr}\,Y$*.*

Proof. We use the notation of Proposition 11 and view U_n as an effective symmetric space endowed with the bi-invariant metric induced by $\langle X,Y\rangle=-\,\mathrm{tr}(XY)$. Consider the Nomizu map

$$\Lambda^{\mu}(X)Y:=i(\mathrm{tr}\,(Y)X-\mathrm{tr}\,(XY)\,\mathrm{Id}),\quad \forall\ X,Y\in\mathfrak{m}\cong\mathfrak{u}(n).$$

By Proposition 11 we know that the bi-invariant metric connection $\nabla^{\mu}_XY=\nabla^{c}_XY+\Lambda^{\mu}(X)Y$ has torsion of vectorial type, associated to the U_n-invariant linear form $\varphi(Z)=-i\,\mathrm{tr}(Z)=i\langle\mathrm{Id},Z\rangle$. The dual vector $\xi\in\mathfrak{m}$ is defined by $\varphi(Z)=\langle Z,\xi\rangle$ for any $Z\in\mathfrak{m}$ and hence we conclude that $\xi=i\,\mathrm{Id}$, in particular $0\neq\langle\xi,\xi\rangle=1=\|\xi\|^2$. Thus, the vectorial structure is non-degenerate and we can apply Corollary 3, i.e.

$$\begin{aligned}\mathrm{Ric}^{\mu}(X,Y)&=\mathrm{Ric}^{g}(X,Y)+(n-2)\big(\langle X,\xi\rangle\langle Y,\xi\rangle-\langle X,Y\rangle\big)\\&=\mathrm{Ric}^{g}(X,Y)+(n-2)\big(\,\mathrm{tr}(XY)-\mathrm{tr}\,X\,\mathrm{tr}\,Y\big).\end{aligned}$$

Now, $\langle\ ,\ \rangle$ is a bi-invariant inner product and hence $\mathrm{Ric}^{g}(X,Y)=-\frac{1}{4}B(X,Y)$ for any $X,Y\in\mathfrak{u}(n)$, where $B(X,Y)=2n\,\mathrm{tr}\,XY-2\,\mathrm{tr}\,X\,\mathrm{tr}\,Y$ is the Killing form of U_n (cf. [B, Arv] where the statement is given for a compact semi-simple Lie group, but notice that Ric^{g} satisfies the same formula for any bi-invariant metric g on a Lie group G). Thus, a small computation in combination with the formula given above yields the result. □

Remark 11. For $n=3$, Corollary 4 gives rise to the remarkable expression

$$\mathrm{Ric}^{\mu}(X,X)=-\frac{1}{2}\big(\,\mathrm{tr}\,X^2+(\mathrm{tr}\,X)^2\big),\quad \forall X\in\mathfrak{u}(3).$$

Thus, in this case we conclude that $\mathrm{Ric}^{\mu}(X,X)>0$ is always positive for any non-zero left-invariant vector field $0\neq X\in\mathfrak{u}(3)$. Recall that $\mathrm{Ric}^{g}(X,X)\geq 0$ for any $0\neq X\in\mathfrak{su}(3)$ with $\mathrm{Ric}^{g}(X,X)=0$, if and only $X\in Z(\mathfrak{u}(3))\cong\mathbb{R}$. Finally, on U_4 the Ricci tensor is degenerate, $\mathrm{Ric}^{\mu}(X,Y)=-\frac{3}{2}\beta(X,Y)$.

2.4 Classification of invariant connections on non-symmetric SII spaces

2.4.1 Strongly isotropy irreducible spaces (SII)

Consider a compact, connected, effective, non-symmetric SII homogeneous space $M=G/K$. Since G is a compact simple Lie group (see [Wo1, p. 62]), any such manifold is a standard homogeneous Riemannian manifold. Passing to a covering $\tilde{G}$ of G, if G/K is not simply-connected but G is connected, then $\tilde{G}$ acts transitively on the universal covering of G/K with connected isotropy group, say K', and it turns out that G/K is SII if and only if $\tilde{G}/K'$ is. Hence, whenever necessary we can assume that G/K is a compact, connected and simply-connected, effective, non-symmetric SII space, with G being compact, connected and simple and $K\subset G$ compact and connected. In this setting, the strongly isotropy irreducible condition is equivalent to an (almost effective) irreducible action of the Lie algebra $\mathfrak{k} = T_eK$ on $\mathfrak{m} \cong T_oG/K$. For a list of non-symmetric SII spaces we refer to [B, Tables 5, 6, p. 203]. We remark however that there are misprints in Table 6 of [B], related to SII homogeneous spaces $M = G/K$ of $G = \mathrm{Sp}_n$ (compare for example with [Wo1, Thm. 7.1]). We correct these errors in our Table 2.5 below.

Proposition 13. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective, non-symmetric (compact) SII homogeneous Riemannian manifold, endowed with a G-invariant affine connection ∇^{μ} compatible with the Killing metric $\langle\ ,\ \rangle = -B|_{\mathfrak{m}}$, where $\mu \in \mathrm{Hom}_K(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$. Then, the torsion T^{μ} of ∇^{μ} does not carry a component of vectorial type.*

Proof. Assume that $M = G/K$ carries a G-invariant metric connection ∇ whose torsion is of vectorial type and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the reductive decomposition with respect to the Killing metric. Then, by Proposition 10 (1), we have that $\hat{\mu}(X, Y, Z) = \frac{1}{2}\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle X, Y\rangle\varphi(Z) - \langle X, Z\rangle\varphi(Y)$, for some G-invariant 1-form φ on $M = G/K$. However, $\mathfrak{m}$ is a self-dual and (strongly) irreducible K-module over $\mathbb{R}$; thus global G-invariant 1-forms do not exist, since dually the isotropy representation needs to preserve some vector field ξ and hence a 1-dimensional subspace of $\mathfrak{m}$, spanned by ξ. $\square$

Corollary 5. *Let $(M = G/K, g)$ be an effective, non-symmetric (compact) SII homogeneous Riemannian manifold, endowed with a non torsion-free G-invariant metric connection ∇. Then, the torsion $0 \neq T$ of ∇ is totally skew-symmetric, $T \in \mathcal{A}_3 \cong \Lambda^3 TM$, or traceless cyclic $T \in \mathcal{A}_2$, or of mixed type $T \in \mathcal{A}_2 \oplus \mathcal{A}_3$, i.e. traceless.*

2.4.2 An application in the spin case

Consider an effective, non-symmetric (compact) SII homogeneous Riemannian manifold $(M^n = G/K, g)$. Assume that $M = G/K$ admits a G-invariant spin structure, i.e. a G-homogeneous $\mathrm{Spin}(\mathfrak{m})$-principal bundle $P \to M$ and a double covering morphism $\Lambda : P \to \mathrm{SO}(M, g)$ compatible with the principal groups' actions. Recall that an invariant spin structure corresponds to a lift of the isotropy representation χ into the spin group $\mathrm{Spin}(\mathfrak{m}) \equiv \mathrm{Spin}_n$, i.e. a homomorphism $\widetilde{\chi} : K \to \mathrm{Spin}(\mathfrak{m})$ such that $\chi = \lambda \circ \widetilde{\chi}$, where $\lambda : \mathrm{Spin}(\mathfrak{m}) \to \mathrm{SO}(\mathfrak{m})$ is the double covering of $\mathrm{SO}(\mathfrak{m}) \equiv \mathrm{SO}_n$. We shall denote by $\kappa_n : C\ell^{\mathbb{C}}(\mathfrak{m}) \xrightarrow{\sim} \mathrm{End}(\Delta_{\mathfrak{m}})$ the Clifford representation and by $C\ell(X \otimes \phi) := \kappa_n(X)\psi = X \cdot \psi$ the Clifford multiplication between vectors and spinors, see [A] for more details. Set $\rho := \kappa \circ \widetilde{\chi} : K \to \mathrm{Aut}(\Delta_{\mathfrak{m}})$, where $\kappa = \kappa_n|_{\mathrm{Spin}(\mathfrak{m})} : \mathrm{Spin}(\mathfrak{m}) \to \mathrm{Aut}(\Delta_{\mathfrak{m}})$ is the spin representation. The spinor bundle $\Sigma \to G/K$ is the homogeneous vector bundle associated to $P := G \times_{\widetilde{\chi}} \mathrm{Spin}(\mathfrak{m})$ via the representation ρ, i.e. $\Sigma = G \times_{\rho} \Delta_{\mathfrak{m}}$. Therefore we may identify sections of Σ with smooth functions $\varphi : G \to \Delta_{\mathfrak{m}}$ such that $\varphi(gk) = \kappa\big(\widetilde{\chi}(k^{-1})\big)\varphi(g) = \rho(k^{-1})\varphi(g)$ for any $g \in G, k \in K$.
Choose a G-invariant metric connection ∇ on G/K, corresponding to a connection map $\Lambda \in \mathrm{Hom}_K(\mathfrak{m}, \mathfrak{so}(\mathfrak{m}))$. The lift $\widetilde{\Lambda} := \lambda_*^{-1} \circ \Lambda : \mathfrak{m} \to \mathfrak{spin}(\mathfrak{m})$ induces a covariant derivative on spinor fields (which we still denote by the same symbol) $\nabla : \Gamma(\Sigma) \to \Gamma(T^*(G/K) \otimes \Sigma)$, given by $\nabla_X \psi = X(\psi) + \widetilde{\Lambda}(X)\psi$. Here, the vector $X \in \mathfrak{m}$ is considered as a left-invariant vector field in G and $\widetilde{\Lambda}(X)\psi$ as an equivariant function $\widetilde{\Lambda}(X)\psi : G \to \mathfrak{m}$. The Dirac operator $D := C\ell \circ \nabla : \Gamma(\Sigma) \to \Gamma(\Sigma)$ associated to ∇ is defined as follows (cf. [A]):

$$D(\psi) := \sum_i \kappa_n(Z_i)\{Z_i(\psi) + \widetilde{\Lambda}(Z_i)\psi\} = \sum_i Z_i \cdot \{Z_i(\psi) + \widetilde{\Lambda}(Z_i)\psi\},$$

where Z_i denotes a $\langle\ ,\ \rangle$-orthonormal basis of $\mathfrak{m}$.

Remark 12. Given a spin Riemannian manifold (M, g) endowed with a metric connection ∇, basic properties of the induced Dirac operator $D = C\ell \circ \nabla$ are reflected in the type of the torsion of ∇. For example, by a result of Th. Friedrich [Fr] (see also [FrS, PS]), it is known that the formal self-adjointness of the Dirac operator $D = C\ell \circ \nabla$ is equivalent to the condition $A \in \mathcal{A}_2 \oplus \mathcal{A}_3$, where $A = \nabla - \nabla^g$. Hence, in our case as an immediate consequence of Corollary 5 we obtain that

Corollary 6. *Let $(M = G/K, g)$ be an effective, non-symmetric (compact) SII homogeneous Riemannian manifold, endowed with a G-invariant metric connection ∇ and a G-invariant spin structure. Then, the Dirac operator D associated to ∇ is formally self-adjoint.*

Note that the classification of invariant spin structures on non-symmetric SII spaces is an open problem (see [CG] for invariant spin structures on symmetric spaces and [AℓC] for a more recent study of spin structures on reductive homogeneous spaces).

2.4.3 Classification results on invariant connections

For the presentation of the classification results, we use the notation of [OV, p. 299]. In particular, for a compact simple Lie group G we shall denote by $R(\pi)$ the complex irreducible representation of highest weight π. We mention that the isotropy representation of a compact, non-symmetric, effective SII space turns out to be of either real or complex type. In fact, fixing a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, whenever the complexification $\mathfrak{m}^{\mathbb{C}}$ splits into two complex submodules, these are never equivalent representations (see also [Wo1]). Hence, by Schur's lemma we have the identification $\mathrm{Hom}_K(\mathfrak{m}, \mathfrak{m}) = \mathbb{C}$ for complex type and $\mathrm{Hom}_K(\mathfrak{m}, \mathfrak{m}) = \mathbb{R}$ for real type. In the first case, the endomorphism J induced by $i \in \mathbb{C}$ makes G/K a homogeneous almost complex manifold. Note that the same conclusions are true for a symmetric space, see [L2, L3] (recall that the adjoint representation of a compact simple Lie group is always of real type).

Remark 13. The multiplicities that we describe below have also been presented in the PhD thesis [Cℓ] (see Tables I.3.1–I.3.4, pp. 77–79), for a different however aim, namely the description of the components of the intrinsic torsion associated to (irreducible) G-structures over non-symmetric compact SII spaces (see also [CℓS]). We remark that there are a few errors/omissions in [Cℓ], related with some low-dimensional cases, namely:

- the case $p = 2, q \geq 3$ of the family $\mathrm{SU}_{pq} / \mathrm{SU}_p \times \mathrm{SU}_q$,
- the case $n = 5$ of the family $\mathrm{SU}_{\frac{n(n-1)}{2}} / \mathrm{SU}_n$,
- the case $n = 6$ of the family $\mathrm{SO}_{\frac{(n-1)(n+2)}{2}} / \mathrm{SO}_n$ (due to isomorphism $\mathfrak{so}(6) = \mathfrak{su}(4)$).

In these mentioned cases, the general decompositions of $\Lambda^2\mathfrak{m}$ or $\mathrm{Sym}^2\,\mathfrak{m}$ change and most times this affects to multiplicities that we are interested in. Notice also that for the manifold $\mathrm{SO}_{4n} / \mathrm{Sp}_n \times \mathrm{Sp}_1$ the enumeration in [Wo1, B] starts for $n \geq 2$ (as we do), but in [Cℓ] it is written $n \geq 3$. We correct these errors in our Table 2.4 (they are indicated by an asterisk). Notice finally that the author of this thesis uses the LiE program (as we do) and for infinite families he is based on stability arguments, see [Cℓ, Rem. I.3.9]. Below we also give examples of how such families can be treated even without the aid of a computer.

Remark 14. Given a reductive homogeneous space $M = G/K$ of a classical simple Lie group G, there is a simple method for the computation of the associated isotropy representation $\chi : K \to \mathrm{Aut}(\mathfrak{m})$, given as follows. Let us denote by $\rho_n : \mathrm{SO}_n \to \mathrm{Aut}(\mathbb{R}^n)$, $\mu_n : \mathrm{SU}_n \to \mathrm{Aut}(\mathbb{C}^n)$ and $\nu_n : \mathrm{Sp}_n \to \mathrm{Aut}(\mathbb{H}^n)$ be the standard representations of SO_n, SU_n (or U_n), and Sp_n, respectively. Recall that the complexified adjoint representation $\mathrm{Ad}_G^{\mathbb{C}} = \mathrm{Ad}_G \otimes \mathbb{C}$, satisfies

$$\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SO}_n} = \Lambda^2\rho_n, \quad \mathrm{Ad}^{\mathbb{C}}_{\mathrm{U}_n} = \mu_n \otimes \mu_n^*, \quad \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_n} \oplus 1 = \mu_n \otimes \mu_n^*, \quad \mathrm{Ad}^{\mathbb{C}}_{\mathrm{Sp}_n} = \mathrm{Sym}^2\nu_n,$$

where μ_n^* is the dual representation of μ_n and 1 denotes the trivial 1-dimensional representation. Let G be one of the Lie groups $\mathrm{SO}_n, \mathrm{SU}_n, \mathrm{Sp}_n$ and let $\pi : K \to G$ be an (almost) faithful representation of a compact connected subgroup K. Using the identity $\mathrm{Ad}\,\big|_K = \mathrm{Ad}_K \oplus \chi$, we see that the isotropy representation χ of $G/\pi(K)$ is determined by $\Lambda^2\pi = \mathrm{ad}_{\mathfrak{k}} \oplus \chi$ in the orthogonal case, by $\pi \otimes \pi^* = 1 \oplus \mathrm{ad}_{\mathfrak{k}} \oplus \chi$ in the unitary case and finally by $\mathrm{Sym}^2\,\pi = \mathrm{ad}_{\mathfrak{k}} \oplus \chi$ in the symplectic case (cf. [Wo1, WZ]).

Theorem 24. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective, non-symmetric (compact) SII homogeneous space. Consider the B-orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then, the complexified isotropy representation $\mathfrak{m}^{\mathbb{C}}$ and the multiplicities* $\mathbf{a}$, $\mathbf{s}$, $\mathcal{N}$ *and* ℓ *are given in Tables 2.4 and 2.5.*

2.4.4 On the Theorems A.1, A.2 and B – Conclusions

The results in Tables 2.4 and 2.5 allows us to deduce that several non-symmetric SII spaces are carrying *new* families of invariant metric connections, in the sense that they are different from the Lie bracket family $\eta^{\alpha}(X, Y) = \frac{1-\alpha}{2}[X, Y]_{\mathfrak{m}}$ (see Lemma 8). In combination with Lemma 12, we also certify the existence of compact, effective, non-symmetric SII quotients $M = G/K$ which are endowed with additional families of G-invariant metric connections with skew-torsion, besides η^{α}. In full details, this occurs in the following two situations:

- when $2 \leq \ell \leq \mathbf{a}$ and the isotropy representation is not of complex type (since for complex type we may have $\ell = 2 = \mathbf{a}$, but due to Schurs's lemma all these invariant connections must be exhausted by the family $\eta^{\alpha}(X, Y) = \frac{1-\alpha}{2}[X, Y]_{\mathfrak{m}}$ with $\alpha \in \mathbb{C}$), or
- when the isotropy representation is of complex type but ℓ (and hence $\mathbf{a}$) is strictly greater than 2.

This observation, in combination with Lemmas 9, 12 and the results in Tables 2.4 and 2.5, yields Theorems A.1 and A.2. Theorem B it is also a direct conclusion of the multiplicity $\mathbf{s}$ given in Tables 2.4 and 2.5 and Lemma 10. In fact, for affine connections induced by symmetric elements $\mu \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$, we also conclude that

Corollary 7. *Let $(M = G/K, g = -B|_{\mathfrak{m}})$ be an effective, non-symmetric, SII homogeneous space associated to the Lie group $G = \mathrm{SU}_n$. Then, there is always a copy of $\mathfrak{m}$ inside $\mathrm{Sym}^2 \mathfrak{m}$, induced by the restriction of the $\mathrm{Ad}(\mathrm{SU}_n)$-invariant symmetric bilinear mapping*

$$\eta : \mathfrak{su}(n) \times \mathfrak{su}(n) \to \mathfrak{su}(n), \quad \eta(X, Y) := i\big\{XY + YX - \frac{2}{n}\,\mathrm{tr}(XY)\,\mathrm{Id}\,\big\}$$

on the corresponding reductive complement $\mathfrak{m}$. If M is isometric to one of the manifolds

$$\mathrm{SU}_{10}\,/\,\mathrm{SU}_5, \quad \mathrm{SU}_{2q}\,/\,\mathrm{SU}_2 \times \mathrm{SU}_q \ (q \geq 3), \quad \mathrm{SU}_{16}\,/\,\mathrm{Spin}_{10},$$

then the 1-parameter family of SU_n-invariant affine connections on $M = \mathrm{SU}_n/K$ associated to the restriction $\eta|_{\mathfrak{m}} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$, exhausts all SU_n-invariant affine connections induced by some $0 \neq \mu \in \mathrm{Hom}_K(\mathrm{Sym}^2 \mathfrak{m}, \mathfrak{m})$.

Proof. The first part is based on [L3, Thm. 6.1]. Notice that η is known by [L2, p. 550]. Now, using the results of Tables 2.4 and 2.5 about the multiplicity $\mathbf{s}$ of $\mathfrak{m}$ inside $\mathrm{Sym}^2 \mathfrak{m}$, we obtain the result. □

2.4.5 Some explicit examples

Let us now compute the desired multiplicities $\mathbf{a}$, $\mathbf{s}$ and ℓ, for general families of (non-symmetric) SII spaces, without the aid of computer. For this we need first to recall some preliminaries of representation theory (for more details we refer to [BT, Sim, Cℓ]).

If π is a complex representation of a compact Lie algebra $\mathfrak{k}$, then $\overline{\pi} \cong \pi^*$, where $\overline{\pi}$ denotes the complex conjugate representation and π^* the dual representation. If π is a complex representation of $\mathfrak{k}$ on V, then there is a symmetric (resp. skew-symmetric) non-degenerate bilinear form on V invariant under π, if and only if there is a anti-linear intertwining map τ with $\tau^2 = \mathrm{Id}$ (resp. $\tau^2 = -\mathrm{Id}$) [BT, Prop. 6.4.]. If π is irreducible then Schur's lemma ensures the uniqueness of such a bilinear form. A complex representation carrying a conjugate linear intertwining map τ with $\tau^2 = \mathrm{Id}$ (resp. $\tau^2 = -\mathrm{Id}$) is called of real type (resp. quaternonic type). Finally we call a complex representation $\pi : \mathfrak{k} \to \mathfrak{gl}(V)$ of complex type if it is not self dual, i.e. $V \not\cong V^*$.

Let (π, V) and (π', W) be representations of a connected (not necessarily compact) Lie group H on two vector spaces V and W, respectively. It is important to note that even if π and π' are irreducible, then the tensor product representation $V \otimes W$, defined by $\pi \otimes \pi' : H \to \mathrm{Aut}(V \otimes W)$, $(\pi \otimes \pi')(h)(u \otimes w) = \pi(h)u \otimes \pi'(h)w$, is always reducible. Let us denote by $\Lambda^k \pi$ and $\mathrm{Sym}^k \pi$ the k-th exterior power and k-th symmetric power, respectively. For $k = 2$, it is easy to prove that

$$\left\{ \begin{array}{lcl} \Lambda^2(V \oplus W) & = & \Lambda^2 V \oplus (V \otimes W) \oplus \Lambda^2 W, \\ \mathrm{Sym}^2(V \oplus W) & = & \mathrm{Sym}^2 V \oplus (V \otimes W) \oplus \mathrm{Sym}^2 W. \end{array} \right. \tag{2.4.1}$$

If (π, V) and (π', W) are representations of two connected Lie groups H and H', respectively, then the vector space $V \otimes W$ carries a representation of the product group $H \times H'$, say $(\pi \hat{\otimes} \pi', V \otimes W)$, given by $\pi \hat{\otimes} \pi'(h, h')(u \otimes w) = \pi(h)u \otimes \pi'(h')w$. This representation is called the external tensor product of π and π'. In the finite-dimensional case, $\pi \hat{\otimes} \pi'$ is an irreducible representation of $H \times H'$,

Table 2.4: The multiplicities $\mathbf{a}$, $\mathbf{s}$, $\mathcal{N}$ and ℓ for (non-symmetric) SII homogeneous spaces–Classical families

Classical families and their associated low-dimensional cases

G	$M = G/K$	$\mathfrak{m}^{\mathbb{C}}$	$\mathbf{a}$	$\mathbf{s}$	$\mathcal{N}$	ℓ	Type
SU_n	1) $\mathrm{SU}_{\frac{n(n-1)}{2}} / \mathrm{SU}_n \quad (n \geq 6)$	$R(\pi_2 + \pi_{n-2})$	1	2	3	1	r
	1^*_α) $\mathrm{SU}_{10} / \mathrm{SU}_5$	$R(\pi_2 + \pi_3)$	1	1	2	1	r
	2) $\mathrm{SU}_{\frac{n(n+1)}{2}} / \mathrm{SU}_n \quad (n \geq 3)$	$R(2\pi_1 + 2\pi_{n-1})$	1	2	3	1	r
	3) $\mathrm{SU}_{pq} / \mathrm{SU}_p \times \mathrm{SU}_q \quad (p, q \geq 3)$	$R(\pi_1 + \pi_{p-1}) \hat{\otimes} R(\pi_1 + \pi_{q-1})$	2	2	4	2	r
	3^*_α) $\mathrm{SU}_{2q} / \mathrm{SU}_2 \times \mathrm{SU}_q \quad (q \geq 3)$	$R(2\pi_1) \hat{\otimes} R(\pi_1 + \pi_{q-1})$	1	1	2	1	r
SO_n	4) $\mathrm{SO}_{n^2-1} / \mathrm{SU}_n \quad (n \geq 4)$	$R(2\pi_1 + \pi_{n-2}) \oplus R(\pi_2 + 2\pi_{n-1})$	6	2	8	4	c
	4_α) $\mathrm{SO}_8 / \mathrm{SU}_3$	$R(3\pi_1) \oplus R(3\pi_2)$	2	0	2	2	c
	5) $\mathrm{SO}_{\frac{n(n-1)}{2}} / \mathrm{SO}_n \quad (n \geq 9)$	$R(\pi_1 + \pi_3)$	3	1	4	2	r
	5_α) $\mathrm{SO}_{21} / \mathrm{SO}_7$	$R(\pi_1 + 2\pi_3)$	3	1	4	2	r
	5_β) $\mathrm{SO}_{28} / \mathrm{SO}_8$	$R(\pi_1 + \pi_3 + \pi_4)$	4	3	7	2	r
	6) $\mathrm{SO}_{\frac{(n-1)(n+2)}{2}} / \mathrm{SO}_n \quad (n \geq 7)$	$R(2\pi_1 + \pi_2)$	3	1	4	2	r
	6_α) $\mathrm{SO}_{14} / \mathrm{SO}_5$	$R(2\pi_1 + 2\pi_2)$	3	1	4	2	r
	6^*_β) $\mathrm{SO}_{20} / \mathrm{SO}_6$	$R(2\pi_1 + \pi_2 + \pi_3)$	3	2	5	2	r
	7) $\mathrm{SO}_{(n-1)(2n+1)} / \mathrm{Sp}_n \quad (n \geq 4)$	$R(\pi_1 + \pi_3)$	3	1	4	2	r
	7_α) $\mathrm{SO}_{14} / \mathrm{Sp}_3$	$R(\pi_1 + \pi_3)$	1	0	1	1	r
	8) $\mathrm{SO}_{n(2n+1)} / \mathrm{Sp}_n \quad (n \geq 3)$	$R(2\pi_1 + \pi_2)$	3	1	4	2	r
	8_α) $\mathrm{SO}_{10} / \mathrm{Sp}_2$	$R(2\pi_1 + \pi_2)$	2	1	3	1	r
	9^*) $\mathrm{SO}_{4n} / \mathrm{Sp}_n \times \mathrm{Sp}_1 \quad (n \geq 2)$	$R(\pi_2) \hat{\otimes} R(2\pi_1)$	1	0	1	1	r
Sp_n	10) $\mathrm{Sp}_n / \mathrm{SO}_n \times \mathrm{Sp}_1 \quad (n \geq 5)$	$R(2\pi_1) \hat{\otimes} R(2\pi_1)$	1	0	1	1	r
	10_α) $\mathrm{Sp}_3 / \mathrm{SO}_3 \times \mathrm{Sp}_1$	$R(4\pi_1) \hat{\otimes} R(2\pi_1)$	1	0	1	1	r
	10_β) $\mathrm{Sp}_4 / \mathrm{SO}_4 \times \mathrm{Sp}_1$	$R(2\pi_1 + 2\pi_2) \hat{\otimes} R(2\pi_1)$	1	0	1	1	r

Table 2.5: The multiplicities $\mathbf{a}$, $\mathbf{s}$, $\mathcal{N}$ and ℓ for (non-symmetric) SII homogeneous spaces–Exceptions

Exceptions ("exceptions" in terms of [B, p. 203])

G	$M = G/K$	$\mathfrak{m}^{\mathbb{C}}$	$\mathbf{a}$	$\mathbf{s}$	$\mathcal{N}$	ℓ	Type
SU_n	$\mathrm{SU}_{16}\,/\,\mathrm{Spin}_{10}$	$R(\pi_4+\pi_5)$	1	1	2	1	r
	$\mathrm{SU}_{27}\,/\,\mathrm{E}_6$	$R(\pi_1+\pi_6)$	1	2	3	1	r
SO_n	$\mathrm{SO}_7\,/\,\mathrm{G}_2$	$R(\pi_1)$	1	0	1	1	r
	$\mathrm{SO}_{14}\,/\,\mathrm{G}_2$	$R(3\pi_1)$	2	0	2	2	r
	$\mathrm{SO}_{16}\,/\,\mathrm{Spin}_9$	$R(\pi_3)$	1	0	1	1	r
	$\mathrm{SO}_{26}\,/\,\mathrm{F}_4$	$R(\pi_3)$	2	0	2	2	r
	$\mathrm{SO}_{42}\,/\,\mathrm{Sp}_4$	$R(2\pi_3)$	2	0	2	2	r
	$\mathrm{SO}_{52}\,/\,\mathrm{F}_4$	$R(\pi_2)$	2	0	2	2	r
	$\mathrm{SO}_{70}\,/\,\mathrm{SU}_8$	$R(\pi_3+\pi_5)$	2	1	3	2	r
	$\mathrm{SO}_{248}\,/\,\mathrm{E}_8$	$R(\pi_7)$	2	0	2	2	r
	$\mathrm{SO}_{78}\,/\,\mathrm{E}_6$	$R(\pi_4)$	2	0	2	2	r
	$\mathrm{SO}_{128}\,/\,\mathrm{Spin}_{16}$	$R(\pi_6)$	2	0	2	2	r
	$\mathrm{SO}_{133}\,/\,\mathrm{E}_7$	$R(\pi_3)$	2	0	2	2	r
Sp_n	$\mathrm{Sp}_2\,/\,\mathrm{SU}_2$	$R(6\pi_1)$	1	0	1	1	r
	$\mathrm{Sp}_7\,/\,\mathrm{Sp}_3$	$R(2\pi_3)$	1	0	1	1	r
	$\mathrm{Sp}_{10}\,/\,\mathrm{SU}_6$	$R(2\pi_3)$	1	0	1	1	r
	$\mathrm{Sp}_{16}\,/\,\mathrm{Spin}_{12}$	$R(2\pi_6)$ or $R(2\pi_5)$	1	0	1	1	r
	$\mathrm{Sp}_{28}\,/\,\mathrm{E}_7$	$R(2\pi_7)$	1	0	1	1	r
G_2	$\mathrm{G}_2\,/\,\mathrm{SU}_3$	$R(\pi_1)\oplus R(\pi_2)$	2	0	2	2	c
	$\mathrm{G}_2\,/\,\mathrm{SO}_3$	$R(10\pi_1)$	1	0	1	1	r
F_4	$\mathrm{F}_4\,/(\mathrm{SU}_3^1\times\mathrm{SU}_3^2)$	$(R(2\pi_1)\hat{\otimes}R(\omega_1))\oplus(R(2\pi_2)\hat{\otimes}R(\omega_2))$	2	0	2	2	c
	$\mathrm{F}_4\,/(\mathrm{G}_2\times\mathrm{SU}_2)$	$R(\pi_1)\hat{\otimes}R(4\omega_1)$	1	0	1	1	r
E_6	$\mathrm{E}_6\,/\,\mathrm{SU}_3$	$R(4\pi_1+\pi_2)\oplus R(\pi_1+4\pi_2)$	6	4	10	4	c
	$\mathrm{E}_6\,/(\mathrm{SU}_3\times\mathrm{SU}_3\times\mathrm{SU}_3)$	$(R(\pi_1)\hat{\otimes}R(\omega_1)\hat{\otimes}R(\theta_1))\oplus$ $(R(\pi_2)\hat{\otimes}R(\omega_2)\hat{\otimes}R(\theta_2))$	2	0	2	2	c
	$\mathrm{E}_6\,/\,\mathrm{G}_2$	$R(\pi_1+\pi_2)$	1	1	2	1	r
	$\mathrm{E}_6\,/(\mathrm{G}_2\times\mathrm{SU}_3)$	$R(\pi_1)\hat{\otimes}R(\omega_1+\omega_2)$	1	1	2	1	r
E_7	$\mathrm{E}_7\,/\,\mathrm{SU}_3$	$R(4\pi_1+4\pi_2)$	2	3	5	2	r
	$\mathrm{E}_7\,/(\mathrm{SU}_3\times\mathrm{SU}_6)$	$(R(\pi_1)\hat{\otimes}R(\omega_2))\oplus(R(\pi_2)\hat{\otimes}R(\omega_5))$	2	0	2	2	c
	$\mathrm{E}_7\,/(\mathrm{G}_2\times\mathrm{Sp}_3)$	$R(\pi_1)\hat{\otimes}R(\omega_2)$	1	0	1	1	r
	$\mathrm{E}_7\,/(\mathrm{F}_4\times\mathrm{SU}_2)$	$R(\pi_4)\hat{\otimes}R(2\omega_1)$	1	0	1	1	r
E_8	$\mathrm{E}_8\,/\,\mathrm{SU}_9$	$R(\pi_3)\oplus R(\pi_6)$	2	0	2	2	c
	$\mathrm{E}_8\,/(\mathrm{F}_4\times\mathrm{G}_2)$	$R(\pi_4)\hat{\otimes}R(\omega_1)$	1	0	1	1	r
	$\mathrm{E}_8\,/(\mathrm{E}_6\times\mathrm{SU}_3)$	$(R(\pi_1)\hat{\otimes}R(\omega_1))\oplus(R(\pi_6)\hat{\otimes}R(\omega_2))$	2	0	2	2	c

if and only π and π' are both irreducible. In particular, if H, H' are compact Lie groups, then a representation of $H \times H'$ in $\mathrm{GL}(\mathbb{C}^n)$ is irreducible if and only if it is the tensor product of an irreducible representation of H with one of H'. Finally, one has the following equivariant isomorphisms:

$$\left\{\begin{array}{lcl} (V\hat{\otimes}W)\otimes(V'\hat{\otimes}W') & = & (V\otimes V')\hat{\otimes}(W\otimes W') \\ \Lambda^2(V\hat{\otimes}W) & = & (\Lambda^2 V\hat{\otimes}\operatorname{Sym}^2 W)\oplus(\operatorname{Sym}^2 V\hat{\otimes}\Lambda^2 W) \\ \operatorname{Sym}^2(V\hat{\otimes}W) & = & (\operatorname{Sym}^2 V\hat{\otimes}\operatorname{Sym}^2 W)\oplus(\Lambda^2 V\hat{\otimes}\Lambda^2 W) \end{array}\right. \tag{2.4.2}$$

We finally remark that if V, W are complex irreducible representations of two compact Lie groups H and H' respectively, then $V\hat{\otimes}W$ is of real type if V, W are both of real type or both of quaternionic type, $V\hat{\otimes}W$ is if complex type if at least one of V, W are of complex type and finally, $V\hat{\otimes}W$ is of quaternonic type if one of V, W is of real type and the other one of quaternionic type.

Lemma 13. *Consider the homogeneous space $M_{p,q} := G/K = \mathrm{SU}_{pq}/\mathrm{SU}_p\times\mathrm{SU}_q$ with $p, q > 2, p+q > 4$. Then, the multiplicities of the isotropy representation $\mathfrak{m} = \mathrm{Ad}_{\mathrm{SU}_p}\hat{\otimes}\mathrm{Ad}_{\mathrm{SU}_q}$ inside $\Lambda^2\mathfrak{m}$ and $\operatorname{Sym}^2\mathfrak{m}$ are given as follows: for $p=2, q\geq 3$ it is $\mathbf{a}=\mathbf{s}=1$, while for $p, q\geq 3$ it is $\mathbf{a}=\mathbf{s}=2$. Moreover, the dimension of the trivial submodule $(\Lambda^3\mathfrak{m})^K$ is $\ell=1$ for $p=2, q\geq 3$ and $\ell=2$ for $p, q\geq 3$.*

Proof. The inclusion $\pi : K \to G$ is given by the external tensor product of the standard representations μ_p and μ_q of SU_p and SU_q, respectively. Thus, a short application of Remark 14 in combination with the relation $\mathrm{Ad}^{\mathbb{C}}_{A_n} = R(\pi_1+\pi_n)$, yields that

$$\mathfrak{m}^{\mathbb{C}} = (\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}) = R(\pi_1+\pi_{p-1})\hat{\otimes}R(\pi_1+\pi_{q-1}).$$

Consequently, the isotropy representation $\mathfrak{m} = \mathrm{Ad}_{\mathrm{SU}_p}\hat{\otimes}\mathrm{Ad}_{\mathrm{SU}_q}$ of $M_{p,q} = \mathrm{SU}_{pq}/\mathrm{SU}_p\times\mathrm{SU}_q$ is irreducible over $\mathbb{R}$ and is of real type, since it is the external tensor product of two representations of real type. Now, by [OV] we also know that

$$\Lambda^2\mathrm{Ad}^{\mathbb{C}}_{A_n} = R(2\pi_1+\pi_{n-1})\oplus R(\pi_2+2\pi_n)\oplus\mathrm{Ad}^{\mathbb{C}}_{A_n},\quad n\geq 3 \tag{2.4.3}$$

$$\operatorname{Sym}^2\mathrm{Ad}^{\mathbb{C}}_{A_n} = \left\{\begin{array}{ll} R(2\pi_1+2\pi_n)\oplus R(\pi_2+\pi_{n-1})\oplus\mathrm{Ad}^{\mathbb{C}}_{A_n}\oplus 1, & \text{if } n\geq 3, \\ R(2\pi_1+2\pi_2)\oplus\mathrm{Ad}^{\mathbb{C}}_{A_n}\oplus 1, & \text{if } n=2. \end{array}\right. \tag{2.4.4}$$

Certainly, for $\mathfrak{su}_2 = \mathfrak{so}_3 = \mathfrak{sp}_1$ one gets a 3-dimensional irreducible representation $\Lambda^2(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}) \cong \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2} = R(2\pi_1)$. Moreover, it is $\operatorname{Sym}^2(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}) = R(4\pi_1)\oplus 1$. Notice also that $\Lambda^2(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}) = R(3\pi_1)\oplus R(3\pi_2)\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}$, since $\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3} = R(\pi_1+\pi_2)$. Due to this small speciality of SU_2 and the different decomposition of $\operatorname{Sym}^2\mathrm{Ad}^{\mathbb{C}}_{A_n}$ (for $n=2$ and $n\geq 3$, respectively) one has to separate the examination into two cases:

Case A: $p=2$, $q\geq 3$. Then we have $M_{2q} = \mathrm{SU}_{2q}/\mathrm{SU}_2\times\mathrm{SU}_q$ and

$$\mathfrak{m}^{\mathbb{C}} = \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q} = R(2\pi_1)\hat{\otimes}R(\pi_1+\pi_{q-1}),\quad (q\geq 3).$$

Hence, a combination of (2.4.2), (2.4.3), (2.4.4) and $\Lambda^2(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}) = \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2} = R(2\pi_1)$ shows that

$$\begin{aligned} \Lambda^2(\mathfrak{m}^{\mathbb{C}}) &= \big(R(2\pi_1)\hat{\otimes}\operatorname{Sym}^2\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\oplus\big((R(4\pi_1)\oplus 1)\hat{\otimes}\Lambda^2\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big) \\ &= \big(R(2\pi_1)\hat{\otimes}R(2\pi_1+2\pi_{q-1})\big)\oplus\big(R(2\pi_1)\hat{\otimes}R(\pi_2+\pi_{q-2})\big)\oplus\big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\oplus R(2\pi_1) \\ &\quad\oplus\big(R(4\pi_1)\hat{\otimes}R(2\pi_1+\pi_{q-2})\big)\oplus\big(R(4\pi_1)\hat{\otimes}R(\pi_2+2\pi_{q-1})\big)\oplus\big(R(4\pi_1)\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big) \\ &\quad\oplus R(2\pi_1+\pi_{q-2})\oplus R(\pi_2+2\pi_{q-1})\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}. \end{aligned}$$

We deduce that $\mathfrak{m}^{\mathbb{C}}$ appears once inside $\Lambda^2(\mathfrak{m}^{\mathbb{C}})$ and since $\mathfrak{m}$ is of real type, it follows that $\mathbf{a}=1$. Let us treat the decomposition of the second symmetric power. We start with the low dimensional case $p=2, q=3$, i.e. $\mathfrak{m}^{\mathbb{C}} = \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}$. A combination of (2.4.2), (2.4.3) and (2.4.4), yields

$$\begin{aligned} \operatorname{Sym}^2(\mathfrak{m}^{\mathbb{C}}) &= \Big((R(4\pi_1)\oplus 1)\hat{\otimes}\big(R(2\pi_1+2\pi_2)\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}\oplus 1\big)\Big)\oplus\Big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\big(R(3\pi_1)\oplus R(3\pi_2)\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}\big)\Big) \\ &= \big(R(4\pi_1)\hat{\otimes}R(2\pi_1+2\pi_2)\big)\oplus\big(R(4\pi_1)\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}\big)\oplus R(4\pi_1)\oplus R(2\pi_1+2\pi_2)\oplus\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}\oplus 1 \\ &\quad\oplus\big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}R(3\pi_1)\big)\oplus\big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}R(3\pi_2)\big)\oplus\big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_3}\big). \end{aligned}$$

Hence, there is a copy of $\mathfrak{m}^{\mathbb{C}}$ inside $\mathrm{Sym}^2(\mathfrak{m}^{\mathbb{C}})$ and as above we conclude that $\mathbf{s}=1$. In a similar way, for $p=2$ and $q\leq 4$, we get that

$$\begin{aligned}\mathrm{Sym}^2(\mathfrak{m}^{\mathbb{C}}) &= \Big(\big(R(4\pi_1)\oplus 1\big)\hat{\otimes}\big(R(2\pi_1+2\pi_{q-1})\oplus R(\pi_2+\pi_{q-2})\oplus \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\oplus 1\big)\\ &\quad\oplus\Big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\big(R(2\pi_1+\pi_{q-2})\oplus R(\pi_2+2\pi_{q-1})\oplus \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\Big)\\ &= \big(R(4\pi_1)\hat{\otimes}R(2\pi_1+2\pi_{q-1})\big)\oplus\big(R(4\pi_1)\hat{\otimes}R(\pi_2+\pi_{q-2})\big)\oplus\big(R(\pi_2+\pi_{q-2})\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\\ &\quad\oplus R(4\pi_1)\oplus R(2\pi_1+2\pi_{q-1})\oplus R(\pi_2+\pi_{q-2})\oplus \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\oplus 1\\ &\quad\oplus\big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}R(2\pi_1+\pi_{q-2})\big)\oplus\big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}R(\pi_2+2\pi_{q-1})\big)\oplus\big(\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_2}\hat{\otimes}\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big).\end{aligned}$$

Thus again we conclude $\mathbf{s}=1$.

Case B: $3\leq p\leq q$. In this case, a combination of (2.4.2), (2.4.3) and (2.4.4) yields the following decomposition for any $p\geq 3$ and $q\geq p$:

$$\begin{aligned}\Lambda^2(\mathfrak{m}^{\mathbb{C}}) &= \big(\Lambda^2 R(\pi_1+\pi_{p-1})\hat{\otimes}\,\mathrm{Sym}^2 R(\pi_1+\pi_{q-1})\big)\oplus\big(\mathrm{Sym}^2 R(\pi_1+\pi_{p-1})\hat{\otimes}\Lambda^2 R(\pi_1+\pi_{q-1})\big)\\ &= \big(R(2\pi_1+\pi_{p-2})\oplus R(\pi_2+2\pi_{p-1})\oplus \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\big)\hat{\otimes}\big(R(2\pi_1+2\pi_{q-1})\oplus R(\pi_2+\pi_{q-2})\oplus \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\oplus 1\big)\\ &\quad\oplus\big(R(2\pi_1+2\pi_{p-1})\oplus R(\pi_2+\pi_{p-1})\oplus \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\oplus 1\big)\hat{\otimes}\big(R(2\pi_1+\pi_{q-2})\oplus R(\pi_2+2\pi_{q-1})\oplus \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big).\end{aligned}$$

These two external tensor products each contain one copy of $\mathfrak{m}^{\mathbb{C}}$, hence we have $\mathbf{a}=2$ in this case. Passing to the second symmetric power and working in the same way we get that

$$\begin{aligned}\mathrm{Sym}^2(\mathfrak{m}^{\mathbb{C}}) &= \big(\mathrm{Sym}^2\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\hat{\otimes}\,\mathrm{Sym}^2\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\oplus\big(\Lambda^2\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\hat{\otimes}\Lambda^2\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big)\\ &= \big(R(2\pi_1+2\pi_{p-1})\oplus R(\pi_2+\pi_{p-2})\oplus \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\oplus 1\big)\hat{\otimes}\big(R(2\pi_1+2\pi_{q-1})\oplus R(\pi_2+\pi_{q-2})\oplus \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\oplus 1\big)\\ &\quad\oplus\big(R(2\pi_1+\pi_{p-2})\oplus R(\pi_2+2\pi_{p-1})\oplus \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\big)\hat{\otimes}\big(R(2\pi_1+\pi_{q-2})\oplus R(\pi_2+2\pi_{q-1})\oplus \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q}\big).\end{aligned}$$

It follows that there are two instances of $\mathfrak{m}^{\mathbb{C}}$ inside $\mathrm{Sym}^2(\mathfrak{m}^{\mathbb{C}})$, i.e. $\mathbf{s}=2$.
To compute the dimension of the space of invariant three-forms, consider the additional equivariant isomorphism

$$\Lambda^3(V\otimes W)=\big(\Lambda^3 V\otimes \mathrm{Sym}^3 W\big)\oplus\big(P_V(2,1)\otimes P_W(2,1)\big)\oplus\big(\mathrm{Sym}^3 V\otimes\Lambda^3 W\big),$$

where $P_V(2,1)$ and $P_W(2,1)$ are the (2,1)-plethysms of V and W, respectively. From this we deduce that any invariant 3-form on $V\otimes W$ can be projected to component forms induced from the following cases:

- An invariant 3-form on V and an invariant symmetric 3-tensor on W
- The product of two invariant elements of the (2,1)-plethysms of V and of W
- An invariant symmetric 3-tensor on V and an invariant 3-form on W

Let us first compute such objects for $V:=\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}=R(\pi_1+\pi_{p-1})$. We have

$$\Lambda^3 R(\pi_1+\pi_{p-1})\oplus P_{R(\pi_1+\pi_{p-1})}(2,1)=\Lambda^2 R(\pi_1+\pi_{p-1})\otimes R(\pi_1+\pi_{p-1}),$$

and by (2.4.3) we have $\Lambda^2\,\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}=\Lambda^2 R(\pi_1+\pi_{p-1})=R(\pi_1+\pi_{p-1})\oplus R(2\pi_1+\pi_{p-2})\oplus R(\pi_2+2\pi_{p-1})$. The first part of the computation is

$$\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}\otimes\mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_p}=R(\pi_1+\pi_{p-1})\otimes R(\pi_1+\pi_{p-1})=R(2\pi_1+2\pi_{p-1})\oplus R(\pi_1+\pi_{p-1})\oplus\mathbb{R},$$

where the trivial term $\mathbb{R}$ corresponds to the tensor of the Killing form of $\mathfrak{su}(p)$ (see Remark 7). This yields the first invariant 3-form on $R(\pi_1+\pi_{p-1})$. Now we are left with the task of computing unsymmetrized tensor products of irreducible SU_p-modules, and this can be done via the Littlewood-Richardson rule. The result is that the products $R(2\pi_1+\pi_{p-2})\otimes R(\pi_1+\pi_{p-1})$ and $R(\pi_2+2\pi_{p-1})\otimes R(\pi_1+\pi_{p-1})$, do not contain the trivial representation. Hence the (2,1)-plethysm of $R(\pi_1+\pi_{p-1})$ also does not admit any trivial modules. By (2.4.4) we also deduce that the multiplicity of V in $\mathrm{Sym}^2 R(\pi_1+\pi_{p-1})$ is 0 for $p=2$, or 1 for $p\geq 3$. Furthermore, we have

$$\mathrm{Sym}^3 R(\pi_1+\pi_{p-1})\oplus P_{R(\pi_1+\pi_{p-1})}(2,1)=\mathrm{Sym}^2 R(\pi_1+\pi_{p-1})\otimes R(\pi_1+\pi_{p-1})$$

and the tensor product between $R(\pi_1+\pi_{p-1}) \subset \mathrm{Sym}^2 R(\pi_1+\pi_{p-1})$ and the rightmost factor $R(\pi_1+\pi_{p-1})$ contains a trivial submodule, corresponding as before to the tensor of the Killing form. This trivial submodule is contained in $\mathrm{Sym}^3 R(\pi_1+\pi_{p-1})$, since we have already shown that $P_{R(\pi_1+\pi_{p-1})}(2,1)$ contains no such. Note that the computations for $W := \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SU}_q} = R(\pi_1+\pi_{q-1})$ are identical (except for switching the label p to q). Therefore, we finally get a one dimensional trivial submodule in $\Lambda^3(V \otimes W)$ for the case $p = 2$ (or q=2), and when $p, q \geq 3$ we get a two dimensional trivial submodule in $\Lambda^3(V \otimes W)$. In our notation, this means $\ell = 1$ for $p = 2, q \geq 3$ and $\ell = 2$ for $p, q \geq 3$. This completes the proof. $\square$

Lemma 14. *Consider the homogeneous space $M = G/K = \mathrm{Sp}_n/(\mathrm{SO}_n \times \mathrm{Sp}_1)$ with $n \geq 3$. Then, the isotropy representation $\mathfrak{m} = R(2\pi_1)\hat{\otimes}\mathfrak{sp}(1)$ has multiplicity $\mathbf{a} = 1$ inside $\Lambda^2\mathfrak{m}$ and multiplicity $\mathbf{s} = 0$ inside $\mathrm{Sym}^2\mathfrak{m}$. Moreover, the dimension of trivial submodule $(\Lambda^3\mathfrak{m})^K$ is $\ell = 1$.*

Proof. An embedding of a compact Lie group K into Sp_n is equivalent to a (faithful) representation $\phi : K \to \mathrm{GL}(\mathbb{H}^n)$. This is a representation of real dimension $4n$ with an invariant quaternionic structure. Since $K = \mathrm{SO}_n \times \mathrm{Sp}_1$ is compact, the image of ϕ will be inside some conjugacy class of Sp_n. We are looking for the unique isotropy irreducible embedding, which means that ϕ should be an irreducible representation. Let $R(\omega_1, \omega_2) = R(\omega_1)_{\mathrm{SO}_n}\hat{\otimes}_{\mathbb{C}}R(\omega_2)_{\mathrm{Sp}_1}$ denotes the associated real irreducible representation. The obvious candidate is $\phi = R(\pi_1, \pi_1) = \mathbb{R}^n\hat{\otimes}_{\mathbb{R}}\mathbb{H} = \mathbb{R}^n\hat{\otimes}_{\mathbb{R}}\mathbb{C}^2$. This irreducible representation is obviously of quaternionic type. Recall now the adjoint representation of Sp_n is the real submodule inside $\mathrm{Ad}^{\mathbb{C}}_{\mathrm{Sp}_n} = \mathrm{Sym}^2 \nu_n = \mathrm{Sym}^2_{\mathbb{C}} \mathbb{H}^n$. Thus we must take into account the complex structure on ϕ, which is defined by its action on the right tensor factor $\mathbb{H} \simeq \mathbb{C}^2$. By applying (2.4.2), we compute

$$\begin{aligned}
\mathrm{Sym}^2_{\mathbb{C}}\phi &= \mathrm{Sym}^2_{\mathbb{C}}(\mathbb{R}^n\hat{\otimes}_{\mathbb{R}}\mathbb{C}^2) = (\mathrm{Sym}^2_{\mathbb{R}}\mathbb{R}^n\hat{\otimes}_{\mathbb{R}}\mathrm{Sym}^2_{\mathbb{C}}\mathbb{C}^2) \oplus (\Lambda^2_{\mathbb{R}}\mathbb{R}^n\hat{\otimes}\Lambda^2_{\mathbb{C}}\mathbb{C}^2) \\
&= (R(2\pi_1) \oplus R(0)) \otimes \mathrm{Ad}^{\mathbb{C}}_{\mathrm{Sp}_1} \oplus \mathrm{Ad}^{\mathbb{C}}_{\mathrm{SO}_n}.
\end{aligned}$$

This immediately yields the isotropy representation

$$\mathfrak{m} = \mathfrak{sp}(n)/(\mathfrak{so}(n) \oplus \mathfrak{sp}(1)) = R(2\pi_1)\hat{\otimes}\mathfrak{sp}(1) = R(2\pi_1, 2\pi_1),$$

which is irreducible. Since $\mathfrak{m}$ has real type and its tensor factors also have real type, we can apply complex representation theory without performing any extra complexifications. We proceed with the decomposition of $\Lambda^2\mathfrak{m}$ and $\mathrm{Sym}^2\mathfrak{m}$. For any $n > 4$ note the following decompositions of SO_n-modules: $\Lambda^2 R(2\pi_1) = R(2\pi_1+\pi_2) \oplus R(\pi_2)$ and $\mathrm{Sym}^2 R(2\pi_1) = R(4\pi_1) \oplus R(2\pi_1) \oplus R(2\pi_2) \oplus R(0)$. For Sp_1 we have that $\Lambda^2\mathfrak{sp}(1) = \mathfrak{sp}(1)$ and $\mathrm{Sym}^2\mathfrak{sp}(1) = R(4\pi_1) \oplus R(0) = R(4\pi_1) \oplus 1$. Hence we conclude that only those terms in the tensor square that contain a factor of $\mathrm{Sym}^2 R(2\pi_1)$ and $\Lambda^2\mathfrak{sp}(1)$ will yield copies of $\mathfrak{m}$. In particular, the decomposition

$$\Lambda^2\mathfrak{m} = \Lambda^2\big(R(2\pi_1)\hat{\otimes}\mathfrak{sp}(1)\big) = \big(\Lambda^2 R(2\pi_1)\hat{\otimes}\,\mathrm{Sym}^2\mathfrak{sp}(1)\big) \oplus \big(\mathrm{Sym}^2 R(2\pi_1)\hat{\otimes}\Lambda^2\mathfrak{sp}(1)\big)$$

contains precisely one instance of $\mathfrak{m}$, i.e. $\mathbf{a} = 1$. One the other hand,

$$\mathrm{Sym}^2\mathfrak{m} = \mathrm{Sym}^2\big(R(2\pi_1)\hat{\otimes}\mathfrak{sp}(1)\big) = \big(\Lambda^2 R(2\pi_1)\hat{\otimes}\Lambda^2\mathfrak{sp}(1)\big) \oplus \big(\mathrm{Sym}^2 R(2\pi_1)\hat{\otimes}\,\mathrm{Sym}^2\mathfrak{sp}(1)\big),$$

hence $\mathbf{s} = 0$. This proves the claim for $n > 4$. For completeness we examine the low-dimensional cases. Let first $n = 3$. The defining representation ϕ of $K = \mathrm{SO}_3 \times \mathrm{Sp}_1$ must be of real dimension $12 = 3 \times 4$ and hence the only irreducible possibility is $\phi = \mathbb{R}^3\hat{\otimes}\mathbb{H} = R(2\pi_1)\hat{\otimes}R(\pi_1)$. Thus we get

$$\mathrm{Sym}^2_{\mathbb{C}}\phi = \mathrm{Sym}^2_{\mathbb{C}}(\mathbb{R}^3\hat{\otimes}_{\mathbb{R}}\mathbb{C}^2) = \big(R(4\pi_1)\hat{\otimes}\mathfrak{sp}(1)\big)^{\mathbb{C}} \oplus \big(\mathfrak{sp}(1) \oplus \mathfrak{so}(3)\big)^{\mathbb{C}}.$$

Hence in this case $\mathfrak{m} = R(4\pi_1)\hat{\otimes}\mathfrak{sp}(1)$. As $\mathfrak{so}(3)$-modules, we have that

$$\Lambda^2 R(4\pi_1) = R(6\pi_1) \oplus \mathfrak{so}(3), \quad \mathrm{Sym}^2 R(4\pi_1) = R(8\pi_1) \oplus R(4\pi_1) \oplus R(0).$$

Therefore, only products of $\mathrm{Sym}^2 R(4\pi_1)$ and $\Lambda^2\mathfrak{sp}(1)$ yield copies of $\mathfrak{m}$. Consequently, the result is the same as above, the multiplicity of $\mathfrak{m}$ is one in $\Lambda^2\mathfrak{m}$ and zero in $\mathrm{Sym}^2\mathfrak{m}$.
Assume now that $n = 4$. The defining representation of $K = \mathrm{SO}_4 \times \mathrm{Sp}_1$ is $\phi = \mathbb{R}^4\hat{\otimes}\mathbb{H}$, but $\mathbb{R}^4 = R(\pi_1+\pi_2)$ in terms of highest weights, instead of being $R(\pi_1)$ as before, because SO_4 is non-simple. We get

$$\mathrm{Sym}^2_{\mathbb{C}}\phi = \mathrm{Sym}^2_{\mathbb{C}}(\mathbb{R}^4\hat{\otimes}_{\mathbb{R}}\mathbb{C}^2) = \big(R(2\pi_1+2\pi_2)\hat{\otimes}\mathfrak{sp}(1)\big)^{\mathbb{C}} \oplus \big(\mathfrak{sp}(1) \oplus \mathfrak{so}(4)\big)^{\mathbb{C}}$$

and thus $\mathfrak{m} = R(2\pi_1 + 2\pi_2)\hat{\otimes}\mathfrak{sp}(1)$ in this case. As $\mathfrak{so}(4)$-modules, we see that

$$\begin{aligned}\Lambda^2 R(2\pi_1+2\pi_2) &= R(2\pi_1+4\pi_2)\oplus R(4\pi_1+2\pi_2)\oplus\mathfrak{so}(3),\\ \mathrm{Sym}^2 R(2\pi_1+2\pi_2) &= R(4\pi_1+4\pi_2)\oplus R(4\pi_1)\oplus R(4\pi_2)\oplus R(2\pi_1+2\pi_2)\oplus R(0),\end{aligned}$$

and the same argument as previously yields that $\mathbf{a} = 1$ and $\mathbf{s} = 0$.
Now, our assertion for $(\Lambda^3\mathfrak{m})^K$ can be deduced very easily as follows: Any invariant element of $\Lambda^3\mathfrak{m}$ induces an equivariant map in $\mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m})$. For any $n \geq 3$ we have shown that $\mathbf{a} = \dim_{\mathbb{R}} \mathrm{Hom}_K(\mathfrak{m}, \Lambda^2\mathfrak{m}) = 1$. Thus, $\dim_{\mathbb{R}}(\Lambda^3\mathfrak{m})^K \leq 1$, but by Lemma 12 we also get $\dim_{\mathbb{R}}(\Lambda^3\mathfrak{m})^K \geq 1$ and the result follows. Note that this method for the computation of the multiplicity ℓ, applies on any non-symmetric SII space $M = G/K$ whose isotropy representation is of real type and has $\mathbf{a} = 1$ (or whose isotropy representation is of complex type and has $\mathbf{a} = 2$, e.g. $\mathrm{S}^6 \cong \mathrm{G}_2 / \mathrm{SU}_3$). $\square$

2.5 Classification of Homogeneous ∇-Einstein structures on SII spaces

2.5.1 Homogeneous ∇-Einstein structures

Similarly with invariant Einstein metrics on homogeneous Riemannian manifolds, on triples (M^n, g, T) consisting of a homogeneous Riemannian manifold $(M^n = G/K, g)$ endowed with a (non-trivial) invariant 3-form T, on may speak of *homogeneous ∇-Einstein structures.* In particular,

Definition 5. *A triple (M^n, g, T) of a connected Riemannian manifold (M, g) carrying a (non-trivial) 3-form $T \in \Lambda^3 T^*M$, is called a G-homogeneous ∇-Einstein manifold (with skew-torsion) if there is a closed subgroup $G \subseteq \mathrm{Iso}(M, g)$ of the isometry group of (M, g), which acts transitively on M and a G-invariant connection ∇ compatible with g and with skew-torsion T, whose Ricci tensor satisfies the condition (2.1.2).*

In this case, g is a G-invariant metric, the Levi-Civita connection ∇^g is a G-invariant metric connection and since $2(\nabla - \nabla^g) = T$, the torsion T of ∇ is given necessarily by a G-invariant 3-form $0 \neq T \in \Lambda^3(\mathfrak{m})^K$, where $\mathfrak{m} \cong T_oM$ is a reductive complement of $M = G/K$ with $K \subset G$ being the (closed) isotropy group. In particular, the ∇-Einstein condition (2.1.2) is $\mathrm{Ad}(K)$-invariant, in the sense that the Ricci tensor Ric^{∇} is a G-invariant covariant 2-tensor which is described by an $\mathrm{Ad}(K)$-invariant bilinear form on $\mathfrak{m}$, and the same for its symmetric part. Moreover,

Proposition 14. *On a homogeneous Riemannian manifold $(M = G/K, g)$ carrying a G-invariant (non-trivial) 3-form $T \in \Lambda^3(\mathfrak{m})^K$, the scalar curvature $\mathrm{Scal} = \mathrm{Scal}^{\nabla}$ associated to the G-invariant metric connection $\nabla := \nabla^g + \frac{1}{2}T$ is a constant function on M.*

Proof. It is well-known that on a reductive homogeneous space, the scalar curvature Scal^g of the Levi-Civita connection (related to a G-invariant Riemannian metric g, or the corresponding $\mathrm{Ad}(K)$-invariant inner product $\langle\ ,\ \rangle$ on the reductive complement $\mathfrak{m}$) is independent of the point, i.e. it is a constant function on M [B, NRS]. Let ∇ be a G-invariant metric connection on $(M = G/K, g)$ whose skew-torsion coincides with the invariant 3-form $0 \neq T \in \Lambda^3(\mathfrak{m})^K$. Due to the identity $\mathrm{Scal} = \mathrm{Scal}^g - \frac{3}{2}\|T\|^2$ it is sufficient to prove that $\|T\|^2$ is constant, which is obvious since T corresponds to a G-invariant tensor field. Consequently, $\mathrm{Scal}^{\nabla} : G/K \to \mathbb{R}$ is constant. $\square$

2.5.2 On the proofs of Theorems C, D and E

Let us focus now on an effective, non-symmetric (compact) strongly isotropy irreducible homogeneous Riemannian manifold $(M = G/K, g = -B|_{\mathfrak{m}})$, where $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$ is the associated B-orthogonal decomposition. We denote by $\mathcal{M}^{sk}_G(\mathrm{SO}(G/K, g)) \subseteq \mathcal{M}_G(\mathrm{SO}(G/K, g)) \subseteq \mathcal{A}ff_G(F(G/K))$ the space of G-invariant affine connections on $M = G/K$ which are compatible with the Killing metric $g = -B|_{\mathfrak{m}}$ and have invariant 3-forms $0 \neq T \in (\Lambda^3\mathfrak{m})^K$ as their torsion tensors. For the corresponding set of homogeneous ∇-Einstein structures, we will write $\mathcal{E}^{sk}_G(\mathrm{SO}(G/K, g))$. As stated in the introduction, Lemma 12 and Schur's lemma allow us to parametrize $\mathcal{E}^{sk}_G(\mathrm{SO}(G/K, g))$ by the space of global G-invariant 3-forms. Hence, this finally yields the identification $\mathcal{E}^{sk}_G(\mathrm{SO}(G/K, g)) = \mathcal{M}^{sk}_G(\mathrm{SO}(G/K, g))$. With the aim to clarify this identification and give explicit proofs of Theorems C, D and E in introduction, let us recall first the following important result of [C1].

Theorem 25. ([C1, Thm. 4.7]) *Let $(M^n = G/K, g)$ be an effective, compact and simply-connected, isotropy irreducible standard homogeneous Riemannian manifold $(M^n = G/K, g)$ of a compact connected simple Lie group G, which is not a symmetric space of Type I. Then, $(M^n = G/K, g)$ is a ∇^α-Einstein manifold for any parameter $\alpha \neq 0$, where $\nabla^\alpha = \nabla^g + \frac{1}{2}T^\alpha = \nabla^c + \Lambda^\alpha$ is the 1-parameter family of G-invariant metric connections on M, with skew-torsion $0 \neq T^\alpha = \alpha \cdot T^c$ (see Lemma 8).*

Note that for a symmetric space $M = G/K$ of Type I, the associated space of G-invariant affine *metric* connections is always a point, i.e. $\nabla^\alpha \equiv \nabla^c \equiv \nabla^g$ and no torsion appears. On the other hand, Theorem 25 generalises the well-known fact that a compact simple Lie group G is ∇^α-Einstein with (non-trivial) *parallel torsion* for any $0 \neq \alpha \in \mathbb{R}$, with the flat ± 1-connections of Cartan-Schouten being the trivial members (see for example [AF, Lemma 1.8] or [C1, Thm. 1.1]). Notice however, that if $M = G/K$ is not isometric to a compact simple Lie group, then the ∇^α-Einstein structures described in Theorem 25 have parallel torsion only for $\alpha = 1$. We finally remark that both $\mathrm{S}^6 = \mathrm{G}_2 / \mathrm{SU}_3, \mathrm{S}^7 = \mathrm{G}_2 / \mathrm{Spin}_7$ are (strongly) isotropy irreducible and non-symmetric, hence they are ∇^α-Einstein manifolds with skew-torsion, for any $0 \neq \alpha \in \mathbb{C}$, $0 \neq \alpha \in \mathbb{R}$, respectively (due to the type of their isotropy representation). The same applies for any compact, non-symmetric, effective SII homogeneous Riemannian manifold and this gives rise to Theorem C which is an immediate consequence of Theorem A.1 and Theorem 25.
Let us proceed now with a proof of Theorems D and E.

Proof. **(Proof of Theorem D)** If $M = G/K$ is a manifold in Table 2.2 whose isotropy representation is of real type, different than $\mathrm{SO}_{10} / \mathrm{Sp}_2$, then Theorem A.2 (i) ensures the existence of a second (real) 1-parameter family of G-invariant connections $\nabla^t \neq \nabla^a$, compatible with the Killing metric and with skew-torsion T^t such that $T^t \neq T^c \sim T^\alpha$ (for any $t, \alpha \in \mathbb{R}$), where T^c is the torsion of the (unique) canonical connection corresponding to $\mathfrak{m}$. Thus, we can write $\nabla^t = \nabla^g + \frac{1}{2}T^t$ with $\nabla^t \neq \nabla^\alpha$. Since $\mathrm{Ric}^{\nabla^t} \equiv \mathrm{Ric}^t$ is G-invariant, the same is true for $\delta^g T^t = \delta^{\nabla^t} T^t$, in particular we can view the codifferential of $T^t \in (\Lambda^3\mathfrak{m})^K$ as a G-invariant 2-form. However, χ is of real type, hence the trivial representation $\mathbb{R}$ does not appear in $\Lambda^2\mathfrak{m}$, i.e. $(\Lambda^2\mathfrak{m})^K = 0$. Hence, we deduce that $\delta^g T^t = 0 = \delta^t T^t$ and since $\mathfrak{m}$ is (strongly) isotropy irreducible over $\mathbb{R}$ and the Ricci tensor Ric^t is symmetric, by Schur's lemma it must be a multiple of the Killing metric, i.e. $(M = G/K, -B|_{\mathfrak{m}}, \nabla^t)$ is ∇^t-Einstein with skew-torsion. Our final claim follows now in combination with Theorem 25. $\square$

It is well-known that an effective SII homogeneous space $M = G/K$ admits an (integrable) G-invariant complex structure if and only if is a Hermitian symmetric space [Wo1]. Moreover, the existence of an invariant *almost* complex structure $J \in \mathrm{End}(\mathfrak{m})$ on a strongly isotropy irreducible space implies that the isotropy representation is not of real type, hence $\chi = \phi \oplus \overline{\phi}$ for some irreducible complex representation with $\phi \ncong \overline{\phi}$. Consequently, any manifold which appears in Tables 2.4, 2.5 and whose isotropy representation is of complex type, is a G-homogeneous almost complex manifold (see also [Wo1, Cor. 13.2]). Notice also

Lemma 15. *Let $\mathfrak{k}$ be a compact Lie algebra and let $\rho : \mathfrak{k} \to \mathrm{End}(\mathfrak{m})$ be a faithful (irreducible) representation of $\mathfrak{k}$ over $\mathbb{R}$, endowed with an invariant inner product $B_{\mathfrak{m}}$. Assume that $\dim \mathfrak{m} \geq 2$. If $\mathfrak{m}$ admits an $\mathrm{ad}(\mathfrak{k})$-invariant complex structure J (as a vector space), then $\Lambda^2\mathfrak{m}$ contains the trivial representation $\mathbb{R}$.*

Proof. We only mention that since $\mathfrak{m}$ is an irreducible complex type representation of a compact Lie algebra $\mathfrak{k}$, it is unitary, therefore the $\mathrm{ad}(\mathfrak{k})$-invariant Kähler form $\omega(X, Y) = B_{\mathfrak{m}}(JX, Y)$ gives rise to an $\mathrm{ad}(\mathfrak{k})$-invariant element inside $\Lambda^2\mathfrak{m}$. $\square$

Consider now the spaces $\mathrm{SO}_{n^2-1} / \mathrm{SU}_n$ $(n \geq 4)$ and $\mathrm{E}_6 / \mathrm{SU}_3$. Since their isotropy representation is of complex type, Lemma 15 certifies the existence of G-invariant 2-forms. Thus, in contrast to Theorem D, we cannot deduce that the Ricci tensor of all predicted G-invariant metric connections with skew-torsion must be necessarily symmetric (although this is the case always for Ric^α). However, since we are considering the isotropy irreducible case, we obtain Theorem E as follows:

Proof. **(Proof of Theorem E)** Assume that $(M = G/K, g = -B|_{\mathfrak{m}})$ is one of the manifolds $\mathrm{SO}_{n^2-1} / \mathrm{SU}_n$ $(n \geq 4)$ or $\mathrm{E}_6 / \mathrm{SU}_3$. By Theorem A.2 (ii) (see also Table 2.2) we know that $M = G/K$ admits a 4-dimensional space of G-invariant metric connections with skew-torsion. Now, the ∇-Einstein condition is related only with the symmetric part of the Ricci tensor associated to any such connection. Since this tensor is G-invariant, Schur's lemma ensures that the ∇-Einstein equation is satisfied for any available G-invariant metric connection ∇ with skew-torsion. Therefore, the space of G-invariant ∇-Einstein structures has the same dimension with the space of G-invariant metric connections with skew-torsion. This proves Theorem E. $\square$

Classification of regular contact actions of 2D Lie algebras

Christian O'Cadiz Gustad

3.1 Introduction

In this paper we investigate contact equivalence classes of actions of 2 dimensional Lie Algebras on a manifold M equipped with contact structure ξ.

Let $\mathfrak{g}$ be a solvable Lie algebra over $\mathbb{R}$, $\dim \mathfrak{g} = 2$ and $\rho : \mathfrak{g} \to \mathcal{D}(M)$, be an effective representation of this algebra into the Lie algebra of contact vector fields.. We say that two representations ρ_1, ρ_2 are *locally equivalent* at a point $a \in M$, if there exist a local contact diffeomorphism $\phi : M \to M$ where $\phi(a) = a$, such that $\rho_2 = \phi_* \circ \rho_1$.

Moreover we require that the $\mathfrak{g}$-action has a two-dimensional orbit at the point under consideration. The first formulation and treatment of this equivalence problem of Lie algebra actions on manifolds was done by Sophus Lie in [Li76]. Further development has focused on the classification around a fixed point of the group action, for instance in the work of Cartan [Car]. For the case of a one-dimensional action with fixed point, also the work of Poincare [Arn] Sternberg [SSlh]. However the contact case is largely unknown. In [LycS] the normalization of a contact vector field around a elliptic singularities was treated, moreover in [LycSS] normal forms of a Lie algebra actions on general, contact and symplectic manifolds were related to spectral sequences. Thus we initiate the generalization of these results by treating the simplest non-trivial example. However the classification given here will be of particular interest, as it has applications to geometric theory of second order differential equations. Nonetheless the scheme outlined here generalizes naturally to any given Lie algebra structure.
We begin by recalling some fundamental notions in contact geometry.

3.1.1 Contact geometry

Consider a $2n+1-$dimensional manifold M, a contact structure ξ is a $2n$-dimensional distribution such that the curvature form Ω_ξ is non-degenerate. Locally we may express any co-dimension 1 distribution as the kernel of a differential 1-from α called the contact form

$$\alpha = \ker \xi.$$

We define a strict contact manifold as the pair (M, α) when $\xi = \ker \alpha$ is contact.

The non-integrability of the distribution can be expressed through the contact form in by the Forbenious condition

$$\alpha \wedge (d\alpha)^n \neq 0.$$

Let M be a strict contact manifold, then for every $a \in M$ we have the following decomposition:

$$T_a M = \ker \alpha_a \oplus \ker d_a \alpha,$$

where $d_a \alpha$ defines a symplectic structure on ξ_a.
A smooth map ϕ between contact manifolds (M_1, ξ_1) and (M_2, ξ_2) is said to be contact if

$$\phi_*(\xi_1) = \xi_2.$$

Then if ϕ is a diffeomorphism which is also contact, we say that ϕ is a contact diffeomorphism, similarly one defines a contact transformation when $M_1 = M_2$. This condition can equivalently defined for the contact form:

$$\phi^*(\alpha_2) = \lambda\alpha_1,$$

where α_1, α_2 correspond to ξ_1, ξ_2 respectively, and $\lambda \in C^\infty(M_1)$ which we call the conformal factor. A vector field $X \in \Gamma(TM)$ is called a contact vector field, when the flow of X consists of one-parameter groups of contact transformations. This condition is expressed through the Lie derivative:

$$L_X(\alpha) = \lambda_X \alpha, \tag{3.1.1}$$

where L_X denotes the Lie derivative along X, and λ_X is some smooth function.
Equivalently

$$L_X(\alpha) \wedge \alpha = 0.$$

Moreover the pairing $\alpha(X_f) = f$ defines the generating function of a contact vector field.
This definition is justified by the following proposition:

Proposition 15. *Let (M, α) be a strict contact manifold. Then for any function $f \in C^\infty(M)$ there exist a unique contact vector field X_f such that:*

$$\begin{aligned} X_f &= f, \\ i_{X_f} d\alpha &= X_1\alpha - df, \\ L_{X_f} &= X_1(f)\alpha. \end{aligned}$$

where X_1 the contact vector field with generating function 1, also called the Reeb vector field. These properties define the bijection

$$\begin{aligned} \Delta : C^\infty(M) &\to \mathfrak{ct}(M), \\ f &\to X_f, \end{aligned}$$

where Δ is a linear differential operator and $\mathfrak{ct}(M)$ is the Lie algebra of contact vector fields.

Proof. See for reference [CtB] □

This bijection Δ endows $C^\infty(M)$ with a Lie algebra structure by

$$\Delta([f, g]) = [\Delta(f), \Delta(g)].$$

This bracket is called the Lagrange bracket and denote it by $\{f, g\}$.
Moreover from the proposition above, we have that

$$\{f, g\} = d\alpha(X_g, X_f) = X_f(g) - X_1(f)g. \tag{3.1.2}$$

The Darboux theorem states that, two contact manifolds of the same dimension are locally contact equivalent.

This means that locally every contact structure looks like the standard contact structure on $\mathbb{R}^{2n+1}$ with coordinates (q_i, u, p_i) such that the contact form is given by:

$$\alpha = du - \sum_i p_i dq_i.$$

In these (canonical) coordinates we have

$$X_f = \sum_i -f_{p_i} \frac{\partial}{\partial q_i} + \left(\sum_i f - p_i f_{p_i} \right) \frac{\partial}{\partial u} + \sum_i (f_{q_i} + p_i f_u) \frac{\partial}{\partial p_i}. \tag{3.1.3}$$

The Reeb vector field in this case has the form

$$X_1 = \frac{\partial}{\partial u}.$$

It is easy to see that $X_f(a) \neq 0$ if and only if

$$f(a) \neq 0 \text{ or } f(a) = 0, \text{ but } df|_{\xi_a} \neq 0.$$

Let X_f be a vector field such that $X_f(a) \neq 0$, then we have two possibilities,

$$X_f(a) \in \xi(a),$$

or

$$X_f(a) \notin \xi(a).$$

Contact diffeomorphisms necessarily preserve this property since they must preserve ξ. Thus defining an invariant property of a non-zero contact vector field.
Since the values $f(a)$ determines the direction of $X_f(a)$ transversal to ξ_a, we observe that if X_f is a non-zero contact vector field. Then $X_f(a) \notin \xi(a)$ if and only if $f(a) \neq 0$.

Let $\mathfrak{g}$ be a 2-dimensional Lie algebra over $\mathbb{R}$ and let X_f, X_g be a basis in $\mathfrak{g}$ and ρ a representation of this algebra, into the Lie algebra of contact vector fields.
For simplicity we will denote the images $\rho(X_f)$ and $\rho(X_g)$ by X_f, X_g respectively.
It is well known that there are two 2-dimensional Lie algebras up to isomorphism, namely the commutative algebra

$$[X_f, X_g] = 0, \tag{3.1.4}$$

and the non-commutative algebra

$$[X_f, X_g] = X_f. \tag{3.1.5}$$

We will devote our attention to the non-commutative case, as any result in this case will give us an analogous procedure for the commutative algebra.

In the non-commutative case we need to split our investigation into two cases

1. The derived sub-algebra generated by X_f is transversal to the contact distribution at $a \in M$
2. The derived sub-algebra generated by X_f lies in the contact distribution at $a \in M$

Throughout this article we use coordiate charts centered at $a \in M$.

3.1.2 Vector fields transversal to the contact distribution

We start by investigating equivalence classes of a contact vector field X_f using the path-lifting method.
To prove existence of a contact diffeomorphism ϕ such that

$$\phi_*(X_f) = X_h,$$

construct a path X_{f_t} in the space of contact vector fields connecting X_f and X_h, such that $X_{f_0} = X_f$ and $X_{f_1} = X_h$, or equivalently a path in the space $C^\infty(M)$ connecting their respective generating functions.

We seek a one-parameter family of contact diffeomorphisms ϕ_t, which fix the point a for all t and is the identity for $t = 0$, such that

$$\phi_{t,*}(X_{f_t}) = X_{f_0}. \tag{3.1.6}$$

For the value $t = 1$ we get our desired diffeomorphism.

Let X_{λ_t} to be the one-parameter family of contact vector fields corresponding to ϕ_t in the way that

$$\phi^*_{t+\Delta t} = Id + \Delta t X_{\lambda_t} + o(\Delta t).$$

Differentiating (3.1.6) we get

$$\phi_{t,*}\left(L_{\lambda_t}(X_{f_t}) + X_{\dot{f}_t}\right) = 0,$$

where $\dot{f} = \frac{d}{dt} f_t$.
Since ϕ is a diffeomorphism then

$$L_{\lambda_t}(X_{f_t}) + X_{\dot{f}} = [X_{\lambda_t}, X_{f_t}] + X_{\dot{f}} = 0. \tag{3.1.7}$$

The last condition can be expressed through the Lagrange bracket of the generating functions

$$\{\lambda_t, f_t\} + \dot{f} = 0.$$

Using (3.1.2), we arrive at the following PDE for λ_t

$$X_{f_t}(\lambda_t) - X_1(f_t)\lambda_t - \dot{f}_t = 0. \tag{3.1.8}$$

Moreover, since the diffeomorphisms ϕ_t fixes the point we have that $X_{\lambda_t}(a) = 0$, or

$$\lambda_t(a) = 0, \qquad d_a\lambda_t|_{\xi_a} = 0. \tag{3.1.9}$$

It is well known that such a system has unique solution in a neighborhood of a if the following conditions are satisfied

1. The initial data are given on a surface Γ_t which is transversal to X_{f_t} at a.
2. $X_{f_t}(a) \neq 0$.

Since $f_t \neq 0$ the surface Γ_t with $T_a\Gamma_t = \xi_a$ will always be transversal to $X_{f_t}(a)$, and the initial condition (3.1.9) will be satisfied if

$$\lambda_t|_{\Gamma_t} = \lambda_t^0,$$

where $\lambda_t^0 \in \mu_a^2(\Gamma)$.
Therefore each such λ_t^0 produces a unique solution of (3.1.8), and by constructing a path X_{f_t} connecting X_f with $\pm X_1$ imply solubility.

Hence we see that every contact vector field X_f is locally equivalent to $\pm X_1$, however it is easy to see that the vector fields $X_1, -X_1$ are locally equivalent by a contact transformation mapping u coordinate to $-u$, summarizing we have the following

Theorem 2. *Let X be a contact vector field on M and $a \in M$ such that $X_a \notin \xi_a$. Then there are local coordinates $(q_1, \ldots, q_n, u, p_1, \ldots, p_n)$ in a neighborhood of a such that*

$$X = \frac{\partial}{\partial u}$$

Consider now the vector field X_g see (3.1.5) , in the coordinates we have picked above.
The commutator relation

$$[X_1, X_g] = X_1,$$

gives us that

$$g = u + g_0(q_1, \ldots, q_n, p_1, \ldots, p_n).$$

In order for X_1, X_g to define a two-dimensional orbit at a we get in addition the following restriction

$$dg|_{\xi_a} \neq 0.$$

To classify such functions g, we apply the path-lifting method once again.
We seek a contact diffeomorphism ϕ such that

$$\phi_*(X_g) = X_h,$$

and ϕ preserves X_f, that is

$$\phi_*(X_1) = X_1.$$

Construct a path X_{g_t} in the space of contact vector fields connecting X_g and X_h, such that $X_{g_0} = X_g$ and $X_{g_1} = X_h$ and a one-parameter family of contact diffeomorphisms ϕ_t, which fixes the point a for all t ,is the identity for $t = 0$ and its differential fixes X_{g_1},

$$\phi_{t,*}(X_t) = X_{g_1}. \tag{3.1.10}$$

We choose now X_{λ_t} to be the one-parameter family of contact vector fields corresponding to ϕ_t in the sense that

$$\phi^*_{t+\Delta t} = Id + \Delta t X_{\lambda_t} + o(\Delta t).$$

For ϕ_t to preserve X_1 we require additionally that

$$L_{X_{\lambda_t}}(X_1) = [X_{\lambda_t}, X_1] = 0.$$

From the expression of the Lagrange bracket, we see that this means that λ_t is independent of u. As above we arrive at the following PDE for λ_t:

$$X_{g_t}(\lambda_t) - \lambda_t - \dot{g}_t = 0, \tag{3.1.11}$$
$$\lambda_{t,u} = 0. \tag{3.1.12}$$

We require that the diffeomorphism ϕ_t fix the point, therefore the vector field X_{λ_t} should satisfy $X_{\lambda_t}(0) = 0$.

This condition can be expressed through the generating function λ_t as

$$\lambda_t(a) = d\lambda_t|_{\xi_a} = 0. \tag{3.1.13}$$

Since λ_t is independent of u, and we have that $d_0\lambda_t = 0$. From this and (3.1.11) we see that $\dot{g}_t = 0$. Consider the projection

$$\pi : M \to \overline{M},$$

where $\overline{M}$ is the quotient of M by the action of X_1.

(3.1.11) takes the form

$$\overline{X}_{g_t}(\lambda_t) = \lambda_t + \dot{g}_t,$$

where

$$\overline{X}_{g_t} = \pi_*(X_{g_t}).$$

Let Γ_t be any $2n-1$-dimensional hyper-surface in $\overline{M}$ passing though a, such that Γ_t is transversal to $\overline{X}_{g_t}$ at a. Then the initial condition

$$\lambda_t|_\Gamma = \lambda_t^0$$

Thus we have coordinates $(q_1, \ldots, q_n, u, p_1, \ldots, p_n)$ such that

$$X_g = cX_1 + X_{p_1+u},$$

where $c \in \mathbb{R}$. Summarizing we have shown the following

Theorem 3. *Let ρ_1, ρ_2 be contact representations of the non-abelian Lie algebra of the dimension 2 with two-dimensional orbit, such that the representations of their derived sub-algebras do not belong to the contact distribution at $a \in M$. Then ρ_1, ρ_2 are locally contact equivalent.*

The following corollary follows immediately

Corollary 2. *Let ρ be a contact representation of the non-commutative Lie algebra satisfying the conditions above, then there exists a basis X_f, X_g of $\mathfrak{g}$ and a local canonical coordinates $(q_1 \ldots, q_n, u, p_1, \ldots, p_n)$ in a neighborhood of a such that*

$$X_f = \frac{\partial}{\partial u},$$
$$X_g = -\frac{\partial}{\partial q_1} + u\frac{\partial}{\partial u} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i}.$$

The commutative case will be completely analogous, and we arrive at the following

Theorem 4. *Let ρ_1, ρ_2 be contact representations of the commutative Lie algebra of dimension 2 with two-dimensional orbit, such that the image of ρ_1, ρ_2 do not belong to the contact distribution at a. Then ρ_1, ρ_2 are contact equivalent*

Corollary 3. *Let ρ be a contact representation of the commutative Lie algebra satisfying the conditions above, then there exists a basis X_f, X_g of $\mathfrak{g}$ and local canonical coordinates $(q_1, \ldots, q_n, u, p_1, \ldots, p_n)$ of a such that*

$$X_f = \frac{\partial}{\partial u},$$
$$X_g = -\frac{\partial}{\partial q_1}.$$

3.2 Derived sub-algebra tangent to the contact distribution

We consider now contact equivalence classes of contact vector fields X_f at a point a, such that $X_f(a) \in \xi_a$.
Applying the path-lifting method we arrive at the equation

$$X_{f_t}(\lambda_t) - X_1(f_t)\lambda_t - \dot{f}_t = 0, \tag{3.2.1}$$

subject to the initial conditions

$$\lambda_t(a) = 0, \quad d\lambda_t|_{\xi_a} = 0, \tag{3.2.2}$$

where f_t is some path connecting our initial generating function and a desired form.
Since $X_f \in \xi_a$, we see from (3.2.1) that $\dot{f}_t(a) = 0$.

In this case, we cannot let Γ_t coincide with ξ_a since $X_{f_t}(a) \in \xi_a$, thus let Γ_t be an arbitrary surface transversal to ξ_a at a.
The condition that $\dot{f}(a) = 0$ we get if we take the initial data

$$\lambda_t|_\Gamma = \lambda_t^0$$

to be $\lambda_t^0 \in \mu_a^2(\Gamma)$.
Take a path f_t connecting X_f with X_{p_1}, thus we get the following.

Theorem 5. *Let X_f be a contact vector field on M and $a \in M$ such that $f(a) = 0$ and $X_f(a) \neq 0$. Then there are local coordinates $(q_1, \ldots, q_n, u, p_1, \ldots, p_n)$ in a neighborhood of a such that*

$$X_f = X_{p_1} = -\frac{\partial}{\partial q_1}$$

Consider now the vector field X_g in the coordinates constructed above.
Then the commutator relation

$$[X_{p_1}, X_g] = X_{p_1}$$

gives us that

$$g = -q_1 p_1 + g_0(q_2, \ldots, q_n, u, p_1, \ldots, p_n).$$

In order for X_{q_1}, X_g to define a two-dimensional orbit at a, we have two possibilities

1. $X_g(a)$ is transversal to ξ_a, equivalently $g(a) \neq 0$,
2. $X_g(a)$ is tangent to ξ_a, equivalently $g(a) = 0$.

3.2.1 Case 1

As earlier normalization of the field X_g can be expressed as the the equation

$$X_{g_t}(\lambda_t) - X_1(g_t)\lambda_t - \dot{g}_t = 0, \tag{3.2.3}$$

subject to the initial conditions

$$\lambda_t(a) = 0, \quad d\lambda_t|_{\xi_a} = 0, \tag{3.2.4}$$

where g_t is a path connecting our initial generating function and a desired h.

Since the diffeomorphism, corresponding to X_{λ_t} should preserve X_{p_1}, we have that

$$\{\lambda_t, p_1\} = 0.$$

Thus the function λ_t is independent of q_1.
Consider the projection

$$\pi : M \to \overline{M},$$

where $\overline{M}$ denotes the quotient of M by the action of X_f.
On $\overline{M}$ we have

$$\overline{X}_{g_t}(\lambda_t) = X_1(g_t)\lambda_t + \dot{g}, \tag{3.2.5}$$

where,

$$\overline{X}_{g_t} = \pi_*(X_{g_t}).$$

We have from (3.2.5) that

$$g_t(a)C = \dot{g}(a), \tag{3.2.6}$$

where $C = X_1(\lambda_t)(a)$.

Let Γ_t be a hyper-surface in $\overline{M}$ such that $T_a\Gamma_t$ is transversal to $X_{g_t}(a)$. Consider the initial conditions

$$\lambda_t|_\Gamma = \lambda_t^0,$$

where λ_t^0 satisfies (3.2.4).
Again this function is free for us to choose, and each choice gives a unique solution λ_t.
Consider now the path g_t of the form

$$g_t = e^t g(a) + (1-t)g_0.$$

Thus X_{g_t} is locally equivalent to $-X_{p_1q_1} \pm X_1$.
However since linear contact transformations preserve X_{q_1,p_1} we see that there exists a contact transformation fixing the horizontal direction.
Thus we have found coordinates $(q_1, \ldots, q_n, u, p_1, \ldots, p_n)$ such that

$$X_g = X_1 - X_{p_1q_1}.$$

Summarizing we have shown the following

Theorem 6. *Let ρ be a contact representation of the non-abelian Lie algebra of the dimension 2, and let ρ have a two dimensional orbit, and the representation of the derived sub-algebra belongs to the contact distribution at the point a. Then any representations of this type are locally contact equivalent.*

Corollary 4. *Let ρ be a representation of $\mathfrak{g}$ satisfying the above conditions. Then there exists a basis X_f, X_g of $\mathfrak{g}$ and a local canonical coordinates $(q_1 \ldots, q_n, u, p_1, \ldots, p_n)$ in a neighborhood of a such that*

$$X_f = -\frac{\partial}{\partial q_1},$$
$$X_g = q_1\frac{\partial}{\partial q_1} + \frac{\partial}{\partial u} - p_1\frac{\partial}{\partial p_1}.$$

3.2.2 Case 2

It is important to note that in dimension 3 this case is not possible, due to the following. Let X_f, X_g be two contact vector field that are involution and tangent to ξ at a. Since ξ is non-integrable, we have that $[\xi,\xi] \notin \xi$. Thus X_f, X_g cannot define a two-dimensional orbit.
Assume that $n \geq 2$, since λ_t is independent of q_1 we can consider (3.2.3) as an equation in dimension $2n$. That is consider the projection

$$\pi : M \to \overline{M},$$

where $\overline{M}$ denotes the quotient of M by the action of X_f. On $\overline{M}$ we have

$$\overline{X}_{g_t}(\lambda_t) = X_1(g_t)\lambda_t + \dot{g}, \tag{3.2.7}$$

where,

$$\overline{X}_{g_t} = \pi_*(X_{g_t}).$$

and π denotes the projection onto the orthogonal compliment of $X_f(a)$.
On this space we have from (3.2.7) that

$$\dot{g}(a) = 0 \tag{3.2.8}$$

Let Γ_t be a hyper-surface in $\overline{M}$ such that $T_a\Gamma_t$ is transversal to $X_{g_t}(a)$. Then we impose the initial conditions restricted to Γ_t

$$\lambda_t|_\Gamma = \lambda_t^0,$$

where λ_t^0 satisfies the initial data. Construct again a hyper-surface Γ_t and non-zero a path connecting X_{g_0} with X_{p_2}. Thus we have found coordinates $(q_1, \ldots, q_n, u, p_1, \ldots, p_n)$ such that

$$X_g = cX_{p_1} - X_{p_1q_1} + X_{p_2}.$$

Summarizing we have shown the following

Theorem 7. *Let ρ be a contact representation of the non-abelian Lie algebra of the dimension 2, and let ρ have a two dimensional orbit, belongs to the contact distribution at the point a. Then any representations of this type are locally equivalent.*

Corollary 5. *Let ρ be a representation of $\mathfrak{g}$ satisfying the above conditions, then there exists a basis X_f, X_g in $\mathfrak{g}$ and a local canonical coordinates $(q_1 \ldots, q_n, u, p_1, \ldots, p_n)$ in a neighborhood of a such that*

$$\begin{aligned} X_f &= -\frac{\partial}{\partial q_1} \\ X_g &= q_1\frac{\partial}{\partial q_1} - p_1\frac{\partial}{\partial p_1} - \frac{\partial}{\partial q_2} \end{aligned}$$

Analogously for the commutative algebra we have

Theorem 8. *Let ρ be a representation of the non-abelian Lie algebra of the dimension 2, and let ρ have a two dimensional orbit, and the representation is tangent to the contact distribution at the point a. Then any other representation of this type is locally equivalent.*

Corollary 6. *Let ρ be a representation of $\mathfrak{g}$ satisfying the conditions above, then there exists a basis X_f, X_g of $\mathfrak{g}$ and a local coordinates $(q_1 \ldots, q_n, u, p_1, \ldots, p_n)$ in a neighborhood of a such that*

$$\begin{aligned} X_f &= -\frac{\partial}{\partial q_1} \\ X_g &= -\frac{\partial}{\partial q_2} \end{aligned}$$

3.3 Differential invariants

In this section, we find the second order differential invariants of the normal forms found in the previous sections, thus finding contact equivalences of second order differential equations solvable by quadratures. For a complete exposition on this method, we refer the reader to [CtB].

Table 3.1: Differential invariants

$\mathfrak{g}$	X	Y	$n-th$ order invariant	$n-th$ order ODE
$[X,Y]=0$	$\frac{\partial}{\partial y}$	$\frac{\partial}{\partial x}$	$I_n = y_n$	$F(I_1,\ldots,I_n)=0$
$[X,Y]=X$	$\frac{\partial}{\partial y}$	$-\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+y_1\frac{\partial}{\partial y_1}$	$I_n = y_n e^x$	$F(I_1,\ldots,I_n)=0$
$[X,Y]=X$	$-\frac{\partial}{\partial x}$	$x\frac{\partial}{\partial x}+(y+1)\frac{\partial}{\partial y}-y_1\frac{\partial}{\partial y_1}$	$I_n = \frac{y_n}{y+1}$	$F(I_1,\ldots,I_n)=0$

Local structure of 2 dimensional solvable Lie algebra actions on the plane

Christian O'Cadiz Gustad

Abstract

In this paper the local classification of 2-dimensional solvable Lie algebra action on the plane is given. Normal forms of such actions are found.

4.1 Introduction

In this paper we investigate equivalence classes of 2 dimensional solvable Lie algebras acting on the plane.
Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$, $\dim \mathfrak{g} = n$ and let $\rho : \mathfrak{g} \to \mathcal{D}(\mathbb{R}^m)$ be a representation of this algebra into the Lie algebra of vector fields on $\mathbb{R}^m$ such that $\ker \mathfrak{g} = 0$. We say that two representations ρ_1, ρ_2 are *locally equivalent* at a point a, if there exist a local diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ where $\phi(a) = a$, such that $\rho_2 = \phi_* \circ \rho_1$.
Representatives of equivalences classes are called *normal forms*.
Sophus Lie in [SlRM] described finite dimensional Lie algebras acting on the plane. From this classification one can extract (See below Theorem 11) the normal form of the locally transitive action of the 2-dimensional solvable Lie algebra on the plane. Singular, or non-transitive actions of Lie algebras(and Lie groups) is still "Terra incognita". Mainly, they concern to actions of Lie algebras and groups with fixed point. Thus, for actions of compact Lie groups, we have the following result

Theorem 9 (E.Cartan)**.** *Let G be a compact Lie group, which acts on the manifold M in such a way that $g(a) = a$ for all $g \in G$. Then there exists local coordinates such that the action will be linear.*

R. Hermann [HerF] proved that actions of semi-simple Lie algebras in a neighborhood of a fixed point can be linearized on the formal level. Later S.Sternberg and V. Guillem [SsV] proved that this result can not be extended to the analytical case.
For non semi-simple Lie algebras there is the classical S. Sternberg theorem on linearization [SSlh]: Let λ_i be the eigenvalues of the linear part of X_i at zero, i.e. the matrix $||\frac{\partial X_i}{\partial x_j}(0)||$. We say our system has *resonance* ,if there exist an eigenvalue λ_i such that:

$$\lambda_i = \sum_j m_{ij} \lambda_j,$$

where m_{ij} are non-negative integers and $\sum_j m_{ij} \geq 2$.

Theorem 10 (Sternberg)**.** *Let X be a vector field, of the form:*

$$X = \sum_i X_i(x_1, \ldots, x_i) \partial_{x_i},$$

where $X_i(0) = 0$ for all i. If the vector field X has no resonances then there are local coordinates in which X has the linear form

$$X = \sum_i \lambda_i x_i \partial_{x_i}.$$

For several Lie algebras V. Lychagin proposed in [LycSS], [Ly2] some spectral sequences which give formal classifiaction, and formal normal forms in a neighborhood of a singular orbit.
In this paper we analyze in details the case of 2 dimensional non-abelian solvable algebras.
Let $\mathcal{O}(a)$ denote the $\mathfrak{g}$-orbit of the point $a \in \mathbb{R}^2$. We split our consideration into the three cases:

1. $\dim \mathcal{O}(a) = 2$.
   This is the classical case where the action is transitive.

2. $\dim \mathcal{O}(a) = 1$.
   In this case we will call the action weak singular.

3. $\dim \mathcal{O}(a) = 0$.
   In this case we will call the action singular.

Throughout this paper we pick coordinates in such a way that the point under consideration is at the origin.

4.2 Transitive Action

Let $\mathfrak{g}$ be a 2-dimensional non-abelian Lie algebra over $\mathbb{R}$, and let X, Y be a basis in $\mathfrak{g}$ such that

$$[X, Y] = X.$$

We will use the same notations X, Y for the images $\rho(X)$ and $\rho(Y)$.

Assume that the action is transitive at the point $a \in \mathbb{R}^2$, or in other words, assume that the vectors $X_a, Y_a \in T_a\mathbb{R}^2$ are linear independent.

Theorem 11 (Sophus Lie)**.** *Let the solvable non-abelian Lie algebra $\mathfrak{g}$, $\dim \mathfrak{g} = 2$, act transitively in a neighborhood of $O \subset \mathbb{R}^2$. Then there are local coordinates (x, y) such that*

$$X = \partial_x,$$
$$Y = x\partial_x + \partial_y.$$

Proof. First choose coordinates such $X = \partial_x$. Let

$$Y = \alpha(x, y)\partial_x + \beta(x, y)\partial_y.$$

In these coordinates, the commutator relation gives

$$\alpha_x\partial_x + \beta_x\partial_y = \partial_x.$$

Therefore there is a local diffeomorphism $\Theta : (x, y) \to (x, f(y))$ such that

$$\Theta_*(Y) = x\partial_x + \partial_y.$$

□

4.3 Weak singular action

In this section we investigate the case when $\dim \mathcal{O}(a) = 1$.

Let $\mathcal{O}^{(1)}(a)$ denote the $\mathfrak{g}^{(1)}$-orbit at the point a. We split our investigation into the two cases:

1. The orbit of the derived subalgebra is singular i.e. $\dim \mathcal{O}^{(1)}(a) = 0$,
2. The orbit of the derived subalgebra is a curve i.e. $\dim \mathcal{O}^{(1)}(a) = 1$.

Since $\dim \mathcal{O}(a) = 1$, one of the vectors $X_a, Y_a \in T_a\mathbb{R}^2$ is nonzero. Therefore we can choose coordinates (x, y) in a neighborhood of $a \in \mathbb{R}^2$, such that the corresponding vector field equals to ∂_x.
We need the following lemma which describes the local behavior of vector fields on the line $\mathbb{R}$.

Lemma 16. *Let $X = b(x)\partial_x$ be a vector field on $\mathbb{R}$, such that $X(0) = 0$. Then there exists a local diffeomorphism $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(0) = 0$, such that $\phi_*(X)$ has one of the following forms*

- $\lambda x\partial_x$,
  if $b(x)$ has a zero of order 1 at 0,
- $x^k\partial_x$ *if $b(x)$,*
  has a zero of order k at 0 where k is even,
- $\pm x^k\partial_x$
  if $b(x)$, has a zero of orderk at 0 where k is odd,
- $b(x)\partial_x$,
  if $b(x)$, is a flat at 0.

Proof. See for example, [HaO]. □

4.3.1 Non singular orbit of the derived subalgebra

In this section we assume, that $\dim \mathcal{O}^{(1)}(a) = 1$, or that $X_a \neq 0$. Then we can choose local coordinates (x, y) in such a way that $X = \partial_x$, in a neighborhood of the point $a \in \mathbb{R}^2$. Then

$$Y = \alpha\partial_x + \beta\partial_y,$$

in these coordinates, and the commutator relation $[X, Y] = X$ gives the system of differential equations on the functions α and β:

$$\alpha_x = 1 \qquad \beta_x = 0.$$

Therefore, we can assume that in these coordinates:

$$\alpha = x \qquad \beta_x = b(y).$$

Note that transformations of the form

$$(x, y) \to (x, Y(y)),$$

do not change the form of X, and in Y they act on the vector field $b(y)\partial_y$. Therefore, applying lemma 16 we get the theorem:

Theorem 12. *Let $\mathfrak{g}$ act in such a way, that $\dim \mathcal{O}(a) = 1$ $\dim \mathcal{O}^{(1)}(a) = 1$. then there are local coordinates (x, y) at a neighborhood of the point $a \in \mathbb{R}^2$, such that*

$$X = \partial_x,$$

and the vector field Y has one of the following forms:

$$\begin{array}{ll} 1. & x\partial_x + \lambda y\partial_y, \\ 2. & x\partial_x + y^p\partial_y, \\ 3. & x\partial_x \pm y^q\partial_y, \\ 4. & x\partial_x + b(y)\partial_y, \end{array}$$

where p, q are natural numbers, $p \geq 2, q \geq 3$, and $b(y)$ is a flat function at the point 0.

4.3.2 Singular orbit of the derived subalgebra

Consider now the case, when $\dim \mathcal{O}(a) = 1$, and $\dim \mathcal{O}^{(1)}(a) = 0$ i.e. the case when $Y_a \neq 0$, but $X_a = 0$.
Then there are local coordinates (x, y) such that $Y = \partial_x$. The commutator relation $[X, Y] = X$ can be rewritten for the functions α, β, when

$$X = \alpha(x, y)\partial_x + \beta(x, y)\partial_y,$$

as the following system of differential equations:

$$\alpha_x = -\alpha \qquad \beta_x = -\beta.$$

Solving these equations we get

$$X = e^{-x}\left(\alpha(y)\partial_x + \beta(y)\partial_y\right),$$

where $\alpha(0) = \beta(0) = 0$.
Again we apply lemma 16 on the vector field $\beta(y)$ and arrive at the theorem:

Theorem 13. *Let $\dim \mathcal{O}(a) = 1$ and $\dim \mathcal{O}^{(1)}(a) = 0$. Then there are local coordinates (x, y) in a neighborhood of the point $a \in \mathbb{R}^2$, such that*

$$Y = \partial_x,$$

and the vector field X has one of the following forms:

$$\begin{array}{ll} 1. & e^{-x}(\alpha(y)\partial_x + \lambda y \partial_y), \\ 2. & e^{-x}(\alpha(y)\partial_x + y^p \partial_y), \\ 3. & e^{-x}(\alpha(y)\partial_x \pm y^q \partial_y), \\ 4. & e^{-x}(\alpha(y)\partial_x + \beta(y)\partial_y), \end{array}$$

where $\alpha(y)$ is an arbitrary function, $\alpha(0) = 0$, $\lambda \neq 0$ p, q are natural numbers $p \geq 2, q \geq 3$ and β is a flat function at 0.

4.4 Singular action

In this section we investigate the case when $\mathcal{O}(a) = a$, or when $X_a = Y_a = 0$. The general procedure will be divided on the three steps:

1. Find restrictions on the linear term of a representation from the commutator relation $[X, Y] = X$.
2. The commutator relation gives us a differential equation on the coefficients of the vector fields X, Y, which we investigate formally, under the condition that the vector field Y has no resonances.
3. Investigate when the formal solution can be extended to a smooth solution.

Let A, B be the linear parts of X, Y at the point a respectively i.e. $A = [X]^1_a, B = [Y]^1_a$ the 1-st jets of ad$X$ and ad$Y$ at a.
The commutator relations $[X, Y] = X$ of the vector fields, gives the following commutation relation of operators $[A, B] = A$.
We assume that $B \neq 0$ and split our investigation into the two cases:

1. The representation of X has a non-vanishing first jet at a i.e.$A \neq 0$.
2. The representation of X has vanishing first jet at a i.e. $A = 0$.

4.4.1 The vector field X has a non-vanishing first jet

We need the following version of the Lie Theorem on representations of solvable Lie algebras.

Proposition 16. *Let $A = [X]^1_a \neq 0,\ B = [Y]^1_a \neq 0$. Then there is a basis of $T_a\mathbb{R}^2$, such that*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} \lambda - 1 & 0 \\ 0 & \lambda \end{bmatrix},$$

where $\lambda \in \mathbb{R}$.

Proof. Since B is a real operator, we have three possibilities for B:

1. Eigenvectors of the operator B form a basis of $T_a\mathbb{R}^2$.
2. The operator B has complex eigenvalues.
3. The operator B has one real eigenvalue.

Consider the first case.
Choose a basis e_1, e_2 that are eigenvectors for the operator B i.e.

$$Be_1 = \lambda_1 e_1 \qquad Be_2 = \lambda_2 e_2.$$

The commutator relation $[A, B] = A$ acting on e_1 gives us:

$$[A, B]e_1 = \lambda_1 Ae_1 - BAe_1 = Ae_1,$$
$$BAe_1 = A(\lambda_1 - 1)e_1.$$

For e_2 we have:

$$[A, B]e_2 = \lambda_2 Ae_2 - BAe_2 = Ae_2,$$
$$BAe_2 = A(\lambda_2 - 1)e_2.$$

Therefore, if Ae_1 and Ae_2 are non-zero vectors, they are eigenvectors for the operator B, with eigenvalues $(\lambda_1 - 1)$ and $(\lambda_2 - 1)$ respectively.
This implies that $\lambda_1 = \lambda_2 - 1$ and $\lambda_2 = \lambda_1 - 1$. This is a contradiction and the condition that $A \neq 0$, show that either $Ae_1 = 0$, or $Ae_2 = 0$.

Let us say that $Ae_1 = 0$. The commutator relation shows that $tr(A) = 0$. Therefore $Ae_2 = e_1$ and

$$\lambda_1 - \lambda_2 = 1.$$

Consider the second case.

Then the complexification $B^{\mathbb{C}}$ of the operator B has the eigenvector basis $e, \overline{e} \in T_a^{\mathbb{C}}\mathbb{R}^2$. Then we have that

$$B^{\mathbb{C}}(e) = \lambda e \qquad B^{\mathbb{C}}(\overline{e}) = \overline{\lambda}\overline{e}.$$

Then, similarly to the case above, we get that $Ae = \overline{e}$ and $A\overline{e} = 0$, but $A\overline{e} = \overline{Ae} = e$. This contradiction shows that the eigenvalues of B are real.

Finally, Assume that the operator B has a Jordan matrix and let B act on e_1 and e_2 in the following way:

$$Be_1 = \lambda e_1, \qquad Be_2 = \lambda e_2 + e_1.$$

The commutator relation $[A, B] = A$ acting on e_1 gives that

$$\begin{aligned} BAe_1 &= ABe_1 - Ae_1, \\ &= (\lambda - 1)Ae_1, \end{aligned}$$

and

$$\begin{aligned} BAe_2 &= ABe_2 - Ae_2 = A(\lambda e_2 + e_1) - Ae_2, \\ &= (\lambda - 1)Ae_2. \end{aligned}$$

But $\lambda - 1 \neq \lambda$ and one of the vectors Ae_1 or Ae_2 is nonzero. This contradiction proves the proposition □

Remark 15. *Operator B has two eigenvalues λ_1 and λ_2, where $\lambda_1 - \lambda_2 = \pm 1$. In the previous proposition we let λ denote the eigenvalue of the vector which does not belong to* $\ker A$

From this we get the corollary:

Corollary 1. *There are local coordinates (x, y) in the neighborhood of the point $a \in \mathbb{R}^2$, in which the vector fields X and Y have the following form:*

$$\begin{aligned} X &= x\partial_y + \alpha(x, y)\partial_x + \beta(x, y)\partial_y \\ Y &= (\lambda - 1)x\partial_x + \lambda y\partial_y + \tilde{\alpha}(x, y)\partial_x + \tilde{\beta}(x, y)\partial_y, \end{aligned}$$

where functions $\alpha, \tilde{\alpha}, \beta, \tilde{\beta}$ have zeros of second order at 0 i.e.

$$\alpha(0) = \tilde{\alpha}(0) = \beta(0) = \tilde{\beta}(0) = 0,$$

and

$$d_0\alpha(0) = d_0\tilde{\alpha}(0) = d_0\beta(0) = d_0\tilde{\beta}(0) = 0,$$

Resonance conditions

In this subsection, we investigate the resonance conditions for the vector field Y, in order to apply the S.Sternber linearization theorem.

Theorem 14. *Let $\lambda - 1$ and λ be the eigenvalues of the linear part of the operator $B = [Y]^1_a$, and let*

$$\lambda \notin \left(\mathbb{Q}[0,1] \cup \{1 + \frac{1}{n} | n \in \mathbb{N}\} \cup \{-\frac{1}{q} | q \in \mathbb{N}\}\right)$$

Then there are local coordinates (x, y) in a neighborhood of the point $a \in \mathbb{R}^2$, such that

$$Y = (\lambda - 1)x\partial_x + \lambda y \partial_y$$

Proof. The result will follow form the Sternber linearization theorem. We show that the conditions on λ are exactly conditions under which the operator B has no resonances. Thus, we analyze the resonance conditions for B.
Assume that

$$m_1\lambda + m_2(\lambda - 1) = \lambda,$$

where $m_1, m_2 \in \mathbb{Z}_+$ and $m_1 + m_2 \geq 2$.
From this equation we get that

$$\lambda = \frac{m_2}{m_1 + m_2 - 1}.$$

Therefore, λ should be a ration number. Let us put $\lambda = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ are coprime, and $q > 0$. Then,

$$\begin{aligned} m_2 &= kp, \\ m_1 + m_2 - 1 &= kq, \end{aligned}$$

for some $k \in \mathbb{Z}$.
Therefore the resonance conditions on λ are equivalent to the inequalities:

$$kp \geq 0, \quad kq \geq 1, \quad k(q - p) \geq -1.$$

The relations $kq \geq 1$ and $q > 0$ imply that $k > 0$, and therefore $p \geq 0$. Therefore we analyst the final inequality

$$k(q - p) \geq -1.$$

We have the following cases:

$$\begin{aligned} k(q - p) = -1 &\Longrightarrow k = 1, q - p = -1 \\ k(q - p) \geq 0 &\Longrightarrow q \geq p. \end{aligned}$$

This lead us to the following resonance set for λ:

- When $k = 1$, $\lambda = \frac{p}{q} = \frac{q+1}{q} = 1 + \frac{1}{q}$.
- When $q \geq p$, then $0 \leq \lambda = \frac{p}{q} \leq 1$.

We now consider the second resonance condition

$$n_1\lambda + n_2(\lambda - 1) = \lambda - 1,$$

where $n_1, n_2 \in \mathbb{Z}_+$ and $n_1 + n_2 \geq 2$.
From this equation we get that

$$\lambda = \frac{n_2 - 1}{n_1 + n_2 - 1}.$$

And the resonance conditions can be written as

$$\begin{aligned} n_2 &= lp, \\ n_1 + n_2 - 1 &= lq, \end{aligned}$$

for some $l \in \mathbb{Z}$.
Finally, the resonance conditions are equivalent to the following system of inequalities

$$\begin{aligned} n_2 &= lp + 1 \geq 0, \\ n_1 &= l(q-p) \geq 0, \\ n_1 + n_2 &= lq + 1 \geq 2. \end{aligned}$$

Alternatively

$$lq \geq 1, \quad lp \geq -1, \quad l(q-p) \geq 0.$$

Conditions, $lq \geq 1$ and $q > 0$ gives that $l > 0$, and therefore

$$lp \geq -1, \qquad q - p \geq 0$$

They give the following cases:

- $lp = -1$, or $l = 1, p = -1$, then $\lambda = \frac{p}{q} = -\frac{1}{q}$.
- $lp \geq 0$, or $l \geq 0, q \geq p$, then $0 \leq \lambda = \frac{p}{q} \leq 1$.

Therefore, the resonance values of λ for the second resonance condition, are rational number from the interval $[0, 1]$, or $\lambda = -\frac{1}{q}$, where $q \geq 2$.
The union of all these set, will give us all resonant values of λ. Discarding this, Sternberg's linearization conditions proves the theorem. □

Formal Solution

From now on we assume that the conditions in Theorem 14 holds, and therefore there are coordinates (x, y) in a neighborhood such that the vector field Y is linear.

Lemma 17. *In the coordinates (x, y), the vector field X has the following form.*

$$X = ky^{2q}\partial_x + x\partial_y,$$

where q is a natural number, $k \in \mathbb{R}$ and $\lambda = -\frac{2}{2q-1}$

Proof. Let

$$X = \alpha(x, y)\partial_x + \beta(x, y)\partial_y$$

where function α has a 2nd order zero at the point, $\beta_x = 1, \beta_y = 0$, and

$$Y = (\lambda - 1)x\partial_x + \lambda y\partial_y.$$

Then the commutator relation relation $[X, Y] = X$ gives the system of differential equation:

$$\begin{aligned} Y(\alpha) &= (\lambda - 2)\alpha, \\ Y(\beta) &= (\lambda - 1)\beta. \end{aligned} \tag{4.4.1}$$

The formal series for function α and β have the form:

$$\begin{aligned} \alpha &= \sum_{ij} a_{ij} x^i y^j \\ \beta &= x + \sum_{kl} b_{kl} x^k y^l, \end{aligned}$$

where $i, j, k, l \in \mathbb{Z}_+$ and $i + j \geq 2, k + l \geq 2$.

From (4.4.1) we get the following linear system of equations

$$\begin{aligned} ((\lambda - 1)i + \lambda j - \lambda + 2 = 0)a_{ij} = 0, \\ ((\lambda - 1)k + \lambda l - \lambda + 1 = 0)b_{kl} = 0. \end{aligned}$$

Assume that $a_{ij} \neq 0$ and $b_{kl} \neq 0$, some pairs (i,j) and (k,l). Then we get the equations on λ:

$$
\begin{aligned}
i(\lambda - 1) + j\lambda - \lambda + 2 = 0, \\
k(\lambda - 1) + \lambda l - \lambda + 1 = 0.
\end{aligned}
$$

Solving these equations gives

$$
\lambda = \frac{p}{q} = \frac{i-2}{i_1 + j - 1} = \frac{k-1}{k+l-1},
$$

and we should discard the solutions which give resonant λ.
Consider the equation of k, l. Here we have that

$$
\lambda = \frac{k-1}{k+l-1} = 1 - \frac{l}{k+l-1}.
$$

Therefore, $\lambda \in \mathbb{Q}[0,1]$ if $k \geq 1$, and $\lambda = -\frac{l}{l-1}$, when $k = 0$. Thus, we have no nontrivial solutions for non-resonant λ.
Consider the equation of i, j.
We have

$$
\lambda = \frac{i-2}{i+j-1} = 1 - \frac{j+1}{i+j-1}.
$$

Therefore $\lambda \in \mathbb{Q}[0,1]$ if $i \geq 2$.
For the case $i = 1$, we have

$$
\lambda = -\frac{1}{j}, \qquad j \geq 1,
$$

and for the case $i = 0$

$$
\lambda = -\frac{2}{j-1}, \qquad j \geq 2.
$$

Therefore, the only non-resonant λ, correspond to the case $i = 0$, $j = 2q$ and $\lambda = -\frac{2}{2q-1}$, where $q \in \mathbb{N}$. □

This lemma shows that on the formal level, we can transform vector fields X and Y to the following form:

$$
\begin{aligned}
X &= ky^{2q}\partial_x + x\partial_y, \\
Y &= \frac{1+2q}{1-2q}x\partial_x + \frac{2}{1-2q}y\partial_y.
\end{aligned}
$$

Thus we have proved the following theorem:

Theorem 15. *Let the eigenvalues $\lambda, \lambda - 1$ be non-resonant. Then there are local coordinates (x,y) in a neighborhood of the point $a \in \mathbb{R}^2$, such that ∞-jets of X and Y have the canonical form:*

1.

$$
X = x\partial_y, \qquad Y = (\lambda - 1)x\partial_x + \lambda y\partial_y.
$$

2.

$$
X = x\partial_x + y^{2q}\partial_x, \qquad Y = \frac{1+2q}{1-2q}x\partial_x + \frac{2}{1-2q}y\partial_y,
$$

where $q \in \mathbb{N}$.

Remark 16. *Assume that $k \neq 0$. We can then take a diffeomorphism of the form*

$$
\phi(x,y) \longrightarrow (tx, ty),
$$

where $t \neq 0$. This diffeomorphism preserves the vector field Y, and we pick t in such a that we can assume $k = 1$.

4.4.2 The vector field X has a vanishing first jet

In this section we start to investigate the case, when $A = [X]_a^1 = 0$, but $B = [Y]_a^1 \neq 0$. Then for the operator B we the following possibilities:

- Operator B is diagonalizable and has real eigenvalues λ_1, λ_2.
- Operator B has complex eigenvalues $\lambda, \overline{\lambda}$.
- Operator B has eigenvalue λ with multiplicity 2.

We now treat all these cases separately and use the same methods as applied in the previous section.

Operator B is a Jordan form

Let us choose local coordinates (x, y) in such a way, that the operator B takes the Jordan form

$$B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

in the basis ∂_x, ∂_y of $T_a\mathbb{R}^2$.
The resonance conditions for vector field Y has the form:

$$\lambda(m_1 + m_2 - 1) = 0.$$

Since $m_1 + m_2 \geq 2$, we have resonance when $\lambda = 0$ only.
Assuming $\lambda \neq 0$, we can apply the Sternberg theorem to vector field Y and get the following representations of vector fields X and Y:

$$\begin{aligned} X &= \alpha(x, y)\partial_x + \beta(x, y)\partial_y, \\ Y &= \lambda(x\partial_x + y\partial_y) + x\partial_y. \end{aligned}$$

Here α, β have second order zeroes at 0.
The commutator relation $[X, Y] = X$ gives us the following system of equations for functions α, β:

$$\begin{aligned} Y(\alpha) &= (\lambda - 1)\alpha, \\ Y(\beta) &= (\lambda - 1)\beta + \alpha. \end{aligned} \tag{4.4.2}$$

Consider first formal solutions of the system.
Let

$$\begin{aligned} \alpha &= \sum_{ij} a_{ij} x^i y^j, \\ \beta &= \sum_{kl} b_{kl} x^k y^l, \end{aligned}$$

be the formal series with $i + j \geq 2$ and $k + l \geq 2$.
It is easy to check that:

$$Y(x^i y^j) = \lambda(i + j)x^i y^j + jx^{i+1}y^{j-1}.$$

Therefore ,

$$Y(\alpha) - (\lambda - 1)\alpha = \sum_{i+j\geq 2} \left[\lambda(i + j - 1)a_{ij} + a_{ij} + (j + 1)a_{i-1\,j+1}\right] x^i y^j,$$

and

$$Y(\beta) - (\lambda - 1)\beta - \alpha = \sum_{k+l\geq 2} \left[\lambda(k + l - 1)b_{kl} + b_{kl} + (j + 1)b_{k-1\,l+1} - a_{kl}\right] x^k y^l.$$

Thus, the system of differential equation (4.4.2), on the formal level, is equivalent to the following system of linear equations for coefficients a_{ij}, b_{kl}:

$$\begin{aligned} (\lambda(i + j - 1) + 1)a_{ij} + (j + 1)a_{i-1\,j+1} &= 0, \\ (\lambda(k + l - 1) + 1)b_{kl} + (l + 1)b_{k-1\,l+1} - a_{kl} &= 0, \end{aligned} \tag{4.4.3}$$

where i, j, k, l are natural numbers such that $i + j \geq 2$ and $k + l \geq 2$.
Let $i + j = n + 1$, where $n \geq 1$ is fixed.
Then the first part of the system (4.4.3) gives the following linear system for vector $||a_{ij}||$, where $i + j = n + 1$:

$$(n\lambda + 1)a_{ij} + (j + 1)a_{i-1\ j+1} = 0. \tag{4.4.4}$$

Therefore, if $n\lambda + 1 \neq 0$ we will only have a trivial solution of (4.4.4). Then taking $a_{ij} = 0$, we get only a trivial solution for b_{kl}.
Now let λ be a rational number of the form

$$\lambda = -\frac{1}{n},$$

where $n \geq 1$.
Then equation (4.4.4) has solution

$$a_{ij} = 0,$$

where $j \geq 1$ and $a_{i\ 0}$ is arbitrary.
Thus we have a non-trivial solution for α. If α is trivial, investigation of β will be analogous to this case.
Assume α non trivial.

We have that $j = 0$ and i is arbitrary. Equations for $b_{i,j}$, where $i + j = n + 1$, take the form

$$(j + 1)b_{i-1\ j+1} = a_{ij}.$$

Therefore,

$$b_{i-1\ j+1} = 0,$$

if $i \neq 2$ and $b_{1\ n}$ is arbitrary.
Thus, we have found the solutions

$$\alpha = k_1 x^{n+1} \qquad \beta = k_1 x^n y + k_2 x^{n+1},$$

in the case when $n\lambda + 1 = 0$.
Summarizing we get the following theorem:

Theorem 16. *Let $[X]_a^1 = 0$, and $B = [Y]_a^1$ has nonzero eigenvalues of multiplicity 2 and corresponds to the Jordan form. Then ∞-jets of X and Y at the point $a \in \mathbb{R}^2$ has the following form*

1. $$Y = \lambda(x\partial_x + y\partial_y) + x\partial_y,$$
   $$X = 0,$$

   if $n\lambda + 1 \neq 0$, for all $n \in \mathbb{N}$.

2. $$Y = \lambda(x\partial_x + y\partial_y) + x\partial_y,$$
   $$X = k_1 x^{n+1}\partial_x + (k_1 x^n y + k_2 x^{n+1})\partial_y,$$

   where $k_1, k_2, k_3 \in \mathbb{R}$ and $n\lambda + 1 = 0$ for some $n \in \mathbb{N}$.

Remark 17. *Take a diffeomorphism of the form:*

$$\phi : (x, y) \longrightarrow (tx, ty).$$

This will preserve the normal form of Y. If n is odd, we can pick such a t that ϕ will bring one of the nonzero constants k_1, k_2 to 1.
Similarly if n is even, we can pick such a t that ϕ will bring one of the nonzero constants k_1, k_2 to ± 1.

Operator B has complex eigenvalues

The resonance conditions for the vector field Y, will then be:

$$\Re(\lambda)(m_1 + m_2 - 1) = 0,$$
$$\Im(\lambda)(m_1 - m_2 - 1) = 0.$$

From this we see immediately that we have resonance, iff $\Re(\lambda) = 0$.
Consider the action of operator B on the complexification of $T_a\mathbb{R}^2$, and choose coordinates x, y such that the vectors

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y),$$
$$\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y),$$

are eigenvectors for B.

In these coordinates the vector fields X and Y have the form

$$X = \alpha(z, \overline{z})\partial_z + \overline{\alpha(z, \overline{z})}\partial_{\overline{z}},$$
$$Y = \lambda z\partial_z + \overline{\lambda}\overline{z}\partial_{\overline{z}},$$

where α is a complex function of second order at 0.
Viewing the commutator relation $[X, Y] = X$ as a differential equation on the function α we get the following equation:

$$Y(\alpha) = (\lambda - 1)\alpha. \tag{4.4.5}$$

Now we expand α through the formal series

$$\alpha = \sum_{kl} a_{kl} z^k \overline{z}^l,$$

where $k + l \geq 2$.
Then the equation (4.4.5) takes the form

$$\sum_{k+l\geq 2} (k\lambda + l\overline{\lambda})a_{kl} z^k \overline{z}^l = (\lambda - 1) \sum_{k+l\geq 2} a_{kl} z^k \overline{z}^l.$$

Therefore,

$$(k\lambda + l\overline{\lambda} - (\lambda - 1)a_{kl}) = 0,$$

and we get nontrivial solution iff

$$k\lambda + l\overline{\lambda} - (\lambda - 1) = 0 \tag{4.4.6}$$

for some natural numbers k, l, $k + l \geq 2$.
Let $\lambda = \lambda_0 + i\lambda_1$, where $\lambda_0 = \Re(\lambda)$, $\lambda_1 = \Im(\lambda) \neq 0$. Taking the real and imaginary parts of equation (4.4.7), we get the system

$$k\lambda_0 + l\lambda_0 + 1 = 0, \tag{4.4.7}$$
$$k\lambda_1 - l\lambda_1 - \lambda_1 = 0.$$

Since we consider the complex case, that is $\lambda_1 \neq 0$, we have that

$$k = l + 1.$$

Putting this relation into the first equation of the system (4.4.7), we get

$$2l\lambda + 1 = 0.$$

Summarizing, we get the following result;

Theorem 17. *Let $[X]^1_a = 0$, and $B = [Y]^1_a$ have complex eigenvalues $(\lambda, \overline{\lambda})$ where $\Im(\lambda) \neq 0$, $\Re(\lambda) \neq 0$. Then ∞- jets of X and Y can be written in one of the following normal forms:*

1.

$$Y = \lambda z\partial_z + \overline{\lambda}\overline{z}\partial_{\overline{z}},$$
$$X = 0,$$

If $2l\Re(\lambda) + 1 \neq 0$*, for all* $l \geq 1$.

2.

$$Y = \lambda z\partial_z + \overline{\lambda}\overline{z}\partial_{\overline{z}},$$
$$X = a(z, \overline{z})\partial_z + \overline{a(z, \overline{z})}\partial_{\overline{z}},$$

where

$$a(z, \overline{z}) = \alpha z|z|^{2l},$$

$\alpha \in \mathbb{C}\ 0$*, and* $2l\Re(\lambda) + 1 = 0$.

Remark 18. *A diffeomorphism which scales* z *and* $\overline{z}$ *by some complex number* w*, will preserve the normal form of* Y *and act on the arbitrary constant* α *in the way*

$$\alpha \longrightarrow \alpha|w|^{2l}.$$

Therefore, in the normal form (2)*, we can take a parameter* w*, so that*

$$|\alpha| = 1.$$

Operator B has real eigenvalues and is diagonalizeable

Let (λ_1, λ_2) be eigenvalues of operator B, and assume that this pair is not resonant. Then due to Sternberg linearization theorem, we can choose local coordinates (x, y) in a neighborhood of the point $a \in \mathbb{R}^2$ in such a way that:

$$X = \alpha(x, y)\partial_x + \beta(x, y)\partial_y,$$
$$Y = \lambda_1 x\partial_x + \lambda_2 y\partial_y,$$

where α and β are functions of second order.
Viewing the commutator relation $[X, Y] = X$ as a differential equation, we get the following system of equations:

$$Y(\alpha) = (\lambda_1 - 1)\alpha, \qquad (4.4.8)$$
$$Y(\beta) = (\lambda_2 - 1)\beta,$$

on the functions α and β.
Writing the formal series

$$\alpha = \sum_{i+j\geq 2} a_{ij}x^i y^j,$$
$$\beta = \sum_{k+l\geq 2} b_{kl}x^k y^l,$$

From the system (4.4.8) we get the following system of linear equation on coefficients a_{ij}, b_{kl}:

$$(i\lambda_1 + j\lambda_2 - \lambda_1 + 1)a_{ij} = 0, \qquad (4.4.9)$$
$$(k\lambda_1 + l\lambda_2 - \lambda_2 + 1)b_{kl} = 0,$$

where $i + j \geq 2$ and $k + l \geq 2$.

Define the following sets

$$\Sigma_1 = \{(i, j)|(i - 1)\lambda_1 + j\lambda_2 = -1, \ \text{where } i + j \geq 2\},$$
$$\Sigma_2 = \{(k, l)|k\lambda_1 + (l - 1)\lambda_2 = -1, \ \text{where } k + l \geq 2\}.$$

Then the following result holds:

Theorem 18. *Let $[X]_a^1 = 0$ and operator $B = [Y]_a^1$ has is diagonalizable with real eigenvalues which are not resonant.*
Then there is a local system coordinates (x, y) in a neighborhood of the point $a \in \mathbb{R}^2$, such that ∞ jet of X and Y at the point have the following normal forms;

$$Y = \lambda_1 x\partial_x + \lambda_2 y\partial_y,$$
$$X = \left(\sum_{(i,j)\in\Sigma_1} a_{ij}x^i y^j\right)\partial_x + \left(\sum_{(k,l)\in\Sigma_1} b_{kl}x^k y^l\right)\partial_y.$$

The function α has a non-vanishing second jet

In this section we investigate the case when the function α in the representation:

$$X = \alpha(x, y)\partial_x + \beta(x, y)\partial_y,$$
$$Y = \lambda_1 x\partial_x + \lambda_2 y\partial_y,$$

has non trivial second jet.

Theorem 19. *Let the linear terms of Y be diagonalizable, and let the function α have a nontrivial 2nd jet. Then there are local coordinates (x, y) such that ∞-jets of X and Y are one of the 7 following forms:*

#	$\mathfrak{g} = \langle X, Y\rangle$	*Restrictions*
1	$X = \left(\sum_{i+j=2} k_{ij}x^i y^j\right)\partial_x + \left(\sum_{i+j=2} c_{ij}x^i y^j\right)\partial_y$ $Y = -x\partial_x - y\partial_y$	$k_{ij}, c_{ij} \in \mathbb{R}$ $k_{ij} \neq 0$ *for some* i, j
2	$X = \left(k_1x^2 + k_2y^q\right)\partial_x + k_3xy\partial_y$ $Y = -x\partial_x - \frac{2}{q}y\partial_y$	$k_1 \neq 0, k_2, k_3 \in \mathbb{R}$, $q \geq 3$ *odd*
3	$X = k_1x^2\partial_x + k_2xy\partial_y$ $Y = -x\partial_x + \gamma y\partial_y$	$k_1 \neq 0, k_2 \in \mathbb{R}$ $\gamma \in \mathbb{R} - \left(\mathbb{Q}^+ \cup \mathbb{Q}_2^-\right)$ $\mathbb{Q}_2^- = \{-q \mid q \geq 2\} \cup \{-\frac{1}{q} \mid q \geq 2\}$
4	$X = k_1xy\partial_x(k_2y^2 + k_3x^m)\partial_y$ $Y = -\frac{2}{n}x\partial_x - y\partial_y$	$n \geq 3$ *odd*
5	$X = k_1xy\partial_y + k_2y^2\partial_y$ $Y = \gamma_1x\partial_x - y\partial_y$	$k_1 \neq 0, k_2 \in \mathbb{R}$ $\gamma_1 \in \mathbb{R} - \left(\mathbb{Q}^+ \cup \mathbb{Q}_2^- \cup \{-\frac{2}{n} \mid n \geq 3\}\right)$
6	$X = (k_1y^2 + k_2x^{n+1})\partial_x + k_3x^n y\partial_y$ $Y = \left(-\frac{1}{n}x\partial_x - \frac{1+n}{2n}y\partial_y\right)$	$k_1 \neq 0, k_2, k_3 \in \mathbb{R}$ n *even*
7	$X = k_1y^2\partial_x + k_2x^n\partial_y$ $Y = -\frac{3}{2n-1}x\partial_x - \frac{n+1}{2n-1}y\partial_y$	$k_1 \neq 0, k_2 \in \mathbb{R}$, $n \geq 3 \in \mathbb{N} - \mathbb{N}_3$ $\mathbb{N}_3 = \{k \mid k = 3p - 1, p \geq 2\}$

Proof. We have that:

$$X = \alpha(x, y)\partial_x + \beta(x, y)\partial_y,$$
$$Y = \lambda_1 x\partial_x + \lambda_2 y\partial_y.$$

The formal series for α and β has the form:

$$\alpha = \sum_{i+j\geq 2} a_{ij}x^i y^j,$$
$$\beta = \sum_{i+k\geq 2} b_{kl}x^k y^l.$$

Since $a_{ij} \neq= 0$ for some $i + j = 2$, we split the investigation into the three cases:

1. $i = 2$ and $j = 0$,
2. $i = j = 1$,
3. $i = 0$ and $j = 2$.

In each case we assume that the S. Sternberg linearization condition on λ_1, λ_2 holds, such that Y can be brought to the linear form.

Case 1 In this case we have that $i = 2, j = 0$.
Assume that $l \neq 1$ then (4.4.9) gives that

$$\lambda_1 = -1,$$
$$\lambda_2 = \frac{k-1}{l-1}.$$

We now investigate the resonance condition for λ_1:

$$m_1 \frac{k-1}{l-1} - m_2 = -1,$$

where m_1, m_2 are non negative and $m_1 + m_2 \geq 2$.
It is easy to see that the only possibilities for k, l that give non resonant λ_1 are

- $k = 0,\ l = 2$,
- $k = 2,\ l = 0$.

Both these cases give that $\lambda_1 = \lambda_2 = -1$, which does not have any resonances. Thus we have the normal forms

$$X = k_1 x^2 \partial_x + (k_2 x^2 + k_3 y^2)\partial_y,$$
$$Y = -x\partial_x - y\partial_y,$$

where k_i are arbitrary reals.

We now investigate the case when $l = 1$.
From (4.4.9) we get that $k = 1$ and λ_2 is arbitrary. Therefore the resonant set of λ_2 is given by the system:

$$-m_1 + \lambda_2 m_2 = -1,$$
$$-n_2 + \lambda_2 n_2 = \lambda_2,$$

where $m_1 + m_2 \geq 2$ and $n_1 + n_2 \geq 2$.
Solving this system for λ_2 gives that

$$\lambda_2 = \frac{m_1 - 1}{m_2},$$
$$\lambda_2 = \frac{n_1}{n_2 - 1}.$$

It is easy to see that if

$$\lambda_2 \in \mathbb{Q}^+ \cup \mathbb{Q}_2^-,$$

where $\mathbb{Q}_2^- = \{-q | q \geq 2\} \cup \{-\frac{1}{q} | q \geq 2\}$, then the eigenvalues will be resonant. Thus we have the normal form:

$$X = k_1 x^2 \partial_x + k_2 xy\partial_y,$$
$$Y = -x\partial_x + \gamma y \partial_y,$$

where $\gamma \in \mathbb{R} - \left(\mathbb{Q}^+ \cup \mathbb{Q}_2^-\right)$ and k_1, k_2 are arbitrary reals.

Case 2 In this case we have that $i = j = 1$. Assume that $k \neq 0$ then (4.4.9) gives that

$$\lambda_2 = -1,$$
$$\lambda_1 = \frac{l-2}{k}.$$

The resonance conditions for λ_1, λ_2 will now have the form:

$$\lambda_1 = \frac{m_2}{m_1 - 1},$$
$$\lambda_1 = \frac{n_2 - 1}{n_1}.$$

Assume $k \neq 1, l \neq 1$ simultaneously. It is then easy to see that the only possibility for k, l that give a non-resonant λ_1 is when $l = 0$ and $k \geq 3$ is an odd number. Thus we have the normal form:

$$X = k_1 xy\partial_x + k_2 x^n \partial_y,$$
$$Y = -\frac{2}{n} x\partial_x - y\partial_y,$$

where k_1, k_2 are arbitrary reals. When $k = l = 1$ we have a special case where $\lambda_1 = \lambda_2 = -1$ which does not have resonance. Thus we have the normal forms:

$$X = xy(k_1\partial_x + k_2\partial_y),$$
$$Y = -x\partial_x - y\partial_y,$$

where k_1, k_2 are arbitrary reals.

We now treat the case when $k = 0$.
From (4.4.9) we get that λ_1 may be arbitrary and $l = 2$. The restriction on λ_1 will be analogous to the case when $i = 2$ $j = 0$ and $l = 1$. Thus we have the normal forms:

$$X = k_1 xy\partial_x + k_2 y^2 \partial_y,$$
$$Y = \gamma x\partial_x - y\partial_y,$$

where $\gamma \in \mathbb{R} - \left(\mathbb{Q}^+ \cup \mathbb{Q}_2^-\right)$ and k_1, k_2 are arbitrary reals.

Case 3 In this case we have that $i = 0$ and $j = 2$. From (4.4.9) we get that:

$$\lambda_1 = 2\lambda_2 + 1,$$
$$\lambda_2 = \frac{1+k}{1-2k-l}.$$

Therefore the resonant set of λ_1, λ_2 are given by the system:

$$\lambda_2 = \frac{1-m_1}{2m_1 + m_2 - 2},$$
$$\lambda_2 = \frac{n_1}{1 - 2n_1 - n_2}.$$

It is easy to see that we always have a solution to these equations, by the shifting the equation in the following way:

$$\lambda_2 = \frac{(m_1 - 1) + 1}{2(m_1 - 2) + (m_1 + 3) - 1},$$
$$\lambda_2 = \frac{(n_1 - 1) + 1}{2(n_1 - 1) + (n_2 + 2) - 1}.$$

This shows us that we can guarantee λ_1, λ_2 resonant for all k, l expect for the cases when:

1. $l = 2$ and $k = 0$,
2. $k \geq 1$ and $l = 1$,
3. $l = 0$ and $k \geq 2$.

These three cases must be investigated in detail.

Case 1 In this case we have $\lambda_1 = \lambda_2 = -1$, which is non-resonant. Thus we have the normal form:

$$X = y^2(k_1\partial_x + k_2\partial_y),$$
$$Y = -x\partial_x - y\partial_y,$$

where k_1, k_2 are arbitrary reals.

Case 2 Recall that $l = 1$ and $k \geq 1$. We begin by looking to (4.4.9) to see that

$$\lambda_2 = -\frac{1+k}{2k}.$$

We now investigate which values of k that give a resonant pair λ_1, λ_2.
Therefore we express the resonance conditions for λ_1 through k by the equation:

$$-\frac{1+k}{2k} = -\frac{m_1 - 1}{2m_1 + m_2 - 2}.$$

Solving this equation gives us:

$$2m_1 + (1+k)m_2 = 2.$$

Due to our restrictions on m_1, m_2 and k this equation will never have a solution.
Now we investigate resonance condition for λ_2 through the equation:

$$-\frac{1+k}{2k} = -\frac{n_1}{2n_1 + n_2 - 1}.$$

Solving this we arrive to the equation

$$2n_1 + (n_2 - 1)(1+k) = 0.$$

Due to our restrictions on n_1, n_2 and k, the only possibility we have to satisfy this equation is when $n_2 = 0$. This shows that we will have resonance when:

$$k = 2n_1 - 1,$$

for some $n_1 \geq 2$.
Thus we arrive at the normal form:

$$X = k_1 y^2\partial_x + k_2 x^n y\partial_y,$$
$$Y = -\frac{1}{n}(x\partial_x + \frac{(1+n)}{2}y\partial_y),$$

where $n \in (\mathbb{N}_{\text{even}} \cup \{1\})$ and k_1, k_2 are arbitrary reals.

Case 3 Recall that $l = 0$ and $k \geq 2$. We look to equation (4.4.9) and see that

$$\lambda_2 = -\frac{1+k}{2k-1}.$$

We now investigate which values of k that give a resonant pair λ_1, λ_2.
Therefore we express the resonance conditions for λ_1 through k by the equation:

$$-\frac{1+k}{2k-1} = -\frac{m_1 - 1}{2m_1 + m_2 - 2}.$$

Solving this equation gives us resonance when

$$3m_1 + (1+k)m_2 = 1.$$

Due to our restriction on k, m_1 and m_2, this equation will never have a solution.
We now investigate the resonance condition for λ_2

$$-\frac{1+k}{2k} = -\frac{n_1}{2n_1 + n_2 - 1}.$$

Solving this we arrive to the equation

$$3n_1 + (k+1)(n_2 - 1) = 0.$$

Due to our restriction on k, n_1 and n_2 we only have the possibility of $n_2 = 0$. This shows that we have resonance when

$$k = 3n_1 - 1.$$

for some $n_1 \geq 2$. Thus we have the normal forms:

$$\begin{aligned} X &= k_1 y^2 \partial_x + k_2 x^n \partial_y, \\ Y &= \frac{1}{2n-1}(3x\partial_x + (n+1)y\partial_y), \end{aligned}$$

where $n \in \mathbb{N} - (\{1\} \cup \mathbb{N}_3.\ \mathbb{N}_3) = \{k | k = 3n-1, n \geq 2\}$ and $k_1 k_2$ are arbitrary reals.

Superpositions Having found all these representation, we must take a superposition whenever they have the same eigenvalues in additions to finding higher order terms in the function α.We gather our previous results in the table below

Table 4.1: Normal forms

Case	(λ_1, λ_2)	Normal form of X	restrictions
1	$(-1,-1)$	$k_1 x^2 \partial_x + (k_2 x^2 + k_3 y^2)\partial_y$	none
2	$(-1, \gamma)$	$k_1 x^2 \partial_x + k_2 xy \partial_y$	$\gamma \in \mathbb{R} - \left(\mathbb{Q}^+ \cup \mathbb{Q}_2^-\right)$
3	$(-\frac{2}{n}, -1)$	$k_1 xy \partial_x + k_2 x^n \partial_y$	$n \geq 3$ odd
4	$(-1,-1)$	$xy(k_1 \partial_x + k_2 \partial_y)$	none
5	$(\gamma, -1)$	$k_1 xy \partial_x + k_2 y^2 \partial_y$	$\gamma \in \mathbb{R} - \left(\mathbb{Q}^+ \cup \mathbb{Q}_2^-\right)$
6	$(-1,-1)$	$y^2(k_1 \partial_x + k_2 \partial_y)$	none
7	$\left(-\frac{1}{n}, -\frac{n+1}{2n}\right)$	$k_1 y^2 \partial_x + k_2 x^n y \partial_y$	n even or equal 1
8	$\left(-\frac{3}{2n-1}, -\frac{n+1}{2n-1}\right)$	$k_1 y^2 \partial_x + k_2 x^n \partial_y$	$n \in \mathbb{N} - (\mathbb{N}_3 \cup \{1\})$

$\lambda_1 = \lambda_2 = -1$. Here all 8 cases will have some solution. We therefore take a superposition of every possibility to arrive at the normal form

$$\begin{aligned} Y &= -x\partial_x - y\partial_y, \\ X &= \left(k_1x^2 + k_2xy + k_3y^2\right)\partial_x + \left(k_4x^2 + k_5xy + k_6y^2\right)\partial_y, \end{aligned}$$

where k_i are arbitrary reals.
It is easy to see from the equation

$$-i - j = -2$$

that there will be no higher order terms for α. The normal form of Y is preserved under scalings by a real number in x and y. Therefore we can pick such a diffeomorphism that will bring some nonzero k_i to 1.

Case 2 With the exception of $\gamma = -1$, there are no intersection of these eigenvalues with any of the other cases.
It is easy to see from the equation

$$-i + \gamma j = -2,$$

that $\gamma = -\frac{2}{q}$, where q is an odd number will give us the normal forms:

$$\begin{aligned} Y &= -x\partial_x - \frac{2}{q}y\partial_y, \\ X &= (k_1x^2 + k_2y^q)\partial_x + k_3xy\partial_y, \end{aligned}$$

where $k_1 \neq 0, k_2, k_3$ are arbitrary reals,
and

$$\begin{aligned} Y &= -x\partial_x\gamma y\partial_y, \\ X &= k_1x^2\partial_x + k_2xy\partial_y, \end{aligned}$$

where $k_1 \neq 0, k_2$ are arbitrary reals and $\gamma \in \mathbb{R} - \left(\mathbb{Q}^+ \cup \mathbb{Q}_2^-\right)$.
Again it possible to scale x and y by some arbitrary nonzero real to bring some k_i to 1.

Case 3 Here we have an intersection with case 5 when

$$\gamma = -\frac{2}{n}.$$

It is easy to check that the equation:

$$-\frac{2}{n}i - j = -\frac{2}{n} - 1,$$

gives no higher order terms for α. Thus we have the normal form

$$\begin{aligned} Y &= -\frac{2}{n}x\partial_x - y\partial_y, \\ X &= k_1xy\partial_x + (k_2y^2 + k_3x^n)\partial_y, \end{aligned}$$

where n is an odd number greater than 1.
Again we may take some scaling in x and y to bring some nonzero k_i to 1.

Case 5 Assume that $\gamma \neq -1$ and $\gamma \neq -\frac{2}{n}$, we then look to the equation

$$\gamma i - j = \gamma - 1,$$

and find rational solution to this equation by setting $\gamma = -\frac{p}{q}$, where p, q are co-prime integers and $pq \geq 0$.
We can then write the equation as

$$pi + qj = p + q.$$

It is easy to check that this equation has no solution when $i + j > 2$. Therefore there are no higher order terms of α, and have the normal form

$$\begin{aligned} Y &= \gamma_1 x\partial_x - y\partial_y, \\ X &= k_1 xy\partial_x + k_2 y^2\partial_y, \end{aligned}$$

where $k_1 \neq 0, k_2$ are arbitrary reals and $\gamma_1 \in \mathbb{R} - \left(\mathbb{Q}^+ \cup \mathbb{Q}_2^- \cup -\frac{2}{n}|n \geq 3\right)$.
Again we can take some scaling in x and y to bring some nonzero k_i to 1.

Case 7 If $n \neq 1$ this does not intersect with any of the other cases. We look to the equation:

$$-\frac{1}{n}i - \frac{1+n}{2n}j = -n - 1,$$

where n is an even number. We rewrite this equation as

$$i + \frac{1}{2}j = n + 1.$$

It is easy to see that this only has the solution $i = n + 1$ when $i + j \geq 3$. Thus we have the normal form

$$\begin{aligned} X &= (k_1 y^2 + k_2 x^{n+1})\partial_x + k_3 x^n y\partial_y, \\ Y &= -\frac{1}{n}(x\partial_x + \frac{1}{2}(1+n)y\partial_y), \end{aligned}$$

where n is an even number and $k_1 \neq 0, k_2, k_3$ are arbitrary reals.
Again we may take some scaling in x and y to bring some nonzero k_i to 1.

Case 8 If $n \neq 2$ this gives us a unique case. We look to the equation

$$-\frac{3}{2n-1}i - \frac{n+1}{2n-1}j = -\frac{3}{2n-1} - 1,$$

and rewrite it as

$$\frac{3}{n+1}i + j = 2.$$

It is easy to see that this has no solutions for non-negative integers i, j. Thus we have the normal form:

$$\begin{aligned} X &= k_1 y^2\partial_x + k_2 x^n\partial_y, \\ Y &= \frac{1}{2n-1}(3x\partial_x + (n+1)y\partial_y), \end{aligned}$$

where $k_1 \neq 0, k_2$ are arbitrary reals and $n \in \mathbb{N} - (\mathbb{N}_3 \cup \{1\})$.
Again we may take some scaling in x and y to bring some nonzero k_i to 1. □

4.5 Smooth classification

Assume that we take coordinates (x, y) in which vector field Y is linearizable

$$Y = \lambda_1 x\partial_x + \lambda_2 y\partial_y,$$

with the condition $\lambda_1\lambda_2 > 0$.
Let then

$$X = \alpha(x, y)\partial_x + \beta(x, y)\partial_y,$$

be a representation of X.

Theorem 20. *Let the eigenvalues of $[Y]_a^1$ have real eigenvalues λ_1, λ_2 and let*

$$X = \alpha(x,y)\partial_x + \beta(x,y)\partial_y,$$

and

$$\tilde{X} = \alpha(\tilde{x,}y)\partial_x + \beta(\tilde{x,}y)\partial_y,$$

be two vector fields such that the commutator relations $[X, Y] = X$ and $[\tilde{X}, Y] = \tilde{X}$ hold for both. If

$$\lambda_1\lambda_2 > 0,$$

and the ∞-jet of functions (α, β) and $(\tilde{\alpha}, \tilde{\beta})$ coincide:

$$[\alpha]_a^\infty = [\tilde{\alpha}]_a^\infty, \qquad [\beta]_a^\infty = [\tilde{\beta}]_a^\infty.$$

Then $\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta}$ in a neighborhood of the point $a \in \mathbb{R}^2$

Proof. Assume α and $\tilde{\alpha}$ satisfy the differential equation

$$Y(\alpha) = (\lambda_1 - 1)\alpha,$$
$$Y(\tilde{\alpha}) = (\lambda_1 - 1)\tilde{\alpha}.$$

Therefore the difference

$$\epsilon = \alpha - \tilde{\alpha},$$

which is a flat function at the point $a \in \mathbb{R}^2$, also satisfies this equation.
Consider a trajectory

$$(x(t), y(t)) = \left(e^{\lambda_1 t}x_0, e^{\lambda_2 t}y_0\right).$$

of the vector field Y, and let

$$\phi(t) = \epsilon(x(t), y(t)),$$

be the restriction of the function ϵ on this trajectory.
Then we have

$$\dot{\phi} = (\lambda_1 - 1),$$

and therefore

$$\phi(t) = e^{(\lambda_1 - 1)t}\phi(0) = x(t)^{1 - \frac{1}{\lambda_1}}\phi(0).$$

Since $\lambda_1\lambda_2 > 0$ we can always approach the origin by letting $t \to \pm\infty$. We see that ϕ behaves as a power of x as we approach the origin, which contradicts that ϵ is flat. The case for β is analogous to this. $\square$

This shows that for $\lambda \notin [0, 1]$ in Theorem 15 and $\gamma, \gamma_1 \in \mathbb{Q}$ in Theorem 19 give smooth classification of normal forms.

However if $\lambda_1\lambda_2 < 0$, the trajectories of the representation, never approach the singularity. We see immediately that $x^{\lambda_1}y^{-\lambda_2}$(or $x^{-\lambda_2}y^{\lambda_1}$) is an invariant to this action. If we take a flat function f, we can always have a superposition of a flat solution $f(x^{\lambda_2}y^{-\lambda_1})$ as a flat solution to the commutator relation.

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